1\. (1) $\dfrac{1}{z(z^2+1)^2}$

$z=0$ 一级极点. $z=\pm i$, 二级极点.

(3). $\dfrac{1}{z^3-z^2-z+1}=\dfrac{1}{(z^2-1)(z-1)}=\dfrac{1}{(z+1)(z-1)^2}$

$z=-1$ 一级极点. $z=1$ 二级极点.

(5). $\dfrac{z}{(1+z^2)(1+e^{\pi z})}$

由 $1+e^{\pi z}=0$ 得 $z_k=(2k+1)i$ $(k=0,\pm1,\pm2,\pm3,\dots)$

由 $1+z^2=0$ 得 $z=\pm i$ 为 $(1+z^2)$ 的一级零点.

$(1+e^{\pi z})'\Big|_{z_k}=\pi e^{\pi z_k}=-\pi\neq0$. $\therefore z_k$ 为 $(1+e^{\pi z})$ 的一级零点.

$\therefore$ 综上, $\pm i$ 为二级极点. $z_k=(2k+1)i$, $(k=\pm1,\pm2,\pm3,\dots)$ 为一级极点.

(7). $\dfrac{1}{z^2(e^z-1)}$

~~$(e^z-1)'\Big|_{z=0}$~~ $z^2(e^z-1)=\left[\left(1+z+\dfrac{z^2}{2!}+\dots+\dfrac{z^n}{n!}+\dots\right)-1\right]\cdot z^2$

$=\sum\limits_{n=3}^{\infty}\dfrac{z^n}{(n-2)!}=z^3+\dfrac{z^4}{2!}+\dots+\dfrac{z^n}{(n-2)!}+\dots$

$\therefore z=0$ 为三级零点, $\therefore \underset{z=0}{}$ 为 $\dfrac{1}{z^2(e^z-1)}$ 的三级极点.

~~$e^z-1=z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^n}{n!}+\dots$~~

$e^z-1=0$ $z_k=0+2k\pi i$ $(k=\pm1,\pm2,\pm3,\dots)$

$(e^{z-1})'\Big|_{z_k}=e^{z_k-1}\neq0$.

$\therefore z=2k\pi i$ $(k=\pm1,\pm2,\pm3,\dots)$ 为一级极点.

(9). $\dfrac{1}{\sin z^2}$

~~$z=0$ 为 $\sin z^2$~~ $\sin z^2=0$ $z=\pm\sqrt{k\pi}$, $(k=0,1,2,\dots)$

$z=\pm i\sqrt{k\pi}$ $(k=0,1,2,\dots)$

$(\sin z^2)'\Big|_{z=\pm\sqrt{k\pi}\text{ 或 }z=\pm i\sqrt{k\pi}}=2z\cos z^2=\begin{cases}0, & k=0\\ \neq0, & k\neq0\end{cases}$

$(\sin z^2)''\Big|_{z=0}=-\sin z^2\cdot4z^2+2\cos z^2=2\neq0$ $\therefore z=0$ 为二级极点. $z=\pm\sqrt{k\pi}$, $\pm i\sqrt{k\pi}$ $(k=1,2,3,\dots)$ 为一级极点.

2. 存在解析函数 $\varphi(z)$ 使

$f(z)=(z-z_0)^m\varphi(z)$，且 $\varphi(z_0)\neq 0$.

$f'(z)=m(z-z_0)^{m-1}\varphi(z)+\varphi'(z)(z-z_0)^m$.

$=(z-z_0)^{m-1}\left[m\varphi(z)+(z-z_0)\varphi'(z)\right]$

$m\varphi(z)+(z-z_0)\cdot\varphi'(z)$ 当 $z=z_0$ 时，$m\varphi(z_0)\neq 0$，且解析.

$\therefore z=z_0$ 为 $f'(z)$ 的 $m-1$ 级零点.

3. $\operatorname{ch} z=\frac{e^z+e^{-z}}{2}$

$\operatorname{ch}\frac{\pi}{2}i=\frac{1}{2}\left[e^{\frac{\pi}{2}i}+e^{-\frac{\pi}{2}i}\right]=\frac{1}{2}(i-i)=0$

$(\operatorname{ch} z)'\Big|_{z=\frac{\pi}{2}i}=\operatorname{sh}\frac{\pi}{2}i=\frac{1}{2}\left[e^{\frac{\pi}{2}i}-e^{-\frac{\pi}{2}i}\right]=i\neq 0$

$\therefore z=\frac{\pi}{2}i$ 为一级零点.

4. 当 $z=0$ 时 $\sin z+\operatorname{sh} z-2z=0+\frac{1}{2}(e^0-e^0)-0=0$

$(\sin z+\operatorname{sh} z-2z)'\Big|_{z=0}=\cos z+\operatorname{ch} z-2\Big|_{z=0}=0$

$(\sin z+\operatorname{sh} z-2z)''\Big|_{z=0}=-\sin z+\operatorname{sh} z\Big|_{z=0}=1\neq 0$.

$\therefore z=0$ 为 $\sin z+\operatorname{sh} z-2z$ 的三级零点.

$\therefore z=0$ 为 $(\sin z+\operatorname{sh} z-2z)^{-2}$ 的二级极点.

8. (2). $\operatorname{Res}\left[\frac{1-e^{2z}}{z^4},0\right]=\frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}\left[z^4\cdot\frac{1-e^{2z}}{z^4}\right]$

$=\frac{1}{6}\times(-8)e^{2z}=-\frac{4}{3}$

(4) $\operatorname{Res}\left[\frac{z}{\cos z},\frac{\pi}{2}+k\pi\right]\ (k=0,\pm1,\pm2,\cdots)=\lim_{z\to\frac{\pi}{2}+k\pi}\left[z-\left(\frac{\pi}{2}+k\pi\right)\right]\cdot\frac{z}{\cos z}$

$=\left(\frac{\pi}{2}+k\pi\right)\cdot(-1)^{k+1}$，$k=0,\pm1,\pm2,\cdots$

(6) $z^2\sin\frac{1}{z}$.

~~Res ($z^2\sin\frac{1}{z}$, 0)~~

$$z^2\sin\frac{1}{z}=\left(\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}-\cdots+(-1)^n\frac{z^{-2n-1}}{(2n+1)!}+\cdots\right)\cdot z^2$$

$$=z-\frac{1}{3!}z^{-1}-\cdots+(-1)^n\frac{1}{(2n+1)!}z^{-2n+1}+\cdots$$

$\therefore \mathrm{Res}(z^2\sin\frac{1}{z},0)=-\frac{1}{6}$.

(8) $\frac{\operatorname{sh}z}{\operatorname{ch}z}$

$\operatorname{ch}z=0 \Rightarrow e^z+e^{-z}=0$ 解得 $z=(\frac{\pi}{2}+k\pi)i \quad k=0,\pm1,\pm2\cdots)$

$$\mathrm{Res}\left[f(z),\frac{\pi}{2}+k\pi\right]=\frac{\operatorname{sh}(\frac{\pi}{2}+k\pi)i}{\sin(\frac{\pi}{2}+k\pi)i}=1.$$

9. (1) $\oint_{|z|=\frac{3}{2}}\frac{\sin z}{z}dz=2\pi i\,\mathrm{Res}\left[\frac{\sin z}{z},0\right]=0.$

(3). $\oint_{|z|=\frac{3}{2}}\frac{1-\cos z}{z^m}dz=2\pi i\cdot\mathrm{Res}\left[\frac{1-\cos z}{z^m},0\right]$

$$\frac{1-\cos z}{z^m}=\frac{1}{z^m}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\cdots+(-1)^{n+1}\frac{z^{2n}}{(2n)!}+\cdots\right]$$

当 $m\le2$ 时, $\mathrm{Res}[f(z),0]=0$

$m=2n>2$ 时 $\mathrm{Res}[f(z),0]=0$.

当 $m=2n+1>2$ 时 $\mathrm{Res}[f(z),0]=\frac{(-1)^{n+1}}{(2n)!}=\frac{(-1)^{\frac{m+1}{2}}}{(m-1)!}$

$\therefore$ m 为大于等于3的奇数时, 积分为 $\frac{(-1)^{\frac{m+1}{2}}2\pi i}{(m-1)!}$; 当m为其他整数时, 积分为0.

(15). $\oint_{|z|=3} \operatorname{tg}\pi z\, dz = \oint_{|z|=3} \frac{\sin\pi z}{\cos\pi z}\, dz.$

~~$\pi z \neq \frac{\pi}{2}+k\pi \ (k=0,\pm1,\pm2)$~~

$\therefore \operatorname{tg}\pi z$ 有一级极点 $z_k = k+\frac{1}{2}$

$\operatorname{Res}[\operatorname{tg}\pi z, z_k] = \frac{\sin\pi z}{(\cos\pi z)'}\Big|_{z=z_k} = -\frac{1}{\pi}$

当 $k=0,\pm1,\pm2,-3$ 时，这些极点在 $|z|=3$ 内.

$\therefore \oint_{|z|=3} \operatorname{tg}\pi z\, dz = 2\pi i \sum_{k=-3}^{2} \operatorname{Res}[\operatorname{tg}\pi z, z_k] = 2\pi i \cdot 6 \cdot (-\frac{1}{\pi}) = -12i$

10. ~~$\operatorname{Res}[\cos z - \sin z, \infty] = -\operatorname{Res}[\cos\frac{1}{z}$~~

(2) $\cos z - \sin z = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots (-1)^n \frac{z^{2n}}{(2n)!}\right) - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots\right)$

含有无穷多个正幂次项，故 ∞ 是本性奇点.

$\because C_{-1} = 0 \quad \therefore \operatorname{Res}[\cos z - \sin z, \infty] = -C_{-1} = 0$

(3) 将 $\frac{2z}{3+z^2}$ 在解析圆环 $\sqrt{3} < |z| < +\infty$ 内展开.

$f(z) = \frac{2}{z} \cdot \frac{1}{1+\frac{3}{z^2}} = \frac{2}{z} \cdot \left[1 - \frac{3}{z^2} + \left(\frac{3}{z^2}\right)^2 - \cdots + \left(\frac{3}{z^2}\right)^n \cdot (-1)^n\right]$

展开式不含正幂次项，故 ∞ 为可去奇点.

$C_{-1} = 2 \quad \therefore \operatorname{Res}\left[\frac{2z}{3+z^2}, \infty\right] = -C_{-1} = -2$

11. (1) $f(z) = \frac{e^z}{z^2-1}$

它有两个一级极点 $z=\pm1$，由全部留数和为 0 可知

$\operatorname{Res}[f(z), \infty] = -\operatorname{Res}[f(z), 1] - \operatorname{Res}[f(z), -1]$

$= -\lim_{z\to1} \frac{e^z}{z+1} - \lim_{z\to-1} \frac{e^z}{z-1} = -\frac{e}{2} + \frac{e^{-1}}{2} = \operatorname{sh}1$

(2) $f(z)=\frac{1}{z(z+1)^4(z-4)}$

$\mathrm{Res}[f(z),\infty]=-\mathrm{Res}[f(\frac{1}{z})\frac{1}{z^2},0]$

$\frac{1}{z^2}f(\frac{1}{z})=\frac{1}{\frac{1}{z}(\frac{1}{z}+1)^4\cdot(\frac{1}{z}-4)}\cdot\frac{1}{z^2}$

$=\frac{z^4}{(1-4z)(z+1)^4}$

$z=0$ 为可去奇点.

$\therefore \mathrm{Res}[f(z),\infty]=-\mathrm{Res}[f(\frac{1}{z})\frac{1}{z^2},0]=0$

12.(1). 在 $|z|=3$ 的外部，只有 ∞ 点为奇点.

$\therefore \oint_C \frac{z^{15}}{(z^2+1)^2(z^4+2)^3}dz=-2\pi i\,\mathrm{Res}[f(z),\infty]$

$=2\pi i\,\mathrm{Res}[f(\frac{1}{z})\frac{1}{z^2},0]$

$=2\pi i\cdot\mathrm{Res}\left[\frac{1}{z(1+z^2)^2(1+2z^4)^3},0\right]$

$=2\pi i$

(3). $\oint_C \frac{z^{2n}}{1+z^n}dz = -2\pi i \, Res\left[\frac{z^{2n}}{1+z^n}, \infty\right]$

在∞的去心邻域 $1<|z|<\infty$ 内.

$$\frac{z^{2n}}{1+z^n} = z^n \frac{1}{1+z^n} = z^n\left(1-\frac{1}{z^n}+\frac{1}{z^{2n}}+\cdots\right)$$

$$= z^n - 1 + \frac{1}{z^n} - \frac{1}{z^{2n}} + \cdots$$

$$C_{-1} = \begin{cases} 1, & n=1\text{时} \\ 0, & n\neq 1\text{时} \end{cases}$$

$$Res\left[\frac{z^{2n}}{1+z^n}, \infty\right] = -C_{-1} = \begin{cases} -1, & n=1\text{时} \\ 0, & n\neq 1\text{时} \end{cases}$$

$$\therefore \oint \frac{z^{2n}}{1+z^n}dz = -2\pi i \cdot Res\left[\frac{z^{2n}}{1+z^n}, \infty\right] = \begin{cases} 2\pi i, & \text{当}n=1\text{时} \\ 0, & n\neq 1\text{时} \end{cases}$$

13.(2) $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta \quad (a>b>0)$

$\because a>b>0 \quad \therefore a+b\cos\theta$ 在 $0\le\theta\le 2\pi$内不为0，所以积分存在.

$$I = \oint_{|z|=1} \frac{\left(\frac{z^2-1}{2iz}\right)^2}{a+b\frac{z^2+1}{2z}} \cdot \frac{dz}{iz} = \oint_{|z|=1} \frac{i(z^2-1)^2}{2z^2(bz^2+2az+b)}dz$$

$$= 2\pi i\left\{Res[f(z),0] + Res\left[f(z), \frac{-a+\sqrt{a^2-b^2}}{b}\right]\right\}$$

$$Res[f(z),0] = \frac{i(z^2-1)[bz^3+3bz+3az^2+a]}{(b+2az+bz^2)^2}\Bigg|_{z=0}$$

$$= -\frac{ai}{b^2}$$

$$Res\left[f(z), \frac{-a+\sqrt{a^2-b^2}}{b}\right] = \frac{i(z^2-1)^2}{4z(bz^2+3az+b)}\Bigg|_{z=\frac{-a+\sqrt{a^2-b^2}}{b}}$$

$$= \frac{i\sqrt{a^2-b^2}}{b^2}$$

故 $I = \frac{2\pi}{b^2}\cdot\left(a-\sqrt{a^2-b^2}\right)$

(4). $\int_0^{+\infty} \frac{x^2}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}dx$

$f(z)$ 在上半平面有一级极点 $e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}$.

$Res[f(z), e^{\frac{\pi}{4}i}] = \frac{z^2}{4z^3}\Big|_{z=e^{\frac{\pi}{4}i}}$

$= \frac{e^{\frac{\pi}{2}i}}{4e^{\frac{3\pi}{4}i}} = \frac{1}{4}\cdot e^{-\frac{1}{4}\pi i} = \frac{1}{4}(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)$

$Res[f(z), e^{\frac{3\pi}{4}i}] = \frac{z^2}{4z^3}\Big|_{z=e^{\frac{3\pi}{4}i}} = \frac{e^{\frac{3}{2}\pi i}}{4e^{\frac{9\pi}{4}i}}$

$= \frac{1}{4}e^{\frac{5\pi}{4}i} = \frac{1}{4}(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)$

$\therefore I = \frac{1}{2}\cdot 2\pi i \{Res[f(z), e^{\frac{\pi}{4}i}] + Res[f(z), e^{\frac{3\pi}{4}i}]\} = \frac{\pi}{2\sqrt{2}}$

(6). $\int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2}$ 令 $f(z) = \frac{z}{1+z^2}$

在上半平面有 $z=i$ 一个一级极点.

~~$\therefore Res[f(z), i] = \frac{z}{2z}\Big|_{z=i} = \frac{1}{2}$~~

~~$\therefore \int_{-\infty}^{+\infty} \frac{x\sin x}{1+x^2} \sin x = 2\pi i$~~

$\therefore Res[f(z)\cdot e^{iz}, i] = \lim_{z\to i} (z-i)\frac{ze^{iz}}{(z-i)(z+i)}$

$= \frac{1}{2}e^{-1}$

$\therefore I = 2\pi i \, Res[f(z)e^{iz}, i] = \pi i e^{-1}$.

$\therefore$ 原积分为 $Im(I) = \pi e^{-1}$.