2.(1) $\frac{\partial u}{\partial x}=2x$ $\frac{\partial u}{\partial y}=0$ $\frac{\partial v}{\partial x}=0$ $\frac{\partial v}{\partial y}=-1$

当$x=-\frac{1}{2}$时 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

满足柯西黎曼方程

所以 $f(z)=x^2-iy$ 仅在直线 $x=-\frac{1}{2}$上可导，在Z平面处处不解析.

(2). $\frac{\partial u}{\partial x}=6x^2$ $\frac{\partial u}{\partial y}=0$ $\frac{\partial v}{\partial x}=0$ $\frac{\partial v}{\partial y}=9y^2$

要使 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，只有当$6x^2=9y^2$，即 $y=\pm\frac{\sqrt{6}}{3}x$时，才满足柯西-黎曼方程. ∴$f(z)$仅在直线$y=\pm\frac{\sqrt{6}}{3}x$上可导，在Z平面上处处不解析.

3. (1) 在Z平面上处处解析. $f'(z)=5(z-1)^4$

(3) 除$z=\pm1$外，在复平面上处处解析. $f'(z)=-\frac{2z}{(z^2-1)^2}$

4. (2) 函数分母为0的点为奇点，即$z=-1$和$z=\pm i$.

6. (2) 假命题. $h(z)=|z|^2$仅在$z=0$处可导，在其他点都不可导. 所以它处处不解析.

(4). 不成立. 假命题. $z=0$是 $z^2+\frac{1}{z}$和$z^2-\frac{1}{z}$的奇点，但它不是 $z^2+\frac{1}{z}+(z^2-\frac{1}{z})=2z^2$的奇点；

$z=0$是$\frac{1}{z}$和$\frac{1}{z^2}$的奇点，但它不是$\frac{\frac{1}{z}}{\frac{1}{z^2}}=z$的奇点.

(6). 真命题. 证明：若u为实常数，则 $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$

由柯西-黎曼方程可知，又∵在区域D是解析的

∴ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$ $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$

∴ v也是常数. 同理，若v是实常数，u也是常数.

因此 $f(z)$在D内为常数. 得证。

8. $u(x,y) = my^3 + nx^2y$

$v(x,y) = x^3 + lxy^2$

$\frac{\partial u}{\partial x} = 2xny \quad \frac{\partial u}{\partial y} = 3y^2m + nx^2$

$\frac{\partial v}{\partial x} = 3x^2 + ly^2 \quad \frac{\partial v}{\partial y} = 2xyl$

由柯西-黎曼方程知 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

即 $\begin{cases} 2xny = 2xyl \\ 3y^2m + nx^2 = -3x^2 - ly^2 \end{cases}$

解得 $n=-3 \quad l=-3 \quad m=1$

9. 将 $x = r\cos\theta$, $y = r\sin\theta$ 代入 $u = u(x,y)$, $v = v(x,y)$

$r = \sqrt{x^2+y^2} \quad \theta = \arctan\frac{y}{x}$.

~~$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}(-\sin\theta) r$~~

~~$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\cdot\cos\theta$.~~

$u = 3x^2 + y$

$= 3r^2\cos^2\theta + r\sin\theta$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$

$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta = \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta = r\frac{\partial u}{\partial r}$

即 $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$

$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$

$= -\frac{1}{r}\frac{\partial u}{\partial \theta}$

综上, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

12. (2) $\cos z=0$

即 $\dfrac{e^{iz}+e^{-iz}}{2}=0$

即 $e^{iz}=-e^{-iz}$

$e^{2iz}=-1$

$e^{-2y}(\cos 2x+i\sin 2x)=-1$

$e^{-2y}=1 \quad y=0 \quad \cos 2x=-1 \quad \sin 2x=0$

解得 $y=0,\ x=\dfrac{\pi}{2}+n\pi \quad (n=0,\pm1,\pm2,\cdots)$

$\therefore \quad z=\dfrac{\pi}{2}+n\pi \quad (n=0,\pm1,\pm2,\cdots)$

(4) $\sin z+\cos z=0$

$\dfrac{e^{iz}-e^{-iz}}{2i}+\dfrac{e^{iz}+e^{-iz}}{2}=0$

即 $e^{iz}+e^{-iz}+e^{-iz}\cdot i-e^{iz}\cdot i=0$

$e^{2iz}+1+i-ie^{2iz}=0$

$e^{2iz}(1-i)=-1-i$

$e^{2iz}=\dfrac{-1-i}{1-i}=\dfrac{1+i}{i-1}=\dfrac{1-1+2i}{-1-1}=-i$

$z=\dfrac{1}{2i}Ln\left(\dfrac{1+i}{i-1}\right)=\dfrac{1}{2i}Ln(-i)=\dfrac{1}{2i}\left[\ln|-i|+i(\arg(-i)+2k\pi)\right]$

$=\dfrac{i}{2i}\left(-\dfrac{\pi}{2}+2k\pi\right)=-\dfrac{\pi}{4}+k\pi \quad ,k=0,\pm1,\pm2\ldots$

13. $\cancel{\cos(z_1+z_2)}$

$$\cos z_1 \cdot \cos z_2 - \sin z_1 \sin z_2$$

$$= \frac{e^{iz_1}+e^{-iz_2}}{2} \cdot \frac{e^{iz_2}+e^{-iz_2}}{2} - \frac{e^{iz_1}-e^{-iz_1}}{2i} \cdot \frac{e^{iz_2}-e^{-iz_2}}{2i}$$

$$= \cancel{\frac{2(e^{iz_1}+e^{iz_2})}{4}} \frac{\left[e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}\right]\cdot 2}{4}$$

$$= \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2} = \cos(z_1+z_2)$$

同理 $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(3). $\sin 2z = 2\sin z \cos z$.

$$2\sin z \cos z = 2 \cdot \frac{e^{iz}-e^{-iz}}{2i} \cdot \frac{e^{iz}+e^{-iz}}{2}$$

$$= \frac{e^{2iz}-e^{-iz}}{2i} = \sin 2z.$$

(5). $\sin(\frac{\pi}{2}-z) = \cancel{\frac{e^{iz}-e^{-iz}}{2i}} \quad \cancel{\frac{e^{i(\frac{\pi}{2}-z)}-e^{-i(\frac{\pi}{2}-z)}}{2i}}$

$$= \sin\frac{\pi}{2}\cos(z) - \cos\frac{\pi}{2}\cdot\sin(-z)$$

$$= \cos z$$

$$= \frac{e^{-iz}+e^{iz}}{2}$$

$$= \cos z.$$

$$\cos(z+\pi) = \cos z \cdot \cos\pi - \sin z \cdot \sin\pi$$

$$= -\cos z.$$

15. $Ln(-i) = \ln|-i| + i[\arg(-i) + 2k\pi]$

$= -\frac{\pi}{2}i$

$Ln(-3+4i) = \ln|-3+4i| + i[\arg(-3+4i)]$

$\cancel{= \ln 5 + i\arctan(-\frac{4}{3})}$

$= \ln 5 + i[\arctan(-\frac{4}{3}) + \pi]$

16. (1) $Ln(z_1 z_2) = Ln z_1 + Ln z_2$

$Ln z_1 + Ln z_2 = \ln|z_1| + i[\arg(z_1)] + \ln|z_2| + i[\arg(z_2)]$

$= \ln|z_1 z_2| + i[\arg(z_1 \cdot z_2)]$

$= Ln(z_1 z_2)$

(2) $Ln\frac{z_1}{z_2} = \ln\left|\frac{z_1}{z_2}\right| + i\mathrm{Arg}\frac{z_1}{z_2}$

$= \ln|z_1| - \ln|z_2| + i\mathrm{Arg} z_1 - i\mathrm{Arg} z_2$

$= Ln z_1 - Ln z_2.$

18. $e^{1-i\frac{\pi}{2}} = e \cdot e^{-i\frac{\pi}{2}} = e[\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})] = -ei$

$\exp\left[\frac{(1+i\pi)}{4}\right] = e^{\frac{1}{4}}[\cos(\frac{\pi}{4}) - i\sin(\frac{\pi}{4})] = \frac{\sqrt{2}}{2}e^{\frac{1}{4}}(1-i)$

$3^i = e^{i\ln 3} = e^{i[\ln 3 + i(\arg 3 + 2k\pi)]} = e^{i\ln 3} \cdot e^{-2k\pi}$

$= e^{-2k\pi} \cdot [\cos(\ln 3) + i\sin(\ln 3)]$ $k=0, \pm 1, \pm 2 \cdots$

$(1+i)^i = e^{i\ln(1+i)} = e^{i[\ln|1+i| + i(\arg(1+i) + 2k\pi)]} = e^{\frac{i\ln 2}{2} - \frac{\pi}{4} - 2k\pi}$

$= e^{-\frac{\pi}{4} - 2k\pi}(\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2})$ $k=0, \pm 1, \pm 2 \cdots$

20.

(1) $ch^2 z - sh^2 z = 1$

$= \left(\frac{1}{2}\right)^2 (e^z + e^{-z})^2 - \frac{1}{4}(e^z - e^{-z})^2$

$= \frac{1}{4}(e^{2z} + e^{-2z} + 2) - \frac{1}{4}(e^{2z} + e^{-2z} - 2)$

$= 1$

(2) $sh^2 z + ch^2 z = ch\, 2z$.

$= \frac{1}{4}[e^{2z} + e^{-2z} - 2] + \frac{1}{4}[e^{2z} + e^{-2z} + 2]$

$= \frac{1}{2}[e^{2z} + e^{-2z}]$

$= ch\, 2z$.

(3). $\cancel{sh(z_1} \; sh z_1\, ch z_2 + ch z_1 \cdot sh z_2$

$= \frac{e^{z_1} - e^{-z_1}}{2} \cdot \frac{e^{z_2} + e^{-z_2}}{2} + \frac{e^{z_1} + e^{-z_1}}{2} \cdot \frac{e^{z_2} - e^{-z_2}}{2}$

$= \frac{2e^{z_1 + z_2} - 2e^{-z_1 - z_2}}{4}$

$= \frac{e^{z_1 + z_2} - e^{-(z_1 + z_2)}}{2} = sh(z_1 + z_2)$

同理 $\cancel{ch(z_1 + z_2) =}$ $ch z_1\, ch z_2 + sh z_1 \cdot sh z_2$

$= \frac{e^{z_1} + e^{-z_1}}{2} \cdot \frac{e^{z_2} + e^{-z_2}}{2} + \frac{e^{z_1} - e^{-z_1}}{2} \cdot \frac{e^{z_2} - e^{-z_2}}{2}$

$= \frac{e^{(z_1 + z_2)} + e^{-(z_1 + z_2)}}{2}$.

22. $chiy = \frac{e^{-iy}+e^{iy}}{2} = \cos y$

$shiy = \frac{e^{iy}-e^{-iy}}{2} = i\sin y$

2.3.20.

$ch(x+iy) = chx\cos y + i\,shx\sin y$

$$= \frac{e^{x}+e^{-x}}{2}\cdot\frac{e^{iy}+e^{-iy}}{2} + i\,\frac{e^{x}-e^{-x}}{2}\,\frac{e^{iy}-e^{-iy}}{2i}$$

$$= \frac{e^{(x+iy)}+e^{(x-iy)}+e^{(-x+iy)}+e^{-(x+iy)}}{4} + \frac{e^{(x+iy)}-e^{(x-iy)}-e^{(-x+iy)}+e^{-(x+iy)}}{4}$$

$$= \frac{e^{(x+iy)}+e^{-(x+iy)}}{2} = ch(x+iy)$$

$shx\cos y + i\,chx\sin y$

$$= \frac{e^{x}-e^{-x}}{2}\,\frac{e^{iy}+e^{-iy}}{2} + i\,\frac{e^{x}+e^{-x}}{2}\cdot\frac{e^{iy}-e^{-iy}}{2i}$$

$$= \frac{e^{x+iy}-e^{-(x+iy)}-e^{(-x+iy)}+e^{x-iy}}{4} + \frac{e^{(x+iy)}-e^{(x-iy)}+e^{(-x+iy)}-e^{-(x+iy)}}{4}$$

$$= \frac{e^{x+iy}-e^{-(x+iy)}}{2} = sh(x+iy)$$