1. (1). $z_n=\frac{1+ni}{1-ni}$

收敛. 极限为 -1.

(3). $z_n=(-1)^n+\frac{i}{n+1}$

发散.

(3). $z_n=\frac{1}{n}e^{-\frac{n\pi i}{2}}$

$=\frac{1}{n}\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$

收敛. 极限为 0.

3. (2) $\sum_{n=2}^{\infty}\frac{i^n}{\ln n}$

$\left|\frac{i^n}{\ln n}\right|=\frac{1}{\ln n}.$

$\ln n<n$. 当 $n\geq 2$ 时.

$\therefore \frac{1}{\ln n}>\frac{1}{n}$ $\therefore \sum_{n=2}^{\infty}\frac{1}{\ln n}>\sum_{n=2}^{\infty}\frac{1}{n}$.

$\because \sum_{n=2}^{\infty}\frac{1}{n}$ 发散. $\therefore \sum_{n=2}^{\infty}\frac{1}{\ln n}$ 也发散.

$\therefore \sum_{n=2}^{\infty}\frac{i^n}{\ln n}$ 不绝对收敛.

$i^n=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^n=\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}$

$\sum_{n=2}^{\infty}\frac{i^n}{\ln n}=\sum_{n=2}^{\infty}\left(\frac{\cos\frac{n\pi}{2}}{\ln n}+i\frac{\sin\frac{n\pi}{2}}{\ln n}\right)=\sum_{k=1}^{\infty}\frac{(-1)^k}{\ln 2k}+\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{\ln(2k-1)}\right)i$

$=\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln 2n}+i\cdot\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\ln(2n-1)}$

$\frac{(-1)^n}{\ln 2n}$ 与 $\frac{(-1)^{n-1}}{\ln(2n-1)}$ 都是条件收敛, 所以 $\sum_{n=2}^{\infty}\frac{i^n}{\ln n}$ 条件收敛.

(3). $\sum_{n=0}^{\infty}\left|\frac{(6+5i)^n}{8^n}\right| = \sum_{n=0}^{\infty}\left|\frac{6+5i}{8}\right|^n = \sum_{n=0}^{\infty}\left(\frac{\sqrt{61}}{8}\right)^n.$

$\because \frac{\sqrt{61}}{8}<1 \quad \therefore \sum_{n=0}^{\infty}\left|\frac{(6+5i)^n}{8^n}\right|$ 收敛.

$\therefore \sum_{n=0}^{\infty}\frac{(6+5i)^n}{8^n}$ 绝对收敛.

6.(2) $\sum_{n=1}^{\infty}\frac{(n!)^2}{n^n}\cdot z^n$

$c_n=\frac{(n!)^2}{n^n}$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^2}{(n+1)^{n+1}} \div \frac{(n!)^2}{n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2(n!)^2}{(n+1)^n(n+1)}\times\frac{n^n}{(n!)^2}$$

$$= \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^{n-1}}$$

$$= \lim_{n\to\infty}\frac{n+1}{(1+\frac{1}{n})^n} \to \infty.$$

∴收敛半径为 0.

(4). $c_n=e^{i\frac{\pi}{n}} = \cos\frac{\pi}{n}+i\sin\frac{\pi}{n} = (\cos\pi+i\sin\pi)^n$

$\therefore |c_n|=1 \quad \therefore \lim_{n\to\infty}\sqrt{|c_n|}=1$

∴收敛半径为 1.

(6). $\operatorname{Ln} in = \ln|in| + i(\operatorname{Arg} in) = \ln n + \frac{\pi}{2}i$

$$|\operatorname{Ln} in| = \left(\ln^2 n+\frac{\pi^2}{4}\right)^{\frac{1}{2}}$$

$$|c_n| = \left(\frac{1}{\ln^2 n+\frac{\pi^2}{4}}\right)^{\frac{n}{2}}$$

$$\therefore \lim_{n\to\infty}\sqrt[n]{|c_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{\ln^2 n+\frac{\pi^2}{4}}\right)^{\frac{n}{2}}} = \lim_{n\to\infty}\left(\frac{1}{\ln^2 n+\frac{\pi^2}{4}}\right)^{\frac{1}{2}} = 0.$$

∴收敛半径 $R=\infty$.

7.

$\because |(\mathrm{Re}\, C_n) z^n| \le |C_n z^n|$ $|(\mathrm{Re}\, C_n) z^n| < |C_n| \cdot |z|^n$

$\therefore \sum_{n=0}^{\infty} |(\mathrm{Re}\, C_n) z^n| \le \sum_{n=0}^{\infty} |C_n z^n|$.

$\because \sum_{n=0}^{\infty} C_n z^n$ 的收敛半径为 R, $\therefore \sum_{n=0}^{\infty} C_n z^n$ 在圆 $|z| < R$ 内绝对收敛.

$\therefore \sum_{n=0}^{\infty} (\mathrm{Re}\, C_n) \cdot z^n$ 在圆 $|z| < R$ 内绝对收敛.

$\therefore \sum_{n=0}^{\infty} (\mathrm{Re}\, C_n) z^n$ ~~在圆 $|z| < R$ 内绝对收敛~~ 的收敛半径大于或等于 R.

8.

$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lambda$.

则 $\lambda_1 = \frac{C_{n+1}}{C_n}$

$\lambda_2 = \lim_{n\to\infty} \left| \frac{\frac{C_{n+1}}{n+2}}{\frac{C_n}{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n+2} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lambda$

$\lambda_3 = \lim_{n\to\infty} \left| \frac{n+1\ C_{n+1}}{n\ C_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lambda$.

$\therefore$ 它们的收敛半径都为 $\begin{cases} \frac{1}{|\lambda|}, & \lambda \ne 0 \\ \infty, & \lambda = 0 \end{cases}$

$\therefore$ 它们有相同的收敛半径.

9. 已知 $\sum_{n=0}^{\infty} C_n$ 收敛, 则 $\sum_{n=0}^{\infty} C_n z^n$ 在 $z=1$ 处收敛.

对 $|z| < 1$ 的 z, $\sum_{n=0}^{\infty} C_n z^n$ 绝对收敛 $\therefore R \ge 1$.

若 $R > 1$, 则 $\sum_{n=0}^{\infty} C_n z^n$ 在收敛圆 $|z| < R$ 内绝对收敛.

则在 $z=1$ 处也绝对收敛. 即 $\sum_{n=0}^{\infty} |C_n|$ 收敛. 这与题意矛盾. $\therefore R = 1$

10. ~~若 $\sum_{n=0}^{\infty} c_n z^n$ 在它的收敛圆上一点 z_0 绝对收敛.~~

~~则 $\sum_{n=0}^{\infty} c_n z_0^n$~~

设收敛圆半径为 R.

在收敛圆所围的闭区域上.

$|z| \le |z_0|$.

则 $|c_n z^n| = |c_n| |z|^n \le |c_n| |z_0|^n = |c_n z_0^n|$

又 $\because \sum_{n=0}^{\infty} |c_n z_0^n|$ 收敛

$\therefore \sum_{n=0}^{\infty} |c_n z^n|$ 收敛. 得证.

11. (2) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots,\quad |z|<1$

$\frac{1}{1+z^2} = 1 - z^2 + z^4 + \cdots + (-1)^n z^{2n} + \cdots,\quad |z|<1$

$\left(\frac{1}{1+z^2}\right)' = \frac{-2z}{(1+z^2)^2}$

$\therefore \frac{-2z}{(1+z^2)^2} = -2z + 4z^3 + \cdots + 2n\cdot(-1)^n z^{2n-1} + \cdots,\quad |z|<1$

$\therefore \frac{1}{(1+z^2)^2} = 1 - 2z^2 + 3z^4 + \cdots + (-1)^{n-1}\cdot n\cdot z^{2n-2} + \cdots \quad |z|<1$

$R=1$

(4). $\operatorname{sh} z = \frac{e^z - e^{-z}}{2}$ $\quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \frac{z^n}{n!} \cdots,\quad |z| < +\infty$

$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots,\quad |z| < +\infty.$

$\therefore \operatorname{sh} z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots,\quad |z| < +\infty.$

$R = +\infty.$

(6) $e^{z^2} = 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \cdots,\quad |z| < +\infty.$

$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots,\quad |z| < +\infty.$

$e^{z^2}\cdot \sin z^2 = \left(1 + z^2 + \frac{z^4}{2!} + \cdots\right)\cdot\left(z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots\right)$

$= z^2 + z^4 + \frac{z^6}{3} + \cdots,\quad |z| < +\infty.$

(8). $\sin\frac{1}{1-z} = \sin\left(1+\frac{z}{1-z}\right) = \sin 1\cdot\cos\frac{z}{1-z} + \cos 1\cdot\sin\frac{z}{1-z}$

$\frac{z}{1-z} = z+z^2+z^3+\cdots = \sum_{n=0}^{\infty} z^{n+1},\ |z|<1$

$\sin\frac{z}{1-z} = (z+z^2+z^3+\cdots) - \frac{1}{3!}(z+z^2+z^3+\cdots)^3+\cdots$

$= z+z^2+\frac{5}{6}z^3+\cdots\quad,|z|<1$

$\cos\frac{z}{1-z} = 1-\frac{1}{2!}(z+z^2+z^3+\cdots)^2+\frac{1}{4!}(z+z^2+z^3+\cdots)^4+\cdots$

$= 1-\frac{1}{2}z^2-z^3+\cdots\quad,|z|<1.$

$\sin\frac{1}{1-z} = \sin 1+(\cos 1)z+(\cos 1-\frac{1}{2}\sin 1)z^2+(\frac{5}{6}\cos 1-\sin 1)z^3+\cdots,|z|<1$

$R=1.$

12. (2) 原式 $= \frac{2}{z+2} - \frac{1}{z+1}$

$\frac{1}{z+2} = \frac{1}{4+z-2} = \frac{1}{4}\frac{1}{1+\frac{z-2}{4}} = \frac{1}{4}\left[1-\frac{z-2}{4}+\left(\frac{z-2}{4}\right)^2-\cdots\right]\quad |z-2|<4$

$\frac{1}{z+1} = \frac{1}{3+z-2} = \frac{1}{3}\frac{1}{1+\frac{z-2}{3}} = \frac{1}{3}\left[1-\frac{z-2}{3}+\left(\frac{z-2}{3}\right)^2-\cdots\right]\quad |z-2|<3$

故原式 $= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^n}{2^{2n}} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(z-2)^n}{3^n}$

$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2^{2n+1}}-\frac{1}{3^{n+1}}\right)(z-2)^n\quad, |z-2|<3.\quad R=3.$

(4) $\frac{1}{4-3z} = \frac{1}{4-3(z-1-i)-3-3i} = \frac{1}{1-3i-3(z-1-i)}$

$= \frac{1}{1-3i}\frac{1}{1-\frac{3}{1-3i}[z-(1+i)]} = \frac{1}{1-3i}\left[1+\frac{3}{1-3i}[z-(1+i)]+\left(\frac{3}{1-3i}\right)^2[z-(1+i)]^2+\cdots\right]$

$\left|\frac{3}{1-3i}[z-(1+i)]\right|<1\quad\therefore |z-(1+i)| < \left|\frac{1-3i}{3}\right| = \frac{\sqrt{10}}{3}$

$\therefore R=\frac{\sqrt{10}}{3}.$

16). $\operatorname{arctg} z = \frac{1}{1+z^2}$.

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad |z|<1$$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots\cdots \qquad |z|<1$$

$$\therefore \arctan z = \int_0^z \frac{1}{1+z^2}\,dz = \int_0^z \sum_{n=0}^{\infty} (-1)^n z^{2n}\,dz$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} \qquad |z|<1$$

$R=1$.

16.(1). $\frac{1}{(z^2+1)(z-2)} \qquad 1<|z|<2$

令 $\frac{1}{(z^2+1)(z-2)} = \frac{a}{z+i} + \frac{b}{z-i} + \frac{c}{z-2}$

$$1 = a(z-i)(z-2) + b(z+i)(z-2) + c(z^2+1)$$

解得 $c=\frac{1}{5}$ $b=\frac{-1+2i}{10}$ $a=\frac{-1-2i}{10}$

$$原式 = \frac{-1-2i}{10}\frac{1}{(z+i)} + \frac{-1+2i}{10}\frac{1}{z-i} + \frac{1}{5}\frac{1}{z-2}$$

$$= \frac{-1-2i}{10}\frac{1}{z(1+\frac{i}{z})} + \frac{-1+2i}{10}\frac{1}{z(1-\frac{i}{z})} + \frac{1}{5}\left(-\frac{1}{2}\right)\frac{1}{1-\frac{z}{2}}$$

$$= \frac{-1-2i}{10}\sum_{n=0}^{\infty}\frac{(-1)^n i^n}{z^{n+1}} + \frac{-1+2i}{10}\sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} - \frac{1}{10}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} + \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}} - \frac{1}{10}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\frac{1}{5}\left(\cdots + \frac{2}{z^4} + \frac{1}{z^3} - \frac{2}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots\right)$$

$$1<|z|<2$$

(3) $\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2} = -\frac{1}{z-1} + \frac{1}{z-1-1}$

$= -\frac{1}{z-1} - \frac{1}{1-(z-1)}$

$= -\frac{1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n = -\sum_{n=-1}^{\infty} (z-1)^n$

(5). $\frac{1}{z^2(z-i)} = \frac{1}{z-i} \cdot \frac{1}{z^2} = \frac{1}{z-i} \frac{1}{(z-i+i)^2}$ $0<|z-i|<1$

$= \frac{1}{z-i} \frac{1}{i^2[1+\frac{z-i}{i}]^2}$

$= -\frac{1}{z-i} \frac{1}{[1-i(z-i)]^2}$

$= -\frac{1}{z-i} \cdot \sum_{n=1}^{\infty} n i^{n-1} (z-i)^{n-1}$

$= -\sum_{n=1}^{\infty} n i^{n-1} \cdot (z-i)^{n-2}$

$= \sum_{n=1}^{\infty} n \cdot i^2 \cdot i^{n-1} (z-i)^{n-2}$

$= \sum_{n=1}^{\infty} n i^{n+1} (z-i)^{n-2}$.

$1<|z-i|<+\infty$

$\frac{1}{(z-i+i)^2} = \frac{1}{(z-i)^2} \cdot \frac{1}{(1+\frac{i}{z-i})^2}$

$\because \left(\frac{1}{1+\xi}\right)' = -\frac{1}{(1+\xi)^2} = 1-2\xi+3\xi^2+\cdots$

$\therefore \frac{1}{(1+\frac{i}{z-i})^2} = 1-2\left(\frac{i}{z-i}\right)+3\left(\frac{i}{z-i}\right)^2-4\left(\frac{i}{z-i}\right)^3+\cdots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \cdot \left(\frac{i}{z-i}\right)^{n-1}$

$\therefore$ 原式 $= \frac{1}{(z-i)^3} \cdot \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \cdot \left(\frac{i}{z-i}\right)^{n-1}$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \cdot \frac{i^{n-1}}{(z-i)^{n+2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1) i^n}{(z-i)^{n+3}}$ $1<|z-i|<+\infty$.

(7) ① $3<|z|<4$.

$$\frac{(z-1)(z-2)}{(z-3)(z-4)} = 1-\frac{6}{4-z}\ \frac{2}{z-3}$$

$$= 1-\frac{6}{4}\frac{1}{1-\frac{z}{4}}-\frac{2}{z}\frac{1}{1-\frac{3}{z}}$$

$$= 1-\frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n-\frac{2}{z}\sum_{n=-1}^{\infty}\left(\frac{3}{z}\right)^n$$

$$= 1-\frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{4^n}z^n-2\sum_{n=-1}^{\infty}\frac{3^n}{z^{n+1}}$$

② $4<|z|<+\infty$

$$\frac{(z-1)(z-2)}{(z-3)(z-4)} = 1-\frac{6}{4-z}-\frac{2}{z-3}$$

$$= 1+\frac{6}{z-4}-\frac{2}{z-3}$$

$$= 1+\frac{6}{z}\cdot\frac{1}{1-\frac{4}{z}}-\frac{2}{z}\frac{1}{1-\frac{3}{z}}$$

$$= 1+\frac{6}{z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n-\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

$$= 1+\sum_{n=1}^{\infty}\left(3\cdot 2^{2n-1}-2\cdot 3^{n-1}\right)\cdot z^{-n}$$

17. $f(z)$在圆环域内能否展开为洛朗级数关键在于验证$f(z)$在此圆环域内是否解析.

$f(z)=\tan\frac{1}{z}$ $z=0$是一个奇点. 所以关键看是否能找到$z=0$的一个去心邻域 $0<|z|<R$, 将它展成洛朗级数. 除奇点$z=0$外, $z_n=\left(\frac{1}{2}+n\right)\pi$ $(n=0,\pm1,\pm2)$都是它的奇点, 而且$\lim\limits_{n\to\infty} z_n=0$.

∴对任何$R>0$, 去心邻域$0<|z|<R$内都含有该函数的奇点z_n.

即在$z=0$的任何去心邻域内 $f(z)=\tan\frac{1}{z}$ 都不解析.

19. (2). $\frac{z+2}{(z+1)z} = \cancel{\frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right)}$

$$= (z+2)\left(\frac{1}{z} - \frac{1}{z(1+\frac{1}{z})}\right).$$

$$= (z+2)\left(\frac{1}{z} - \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+1}}\right) \quad |z|>1.$$

故 $\int f(z)\,dz = 2\pi i.$

(4). $\frac{z}{(z+1)(z+2)} = z\left[\frac{1}{z(1+\frac{1}{z})} - \frac{1}{z(1+\frac{2}{z})}\right]$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^n}, \quad |z|>2$$

故 $\int_C f(z)\,dz = 2\pi i$

20. 当C的内部不含原点时，级数 $\sum_{n=-2}^{\infty} z^n$ 为C上及C内的解析函数.

$$\oint_C \left(\sum_{n=-2}^{\infty} z^n\right) dz = 0.$$

当C内部含原点时，$\oint_C \left(\sum_{n=-2}^{\infty} z^n\right) dz = \sum_{n=-2}^{\infty} \oint_C z^n\,dz$

$$= \oint_C \frac{1}{z^2}\,dz + \oint_C \frac{1}{z}\,dz.$$

当 $0<|z|<1$ 时. $\oint_C \frac{1}{z^2}dz = 2\pi i \cdot c_{-1} = 0$

$$\oint_C \frac{1}{z}\,dz = 2\pi i \cdot c_{-1} = 2\pi i$$

综上所述，$\oint_C \left(\sum_{n=-2}^{\infty} z^n\right) dz = \begin{cases} 2\pi i, & C包含原点. \\ 0, & C不含原点. \end{cases}$