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高等代数期中模拟考试.

一、选择题.

1. False. If $\vec{u}$ and $\vec{v}$ are multiples, the Span$\{\vec{u}, \vec{v}\}$ is a line, and $\vec{w}$ need not be on that line.

2. True. For the transformation $\vec{x} \mapsto A\vec{x}$ to map $\mathbb{R}^5$ onto $\mathbb{R}^6$, the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.

3. False. Every equation $A\vec{x} = \vec{0}$ has the trivial solution whether or not some variables are free.

4. True. An $n \times n$ elementary matrix is obtained by a row operation on I_n.

5. False. A must be square in order to conclude from the equation $AB = I$ that A is invertible.

6. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\vec{x} = \vec{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $(C-D)$ satisfied the equation $B\vec{x} = \vec{0}$. Then $B(C-D) = 0$ and $BC = BD$.

7. True. $\det A^T A = (\det A)^2 \geq 0$

8. True $\det A^3 = 0 \Rightarrow (\det A)^3 = 0 \Rightarrow \det A = 0$.

9. True. If A is a square matrix and a multiply of one row of A is added to another row to produce a matrix B, the $\det B = \det A$.

10. False. Nonpivot columns need not to be linearly dependent as a subset of the matrix columns but nonpivot columns are linear combinations of the pivot columns.

二、填空题

11. $AB=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 9\\ -3 & k\end{bmatrix}=\begin{bmatrix}-7 & 18+3k\\ -4 & -9+k\end{bmatrix}$

while $BA=\begin{bmatrix}1 & 9\\ -3 & k\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}=\begin{bmatrix}-7 & 12\\ -6-k & -9+k\end{bmatrix}$

Then $AB=BA$ if and only if $18+3k=12$ and $-4=-6-k$, which happens if and only if $k=-2$.

12. It's easy to know that the change-of-coordinates matrix form I to B is $[\vec{b_1}\ \vec{b_2}]^{-1}$. And the B-cordinate vector of $\vec{x}$ is $[\vec{b_1}\ \vec{b_2}]^{-1}\vec{x}$.

13. Expanding along the first row:

$$\begin{vmatrix}3 & 0 & 4\\ 2 & 3 & 2\\ 0 & 5 & -1\end{vmatrix} = 3\begin{vmatrix}3 & 2\\ 5 & -1\end{vmatrix}+4\begin{vmatrix}2 & 3\\ 0 & 5\end{vmatrix} = 3(-13)+4(10)=1.$$

14. T is one-to-one

$\Leftrightarrow$ The columns of A are linearly independent.

$\Leftrightarrow$ A has n pivot columns.

15. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

$\Leftrightarrow$ The columns of A span $\mathbb{R}^m$.

$\Leftrightarrow$ A has a pivot position in each row.

$\Leftrightarrow$ A has m pivot columns.

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16. Consider the augmented matrix of corresponding system:

$$\begin{bmatrix} 2 & -4 & -2 & b_1 \\ -5 & 1 & 1 & b_2 \\ 7 & -5 & -3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & b_1 \\ 0 & -9 & -4 & \frac{5}{2}b_1+b_2 \\ 0 & 9 & 4 & b_3-\frac{7}{2}b_1 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & b_1 \\ 0 & -9 & -4 & \frac{5}{2}b_1+b_2 \\ 0 & 0 & 0 & b_2+b_3-b_1 \end{bmatrix}$$

The system is consistent if and only if $b_2+b_3-b_1=0$.

17. Since $A^{-1}B$ is the solution of $AX=B$, row reduction of $[A\ B]$ to $[I\ X]$ will produce $X=A^{-1}B$.

$$[A\ B] = \left[\begin{array}{ccc|cc} 1 & 3 & 8 & \cancel{-3}\,3 & 5 \\ 2 & 4 & 11 & 1 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & \overset{-2}{\cancel{10}} & -1 \\ 0 & 1 & 0 & 15\,\cancel{9} & 10 \\ 0 & 0 & 1 & -5 & -3 \end{array}\right] = [I\ X].$$

Thus $A^{-1}B$ is $\begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}$.

18. $$\det T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a, & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b).$$

19. $$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B \sim \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It's easily to see from B that:

Col A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$

Row A: $\{(1,3,4,-1,2),\ (0,0,1,-1,1),\ (0,0,0,0,-5)\}$.

It's easily to see from C that: $x_1=-3x_2-3x_4$, $x_3=x_4$, $x_5=0$. Thus:

Nul A: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

20. $A=\begin{bmatrix} -5 & 3 & 2 & -6 & -8 \\ 4 & 1 & 3 & -8 & -7 \\ 5 & 1 & 4 & 5 & 19 \\ -7 & -5 & -2 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The pivot columns of A form the basis for Col A: $\begin{bmatrix} -5 \\ 4 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} -8 \\ -7 \\ 19 \\ 5 \end{bmatrix}$.

For Nul A, solve $A\vec{x}=\vec{0}$ and get that:

$x_1=-x_3$, $x_2=x_3$, x_3 is free, $x_4=0$, $x_5=0$.

So $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_3\vec{u}$, and $\vec{u}$ is the basis for Nul A.

四. 证明题.

21. $(A-AX)^{-1}=X^{-1}B \Rightarrow X(A-AX)^{-1}=B$.

Since X, $A-AX$ are invertible, B is invertible.

22. Suppose that the subspace $H=\text{Span}\{\vec{v_1},\dots,\vec{v_5}\}$ is four demensional. If $\{\vec{v_1},\dots,\vec{v_5}\}$ were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of demension in Section 2.9, which essentially says that every basis of a p-demensional subspace consists of p vectors. Thus, $\{\vec{v_1},\dots,\vec{v_5}\}$ must be linearly dependent.