考试科目：《高等代数》（A 卷）

学年学期：**2020** 学年第 **1** 学期　　　　姓　　名：________

学　院/系：　　　　　　　　　　　　　　　学　　号：________

考试方式：闭卷　　　　　　　　　　　　　　年级专业：________

考试时长：**120** 分钟　　　　　　　　　　班　　别：________

警示

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-----------以下为试题区域，共 4 道大题，总分 100 分，考生请在答题纸上作答-----------

Notes: we use lowercase letter (e.g. a, b, c) to represent scalar, lowercase letter with arrow above (e.g. $\vec{a}, \vec{b}, \vec{c}$) to represent vector and uppercase letter (e.g. A, B, C) to represent matrix. rank(A) is the rank of the matrix A, $A^* = \text{adj}\ A$ is the adjugate matrix of A, $\det(A)$ is the determinant of A, and A^T is the transpose of A.

一、填空题（共 2 小题，第1小题 15 分，第2小题 6 分，共 21 分）

1. (15 分) If A is a 5×3 matrix with linearly independent columns, find each of these **explicitly**:

   (a) The nullspace of A, i.e., Nul A = ________.

   (b) The dimension of the nullspace of A^T, i.e., dim Nul A^T = ________.

   (c) One particular solution $\vec{x}_p$ to $A\vec{x}_p$ = column 2 of A is ________.

   (d) The general (complete) solution to $A\vec{x}$ = column 2 of A is ________.

   (e) The reduced row echelon form R of A is ________.

2. (6 分) If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$, please compute the determinants of the following matrices:

   (a) $\det(A)$ = ________.

   (b) Let O and I_3 be the 3×3 zero matrix and the 3×3 identity matrix, respectively. If $B = \begin{bmatrix} O & -A \\ I_3 & -I_3 \end{bmatrix}$, then $\det(B)$ = ________.

(c) If $C = \begin{bmatrix} A & -A \\ I_3 & -I_3 \end{bmatrix}$, then $\det(C)$ = ________.

二、选择题（共 2 小题，每小题 5 分，共 10 分）

1. (5 分) The columns of an $m \times n$ matrix A span $\mathbb{R}^m$ if and only if ______.

(A) The columns of A are linearly dependent.　　(B) The rows of A are linearly dependent.

(C) A has full row rank.　　(D) A has full column rank.

2. (5 分) Let N be a matrix whose columns are a basis for the null space of an $m \times n$ matrix A. Denote B as a matrix whose columns are a basis for the null space of N^T. If the rank of A is r, then ______.

(A) B is an $m \times (m-r)$ matrix.　　(B) B is an $m \times r$ matrix.

(C) B is an $n \times (n-r)$ matrix.　　(D) B is an $n \times r$ matrix.

三、计算题（共 4 小题，第1、2小题各 10 分，第3、4小题各 15 分，共 50 分）

1. (10分) Consider the following linear system and answer the questions.

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 + x_2 + bx_3 = 2 \\ ax_1 + bx_2 + (b-a)x_3 = 1 + 3a \end{cases}$$

(a) Choose a and b such that the system has *i*) no solution, *ii*) a unique solution, and *iii*) infinitely many solutions.

(b) Calculate the solution for $a = 2, b = 1$.

2. (10分) Let $A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Answer the following questions.

(a) Is the matrix equation $A\vec{x} = \vec{b}$ consistent? If the answer is "yes", please find the general solution. Otherwise, please find the least-squares solution.

(b) Please calculate the orthogonal projection of $\vec{b}$ onto Col A (i.e., the column space of A) and compute the distance from $\vec{b}$ to Col A.

3. (15分) Denote $\mathbb{P}_3$ as the vector space of polynomials of degree at most 3 with real coefficients. Let T be a mapping from $\mathbb{P}_3$ to $\mathbb{P}_3$ defined by

$$T(f) = \frac{d^2 f}{dx^2} + 2\frac{df}{dx}$$

for all $f(x) \in \mathbb{P}_3$.

(a) Show that T is a linear transformation.

(b) Find the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ for T relative to $\mathcal{B} = \{1, x, x^2, x^3\}$ and $\mathcal{C} = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$, respectively.

(c) Denote $P_{\mathcal{C} \leftarrow \mathcal{B}}$ as the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. Compute the change-of-coordinates matrices $P_{\mathcal{C} \leftarrow \mathcal{B}}, P_{\mathcal{B} \leftarrow \mathcal{C}}$ and verify that $[T]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[T]_{\mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{C}}$.

4. (15分) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Answer the following questions.

(a) The matrix A can be written as $A = I_3 + B$, where $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and I_3 is the 3×3 identity matrix. Find all the eigenvalues of A and B.

(b) If possible, orthogonally diagonalize A, i.e., find an orthogonal matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

(c) Write the vector $\vec{x}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a combination of eigenvectors of A, and compute the vector $\vec{x}_{100} = A^{100}\vec{x}_0$.

四、证明题（共 2 小题，第1小题 10 分，第2小题 9 分，共 19 分）

1. (10分) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.

(a) Prove that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

(b) Prove that $\text{rank}(AB) = \text{rank}(A)$ if and only if there is a $p \times n$ matrix X such that $ABX = A$.

2. (9分) If A is an $n \times n$ matrix of rank $n-1$, prove that $\text{rank}(A^*) = 1$ and the null space of A is the same as the column space of A^*, i.e., $\text{Nul}\, A = \text{Col}\, A^*$.