习题 1.1

1.证明$\sqrt{3}$为无理数.

证 若$\sqrt{3}$不是无理数, 则$\sqrt{3}=\dfrac{p}{q}$, p,q为互素自然数.$3=\dfrac{p^2}{q^2}$, $p^2=3q^2$.3除尽p^2,

必除尽p,否则$p=3k+1$或$p=3k+2$.$p^2=9k^2+6k+1$, $p^2=9k^2+12k+4$,3除

p^2将余1.故$p=3k,9k^2=3q^2,q^2=3k^2$,类似得3除尽q.与p,q互素矛盾.

2.设p是正的素数,证明$\sqrt{p}$是无理数.

证 设$\sqrt{p}=\dfrac{a}{b}$,a,b为互素自然数, 则$p=\dfrac{a^2}{b^2}$,$a^2=pb^2$,素数p除尽a^2,故p除尽a,

$a=pk.p^2k^2=pb^2,pk^2=b^2$.类似得$p$除尽$b$.此与$a,b$为互素自然数矛盾.

3.解下列不等式:

(1)$|x|+|x-1|<3$.\;(2)$|x^2-3|<2$.

解 (1)若$x<0$,则$-x+1-x<3,2x>-2,x>-1,(-1,0)$;

若$0<x<1$,则$x+1-x<3,1<3,(0,1)$;

若$x>1$,则$x+x-1<3,x<3/2,(1,3/2)$.

$X=(-1,0)\cup(0,1)\cup(1,3/2)$.

(2)$-2<x^2-3<2,1<x^2<5,1<|x|^2<5,1<|x|<\sqrt{5},x=(1,\sqrt{5})\cup(-\sqrt{5},-1)$.

4.设a,b为任意实数,(1)证明$|a+b|\geq|a|-|b|$;(2)设$|a-b|<1$,证明$|a|<|b|+1$.

证(1)$|a|=|a+b+(-b)|\leq|a+b|+|-b|=|a+b|+|b|,|a+b|\geq|a|-|b|$.

(2)$|a|=|b+(a-b)|\leq|b|+|a-b|<|b|+1$.

5.解下列不等式:

(1)$|x+6|>0.1$;(2)$|x-a|>l$.

解(1)$x+6>0.1$或$x+6<-0.1$.$x>-5.9$或$x<-6.1$.$X=(-\infty,-6.1)\cup(-5.9,+\infty)$.

(2)若$l>0,X=(a+l,+\infty)\cup(-\infty,a-l)$;若$l=0,x\neq a$;若$l<0,X=(-\infty,+\infty)$.

6.若$a>1$,证明$0<\sqrt[n]{a}-1<\dfrac{a-1}{n}$,其中$n$为自然数.

证若$a>1$,显然$\sqrt[n]{a}=b>1$.$a-1=\sqrt[n]{a}^n-1=(\sqrt[n]{a}-1)(b^{n-1}+b^{n-2}+\cdots+1)>n(\sqrt[n]{a}-1)$.

7.设(a,b)为任意一个开区间,证明(a,b)中必有有理数.

证取自然数n 满足$1/10^n<b-a$.考虑有理数集合

A=$A_n=\{\dfrac{m}{10^n}\,|\,m\in\mathbf{Z}\}$. 若$A_n\cap(a,b)=\varnothing$,则$A=B\cup C,B=A\cap\{x\,|\,x\geq b\}$,

$C=A\cap\{x\,|\,x\leq a\}$.B中有最小数$m_0/10^n,(m_0-1)/10^n\in C$,

$b-a\leq m_0/10^n-(m_0-1)/10^n=1/10^n$, 此与$n$的选取矛盾.

8.设(a,b)为任意一个开区间,证明(a,b)中必有无理数.

证取自然数n 满足$1/10^n<b-a$.考虑无理数集合$A_n=\{\sqrt{2}+\dfrac{m}{10^n}\,|\,m\in\mathbf{Z}\}$. 以下仿8题.

习题 1.2

1.求下列函数的定义域:

$(1)y=\ln(x^2-4);(2)y=\ln\sqrt{\dfrac{1+x}{1-x}};(3)y=\sqrt{\ln\dfrac{5x-x^2}{4}};(4)y=\dfrac{1}{\sqrt{2x^2+5x-3}}.$

解 $(1)x^2-4>0,|x|^2>4,|x|>2,D=(-\infty,-2)\cup(2,+\infty).$

$(2)\dfrac{1+x}{1-x}>0.\begin{cases}1-x>0\\1+x>0\end{cases}$或$\begin{cases}1-x<0\\1+x<0\end{cases}.-1<x<1,D=(-1,1).$

$(3)\dfrac{5x-x^2}{4}>1,x^2-5x-4<0.x^2-5x+4=0,(x-1)(x-4)=0,x_1=1,x_2=4.$

$D=(1,4).$

$(4)2x^2+5x-3>0.(2x-1)(x+3)=0,x_1=-3,x_2=1/2.D=(-\infty,-3)\cup(1/2,+\infty).$

2.求下列函数的值域$f(X)$,其中X为题中指定的定义域.

$(1)\ f(x)=x^2+1,X=(0,3).f(X)=(1,10).$

$(2)\ f(x)=\ln(1+\sin x),X=(-\pi/2,\pi],\ f(X)=(-\infty,\ln 2].$

$(3)\ f(x)=\sqrt{3+2x-x^2},X=[-1,3],3+2x-x^2=0,x^2-2x-3=0,(x+1)(x-3)=0,$

$x_1=-1,x_2=3,\ f(X)=[0,\ f(1)]=[0,4].$

$(4)\ f(x)=\sin x+\cos x,X=(-\infty,+\infty).$

$f(x)=\sqrt{2}(\sin x\cos(\pi/4)+\cos x\sin(\pi/3))=\sqrt{2}\sin(x+\pi/4),\ f(X)=[-\sqrt{2},\sqrt{2}].$

3.求函数值:

(1)设$f(x)=\dfrac{\ln x^2}{\ln 10}$,求$f(-1),\ f(-0.001),\ f(100);$

(2)设$f(x)=\arcsin\dfrac{x}{1+x^2}$,求$f(0),\ f(1),\ f(-1);$

(3)设$f(x)=\begin{cases}\ln(1-x),-\infty<x\le 0,\\-x,\qquad 0<x<+\infty,\end{cases}$求$f(-3),\ f(0),\ f(5).$

(4)设$f(x)=\begin{cases}\cos x,0\le x<1,\\1/2,\quad x=1,\\2^x,\quad 1<x\le 3\end{cases}$ 求$f(0),\ f(1),\ f(3/2),\ f(2).$

解(1) $f(x)=\log x^2,\ f(-1)=\log 1=0,\ f(-0.001)=\log(10^{-6})=-6,\ f(100)=\log 10^4=4.$

$(2)\ f(0)=0,\ f(1)=\arcsin(1/2)=\pi/6,\ f(-1)=\arcsin(-1/2)=-\pi/6.$

$(3)\ f(-3)=\ln 4,\ f(0)=0,\ f(5)=-5.$

$(4)\ f(0)=\cos 0=1,\ f(1)=1/2,\ f(3/2)=2\sqrt{2},\ f(2)=4.$

4.设函数$f(x)=\dfrac{2+x}{2-x},x\ne\pm 2$,求$f(-x),\ f(x+1),\ f(x)+1,\ f\left(\dfrac{1}{x}\right),\dfrac{1}{f(x)}.$

解$f(-x)=\dfrac{2-x}{2+x},x\ne\pm 2;\ f(x+1)=\dfrac{2+x+1}{2-x-1}=\dfrac{3+x}{1-x},x\ne 1,x\ne -3,$

$f(x)+1=\dfrac{2+x}{2-x}+1=\dfrac{4}{2-x}, x\neq\pm 2;\ f\left(\dfrac{1}{x}\right)=\dfrac{2-1/x}{2+1/x}=\dfrac{2x-1}{2x+1}, x\neq 0, x\neq\pm 1/2,$

$\dfrac{1}{f(x)}=\dfrac{2+x}{2-x}, x\neq\pm 2.$

5.设$f(x)=x^3$,求$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$，其中$\Delta x$为一个不等于零的量.

解$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{(x+\Delta x)^3-x^3}{\Delta x}=\dfrac{x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3-x^3}{\Delta x}=3x^2+3\Delta x+\Delta x^2.$

6.设$f(x)=\ln x, x>0, g(x)=x^2, -\infty<x<+\infty$,试求$f(f(x)), g(g(x)), f(g(x)), g(f(x))$.

解$f(f(x))=f(\ln x)=\ln\ln x, x>1; g(g(x))=g(x^2)=x^4, -\infty<x<+\infty;$

$f(g(x))=f(x^2)=\ln x^2, x\neq 0; g(f(x))=g(\ln x)=\ln^2 x, x>0.$

7.设$f(x)=\begin{cases}0, & x\geq 0,\\ -x, & x<0;\end{cases}$ $g(x)=\begin{cases}x, & x\geq 0;\\ 1-x, & x<0,\end{cases}$求$f(g(x)), g(f(x))$.

解$\forall x, g(x)\geq 0, f(g(x))=0.$

$g(f(x))=\begin{cases}g(0), & x\geq 0,\\ g(-x), & x<0.\end{cases}=\begin{cases}0, & x\geq 0,\\ -x, & x<0.\end{cases}$

8.作下列函数的略图:

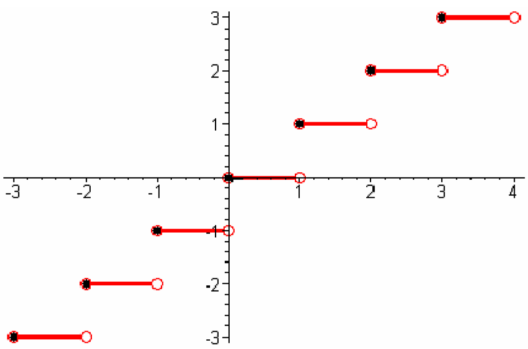
(1)$y=[x]$,其中$[x]$为不超过x的最大整数;

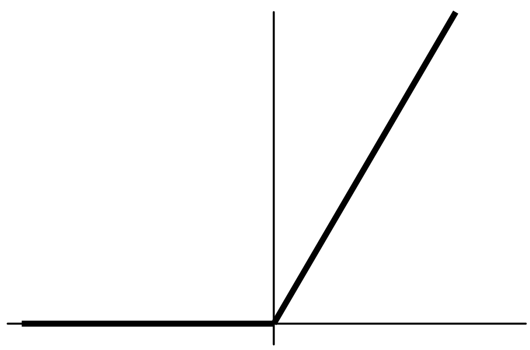
(2)$y=[x]+x$;

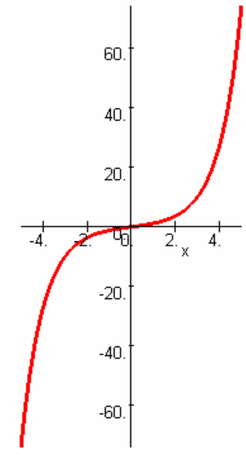
(3)$y=\sinh x=\dfrac{1}{2}(e^x-e^{-x})(-\infty<x<+\infty)$;

(4)$y=\cosh x=\dfrac{1}{2}(e^x+e^{-x})(-\infty<x<+\infty)$;

(5)$y=\begin{cases}x^2, & 0\leq x<0,\\ x-1, & -1\leq x<0.\end{cases}$







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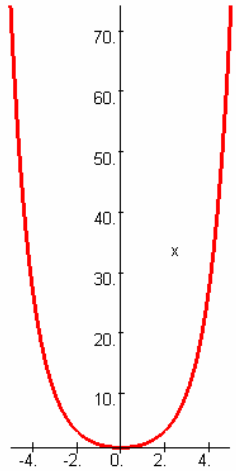

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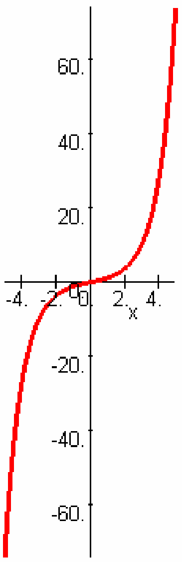

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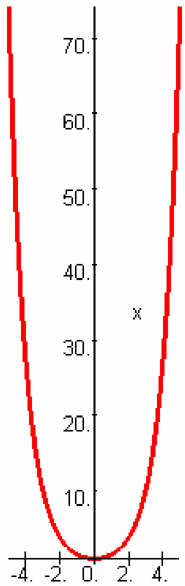

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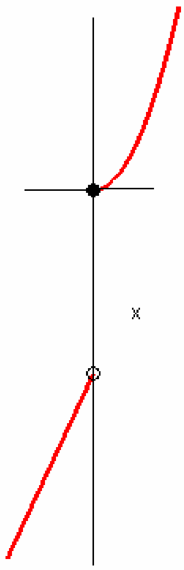


（5）

9.设$f(x)=\begin{cases}x^2, x\geq 0,\\ x, \quad x<0,\end{cases}$求下列函数并且作它们的图形：

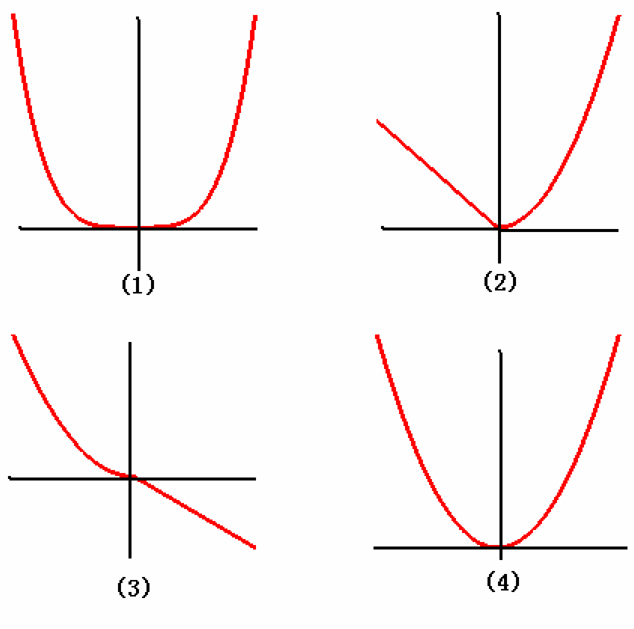
(1)$y=f(x^2)$;(2)$y=|f(x)|$;(3)$y=f(-x)$;(4)$y=f(|x|)$.

解(1)$y=x^4, -\infty<x<+\infty$.

(2)$y=|f(x)|=\begin{cases}x^2, x\geq 0,\\ -x, \quad x<0.\end{cases}$

(3)$y=f(-x)=\begin{cases}x^2, -x\geq 0,\\ -x, \quad -x<0\end{cases}=\begin{cases}x^2, x\leq 0,\\ -x, \ x>0.\end{cases}$

(4)$y=f(|x|)=x^2, -\infty<x<+\infty$.



10.求下列函数的反函数:

$(1) y=\frac{x}{2}-\frac{2}{x}(0<x<+\infty);$

$(2) y=\sinh x(-\infty<x<+\infty);$

$(3) y=\cosh x(0<x<+\infty).$

解$(1) \frac{x}{2}-\frac{2}{x}=y, x^2-2yx-4=0, x=y+\sqrt{y^2+4}, y=x+\sqrt{x^2+4}(-\infty<x<+\infty).$

$(2) \frac{e^x-e^{-x}}{2}=y, z=e^x, z^2-2yz-1=0, e^x=z=y+\sqrt{y^2+1}, x=\ln(y+\sqrt{y^2+1}),$

$y=\ln(x+\sqrt{x^2+1}),(-\infty<x<+\infty).$

$(3) \frac{e^x+e^{-x}}{2}=y, z=e^x, z^2-2yz+1=0, e^x=z=y+\sqrt{y^2-1}, x=\ln(y+\sqrt{y^2-1}),$

$y=\ln(x+\sqrt{x^2-1}),(x\geq 1).$

11.证明$\cosh^2 x-\sinh^2 x=1.$

证$\cosh^2 x-\sinh^2 x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=\frac{(e^{2x}+e^{-2x}+2)-(e^{2x}+e^{-2x}-2)}{4}=1.$

12.下列函数在指定区间内是否是有界函数?

$(1) y=e^{x^2}, x\in(-\infty,+\infty);$否

$(2) y=e^{x^2} x\in(0,10^{10});$是

$(3) y=\ln x, x\in(0,1);$否

$(4) y=\ln x, x\in(r,1),$其中$r>0.$是

$(5) y=\frac{e^{-x^2}}{2+\sin x}+\cos(2^x), x\in(-\infty,+\infty);$是$|y|\leq\frac{1}{2-1}+1=2.$

(6) $y=x^2\sin x, x\in(-\infty,+\infty)$; 否.

(7) $y=x^2\cos x, x\in(-10^{10},10^{10})$.是

13.证明函数$y=\sqrt{1+x}-\sqrt{x}$在$(1,+\infty)$内是有界函数.

证$y=\sqrt{1+x}-\sqrt{x}=\dfrac{(\sqrt{1+x}-\sqrt{x})(\sqrt{1+x}+\sqrt{x})}{\sqrt{1+x}+\sqrt{x}}=\dfrac{1}{\sqrt{1+x}+\sqrt{x}}<\dfrac{1}{\sqrt{2}+1}(x>1)$.

13.研究函数$y=\dfrac{x^6+x^4+x^2}{1+x^6}$在$(-\infty,+\infty)$内是否有界.

解$|x|\le 1$时,$\dfrac{x^6+x^4+x^2}{1+x^6}\le 3$,$|x|>1$时,$\dfrac{x^6+x^4+x^2}{1+x^6}\le\dfrac{3x^6}{x^6}=3$,

$|y|=y\le 3, x\in(-\infty,+\infty)$.

习题 1.3

1.设$x_n = \dfrac{n}{n+2}(n=1,2,\cdots)$,证明$\lim\limits_{n\to\infty} x_n = 1$,即对于任意$\varepsilon>0$,求出正整数$N$,使得当$n>N$时有 $|x_n-1|<\varepsilon$,并填下表:

ε	0.1	0.01	0.001	0.0001
$\boldsymbol{N}$	18	198	1998	19998

证$\forall\varepsilon>0$,不妨设$\varepsilon<1$,要使$|x_n-1| = \left|\dfrac{n}{n+2}-1\right| = \dfrac{2}{n+2}<\varepsilon$,只需$n>\dfrac{2}{\varepsilon}-2$,取$N=\left[\dfrac{2}{\varepsilon}-2\right]$,则当$n>N$时,就有$|x_n-1|<\varepsilon$.

2.设$\lim\limits_{n\to\infty} a_n = l$,证明$\lim\limits_{n\to\infty}|a_n| = |l|$.

证$\forall\varepsilon>0,\exists N$,使得当$n>N$时,$|a_n-l|<\varepsilon$,此时$||a_n|-|l||\le|a_n-l|<\varepsilon$,故$\lim\limits_{n\to\infty}|a_n|=|l|$.

3.设$\{a_n\}$有极限l,证明

(1)存在一个自然数$N,n<N\ |a_n|<|l|+1$;

(2)$\{a_n\}$是一个有界数列,即存在一个常数M,使得$|a_n|\le M(n=1,2,\cdots)$.

证(1)对于$\varepsilon=1,\exists N$,使得当$n>N$时,$|a_n-l|<1$,此时$|a_n|=|a_n-l+l|\le|a_n-l|+|l|<|l|+1$.

(2)令$M=\max\{|l|+1,|a_1|,\cdots,|a_N|\}$,则$|a_n|\le M(n=1,2,\cdots)$.

4. 用ε-N说法证明下列各极限式:

(1)$\lim\limits_{n\to\infty}\dfrac{3n+1}{2n-3}=\dfrac{3}{2}$;　　　(2)$\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2}\sin n}{n+1}=0$;

(3)$\lim\limits_{n\to\infty} n^2q^n = 0(|q|<1)$;　(4)$\lim\limits_{n\to\infty}\dfrac{n!}{n^n}=0$;

(5)$\lim\limits_{n\to\infty}\left(\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)\cdot n}\right)=1$;

(6)$\lim\limits_{n\to\infty}\left(\dfrac{1}{(n+1)^{3/2}}+\cdots+\dfrac{1}{(2n)^{3/2}}\right)=0$.

证(1)$\forall\varepsilon>0$,不妨设$\varepsilon<1$,要使$\left|\dfrac{3n+1}{2n-3}-\dfrac{3}{2}\right|=\dfrac{11}{2(2n-3)}<\varepsilon$,只需$n>\dfrac{11}{2\varepsilon}+3$,取$N=\left[\dfrac{11}{2\varepsilon}+3\right]$,当$n>N$时,$\left|\dfrac{3n+1}{2n-3}-\dfrac{3}{2}\right|<\varepsilon$,故$\lim\limits_{n\to\infty}\dfrac{3n+1}{2n-3}=\dfrac{3}{2}$.

(2)$\forall\varepsilon>0$,要使$\left|\dfrac{\sqrt[3]{n^2}\sin n}{n+1}\right|<\varepsilon$,由于$\left|\dfrac{\sqrt[3]{n^2}\sin n}{n+1}\right|\le\sqrt[3]{n}$,只需$\sqrt[3]{n}<\varepsilon,n>\dfrac{1}{\varepsilon^3}$,

$$\text{取}N=\left[\frac{1}{\varepsilon^3}\right],\text{当}n>N\text{时}\left|\frac{\sqrt[3]{n^2}\sin n}{n+1}\right|<\varepsilon.$$

$$(3)\ |q|=\frac{1}{1+\alpha}(\alpha>0).n>4$$

$$|n^2q^n|=\frac{n^2}{(1+\alpha)^n}=\frac{n^2}{1+n\alpha+\frac{n(n-1)}{2}\alpha^2+\frac{n(n-1)(n-2)}{6}\alpha^3+\cdots+\alpha^n}$$

$$<\frac{6n}{(n-1)(n-2)\alpha^3}<\frac{24}{n\alpha^3}<\varepsilon,n>\frac{24}{\varepsilon\alpha^3},N=\max\{4,\left[\frac{24}{\varepsilon\alpha^3}\right]\}.$$

$$(4)\ \frac{n!}{n^n}\le\frac{1}{n}<\varepsilon,n>\frac{1}{\varepsilon},N=\left[\frac{1}{\varepsilon}\right].$$

$$(5)\ \left|\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}\right)-1\right|$$

$$=\left|\left(\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{(n-1)}-\frac{1}{n}\right)\right)-1\right|=\frac{1}{n}<\varepsilon,n>\frac{1}{\varepsilon},N=\left[\frac{1}{\varepsilon}\right].$$

$$(6)\ \frac{1}{(n+1)^{3/2}}+\cdots+\frac{1}{(2n)^{3/2}}\le\frac{n}{(n+1)^{3/2}}<\frac{1}{\sqrt{n}}<\varepsilon,n>\frac{1}{\varepsilon^2},N=\left[\frac{1}{\varepsilon^2}\right].$$

5.设$\lim\limits_{n\to\infty}a_n=0,\{b_n\}$是有界数列,即存在常数$M$,使得$|b_n|<M\ (n=1,2,\cdots)$,证明$\lim\limits_{n\to\infty}a_nb_n=0.$

证$\forall\varepsilon>0,\exists$正整数 N,使得 $|a_n|<\dfrac{\varepsilon}{M},|a_nb_n|=|a_n||b_n|\le\dfrac{\varepsilon}{M}\cdot M=\varepsilon,$

故$\lim\limits_{n\to\infty}a_nb_n=0.$

6.证明$\lim\limits_{n\to\infty}\sqrt[n]{n}=1.$

证$\forall\varepsilon>0,$要使$|\sqrt[n]{n}-1|=\sqrt[n]{n}-1<\varepsilon,$只需$\dfrac{n}{(1+\varepsilon)^n}<1.$

$$\text{而}\frac{n}{(1+\varepsilon)^n}=\frac{n}{1+n\varepsilon+\frac{n(n-1)}{2}\varepsilon^2}<\frac{2}{(n-1)\varepsilon^2}<\frac{4}{n\varepsilon^2},\text{只需}\frac{4}{n\varepsilon^2}<1,n>\frac{4}{\varepsilon^2},N=\left[\frac{4}{\varepsilon^2}\right].$$

7.求下列各极限的值:

$$(1)\ \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.$$

$$(2)\ \lim_{n\to\infty}\frac{n^3+3n^2-100}{4n^3-n+2}=\lim_{n\to\infty}\frac{1+3/n-100/n^2}{4-1/n^2+2/n^2}=\frac{1}{4}.$$

$$(3)\ \lim_{n\to\infty}\frac{(2n+10)^4}{n^4+n^3}=\lim_{n\to\infty}\frac{(2+10/n)^4}{1+1/n}=16.$$

$$(4)\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-2n}=\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^{-2}=e^{-2}.$$

(5) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n}=\lim\limits_{n\to\infty}\dfrac{1}{\left(1+\dfrac{1}{n-1}\right)^{n-1}\left(1+\dfrac{1}{n-1}\right)}$

$=\dfrac{1}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n-1}\right)^{n-1}\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n-1}\right)}=\dfrac{1}{e}.$

(6) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n^2}=\lim\limits_{n\to\infty}\left[\left(1-\dfrac{1}{n}\right)^{n}\right]^{n}$，取 $q\in(\dfrac{1}{e},1)$，$\exists N$，当 $n>N$ 时，$\left(1-\dfrac{1}{n}\right)^{n}<q$

$0<\left[\left(1-\dfrac{1}{n}\right)^{n}\right]^{n}<q^{n}$，$\lim\limits_{n\to\infty}q^{n}=0$，$\lim\limits_{n\to\infty}\left[\left(1-\dfrac{1}{n}\right)^{n}\right]^{n}=0$，即 $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n^2}=0.$

(7) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n^2}\right)^{n}=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n}\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n}=e\cdot\dfrac{1}{e}=1.$

8.利用单调有界序列有极限证明下列序列极限的存在性:

(1) $x_n=\dfrac{1}{1}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{n^2}$，$x_{n+1}=x_n+\dfrac{1}{(n+1)^2}>x_n$，

$x_n<1+\dfrac{1}{1\cdot 2}+\cdots+\dfrac{1}{(n-1)n}=2-\dfrac{1}{n}<2.$ x_n单调增加有上界，故有极限.

(2) $x_n=\dfrac{1}{2+1}+\dfrac{1}{2^2+1}+\cdots+\dfrac{1}{2^n+1}$，$x_{n+1}=x_n+\dfrac{1}{2^{n+1}+1}>x_n$，

$x_n=\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^n}=\dfrac{1}{2}\left(1+\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^{n-1}}\right)=\dfrac{1}{2}\cdot\dfrac{1-\dfrac{1}{2^n}}{1-\dfrac{1}{2}}<1.$

x_n单调增加有上界，故有极限.

(3) $x_n=\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}.$ $x_{n+1}-x_n=\dfrac{1}{2n+2}-\dfrac{1}{n+1}=-\dfrac{1}{2n+2}<0,$

$x_{n+1}<x_n$，$x_n>0$，x_n单调减少有下界，故有极限.

(4) $x_n=1+1+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}.$ $x_{n+1}-x_n=\dfrac{1}{(n+1)!}>0,$

$x_n\le 2+\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\cdots+\left(\dfrac{1}{n-1}-\dfrac{1}{n}\right)=3-\dfrac{1}{n}<3.$

x_n单调增加有上界，故有极限.

9. 证明 $e=\lim\limits_{n\to\infty}\left(1+1+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right).$

证 $\left(1+\frac{1}{n}\right)^n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}+$

$\cdots+\frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^n}$

$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$<1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}.e=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\le\lim\limits_{n\to\infty}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right).$

对于固定的正整数k, 由上式, 当$n>k$时,

$\left(1+\frac{1}{n}\right)^n>2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right),$

令$n\to\infty$得$e\ge\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{k!}\right),$

$e\ge\lim\limits_{k\to\infty}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{k!}\right)=\lim\limits_{n\to\infty}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right).$

10.设满足下列条件:$|x_{n+1}|\le k|x_n|, n=1,2,\cdots$,其中是小于1的正数.证明 $\lim\limits_{n\to\infty}x_n=0.$

证由$|x_{n+1}|\le k|x_n|\le k^2|x_{n-1}|\le\cdots k^{n-1}|x_1|\to 0(n\to\infty)$,得$\lim\limits_{n\to\infty}x_n=0.$

习题 1.4

1.直接用ε-δ说法证明下列各极限等式:

(1)$\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}(a>0)$;(2)$\lim\limits_{x\to a}x^2=a^2$;(3)$\lim\limits_{x\to a}e^x=e^a$;(4)$\lim\limits_{x\to a}\cos x=\cos a$.

证 (1)$\forall\varepsilon>0$,要使$|\sqrt{x}-\sqrt{a}|=\dfrac{|x-a|}{\sqrt{x}-\sqrt{a}}<\varepsilon$,由于$\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}}<\dfrac{|x-a|}{\sqrt{a}}$,

只需$\dfrac{|x-a|}{\sqrt{a}}<\varepsilon,|x-a|<\sqrt{a}\varepsilon$.取$\delta=\sqrt{a}\varepsilon$,则当$|x-a|<\delta$时,$|\sqrt{x}-\sqrt{a}|<\varepsilon$,故$\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}$.

(2)$\forall\varepsilon>0$,不妨设$|x-a|<1$.要使$|x^2-a^2|=|x+a||x-a|<\varepsilon$,由于

$|x+a|\le|x-a|+|2a|<1+|2a|$,

只需$(1+|2a|)|x-a|<\varepsilon,|x-a|<\dfrac{\varepsilon}{1+|2a|}$.取$\delta=\min\{\dfrac{\varepsilon}{1+|2a|},1\}$,则当$|x-a|<\delta$时,

$|x^2-a^2|<\varepsilon$,故$\lim\limits_{x\to a}x^2=a^2$.

(3) $\forall\varepsilon>0$,设$x>a$.要使$|e^x-e^a|=e^a(e^{x-a}-1)<\varepsilon$,即$0<(e^{x-a}-1)<\dfrac{\varepsilon}{e^a},1<e^{x-a}<1+\dfrac{\varepsilon}{e^a}$,

$0<x-a<\ln\left(1+\dfrac{\varepsilon}{e^a}\right)$,取$\delta=\min\{\dfrac{\varepsilon}{1+|2a|},1\}$,则当$0<x-a<\delta$时,$|e^x-e^a|<\varepsilon$,

故$\lim\limits_{x\to a+}e^x=e^a$. 类似证$\lim\limits_{x\to a-}e^x=e^a$.故$\lim\limits_{x\to a}e^x=e^a$.

(4)$\forall\varepsilon>0$,要使$|\cos x-\cos a|=2\left|\sin\dfrac{x+a}{2}\sin\dfrac{x-a}{2}\right|=2\left|\sin\dfrac{x+a}{2}\right|\left|\sin\dfrac{x-a}{2}\right|\le|x-a|$,

取$\delta=\varepsilon$,则当$|x-a|<\delta$时,$|\cos x-\cos a|<\varepsilon$,故$\lim\limits_{x\to a}\cos x=\cos a$.

2.设$\lim\limits_{x\to a}f(x)=l$,证明存在$a$的一个空心邻域$(a-\delta,a)\cup(a,a+\delta)$,使得函数$u=f(x)$在该邻域内使有界函数.

证对于$\varepsilon=1$,存在$\delta>0$,使得当 $0<|x-a|<\delta$时,$|f(x)-l|<1$,从而

$|f(x)|=|f(x)-l+l|\le|f(x)-l|+|l|<1+|l|=M$.

3.求下列极限:

(1)$\lim\limits_{x\to0}\dfrac{(1+x)^2-1}{2x}=\lim\limits_{x\to0}\dfrac{2x+x^2}{2x}=\lim\limits_{x\to0}(1+\dfrac{x}{2})=1$.

(2)$\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to0}\dfrac{2\sin^2\left(\dfrac{x}{2}\right)}{x^2}=\dfrac{1}{2}\lim\limits_{x\to0}\left(\dfrac{\sin\left(\dfrac{x}{2}\right)}{\dfrac{x}{2}}\right)^2=\dfrac{1}{2}\cdot1^2=\dfrac{1}{2}$.

(3)$\lim\limits_{x\to0}\dfrac{\sqrt{x+a}-\sqrt{a}}{x}=\lim\limits_{x\to0}\dfrac{x}{x(\sqrt{x+a}+\sqrt{a})}=\dfrac{1}{2\sqrt{a}}(a>0)$.

(4)$\lim\limits_{x\to1}\dfrac{x^2-x-2}{2x^2-2x-3}=\dfrac{-2}{-3}$.

(5)$\lim\limits_{x\to0}\dfrac{x^2-x-2}{2x^2-2x-3}=\dfrac{-2}{-3}$.

$$(6)\lim_{x\to\infty}\frac{(2x-3)^{20}(2x+2)^{10}}{(2x+1)^{30}}=\frac{2^{30}}{2^{30}}=1.$$

$$(7)\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}=1.$$

$$(8)\lim_{x\to -1}\left(\frac{1}{x+1}-\frac{3}{x^3+1}\right)=\lim_{x\to -1}\frac{x^2-x+1-3}{(x+1)(x^2-x+1)}=\lim_{x\to -1}\frac{x^2-x-2}{(x+1)(x^2-x+1)}$$

$$=\lim_{x\to -1}\frac{(x+1)(x-2)}{(x+1)(x^2-x+1)}=\lim_{x\to -1}\frac{(x-2)}{(x^2-x+1)}=\frac{-3}{3}=-1.$$

$$(9)\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)(\sqrt{1+2x}+3)}{(\sqrt{x}-2)(\sqrt{x}+2)(\sqrt{1+2x}+3)}$$

$$=\lim_{x\to 4}\frac{(2x-8)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)}=\frac{2\cdot 4}{6}=\frac{4}{3}.$$

$$(10)\lim_{x\to 1}\frac{x^n-1}{x-1}=\lim_{y\to 0}\frac{(1+y)^n-1}{y}=\lim_{y\to 0}\frac{ny+\frac{n(n-1)}{2}y^2+\cdots+y^n}{y}=n.$$

$$(11)\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}=0.$$

$$(12)\lim_{x\to 0}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}(b_n\neq 0)=\frac{a_m}{b_n}.$$

$$(13)\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}(a_0\cdot b_0\neq 0)=\begin{cases}a_0/b_0, & m=n\\ 0, & n>m\\ \infty, & m>n.\end{cases}$$

$$(14)\lim_{x\to\infty}\frac{\sqrt{x^4+8}}{x^2+1}=\lim_{x\to\infty}\frac{\sqrt{1+8/x^4}}{1+1/x^2}=1.$$

$$(15)\lim_{x\to 0}\frac{\sqrt[3]{1+3x}-\sqrt[3]{1-2x}}{x+x^2}$$

$$=\lim_{x\to 0}\frac{(\sqrt[3]{1+3x}-\sqrt[3]{1-2x})(\sqrt[3]{1+3x}^2+\sqrt[3]{1+3x}\cdot\sqrt[3]{1-2x}+\sqrt[3]{1-2x}^2)}{(x+x^2)(\sqrt[3]{1+3x}^2+\sqrt[3]{1+3x}\cdot\sqrt[3]{1-2x}+\sqrt[3]{1-2x}^2)}$$

$$=\lim_{x\to 0}\frac{5x}{x(1+x)(\sqrt[3]{1+3x}^2+\sqrt[3]{1+3x}\cdot\sqrt[3]{1-2x}+\sqrt[3]{1-2x}^2)}$$

$$=\lim_{x\to 0}\frac{5}{(1+x)(\sqrt[3]{1+3x}^2+\sqrt[3]{1+3x}\cdot\sqrt[3]{1-2x}+\sqrt[3]{1-2x}^2)}=\frac{5}{3}.$$

$$(16)a>0,\ \lim_{x\to a+0}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\lim_{x\to a+0}\left(\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}+\frac{1}{\sqrt{x+a}}\right)$$

$$=\lim_{x\to a+0}\left(\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{\sqrt{x+a}\sqrt{x-a}(\sqrt{x}+\sqrt{a})}+\frac{1}{\sqrt{x+a}}\right)$$

$$=\lim_{x\to a+0}\left(\frac{(x-a)}{\sqrt{x+a}\sqrt{x-a}(\sqrt{x}+\sqrt{a})}+\frac{1}{\sqrt{x+a}}\right)$$

$$=\lim_{x\to a+0}\left(\frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})}+\frac{1}{\sqrt{x+a}}\right)=\frac{1}{\sqrt{2a}}.$$

4.利用$\lim\limits_{x\to}\frac{\sin x}{x}=1$及$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$求下列极限:

(1) $\lim\limits_{x\to 0}\frac{\sin\alpha x}{\tan\beta x}=\lim\limits_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}\lim\limits_{x\to 0}\cos\beta x=\frac{\alpha}{\beta}.$

(2) $\lim\limits_{x\to}\frac{\sin(2x^2)}{3x}=\lim\limits_{x\to 0}\frac{\sin(2x^2)}{2x^2}g\lim\limits_{x\to 0}\frac{2x^2}{3x}=1g0=0$

(3) $\lim\limits_{x\to 0}\frac{\tan 3x-\sin 2x}{\sin 5x}=\lim\limits_{x\to 0}\frac{\tan 3x}{\sin 5x}-\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 5x}=\frac{3}{5}-\frac{2}{5}=\frac{1}{5}.$

(4) $\lim\limits_{x\to 0+}\frac{x}{\sqrt{1-\cos x}}=\lim\limits_{x\to 0+}\frac{x}{\sqrt{2}\sin\frac{x}{2}}=\sqrt{2}.$

(5) $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\frac{\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{\frac{x-a}{2}}=\cos a.$

(6) $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{-x}=\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{\frac{x}{k}(-k)}=\left[\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{\frac{x}{k}}\right]^{-k}=e^{-k}.$

(7) $\lim\limits_{y\to 0}(1-5y)^{1/y}=\left[\lim\limits_{y\to 0}(1-5y)^{1/(5y)}\right]^{-5}=e^{-5}.$

(8) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+100}=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left[\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)\right]^{100}=e.$

5.给出$\lim\limits_{x\to a}f(x)=+\infty$及$\lim\limits_{x\to-\infty}f(x)=-\infty$的严格定义.

$\lim\limits_{x\to a}f(x)=+\infty$: 对于任意给定的$A>0$, 存在$\delta>0$, 使得当$0<|x-a|<\delta$时$f(x)>A$.

$\lim\limits_{x\to-\infty}f(x)=-\infty$: 对于任意给定的$A>0$, 存在$\Delta>0$, 使得当$x<-\Delta$时$f(x)<-A$.

习题 1.5

1.试用$\varepsilon-\delta$说法证明

(1)$\sqrt{1+x^2}$在$x=0$连续

(2)$\sin 5x$在任意一点$x=a$连续.

证(1)$\forall\varepsilon>0$,要使$|\sqrt{1+x^2}-\sqrt{1+0^2}|=\dfrac{x^2}{\sqrt{1+x^2}+1}<\varepsilon$.由于$\dfrac{x^2}{\sqrt{1+x^2}+1}\le x^2$,只需

$x^2<\varepsilon,|x|<\sqrt{\varepsilon}$,取$\delta=\sqrt{\varepsilon}$,则当$|x|<\delta$时有$|\sqrt{1+x^2}-\sqrt{1+0^2}|<\varepsilon$,故$\sqrt{1+x^2}$在$x=0$连续.

(2)(1)$\forall\varepsilon>0$,要使$|\sin 5x-\sin 5a|=2|\cos\dfrac{5x+5a}{2}||\sin\dfrac{5(x-a)}{2}|<\varepsilon$.

由于$2|\cos\dfrac{5x+5a}{2}||\sin\dfrac{5(x-a)}{2}|\le 5|x-a|$,只需$5|x-a|<\varepsilon,|x-a|<\dfrac{\varepsilon}{5}$,

取$\delta=\dfrac{\varepsilon}{5}$,则当$|x-a|<\delta$时有$|\sin 5x-\sin 5a|<\varepsilon$,故$\sin 5x$在任意一点$x=a$连续.

2. 设$y=f(x)$在x_0处连续且$f(x_0)>0$,证明存在$\delta>0$使得当$|x-x_0|<\delta$时$f(x)>0$.

证由于$f(x)$在x_0处连续,对于$\varepsilon=f(x_0)/2$,存在存在$\delta>0$使得当$|x-x_0|<\delta$时

$f(x)-f(x_0)|<f(x_0)/2$, 于是$f(x)>f(x_0)-f(x_0)/2=f(x_0)/2>0$.

3.设$f(x)$在(a,b)上连续,证明$|f(x)|$在(a,b)上也连续,并且问其逆命题是否成立?

证任取 $x_0\in(a,b)$, f在x_0连续.任给$\varepsilon>0$,存在$\delta>0$使得当$|x-x_0|<\delta$时

$|f(x)-f(x_0)|<\varepsilon$,此时$||f(x)|-|f(x_0)||\le|f(x)-f(x_0)|<\varepsilon$,故$|f|$在$x_0$连续.其逆命题

不真,例如$f(x)=\begin{cases}1, x\text{是有理数}\\-1, x\text{是无理数}\end{cases}$处处不连续,但是$|f(x)|\equiv 1$处处连续.

4. 适当地选取a, 使下列函数处处连续:

(1) $f(x)=\begin{cases}\sqrt{1+x^2}, x<0,\\ a+x \quad x\ge 0;\end{cases}$ (2) $f(x)=\begin{cases}\ln(1+x), \quad x\ge 1,\\ a\arccos\pi x, x<1.\end{cases}$

解(1) $\lim\limits_{x\to 0-}f(x)=\lim\limits_{x\to 0-}\sqrt{1+x^2}=1=f(0)$, $\lim\limits_{x\to 0+}f(x)=f(0)=a=1$.

(2) $\lim\limits_{x\to 1+}f(x)=\lim\limits_{x\to 1+}\ln(1+x)=\ln 2=f(1)$, $\lim\limits_{x\to 1-}f(x)=\lim\limits_{x\to 1-}a\arccos\pi x=-a=f(1)=\ln 2$,

$a=-\ln 2$.

5.利用初等函数的连续性及定理3求下列极限:

(1) $\lim\limits_{x\to+\infty}\cos\dfrac{\sqrt{1+x}-\sqrt{x}}{x}=\cos\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+x}-\sqrt{x}}{x}=\cos 0=1$.

(2) $\lim\limits_{x\to 2}x^{\sqrt{x}}=2^{\sqrt{2}}$.

(3) $\lim\limits_{x\to 0}e^{\frac{\sin 2x}{\sin 3x}}=e^{\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x}}=e^{\frac{2}{3}}$.

(4) $\lim\limits_{x\to\infty}\arctan\dfrac{\sqrt{x^4+8}}{x^2+1}=\arctan\lim\limits_{x\to\infty}\dfrac{\sqrt{x^4+8}}{x^2+1}=\arctan 1=\dfrac{\pi}{4}$.

(5) $\lim\limits_{x\to\infty}\sqrt{(\sqrt{x^2+1}-\sqrt{x^2-2})\,|x|}=\sqrt{\lim\limits_{x\to\infty}\left[(\sqrt{x^2+1}-\sqrt{x^2-2})\,|x|\right]}$

$=\sqrt{\lim\limits_{x\to\infty}\left[\dfrac{3|x|}{\sqrt{x^2+1}+\sqrt{x^2-2}}\right]}=\sqrt{\lim\limits_{x\to\infty}\left[\dfrac{3}{\sqrt{1+1/x^2}+\sqrt{1-2/x^2}}\right]}=\sqrt{\dfrac{3}{2}}.$

6.设 $\lim\limits_{x\to x_0}f(x)=a>0,\lim\limits_{x\to x_0}g(x)=b,$ 证明 $\lim\limits_{x\to x_0})f(x)^{g(x)}=a^b.$

证 $\lim\limits_{x\to x_0})f(x)^{g(x)}=\lim\limits_{x\to x_0})e^{(\ln f(x))g(x)}=e^{\lim\limits_{x\to x_0}[(\ln f(x))g(x)]}=e^{b\ln a}=a^b.$

7.指出下列函数的间断点及其类型,若是可去间断点,请修改函数在该点的函数值,使之称为连续函数:

(1) $f(x)=\cos\pi(x-[x])$,间断点 $n\in\mathbf{Z}$,第一类间断点.

(2) $f(x)=\operatorname{sgn}(\sin x)$,间断点 $n\pi,n\in\mathbf{Z}$,第一类间断点.

(3) $f(x)=\begin{cases}x^2,x\neq 1,\\ 1/2,x=1.\end{cases}$ 间断点 $x=1$,第一类间断点.

(4) $f(x)=\begin{cases}x^2+1,0\le x\le 1\\ \sin\dfrac{\pi}{x-1},1<x\le 2,\end{cases}$ 间断点 $x=1$,第二类间断点.

(5) $f(x)=\begin{cases}\dfrac{1}{2-x},0\le x\le 1,\\ x,1<x\le 2,\\ \dfrac{1}{1-x},2<x\le 3.\end{cases}$ 间断点 $x=2$,第一类间断点.

8.设 $y=f(x)$ 在 $\mathbf{R}$ 上是连续函数,而 $y=g(x)$ 在 $\mathbf{R}$ 上有定义,但在一点 x_0 处间断.
问函数 $h(x)=f(x)+g(x)$ 及 $\varphi(x)=f(x)g(x)$ 在 x_0 点是否一定间断?

解 $h(x)=f(x)+g(x)$ 在 x_0 点一定间断. 因为如果它在 x_0 点连续,
$g(x)=(f(x)+g(x))-f(x)$ 将在 x_0 点连续, 矛盾. 而 $\varphi(x)=f(x)g(x)$ 在 x_0 点
未必间断. 例如 $f(x)\equiv 0,g(x)=D(x)$.

习题 1.6

1.证明:任一奇数次实系数多项式至少有一实根.

证设$P(x)$是一奇数次实系数多项式, 不妨设首项系数是正数, 则$\lim\limits_{x\to+\infty} P(x)=+\infty$, $\lim\limits_{x\to-\infty} P(x)=-\infty$,存在$A,B,A<B,P(A)<0,P(B)>0,P$在$[A,B]$连续,根据连续函数的中间值定理,存在$x_0\in(A,B)$,使得$P(x_0)=0$.

2.设$0<\varepsilon<1$,证明对于任意一个$y_0\in\mathbf{R}$,方程$y_0=x-\varepsilon\sin x$有解,且解是唯一的.

证令$f(x)=x-\varepsilon\sin x$, $f(-|y_0|-1)=-|y_0|-1+\varepsilon<-|y_0|\le y_0$,

$f(|y_0|+1)\ge|y_0|+1-\varepsilon>|y_0|\ge y_0$, f在$[-|y_0|-1,|y_0|+1]$连续,由中间值定理,存在$x_0\in[-|y_0|-1,|y_0|+1]$, $f(x_0)=y_0$.设$x_2>x_1$,

$f(x_2)-f(x_1)=x_2-x_1-\varepsilon(\sin x_2-\sin x_1)\ge x_2-x_1-\varepsilon|x_2-x_1|>0$,故解唯一.

3.设$f(x)$在(a,b)连续,又设$x_1,x_2\in(a,b),m_1>0,m_2>0$,证明存在$\xi\in(a,b)$使得

$$f(\xi)=\frac{m_1f(x_1)+m_2f(x_2)}{m_1+m_2}.$$

证如果$f(x_1)=f(x_2)$,取$\xi=x_1$即可.设$f(x_1)<f(x_2)$,则

$$f(x_1)=\frac{m_1f(x_1)+m_2f(x_1)}{m_1+m_2}\le\frac{m_1f(x_1)+m_2f(x_2)}{m_1+m_2}\le\frac{m_1f(x_2)+m_2f(x_2)}{m_1+m_2}=f(x_2),$$

在$[x_1,x_2]$上利用连续函数的中间值定理即可.

4.设$y=f(x)$在$[0,1]$上连续且$0\le f(x)\le1,\forall x\in[0,1]$.证明在存在一点$t\in[0,1]$使得$f(t)=t$.

证$g(t)=f(t)-t,g(0)=f(0)\ge0,g(1)=f(1)-1\le0$.如果有一个等号成立,取$t$为0或1.如果等号都不成立,则由连续函数的中间值定理,存在$t\in(0,1)$,使得$g(t)=0$,即$f(t)=t$.

5.设$y=f(x)$在$[0,2]$上连续,且$f(0)=f(2)$.证明在$[0,2]$存在两点x_1与x_2,使得$|x_1-x_2|=1$,且$f(x_1)=f(x_2)$.

证令$g(x)=f(x+1)-f(x),x\in[0,1]$.

$g(0)=f(1)-f(0),g(1)=f(2)-f(1)=f(0)-f(1)=-g(0)$.如果$g(0)=0$,则$f(1)=f(0)$,取$x_1=0,x_2=1$.如果$g(0)\ne0$,则$g(0),g(1)$异号,由连续函数的中间值定理,存在$\xi\in(0,1)$使得$g(\xi)=f(\xi+1)-f(\xi)=0$,取$x_1=\xi,x_2=\xi+1$.

第一章总练习题

1.求解下列不等式：

(1) $\left|\dfrac{5x-8}{3}\right| \ge 2.$

解 $\dfrac{|5x-8|}{3} \ge 2. |5x-8| \ge 6, 5x-8 \ge 6$或$5x-8 \le -6, x \ge \dfrac{14}{5}$或$x \le \dfrac{2}{5}$.

(2) $\left|\dfrac{2}{5}x-3\right| \le 3,$

解 $-3 \le \dfrac{2}{5}x-3 \le 3, 0 \le x \le 15.$

(3) $|x+1| \ge |x-2|$

解$(x+1)^2 \ge (x-2)^2, 2x+1 \ge -4x+4, x \ge \dfrac{1}{2}.$

2.设$y = 2x+|2-x|$，试将x表示成y的函数.

解 当$x \le 2$时，$y = x+2, y \le 4, x = y-2$；当$x > 2$时，$y = 3x-2, y > 4, x = \dfrac{1}{3}(y-2).$

$$x = \begin{cases} y-2, y \le 4 \\ \dfrac{1}{3}(y-2), y > 4. \end{cases}$$

3.求出满足不等式$\sqrt{1+x} < 1+\dfrac{1}{2}x$的全部$x$.

解$x \ge -1$. $2\sqrt{1+x} < x+2, 4(1+x) < x^2+4x+4, x^2 > 0. x \ge -1, x \ne 0.$

4.用数学归纳法证明下列等式：

(1) $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\cdots+\dfrac{n}{2^n} = 2-\dfrac{n+2}{2^n}.$

①代入
②设对n
成立

证 当$n=1$时，$2-\dfrac{1+2}{2^1} = \dfrac{1}{2}$，等式成立.设等式对于$n$成立，则

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n+1}{2^{n+1}} = \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}+\frac{n+1}{2^{n+1}}$$
$$= 2-\frac{n+2}{2^n}+\frac{n+1}{2^{n+1}} = 2-\frac{2n+4-(n+1)}{2^{n+1}} = 2-\frac{(n+1)+3}{2^{n+1}},$$

即等式对于$n+1$也成立.故等式对于任意正整数皆成立.

(2) $1+2x+3x^2+\cdots+nx^{n-1} = \dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} (x \ne 1).$

证 当$n=1$时，$\dfrac{1-(1+1)x^n+1x^{1+1}}{(1-x)^2} = \dfrac{(1-x)^2}{(1-x)^2} = 1$，等式成立.

设等式对于n成立，则

$$1+2x+3x^2+\cdots+nx^{n-1}+(n+1)x^n = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}+(n+1)x^n$$

$$=\frac{1-(n+1)x^n+nx^{n+1}+(1-x)^2(n+1)x^n}{(1-x)^2}$$

$$=\frac{1-(n+1)x^n+nx^{n+1}+(1-2x+x^2)(n+1)x^n}{(1-x)^2}$$

$$=\frac{1-(n+1)x^n+nx^{n+1}+(x^n-2x^{n+1}+x^{n+2})(n+1)}{(1-x)^2}$$

$$=\frac{1-(n+1)x^n+nx^{n+1}+(x^n-2x^{n+1}+x^{n+2})(n+1)}{(1-x)^2}$$

$$=\frac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2},$$

即等式对于$n+1$成立.由归纳原理,等式对于所有正整数都成立.

5.设$f(x)=\dfrac{|2+x|-|x|-2}{x}$

(1)求$f(-4), f(-1), f(-2), f(2)$的值;

(2)将$f(x)$表成分段函数;

(3)当$x\to 0$时$f(x)$是否有极限:

(4)当$x\to -2$时是否有极限?

解(1) $f(-4)=\dfrac{2-4-2}{-4}=-1,\ f(-1)=\dfrac{1-1-2}{-1}=2,\ f(-2)=\dfrac{-2-2}{-2}=2,\ f(2)=\dfrac{4-2-2}{2}=0.$

(2) $f(x)=\begin{cases}-4/x, x\le -2;\\ 2, -2<x\le 0;\\ 0, x>0.\end{cases}$

(3)无.因为$\lim\limits_{x\to 0-}f(x)=2, \lim\limits_{x\to 0+}f(x)=0\ne\lim\limits_{x\to 0-}f(x).$

(4)有.$\lim\limits_{x\to -2-}f(x)=\lim\limits_{x\to -2-}(-4/x)=2, \lim\limits_{x\to -2+}f(x)=\lim\limits_{x\to -2+}2=2=\lim\limits_{x\to -2-}f(x), \lim\limits_{x\to -2}f(x)=2.$

6.设$f(x)=[x^2-14]$,即$f(x)$是不超过x^2-14的最大整数.

(1)求$f(0), f\left(\dfrac{3}{2}\right), f(\sqrt{2})$的值;

(2) $f(x)$在$x=0$处是否连续?

(3) $f(x)$在$x=\sqrt{2}$处是否连续?

解(1) $f(0)=[-14]=-14, f\left(\dfrac{3}{2}\right)=\left[\dfrac{9}{4}-14\right]=\left[-6+\dfrac{1}{4}\right]=-7. f(\sqrt{2})=[-12]=-12.$

(2)连续.因为$\lim\limits_{x\to 0}f(x)=\lim\limits_{y\to 0+}[y-14]=-14=f(0).$

(3)不连续.因为$\lim\limits_{x\to\sqrt{2}+}f(x)=-12, \lim\limits_{x\to\sqrt{2}-}f(x)=-11.$

7.设两常数a,b满足$0\le a<b$,对一切自然数n,证明:

(1) $\dfrac{b^{n+1}-a^{n+1}}{b-a}<(n+1)b^n$; (2) $(n+1)a^n<\dfrac{b^{n+1}-a^{n+1}}{b-a}.$

证 $\dfrac{b^{n+1}-a^{n+1}}{b-a}=\dfrac{(b-a)(b^n+b^{n-1}a+\cdots+a^n)}{b-a}<b^n+b^{n-1}b+\cdots+b^n=(n+1)b^n$,

类似有 $\dfrac{b^{n+1}-a^{n+1}}{b-a}>(n+1)a^n$.

8.对 $n=1,2,3,\cdots$, 令 $a_n=\left(1+\dfrac{1}{n}\right)^n, b_n=\left(1+\dfrac{1}{n}\right)^{n+1}$.

证明:序列$\{a_n\}$单调上升,而序列$\{b_n\}$单调下降, 并且$a_n<b_n$.

证令 $a=1+\dfrac{1}{n+1}, b=1+\dfrac{1}{n}$,则由7题中的不等式,

$$\frac{\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n+1}\right)^{n+1}}{\frac{1}{n}-\frac{1}{n+1}}<(n+1)\left(1+\frac{1}{n}\right)^n,$$

$$\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n+1}\right)^{n+1}<(n+1)\left(1+\frac{1}{n}\right)^n\frac{1}{n(n+1)}$$

$$\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n\frac{1}{n}<\left(1+\frac{1}{n+1}\right)^{n+1},$$

$$\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}.$$

$$(n+1)\left(1+\frac{1}{n+1}\right)^n<\frac{\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n+1}\right)^{n+1}}{\frac{1}{n}-\frac{1}{n+1}}$$

$$(n+1)\left(1+\frac{1}{n+1}\right)^n\frac{1}{n(n+1)}<\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n+1}\right)^n\frac{1}{n}<\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n+1}\right)^n\left(\frac{1}{n}+1+\frac{1}{n+1}\right)<\left(1+\frac{1}{n}\right)^{n+1}.$$

我们证明 $\dfrac{1}{n}+1+\dfrac{1}{n+1}>\left(1+\dfrac{1}{n+1}\right)^2$.

$$\Leftrightarrow\frac{1}{n}+1+\frac{1}{n+1}>1+\frac{2}{n+1}+\frac{1}{(n+1)^2}$$

$\Leftrightarrow\dfrac{1}{n(n+1)}>\dfrac{1}{(n+1)^2}$.最后不等式显然成立.

当 $n\to\infty$ 时, $\left(1+\dfrac{1}{n}\right)^n\to e, \left(1+\dfrac{1}{n}\right)^{n+1}\to e$, 故 $\left(1+\dfrac{1}{n}\right)^n<e<\left(1+\dfrac{1}{n}\right)^{n+1}$.

9.求极限

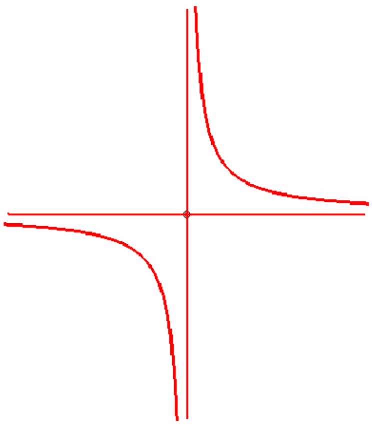
$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

解$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$

$$=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdots\frac{n}{n}\cdot\frac{n+1}{n}=\frac{1}{n}\cdot\frac{n+1}{2}\to\frac{1}{2}(n\to\infty).$$

10.作函数$f(x)=\lim\limits_{n\to\infty}\frac{nx}{nx^2+a}(a\neq 0)$的图形.

解$f(x)=\lim\limits_{n\to\infty}\frac{nx}{nx^2+a}=\begin{cases}0, & x=0;\\ 1/x, & x\neq 0.\end{cases}$



11.在？ 关于有界函数的定义下,证明函数$f(x)$在区间$[a,b]$上为有界函数的充要条件为存在一个正的常数M使得$|f(x)|<M,\forall x\in[a,b]$.

证设存在常数M_1,N使得$M_1\le f(x)\le N,\forall x\in[a,b]$,取$M=\max\{|M_1|,|N|\}+1$,则有$|f(x)|<M,\forall x\in[a,b]$.

反之,若存在一个正的常数M使得$|f(x)|<M,\forall x\in[a,b]$,则$-M<f(x)<M,\forall x\in[a,b]$.

12.证明:若函数$y=f(x)$及$y=g(x)$在$[a,b]$上均为有界函数,则$f(x)+g(x)$及$f(x)g(x)$也都是$[a,b]$上的有界函数.

证存在$M_1,M_2,|f(x)|<M_1,|g(x)|<M_2,\forall x\in[a,b]$.$|f(x)+g(x)|\le|f(x)|+|g(x)|<M_1+M_2$,
$|f(x)g(x)|=|f(x)||g(x)|<M_1M_2,\forall x\in[a,b]$.

13.证明: $f(x)=\frac{1}{x}\cos\frac{\pi}{x}$在$x=0$的任一邻域内都是无界的,但当$x\to 0$时$f(x)$不是无穷大量.

证任取一个邻域$(-\delta,\delta),\delta>0$和$M>0$,取正整数$n$,满足$\frac{1}{n}<\delta$和$n>M$,则$\left|f(\frac{1}{n})\right|=n>M$,

故$f(x)$在$(-\delta,\delta)$无界. 但是$x_n=\frac{1}{2n+1/2}\to 0$, $f(x_n)=(2n+1/2)\cos(2n+1/2)\pi=0\not\to\infty$,

故当$x\to 0$时$f(x)$不是无穷大量.

14.证明$\lim\limits_{n\to\infty} n(x^{\frac{1}{n}}-1)=\ln x(x>0).$

证令$x^{\frac{1}{n}}-1=y_n$,则$\frac{1}{n}\ln x=\ln(1+y),n=\frac{\ln x}{\ln(1+y)}.\lim\limits_{n\to\infty} y_n=\lim\limits_{n\to\infty} x^{\frac{1}{n}}-1=0.$

注意到$\lim\limits_{y\to 0}\frac{\ln(1+y)}{y}=\lim\limits_{y\to 0}\ln(1+y)^{\frac{1}{y}}=\ln\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}=\ln e=1,$

我们有$n(x^{\frac{1}{n}}-1)=\frac{y_n\ln x}{\ln(1+y_n)}\to\ln x(n\to\infty).$

15.设$f(x)$及$g(x)$在实轴上有定义且连续.证明:若$f(x)$与$g(x)$在有理数集合处处相等,则它们在整个实轴上处处相等.

证任取一个无理数x_0,取有理数序列$x_n\to x_0$, $f(x_0)=\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty} g(x_n)=g(x_0).$

16.证明$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}.$

证$\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim\limits_{y\to 0}\frac{2\sin^2 y}{4y^2}=\frac{1}{2}\left(\lim\limits_{y\to 0}\frac{\sin y}{y}\right)^2=\frac{1}{2}\cdot 1^2=\frac{1}{2}.$

17.证明:(1)$\lim\limits_{y\to 0}\frac{\ln(1+y)}{y}=1$;(2)$\lim\limits_{x\to 0}\frac{e^{x+a}-e^x}{x}=e^a.$

证(1)$\lim\limits_{y\to 0}\frac{\ln(1+y)}{y}=\lim\limits_{y\to 0}\ln(1+y)^{\frac{1}{y}}=\ln\lim\limits_{y\to 0}(1+y)^{\frac{1}{y}}=\ln e=1.$

(2)$\lim\limits_{x\to 0}\frac{e^{x+a}-e^a}{x}=\lim\limits_{x\to 0}\frac{e^a(e^x-1)}{x}=e^a\lim\limits_{x\to 0}\frac{e^x-1}{x}=e^a\lim\limits_{y\to 0}\frac{y}{\ln(1+y)}=e^a\frac{1}{\lim\limits_{y\to 0}\frac{\ln(1+y)}{y}}$

$=e^a\cdot\frac{1}{1}=e^a.$

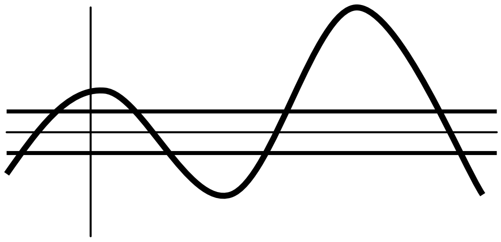
18.设$y=f(x)$在a点附近有定义且有极限$\lim\limits_{x\to a} f(x)=0$,又设$y=g(x)$在$a$点附近有定义,且是有界函数.证明$\lim\limits_{x\to a} f(x)g(x)=0.$

证设$|g(x)|<M,0<|x-a|<\delta_0$.对于任意$\varepsilon>0$,存在$\delta_1>0$,使得当$0<|x-a|<\delta_1$时$|f(x)|<\varepsilon/M$.令$\delta=\min\{\delta_1,\delta_0\}$,则$0<|x-a|<\delta$时,$|f(x)g(x)|=|f(x)||g(x)|<\frac{\varepsilon}{M}\cdot M=\varepsilon$,故$\lim\limits_{x\to a} f(x)g(x)=0.$

19.设$y=f(x)$在$(-\infty,+\infty)$中连续,又设c为正的常数,定义$g(x)$如下

$$g(x)=\begin{cases} f(x) & 当|f(x)|\le c \\ c & 当f(x)>c \\ -c & 当f(x)<-c \end{cases}$$

试画出$g(x)$的略图,并证明$g(x)$在$(-\infty,+\infty)$上连续.



证(一)若$|f(x_0)|<c$,则存在$\delta_0>0$,当$|x-x_0|<\delta_0$时$|f(x)|<c$, $g(x)=f(x)$,
$\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0} f(x)=f(x_0)=g(x_0)$.
若$f(x_0)>c$,则存在$\delta_0>0$,当$|x-x_0|<\delta_0$时$f(x)>c$, $g(x)=c$,
$\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0} c=c=g(x_0)$.
若$f(x_0)=c$,则$g(x_0)=c$.对于任意$\varepsilon>0$,不妨设$\varepsilon<c$,存在$\delta>0$,使得当$|x-x_0|<\delta$时
$|f(x)-c|<\varepsilon$.设$|x-x_0|<\delta$.若$f(x)\le c$,则$g(x)=f(x)$,$|g(x)-g(x_0)|=|f(x)-c|<\varepsilon$,
若$f(x)>c$, 则$g(x)=c$,$|g(x)-g(x_0)|=0<\varepsilon$.
证(二)利用$g(x)=\min\{f(x),c\}+\max\{f(x),-c\}-f(x)$.
$\max\{f_1(x),f_2(x)\}=(|f_1(x)-f_2(x)|+f_1(x)+f_2(x))/2$.
$\min\{f_1(x),f_2(x)\}=(-|f_1(x)-f_2(x)|+(f_1(x)+f_2(x))/2$.
20.设$f(x)$在$[a,b]$上连续,又设$\eta=\dfrac{1}{3}[f(x_1)+f(x_2)+f(x_3)]$,
其中$x_1,x_2,x_3\in[a,b]$.证明存在一点$c\in[a,b]$,使得$f(c)=\eta$.
证若$f(x_1)=f(x_2)=f(x_3)$,则$\eta=f(x_1)$,取$c=x_1$即可.
否则设$f(x_1)=\min\{f(x_1),f(x_2),f(x_3)\}$, $f(x_3)=\min\{f(x_1),f(x_2),f(x_3)\}$,
$f(x_1)<\eta<f(x_3)$, f在$[x_1,x_3]$连续,根据连续函数的中间值定理,存在一点$c\in[a,b]$,
使得$f(c)=\eta$.
21.设 $y=f(x)$在点x_0连续而$g(x)$在点x_0附近有定义,但在x_0不连续问$kf(x)+l\,g(x)$
是否在x_0连续,其中k,l为常数.
解如果$l=0$,$kf(x)+l\,g(x)$在x_0连续;如果$1\ne 0$,$kf(x)+l\,g(x)$在x_0不连续, 因否则
$g(x)=[[kf(x)+\lg(x)]-kf(x)]/l$将在$x_0$连续.
22.证明Dirichlet函数处处不连续.
证任意取x_0.取有理数列$x_n\to x_0$,则$D(x_n)\to 1$;取无理数列$x_n'\to x_0$,则$D(x_n')\to 0$;
故$\lim\limits_{x\to x_0} D(x)$不存在,$D(x)$在$x_0$不连续.
23. 求下列极限:
(1)$\lim\limits_{x\to\infty}\left(\dfrac{1+x}{1+2x}\right)^{|x|}=0$;(2)$\lim\limits_{x\to+\infty}(\arctan x)\sin\dfrac{1}{x}=\dfrac{\pi}{2}g0=0$;
(3)$\lim\limits_{x\to 0}\dfrac{\tan 5x}{\ln(1+x^2)+\sin x}=\lim\limits_{x\to 0}\dfrac{\tan 5x/x}{x[[\ln(1+x^2)]/x^2]+\sin x/x}=\dfrac{5}{1}=5$.
(4)$\lim\limits_{x\to 1}(\sqrt{x})^{\frac{1}{\sqrt{x}-1}}=\lim\limits_{y\to 0}(1+y)^{1/y}=e$.
24.设函数$y=f(x)$在$[0,+\infty)$内连续,且满足$0\le f(x)\le x$.设$a_1\ge 0$是一任意数,并假定
$a_2=f(a_1),a_3=f(a_2),\text{L}$,一般地$a_{n+1}=f(a_n)$.试证明$\{a_n\}$单调递减,且极限$\lim\limits_{n\to\infty}a_n$存在.
若$l=\lim\limits_{n\to\infty}a_n$,则$l$是方程$f(x)=x$的根,即$f(l)=l$.
证$a_{n+1}=f(a_n)\le a_n$,$\{a_n\}$单调递减. 又$a_{n+1}=f(a_n)\ge 0(n=1,2,)$,$\{a_n\}$单调递减有下界,

故a_n有极限. 设$l=\lim\limits_{n\to\infty}a_n$,则$l=\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}f(a_n)=f(\lim\limits_{n\to\infty}a_n)=f(l)$.

25.设函数$y=E(x)$在$(-\infty,+\infty)$内有定义且处处连续,并且满足下列条件:

$E(0)=1, E(1)=e, E(x+y)=E(x)\cdot E(y)$.

证明$E(x)=e^x(\forall x\in(-\infty,+\infty))$.

证用数学归纳法易得$E(x_1+\cdots+x_n)=E(x_1)\cdot\cdots\cdot E(x_n)$.于是$E(nx)=E(x)^n$.

设n是正整数,则$E(n)=E(1+\cdots+1)=E(1)^n=e^n$.

$1=E(0)=E(n+(-n))=E(n)\cdot E(-n)=e^n\cdot E(-n), E(-n)=e^{-n}$.于对于任意整数

$E(n)=e^n$.

对于任意整数$n, E(1)=E(n\cdot\frac{1}{n})=E(n)\cdot E(\frac{1}{n})=e^n\cdot E(\frac{1}{n}),\cdot E(\frac{1}{n})=e^{\frac{1}{n}}$.

$E(\frac{m}{n})=E(m\cdot\frac{1}{n})=\left(E(\frac{1}{n})\right)^m=\left(e^{\frac{1}{n}}\right)^m=e^{\frac{m}{n}}$.即对于所有有理数$r, E(r)=e^r$.

对于无理数x,取有理数列$\mathrm{x_n}\to x$,由$E(x)$的连续性,

$E(x)=\lim\limits_{n\to\infty}E(x_n)=\lim\limits_{n\to\infty}e^{x_n}=e^{\lim\limits_{n\to\infty}x_n}(e^x$的连续性$)=e^x$.

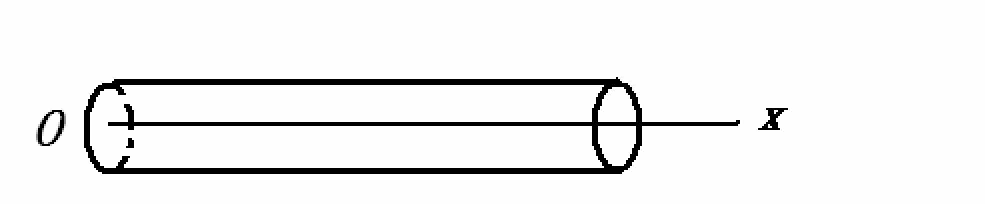
习题 2.1

1.设一物质细杆的长为l,其质量在横截面的分布上可以看作均匀的.现取杆的左端点为坐标原点O,杆所在直线为x轴.设从左端点到细杆上任一点x之间那一段的质量为$m(x)=2x^2(0\le x\le l)$

(1)给自变量x一个增量Δx,求的相应增量Δm;

(2)求比值$\dfrac{\Delta m}{\Delta x}$,问它的物理意义是什么?

(3)求极限$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}$,问它的物理意义是什么?



解(1)$\Delta m=2(x+\Delta x)^2-2x^2=2(x^2+2x\Delta x+\Delta x^2)-2x^2=2(2x\Delta x+\Delta x^2)$.

(2)$\dfrac{\Delta m}{\Delta x}=\dfrac{2(2x\Delta x+\Delta x^2)}{\Delta x}=2(2x+\Delta x)$.$\dfrac{\Delta m}{\Delta x}$是$x$到$x+\Delta x$那段细杆的平均线密度.

(3)$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}=\lim\limits_{\Delta x\to 0}2(2x+\Delta x)=4x$.$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}$是细杆在点$x$的线密度.

2.根据定义,求下列函数的导函数:

(1)$y=ax^3$;(2)$y=\sqrt{2px},p>0$;(3)$y=\sin 5x$.

解(1)$y'=\lim\limits_{\Delta x\to 0}\dfrac{a(x+\Delta x)^3-ax^3}{\Delta x}$

$=a\lim\limits_{\Delta x\to 0}\dfrac{(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3)-x^3}{\Delta x}=a\lim\limits_{\Delta x\to 0}(3x^2+3x\Delta x+\Delta x^2)=3ax^2$.

(2)$y'=\lim\limits_{\Delta x\to 0}\dfrac{\sqrt{2p(x+\Delta x)}-\sqrt{2px}}{\Delta x}=\sqrt{2p}\lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

$=\sqrt{2p}\lim\limits_{\Delta x\to 0}\dfrac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\sqrt{2p}\lim\limits_{\Delta x\to 0}\dfrac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$

$=\sqrt{2p}\lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\dfrac{\sqrt{2p}}{2\sqrt{x}}$.

(3)$y'=\lim\limits_{\Delta x\to 0}\dfrac{\sin 5(x+\Delta x)-\sin 5x}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{2\cos\dfrac{5(2x+\Delta x)}{2}\sin\dfrac{5\Delta x}{2}}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\dfrac{2g\dfrac{5}{2}\cos\dfrac{5(2x+\Delta x)}{2}\sin\dfrac{5\Delta x}{2}}{\dfrac{5\Delta x}{2}}=5\lim\limits_{\Delta x\to 0}\cos\dfrac{5(2x+\Delta x)}{2}\lim\limits_{\Delta x\to 0}\dfrac{\sin\dfrac{5\Delta x}{2}}{\dfrac{5\Delta x}{2}}=5\cos 5x$.

3.求下列曲线$y=f(x)$在指定点$M(x_0,f(x_0))$处的切线方程:

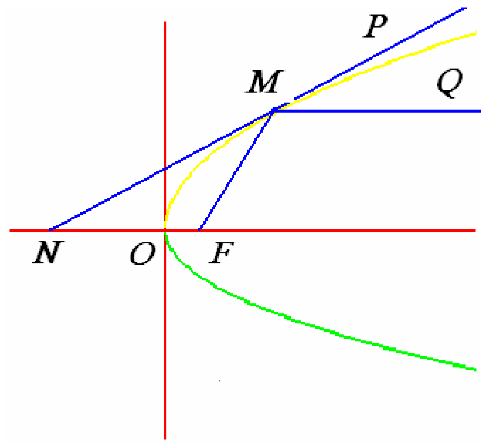
(1)$y=2^x, M(0,1)$; (2)$y=x^2+2, B(3,11)$.

解(1)$y'=2^x\ln 2, y'(0)=\ln 2$,切线方程$y-1=\ln 2(x-0), y=(\ln 2)x+1$.

(2)$y'=2x, y'(3)=6$,切线方程:$y-11=6(x-3)$.

4.试求抛物线$y^2=2px(p>0)$上任一点$M(x,y)(x>0,y>0)$处的切线斜率,

并证明:从抛物线的焦点$F\left(\frac{p}{2},0\right)$发射光线时,其反射线一定平行于$x$轴.



证$y=\sqrt{2px}, y'=\frac{2p}{2\sqrt{2px}}=\frac{p}{y}$,过点$M$的切线$PMN$方程:$Y-y=\frac{p}{y}(X-x)$.

切线与x轴交点$N(X_0,0)$,$-y=\frac{p}{y}(X_0-x), X_0=x-\frac{y^2}{p}=-x$.

$$FN=\frac{p}{2}+x, FM=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}=\sqrt{\left(x-\frac{p}{2}\right)^2+2px}$$

$$=\sqrt{x^2+px+\left(\frac{p}{2}\right)^2}=\sqrt{\left(x+\frac{p}{2}\right)^2}=x+\frac{p}{2}=FN$$,故$\angle FNM=\angle FMN$.

过M作PQ平行于x轴,则$\angle PMQ=\angle FNM=\angle FMN$.

5.曲线$y=x^2+2x+3$上哪一点的切线与直线$y=4x-1$平行,并求曲线在该点的切线和法线方程.

解 $y'=2x+2=4, x_0=1, y_0=6, k=4$

切线方程:$y-6=4(x-1), y=4x+2$.法线方程:$y-6=\left(-\frac{1}{4}\right)(x-1), y=-\frac{1}{4}x+\frac{25}{4}$.

6.离地球中心r处的重力加速度g是r的函数,其表达式为

$$g(r)=\begin{cases}\dfrac{GMr}{R^3}, r<R;\\ \dfrac{GM}{r^2},\ r\ge R\end{cases}$$ 其中R是地球的半径,M是地球的质量,G是引力常数.

(1)问$g(r)$是否为r的连续函数:

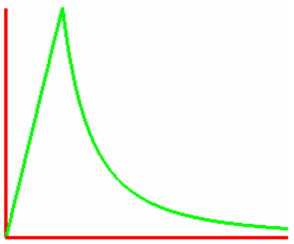
(2)作$g(r)$的草图;

(3)$g(r)$是否是r的可导函数.

解 明显地,$r\ne R$时$g(r)$连续. $\lim\limits_{r\to R-}g(r)=\lim\limits_{r\to R-}\dfrac{GMr}{R^3}=\dfrac{GM}{R^2}$,

$\lim\limits_{r\to R+}g(r)=\lim\limits_{r\to R+}\dfrac{GM}{r^2}=\dfrac{GM}{R^2}=\lim\limits_{r\to R-}g(r)$, $g(r)$在$r=R$连续.

(2)



(3)$r\ne R$时$g(r)$可导.

$g'_-(R)=\dfrac{GM}{R^3}, g'_+(R)=-\dfrac{2GM}{R^3}\ne g'_-(R)$, $g(r)$在$r=R$不可导.

7.求二次函数$P(x)$,已知:点$(1,3)$在曲线$y=P(x)$上,且$P'(0)=3, P'(2)=1$.

解 $P(x)=ax^2+bx+c, P'(x)=2ax+b.$ $\begin{cases}a+b+c=3\\ b=3\\ 4a+b=1\end{cases}$

$b=3, a=-\dfrac{1}{2}, c=3-(a+b)=\dfrac{1}{2}, P(x)=-\dfrac{1}{2}x^2+3x+\dfrac{1}{2}.$

8.求下列函数的导函数:

(1)$y=8x^3+x+7, y'=24x^2+1.$

(2)$y=(5x+3)(6x^2-2), y'=5(6x^2-2)+12x(5x+3)=90x^2+36x-10.$

(3)$y=(x+1)(x-1)\tan x=(x^2-1)\tan x, y'=(2x)\tan x+(x^2-1)\sec^2 x.$

(4)$y=\dfrac{9x+x^2}{5x+6}, y'=\dfrac{(9+2x)(5x+6)-5(9x+x^2)}{(5x+6)^2}=\dfrac{5x^2+12x+54}{(5x+6)^2}.$

(5)$y=\dfrac{1+x}{1-x}=-1+\dfrac{2}{1-x}(x\ne 1), y'=\dfrac{2}{(1-x)^2}.$

$(6) y=\dfrac{2}{x^3-1}(x\neq 1), y'=\dfrac{-6x^2}{(x^3-1)^2}.$

$(7) y=\dfrac{x^2+x+1}{e^x}, y'=\dfrac{(2x+1)e^x-e^x(x^2+x+1)}{e^{2x}}=\dfrac{-x^2+x-1}{e^x}.$

$(8) y=x g 10^x, y'=10^x+x g 10^x \ln 10=10^x(1+x\ln 10).$

$(9) y=x\cos x+\dfrac{\sin x}{x}, y'=\cos x-x\sin x+\dfrac{x\cos x-\sin x}{x^2}.$

$(10) y=e^x\sin x, y'=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x).$

9.定义:若多项式$P(x)$可表为$P(x)=(x-x_0)^m g(x), g(x_0)\neq 0$

则称x_0是$P(x)$的m重根.今若已知x_0是$P(x)$的k重根, 证明x_0是$P'(x)$的$(k-1)$重根

$(k>2)$.

证$P(x)=(x-x_0)^k g(x), g(x_0)\neq 0$

$P'(x)=k(x-x_0)^{k-1}g(x)+(x-x_0)^k g'(x)$

$=(x-x_0)^{k-1}(kg(x)+(x-x_0)g'(x))=(x-x_0)^{k-1}h(x),$

$h(x_0)=kg({}_0x)\neq 0,$由定义x_0是$P'(x)$的$(k-1)$重根.

10.若$f(x)$在$(-a,a)$中有定义,且满足$f(-x)=f(x)$,则称$f(x)$为偶函数.设$f(x)$

是偶函数, 且$f'(0)$存在,试证明$f'(0)=0$.

证$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(-x)-f(0)}{x}=-\lim\limits_{x\to 0}\dfrac{f(-x)-f(0)}{-x}=-f'(0), f'(0)=0.$

11.设$f(x)$在x_0处可导,证明$\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}=2f'(x_0)$.

证 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}=\dfrac{1}{2}\lim\limits_{\Delta x\to 0}\left[\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}-\dfrac{f(x_0-\Delta x)-f(x_0)}{\Delta x}\right]$

$=\dfrac{1}{2}\lim\limits_{\Delta x\to 0}\left[\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+\dfrac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\right]$

$=\dfrac{1}{2}\left[\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+\lim\limits_{\Delta x\to 0}\dfrac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\right]=\dfrac{1}{2}[f'(x_0)+f'(x_0)]=f'(x_0).$

12.一质点沿曲线$y=x^2$运动,且已知时刻$t(0<t<\pi/2)$时质点所在位置

$P(t)=(x(t),y(t))$满足:直线$\overline{OP}$与x轴的夹角恰为t.求时刻t时质点的位置速度及

加速度.

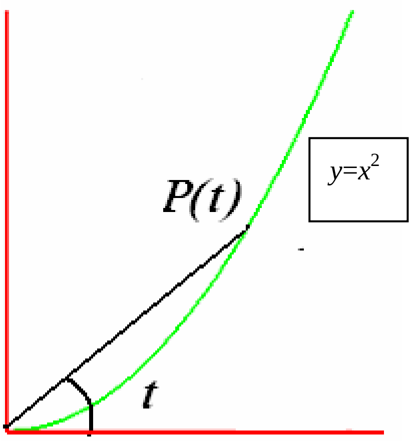
解$\dfrac{y(t)}{x(t)}=\dfrac{x^2(t)}{x(t)}=x(t)=\tan t, y(t)=\tan^2 t,$

位置$(\tan t,\tan^2 t),$

$v'(t)=(\sec^2 t, 2\tan t\sec^2 t),$

$v''(t)=(2\sec^2 t\tan t, 2\sec^4 t+4\tan^2 t\sec^2 t)$

$=2\sec^2 t(\sec^2 t, 2\tan^2 t).$



13.求函数

$$f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, x\neq 0\\ 0, \qquad x=0\end{cases}$$

在$x=0$的左右导数.

解$f'_-(0)=\lim\limits_{x\to 0-}\dfrac{\frac{x}{1+e^{1/x}}}{x}=\lim\limits_{x\to 0-}\dfrac{1}{1+e^{1/x}}=1,\ f'_+(0)=\lim\limits_{x\to 0+}\dfrac{\frac{x}{1+e^{1/x}}}{x}=\lim\limits_{x\to 0+}\dfrac{1}{1+e^{1/x}}=0.$

14.设$f(x)=|x-a|\varphi(x)$,其中$\varphi(x)$在$x=a$处连续且$\varphi(a)\neq 0$.证明$f(x)$在$x=a$不可导.

证$f'_-(a)=\lim\limits_{x\to a-}\dfrac{(a-x)\varphi(x)}{x-a}=-\varphi(a),\mathrm{f}'_+(a)=\lim\limits_{x\to a-}\dfrac{(x-a)\varphi(x)}{x-a}=\varphi(a)\neq f'_-(a).$

习题 2.2

1.下列各题的计算是否正确,指出错误并加以改正:

(1)$(\cos\sqrt{x})'=-\sin\sqrt{x}$,错.$(\cos\sqrt{x})'=-\sin\sqrt{x}\sqrt{x}'=-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$.

(2)$[\ln(1-x)]'=\dfrac{1}{1-x}$,错.$[\ln(1-x)]'=\dfrac{1}{1-x}(1-x)'=\dfrac{1}{x-1}$.

(3)$\left[x^2\sqrt{1+x^2}\right]'=\left(x^2\right)'\left(\sqrt{1+x^2}\right)'=2x\dfrac{x}{\sqrt{1+x^2}}$,错.

$$\left[x^2\sqrt{1+x^2}\right]'=\left(x^2\right)'\left(\sqrt{1+x^2}\right)+\left(x^2\right)\left(\sqrt{1+x^2}\right)'=2x\sqrt{1+x^2}+x^2\frac{x}{\sqrt{1+x^2}}$$

$$=2x\sqrt{1+x^2}+\frac{x^3}{\sqrt{1+x^2}}=\frac{2x+3x^3}{\sqrt{1+x^2}}.$$

(4)$\left[\ln|x+2\sin^2 x|\right]'=\dfrac{1}{x+2\sin^2 x}(1+4\sin x)\cos x$,错.

$$\left[\ln|x+2\sin^2 x|\right]'=\frac{1}{x+2\sin^2 x}(1+4\sin x\cos x).$$

2.记$f'(g(x))=f'(u)|_{u=g(x)}$.现设$f(x)=x^2+1$.

(1)求$f'(x), f'(0), f'(x^2), f'(\sin x)$;

(2)求$\dfrac{d}{dx}f(x^2), \dfrac{d}{dx}f(\sin x)$;

(3)$f'(g(x))$与$\left[f(g(x))\right]'$是否相同?指出两者的关系.

解(1)$f'(x)=2x, f'(0)=0, f'(x^2)=2x^2, f'(\sin x)=2\sin x$.

(2)$\dfrac{d}{dx}f(x^2)=f'(x^2)\left(x^2\right)'=2x^2\cdot 2x=4x^3$.

$$\frac{d}{dx}f(\sin x)=f'(\sin x)(\sin x)'=2\sin x\cos x=\sin 2x.$$

(3)$f'(g(x))$与$\left[f(g(x))\right]'$不同,$\left[f(g(x))\right]'=f'(g(x))g'(x)$.

3.求下列函数的导函数:

(1)$y=\dfrac{2}{x^3-1}, y'=-\dfrac{2\cdot 3x^2}{\left(x^3-1\right)^2}=-\dfrac{6x^2}{\left(x^3-1\right)^2}$.

(2)$y=\sec x, y'=\left((\cos x)^{-1}\right)'=-(\cos x)^{-2}(\cos x)'=-(\cos x)^{-2}(-\sin x)=\tan x\sec x$.

(3)$y=\sin 3x+\cos 5x, y'=3\cos 3x-5\sin 5x$.

(4)$y=\sin^3 x\cos 3x, y'=3\sin^2 x\cos x\cos 3x-3\sin^3 x\sin 3x$

$=3\sin^2 x(\cos x\cos 3x-\sin x\sin 3x)=3\sin^2 x\cos 4x$.

$$(5)y=\frac{1+\sin^2 x}{\cos x^2}, y'=\frac{2\sin x\cos x\cos x^2-(1+\sin^2 x)(-\sin x^2)2x}{\cos^2 x^2}$$
$$=\frac{\sin 2x\cos x^2+2x(1+\sin^2 x)(\sin x^2)}{\cos^2 x^2}.$$
$$(6)y=\frac{1}{3}\tan^3 x-\tan x+x, y'=\tan^2 x\sec^2 x-\sec^2 x+1$$
$$=\tan^2 x\sec^2 x-\tan^2 x=\tan^2 x(\sec^2 x-1)=\tan^4 x.$$
$$(7)y=e^{ax}\sin bx, y'=ae^{ax}\sin bx+be^{ax}\cos bx=e^{ax}(a\sin bx+b\cos bx).$$
$$(8)y=\cos^5\sqrt{1+x^2}, y'=5\cos^4\sqrt{1+x^2}(-\sin\sqrt{1+x^2})\frac{x}{\sqrt{1+x^2}}$$
$$=-\frac{5x\cos^4\sqrt{1+x^2}\sin\sqrt{1+x^2}}{\sqrt{1+x^2}}.$$
$$(9)y=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|, y'=\frac{1}{2}\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right)$$
$$=\frac{1}{2}\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}=\frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$
$$=\frac{1}{\sin(x+\frac{\pi}{2})}=\frac{1}{\cos x}=\sec x.$$
$$(10)y=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|(a>0, x\neq\pm a), y'=\frac{1}{2a}\frac{x+a}{x-a}\frac{(x+a)-(x-a)}{(x+a)^2}=\frac{1}{x^2-a^2}.$$

4.求下列函数的导函数:

$$(1)y=\arcsin\frac{x}{a}(a>0), y'=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}g\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}.$$
$$(2)y=\frac{1}{a}\arctan\frac{x}{a}(a>0), y'=\frac{1}{a}\frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{1}{a}=\frac{1}{a^2+x^2}.$$
$$(3)y=x^2\arccos x(|x|<1), y'=2x\arccos x-\frac{x^2}{\sqrt{1-x^2}}.$$
$$(4)y=\arctan\frac{1}{x}, y'=\frac{1}{1+\frac{1}{x^2}}\frac{-1}{x^2}=-\frac{1}{1+x^2}.$$
$$(5)y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}(a>0),$$

$$y' = \frac{1}{2}\sqrt{a^2 - x^2} + \frac{x}{2}\frac{-2x}{\sqrt{a^2 - x^2}} + \frac{a^2}{2}\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\frac{1}{a}$$

$$= \frac{1}{2}\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2}.$$

(6) $y = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\frac{x + \sqrt{x^2 + a^2}}{a}\ (a > 0)$

$$y' = \frac{1}{2}\sqrt{x^2 + a^2} + \frac{x}{2}\frac{x}{\sqrt{x^2 + a^2}} + \frac{a^2}{2}\frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$= \frac{1}{2}\sqrt{x^2 + a^2} + \frac{x^2}{2\sqrt{x^2 + a^2}} + \frac{a^2}{2\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2}.$$

(7) $y = \arcsin\frac{2x}{x^2 + 1}, x \neq \pm 1.$

$$y' = \frac{1}{\sqrt{1 - \frac{4x^2}{(x^2 + 1)^2}}}\frac{2(x^2 + 1) - 2xg2x}{(x^2 + 1)^2} = 2\frac{1}{|x^2 - 1|}\frac{1 - x^2}{x^2 + 1} = \frac{2\,\mathrm{sgn}(1 - x^2)}{x^2 + 1}.$$

(8) $y = \frac{2}{\sqrt{a^2 - b^2}}\arctan\left(\sqrt{\frac{a - b}{a + b}}\tan\frac{x}{2}\right)(a > b \geq 0).$

$$y' = \frac{2}{\sqrt{a^2 - b^2}}\frac{1}{1 + \frac{a - b}{a + b}\tan^2\frac{x}{2}}\sqrt{\frac{a - b}{a + b}}\sec^2\frac{x}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{a + b + (a - b)\tan^2\frac{x}{2}}\sec^2\frac{x}{2} = \frac{1}{(a + b)\cos^2\frac{x}{2} + (a - b)\sin^2\frac{x}{2}}$$

$$= \frac{1}{a + b\cos x}.$$

(9) $y = (1 + \sqrt{x})(1 + \sqrt{2x})(1 + \sqrt{3x}), \ln y = \ln(1 + \sqrt{x}) + \ln(1 + \sqrt{2x}) + \ln(1 + \sqrt{3x})$

$$y'/y = \frac{1}{2(1 + \sqrt{x})\sqrt{x}} + \frac{2}{2(1 + \sqrt{2x})\sqrt{2x}} + \frac{3}{2(1 + \sqrt{3x})\sqrt{3x}},$$

$$y' = y\left[\frac{1}{2(1 + \sqrt{x})\sqrt{x}} + \frac{2}{2(1 + \sqrt{2x})\sqrt{2x}} + \frac{3}{2(1 + \sqrt{3x})\sqrt{3x}}\right].$$

(10) $y = \sqrt{1 + x + 2x^2}, y' = \frac{1 + 4x}{2\sqrt{1 + x + 2x^2}}.$

(11) $y = \sqrt{x^2 + a^2}, y' = \frac{x}{\sqrt{x^2 + a^2}}.$

(12) $y = \sqrt{a^2 - x^2}, y' = \frac{-x}{\sqrt{a^2 - x^2}}.$

$(13)\, y=\ln(x+\sqrt{x^2+a^2}),\ y'=\dfrac{1}{x+\sqrt{x^2+a^2}}\left(1+\dfrac{x}{\sqrt{x^2+a^2}}\right)=\dfrac{1}{\sqrt{x^2+a^2}}.$

$(14)\, y=(x-1)\sqrt[3]{(3x+1)^2(2-x)}.\ln y=\ln(x-1)+\dfrac{2}{3}\ln(3x+1)+\dfrac{1}{3}\ln(2-x),$

$$\frac{y'}{y}=\frac{1}{x-1}+\frac{2}{3x+1}+\frac{1}{3}\frac{-1}{2-x}$$

$$y'=y\left[\frac{1}{x-1}+\frac{2}{3x+1}+\frac{1}{3}\frac{-1}{2-x}\right].$$

$(15)\, y=e^x+e^{e^x},\ y'=e^x+e^{e^x}\cdot e^x=e^x(1+e^{e^x}).$

$(16)\, y=x^{a^a}+a^{x^a}+a^{a^x}\ (a>0).$

$$y'=a^a x^{a^a-1}+a^{x^a}\ln a(ax^{a-1})+a^{a^x}\ln a a^x\ln a$$

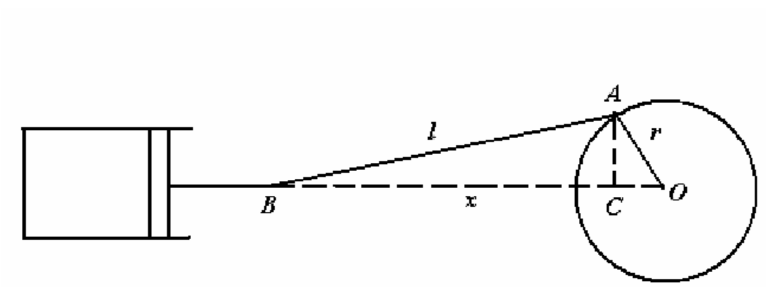
$$=a^a x^{a^a-1}+a\ln a a^{x^a}x^{a-1}+a^{a^x}a^x\ln^2 a.$$

5.一雷达的探测器瞄准着一枚安装在发射台上的火箭,它与发射台之间的距离是400m.设t=0时向上垂直地发射火箭,初速度为0,火箭以的匀加速度8m/s^{2}垂直地向上运动;若雷达探测器始终瞄准着火箭.问:自火箭发射后10秒钟时,探测器的仰角$\theta(t)$的变化速率是多少?

解$x(t)=\dfrac{1}{2}\cdot 8\cdot t^2=4t^2,\tan\theta(t)=\dfrac{x(t)}{400}=\dfrac{t^2}{100},$

$$\theta(t)=\arctan\frac{t^2}{100},\theta'(t)=\frac{1}{1+\left(\frac{t^2}{100}\right)^2}\cdot\frac{t}{50},\theta'(10)=\frac{1}{1+\left(\frac{10^2}{100}\right)^2}\cdot\frac{10}{50}=0.1(\text{弧度}/s).$$

6.在图示的装置中,飞轮的半径为2m且以每秒旋转4圈的匀角速度按顺时针方向旋转.问:当飞轮的旋转角为$\theta=\dfrac{\pi}{2}$时,活塞向右移动的速率是多少?



解$x(t)=2\cos 8\pi t+\sqrt{36-4\sin^{2}8\pi t}$,

$x'(t)=-16\pi\sin 8\pi t+\dfrac{-8\sin 8\pi t\cos 8\pi t g(8\pi)}{2\sqrt{36-4\sin^{2}8\pi t}}$,

$\alpha(t)=8\pi t=\dfrac{\pi}{2},,t_0=\dfrac{1}{16},x'(\dfrac{1}{16})=-16\pi.$

活塞向右移动的速率是16πm/s.

习题 2.3

1.当$x\to 0$时,下列各函数是x的几阶无穷小量?

(1)$y=x+10x^2+100x^3$.1阶.

(2)$y=(\sqrt{x+2}-\sqrt{2})\sin x=\dfrac{x\sin x}{\sqrt{x+2}+\sqrt{2}}$,2阶.

(3)$y=x(1-\cos x)=xg2\sin^2\dfrac{x}{2}$,2阶.

2.已知:当$x\to 0$时,$\alpha(x)=o(x^2)$.试证明$\alpha(x)=o(x)$.

证$\dfrac{\alpha(x)}{x}=\dfrac{\alpha(x)}{x^2}x=o(1)x=o(1).$

3.设$\alpha(x)=o(x)(x\to 0),\beta(x)=o(x)(x\to 0)$.试证明:$\alpha(x)+\beta(x)=o(x)(x\to 0)$.

证$\dfrac{\alpha(x)+\beta(x)}{x}=\dfrac{\alpha(x)}{x}+\dfrac{\beta(x)}{x}=o(1)+o(1)=o(1).$

上述结果有时可以写成$o(x)+o(x)=o(x)$.

4.计算下列函数在指定点x_0处的微分:

(1)$y=x\sin x,x_0=\pi/4.y'=\sin x+x\cos x,y'\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}\left(1+\dfrac{\pi}{4}\right),dy=\dfrac{1}{\sqrt{2}}\left(1+\dfrac{\pi}{4}\right)dx.$

(2)$y=(1+x)^{\alpha}(\alpha>0$是常数$)$.

$y'=\alpha(1+x)^{\alpha-1},y'(0)=\alpha,dy=\alpha dx.$

5.求下列各函数的微分:

(1)$y=\dfrac{1-x}{1+x}=-1+\dfrac{2}{1+x},y'=-\dfrac{2}{(1+x)^2},dy=-\dfrac{2dx}{(1+x)^2}.$

(2)$y=xe^x,y'=e^x+xe^x=e^x(1+x).dy=e^x(1+x)dx.$

6.设$y=\dfrac{2}{x-1}(x\neq 1)$,计算当$x$由3变到3.001时,函数的增量和向相应的微分.

解 $y'=-\dfrac{2}{(x-1)^2},y'(3)=-\dfrac{1}{2}.$

$\Delta y=\dfrac{2}{2.001}-1=-\dfrac{0.001}{2.001},dy=-\dfrac{0.001}{2}.$

7.试计算$\sqrt[5]{32.16}$的近似值.

解$\sqrt[5]{32.16}=2\sqrt[5]{1+.16/32}=2g(1+\dfrac{1}{5}g\dfrac{.16}{32})=2.002.$

8.求下列方程所确定的隐函数的导函数:

(1)$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}(a>0).\dfrac{1}{3}x^{-\frac{1}{3}}+\dfrac{1}{3}y^{-\frac{1}{3}}y'=0,y'=-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}.$

(2)$(x-a)^2+(y-b)^2=c^2(a,b,c$为常数$)$.

$2(x-a)+2(y-b)y'=0,y'=-\dfrac{x-a}{y-b}.$

$(3)\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$.

$$\frac{-\frac{y}{x^2}+\frac{y'}{x}}{1+\left(\frac{y}{x}\right)^2}=\frac{x+yy'}{x^2+y^2},\frac{xy'-y}{x^2+y^2}=\frac{x+yy'}{x^2+y^2},xy'-y=x+yy',y'=\frac{x+y}{x-y}.$$

$(4)y\sin x-\cos(x-y)=0$

$y'\sin x+y\cos x+\sin(x-y)(1-y')=0,$

$$y'=\frac{y\cos x+\sin(x-y)}{\sin(x-y)-\sin x}.$$

9.求下列隐函数在指定的点M的导数:

$(1)y^2-2xy-x^2+2x-4=0,M(3,7)$

$$2yy'-2y-2xy'-2x+2=0,y'=\frac{y+x-1}{y-x},y'(3)=\frac{7+3-1}{7-3}=\frac{9}{4}.$$

$(2)e^{xy}-5x^2y=0,M\left(\frac{e^2}{10},\frac{20}{e^2}\right)$.

$$e^{xy}(y+xy')-10xy-5x^2y'=0,y'=\frac{10xy-ye^{xy}}{xe^{xy}-5x^2},y'\left(\frac{e^2}{10}\right)=\frac{20-\frac{20}{e^2}e^2}{\frac{e^2}{10}e^2-5\cdot\frac{e^4}{100}}=0.$$

10.设$y=f(x)$由下列参数方程给出,求$y'=\frac{dy}{dx}$:

$$(1)\begin{cases}x=2t-t^2\\y=3t-t^3\end{cases}$$

$$\frac{dy}{dx}=\frac{3-3t^2}{2-2t}=\frac{3}{2}(1+t).(t\neq1).$$

$$(2)\begin{cases}x=t\ln t\\y=e^t\end{cases}\frac{dy}{dx}=\frac{e^t}{\ln t+1},t\neq1/e.$$

$$(3)\begin{cases}x=\arccos\frac{1}{\sqrt{1+t^2}}\\y=\arcsin\frac{t}{\sqrt{1+t^2}}\end{cases}$$

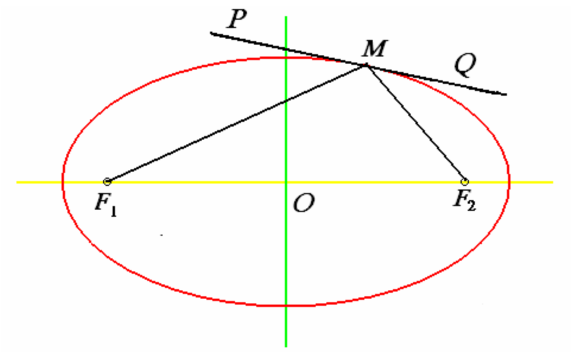
$$\frac{dy}{dx}=\frac{-\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left(-\frac{1}{2}\right)\frac{2t}{(1+t^2)^{3/2}}}{\frac{1}{\sqrt{1-\left(\frac{t^2}{1+t^2}\right)}}\frac{\sqrt{1+t^2}-t\cdot\frac{t}{\sqrt{1+t^2}}}{1+t^2}}=\operatorname{sgn}(t),t\neq0.$$

11.试求椭圆周$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$上一点$M_0(x_0,y_0)$处的切线方程与法线方程. 并证明:从椭圆的一个焦点向椭圆周上任一点M发射的光线,其反射线必通过椭圆的另一个焦点.

$$\frac{2x}{a^2}+\frac{2yy'}{b^2},y'=-\frac{b^2x}{a^2y}.$$

切线方程：$y-y_0=\left(-\dfrac{b^2x_0}{a^2y_0}\right)(x-x_0),\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1.$

法线方程：$y-y_0=\left(\dfrac{a^2y_0}{b^2x_0}\right)(x-x_0),a^2y_0x-b^2x_0y=(a^2-b^2)x_0y_0$



焦点$F_1(-c,0),F_1(c,0),c^2=a^2-b^2(a>b)$.设$y_0\neq 0$.切线斜率$k=-\dfrac{b^2x_0}{a^2y_0}$.

MF_1的斜率$k_1=\dfrac{y_0}{x_0+c}$,MF_2的斜率$k_2=\dfrac{y_0}{x_0-c}$.

$$\tan\angle F_2MQ=\frac{k-k_2}{1+kk_2}=\frac{-\dfrac{b^2x_0}{a^2y_0}-\dfrac{y_0}{x_0-c}}{1-\dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0-c}}=-\frac{b^2x_0(x_0-c)+a^2y_0^2}{a^2y_0(x_0-c)-b^2x_0y_0}$$

$$=-\frac{a^2b^2-b^2cx_0}{(a^2-b^2)x_0y_0-a^2cy_0}=-\frac{b^2(a^2-cx_0)}{c^2x_0y_0-a^2cy_0}=\frac{b^2(a^2-cx_0)}{cy_0(a^2-cx_0)}=\frac{b^2}{cy_0};$$

$$\tan\angle PMF_1=\frac{k_1-k}{1+kk_1}=\frac{\dfrac{b^2x_0}{a^2y_0}+\dfrac{y_0}{x_0+c}}{1-\dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0+c}}=-\frac{b^2x_0(x_0+c)+a^2y_0^2}{a^2y_0(x_0+c)-b^2x_0y_0}$$

$$=\frac{a^2b^2+b^2cx_0}{(a^2-b^2)x_0y_0+a^2cy_0}=\frac{b^2(a^2+cx_0)}{c^2x_0y_0+a^2cy_0}=\frac{b^2(a^2+cx_0)}{cy_0(a^2+cx_0)}=\frac{b^2}{cy_0}=\tan\angle F_2MQ.$$

$\angle PMF_1$和$\angle F_2MQ$都在区间$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$,故$\angle PMF_1=\angle F_2MQ$.

习题 2.4

n

$(1) y = x^n, y^{(n)} = n!.$

$(2) y = e^x, y^{(n)} = e^n.$

$$(3) y = \frac{1}{1+x} = (1+x)^{-1}(x \neq -1). y^{(n)} = (-1)(-1-1)\text{L}\ \ (-1-n+1)(1+x)^{-1-n} = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

$$(4) y = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}, y^{(n)} = (-1)^n n!\left(\frac{1}{x^{n+1}} - \frac{1}{(1+x)^{n+1}}\right).$$

2.设$y(x) = e^x \cos x$,证明$y'' - 2y' + 2y = 0.$

证$y' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$,

$y'' = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = e^x(-2\sin x)$,

$y'' - 2y' + 2y = e^x(-2\sin x) - 2e^x(\cos x - \sin x) + 2e^x \cos x = 0$,

3.设$y = \dfrac{x-3}{x+4}(x \neq -4)$,证明$2y'^2 = (y-1)y''.$

证$y = \dfrac{x-3}{x+4} = 1 - \dfrac{7}{x+4}, y' = \dfrac{7}{(x+4)^2}, y'' = -\dfrac{14}{(x+4)^3}.$

$$2y'^2 = \frac{98}{(x+4)^4}, (y-1)y'' = \left(-\frac{7}{x+4}\right)\left(-\frac{14}{(x+4)^3}\right) = \frac{98}{(x+4)^4} = 2y'^2.$$

4.设$y = (1-x)(2x+1)^2(3x-1)^3$,求$y^{(6)}, y^{(7)}.$

解$y^{(6)} = 6!\text{g}(-108), y^{(7)} = 0.$

5.要使$y = e^{\lambda x}$满足方程$y'' + py' + qy = 0$(其中p, q为常数),λ该取哪些值?

解$y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}, y'' + py' + qy = (\lambda^2 + p\lambda + q)e^{\lambda x} = 0, e^{\lambda x} \neq 0$,

λ该取方程$\lambda^2 + p\lambda + q = 0$的根.

6.飞轮绕一定轴转动,转过的角度θ与时间t的关系为$\theta = t^3 - 2t^2 + 3t - 1$,求飞轮转动的角速度与角加速度.

解角速度$\theta' = 3t^2 - 4t + 3$,角加速度$\theta'' = 6t - 4.$

7.设$f(x) = \dfrac{1}{(1-x)^n}$,其中$n$为一个正整数,求$f^{(k)}(x)$,$k$为一个正整数.

$$解 f(x) = \frac{1}{(1-x)^n} = (1-x)^{-n}, f^{(k)}(x) = (-n)(-n-1)\text{L}\ \ (-n-k+1)(1-x)^{-n-k}(-1)^k$$

$$= \frac{n(n+1)\text{L}\ \ (n+k-1)}{(1-x)^{n+k}}, f^{(k)}(0) = n(n+1)\text{L}\ \ (n+k-1).$$

8.设$y = x^2 \ln(1+x)$,求$y^{(50)}.$

解由 Leibniz公式,

$$y^{(50)} = x^2 \left(\ln(1+x)\right)^{(50)} + 50\cdot(2x)\cdot\left(\ln(1+x)\right)^{(49)} + \frac{50\cdot 49}{2}\cdot 2\cdot\left(\ln(1+x)\right)^{(48)}$$

$$= x^2 \left((1+x)^{-1}\right)^{(49)} + 50\cdot(2x)\cdot\left((1+x)^{-1}\right)^{(48)} + \frac{50\cdot 49}{2}\cdot 2\cdot\left((1+x)^{-1}\right)^{(47)}$$

$$= x^2(-1)(-2)\cdots(-1-49+1)(1+x)^{-50} + 100x(-1)(-2)\cdots(-1-48+1)(1+x)^{-49} +$$
$$2450(-1)(-2)\cdots(-1-47+1)(1+x)^{-48}$$

$$= -x^2\, 49!(1+x)^{-50} + 100\cdot 48!(1+x)^{-49} - 2450\cdot 47!(1+x)^{-48} = \frac{-2\cdot 47!}{(1+x)^{50}}(x^2+50x+1225).$$

9.验证函数$y = C_1 e^{ax} + C_2 e^{bx}$(其中$C_1$与$C_2$为任意常数)是微分方程$y'' - (a+b)y' + aby = 0$的解.

证$y' = (C_1 e^{ax} + C_2 e^{bx})' = C_1 a e^{ax} + C_2 b e^{bx}$, $y'' = (C_1 a e^{ax} + C_2 b e^{bx})' = C_1 a^2 e^{ax} + C_2 b^2 e^{bx}$,

$y'' - (a+b)y' + aby = C_1 a^2 e^{ax} + C_2 b^2 e^{bx} - (a+b)(C_1 a e^{ax} + C_2 b e^{bx}) + ab(C_1 e^{ax} + C_2 e^{bx}) = 0.$

10.验证函数$y = (C_1 x + C_2)e^{ax}$(其中C_1与C_2为任意常数)是微分方程$y'' - 2ay' + a^2 y = 0$的解.

证$y' = \left((C_1 x + C_2)e^{ax}\right)' = C_1 e^{ax} + a(C_1 x + C_2)e^{ax} = e^{ax}(aC_1 x + C_1 + aC_2)$,

$y'' = e^{ax} a(aC_1 x + C_1 + aC_2) + e^{ax}(aC_1) = e^{ax}(a^2 C_1 x + a^2 C_2 + 2aC_1)$,

$y'' - 2ay' + a^2 y$

$= e^{ax}(a^2 C_1 x_1 + a^2 C_2 + 2aC_1) - 2ae^{ax}(aC_1 x + C_1 + aC_2) + a^2(C_1 x + C_2)e^{ax} = 0.$

验证函数$y = C_1 \cos\omega t + C_2 \sin\omega t$(其中$C_1$与$C_2$为任意常数)是微分方程$y'' + \omega^2 y = 0$的解.

证$y' = -C_1 \omega \sin\omega t + C_2 \omega \cos\omega t$, y''

$= -C_1 \omega^2 \cos\omega t - C_2 \omega^2 \sin\omega t = -\omega^2 (C_1 \cos\omega t + C_2 \sin\omega t) = -\omega^2 y.$

习题 2.5

求下列不定积分:

$$1.\int\left(\frac{a}{\sqrt{x}}-\frac{b}{x^2}+3C\sqrt[3]{x^2}\right)dx=2a\sqrt{x}+\frac{b}{x}+\frac{9C}{5}x^{5/3}+C.$$

$$2.\int(1+\sqrt{x})^2dx=\int(1+2\sqrt{x}+x)dx=x+\frac{4}{3}x^{3/2}+\frac{1}{2}x^2+C.$$

$$3.\int a\sec^2 xdx=a\tan x+C.$$

$$4.\int\tan^2 xdx=\int(\sec^2 x-1)dx=\tan x-x+C.$$

$$5.\int\cot^2\varphi d\varphi=\int(\csc^2\varphi-1)d\varphi=-\cot^2\varphi-1+C.$$

$$6.\int\frac{x^2+3}{1+x^2}dx=\int\left(1+\frac{2}{1+x^2}\right)dx=x+2\arctan x+C.$$

$$7.\int\left(\frac{3}{\sqrt{x}}+\frac{4}{\sqrt{1-x^2}}\right)dx=6\sqrt{x}+4\arcsin x+C.$$

$$8.\int(1+\cos^2 x)\sec^2 xdx=\int(\sec^2 x+1)dx=\tan x+x+C.$$

$$9.\int\frac{1-x}{1-\sqrt[4]{x}}dx=\int(1+\sqrt[4]{x}+\sqrt[4]{x}^2+\sqrt[4]{x}^3)dx=x+\frac{4}{5}x^{5/4}+\frac{2}{3}x^{3/2}+\frac{4}{7}x^{7/4}+C$$

$$10.\int\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\right)dx=\ln|x|-\frac{2}{x}-\frac{6}{x^2}+C.$$

$$11.\int\frac{(1-x)^2}{x\text{g}\sqrt[3]{x}}dx=\int\frac{1-2x+x^2}{x^{4/3}}dx=\int\left(x^{-4/3}-2x^{-1/3}+x^{2/3}\right)dx$$

$$=-3x^{-1/3}-3x^{2/3}+\frac{3}{5}x^{5/3}+C.$$

$$12.\int(2\cosh x-\sinh x)dx=2\sinh x-\cosh x+C.$$

$$13.\int\left(\frac{3x^2-1}{x^2}+\frac{(x+1)^2}{\sqrt{x}}\right)dx=\int\left(3-\frac{1}{x^2}+x^{3/2}+2x^{1/2}+x^{-1/2}\right)dx$$

$$=3x+\frac{1}{x}+\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+2\sqrt{x}+C.$$

$$14.\int\frac{1}{\sin^2 x\cos^2 x}dx=\int\left(\frac{1}{\sin^2}+\frac{1}{\cos^2 x}\right)dx=-\cot x+\tan x+C.$$

$$15.\int\frac{2^{x+1}+3^{x-2}}{6^x}dx=\int\left(2\left(\frac{1}{3}\right)^x+\frac{1}{9}\left(\frac{1}{2}\right)^x\right)dx$$

$$=-2\left(\frac{1}{3}\right)^x/\ln 3-\frac{1}{9}\left(\frac{1}{2}\right)^x/\ln 2+C.$$

$$16.\int\frac{1}{x^2(1+x^2)}dx=\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)dx=-\frac{1}{x}-\arctan x+C.$$

17.求解微分方程$y''(x)=a+be^{-x}(a,b$为常数).

解$y'=\int(a+be^{-x})dx=ax-be^{-x}+C_1, y$

$=\int(ax-be^{-x}+C_1)dx=\frac{1}{2}ax^2+be^{-x}+C_1x+C_2.$

18.设$f(x)$满足方程$xf'(x)+f(x)=x^2+1$,求$f(x)$.

解$(xf(x))'=x^2+1, xf(x)=\int x^3+1dx=\frac{1}{4}x^4+x+C,$

$f(x)=\frac{1}{4}x^3+1+\frac{C}{x}.$

习题 2.6

1.根据定积分的定义直接求下列积分:

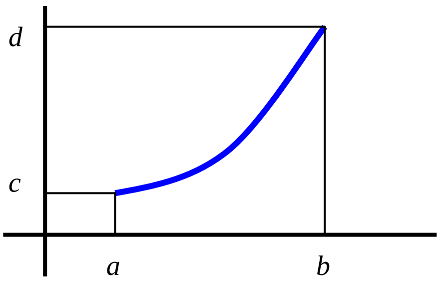
$$(1)\int_a^b kdx=\lim_{\lambda\to 0}\sum_{i=1}^{n}k\Delta x_i=\lim_{\lambda\to 0}k(b-a)=k(b-a).$$

$$(2)\lim_{n\to\infty}\sum_{i=1}^{n}(a+\frac{i(b-a)}{n})\frac{b-a}{n}=\lim_{n\to\infty}(a(b-a)+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i)$$

$$=a(b-a)+(b-a)^2\lim_{n\to\infty}(\frac{1}{n^2}\sum_{i=1}^{n}i)=a(b-a)+(b-a)^2\lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2}$$

$$=a(b-a)+(b-a)^2\lim_{n\to\infty}\frac{(1+1/n)}{2}=a(b-a)+\frac{(b-a)^2}{2}=\frac{b^2-a^2}{2}.$$

2.设函数$x=\varphi(y)$在$[c,d]$上连续且$\varphi(y)>0$.试用定积分表示曲线$x=\varphi(y),y=c,$ $y=d$及y轴所围的图形的面积;又设$c\geq 0$,函数$x=\varphi(y)$在$[c,d]$上严格递增,试求积分和$\int_c^d\varphi(y)dy+\int_a^b\psi(x)dx$,其中$y=\psi(x)$是$x=\varphi(y)$的反函数,$a=\varphi(c),b=\varphi(b)$.



解$\int_c^d\varphi(y)dy+\int_a^b\psi(x)dx=bd-ac.$

3.写出函数$y=x^2$在区间$[0,1]$上的Riemann和,其中分割为n等分,中间点ξ_i为分割小区间的左端点.求出当$n\to\infty$时Riemann和的极限.]

解$s_n=\sum_{i=0}^{n-1}\frac{i^2}{n^2}g\frac{1}{n}=\frac{1}{6n^3}(n-1)n(2n-1)=\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\to\frac{1}{3}(n\to\infty).$

4.求定积分$\int_0^1\sqrt{x}dx.$

解y $=\sqrt{x}$的反函数$x=y^2$,当$x=0$时,$y=0$,当$x=1$时,$y=1$.由2,3题

$$\int_0^1\sqrt{x}dx=1-\int_0^1 y^2dy=1-\frac{1}{3}=\frac{2}{3}.$$

5.证明下列不等式

$$(1)\frac{\pi}{2}<\int_0^{\pi/2}(1+\sin x)dx<\pi.$$

证$\int_0^{\pi/2}(1+\sin x)dx>\int_0^{\pi/2}(1)dx=\frac{\pi}{2}.\int_0^{\pi/2}(1+\sin x)dx<\int_0^{\pi/2}(2)dx=\pi.$

$$(2)\sqrt{2}<\int_0^1\sqrt{2+x-x^2}dx<\frac{3}{2}.$$

证$2+x-x^2=(1+x)(2-x)=0,x_1=-1,x_2=2$.当$x\in(-\infty,1/2)$时,$\sqrt{2+x-x^2}$递增,

.当$x \in (1/2, +\infty)$时,$\sqrt{2+x-x^2}$递减, 故

$$\sqrt{2} = \int_0^1 \sqrt{2}dx < \int_0^1 \sqrt{2+x-x^2}dx < \int_0^1 \sqrt{2+1/2-1/4}dx = \frac{3}{2}.$$

6.判断下列各题中两个积分值之大小:

(1)$\int_0^1 e^x dx > \int_0^1 e^{x^2} dx.$

(2)$\int_0^{\pi/2} x^2 dx > \int_0^{\pi/2} (\sin x)^2 dx.$

(3)$\int_0^1 x dx < \int_0^1 \sqrt{1+x^2} dx.$

7.设函数$y = f(x)$在$[a,b]$上有定义,并且假定$y = f(x)$在任何闭子区间上有最大值和最小值.对于任意一个分割:$T: x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

记m_i为$f(x)$在$[x_{i-1}, x_i]$中的最小值,M_i为$f(x)$在$[x_{i-1}, x_i]$中的最大值. 证明

$y = f(x)$在$[a,b]$上可积的充要条件是极限$\lim\limits_{\lambda(T)\to} \sum\limits_{i=1}^n m_i \Delta x_i$与$\lim\limits_{\lambda(T)\to} \sum\limits_{i=1}^n M_i \Delta x_i$存在并且相等.

证设$y = f(x)$在$[a,b]$上可积, 则$\lim\limits_{\lambda(T)\to 0} \sum\limits_{i=1}^n m_i \Delta x_i = \lim\limits_{\lambda(T)\to 0} \sum\limits_{i=1}^n f(\xi_i) \Delta x_i = \int_a^b f(x)dx,$

$$\lim_{\lambda(T)\to 0} \sum_{i=1}^n M_i \Delta x_i = \lim_{\lambda(T)\to 0} \sum_{i=1}^n f(\eta_i) \Delta x_i = \int_a^b f(x)dx.$$

设$\lim\limits_{\lambda(T)\to 0} \sum\limits_{i=1}^n m_i \Delta x_i = \lim\limits_{\lambda(T)\to 0} \sum\limits_{i=1}^n M_i \Delta x = I$,则

$$\sum_{i=1}^n m_i \Delta x_i \le \sum_{i=1}^n f(\xi_i) \Delta x_i \le \sum_{i=1}^n M_i \Delta x,$$

由夹挤定理, $\lim\limits_{\lambda(T)\to 0} \sum\limits_{i=1}^n f(\xi_i) \Delta x_i = I.$

习题 2.7

1.求下列变上(下)限积分所定义的函数的导函数:

$(1)F(x)=\int_1^{x^2}\frac{dt}{1+t^2},F'(x)=\frac{1}{1+x^2}.$

$(2)G(x)=\int_0^{1+x^2}\sin t^2dt,G'(x)=2x\sin(1+x^2)^2.$

$(3)H(x)=\int_x^1 t^2\cos tdt,H'(x)=-x^2\cos x.$

$(4)L(x)=\int_x^{x^2}e^{-t^2}dt,L'(x)=2xe^{-x^2}-e^{-x^2}.$

2.设$y=f(x)$在$[a,b]$上连续.证明$F_0(x)=\int_a^x f(t)dt$在a处有右导数,且$F_+'(a)=f(a)$.

证$\dfrac{F_0(a+\Delta x)-F_0(a)}{\Delta x}=\dfrac{1}{\Delta x}\int_a^{a+\Delta x}f(t)dt=\dfrac{1}{\Delta x}f(\xi)\Delta x(a\le\xi\le a+\Delta x)$

$=f(\xi)\to f(a)(\Delta x\to 0+)$故$F_+'(a)=f(a)$.

3.设f(x)在$[a,b]$上连续.假定$f(x)$有一个原函数$F(x)$且$F((a)=0$.证明

当$a\le x\le b$时$F(x)=\int_a^x f(t)dt$.

证G(x)$=\int_a^x f(t)dt.G(a)=0$,由变上限积分求导定理,$G'(x)=f(x).F'(x)=f(x)$,

$F(a)=0$.

$(G(x)-F(x))'=G'(x)-F'(x)=f(x)-f(x)=0,G(x)-F(x)=C,x\in[a,b]$.

$C=F(a)-G(a)=0,F(x)=G(x)=\int_a^x f(t)dt,x\in[a,b]$.

4.证明:当$x\in(0,+\infty)$时,$\ln x=\int_1^x\dfrac{dt}{t}dt$.

证由于$(\ln x)'=\dfrac{1}{x},\left(\int_1^x\dfrac{dt}{t}dt\right)'=\dfrac{1}{x},x\in(0,+\infty),\ln1=\int_1^1\dfrac{dt}{t}dt=0$,故$\ln x=\int_1^x\dfrac{dt}{t}dt$.

5.设$y=f(x)$在$[a,b]$上可积, 且$|f(x)|\le L,(\forall x\in[a,b]),uqz$其中$L$为常数.证明

变上限积分$F(x)=\int_a^x f(t)dt$在$[a,b]$上满足Lipschiz 条件:

$|F(x_1)-F(x_2)|\le L|x_1-x_2|,(x_1,x_2\in[a,b])$.

证不妨设$x_1<x_2$,

$|F(x_1)-F(x_2)|=\left|\int_a^{x_2}f(t)dt-\int_a^{x_1}f(t)dt\right|=\left|\int_{x_1}^{x_2}f(t)dt\right|\le\int_{x_1}^{x_2}|f(t)|dt\le\int_{x_1}^{x_2}Ldt=x_2=x_1.$

6.求函数$G(x)=\int_0^x e^t\int_0^t\sin zdzdt$的二阶导数.

解$G'(x)=e^x\int_0^x\sin zdz,G''(x)=\mathrm{e}^x\int_0^x\sin zdz+e^x\sin x=\mathrm{e}^x(1-\cos x)+e^x\sin x.$

习题 2.8

4. 将下列积分改成若干个区间上定积分之和, 然后分别使用Newton-Leibniz公式求处其值:

(1)

1.用Newton-Leibniz公式计算下列定积分:

(1)$\int_0^1 x^3 dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}.$

(2)$\int_a^b e^x dx = \left.e^x\right|_a^b = e^b - e^a.$

(3)$\int_0^{3\pi} \sin x dx = -\cos x|_0^{3\pi} = 2.$

(4)$\int_1^2 \frac{dx}{x} = \ln x|_1^2 = \ln 2.$

(5)$\int_0^{\pi} (2\sin x + x^3)dx = \left[\left[-2\cos x + \frac{x^4}{4}\right]\right]_0^{\pi} = 4 + \frac{\pi^4}{4}.$

(6)$\int_0^1 (x^5 + \frac{1}{3}x^3 + \frac{1}{2}x + 1)dx = \left[\frac{x^6}{6} + \frac{x^4}{12} + \frac{x^2}{4} + x\right]_0^1 = \frac{3}{2}.$

2.验证$\frac{1}{2}x^2 - \frac{1}{x}$是$x + \frac{1}{x^2}$的一个原函数并计算定积分$\int_2^4 \left(x + \frac{1}{x^2}\right)dx$.试问下式

$\int_{-1}^1 \left(x + \frac{1}{x^2}\right)dx = \left.\left(\frac{1}{2}x^2 - \frac{1}{x}\right)\right|_{-1}^1$ 是否成立:为什么?

解$\left(\frac{1}{2}x^2 - \frac{1}{x}\right)' = \left(\frac{1}{2}x^2\right)' - \left(x^{-1}\right)' = x + x^{-2} = x + \frac{1}{x^2}$,故$\frac{1}{2}x^2 - \frac{1}{x}$是$x + \frac{1}{x^2}$的一个原函数.

$\int_2^4 \left(x + \frac{1}{x^2}\right)dx = \left.\left(\frac{1}{2}x^2 - \frac{1}{x}\right)\right|_2^4 = \frac{25}{4}.$

$\int_{-1}^1 \left(x + \frac{1}{x^2}\right)dx = \left.\left(\frac{1}{2}x^2 - \frac{1}{x}\right)\right|_{-1}^1$ 不成立.因为$x + \frac{1}{x^2}$在$[-1,1]$不可积.

3.将下列极限中的和式视作适当函数的Riemann和, 然后使用Newton-Leibniz公式求出其值:

(1)$\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n}\sin\frac{k}{n} = \int_0^1 \sin x dx = -\cos x|_0^1 = 1 - \cos 1.$

(2)$\lim_{n\to\infty} \sum_{k=1}^n \frac{k^3}{n^4} = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n}\left(\frac{k}{n}\right)^3 = \int_0^1 x^3 dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}.$

(3)$\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n}\frac{1}{1+k/n} = \int_0^1 \frac{dx}{1+x} = \ln(1+x)|_0^1 = \ln 2.$

4.将下列积分改成若干个区间上定积分之和, 然后分别使用Newton-Leibniz公式求处其值:

(1)$\int_{-1}^{1}|x|dx=\int_{0}^{1}xdx-\int_{-1}^{0}xdx=\frac{x^2}{2}\bigg|_0^1-\frac{x^2}{2}\bigg|_{-1}^0=1.$

(2)$\int_{-1}^{1}\operatorname{sgn} xdx=\int_{0}^{1}1dx+\int_{-1}^{0}(-1)dx=1-1=0.$

(3)$\int_{0}^{1}x\left|\frac{1}{2}-x\right|dx=\int_{0}^{1/2}x\left(\frac{1}{2}-x\right)dx+\int_{1/2}^{1}x\left(x-\frac{1}{2}\right)dx$

$=\left(\frac{x^2}{4}-\frac{x^3}{3}\right)\bigg|_0^{1/2}+\left(\frac{x^3}{3}-\frac{x^2}{4}\right)\bigg|_{1/2}^1=\frac{1}{16}-\frac{1}{24}+\frac{1}{3}-\frac{1}{4}-\frac{1}{24}+\frac{1}{16}=\frac{1}{8}.$

(4)$\int_{0}^{2\pi}|\sin x|dx=\int_{0}^{\pi}\sin xdx-\int_{\pi}^{2\pi}\sin xdx=-\cos x|_0^{\pi}+\cos x|_{\pi}^{2\pi}=2+2=4.$

(5)$\int_{0}^{2}(x-[x])dx=\int_{0}^{1}xdx+\int_{1}^{2}(x-1)dx=\frac{x^2}{2}\bigg|_0^1+\left(\frac{x^2}{2}-x\right)\bigg|_1^2$

$=\frac{1}{2}-(-\frac{1}{2})=1.$

5.设$F(x)$在$[a,b]$上有连续的导函数$F'(x)$.试证明:存在一点$c\in[a,b]$,使得$F(b)-F(a)=F'(c)(b-a)$.

证$F(b)-F(a)=\int_{a}^{b}F'(x)dx$(Newton-Leibniz公式)

$=F'(c)(b-a)$(定积分中指中值公式).

第二章总练习题

1.讨论函数$f(x)=\begin{cases}|x-3| & x\ge 1\text{时}\\ \dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}, & x<1\text{时}\end{cases}$的连续性和可导性.

解$x\ne 1$时$f(x)$可导. $f(1-0)=\lim\limits_{x\to 1}\left(\dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}\right)=2;$

$f(1+0)=\lim\limits_{x\to 1}|x-3|=2=f(1-0)=f(1)$, f在$x=1$连续.

$$f'_+(1)=(3-x)'|_{x=1}=-1,\ f'_-(1)=\left(\frac{x^2}{4}-\frac{3}{2}x+\frac{13}{4}\right)'\Bigg|_{x=1}=\left(\frac{x}{2}-\frac{3}{2}\right)\Bigg|_{x=1}=-1=f'_+(1),\ f'(1)=-1.$$

f在$x=1$可导.

2.设函数$f(x)=\begin{cases}2x-2 & x<-1\text{时}\\ Ax^3+Bx^2+Cx+D, & -1\le x\le 1\text{时}\\ 5x+7 & x>1\text{时}\end{cases}$

试确定常数A,B,C,D的值,使$f(x)$在$(-\infty,+\infty)$可导.

解$f(-1-0)=\lim\limits_{x\to -1}(2x-2)=-4=f(-1)=-A+B-C+D.$

$f'_-(-1)=(2x-2)'|_{x=-1}=2=f'_+(-1)=(Ax^3+Bx^2+Cx+D)'|_{x=-1}$

$=(3Ax^2+2Bx+C)|_{x=-1}=3A-2B+C.$

$f(1-0)=A+B+C+D=f(1+0)=12,$

$f'_-(1)=3A+2B+C=f'_+(1)=5.$

$$\begin{cases}-A+B-C+D=-4\\ 3A-2B+C=2\\ A+B+C+D=12\\ 3A+2B+C=5.\end{cases}$$

$\{A=-9/4,\ B=3/4,\ C=41/4, D=13/4\}.$

3.设函数$g(x)=(\sin 2x)f(x)$,其中$f(x)$在$x=0$连续.问$g(x)$在$x=0$是否可导,若可导,求出$g'(0)$.

解$\dfrac{g(\Delta x)-g(0)}{\Delta x}=2\dfrac{f(\Delta x)\sin 2\Delta x}{2\Delta x}\to 2f(0)(\Delta x\to 0)$, $g'(0)=2f(0).$

4.问函数$f(x)=\dfrac{x^2+\sin^2 x}{1+x^2}$与$g(x)=\dfrac{-\cos^2 x}{1+x^2}$为什么有相同得导数?

解因为$f(x)-g(x)=1.$

5,.设函数$f(x)$在$[-1,1]$上有定义,且满足$x\le f(x)\le x^2+x, x\in[-1,1]$.证明存在且等于1.

证$0\le f(0)\le 0$, $f(0)=0.\Delta x>0,$

$\dfrac{f(\Delta x)-f(0)}{\Delta x}=\dfrac{f(\Delta x)}{\Delta x}\le\dfrac{\Delta x^2+\Delta x}{\Delta x}=\Delta x+1\to 1(\Delta x\to 0+0)$, $f'_+(0)=1$,类似$f'_-(0)=1$,

故$f'(0)=1.$

6.设$f(x)=|x^2-4|$,求$f'(x)$.

解$|x|>2$ 时, $f(x)=x^2-4$, $f'(x)=2x$. $f'_+(2)=(x^2-4)'|_{x=2}=4$,

$f'_-(2)=(4-x^2)'|_{x=2}=-4$, $f'(2)$不存在,同理$f'(-2)$不存在.

7. 设$y=\dfrac{1+x}{1-x}$,求$\dfrac{d^2y}{dx^2}$.

解$y=-1+\dfrac{2}{1-x}$, $\dfrac{dy}{dx}=\dfrac{2}{(1-x)^2}$, $\dfrac{d^2y}{dx^2}=-\dfrac{4}{(1-x)^3}$.

8.设函数$f(x)$在$(-\infty,+\infty)$上有定义,且满足下列性质:

(1) $f(a+b)=f(a)f(b)$(a,b为任意实数);(2) $f(0)=1$;(3)在$x=0$处可导.证明:对于任意$x\in(-\infty,+\infty)$都有$f'(x)=f'(0)gf(x)$.

证$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{f(x)f(\Delta x)-f(x)f(0)}{\Delta x}$

$=f(x)\dfrac{f(\Delta x)-f(0)}{\Delta x}\to f'(0)gf(x)(\Delta x\to 0)$, $f'(x)=f'(0)gf(x)$.

9.设$f(x)=\begin{cases}1/2^{2n}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\text{L}\ )$; $g(x)=\begin{cases}1/2^{n+1}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\text{L}\ )$;

问$f(x)$在$x=0$处是否可导?$g(x)$在$x=0$处是否可导?

解$\dfrac{f(1/2^n)-f(0)}{1/2^n}=\dfrac{1/2^{2n}}{1/2^n}=\dfrac{1}{2^n}\to 0(n\to\infty)$,

$\dfrac{f(x)-f(0)}{x}=0\to 0(x\neq 1/2^n, x\to 0)$.$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=0$, $f'(0)=0$.

$\dfrac{g(1/2^n)-g(0)}{1/2^n}=\dfrac{1/2^{n+1}}{1/2^n}=\dfrac{1}{2}\to\dfrac{1}{2}(n\to\infty)$,

$\dfrac{g(x)-g(0)}{x}=0\to 0(x\neq 1/2^n, x\to 0)$, $\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}$.$g'(0)$不存在.

10.设$y=f(x)$及$y=g(x)$在$[a,b]$上连续,证明:

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

证$\int_a^b[f(x)+tg(x)]^2dx=\left(\int_a^b g^2(x)dx\right)t^2+\left(2\int_a^b f(x)g(x)dx\right)t+\int_a^b f^2(x)dx\ge 0(*)$,

如果$\int_a^b g^2(x)dx=0$,则由g的连续性$g(x)=0, x\in[a,b]$,不等式两端都是0.

如果$\int_a^b g^2(x)dx>0$,(*) 左端的二次函数恒非负, 故其判别式非正,

$$\left(2\int_a^b f(x)g(x)dx\right)^2-4\left(\int_a^b g^2(x)dx\right)\int_a^b f^2(x)dx\le 0,$$

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

11.求出函数

$$f(x)=\frac{1}{2}x+\frac{1}{2^2}x^2+\cdots+\frac{1}{2^n}x^n$$

在点$x=1$的导数,再将函数$f(x)$写成$f(x)=\dfrac{x/2-(x/2)^{n+1}}{1-x/2}$的形式,再求$f'(1)$,由此证明下列等式:

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

证$f'(x)=\dfrac{1}{2}+\dfrac{2}{2^2}x+\cdots+\dfrac{n}{2^n}x^{n-1}$, $f'(1)=\dfrac{1}{2}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}$.

$$f(x)=\frac{x/2-(x/2)^{n+1}}{1-x/2},$$

$$f'(x)=\frac{(1/2-(n+1)(x/2)^n(1/2))(1-x/2)+(1/2)(x/2-(x/2)^{n+1})}{(1-x/2)^2},$$

$$f'(1)=\frac{(1/2-(n+1)(1/2^{n+1}))(1/2)+(1/2)(1/2-1/2^{n+1})}{1/2^2}$$

$$=(1-(n+1)/2^n)+1-1/2^n=2-\frac{n+2}{2^n}.$$

12.由类似上题的办法证明$1+2x+3x^2+\cdots+nx^{n-1}=\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}(x\neq 1)$.

证由等比级数求和公式$x+x^2+\cdots+x^n=\dfrac{x-x^{n+1}}{1-x}$,

两端求导得$1+2x+3x^2+\cdots+nx^{n-1}$

$$=\frac{(1-(n+1)x^n)(1-x)+(x-x^{n+1})}{(1-x)^2}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}(x\neq 1).$$

13.设$y=f(x)$在$[0,1]$连续且$f(x)>0$证明$\displaystyle\int_0^1\frac{1}{f(x)}dx\geq\frac{1}{\int_0^1 f(x)dx}$.

证$\displaystyle 1=\int_0^1 1dx=\int_0^1\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\leq\int_0^1 f(x)dx\int_0^1\frac{1}{f(x)}dx.$

14.$\displaystyle\ln x=\int_1^n\frac{dt}{t}$

$(a)\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}(n>0)$

$(b)\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n-1};(c)e^{1-\frac{1}{n+1}}<\left(1+\dfrac{1}{n}\right)^n<e.$

证$(1)\displaystyle\frac{1}{n+1}=\int_1^{1+1/n}\frac{dt}{1+1/n}\ln\left(1+\frac{1}{n}\right)=\int_1^{1+1/n}\frac{dt}{t}<\int_1^{1+1/n}\frac{dt}{1}=\frac{1}{n}.$

$(2)\displaystyle\ln n=\ln\frac{2}{1}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{n}{n-1}=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)<1+\frac{1}{2}+\cdots+\frac{1}{n},$

$$\ln n = \ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)>\frac{1}{2}+\cdots+\frac{1}{n}.$$

(3) $\left(1+\frac{1}{n}\right)^{n}=e^{n\ln\left(1+\frac{1}{n}\right)}>e^{n\cdot\frac{1}{n+1}}=e^{1-\frac{1}{n+1}}.$

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解$x\ne 1$时$f(x)$可导. $f(1-0)=\lim\limits_{x\to 1}\left(\dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}\right)=2;$

$f(1+0)=\lim\limits_{x\to 1}|x-3|=2=f(1-0)=f(1)$, f在$x=1$连续.

$$f'_+(1)=(3-x)'|_{x=1}=-1,\ f'_-(1)=\left.\left(\frac{x^2}{4}-\frac{3}{2}x+\frac{13}{4}\right)'\right|_{x=1}=\left.\left(\frac{x}{2}-\frac{3}{2}\right)\right|_{x=1}=-1=f'_+(1),\ f'(1)=-1.$$

f在$x=1$可导.

2.设函数$f(x)=\begin{cases}2x-2 & x<-1\text{时}\\ Ax^3+Bx^2+Cx+D, & -1\le x\le 1\text{时}\\ 5x+7 & x>1\text{时}\end{cases}$

试确定常数A,B,C,D的值,使$f(x)$在$(-\infty,+\infty)$可导.

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$f(1-0)=A+B+C+D=f(1+0)=12,$

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解$\dfrac{g(\Delta x)-g(0)}{\Delta x}=2\dfrac{f(\Delta x)\sin 2\Delta x}{2\Delta x}\to 2f(0)(\Delta x\to 0), g'(0)=2f(0).$

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证$0\le f(0)\le 0$, $f(0)=0.\Delta x>0,$

$\dfrac{f(\Delta x)-f(0)}{\Delta x}=\dfrac{f(\Delta x)}{\Delta x}\le\dfrac{\Delta x^2+\Delta x}{\Delta x}=\Delta x+1\to 1(\Delta x\to 0+0)$, $f'_+(0)=1$,类似$f'_-(0)=1$,

故$f'(0)=1.$

6.设$f(x)=|x^2-4|$,求$f'(x)$.

解$|x|>2$ 时, $f(x)=x^2-4$, $f'(x)=2x$. $f'_+(2)=(x^2-4)'|_{x=2}=4$,

$f'_-(2)=(4-x^2)'|_{x=2}=-4$, $f'(2)$不存在,同理$f'(-2)$不存在.

7. 设$y=\dfrac{1+x}{1-x}$,求$\dfrac{d^2y}{dx^2}$.

解$y=-1+\dfrac{2}{1-x}$, $\dfrac{dy}{dx}=\dfrac{2}{(1-x)^2}$, $\dfrac{d^2y}{dx^2}=-\dfrac{4}{(1-x)^3}$.

8.设函数$f(x)$在$(-\infty,+\infty)$上有定义,且满足下列性质:

(1) $f(a+b)=f(a)f(b)$(a,b为任意实数); (2) $f(0)=1$; (3)在$x=0$处可导.证明:对于任意$x\in(-\infty,+\infty)$都有$f'(x)=f'(0)gf(x)$.

证$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{f(x)f(\Delta x)-f(x)f(0)}{\Delta x}$

$=f(x)\dfrac{f(\Delta x)-f(0)}{\Delta x}\to f'(0)gf(x)(\Delta x\to 0)$, $f'(x)=f'(0)gf(x)$.

9.设$f(x)=\begin{cases}1/2^{2n}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,L\ )$; $g(x)=\begin{cases}1/2^{n+1}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,L\ )$;

问$f(x)$在$x=0$处是否可导? $g(x)$在$x=0$处是否可导?

解$\dfrac{f(1/2^n)-f(0)}{1/2^n}=\dfrac{1/2^{2n}}{1/2^n}=\dfrac{1}{2^n}\to 0(n\to\infty)$,

$\dfrac{f(x)-f(0)}{x}=0\to 0(x\neq 1/2^n, x\to 0)$. $\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=0$, $f'(0)=0$.

$\dfrac{g(1/2^n)-g(0)}{1/2^n}=\dfrac{1/2^{n+1}}{1/2^n}=\dfrac{1}{2}\to\dfrac{1}{2}(n\to\infty)$,

$\dfrac{g(x)-g(0)}{x}=0\to 0(x\neq 1/2^n, x\to 0)$, $\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}$. $g'(0)$不存在.

10.设$y=f(x)$及$y=g(x)$在$[a,b]$上连续,证明:

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

证$\int_a^b[f(x)+tg(x)]^2dx=\left(\int_a^b g^2(x)dx\right)t^2+\left(2\int_a^b f(x)g(x)dx\right)t+\int_a^b f^2(x)dx\ge 0(*)$,

如果$\int_a^b g^2(x)dx=0$,则由g的连续性$g(x)=0, x\in[a,b]$,不等式两端都是0.

如果$\int_a^b g^2(x)dx>0$,(*) 左端的二次函数恒非负, 故其判别式非正,

$$\left(2\int_a^b f(x)g(x)dx\right)^2-4\left(\int_a^b g^2(x)dx\right)\int_a^b f^2(x)dx\le 0,$$

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

11.求出函数

$$f(x)=\frac{1}{2}x+\frac{1}{2^2}x^2+\cdots+\frac{1}{2^n}x^n$$

在点 $x=1$ 的导数,再将函数 $f(x)$ 写成 $f(x)=\dfrac{x/2-(x/2)^{n+1}}{1-x/2}$ 的形式,再求 $f'(1)$,

由此证明下列等式:

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

证 $f'(x)=\dfrac{1}{2}+\dfrac{2}{2^2}x+\cdots+\dfrac{n}{2^n}x^{n-1}$, $f'(1)=\dfrac{1}{2}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}$.

12.由类似上题的办法证明

$$f(x)=\frac{x/2-(x/2)^{n+1}}{1-x/2},$$

$$f'(x)=\frac{(1/2-(n+1)(x/2)^n(1/2))(1-x/2)+(1/2)(x/2-(x/2)^{n+1})}{(1-x/2)^2},$$

$$f'(1)=\frac{(1/2-(n+1)(1/2^{n+1}))(1/2)+(1/2)(1/2-1/2^{n+1})}{1/2^2}$$

$$=(1-(n+1)/2^n)+1-1/2^n=2-\frac{n+2}{2^n}.$$

$$\frac{1-(n+1)x^n+nx^{n+1}\,{}^2}{(1-x)^2}\ (x\neq 1)$$

13.设 $y=f(x)$ 在 $[0,1]$ 连续且 $f(x)>0$ 证明 $\displaystyle\int_0^1\frac{1}{f(x)}dx\geq\frac{1}{\int_0^1 f(x)dx}$.

证 $\displaystyle 1=\int_0^1 1dx=\int_0^1\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\leq\int_0^1 f(x)dx\int_0^1\frac{1}{f(x)}dx.$

14. $\displaystyle\ln x=\int_1^n\frac{dt}{t}$

$(a)\ \dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}(n>0)$

$(b)\ \dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n-1};$

$(c)\ e^{1-\frac{1}{n+1}}<\left(1+\dfrac{1}{n}\right)^n<e.$

证(1) $\displaystyle\frac{1}{n+1}=\int_1^{1+1/n}\frac{dt}{1+1/n}\ln\left(1+\frac{1}{n}\right)=\int_1^{1+1/n}\frac{dt}{t}<\int_1^{1+1/n}\frac{dt}{1}=\frac{1}{n}.$

(2) $\displaystyle\ln n=\ln\frac{2}{1}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{n}{n-1}=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)<1+\frac{1}{2}+\cdots+\frac{1}{n},$

$\displaystyle\ln n=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)>\frac{1}{2}+\cdots+\frac{1}{n}.$

$$(c)\, e^{1-\frac{1}{n+1}} < \left(1+\frac{1}{n}\right)^{n} < e.$$

证 $(1)\ \dfrac{1}{n+1} = \displaystyle\int_{1}^{1+1/n} \frac{dt}{1+1/n} \ln\left(1+\frac{1}{n}\right) = \int_{1}^{1+1/n} \frac{dt}{t} < \int_{1}^{1+1/n} \frac{dt}{1} = \frac{1}{n}.$

$(2)\ \ln n = \ln \dfrac{2}{1}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{n}{n-1} = \ln\left(1+\dfrac{1}{1}\right)+\cdots+\left(1+\dfrac{1}{n}\right) < 1+\dfrac{1}{2}+\cdots+\dfrac{1}{n},$

$\ln n = \ln\left(1+\dfrac{1}{1}\right)+\cdots+\left(1+\dfrac{1}{n}\right) > \dfrac{1}{2}+\cdots+\dfrac{1}{n}.$

$(3)\left(1+\dfrac{1}{n}\right)^{n} = e^{n\ln\left(1+\frac{1}{n}\right)} > e^{n\cdot\frac{1}{n+1}} = e^{1-\frac{1}{n+1}}.$

习题 3.1

换元

求下列不定积分：

$$1.\int\sqrt{1+2x}\,dx=\frac{1}{2}\int\sqrt{1+2x}\,d(1+2x)=\frac{1}{3}(1+2x)^{3/2}+C.$$

$$2.\int\frac{3x}{(x^2+1)^2}dx=\frac{1}{2}\int\frac{3}{(x^2+1)^2}d(x^2+1)=-\frac{3}{2(x^2+1)}+C.$$

(换元基础)

$$3.\int x\sqrt{2x^2+7}\,dx=\frac{1}{4}\int\sqrt{2x^2+7}\,d(2x^2+7)=\frac{1}{6}(2x^2+7)^{3/2}+C.$$

$$4.\int(2x^{3/2}+1)^{2/3}\sqrt{x}\,dx=\frac{2}{3}\int(2x^{3/2}+1)^{2/3}dx^{3/2}$$

$$=\frac{2}{3}\cdot\frac{1}{2}\int(2x^{3/2}+1)^{2/3}d(2x^{3/2}+1)=\frac{1}{5}(2x^{3/2}+1)^{5/3}+C.$$

$$5.\int\frac{e^{1/x}}{x^2}dx=-\int e^{1/x}d(1/x)=-e^{1/x}+C.$$

$$6.\int\frac{dx}{(2-x)^{100}}=-\int\frac{d(2-x)}{(2-x)^{100}}=\frac{1}{99(2-x)^{99}}+C.$$

$\int\frac{1}{1+x^2}=\arctan x$　d 拼出可换元的。

$$7.\int\frac{dx}{3+5x^2}=\frac{1}{3}\int\frac{dx}{1+[(5/3)x]^2}=\frac{1}{3}\cdot\sqrt{\frac{3}{5}}\int\frac{d\sqrt{5/3}x}{1+[\sqrt{5/3}x]^2}=\frac{1}{\sqrt{15}}\arctan\sqrt{\frac{5}{3}}x+C.$$

$$8.\int\frac{dx}{\sqrt{7-3x^2}}=\int\frac{dx}{\sqrt{7}\sqrt{1-3/7x^2}}=\frac{1}{\sqrt{7}}\cdot\sqrt{\frac{7}{3}}\int\frac{d\sqrt{3/7}x}{\sqrt{7}\sqrt{1-\sqrt{3/7}x^2}}=\frac{1}{\sqrt{3}}\arcsin\sqrt{\frac{3}{7}}x+C.$$

$\arcsin=\frac{1}{\sqrt{1-x^2}}$

$$9.\int\frac{dx}{\sqrt{x}(1+x)}=2\int\frac{d\sqrt{x}}{(1+x)}=2\arctan\sqrt{x}+C.$$

$\arccos=-\frac{1}{\sqrt{1-x^2}}$

$$10.\int\frac{e^x}{2+e^{2x}}dx=\int\frac{1}{2+(e^x)^2}de^x=\frac{1}{\sqrt{2}}\arctan e^x+C.$$

$\arctan=\frac{1}{1+x^2}$

☆巧妙变换!

$$11.\int\frac{dx}{\sqrt{e^{-2x}-1}}=\int\frac{de^x}{\sqrt{1-(e^x)^2}}=\arcsin e^x+C.$$

$$12.\int\frac{dx}{e^x-e^{-x}}=\int\frac{de^x}{e^{2x}-1}=\int\frac{du}{(u-1)(u+1)}=\frac{1}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$\frac{1}{(u+1)(u-1)}$ 裂成法

$$=\frac{1}{2}\ln\frac{u-1}{u+1}+C=\frac{1}{2}\ln\frac{e^x-1}{e^x+1}+C.$$

☆二次换元

$$13.\int\frac{\ln\ln x}{x\ln x}dx=\int\frac{\ln\ln x}{\ln x}d\ln x=\int\ln\ln x\,d\ln\ln x=\frac{1}{2}(\ln\ln x)^2+C.$$

$$14.\int\frac{dx}{1+\cos x}=\int\frac{dx}{2\sin^2\frac{x}{2}}=\int\frac{d\frac{x}{2}}{\sin^2\frac{x}{2}}=-\cot\frac{x}{2}+C.$$

$\int\csc^2 x \nearrow -\cot\frac{x}{2}+C$

$$15.\int\frac{dx}{1-\sin x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{1+\cos\left(x+\frac{\pi}{2}\right)}=-\cot^2\left(\frac{x}{2}+\frac{\pi}{4}\right)+C.$$

$$16.\int\frac{x^{14}}{(x^5+1)^4}dx=\frac{1}{5}\int\frac{x^{10}}{(x^5+1)^4}dx^5=\frac{1}{5}\int\frac{u^2}{(u+1)^4}du(u=x^5)$$

$$=\frac{1}{5}\int\frac{u^2-1+1}{(u+1)^4}du=\frac{1}{5}\int\frac{(v-1)^2}{v^4}dv(v=u+1)$$

$$=\frac{1}{5}\int\frac{v^2-2v+1}{v^4}dv=\frac{1}{5}\int\left(v^{-2}-2v^{-3}+v^{-4}\right)dv$$

$$=\frac{1}{5}\left(-v^{-1}+v^{-2}-\frac{1}{3}v^{-3}\right)+C=\frac{1}{5}\left(-(x^5+1)^{-1}+(x^5+1)^{-2}-\frac{1}{3}(x^5+1)^{-3}\right)+C.$$

$$17.\int\frac{x^{2n-1}}{x^n-1}dx=\frac{1}{n}\int\frac{x^n}{x^n-1}dx^n=\frac{1}{n}\int\frac{u}{u-1}du(u=x^n)$$

$$=\frac{1}{n}\int\left(1+\frac{1}{u-1}\right)du=\frac{1}{n}(u+\ln|u-1|)+C=\frac{1}{n}(x^n+\ln|x^n-1|)+C.$$

$$18.\int\frac{dx}{x(x^5+2)}=\int\frac{x^4dx}{x^5(x^5+2)}=\frac{1}{5}\int\frac{du}{u(u+2)}(u=x^5)$$

$$=\frac{1}{5}g\frac{1}{2}\int\left(\frac{1}{u}-\frac{1}{u+2}\right)du=\frac{1}{10}(\ln|u|-\ln|u+2|)+C=\frac{1}{10}\ln\left|\frac{u}{u+2}\right|+C.$$

妙☆！

$$19.\int\frac{\ln(x+1)-\ln x}{x(x+1)}dx=\int(\ln(x+1)-\ln x)\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}\Rightarrow d(\ln x-\ln(x+1))$

$$=\int(\ln(x+1)-\ln x)d(\ln x-\ln(x+1))=-\int(\ln(x+1)-\ln x)d(\ln(x+1)-\ln x)$$

$$=-\frac{1}{2}\ln^2\frac{x+1}{x}+C.$$

$$20.\int\frac{e^{\arctan x}+x\ln(1+x^2)}{1+x^2}dx=\int\frac{e^{\arctan x}}{1+x^2}dx+\int\frac{x\ln(1+x^2)}{1+x^2}dx$$

$$=\int e^{\arctan x}d\arctan x+\frac{1}{2}\int\ln(1+x^2)d\ln(1+x^2)$$

$$=e^{\arctan x}+\frac{1}{4}\ln^2(1+x^2)+C.$$

$$=\int e^{\arctan x}d\arctan x+\frac{1}{2}\ln(1+x^2)d\ln(1+x^2)$$

$$=e^{\arctan x}\quad+\underbrace{\frac{1}{4}[\ln(1+x)]^2}+C$$

$$\frac{1}{4}\ln^2(1+x)^2$$

$$21.\int \sin 2x\cos 2x dx=\frac{1}{2}\int \sin 2x d\sin 2x=\frac{1}{4}\sin^2 2x+C.$$

$$22.\int \sin^2\frac{x}{2}\cos\frac{x}{2}dx=2\int \sin^2\frac{x}{2}d\sin\frac{x}{2}=\frac{2}{3}\sin^3\frac{x}{2}+C.$$

???

$$23.\int \sin 5x\sin 6x dx=\frac{1}{2}\int(\cos x-\cos 11x)dx=\frac{1}{2}\left(\sin x-\frac{1}{11}\sin 11x\right)+C.$$

$$24.\int\frac{2x-1}{\sqrt{1-x^2}}dx=\int\frac{2x}{\sqrt{1-x^2}}dx-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\int\frac{d(1-x^2)}{\sqrt{1-x^2}}-\arcsin x+C=-2\sqrt{1-x^2}-\arcsin x+C.$$

$$25.\int\frac{x^3+x}{\sqrt{1-x^2}}dx=\int\frac{x^3}{\sqrt{1-x^2}}dx+\int\frac{x}{\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx^2-\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}$$

这步

?

$$=\frac{1}{2}\int\sqrt{1-x^2}d(1-x^2)+\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx^2-\sqrt{1-x^2}$$

$$=\frac{1}{3}(1-x^2)^{3/2}-2\sqrt{1-x^2}+C.$$

$$26.\int\frac{dx}{(a^2-x^2)^{3/2}}(a>0)$$

$$x=a\sin t,t\in(-\pi/2,\pi/2),dx=a\cos t dt,$$

$$(a^2-x^2)^{3/2}=a^3\cos^3 t,$$

$$\int\frac{dx}{(a^2-x^2)^{3/2}}=\int\frac{dt}{a^2\cos^2 t}dx=\frac{1}{a^2}\tan t+C$$

$$=\frac{1}{a^2}\frac{x/a}{\sqrt{1-(x/a)^2}}+C=\frac{x}{a^2\sqrt{a^2-x^2}}+C.$$

$x<0$时，令$x=-y,y>0$,

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{\sqrt{y^2-a^2}}{y}dy=\sqrt{y^2-a^2}-a\arccos\frac{a}{y}+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{-x}+C=\sqrt{x^2-a^2}-\left(\pi-a\arccos\frac{a}{x}\right)+C$$

$$=\sqrt{x^2-a^2}+a\arccos\frac{a}{x}+C'.$$

$$-\int\frac{1}{\sqrt{1-x^2}}dx.$$

$$\frac{1}{\sqrt{1-(\frac{a}{y})^2}}$$

$$\sqrt{\frac{y^2-a^2}{y^2}}$$

$$-\frac{1}{\frac{\sqrt{y^2-a^2}}{y}}$$

$$27.\int\frac{\sqrt{x^2-a^2}}{x}dx(a>0).x>0\text{时},\text{令}x=a\sec t,t\in(0,\pi/2).$$

$$dx=a\tan t\sec t dt,\sqrt{x^2-a^2}=a\tan t,$$

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=a\int\tan^2 tdt=a\int(\sec^2 t-1)dt=a(\tan t-t)+C$$

$$=a(\sqrt{\sec^2 t-1}-\arccos\frac{a}{x})+C=a(\sqrt{\left(\frac{x}{a}\right)^2-1}-\arccos\frac{a}{x})+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C.$$

$$28.\int\frac{x^2}{\sqrt{a^2-x^2}}dx=-\int\sqrt{a^2-x^2}dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$=-\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+C.$$

$$29.\int\frac{dx}{\sqrt{1+e^{3x}}}=\int\frac{e^{-3x/2}dx}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\int\frac{de^{-3x/2}}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\ln(e^{-3x/2}+\sqrt{1+e^{-3x}})+C$$

$$=-\frac{2}{3}\ln(1+\sqrt{1+e^{3x}})+x+C=-\frac{2}{3}\ln\frac{(\sqrt{1+e^{3x}}+1)(\sqrt{1+e^{3x}}-1)}{\sqrt{1+e^{3x}}-1}+x+C$$

$$=\frac{2}{3}\ln(\sqrt{1+e^{3x}}-1)-x+C.$$

$$30.\int\frac{x^3}{\sqrt{1+x^8}}dx=\frac{1}{4}\int\frac{dx^4}{\sqrt{1+x^8}}=\frac{1}{4}\int\frac{du}{\sqrt{1+u^2}}(u=x^4)$$

$$=\frac{1}{4}\ln(u+\sqrt{1+u^2})+C=\frac{1}{4}\ln(x^4+\sqrt{1+x^8})+C.$$

$$31.\int\frac{dx}{x^6\sqrt{1+x^2}}=\int\frac{dx}{x^7\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{dx^{-2}}{x^4\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{u^2du}{\sqrt{1+u}}(u=\frac{1}{x^2})$$

$$=-\frac{1}{2}\int\frac{(v-1)^2}{v^{1/2}}dv=-\frac{1}{2}\int\frac{v^2-2v+1}{v^{1/2}}dv(v=1+u)$$

$$=-\frac{1}{2}\int(v^{3/2}-2v^{1/2}+v^{-1/2})dx$$

$$=-\frac{1}{2}\left(\frac{2}{5}v^{\frac{5}{2}}-2g\frac{2}{3}v^{\frac{3}{2}}+2gv^{\frac{1}{2}}\right)$$

$$=-\frac{1}{5}\left(1+\frac{1}{x^2}\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}-\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}+C$$

$$=-\frac{\sqrt{1+x^2}^5}{5x^5}+\frac{\sqrt{1+x^2}^3}{3x^3}-\frac{\sqrt{1+x^2}}{x}+C.$$

$$32.\int\frac{e^{2x}}{\sqrt[3]{1+e^x}}dx=\int\frac{e^x}{\sqrt[3]{1+e^x}}de^x=\int\frac{u}{\sqrt[3]{1+u}}du(u=e^x)(\sqrt[3]{u+1}=v,u=v^3-1)$$

$$=\int\frac{u}{\sqrt[3]{1+u}}du=\int\frac{v^3-1}{v}3v^2dv=3\int(v^4-v)dv=3\left(\frac{v^5}{5}-\frac{v^2}{2}\right)+C$$

$$=\frac{3}{5}(e^x+1)^{5/3}-\frac{3}{2}(e^x+1)^{2/3}+C.$$

$$33.\int\frac{dx}{\sqrt{3+x-x^2}}=\int\frac{dx}{\sqrt{3-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}}=\int\frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\frac{13}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$=\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{13}}{2}}+C=\arcsin\frac{2x-1}{\sqrt{13}}+C.$$

$$34.\int\sqrt{7+x-x^2}dx=\int\sqrt{7-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx=\int\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}d\left(x-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}+\frac{29}{8}\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{29}}{2}}+C$$

$$=\frac{2x-1}{4}\sqrt{7+x-x^2}+\frac{29}{8}\arcsin\frac{2x-1}{\sqrt{29}}+C.$$

$$35. \int \frac{dx}{1+\sqrt{x-1}}, 1+\sqrt{x-1}=u, x=1+(u-1)^2, dx=2(u-1)du,$$

$$\int \frac{dx}{1+\sqrt{x-1}}, 1+\sqrt{x-1}=u, x=1+(u-1)^2, dx=2(u-1)du,$$

$$\int \frac{dx}{1+\sqrt{x-1}} == \int \frac{2(u-1)du}{u} = 2(u-\ln u)+C = 2(1+\sqrt{x-1})-\ln(1+\sqrt{x-1}))+C$$

$$= 2\sqrt{x-1}-\ln(1+\sqrt{x-1}))+C'.$$

习题 3.2

求下列不定积分：

$$1.\int x\ln xdx=\frac{1}{2}\int \ln xdx^2=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2 d\ln x$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2 \cdot \frac{1}{x}dx=\frac{x^2}{2}\ln x-\frac{1}{2}\int xdx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

$$2.\int x^2e^{ax}dx=\frac{1}{a}\int x^2de^{ax}=\frac{1}{a}x^2e^{ax}-\frac{1}{a}\int e^{ax}dx^2=\frac{1}{a}x^2e^{ax}-\frac{2}{a}\int xe^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^2}\int e^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^3}e^{ax}+C$$

$$=e^{ax}\left(\frac{1}{a}x^2-\frac{2x}{a^2}+\frac{2}{a^3}\right)+C.$$

$$3.\int x\sin 2xdx=-\frac{1}{2}\int xd\cos 2x=-\frac{1}{2}x\cos 2x+\frac{1}{2}\int \cos 2xdx$$

$$=-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C.$$

$$4.\int \arcsin xdx=x\arcsin x-\int xd\arcsin x=x\arcsin x-\int \frac{xdx}{\sqrt{1-x^2}}$$

$$=x\arcsin x+\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}}=x\arcsin x+\sqrt{1-x^2}+C.$$

$$5.\int \arctan xdx=x\arctan x-\int xd\arctan x=x\arctan x-\int \frac{xdx}{1+x^2}$$

$$=x\arctan x-\frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}=x\arctan x-\frac{1}{2}\ln(1+x^2)+C.$$

$$6.I=\int e^{2x}\cos 3xdx=\frac{1}{2}\int \cos 3xde^{2x}=\frac{1}{2}e^{2x}\cos 3x-\frac{1}{2}\int e^{2x}d\cos 3x$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\int e^{2x}\sin 3xdx=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\int \sin 3xde^{2x}$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\left(e^{2x}\sin 3x-3\int e^{2x}\cos 3xdx\right)$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I,$$

$$I=\frac{4}{13}\left(\frac{1}{2}\cos 3x+\frac{3}{4}\sin 3x\right)e^{2x}+C=\frac{1}{13}\left(2\cos 3x+3\sin 3x\right)e^{2x}+C.$$

$$7.I=\int \frac{\sin 3x}{e^x}dx=-\int \sin 3xde^{-x}=-e^{-x}\sin 3x+3\int e^{-x}\cos 3xdx$$

$$=-e^{-x}\sin 3x-3\int \cos 3xde^{-x}=-e^{-x}\sin 3x-3\left(e^{-x}\cos 3x+3\int e^{-x}\sin 3xdx\right)$$

$$= -e^{-x}\sin 3x - 3(e^{-x}\cos 3x + 3I),$$

$$I = \frac{1}{10}\left(-e^{-x}\sin 3x - 3e^{-x}\cos 3x\right) + C = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) + C.$$

$$8. I = \int e^{ax}\sin bx dx = \frac{1}{a}\int \sin bx de^{ax} = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\int \cos bx de^{ax}$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + b\int e^{ax}\sin bx dx\right)$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + bI\right).$$

$$I = \frac{1}{1+\frac{b^2}{a^2}}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right),$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

$$9. I = \int \sqrt{1+9x^2}dx = x\sqrt{1+9x^2} - \int x d\sqrt{1+9x^2}$$

$$= x\sqrt{1+9x^2} - \int \frac{x\cdot 18x dx}{2\sqrt{1+9x^2}}$$

$$= x\sqrt{1+9x^2} - \left(\int \sqrt{1+9x^2}dx - \int \frac{dx}{\sqrt{1+9x^2}}\right)$$

$$= x\sqrt{1+9x^2} - \left(I - \int \frac{dx}{\sqrt{1+9x^2}}\right),$$

$$I = \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{2}\cdot\frac{1}{3}\ln(3x + \sqrt{1+9x^2}) + C$$

$$= \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{6}\ln(3x + \sqrt{1+9x^2}) + C.$$

$$10. \int x\cosh x dx = \int x d\sinh x = x\sinh x - \int \sinh x dx$$

$$= x\sinh x - \cosh x + C.$$

$$11. \int \ln(x+\sqrt{1+x^2})dx = x\ln(x+\sqrt{1+x^2}) - \int x d\ln(x+\sqrt{1+x^2})$$

$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{xdx}{\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

$$12. \int (\arccos x)^2 dx = x(\arccos x)^2 + 2\int \frac{x\arccos x}{\sqrt{1-x^2}}dx$$

$$= x(\arccos x)^2 - 2\int \arccos x d\sqrt{1-x^2}$$

$$= x(\arccos x)^2 - 2\left(\sqrt{1-x^2}\arccos x + \int 1 dx\right)$$

$$= x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x - 2x + C.$$

$$13.\int \frac{x\arccos x dx}{(1-x^2)^2} = \frac{1}{2}\int \arccos x d\frac{1}{1-x^2}$$

$$= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2}\int \frac{dx}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2}\frac{x}{\sqrt{1-x^2}} + C.$$

$$14.\int \arctan\sqrt{x}dx = x\arctan\sqrt{x} - \int \frac{xdx}{2(1+x)\sqrt{x}}$$

$$= x\arctan\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}dx}{1+x}.\sqrt{x} = u, x = u^2, dx = 2udu$$

$$\int \frac{\sqrt{x}dx}{1+x} = \int \frac{u2udu}{1+u^2} = 2\left(u - \arctan u\right) + C,$$

$$\int \arctan\sqrt{x}dx = x\arctan\sqrt{x} - \frac{1}{2}g2(\sqrt{x} - \arctan\sqrt{x}) + C$$

$$= x\arctan\sqrt{x} - (\sqrt{x} - \arctan\sqrt{x}) + C$$

$$= (x+1)\arctan\sqrt{x} - \sqrt{x} + C.$$

$$15.\int \frac{\arcsin x}{x^2}dx = -\int \arcsin x d\left(\frac{1}{x}\right) = -\frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}}$$

$$= -\frac{\arcsin x}{x} + \int \frac{dx}{x^2\sqrt{1/x^2-1}}(x>0)$$

$$= -\frac{\arcsin x}{x} - \int \frac{d(1/x)}{\sqrt{1/x^2-1}} = -\frac{\arcsin x}{x} - \ln|1/x + \sqrt{1/x^2-1}| + C$$

$$= -\frac{\arcsin x}{x} + \ln(1-\sqrt{1-x^2}) - \ln x + C$$

$$= -\frac{\arcsin x}{x} + \ln(1-\sqrt{1-x^2}) - \ln|x| + C(x \neq 0)(\text{原函数为偶函数}).$$

$$16.\int x^3(\ln x)^2 dx = \frac{1}{4}\int (\ln x)^2 dx^4 = \frac{x^4(\ln x)^2}{4} - \frac{1}{4}\int \frac{x^4 g2\ln x dx}{x}$$

$$= \frac{x^4(\ln x)^2}{4} - \frac{1}{2}\int x^3\ln x dx = \frac{x^4(\ln x)^2}{4} - \frac{1}{8}\int \ln x dx^4$$

$$= \frac{x^4(\ln x)^2}{4} - \frac{x^4}{8}\ln x + \frac{1}{2}\int x^3 dx = \frac{x^4(\ln x)^2}{4} - \frac{x^4}{8}\ln x + \frac{1}{8}x^4 + C.$$

$$17.\int \frac{x\arctan x dx}{(1+x^2)^{5/2}} = \frac{1}{2}\int \frac{\arctan x d(1+x^2)}{(1+x^2)^{5/2}} = \frac{1}{2}g\left(-\frac{2}{3}\right)\int \arctan x d(1+x^2)^{-3/2}$$

$$= -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3}\int \frac{dx}{(1+x^2)^{5/2}}. x = \tan u, u \in (-\pi/2, \pi/2). dx = \sec^2 u du,$$

$$\int \frac{dx}{(1+x^2)^{5/2}} = \int \cos^3 u du = \int (1-\sin^2 u) d \sin u =$$

$$= \sin u - \frac{1}{3}\sin^3 u + C = \frac{x}{\sqrt{1+x^2}} - \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3 + C,$$

$$\int \frac{x \arctan x dx}{(1+x^2)^{5/2}} = -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}} - \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3\right) + C$$

$$= -\frac{\arctan x}{3(1+x^2)^{3/2}} + \frac{1}{3}\frac{x}{\sqrt{1+x^2}} - \frac{1}{9}\frac{x^3}{(1+x^2)^{3/2}} + C.$$

$$18. \int x \ln(x+\sqrt{1+x^2}) dx = \frac{1}{2}\int \ln(x+\sqrt{1+x^2}) dx^2$$

$$= \frac{1}{2}x^2 \ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int \frac{x^2 dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}x^2 \ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int \frac{(x^2+1)-1 dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}x^2 \ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int \sqrt{1+x^2} dx + \frac{1}{2}\int \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}x^2 \ln(x+\sqrt{1+x^2}) - \frac{1}{2}\left(\frac{x\sqrt{1+x^2}}{2} + \frac{\ln(x+\sqrt{1+x^2})}{2}\right) + \frac{1}{2}\ln(x+\sqrt{1+x^2}) + C$$

$$= \frac{1}{2}x^2 \ln(x+\sqrt{1+x^2}) - \frac{1}{4}x\sqrt{1+x^2} + \frac{1}{4}\ln(x+\sqrt{1+x^2}) + C.$$

习题 3.3

求下列不定积分:

$$1.\int\frac{x-1}{x^2+6x+8}dx=\int\frac{x-1}{(x+2)(x+4)}dx,$$

$$\frac{x-1}{(x+2)(x+4)}=\frac{A}{x+2}+\frac{B}{x+4},$$

$$A=\frac{-2-1}{-2+4}=-\frac{3}{2},B=\frac{-4-1}{-4+2}=\frac{5}{2},$$

$$\frac{x-1}{(x+2)(x+4)}=\frac{-3/2}{x+2}+\frac{5/2}{x+4},$$

$$\int\frac{x-1}{x^2+6x+8}dx=-\frac{3}{2}\ln|x+2|+\frac{5}{2}\ln|x+4|+C.$$

$$2.I=\int\frac{3x^4+x^2+1}{x^2+x-6}dx.$$

$$\frac{3x^4+x^2+1}{x^2+x-6}=3x^2-3x+22+\frac{-40x+133}{x^2+x-6},$$

$$\frac{-40x+133}{x^2+x-6}=\frac{-40x+133}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2},$$

$$A=\frac{-40g(-3)+133}{-3-2}=-\frac{253}{5},B=\frac{-40g2+133}{2+3}=\frac{53}{5}.$$

$$I=x^3-\frac{3x^2}{2}+22x-\frac{253}{5}\ln|x+3|+\frac{53}{5}\ln|x-2|+C.$$

$$3.I=\int\frac{2x^2-5}{x^4-5x^2+6}dx$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{2u-5}{u^2-5u+6}(u=x^2)$$

$$=\frac{2u-5}{(u-2)(u-3)}=\frac{A}{u-2}+\frac{B}{u-3},$$

$$A=\frac{2g2-5}{2-3}=1,B=\frac{2g3-5}{3-2}=1.$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{1}{x^2-\sqrt{2}^2}+\frac{1}{x^2-\sqrt{3}^2},$$

$$I=\frac{1}{2\sqrt{2}}\ln\frac{x-\sqrt{2}}{x+\sqrt{2}}+\frac{1}{2\sqrt{3}}\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}+C.$$

$$4.I=\int\frac{dx}{(x-1)^2(x-2)}.$$

$$\frac{1}{(x-1)^2(x-2)}=\frac{1}{x-2}\left(\frac{1}{x-2}-\frac{1}{x-1}\right)$$

$\frac{1}{(x-1)}\left(\frac{1}{x-2}-\frac{1}{x-1}\right)$

$\left(\frac{1}{x-2}-\frac{1}{x-1}\right)-\frac{1}{(x-1)^2}$

$\left(\frac{1}{x-1}\right)'=\frac{-1}{(\ \ )^2}$

$$=\frac{1}{(x-2)^2}-\left(\frac{1}{x-2}-\frac{1}{x-1}\right),$$

$$I=-\frac{1}{x-2}+\ln\left|\frac{x-1}{x-2}\right|+C.$$

$$5.I=\int\frac{x^2}{1-x^4}dx.$$

$$\frac{x^2}{1-x^4}=\frac{x^2}{(1-x^2)(1+x^2)}=\frac{1}{2}\cdot\frac{(1+x^2)-(1-x^2)}{(1-x^2)(1+x^2)}$$

$$=\frac{1}{2}\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right),$$

$$I=\frac{1}{4}\ln\frac{1+x}{1-x}-\frac{1}{2}\arctan x+C.$$

$$6.I=\int\frac{dx}{x^3+1}.$$

$$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1},$$

$$A=\frac{1}{1^2+1+1}=\frac{1}{3},$$

$$1=\frac{x^2-x+1}{3}+(x+1)(Bx+C)=(B+\frac{1}{3})x^2+(B+C-\frac{1}{3})x+C+\frac{1}{3},$$

$$C+\frac{1}{3}=1,C=\frac{2}{3},B+\frac{1}{3}=0,B=-\frac{1}{3}.$$

$$\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$$

$$=\frac{1}{3(x+1)}-\frac{2x-4}{6(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{(2x-1)-3}{(x^2-x+1)}.$$

$$=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{2x-1}{(x^2-x+1)}+\frac{1}{2}\cdot\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2},$$

$$I=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

$$7. I=\int \frac{dx}{1+x^4}. \frac{1}{1+x^4}=\frac{1}{(1+2x^2+x^4)-2x^2}=\frac{1}{(x^2+1)^2-2x^2}$$

$$=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1},$$

$$1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1),$$

$$1=(A+C)x^3+(B-\sqrt{2}A+D+\sqrt{2}C)x^2+(A-\sqrt{2}B+C+\sqrt{2}D)x+B+D.$$

$$\begin{cases} A+C=0 \\ B-\sqrt{2}A+D+\sqrt{2}C=0, \\ A-\sqrt{2}B+C+\sqrt{2}D=0, \\ B+D=1. \end{cases}$$

$$A=\frac{1}{2\sqrt{2}}, B=\frac{1}{2}, C=-\frac{1}{2\sqrt{2}}, D=\frac{1}{2}.$$

$$\frac{1}{1+x^4}=\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}+\frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})}{x^2-\sqrt{2}x+1}\right)+\frac{1}{4} g \frac{1}{(x+\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$+\frac{1}{4} g \frac{1}{(x-\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}.$$

$$I=\frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{4}\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)+C.$$

$$8. I=\int \frac{x^3+x^2+2}{(x^2+2)^2}dx.$$

$$\frac{x^3+x^2+2}{(x^2+2)^2}=\frac{x(x^2+2)}{(x^2+2)^2}+\frac{x^2-2x+2}{(x^2+2)^2}$$

$$=\frac{x}{(x^2+2)}+\frac{1}{(x^2+2)}-\frac{2x}{(x^2+2)^2}.$$

$$I=\frac{1}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{x^2+2}+C.$$

$$9.\int\frac{e^x dx}{e^{2x}+3e^x+2}=\int\frac{de^x}{e^{2x}+3e^x+2}=\int\frac{du}{u^2+3u+2}=$$
$$=\int\frac{du}{(u+1)(u+2)}=\int\left(\frac{1}{u+1}-\frac{1}{u+2}\right)du=\ln\frac{u+1}{u+2}+C=\ln\frac{e^x+1}{e^x+2}+C.$$

$$10.\int\frac{\cos x dx}{\sin^2 x+\sin x-6}=\int\frac{d\sin x}{\sin^2 x+\sin x-6}=\int\frac{du}{u^2+u-6}(u=\sin x)=$$
$$\int\frac{du}{(u+3)(u-2)}=\frac{1}{5}\int\left(\frac{1}{u-2}-\frac{1}{u+3}\right)du=\ln\left|\frac{u-2}{u+3}\right|+C=\ln\left|\frac{\sin x-2}{\sin x+3}\right|+C.$$

$$11.\int\frac{x^3dx}{x^4+x^2+2}=\frac{1}{2}\int\frac{x^2dx^2}{x^4+x^2+2}=\frac{1}{2}\int\frac{udu}{u^2+u+2}$$
$$=\frac{1}{4}\int\frac{2udu}{u^2+u+2}=\frac{1}{4}\int\frac{(2u+1)-1}{u^2+u+2}du=$$
$$=\frac{1}{4}\int\frac{d(u^2+u+2)}{u^2+u+2}du-\frac{1}{4}\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{7}{4}}du$$
$$=\frac{1}{4}\ln(u^2+u+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2u+1}{\sqrt{7}}+C$$
$$=\frac{1}{4}\ln(x^4+x^2+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2x^2+1}{\sqrt{7}}+C.$$

$$12.I=\int\frac{dx}{(x+2)(x^2-2x+2)}.$$
$$\frac{1}{(x+2)(x^2-2x+2)}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+2}$$
$$A=\frac{1}{(-2)^2-2(-2)+2}=\frac{1}{10}.$$
$$\frac{1}{(x+2)(x^2-2x+2)}-\frac{1}{10(x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{10-(x^2-2x+2)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x^2-2x-8)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x+2)(x-4)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x-4)}{10(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2},B=-\frac{1}{10},C=\frac{2}{5}.$$
$$I=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{x-4}{x^2-2x+2}dx$$

$$=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{2x-8}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\int\frac{(2x-2)-6}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\int\frac{dx}{(x-1)^2+1}$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\arctan(x-1)+C$$

$$13. I=\int\frac{dx}{2+\sin x}. \tan\frac{x}{2}=u, x=2\arctan u, dx=\frac{2du}{1+u^2}, \sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2u}{1+u^2}.$$

$$I=\int\frac{\frac{2du}{1+u^2}}{2+\frac{2u}{1+u^2}}=\int\frac{1}{u^2+u+1}du=\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du$$

$$=\frac{2}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}}+C=\frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}+C.$$

$$14. I=\int\frac{dx}{1+\sin x+\cos x}. \tan\frac{x}{2}=u, x=2\arctan u, dx=\frac{2du}{1+u^2},$$
$$\sin x=\frac{2u}{1+u^2}, \cos x=\frac{1-u^2}{1+u^2}.$$
$$I=\int\frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}=2\int\frac{1}{1+u^2+2u+1-u^2}du=\int\frac{1}{u+1}du$$
$$=\ln|u+1|+C=\ln|\tan\frac{x}{2}+1|+C.$$

$$15. \int\cot^4 x dx$$
$$=\int\cot^2 x(\csc^2 x-1)dx$$
$$=\int\cot^2 x\csc^2 x dx-\int\cot^2 x dx$$
$$=-\int\cot^2 x d\cot x-\int(\csc^2 x-1)dx$$
$$=-\frac{1}{3}\cot^3 x+\cot x+x+C.$$

$$16. \int\sec^4 x dx=\int(1+\tan^2 x)d\tan x=\tan x+\frac{1}{3}\tan^3 x+C.$$

$$17. I=\int\frac{\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{-3\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{(-3\cos x+5)-5dx}{5-3\cos x}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{dx}{5-3\cos x}.$$

$$\tan\frac{x}{2}=u, dx=\frac{2du}{1+u^2}, \cos x=\frac{1-u^2}{1+u^2},$$

$$I=-\frac{x}{3}+\frac{5}{3}\int\frac{\frac{2du}{1+u^2}}{5-\frac{3(1-u^2)}{1+u^2}}=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{5(1+u^2)-3(1-u^2)}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{8u^2)+2}=-\frac{x}{3}+\frac{5}{3}\int\frac{du}{4u^2+1}=-\frac{x}{3}+\frac{5}{3}g\frac{1}{2}\int\frac{d2u}{4u^2+1}$$

$$=-\frac{x}{3}+\frac{5}{6}\arctan 2u+C=-\frac{x}{3}+\frac{5}{6}\arctan\left(2\tan\frac{x}{2}\right)+C.$$

$$18. I=\int\frac{\cos^3 xdx}{\sin x+\cos x}=\int\frac{\cos^2 xdx}{1+\tan x}=\int\frac{dx}{(1+\tan x)(1+\tan^2 x)}.$$

$$\tan x=u, x=\arctan u, dx=\frac{du}{1+u^2},$$

$$I=\int\frac{\frac{du}{1+u^2}}{(1+u)(1+u^2)}=\int\frac{du}{(1+u)(1+u^2)^2},$$

$$\frac{1}{(1+u)(1+u^2)^2}=\frac{1}{2(1+u^2)}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)$$

$$=\frac{1}{4}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)+\frac{1-u}{2(1+u^2)^2},$$

$$I=\frac{1}{4}\ln|1+\tan x|+\frac{1}{4}\arctan u-\frac{1}{8}\ln(1+u^2)+\frac{1}{4(1+u^2)}+\frac{1}{2}\left(\frac{1}{2}\arctan u+\frac{u}{2(1+u^2)}\right)+C$$

$$=\frac{1}{4}\ln|1+\tan x|+\frac{x}{2}+\frac{1}{4}\ln|\cos u|++\frac{1}{4}\cos^2 x+\frac{1}{4}\tan x\cos^2 x+C.$$

$$19. \int\sin^5 x\cos^2 xdx=-\int\sin^4 x\cos^2 xd\cos x=-\int(1-u^2)^2u^2du$$

$$=-\int(u^2-2u^4+u^6)dx=-\frac{1}{3}u^3+\frac{2}{5}u^5-\frac{1}{7}u^7+C$$

$$=-\frac{1}{3}(\cos x)^3+\frac{2}{5}(\cos x)^5-\frac{1}{7}(\cos x)^7+C.$$

$$20. \int\sin^6 xdx=\int\left(\frac{1-\cos 2x}{2}\right)^3dx$$

$$=\frac{1}{8}\int(1-3\cos 2x+3\cos^2 2x-\cos^3 2x)dx$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=+C.$$

$$21.\int\sin^2 x\cos^4 xdx=\frac{1}{4}\int\sin^2 2x\cos^2 xdx=\frac{1}{4}\int\left(\frac{\sin 3x+\sin x}{2}\right)^2dx$$

$$=\frac{1}{16}\int\left(\sin^2 3x+\sin^2 x+2\sin 3x\sin x\right)dx$$

$$=\frac{1}{16}\int\left(\frac{1-\cos 6x}{2}+\frac{1-\cos 2x}{2}+\cos 2x-\cos 4x\right)dx$$

$$=\frac{1}{16}\left(x+\frac{1}{4}\sin 2x-\frac{1}{4}\sin 4x-\frac{1}{12}\sin 6x\right)+C.$$

另解:$\int\sin^2 x\cos^4 xdx=\int\frac{1-\cos 2x}{2}g\left(\frac{1+\cos 2x}{2}\right)^2dx$

$$=\frac{1}{8}\int(1+\cos^2 2x+2\cos 2x)(1-\cos 2x)dx$$

$$=\frac{1}{8}\int\left(1+\cos^2 2x+2\cos 2x-\cos 2x-\cos^3 2x-2\cos^2 2x\right)dx$$

$$=\frac{1}{8}\int\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=\frac{1}{16}x-\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C.$$

$$22.I=\int\frac{dx}{\sin x+2\cos x}.\tan\frac{x}{2}=u,x=2\arctan u,dx=\frac{2du}{1+u^2}.$$

$$I=\int\frac{\frac{2du}{1+u^2}}{\frac{2u}{1+u^2}+\frac{2(1-u^2)}{1+u^2}}=\int\frac{2du}{-2u^2+2u+2}=-\int\frac{du}{u^2-u-1}=-\int\frac{du}{\left(u-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}=$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{u-\frac{1}{2}+\frac{\sqrt{5}}{2}}{u-\frac{1}{2}-\frac{\sqrt{5}}{2}}\right|+C=\ln\left|\frac{2u+\sqrt{5}-1}{2u-\sqrt{5}-1}\right|+C.$$

$$23.\int\frac{\sin x\cos x}{\sin^2 x+\cos^4 x}dx=$$

$$=\int\frac{\tan x}{\tan^2 x(1+\tan^2 x)+1}d\tan x=\int\frac{u}{u^2(1+u^2)+1}du(u=\tan x)$$

$$=\frac{1}{2}\int\frac{du^2}{u^2(1+u^2)+1}=\frac{1}{2}\int\frac{dv}{v(1+v)+1}(v=u^2)$$

$$=\frac{1}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2v+1}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{2\tan^2 x+1}{\sqrt{3}}+C.$$

另解$:I=\frac{1}{2}\int\frac{d\sin^2 x}{\sin^2 x+(1-\sin^2 x)^2}=\frac{1}{2}\int\frac{dw}{w+(1-w)^2}(w=\sin^2 x)$

$$=\frac{1}{2}\int\frac{dw}{(w-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\sqrt{3}}\arctan\frac{2w-1}{\sqrt{3}}+C=\arctan\frac{2\sin^2 x-1}{\sqrt{3}}+C.$$

24.$\int\frac{dx}{\sin^4 x}=-\int(1+\cot^2 x)d\cot x=-\cot x-\frac{1}{3}\cot^3 x+C.$

25.$\int\sqrt{\frac{1-x}{1+x}}dx=\int\frac{1-x}{\sqrt{1-x^2}}dx=\arcsin x+\sqrt{1-x^2}+C.$

26.$I=\int\frac{1-\sqrt{x-1}}{1+\sqrt[3]{x-1}}dx.\sqrt[6]{x-1}=u,x=1+u^6,dx=6u^5du,$

$$I=6\int\frac{(1-u^3)u^5du}{1+u^2}=6\int\frac{u^5-u^8}{1+u^2}du=-6\int(u^6-u^4-u^3+u^2+u+1+\frac{-u+1}{1+u^2})dx$$

$$=-6\left(\frac{1}{7}u^7-\frac{1}{5}u^5-\frac{1}{4}u^4+\frac{1}{3}u^3+\frac{1}{2}u^2+u-\frac{1}{2}\ln(1+u^2)+\arctan u\right)+C.$$

27.$\int\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx=\int\frac{\left(\sqrt{x+1}+\sqrt{x-1}\right)^2}{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)}dx$

$$=\int\frac{2x+2\sqrt{x^2-1}}{2}dx=\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln(x+\sqrt{x^2-1})+C.$$

28.$I=\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{dx}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}}.\sqrt[3]{\frac{x-1}{x+1}}=u,\frac{x-1}{x+1}=u^3,$

$$x-1=(x+1)u^3,x=\frac{1+u^3}{1-u^3}=-1+\frac{2}{1-u^3},dx=\frac{6u^2du}{(1-u^3)^2},$$

$$I=\int\frac{\dfrac{6u^2du}{(1-u^3)^2}}{\left(\left(\dfrac{1+u^3}{1-u^3}\right)^2-1\right)u}=6\int\frac{u}{2(2u^3)}du=\frac{3}{2}\left(-\frac{1}{u}\right)+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C.$$

$$29.\int\frac{xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2x-1+1dx}{\sqrt{x^2-x+3}}=$$

$$=\frac{1}{2}\int\frac{d(x^2-x+3)}{\sqrt{x^2-x+3}}+\frac{1}{2}\int\frac{dx}{\sqrt{\left(x-\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{11}}{2}\right)^2}}=$$

$$=\sqrt{x^2-x+3}+\frac{1}{2}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+3}\right)+C.$$

$$30.I=\int\frac{x}{(1+x^{1/3})^{1/2}}dx.(1+x^{1/3})^{1/2}=u,x=(u^2-1)^3,dx=3(u^2-1)^2(2u)du,$$

$$I=6\int\frac{(u^2-1)^3(u^2-1)^2(u)du}{u}=6\int(u^6-3u^4+3u^2-1)(u^4-2u^2+1)du$$

$$=6\int(u^{10}-5u^8+10u^6-10u^4+5u^2-1)du$$

$$=6\left(\frac{1}{11}u^{11}-\frac{5}{9}u^9+\frac{10}{7}u^7-2u^5+\frac{5}{3}u^3-u\right)+C.$$

$$31.I=\int\frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}.\sqrt[4]{x}=u,x=u^4,dx=4u^3du.$$

$$I=\int\frac{u^2 4u^3du}{u^3+1}=4\int\frac{u^5}{u^3+1}dx=4\int\frac{(u^5+u^2)-u^2}{u^3+1}du$$

$$=4\int\left(u^2-\frac{u^2}{u^3+1}\right)du=\frac{4}{3}u^3-\frac{4}{3}\ln(u^3+1)+C=\frac{4}{3}\sqrt[4]{x}^3-\frac{4}{3}\ln(\sqrt[4]{x}^3+1)+C.$$

$$32.\int\frac{2x+3}{\sqrt{x^2+x}}dx=\int\frac{(2x+1)+2}{\sqrt{x^2+x}}dx=\int\frac{1}{\sqrt{x^2+x}}d(x^2+x)+2\int\frac{1}{\sqrt{x^2+x}}dx$$

$$=2\sqrt{x^2+x}+2\int\frac{1}{\sqrt{\left(x+\dfrac{1}{2}\right)^2-\left(\dfrac{1}{2}\right)^2}}dx$$

$$=2\sqrt{x^2+x}+2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C.$$

$$33.\int\frac{2+x}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{16+8x}{\sqrt{4x^2-4x+5}}dx$$

$$=\frac{1}{8}\int\frac{8x-4+20}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{d(4x^2-4x+5)}{\sqrt{4x^2-4x+5}}dx+\frac{5}{2}\int\frac{dx}{\sqrt{4x^2-4x+5}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{x^2-x+5/4}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+1}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+5/4}\right)+C$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(2x-1+\sqrt{4x^2-4x+5}\right)+C'.$$

$$34.\int\sqrt{5-2x+x^2}\,dx=\int\sqrt{2^2+(x-1)^2}\,dx$$

$$=\frac{(x-1)}{2}\sqrt{5-2x+x^2}+2\ln(\sqrt{5-2x+x^2})+C.$$

习题 3.4

求下列各定积分:

$$1. I=\int_{-1}^{1}\frac{xdx}{\sqrt{5-4x}}.\sqrt{5-4x}=u,-1\to 3,1\to 1.5-4x=u^2,x=\frac{1}{4}(5-u^2),dx=-\frac{1}{2}udu,$$

$$I=\int_{3}^{1}\frac{\frac{1}{4}(5-u^2)}{u}\cdot\left(-\frac{1}{2}udu\right)=\frac{1}{8}\int_{1}^{3}(5-u^2)dx=\frac{1}{8}\left(5u-\frac{1}{3}u^3\right)\Big|_{1}^{3}=\frac{1}{6}.$$

$$2.\int_{0}^{\ln 2}xe^{-x}dx=-\int_{0}^{\ln 2}xde^{-x}=-xe^{-x}\Big|_{0}^{\ln 2}+\int_{0}^{\ln 2}e^{-x}dx=-\frac{\ln 2}{2}-e^{-x}\Big|_{0}^{\ln 2}=\frac{1}{2}(1-\ln 2).$$

$$3.\int_{0}^{1}x^2\sqrt{1-x^2}dx=\int_{0}^{\pi/2}\sin^2 t\cos^2 tdt(x=\sin t)$$

$$=\int_{0}^{\pi/2}\sin^2 t(1-\sin^2 t)dt=I_2-I_4=\left(\frac{1}{2}-\frac{3\cdot 1}{4\cdot 2}\right)\frac{\pi}{2}=\frac{\pi}{16}.$$

$$4.\int_{0}^{\pi}x\sin xdx=-\int_{0}^{\pi}xd\cos x=-x\cos x\Big|_{0}^{\pi}+\int_{0}^{\pi}\cos xdx=\pi+\sin x\Big|_{0}^{\pi}=\pi.$$

$$5.\int_{0}^{4}\sqrt{x^2+9}dx=\left(\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\ln(x+\sqrt{x^2+9})\right)\Bigg|_{0}^{4}=10+\frac{9}{2}\ln 3.$$

$$6.\int_{0}^{\frac{1}{2}}\frac{x^2}{\sqrt{1-x^2}}dx=\int_{0}^{\frac{\pi}{6}}\sin^2 tdt=\frac{1}{2}\int_{0}^{\frac{\pi}{6}}(1-\cos 2t)dt=\frac{1}{2}(t-\frac{1}{2}\sin 2t)\Big|_{0}^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right).$$

$$7.\int_{0}^{1}\sqrt{4-x^2}dx=\left(\frac{x}{2}\sqrt{4-x^2}+2\arcsin\frac{x}{2}\right)\Bigg|_{0}^{1}=\frac{\sqrt{3}}{2}+\frac{\pi}{3}.$$

$$8.\int_{0}^{3}x\sqrt[3]{1-x^2}dx=\frac{1}{2}\int_{0}^{3}\sqrt[3]{1-x^2}dx^2=\frac{1}{2}\int_{0}^{9}\sqrt[3]{1-u}du=-\frac{3}{8}(1-u)^{\frac{4}{3}}\Bigg|_{0}^{9}=-\frac{45}{8}.$$

$$9.\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_{0}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_{0}^{\frac{\pi}{2}}\sqrt{\cos x}\sin xdx$$

$$=-2\int_{0}^{\frac{\pi}{2}}\sqrt{\cos x}d\cos x=-\frac{4}{3}\cos^{\frac{3}{2}}x\Bigg|_{0}^{\frac{\pi}{2}}=\frac{4}{3}.$$

$$10.\int_{0}^{\frac{\pi}{2}}\cos^n 2xdx=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\cos^n 2xd2x=\frac{1}{2}\int_{0}^{\pi}\cos^n udu=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^n(t+\frac{\pi}{2})dt$$

$$=\frac{(-1)^n}{2}\int_{-\pi/2}^{\pi/2}\sin^n(t)dt=\begin{cases}0,n=2k-1;\\ \int_{0}^{\pi/2}\sin^n(t)dt=\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}.\end{cases}$$

$$11.\int_{0}^{a}(a^2-x^2)^{\frac{n}{2}}dx(x=a\sin t)=\int_{0}^{\frac{\pi}{2}}\cos^{n+1}tdt=\begin{cases}\dfrac{n!!}{(n+1)!!} & n\text{是偶数};\\ \dfrac{n!!}{(n+1)!!}\dfrac{\pi}{2} & n\text{是奇数}.\end{cases}$$

$$12.\int_0^{\pi/2}\sin^{11}xdx=\frac{10!!}{11!!}=\frac{156}{693}.$$

$$13.\int_0^{\pi}\sin^6\frac{x}{2}dx=2\int_0^{\pi/2}\sin^6udu=2\cdot\frac{5\cdot3}{6\cdot4\cdot2}\cdot\frac{\pi}{2}=\frac{5\pi}{16}.$$

$$14.\int_0^{\pi}(x\sin x)^2dx=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^{\pi}-\frac{1}{4}\int_0^{\pi}x^2d\sin 2x$$

$$=\frac{\pi^3}{6}-\frac{1}{4}x^2\sin 2x\Big|_0^{\pi}+\frac{1}{2}\int_0^{\pi}x\sin 2xdx$$

$$=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}xd\cos 2x=\frac{\pi^3}{6}-\frac{1}{4}x\cos 2x\Big|_0^{\pi}+\frac{1}{4}\int_0^{\pi}\cos 2xdx$$

$$=\frac{\pi^3}{6}-\frac{\pi}{4}+\frac{1}{8}\sin 2x\Big|_0^{\pi}=\frac{\pi^3}{6}-\frac{\pi}{4}.$$

$$15.\int_0^{\pi/4}\tan^4 xdx=\int_0^{\pi/4}\tan^2 x(\sec^2 x-1)dx$$

$$=\int_0^{\pi/4}\tan^2 x\sec^2 xdx-\int_0^{\pi/4}\tan^2 xdx$$

$$=\int_0^{\pi/4}\tan^2 xd\tan x-\int_0^{\pi/4}(\sec^2 x-1)dx$$

$$=\frac{1}{3}\tan^3 x\Big|_0^{\pi/4}-\tan x\Big|_0^{\pi/4}+\frac{\pi}{4}=\frac{1}{3}-1+\frac{\pi}{4}=-\frac{2}{3}+\frac{\pi}{4}.$$

$$16.\int_0^1\arcsin xdx=x\arcsin x\Big|_0^1-\int_0^1 xd\arcsin x$$

$$=\frac{\pi}{2}-\int_0^1\frac{x}{\sqrt{1-x^2}}dx=\frac{\pi}{2}+\sqrt{1-x^2}\Big|_0^1=\frac{\pi}{2}-1.$$

$$17.\int_0^{\pi}\ln(x+\sqrt{x^2+a^2})dx=x\ln(x+\sqrt{x^2+a^2})\Big|_0^{\pi}-\int_0^{\pi}xd\ln(x+\sqrt{x^2+a^2})$$

$$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\int_0^{\pi}\frac{x}{\sqrt{x^2+a^2}}dx=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{x^2+a^2}\Big|_0^{\pi}$$

$$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{\pi^2+a^2}+|a|.$$

18.设$f(x)$在$[a,b]$连续.证明$\int_a^b f(x)dx=(b-a)\int_0^1 f(a+(b-a)x)dx.$

证 令$x=a+(b-a)t$,则$0\to a,1\to b,dx=(b-a)dt$,故

$$\int_a^b f(x)dx=(b-a)\int_0^1 f(a+(b-a)t)dt=(b-a)\int_0^1 f(a+(b-a)x)dx.$$

19.证明$\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^{a^2}xf(x)dx.$

证令$x^2=t$,则$x=0$时,$t=0,x=a$时,$t=a^2$故

$$\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^a x^2f(x^2)dx^2=\frac{1}{2}\int_0^{a^2}tf(t)dt=\frac{1}{2}\int_0^{a^2}xf(x)dt.$$

20.证明$\int_0^1 x^m(1-x)^ndx=\int_0^1 x^n(1-x)^mdx.$

证令$x=1-t$, 则$x=0$时,$x=1$时,$t=0.dx=-dt$,故

$$\int_0^1 x^m(1-x)^n dx = -\int_1^0 (1-t)^m t^n dt = \int_0^1 (1-t)^m t^n dt = \int_0^1 x^n(1-x)^m dx.$$

21.利用分部积分公式证明,若$f(x)$连续,则

$$\int_0^x \int_0^t f(x)dxdt = \int_0^x f(t)(x-t)dx.$$

证$\int_a^x \int_0^t f(x)dxdt = t\int_0^t f(x)dx\Big|_0^x - \int_0^x t\left(\int_0^t f(x)dx\right)' dt$

$= \int_0^x xf(x)dx - \int_0^x tf(t)dt = \int_0^x xf(t)dt - \int_0^x tf(t)dt$

$= \int_0^x f(t)(x-t)dt.$

22.利用换元积分法证明$\int_0^\pi xf(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx.$

证$x=\pi-t, x=0$ 时$, t=\pi, x=\pi$时$, dx=-dt,$故

$\int_0^\pi xf(\sin x)dx = -\int_\pi^0 (\pi-t)f(\sin(\pi-t))dt$

$= \int_0^\pi (\pi-t)f(\sin t)dt = \pi\int_0^\pi f(\sin t)dt - \int_0^\pi tf(\sin t)dt$

$= \pi\int_0^\pi f(\sin t)dt - \int_0^\pi xf(\sin x)dx.$

$2\int_0^\pi xf(\sin x)dx = \pi\int_0^\pi f(\sin t)dt,$

$\int_0^\pi xf(\sin x)dx = \dfrac{1}{2}\pi\int_0^\pi f(\sin t)dt$

$= \dfrac{1}{2}\pi\int_0^{\pi/2} f(\sin t)dt + \dfrac{1}{2}\pi\int_{\pi/2}^\pi f(\sin t)dt$

令$u=\pi-t$,则$t=\pi/2$时$, u=\pi/2, t=\pi$时$, u=0, du=-dt,$

$\int_{\pi/2}^\pi f(\sin t)dt = -\int_{\pi/2}^0 f(\sin(\pi-u))du = \int_0^{\pi/2} f(\sin u)du,$

$\int_0^\pi xf(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx.$

23.利用上题结果求$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx.$

解$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx=-\int_0^{\pi/2}\frac{d\cos x}{1+\cos^2 x}$

$=-\arctan\cos x\big|_0^{\pi/2}=\frac{\pi}{4}.$

24.设函数$f(x)$在$(-\infty,+\infty)$上连续,以T为周期,证明:

(1)函数$F(x)=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt$也以$T$为周期;

(2) $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

证(1)$F(x+T)=\frac{x+T}{T}\int_0^T f(x)dx-\int_0^{x+T} f(t)dt$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_x^{x+T} f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_0^T f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt=F(x).$

(2)$\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx$

$=-\frac{1}{x}\left(\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt\right)=-\frac{F(x)}{x}.$

$F(x)$在$(-\infty,+\infty)$上连续,以T为周期, 故有界,

$\lim\limits_{x\to+\infty}\left(\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx\right)=\lim\limits_{x\to+\infty}\frac{F(x)}{x}=0.$

于是$\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

25.设$f(x)$是以T为周期的连续函数, $f(x_0)\neq 0$,且$\int_0^T f(x)dx=0$,证明:

$f(x)$在区间(x_0,x_0+T)内至少有两个根.

证为明确起见,设$f(x_0)>0$.如果f在(x_0,x_0+T)没有根,则由连续函数的

中间值定理, f在(x_0,x_0+T)恒正,设其最小值为m.则$m>0$,

$\int_{x_0}^{x_0+T} f(x)dx\geq\int_{x_0}^{x_0+T} mdx=mT>0$.由周期性和假设$\int_{x_0}^{x_0+T} f(x)dx=\int_0^T f(x)dx=0$,

矛盾.故f在(x_0,x_0+T)至少有一个根x_1.若f在(x_0,x_0+T)再无其它根,由于

$f(x_0+T)=f(x_0)>0$, f在(x_0,x_1)和(x_1,x_0+T)恒正,

$\int_{x_0}^{x_0+T} f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_0+T} f(x)dx>0$,矛盾.故$f$在$(x_0,x_1)$或$(x_1,x_0+T)$至少

还有一个根根, 即$f(x)$在区间(x_0,x_0+T)内至少有两个根.

26.求定积分

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx$$
其中m为正整数.

解被积函数以2π为周期,故$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=m\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx.$

$$\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=\int_0^{2\pi}\frac{dx}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}dx$$

$$=\int_0^{2\pi}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx=4\int_0^{\pi/2}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx(\sin^2 2x\text{周期为}\frac{\pi}{2})$$

$$=8\int_0^{\pi/2}\frac{dx}{2-\sin^2 2x}dx=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\csc^2 2x-1}$$

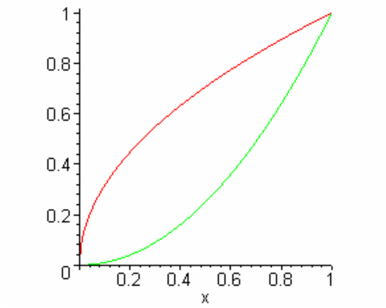
$$=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\cot^2 2x+1}=4\int_{-\infty}^{+\infty}\frac{du}{1+2u^2}=\frac{4}{\sqrt{2}}\arctan\sqrt{2}u\,\Big|_{-\infty}^{+\infty}=2\sqrt{2}\pi.$$

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=2m\sqrt{2}\pi.$$

习题 3.

求下列曲线所围成的的图形的面积：

1.$y=x^2$与$x=y^2$.

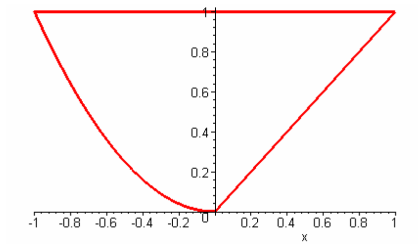


解求交点：$\begin{cases} y=x^2 \\ x=y^2 \end{cases}, x=x^4,$

$x(1-x)(1+x+x^2)=0, x=0, x=1.$

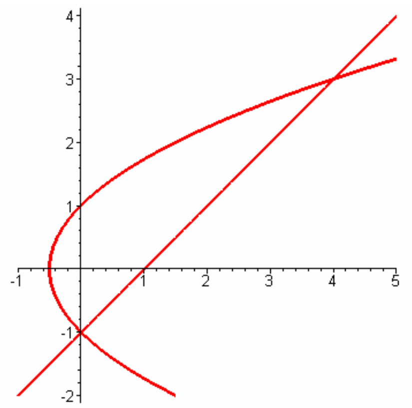
$S=\int_0^1(\sqrt{x}-x^2)dx=\left(\frac{2}{3}x^{3/2}-\frac{1}{3}x^2\right)\Big|_0^1=\frac{1}{3}.$

2.$y=x, y=1$与$y=\frac{x^2}{4}$.



解$S=\int_0^1(y+2\sqrt{y})dy=\left(\frac{y^2}{2}+y^{\frac{3}{2}}\right)\Big|_0^1=\frac{3}{2}.$

3.$y^2=2x+1$与$x-y=1$.



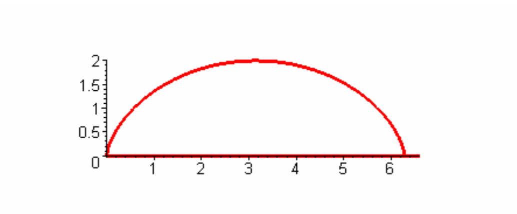
解$\begin{cases} y^2=2x+1 \\ x-y=1 \end{cases} (x-1)^2=2x+1,$

$x^2-4x=0, x=0, y=-1; x=4, y=3.$

$S=\int_{-1}^3\left(y+1-\frac{1}{2}(y^2-1)\right)dx$

$=\left(\frac{3}{2}y+\frac{y^2}{2}-\frac{1}{6}y^3\right)\Big|_{-1}^3=\frac{16}{3}.$

4.$y=0$与$\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $0\le t\le 2\pi$(a>0)



$S=\int_0^{2\pi}a(1-\cos t)da(t-\sin t)$

$=a^2\int_0^{2\pi}(1-\cos t)^2dt$

$=4a^2\int_0^{2\pi}\sin^4\frac{t}{2}dt=8a^2\int_0^{\pi}\sin^4 udu$

$=16a^2\int_0^{\pi/2}\sin^4 udu=16a^2\frac{3}{4g2}g\frac{\pi}{2}$

$=3\pi a^2.$

5.$y=x^2-4$与$y=-x^2-2x.$

解$\begin{cases} y=x^2-4 \\ y=-x^2-2x \end{cases} x^2-4=-x^2-2x,$

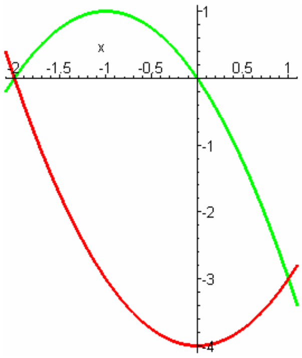
$2x^2+2x-4=0,(2x-2)(x+2)=0,$

$x_1=-2,x_2=1.$

$S=\int_{-2}^{1}(-x^2-2x-x^2+4)dx$

$=\int_{-2}^{1}(-2x^2-2x+4)dx$

$=\left(-\frac{2}{3}x^3-x^2+4x\right)\Big|_{-2}^{1}=9.$



$6.x^2+y^2=8$与$y=\frac{1}{2}x^2$(分上下两部分).

解$\begin{cases}x^2+y^2=8\\ y=\frac{1}{2}x^2\end{cases}$ $x^2+\frac{1}{4}x^4=8$

$x^4+4x^2-32=0,x^2=u$

$u^2+4u-32=0,(u+8)(u-4)=0$

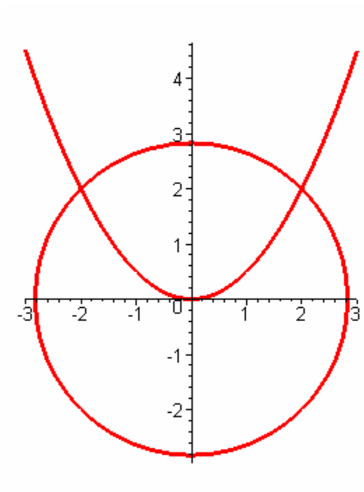
$u_2=-8$(舍)$u_2=4,x^2=4,x_1=-2,x_2=2$

$S_1=\int_{-2}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx$

$=2\int_{0}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx$

$=2\left(\frac{x}{2}\sqrt{8-x^2}+4\arcsin\frac{x}{2\sqrt{2}}\right)\Big|_{0}^{2}=2\pi+\frac{4}{3}$

$S_2=8\pi-\left(2\pi+\frac{4}{3}\right)=6\pi-\frac{4}{3}.$



$7.y=4-x^2$与$y=x+2.$

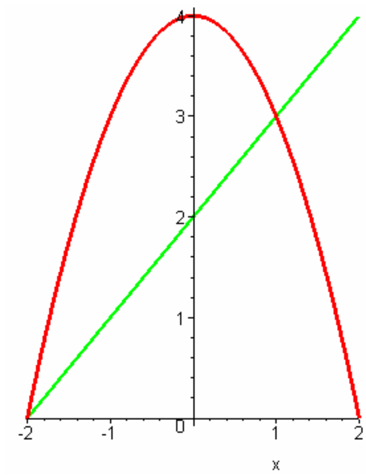
解$\begin{cases}y=4-x^2\\ y=x+2\end{cases}$ $4-x^2=x+2$

$x^2+x-2=0,(x+2)(x-1)=0,$

$x_1=-2,x_2=1.$

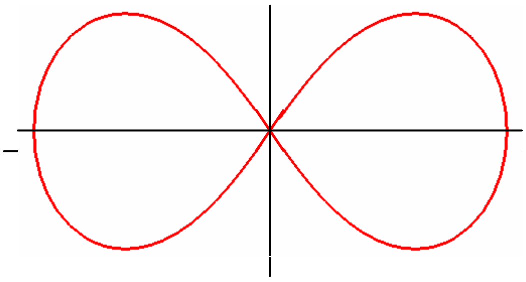
$S=\int_{-2}^{1}(4-x^2-x-2)dx=6-\int_{-2}^{1}(x^2+x)dx$

$=6-\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_{-2}^{1}=\frac{9}{2}.$

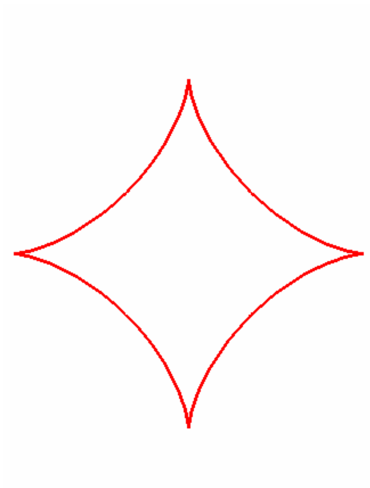


8.其求双纽线$r^2=a^2\cos 2\varphi(a>0)$所围图形的面积.

解$S=4\cdot\frac{1}{2}\int_0^{\pi/4}a^2\cos 2\varphi d\varphi=2a^2\cdot\frac{1}{2}\sin 2\varphi\big|_0^{\pi/4}=a^2.$



求下列曲线围成的平面图形绕轴旋转所成旋转体的体积：



9.$x^{2/3}+y^{2/3}=a^{2/3}(a>0).$

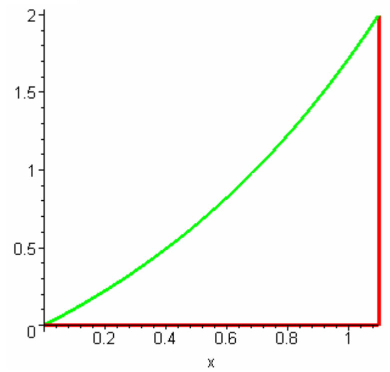
解$\begin{cases}x=a\cos^3 t\\ y=a\sin^3 t\end{cases},0\le t\le 2\pi.$

$V=2\pi\int_0^a y^2dx=2\pi\int_0^{\pi/2}a^2\sin^6 ta\cdot3\cos^2 t\sin tdt$

$=6\pi a^3\int_0^{\pi/2}\sin^7 t\cos^2 tdt$

$=6\pi a^3\int_0^{\pi/2}\sin^7 t(1-\sin^2 t)dt$

$=6\pi a^3\left(\frac{6\cdot4\cdot2}{7\cdot5\cdot3}\left(1-\frac{8}{9}\right)\right)=\frac{32}{105}\pi a^3.$

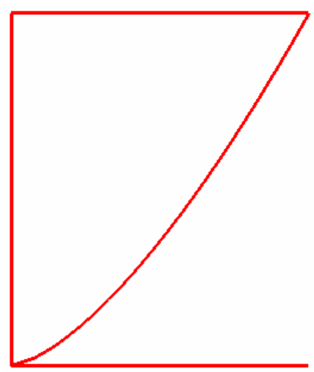


10.$y=e^x-1,x=\ln 3,y=e.$

$V=\pi\int_0^{\ln 3}(e^x-1)^2dx=\pi\int_0^{\ln 3}(e^{2x}-2e^x+1)dx$

$=\pi\left(\frac{1}{2}e^{2x}-2e^x+x\right)\Big|_0^{\ln 3}=\pi\ln 3.$

求下列平面曲线围成的平面图形绕轴旋转所成旋转体的体积：



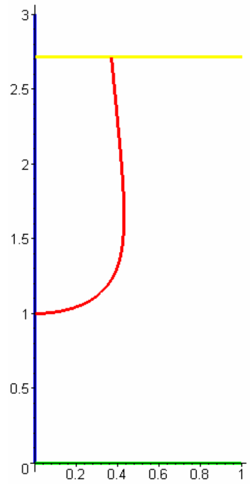
11.$ay^2=x^3,x=0$及$y=b(a>0,b>0).$

解$x=a^{1/3}y^{2/3},$

$V=\pi\int_0^b(a^{1/3}y^{2/3})^2dy$

$=\pi a^{2/3}\cdot\frac{3}{7}y^{7/3}\big|_0^b=\frac{3}{7}\pi a^{2/3}b^{7/3}.$

12. $x=\dfrac{\sqrt{8\ln y}}{y}, x=0, y=e.$



解 $V=\pi\int_1^e \dfrac{8\ln y}{y^2}dy$

$=8\pi\left[-\int_1^e \ln y dy^{-1}\right]$

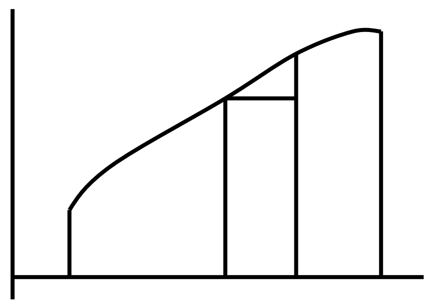
$=8\pi\left[-y^{-1}\ln y|_1^e+\int_1^e y^{-1}d\ln y\right]$

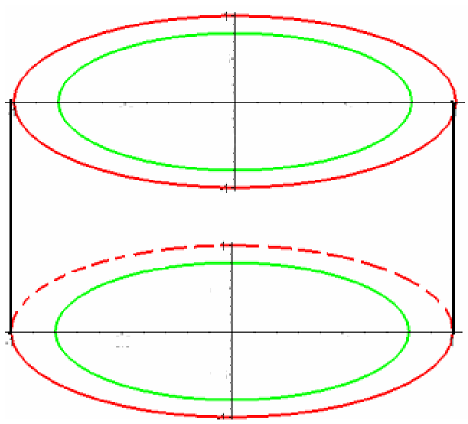
$=8\pi\left[-\dfrac{1}{e}+\int_1^e y^{-2}dy\right]$

$=8\pi\left[-\dfrac{1}{e}-y^{-1}|_1^e\right]=8\pi\left[1-\dfrac{2}{e}\right].$

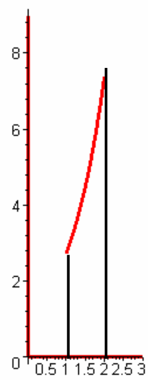
13. 设$y=f(x)$在区间$[a,b](a>0)$上连续且不取负值，试用微元法推导：由曲线$y=f(x)$，直线$x=a$，$x=b$及轴围成的平面图形绕y轴旋转所成立体的体积为$V=2\pi\int_a^b xf(x)dx.$

解 厚度dx的圆筒的体积$dV=2\pi xf(x)dx, V=2\pi\int_a^b xf(x)dx.$





14. 求曲线$y=e^x, x=1, x=2$及x轴所围成的平面图形绕y轴旋转所成的立体的体积.



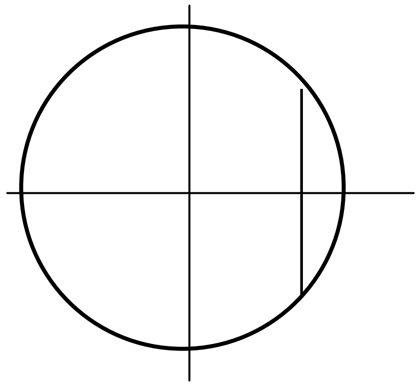
解 $V=2\pi\int_1^2 xe^x dx=2\pi\left[\int_1^2 xde^x\right]$

$=2\pi\left[xe^x|_1^2-\int_1^2 e^x dx\right]=2\pi\left[2e^2-e-e^x|_1^2\right]$

$=2\pi\left[2e^2-e-(e^2-e)\right]=2\pi e^2.$

15.证明:半径为a高为h的球缺的体积为

$$V=\pi h^2\left(a-\frac{h}{3}\right).$$

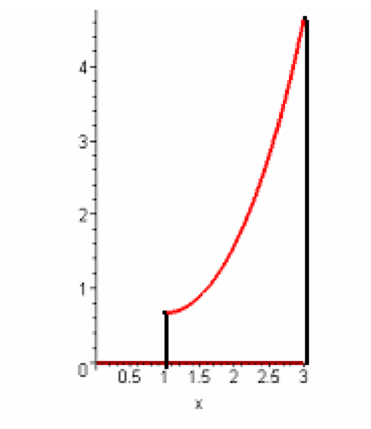


证$y=f(x)=\sqrt{a^2-x^2},a-h\le x\le a.$

$$V=\pi\int_{a-h}^{a}(a^2-x^2)dx=\pi\left[a^2h-\frac{1}{3}x^3\Big|_{a-h}^{a}\right]$$

$$=\pi\left[a^2h-\frac{1}{3}(a^3-(a-h)^3)\right]=\pi h^2\left(a-\frac{h}{3}\right)$$

16.求曲线$y=\dfrac{x^3}{6}+\dfrac{1}{2x}$在$x=1$到$x=3$之间的弧长.

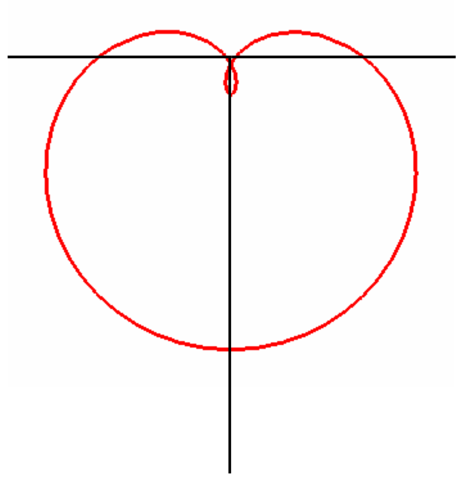


解$y'=\dfrac{x^2}{2}-\dfrac{1}{2x^2}.$

$$s=\int_1^3\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}dx$$

$$=\int_1^3\frac{x^4+1}{2x^2}dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^3=\frac{14}{3}.$$

17.求曲线$r=a\sin^3\dfrac{\theta}{3}$的全长.

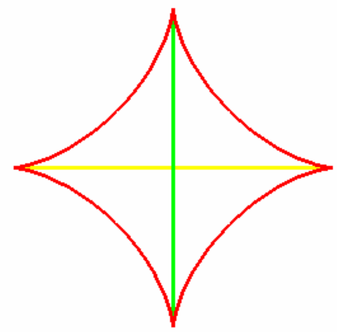


解$r'=a\sin^2\dfrac{\theta}{3}\cos\dfrac{\theta}{3},$

$$s=a\int_0^{3\pi}\sqrt{\sin^6\frac{\theta}{3}+\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}d\theta$$

$$=a\int_0^{3\pi}\sin^2\frac{\theta}{3}d\theta$$

$$=6a\int_0^{\pi/2}\sin^2\theta d\theta=6a\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{2}\pi a.$$

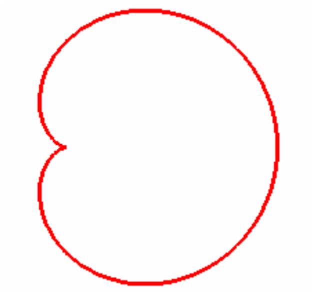


18.求向星形线$x=a\cos^3t,y=a\sin^3t$的弧长.

解$x'=3a\cos^2t(-\sin t),y'=3a\sin^2t\cos t$

$$s=4\cdot 3a\int_0^{\pi/2}\sin t\cos tdx=12a\cdot\frac{1}{2}\sin^2t\Big|_0^{\pi/2}=6a.$$

19.求心脏线$r=a(1+\cos\theta)$的全长.



解$r'=a(-\sin\theta)$

$$s=2a\int_0^{\pi}\sqrt{(1+\cos\theta)^2+\sin^2\theta}d\theta$$

$$=4a\int_0^{\pi}\cos\frac{\theta}{2}dx=8a\sin\frac{\theta}{2}\Big|_0^{\pi}=8a.$$

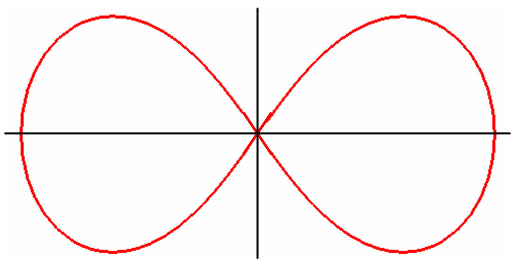
20.试证双纽线$r^2=2a^2\cos 2\theta(a>0)$的全长$L$可表为$L=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}$.20

证$2rr'=-4a^2\sin 2\theta, r'=-2a^2\sin 2\theta/r$,

$$s=4\int_0^{\pi/4}\sqrt{2a^2\cos 2\theta+\frac{4a^4\sin^2 2\theta}{2a^2\cos 2\theta}}d\theta$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{1}{\sqrt{\cos 2\theta}}d\theta$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos^2\theta-\sin^2\theta}}$$



$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{(\cos^2\theta-\sin^2\theta)(\cos^2\theta-\sin^2\theta)}}$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos^4\theta-\sin^4\theta}}$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\tan\theta}{\sqrt{1-\tan^4\theta}}(\tan\theta=x)$$

$$=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}.$$

21.求抛物线$y=1+\frac{x^2}{4}(0\le x\le 2)$绕$x$旋转所得的旋转体的侧面积.

解 $y'=\frac{x}{2}$.

$$S=2\pi\int_0^2\left(1+\frac{x^2}{4}\right)\sqrt{1+\left(\frac{x}{2}\right)^2}dx$$

$$=\frac{1}{4}\pi\int_0^2\sqrt{4+x^2}^3dx=4\pi\int_0^{\pi/4}\frac{dx}{\cos^5 x}$$

$$I_n=\int\sec^n xdx=\int\sec^{n-2}xd\tan x$$

$$=\sec^{n-2}x\tan x-\int(n-2)\sec^{n-2}x\tan^2 xdx$$

$$=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2},$$

$$I_n=\frac{1}{n-1}\sec^{n-2}x\tan x+\frac{n-2}{n-1}I_{n-2}.$$

$$I_5=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}I_3=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}\left(\frac{1}{2}\sec x\tan x+\frac{1}{2}I_1\right)$$

$$=\frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\ln(\tan x+\sec x)+C.$$

$$S=4\pi\left(\frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\ln(\tan x+\sec x)\right)\Big|_0^{\pi/4}$$

$$=\frac{\pi}{2}[7\sqrt{2}+3\ln(1+\sqrt{2})].$$

22.求$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(0<b\le a)$分别绕长,短轴旋转而成的椭球面的面积.

解$\begin{cases}x=a\cos t\\ y=b\sin t\end{cases},0\le t\le 2\pi,x'=-a\sin t,y'=b\cos t.$

$$S_a=2g2\pi b\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\sin t dt=$$

$$=-4\pi b\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\cos^2 t}d\cos t=$$

$$=4\pi b\int_0^1\sqrt{a^2-(a^2-b^2)u^2}du$$

$$=4\pi b\sqrt{a^2-b^2}\int_0^1\sqrt{\varepsilon^{-2}-u^2}du$$

$$=4\pi ab\frac{\sqrt{a^2-b^2}}{a}\left[\frac{u}{2}\sqrt{\varepsilon^{-2}-u^2}+\frac{\varepsilon^{-2}}{2}\arcsin\varepsilon u\right]\Bigg|_0^1$$

$$=2\pi ab\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right).$$

$$S_b=2g2\pi a\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\cos t dt$$

$$=4\pi a\int_0^{\pi/2}\sqrt{b^2+(a^2-b^2)\sin^2 t}d\sin t$$

$$=4\pi a\int_0^1\sqrt{b^2+(a^2-b^2)u^2}du$$

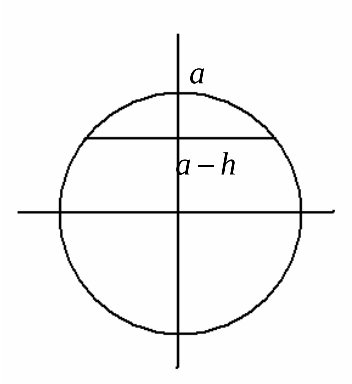
$$=4\pi a\sqrt{a^2-b^2}\int_0^1\sqrt{\frac{b^2}{a^2-b^2}+u^2}du$$

$$=4\pi a\sqrt{a^2-b^2}\left[\frac{u}{2}\sqrt{\frac{b^2}{a^2-b^2}+u^2}+\frac{b^2}{2(a^2-b^2)}\ln\left(u+\sqrt{\frac{b^2}{a^2-b^2}+u^2}\right)\right]\Bigg|_0^1$$

$$=2\pi a^2+\frac{2\pi b^2}{\varepsilon}\ln\left[\frac{a}{b}(1+\varepsilon)\right].$$

23.计算圆弧$x^2+y^2=a^2(a-h\le y\le a,0<h<a)$绕$y$轴
旋转所得球冠的面积.

解$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $\arcsin\dfrac{a-h}{a} \le t \le \dfrac{\pi}{2}$.



$$S = 2\pi\int_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}} x\sqrt{x'^2 + y'^2}\,dt$$

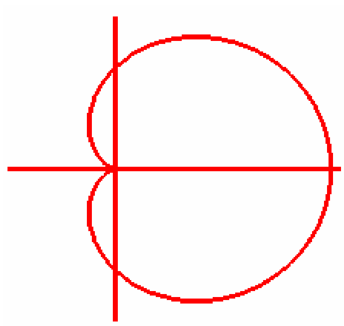
$$= 2\pi a^2\int_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}} \cos t\,dt$$

$$= \pi a^2\left[\sin t\right]\Big|_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}}$$

$$= 2\pi a^2\left[1 - \frac{a-h}{a}\right] = 2\pi ah.$$

24.求心脏线$r = a(1+\cos\theta)$绕极轴旋转所成的旋转体的侧面积.

解$r' = -a\sin\theta$.



$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{3/2}\sin\theta\,d\theta$$

$$= -2\pi a^2\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{3/2}d\cos\theta$$

$$= 2\pi a^2\sqrt{2}\int_{-1}^{1}(1+x)^{3/2}dx$$

$$= 2\pi a^2\sqrt{2}\,\frac{2}{5}(1+x)^{5/2}\Big|_{-1}^{1}$$

$$= \frac{32}{5}\pi a^2.$$

25.有一细棒长10m已知距左端点x处的线密度是$\rho(x) = (7+0.2x)$kg/m求这细棒的质量.

解$m = \int_0^{10}(7+0.2x)dx = \left[7x + 0.1x^2\right]\Big|_0^{10} = 80(\text{kg})$.

26.求半径为a的均匀半圆周的重心坐标.

解由对称性, $x_0 = 0$. $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}, 0 \le t \le \pi$

$$y_0 = \frac{\int_0^{\pi} a\sin t\,a\,dt}{\pi a} = \frac{a}{\pi}[-\cos t]\big|_0^{\pi} = \frac{2a}{\pi}.$$

重心坐标$(0, \dfrac{2a}{\pi})$.

27.有一均匀细杆,长为l.质量为M.计算细杆绕距离一端$l/5$处的转动惯量.

解$\rho = M/l. J = \int_0^{l/5}\frac{M}{l}x^2dx + \int_0^{4l/5}\frac{M}{l}x^2dx$

$$= \frac{M}{l}\frac{x^3}{3}\Big|_0^{l/5} + \frac{M}{l}\frac{x^3}{3}\Big|_0^{4l/5} = \frac{13}{75}Ml^2.$$

28.设有一均匀圆盘,半径为a,质量为M,求它对于通过其圆心且与盘垂直的轴之转动惯量.

解$\rho=\dfrac{M}{\pi a^2}.dm=\dfrac{M}{\pi a^2}2\pi x dx=\dfrac{2Mxdx}{a^2}.$

$$J=\int_0^a x^2\frac{2Mxdx}{a^2}=\frac{2M}{a^2}\frac{x^4}{4}\bigg|_0^a=\frac{1}{2}Ma^2.$$

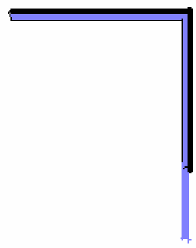
29.有一均匀的圆锥形陀螺,质量为M,底半径为a,高为h,试求此陀螺关于其对称轴的转动惯量.

解$y=\dfrac{a}{h}x,\rho=\dfrac{M}{\frac{1}{3}\pi a^2 h}=\dfrac{3M}{\pi a^2 h},dm=\rho\pi\left(\dfrac{a}{h}x\right)^2dx=\dfrac{3M}{h^3}x^2dx$

$$dJ=\frac{1}{2}dm\left(\frac{a}{h}x\right)^2=\frac{1}{2}\frac{3a^2M}{h^5}x^4dx$$

$$J=\int_0^h\frac{1}{2}\frac{3a^2M}{h^5}x^4dx=\frac{1}{2}\frac{3a^2M}{h^5}\frac{x^5}{5}\bigg|_0^h=\frac{3}{10}Ma^2.$$

30.楼顶上有一绳索沿墙壁下垂,该绳索的密度为2kg/m.若绳索下垂部分长为5m,求将下垂部分全部拉到楼顶所需做的功.



解$dW=2\cdot 9.8xdx.$

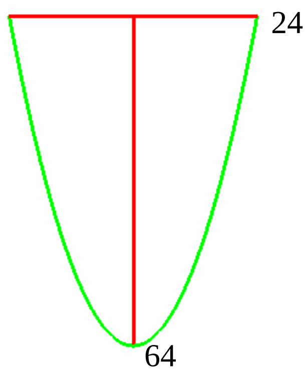
$$W=\int_0^5 2\cdot 9.8xdx=9.8x^2\Big|_0^5=25\cdot 9.8(J).$$

31.设$y=f(x)$在$[a,b]$上连续,非负,将由$y=f(x)x=a,x=b$及x轴围成的曲边梯形垂直放置于水中,使y轴与水平面相齐,求水对此曲边梯形的压力.

解$dS=f(x)dx,dF=\rho pdS=g\rho xf(x)dx,$

$$F=g\rho\int_a^b xf(x)dx.$$

32.一 水闸门的边界线为一抛物线,沿水平面的宽度为48m,最低处在水面下64m,求水对闸门的的压力.



解$y=64-ax^2,0=64-a\cdot 24^2,a=\dfrac{1}{9},x=\pm 3\sqrt{(64-y)}.$

$$F=6g\rho\int_0^{64}y\sqrt{64-y}dy.\sqrt{64-y}=u,y=64-u^2,$$

$y=0$时$u=8,y=64$时$,u=0.$

$$F=6g\rho\int_0^8(64-u^2)u(2u)dy$$

$$=12g\rho\left[64\cdot\frac{u^3}{3}-\frac{u^5}{5}\right]\bigg|_0^8=52428.8g\rho.$$

习题 3.6

1.利用定积分近似计算π的值：

(1)证明公式$\frac{\pi}{4}=\int_0^1 \frac{dx}{1+x^2}$;

(2)令$f(x)=\frac{1}{1+x^2}$,给出$|f'(x)|$在(0,1)的上界;

(3)在使用矩形法近似计算上述积分时,欲使公式误差小于5×10^{-5},应取矩形法中分点个数$n>$?

(4)用电脑计算π到小数点后4位.

解 (1)$\int_0^1 \frac{dx}{1+x^2}=atc\tan x|_0^1=\frac{\pi}{4}$.

(2)$f'(x)=-\frac{2x}{(1+x^2)^2}, |f'(x)|\le 2$.

(3)$|R_n|\le \frac{1}{2n} g2=\frac{1}{n}<5\times10^{-5}, n>\frac{1}{5\times10^{-5}}=2\times10^4$.

(4)n:=2.0*10^4:assume(m,integer):J:=4*sum(1/(1.0+(m/n)^2),m=1..n)/n;

J := 3.141542654 + 0. I

2.自河的一岸开始沿河的横截面方向,每隔5m测量一次水深,一直测到河对岸,依次得到如下21个数据(单位:m)

0, 0. 9, 1. 2, 3. 5, 2. 8, 4. 6, 8. 8, 7. 5, 9. 6, 12. 1, 13. 8,

20. 1, 18. 2, 15. 6, 11. 9, 9. 2, 7. 6, 5. 3, 4. 5, 2. 7, 0.

假定河宽为100m. 试用simpson法计算河床的横截面面积.

解n:=10:d:=[0,0.9,1.2,3.5,2.8,4.6,8.8,7.5,9.6,12.1,13.8,20.1,18.2,15.6,11.9,9.2,7.6,5.3,4.5,2.7,0]:

S:=((100)/(6*n))*(d[1]+d[21]+2*sum(d[2*i+1],i=1..n –1)+4*sum(d[2*i],i=1..n));

(Maple程序)

S := 804.6666667

第三章总练习题

1.为什么用Newton-Leibniz公式于下列积分会得到不正确结果?

(1)$\int_{-1}^{1}\frac{d}{dx}\left(e^{\frac{1}{x}}\right)dx.\frac{d}{dx}\left(e^{\frac{1}{x}}\right)=-\left(e^{\frac{1}{x}}\right)\frac{1}{x^2}[-1,1]$无界,从而不可积.

(2)$\int_{0}^{2\pi}\frac{d\tan x}{2+\tan^2 x}dx.u=\tan x$在$(0,2\pi)$的一些点不可导.

2.证明奇连续函数的原函数为偶函数,而偶连续函数的原函数之一为奇函数.

证设奇连续函数f的原函数为F,现在证明F是偶函数.

$F'(x)=f(x).(F(-x)-F(x))'=-F'(-x)-F'(x)=-f(-x)-f(x)=0,$

$F(-x)-F(x)=C,C=F(-0)-F(0)=0.F(-x)-F(x)=0.$

设偶连续函数f的原函数为F,现在证明F是奇函数.

$F'(x)=f(x).(F(-x)+F(x))'=-F'(-x)+F'(x)=-f(-x)+f(x)=0,$

$F(-x)-F(x)=C.$设$F(0)=0$,则$C=F(-0)-F(0)=0.F(-x)+F(x)=0.$

3.$f(x)\,f(x)=\begin{cases}\sin x, x\ge 0,\\ x^3,\quad x<0,\end{cases}$求定积分$\int_a^b f(x)dx=?$其中$a<0,b>0.$

解$\int_a^b f(x)dx=\int_a^0 f(x)dx+\int_0^b f(x)dx=\int_a^0 x^3dx+\int_0^b\sin xdx$

$=\left.\frac{x^4}{4}\right|_0^a-\cos x\big|_0^b=1+\frac{a^4}{4}-\cos b.$

4.求微商$\frac{d}{dx}\int_0^1\sin(x+t)dx.$

解$\frac{d}{dx}\int_0^1\sin(x+t)dx=\frac{d}{dx}\int_x^{x+1}\sin(u)du=\sin(x+1)-\sin(x).$

5.试证明$\lim\limits_{h\to 0}\int_0^1 f(x+ht)dx=f(x)$,其中$f(x)$是实轴上的连续函数.

证$\lim\limits_{h\to 0}\frac{1}{h}\int_x^{x+h}f(x+ht)du=\left.\left(\int_0^u f(t)dt\right)'\right|_{u=x}=f(x).$

6.求极限$\lim\limits_{n\to\infty}\int_0^1(1-x^2)^n dx.$

解$\int_0^1(1-x^2)^n dx=\int_0^{\pi/2}\cos^{2n+1}tdt=I_{2n+1}=\frac{(2n)!!}{(2n+1)!!}.$

$\left(I_{2n+1}\right)^2<\frac{2(2n)!!}{(2n+1)!!}\text{g}\frac{(2n+1)!!}{(2n+2)!!}=\frac{1}{n+1},$

$0<I_{2n+1}<\frac{1}{\sqrt{n+1}}\to 0(n\to\infty),\lim\limits_{n\to\infty}\int_0^1(1-x^2)^n dx=0.$

7.$\int\frac{\sin x+\cos x}{2\sin x-3\cos x}dx.$

解令$\sin x+\cos x=A(2\sin x-3\cos x)+B(2\sin x-3\cos x)'$

$=A(2\sin x-3\cos x)+B(2\cos x+3\sin x)=(2A+3B)\sin x+(-3A+2B)\cos x,$

$$\begin{cases}2A+3B=1\\-3A+2B=1\end{cases}, A=-\frac{1}{13}, B=\frac{5}{13}.$$

$$\int\frac{\sin x+\cos x}{2\sin x-3\cos x}dx=$$

$$=\int\frac{A(2\sin x-3\cos x)+B(2\sin x-3\cos x)'}{2\sin x-3\cos x}dx$$

$$=Ax+B\ln|2\sin x-3\cos x|+C$$

$$=-\frac{1}{13}x+\frac{5}{13}\ln|2\sin x-3\cos x|+C.$$

8.通过适当的有理化或变量替换求下列积分:

(1)$\int\sqrt{e^x-2}dx.\sqrt{e^x-2}=u, x=\ln(2+u^2), dx=\dfrac{2udu}{2+u^2}.$

$$\int\sqrt{e^x-2}dx=2\int\frac{u^2du}{2+u^2}=2\left(u-2\int\frac{du}{2+u^2}\right)$$

$$=2\left(u-\sqrt{2}\arctan\frac{u}{\sqrt{2}}\right)+C=2\left(\sqrt{e^x-2}-\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}\right)+C.$$

(2)$\int\dfrac{xe^x}{\sqrt{e^x-2}}dx=\int\dfrac{x}{\sqrt{e^x-2}}d(e^x-2)=2\int xd\sqrt{e^x-2}$

$$=2x\sqrt{e^x-2}-2\int\sqrt{e^x-2}dx$$

$$=2x\sqrt{e^x-2}-4\left(\sqrt{e^x-2}-\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}\right)+C.$$

$$=2\sqrt{e^x-2}(x-2)+4\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C.$$

(3)$\int\sqrt{\dfrac{x}{1-x\sqrt{x}}}dx=\dfrac{2}{3}\int\dfrac{dx\sqrt{x}}{\sqrt{1-x\sqrt{x}}}=-\dfrac{2}{3}\times2\sqrt{1-x\sqrt{x}}+C$

$$=-\frac{4}{3}\sqrt{1-x\sqrt{x}}+C.$$

(4)$\int\dfrac{dx}{1+\sqrt{x}+\sqrt{1+x}}=\int\dfrac{(1+\sqrt{x}-\sqrt{1+x})dx}{(1+\sqrt{x}+\sqrt{1+x})(1+\sqrt{x}-\sqrt{1+x})}$

$$=\int\frac{(1+\sqrt{x}-\sqrt{1+x})dx}{2\sqrt{x}}=\frac{1}{2}\left(2\sqrt{x}+x-\sqrt{x(1+x)}+\ln(\sqrt{x}+\sqrt{1+x})\right)+C.$$

$$9.\int\frac{dx}{\sin^4 x+\cos^4 x}=\int\frac{\sec^2 xd\tan x}{1+\tan^4 x}=\int\frac{(1+u^2)du}{1+u^4}.$$

$$\frac{1+u^2}{1+u^4}=\frac{1+u^2}{(1+u^2+\sqrt{2}u)(1+u^2-\sqrt{2}u)}=\frac{1}{2}\left(\frac{1}{1+u^2+\sqrt{2}u}+\frac{1}{1+u^2-\sqrt{2}u}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(u+\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{1}{(u-\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right).$$

$$\int\frac{dx}{\sin^4 x+\cos^4 x}=\frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}u+1)+\arctan(\sqrt{2}u-1)\right)+C.$$

10.设函数$f(x)$在$(-\infty,+\infty)$上连续,以T为周期,令$g(x)=f(x)-\frac{1}{T}\int_0^T f(x)dx$,证明:

函数$h(x)=\int_0^x g(t)dt$也以T为周期.

证(此即习题3.4第24题)

11.设函数$f(x)$在区间$[a,b]$上连续,且$\int_a^b f(x)dx=0$.证明:在(a,b)内至少存在一点c,

使$f(c)=0$.

证若不然, $f(x)$在(a,b)没有 零点,由f的连续性和连续函数的中间值定理,

f在(a,b)不变号.不妨设$f(x)>0, x\in(a,b)$.取c,d满足,$a<c<d<b$,则f在$[c,d]$取最小值

$m>0$.于是

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^d f(x)dx+\int_d^b f(x)dx\ge m(d-c)>0.$$

矛盾.

12.设函数f在区间$[a,b]$上连续,且$\int_a^b f^2(x)dx=0$,证明: $f(x)\equiv 0, x\in[a,b]$.

证若不然, 存在$c\in[a,b]$, $f(c)\ne 0$. 由f在c的连续性, 存在区间 $[d,e]\subseteq[a,b]$,

$$|f(x)|^2>\frac{|f(c)|^2}{2}, x\in[d,e].$$

$$\int_a^b f^2(x)dx\ge\int_d^e f^2(x)dx>\frac{|f(c)|^2}{2}(d-e)>0.$$

矛盾.

13.设$f(x)$在$(-\infty,+\infty)$上可积,证明

(1)对于任意实数a,有$\int_0^a f(x)dx=\int_0^a f(a-x)dx$;

(2)$\int_0^\pi\frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi^2}{4}$;

(3)$\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx=\frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$.

证(1)$\int_0^a f(x)dx(x=a-t)=-\int_a^0 f(a-t)dt=\int_0^a f(a-t)dt=\int_0^a f(a-x)dx.$

$$(2)I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=-\int_0^{\pi}\frac{(x-\pi)\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx-I,$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin xdx}{1+\cos^2 x}=-\frac{\pi}{2}\int_0^{\pi}\frac{d\cos x}{1+\cos^2 x}=\int_0^{1}\frac{\pi du}{1+u^2}=\pi\arctan u\,|_0^1=\frac{\pi^2}{4}.$$

$$(3)I=\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{\sin^2(\pi/2-x)}{\cos(\pi/2-x)+\sin(\pi/2x)}dx$$

$$=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx, 2I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx+\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx$$

$$=\int_0^{\pi/2}\frac{dx}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{dx}{\sqrt{2}\sin(x+\pi/4)}=\frac{1}{\sqrt{2}}\ln|\csc(x+\pi/4)-\cot(x+\pi/4)|\,|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left[\ln\left(\frac{1}{\cos\frac{\pi}{4}}+1\right)\right]-\ln\left(\frac{1}{\sin\frac{\pi}{4}}-1\right)=\sqrt{2}\ln(\sqrt{2}+1),$$

$$I=\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1).$$

14.一质点作直线运动, 其加速度$a(t)=(2t-3)\text{m/s}^2$.若$t=0$时$x=0$且$v=-4\text{m/s}$,求
(1)质点改变动方向的时刻;
(2)头5秒钟内质点所走的总路程.

解$(1)x''(t)=2t-3, x'=t^2-3t+C_1, -4=C_1, x'=t^2-3t-4, x=\frac{t^3}{3}-\frac{3}{2}t^2-4t+C_2,$

$0=C_2.x(t)=\frac{t^3}{3}-\frac{3}{2}t^2-4t.x'=t^2-3t-4=(t-4)(t+1)=0, t_0=4.$

$$s=x(5)-x(4)+|x(4)|=\left(\frac{t^3}{3}-\frac{3}{2}t^2-4t\right)\Bigg|_{t=5}-2\left(\frac{t^3}{3}-\frac{3}{2}t^2-4t\right)\Bigg|_{t=4}=\frac{43}{2}\text{m}.$$

15.一运动员跑完100m,共用了10.2s,在跑头25m时以等加速度进行,然后保持等速运动跑完了剩余路程. 求跑头25m时的加速度.

解$v(t)=\begin{cases}at, & 0\le t\le t_0;\\ at_0, & t_0\le t\le 10.2.\end{cases}$

$$s(t)=\begin{cases}\frac{at^2}{2}, & 0\le t\le t_0;\\ at_0t+C & t_0\le t\le 10.2.\end{cases}$$

$$\begin{cases}at_0^2/2=at_0^2+C\\ at_0^2/2=25\\ 100=10.2at_0+C_2\end{cases}\qquad a\approx 3\text{m/s}^2.$$

16.(1)利用积分的几何意义证明:

$$\frac{1}{n+1}<\ln\frac{n+1}{n}<\frac{1}{n}, n=1,2,\cdots$$

(2)令$x_n=1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln n,$

$y_n=1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}-\ln n,$

证明序列x_n单调上升,而序列y_n单调下降.

(3)证明极限$\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}-\ln n\right)$存在(此极限称为Euler常数).

证 (1)$\frac{1}{n+1}=\int_n^{n+1}\frac{dx}{n+1}<\int_n^{n+1}\frac{dx}{x}=\ln x\big|_n^{n+1}$

$=\ln(n+1)-\ln n=\ln\frac{n+1}{n}<\int_n^{n+1}\frac{dx}{n}=\frac{1}{n}.$

(2)$x_{n+1}-x_n=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln n\right)$

$=\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)>0$(由(1)).

$y_{n+1}-y_n=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$

$=\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)<0$(由(1)).

(3)$y_n>x_n>x_2=1-\ln 2>0(n>2)$.$y_n$单调下降有下界,故有极限$\lim\limits_{n\to\infty}y_n$.

17.证明:当$x>0$时,

$\int_x^1\frac{1}{1+t^2}dt=\int_1^{1/x}\frac{1}{1+t^2}dt.$

证$\int_x^1\frac{1}{1+t^2}dt(x=1/u)=\int_1^{1/x}\frac{1}{1+1/u^2}\times\frac{1}{u^2}dx=\int_1^{1/x}\frac{1}{1+t^2}dt.$

18.设$f(x)$在$(-\infty,+\infty)$上连续(书上为可积,欠妥),且对一切实数x,均有

$f(2-x)=-f(x)$.求实数$a\neq 2$,使$\int_a^2 f(x)dx=0$.

解(条件$f(2-x)=-f(x)$相当f关于$x=1$为奇函数$f(1+1-x)=-f(1+x-1)$)

$\int_0^2 f(x)dx=\int_0^2 f(2-u)du=-\int_0^2 f(u)du,\int_0^2 f(x)dx=0$.取$a=0$即可.

19.利用定积分的性质,证明不等式$\ln(1+x)\le\arctan x, 0\le x\le 1$.

证$\frac{1}{1+t}\le\frac{1}{1+t^2}, t\in[0,1]$,在$[0,x]$上积分得$\int_0^x\frac{dt}{1+t}\le\int_0^x\frac{dt}{1+t^2},$

$\ln(1+x)\le\arctan x, 0\le x\le 1.$

20.(1)设$f(x)$在$[0,a]$上可积,证明$\int_0^a\frac{f(x)dx}{f(x)+f(a-x)}dx=\frac{a}{2}$;

(2)利用(1)中的公式求下列积分的值:

$\int_0^2 \frac{x^2}{x^2-2x+2}dx; \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx$

证(1)$I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx=\int_0^a \frac{f(a-u)}{f(u)+f(a-u)}du$

$2I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-u)}{f(u)+f(a-u)}du$

$=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx=\int_0^a 1dx=a, I=\frac{a}{2}.$

解 (2)$\int_0^2 \frac{x^2}{x^2-2x+2}dx=2\int_0^2 \frac{x^2}{x^2+(2-x)^2}dx=2\times\frac{2}{2}=2.$

$\int_0^{\pi/2} \frac{\sin x}{\sin x+\sin(\pi/2-x)}dx=\frac{\pi/2}{2}=\frac{\pi}{4}.$

21.设$f(x)=\int_{\sin x}^{\tan x}(1+xt^2)dtdx$求$\frac{df(x)}{dx}$.

解$f(x)=\int_{\sin x}^{\tan x}(1+xt^2)dt=\tan x-\sin x+x\int_{\sin x}^{\tan x}t^2dt,$

$\frac{df(x)}{dx}=\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\int_{\sin x}^{\tan x}t^2dt$

$=\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\left.\frac{t^3}{3}\right|_{\sin x}^{\tan x}$

$==\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\frac{1}{3}\left(\tan^3 x-\sin^3 x\right)$

$=\sec^2 x(1+x\tan^2 x)-\cos x(1+x\sin^2 x)+\frac{1}{3}\left(\tan^3 x-\sin^3 x\right).$

22.求定积分$I=\int_0^{\pi/2}\cos^2 3\theta d\theta$的值.

解$I=\int_0^{\pi/2}\cos^2 3\theta d\theta=\frac{1}{2}\int_0^{\pi/2}(1+\cos 6\theta)d\theta=\frac{\pi}{4}+\frac{1}{12}\sin 6\theta\Big|_0^{\pi/2}=\frac{\pi}{4}.$

23.求定积分$I=\int_0^{2\pi}|\sin x-\cos x|dx$的值.

解$\mathrm{I}=2\int_0^{\pi}|\sin x-\cos x|dx$

$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_{\pi/2}^{\pi}|\sin x-\cos x|dx\right)$

$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_0^{\pi/2}|\sin(\frac{\pi}{2}+t)-\cos(\frac{\pi}{2}+t)|dx\right)$

$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_0^{\pi/2}|\cos t+\sin t|dx\right)$

$=2\left(\int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx+\int_0^{\pi/2}(\cos t+\sin t)dx\right)$

$=2\left((\sin x+\cos x)\Big|_0^{\pi/4}+(-\cos x-\sin x)\Big|_{\pi/4}^{\pi/2}+(\sin x-\cos x)\Big|_0^{\pi/2}\right)=4\sqrt{2}.$

24.设$0<x_0<x_1$,求定积分$I=\int_{x_0}^{x_1}\sqrt{(x-x_0)(x_1-x)}dx$的值.

解$I=\int_{x_0}^{x_1}\sqrt{(x-x_0)(x_1-x)}dx$

$=\int_{x_0}^{x_1}\sqrt{-x^2+(x_1+x_0)x-x_0x_1}dx$

$=\int_{x_0}^{x_1}\sqrt{-\left(x-\frac{x_1+x_0}{2}\right)^2+\frac{(x_1+x_0)^2}{4}-x_0x_1}dx$

$=\int_{x_0}^{x_1}\sqrt{-\left(x-\frac{x_1+x_0}{2}\right)^2+\frac{(x_1-x_0)^2}{4}}dx\left(u=x-\frac{x_1+x_0}{2}\right)$

$=\int_{-(x_1-x_0)/2}^{(x_1-x_0)/2}\sqrt{-(u)^2+\frac{(x_1-x_0)^2}{4}}du$

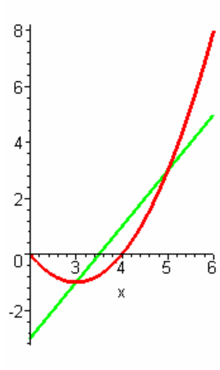
$=2\int_0^a\sqrt{a^2-u^2}dx(a=\frac{x_1-x_0}{2})$

$=\left[u\sqrt{a^2-u^2}+a^2\arcsin\frac{u}{a}\right]\Big|_0^a$

$=\frac{\pi a^2}{2}=\frac{\pi(x_1-x_0)^2}{8}$.

25.求下列曲线所围图形的面积:

(1)$y=x^2-6x+8$与$y=2x-7$.



解$\begin{cases}y=x^2-6x+8\\y=2x-7\end{cases}2x-7=x^2-6x+8,$

$x^2-8x+15=0,(x-3)(x-5)=0.$

$x_1=3,x_2=5.$

$S=\int_3^5(2x-7-(x^2-6x+8))dx=\int_3^5(-x^2+8x-15)dx$

$=\left(-\frac{x^3}{3}+4x^2-15x\right)\Big|_3^5=\frac{4}{3}.$

(2)$y=x^4+x^3+16x-4$与$y=x^4+6x^2+8x-4$.

解$\begin{cases}y=x^4+x^3+16x-4\\y=x^4+6x^2+8x-4\end{cases}x^3+16x-4=6x^2+8x-4,x^3-6x^2+8x=0,$

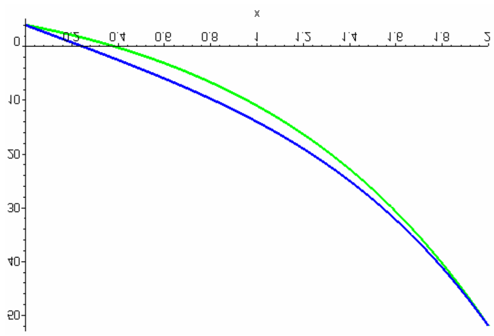
$x=0,x^2-6x+8=0,(x-2)(x-4)=0,x=2,4.$

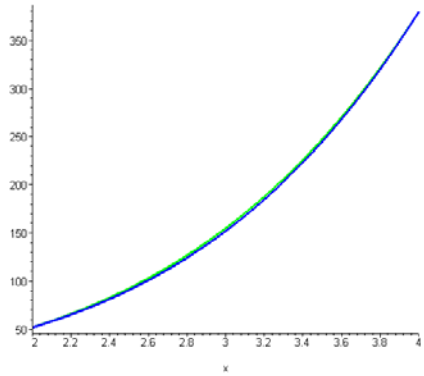
$S=\int_0^2\{(x^4+x^3+16x-4)-(x^4+6x^2+8x-4)]dx$

$+\int_2^4[(x^4+6x^2+8x-4)-(x^4+x^3+16x-4)]dx$

$$=\int_0^2 (x^3-6x^2+8x)dx+\int_2^4(-x^3+6x^2-8x)dx$$

$$=\left[\frac{x^4}{4}-2x^3+4x^2\right]\Bigg|_0^2+\left[-\frac{x^4}{4}+2x^3-4x^2\right]\Bigg|_2^4=8.$$





$(3)y^2=x-1$与$y=x-3.$

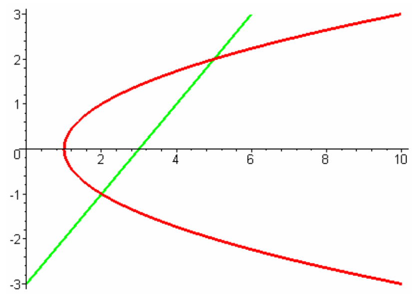
解$(x-3)^2=x-1, x^2-7x+10=0,$

$(x-2)(x-5)=0,$

$x=2,5.y=-1,2.$

$$S=\int_{-1}^2[(y+3)-(1+y^2)]dx$$

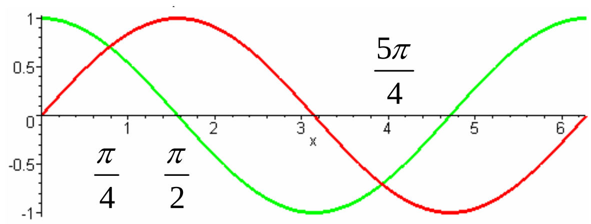
$$=\left(-\frac{y^3}{3}+\frac{y^2}{2}+2y\right)\Bigg|_{-1}^2=\frac{9}{2}.$$



$(4)y=\sin x, y=\cos x$与$x=\pi/2.$

解$S=\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx=(-\cos x-\sin x)\mid_{\pi/4}^{\pi/2}=\sqrt{2}-1;$

$S=\int_{\pi/2}^{\pi 5/4}(\sin x-\cos x)dx=(-\cos x-\sin x)\mid_{\pi/2}^{5\pi/4}=\sqrt{2}+1.$



26.设区域σ由曲线$y=\cos x, y=1$及$x=\pi/2$所围成,将σ绕x轴旋转一周,得一旋转体V.试用两种不同的积分表示体积V,并且求V的值.

解$V=\pi\int_0^{\pi/2}(1-\cos^2 x)dx=2\pi\int_0^1 y\left(\frac{\pi}{2}-\arccos y\right)dy=2\pi\int_0^1 y\arcsin ydy=$

$$V=\frac{\pi}{2}\int_0^{\pi/2}(1-\cos 2x)dx=\frac{\pi^2}{4}.$$

$$V=2\pi\int_0^1 y\arcsin ydy==\pi\int_0^1 \arcsin ydy^2$$

$$=\pi\arcsin y(y^2)\Big|_0^1-\pi\int_0^1 y^2\times\frac{1}{\sqrt{1-y^2}}dx$$

$$=\frac{\pi^2}{2}-\pi\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\arcsin y\right]\Bigg|_0^1=\frac{\pi^2}{2}-\frac{\pi^2}{4}=\frac{\pi^2}{4}.$$

27.求下列定积分的值:

(1)$\int_{\sqrt{2}}^{2}\frac{du}{u\sqrt{u^2-1}}=\int_{\sqrt{2}}^{2}\frac{du}{u^2\sqrt{1-1/u^2}}=\int_{1/2}^{1/\sqrt{2}}\frac{dx}{\sqrt{1-x^2}}=\arcsin x\big|_{1/2}^{1/\sqrt{2}}=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}.$

(2)$\int_{-200}^{200}(91x^{21}-80x^{33}+5580x^{97}+1)dx=400.$

28.设$f(x)$在$[0,7]$上可积,且一直已知$\int_0^2 f(x)dx=5,\int_2^5 f(x)dx=6,\int_0^7 f(x)dx=3.$

(1)求$\int_0^5 f(x)dx$的值;

(2)求$\int_5^7 f(x)dx$的值.

(3)证明:在$(5,7)$内至少存在一点,使$f(x)<0.$

解(1)$\int_0^5 f(x)dx=\int_0^2 f(x)dx+\int_2^5 f(x)dx=5+6=11.$

(2)$\int_5^7 f(x)dx=\int_0^7 f(x)dx-\int_0^5 f(x)dx=3-11=-8.$

证(3)若不然, $f(x)\ge 0, x\in(5,7),$

$\int_5^7 f(x)dx\ge 0,$但是$\int_5^7 f(x)dx=-8<0,$矛盾.

29.设$f(x)=\sin x, h(x)=\frac{1}{x^2}, g(x)=\begin{cases}1, & -\pi\le x\le 2,\\ 2, & 2<x\le\pi.\end{cases}$试求下列定积分的值或表达式:

(1)$\int_{-\pi/2}^{\pi/2} f(x)g(x)dx;(2)\int_1^3 g(x)h(x)dx;(3)\int_{\pi/2}^{x} f(t)g(t)dx.$

解(1)$\int_{-\pi/2}^{\pi/2} f(x)g(x)dx=\int_{-\pi/2}^{\pi/2}\sin xdx=0.$

(2)$\int_1^3 g(x)h(x)dx=\int_1^2 g(x)h(x)dx+\int_2^3 g(x)h(x)dx$

$$=\int_1^2\frac{1}{x^2}dx+\int_2^3\frac{2}{x^2}dx=-\frac{1}{x}\Big|_1^2-\frac{2}{x}\Big|_2^3=\frac{5}{6}.$$

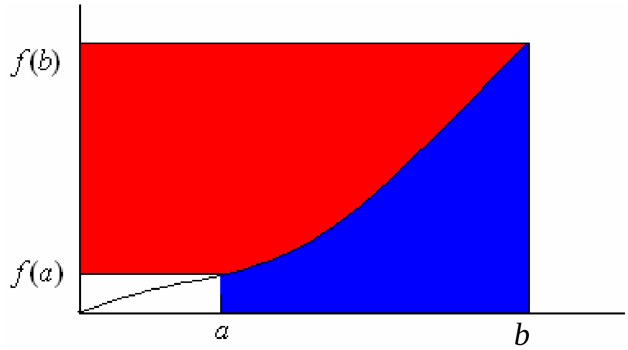
(3)$\int_{\pi/2}^{x} f(t)g(t)dx=\begin{cases}\int_{\pi/2}^{x}\sin tdt=-\cos x, t-\pi\le x\le 2\\ \int_{\pi/2}^{2}\sin tdt+\int_2^x 2\sin tdt=\cos 2-2\cos x, 2<x\le\pi.\end{cases}$

30设函数$f(x)$在区间$[a,b]$上连续，严格单调递增$(a>0), g(y)$是$f(x)$的反函数,
利用定积分的几何意义证明下列公式

$$\int_a^b f(x)dx=bf(b)-af(a)-\int_{f(a)}^{f(b)}g(y)dx.$$

并作图解释这一公式.

解



31.(1)设函数$\varphi(x)$在$[0,+\infty)$上连续且严格单调递增,又设当$x\to+\infty$时$\varphi(x)\to+\infty$且$\varphi(0)$=0. 证明:对于任意实数$a\ge 0, B\ge 0$, 下列不等式成立:

$$aB\le\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$$

其中$\varphi^{-1}(x)$是$\int_0^a\varphi(x)$的反函数.

证由30题,$\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx=a\varphi(a)(*)$.

$B=0$时不等式显然成立. 设$B>0=\varphi(0)$,由于$x\to+\infty$时$\varphi(x)\to+\infty$,存在$a'>0$, $\varphi(a')>B$,φ在$[0,a']$连续,根据连续函数的中间值定理,存在$a_1>0,\varphi(a_1)=B$.

若$a_1=a$,则由(*)得$aB=\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx.$

若$a_1>a$,则$\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$

$$=\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx+\int_{\varphi(a)}^B\varphi^{-1}(x)dx$$

$$=a\varphi(a)++\int_{\varphi(a)}^B\varphi^{-1}(x)dx$$

$$\ge a\varphi(a)+\varphi^{-1}(\varphi(a))(B-\varphi(a))=aB.$$

若$a_1<a$,则$\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$

$$=\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx-\int_B^{\varphi(a)}\varphi^{-1}(x)dx$$

$$=a\varphi(a)-\int_B^{\varphi(a)}\varphi^{-1}(x)dx$$

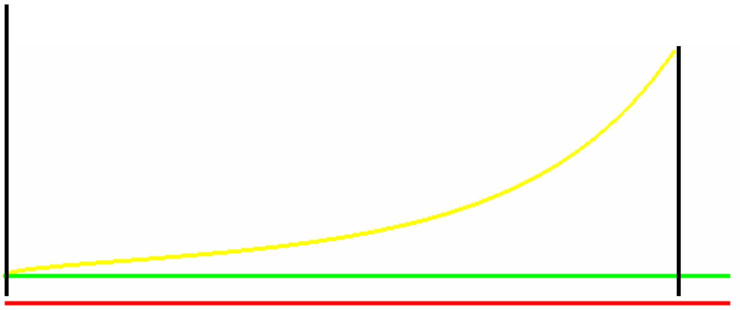
$$\ge a\varphi(a)-\varphi^{-1}(\varphi(a))(\varphi(a)-B)=aB.$$

(2)利用(1)中的不等式, 对于任意实数$a,b\ge 0$, $p,q\ge 1\frac{1}{p}+\frac{1}{q}=1$,证明下列Minkowski不等式$ab\le\frac{a^p}{p}+\frac{b^q}{q}$.

证不妨设$p>1$.在(1)中取$\varphi(x)=x^{p-1}$,则$\varphi^{-1}(x)=x^{1/(p-1)}$.

$$ab\le\int_0^a x^p dx+\int_0^b x^{1/p}dx=\frac{a^p}{p}+\frac{b^{1/(p-1)+1}}{1/(p-1)+1}=\frac{a^p}{p}+\frac{b^{p/(p-1)}}{p/(p-1)}=\frac{a^p}{p}+\frac{b^q}{q}.$$

32.设$a>0$,求a的值,使由曲线$y=1+\sqrt{x}e^{x^2}$, $y=1$及$x=a$所围成的区域绕直线$y=1$旋转所得之旋转体的体积等于2π.

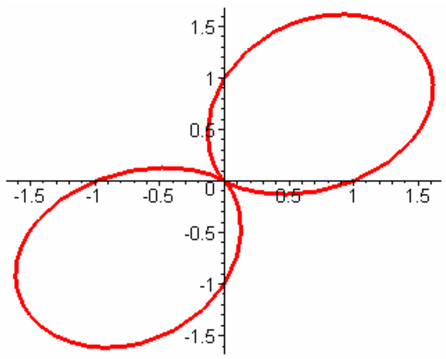


解$\pi\int_0^a (\mathrm{y}-1)^2 dx = 2\pi.\int_0^a (\sqrt{x}e^{x^2})^2 dx = 2,$

$\int_0^a xe^{2x^2} dx = 2, \frac{1}{4}\int_0^a e^{2x^2} d2x^2 = 2, \frac{1}{4}\int_0^{2a^2} e^u du = 2, e^{2a^2} - 1 = 8, 2a^2 = \ln 9, a = \sqrt{\ln 3}.$

33.作由极坐标方程$r = 1+\sin 2\theta$所确定的函数的图形, 并求它所围区域的面积.

解$S = \int_0^\pi (1+\sin 2\theta)^2 d\theta = \int_0^\pi (1+2\sin 2\theta + \frac{1-\cos 4\theta}{2}) d\theta = \frac{3\pi}{2}.$



习题 4.1

1.验证函数$f(x)=x^3-3x^2+2x$在区间$[0,1]$及$[1,2]$上满足Rolle定理的条件并分别求出导数为0的点.

解f处处可导, $f(0)=f(1)=f(2)=0$,故$f(x)$在区间$[0,1]$及$[1,2]$上满足Rolle定理的条件. $f'(x)=3x^2-6x+2=0, x=\dfrac{6\pm\sqrt{36-24}}{6}=\dfrac{3\pm\sqrt{3}}{3}, x$

$x_1=\dfrac{3-\sqrt{3}}{3}\in(0,1), x_2=\dfrac{3+\sqrt{3}}{3}\in(1,2), f'(x_1)=f'(x_2)=0.$

2.讨论下列函数$f(x)$在区间$[-1,1]$上是否满足Rolle定理的条件,若满足,求$c\in(-1,1)$,使$f'(c)=0$.

(1) $f(x)=(1+x)^m(1-x)^n, m,n$为正整数;

(2) $f(x)=1-\sqrt[3]{x^2}$.

解(1) $f'(x)=m(1+x)^{m-1}(1-x)^n-n(1+x)^m(1-x)^{n-1}$

$=(1+x)^{m-1}(1-x)^{n-1}(m-mx-n-nx)=0, c=\dfrac{m-n}{m+1}\in(-1,1), f'(c)=0.$

(2) $f'(x)=-\dfrac{2}{3}x^{-1/3}, f'(0)$不存在.

3.写出函数$f(x)=\ln x$在区间$[1,e]$上的微分中值公式,并求出其中的$c=?$

解$f'(x)=\dfrac{1}{x}, f(e)-f(1)=\ln e-\ln 1=1=\dfrac{1}{c}(e-1), c=e-1.$

4.应用Lagrange中值定理, 证明下列不等式:

(1) $|\sin y-\sin x|\le|x-y|$;

(2) $|\tan x-\tan y|\ge|y-x|, x,y\in(-\pi/2,\pi/2)$;

(3) $\dfrac{b-a}{b}<\ln\dfrac{b}{a}<\dfrac{b-a}{a}(0<a<b)$.

证(1) $|\sin x-\sin y|=|(\sin x)'|_{x=c}(x-y)|=|\cos c||x-y|\le|x-y|$.

(2) $|\tan y-\tan x|=|(\tan x)'|_{x=c}(y-x)|=\sec^2 c|y-x|\ge|y-x|$.

(3) $\dfrac{b-a}{a}<\ln\dfrac{b}{a}=\ln b-\ln a=(\ln x)'|_{x=c}(b-a)=\dfrac{b-a}{c}(c\in(a,b))<\dfrac{b-a}{a}.$

5.证明多项式$P(x)=(x^2-1)(x^2-4)$的导函数的三个根都是实根,并指出它们的范围.

证$P(x)$有四个实根根$\pm1,\pm2$,根据Rolle定理,它的导函数有三个实根,又作为四次多项式的导函数,是三次多项式, 最多三个实根, 故$P(x)$的导函数的三个根都是实根, 分别在区间$(-2,-1),(-1,1),(1,2)$.

6.设$c_1, c_2, \cdots, c_n$为任意实数,证明:函数$f(x)=c_1\cos x+c_2\cos 2x+\cdots+c_n\cos nx$在$(0,\pi)$内必有根.

证$g(x)=c_1\sin x+\frac{1}{2}c_2\sin 2x+\cdots+\frac{1}{n}c_n\sin nx$在$[0,\pi]$满足定理的条件
$(g(0)=g(\pi)=0)$,故其导函数$f(x)$在$(0,\pi)$内必有根.

7.设函数$f(x)$与$g(x)$在(a,b)内可微,$g(x)\neq 0$,且$\begin{vmatrix} f((x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}=0, x\in(a,b)$.
证明:存在常数k,使$f(x)=kg(x), x\in(a,b)$.

$$证\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}=\frac{\begin{vmatrix} f((x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}}{g^2(x)}=0,$$

根据公式的一个推论,存在常数k,使$\frac{f(x)}{g(x)}=k$,即$f(x)=kg(x), x\in(a,b)$.

8.设$f(x)$在$(-\infty,+\infty)$上可微且$f'(x)=k, -\infty<x<+\infty$.证明: $f(x)=kx+b, -\infty<x<+\infty$,其中$k,b$为常数.

证$(f(x)-kx)'=f'(x)-k=k-k=0, -\infty<x<+\infty$., $f(x)-kx=b, -\infty<x<+\infty$.

9.证明下列等式:

(1) $\arcsin x+\arccos x=\pi/2, -1\le x\le 1$;

(2) $\arctan x=\arcsin\frac{x}{\sqrt{1+x^2}}, -\infty<x<+\infty$.

证(1) $(\arcsin x+\arccos x)'=(\arcsin x)'+(\arccos x)'$

$=\frac{1}{\sqrt{1-x^2}}+\left(-\frac{1}{\sqrt{1-x^2}}\right)=0, x\in(-1,1)$, $\arcsin x+\arccos x$在$[-1,1]$连续,故

$\arcsin x+\arccos x=C, C=\arcsin 0+\arccos 0=\frac{\pi}{2}$, $\arcsin x+\arccos x=\frac{\pi}{2}$.

$$(2)\left(\arctan x-\arcsin\frac{x}{\sqrt{1+x^2}}\right)'$$

$$=\frac{1}{1+x^2}-\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}}\frac{\sqrt{1+x^2}-x\times\frac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$=\frac{1}{1+x^2}-\frac{\sqrt{1+x^2}\left(\sqrt{1+x^2}-x\times\frac{x}{\sqrt{1+x^2}}\right)}{1+x^2}=0,$$

$\arctan x-\arcsin\frac{x}{\sqrt{1+x^2}}=C$,以$x=0$代入得$C=0$,故$\arctan x-\arcsin\frac{x}{\sqrt{1+x^2}}=0$,
$x\in(-\infty,+\infty)$.

10.证明不等式: $\frac{2}{\pi}x < \sin x < x, 0 < x < \pi/2.$

证 $f(x)=\frac{\sin x}{x}(0 < x \le \pi/2), f(0)=1, f$在$[0,\pi/2]$连续,

f在$(0,\pi/2)$可导, $f'(x)=\frac{x\cos x-\sin x}{x^2}=\frac{\cos x(x-\tan x)}{x^2}<0.$

f在$[0,\pi/2]$严格单调递减, $=\frac{2}{\pi} f(\frac{\pi}{2}) < f(x) < f(0)=1, 0<x<\pi/2.$

11.设函数$f(x)$在(a,b)内可微, 对于任意一点$x_0 \in (a,b)$, 若$\lim_{x\to x_0} f'(x)$存在, 则

$\lim_{x\to x_0} f'(x) = f'(x_0).$

证$f'(x_0)=\lim_{\Delta x\to 0}\frac{\mathrm{f}(\mathrm{x}_0+\Delta \mathrm{x})-\mathrm{f}(\mathrm{x}_0)}{\Delta \mathrm{x}}=\lim_{\Delta x\to 0}\frac{f'(x_0+\theta\Delta x)\Delta x}{\Delta x}(0<\theta<1)$

$=\lim_{\Delta x\to 0} f'(x_0+\theta\Delta x)=\lim_{x\to x_0} f'(x).$

12.(Darboux中值定理)设$y=f(x)$在(A,B)区间中可导, 又设$[a,b]\subset(A,B)$, 且

$f'(a)<f'(b)$.证明: 对于任意给定的$\eta: f'(a)<\eta<f'(b)$, 都存在$c\in(a,b)$使得

$f'(c)=\eta.$

证先设$f'(a)<0<f'(b). f'(a)=\lim_{\Delta x\to 0+}\frac{f(a+\Delta x)-f(a)}{\Delta x}<0$, 存在$(b-a)/2>\delta_1>0$,

使得$0<\Delta x\le\delta_1$时$\frac{f(a+\Delta x)-f(a)}{\Delta x}<0$, 即$f(a+\Delta x)-f(a)<0$.特别$f(a+\delta_1)<f(a)$.

类似存在$\delta_2: 0<\delta_2<(b-a)/2, f(b-\delta_2)<f(b). f[a,b]$某点$c$取最小值f(c),

$f(c)\le f(a+\delta_1)<f(a), c\ne a$,同理, $c\ne b. c\in(a,b)$, c是极小值点, 由Fermat引理,

$f'(c)=0$.再设$\eta: f'(a)<\eta<f'(b)$.考虑$g(x)=f(x)-\eta x. g'(x)=f'(x)-\eta$,

$g'(a)=f'(a)-\eta<0, g'(b)=f'(b)-\eta>0$,由前面的结果, 存在$c\in(a,b)$使得

$g'(c)=f'(c)-\eta=0$, 即$f'(c)=\eta.$

习题 4.2

用L'Hospital法则求下列极限:

$$1.\lim_{x\to 0}\frac{2^x-1}{3^x-1}=\lim_{x\to 0}\frac{2^x\ln 2}{3^x\ln 3}=\frac{\ln 2}{\ln 3}.$$

$$2.\lim_{x\to 0}\frac{\cos x-1}{x-\ln(1+x)}=\lim_{x\to 0}\frac{-\sin x}{1-1/(1+x)}=-\lim_{x\to 0}\frac{\sin x}{x}=-1.$$

$$3.\lim_{x\to 0}\left(\frac{1}{\ln(x+\sqrt{1+x^2})}-\frac{1}{\ln(1+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{\ln(1+x)-\ln(x+\sqrt{1+x^2})}{\ln(x+\sqrt{1+x^2})\ln(1+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{1/(1+x)-1/\sqrt{1+x^2}}{1/\sqrt{1+x^2}\times\ln(1+x)+\ln(x+\sqrt{1+x^2})1/(1+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sqrt{1+x^2}-1-x}{(1+x)\ln(1+x)+\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})}\right)$$

$$=\lim_{x\to 0}\left(\frac{x/\sqrt{1+x^2}-1}{\ln(1+x)+1+(x/\sqrt{1+x^2})\ln(x+\sqrt{1+x^2})+1}\right)=-\frac{1}{2}.$$

$$4.\lim_{x\to\pi/2}\frac{\tan 3x}{\tan x}=\lim_{x\to\pi/2}\frac{3\sec^2 3x}{\sec^2 x}=3.$$

$$5\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}=\lim_{x\to 0}\frac{(1/(\cos ax))(-\sin ax)a}{(1/(\cos bx))(-\sin bx)b}=\frac{a^2}{b^2}.$$

$$6.\lim_{x\to 0+0}x^\alpha\ln x(\alpha>0)=\lim_{x\to 0+0}\frac{\ln x}{x^{-\alpha}}=\lim_{x\to 0+0}\frac{1/x}{(-\alpha)x^{-a-1}}=-\frac{1}{\alpha}\lim_{x\to 0+0}x^\alpha=0.$$

$$7.\lim_{x\to 0}\frac{e^{-1/x^2}}{x^{100}}=\lim_{y\to+\infty}\frac{y^{50}}{e^y}=\lim_{y\to+\infty}\left(\frac{y}{e^{y/50}}\right)^{50}=\left(\lim_{y\to+\infty}\frac{y}{e^{y/50}}\right)^{50}=\left(\lim_{y\to+\infty}\frac{50}{e^{y/50}}\right)^{50}=0.$$

$$8.\lim_{x\to\frac{\pi}{2}-0}(\tan x)^{2x-\pi}.y=(\tan x)^{2x-\pi},\ \lim_{x\to\frac{\pi}{2}-0}\ln y=\lim_{x\to\frac{\pi}{2}-0}(2x-\pi)\ln\tan x$$

$$=\lim_{x\to\frac{\pi}{2}-0}\frac{\ln\tan x}{\frac{1}{2x-\pi}}=\lim_{x\to\frac{\pi}{2}-0}\frac{\sec^2 x/\tan x}{-\frac{2}{(2x-\pi)^2}}=-2\lim_{z\to 0-0}\frac{z^2\tan z}{\sin^2 z}=0,\ \lim_{x\to\frac{\pi}{2}-0}y=\lim_{x\to\frac{\pi}{2}-0}e^{\ln y}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}-0}\ln y}=e^0=1.$$

$$9.\lim_{x\to\infty}\left(a^{1/x}-1\right)x(a>0)=\lim_{y\to 0}\frac{a^y-1}{y}=\lim_{y\to 0}\frac{a^y\ln a}{1}=\ln a.$$

$$10.\lim_{y\to 0}\frac{y-\arcsin y}{\sin^3 y}=\lim_{y\to 0}\frac{y-\arcsin y}{y^3}=\lim_{y\to 0}\frac{1-\frac{1}{\sqrt{1-y^2}}}{3y^2}$$

$$=\frac{1}{3}\lim_{y\to 0}\frac{\sqrt{1-y^2}-1}{y^2}=-\frac{1}{3}\lim_{y\to 0}\frac{\dfrac{y}{\sqrt{1-y^2}}}{2y}=-\frac{1}{6}.$$

$$11.\lim_{y\to 1}\left(\frac{y}{y-1}-\frac{1}{\ln y}\right)=\lim_{y\to 1}\left(\frac{y\ln y-y+1}{(y-1)\ln y}\right)$$

$$=\lim_{y\to 1}\left(\frac{\ln y+1-1}{\ln y+(y-1)/y}\right)=\lim_{y\to 1}\left(\frac{\ln y}{y\ln y+(y-1)}\right)$$

$$=\lim_{y\to 1}\left(\frac{1/y}{\ln y+2}\right)=\frac{1}{2}.$$

$$12.\lim_{x\to 0}\frac{1-x^2-e^{-x^2}}{x\sin^3 x}=\lim_{x\to 0}\frac{1-x^2-e^{-x^2}}{x^4}=\lim_{y\to 0}\frac{1-y-e^{-y}}{y^2}$$

$$=\lim_{y\to 0}\frac{-1+e^{-y}}{2y}=\lim_{y\to 0}\frac{-e^{-y}}{2}=-\frac{1}{2}.$$

$$13.\lim_{x\to 0}\left(\frac{\arctan x}{x}\right)^{1/x^2},\ y=\left(\frac{\arctan x}{x}\right)^{1/x^2},$$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln\dfrac{\arctan x}{x}}{x^2}=\lim_{x\to 0}\frac{(x/\arctan x)\times\dfrac{\dfrac{x}{1+x^2}-\arctan x}{x^2}}{2x}$$

$$=\lim_{x\to 0}\frac{x-(1+x^2)\arctan x}{2x^3}=\lim_{x\to 0}\frac{1-1-2x\arctan x}{6x^2}=-\frac{1}{3}\lim_{x\to 0}\frac{\arctan x}{x}=-\frac{1}{3},$$

$$\lim_{x\to 0}\left(\frac{\arctan x}{x}\right)^{1/x^2}=e^{-1/3}.$$

$$14.\lim_{x\to+\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}.\ y=\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}.$$

$$\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\frac{\ln\left(\dfrac{\pi}{2}-\arctan x\right)}{\ln x}=-\lim_{x\to+\infty}\frac{x}{\left(\dfrac{\pi}{2}-\arctan x\right)(1+x^2)}$$

$$=-\lim_{x\to+\infty}\frac{x}{\left(\arctan\dfrac{1}{x}\right)(1+x^2)}=-\lim_{x\to+\infty}\frac{x}{\left(\dfrac{1}{x}\right)(1+x^2)}=-1,\ \lim_{x\to+\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}=e^{-1}.$$

$$15.\lim_{x\to 0}\frac{\tan x-x}{x-\sin x}=\lim_{x\to 0}\frac{\sec^2 x-1}{1-\cos x}=\lim_{x\to 0}\frac{\tan^2 x}{1-\cos x}=\lim_{x\to 0}\frac{x^2}{1-\cos x}=\lim_{x\to 0}\frac{2x}{\sin x}=2.$$

$$16.\lim_{x\to 0}\frac{\cosh x-\cos x}{x^2}=\lim_{x\to 0}\frac{\sinh x+\sin x}{2x}=\lim_{x\to 0}\frac{\cosh x+\cos x}{2}=1.$$

$$17.\lim_{x\to 1}\frac{x^x-x}{\ln x-x+1}=\lim_{x\to 1}\frac{x^x(\ln x+1)-1}{1/x-1}=\lim_{x\to 1}\frac{x^x(\ln x+1)-1}{1-x}$$

$$=\lim_{x\to 1}\frac{x^x(\ln x+1)^2+x^{x-1}}{-1}=-2.$$

18. $\lim\limits_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^x$.$y=\left(\frac{2}{\pi}\arctan x\right)^x$.

$$\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\frac{\ln(\frac{2}{\pi}\arctan x)}{\frac{1}{x}}=\lim_{x\to+\infty}\frac{(1/\arctan x)\times\frac{1}{1+x^2}}{-\frac{1}{x^2}}=-\frac{2}{\pi},$$

$$\lim_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^x=e^{-2/\pi}.$$

习题 4.3

1.求下列函数再$x=0$点的的局部Taylor公式:

$$(1)\sinh x=\frac{e^{x}-e^{-x}}{2}$$

$$=\frac{1}{2}\left(\left(1+x+\frac{x^{2}}{2!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}\right)-\left(1-x+\frac{x^{2}}{2!}+\cdots-\frac{x^{2n+1}}{(2n+1)!}\right)\right)+o(x^{2n+2})$$

$$=x+\frac{x^{3}}{3!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}++o(x^{2n+2}).$$

$$(2)\frac{1}{2}\ln\frac{1-x}{1+x}=\frac{1}{2}\left(\left(-x-\frac{x^{2}}{2}+\cdots-\frac{x^{2n}}{2n}-\frac{x^{2n-1}}{2n-1}\right)-\left(x-\frac{x^{2}}{2}+\cdots-\frac{x^{2n}}{2n}+\frac{x^{2n-1}}{2n-1}\right)\right)+o(x^{2n})$$

$$=-\left(x+\frac{x^{3}}{3}+\cdots+\frac{x^{2n-1}}{2n-1}\right)+o(x^{2n}).$$

$$(3)\sin^{2}x=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}\left(\frac{(2x)^{2}}{2!}-\frac{(2x)^{4}}{4!}+\cdots+(-1)^{n-1}\frac{(2x)^{2n}}{(2n)!}\right)+o(x^{2n+1}).$$

$$(4)\frac{x^{2}+2x-1}{x-1}=-(x^{2}+2x-1)(1+x+\cdots+x^{n}+o(x^{n}))$$

$$=-(x^{2}+x^{3}+\cdots+x^{n+2}+o(x^{n+2}))-2(x+x^{2}+\cdots+x^{n+1}+o(x^{n+1}))+(1+x+\cdots+x^{n}+o(x^{n}))$$

$$=1-x-2x^{2}-2x^{3}-\cdots-2x^{n}+o(x^{n}).$$

$$(5)\cos x^{3}=1-\frac{x^{6}}{2!}+\cdots+(-1)^{n}\frac{x^{6n}}{(2n)!}+o(x^{6n+3}).$$

2.求下列函数再$x=0$点的的局部Taylor公式至所指定的阶数:

$(1)e^{x}\sin x(x^{4})$

解$e^{x}\sin x=\left(1+x+\frac{x^{2}}{2}+\frac{x^{3}}{6}+o(x^{3})\right)\left(x-\frac{x^{3}}{6}+o(x^{4})\right)=x+x^{2}+\frac{x^{3}}{3}+o(x^{4}).$

$(2)\sqrt{1+x}\cos x(x^{4})$

解$\sqrt{1+x}\cos x=\left(1+\frac{1}{2}x-\frac{1}{8}x^{2}+\frac{1}{16}x^{3}-\frac{5}{128}x^{4}+o(x^{4})\right)\left(1-\frac{x^{2}}{2}+\frac{x^{4}}{24}+o(x^{5})\right)$

$$=1+\frac{1}{2}x-\frac{5}{8}x^{2}-\frac{3}{16}x^{3}+\frac{25}{384}x^{4}+o(x^{4}).$$

$(3)\sqrt{1-2x+x^{3}}-\sqrt{1-3x+x^{2}}(x^{3})$

解$\sqrt{1-2x+x^{3}}-\sqrt{1-3x+x^{2}}$

$$=\left(1+\frac{1}{2}(-2x+x^{3})-\frac{1}{8}(-2x+x^{3})^{2}+\frac{1}{16}(-2x+x^{3})^{3}\right)$$

$$-\left(1+\frac{1}{2}(-3x+x^{2})-\frac{1}{8}(-3x+x^{2})^{2}+\frac{1}{16}(-3x+x^{2})^{3}\right)$$

$$=\left(1+\frac{1}{2}(-2x+x^3)-\frac{1}{8}(4x^2)+\frac{1}{16}(-8x^3)\right)$$

$$-\left(1+\frac{1}{2}(-3x+x^2)-\frac{1}{8}(9x^2-6x^3)+\frac{1}{16}(-27x^3)\right)+o(x^3)$$

$$=\frac{1}{2}x+\frac{1}{8}x^2+\frac{15}{16}x^3+o(x^3).$$

3.求下列函数在点$x=0$的局部Taylor公式:

(1) $\arctan x$.

解 $\dfrac{1}{1+x^2}=1-x^2+\cdots+(-1)^n x^{2n}+o(x^{2n})$

$$(2)\arcsin x=\int_0^x\frac{1}{1+t^2}dt=\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}x^{2k+1}+o(x^{2n+1})$$

$$=x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+o(x^{2n+1}).$$

解 $\dfrac{1}{\sqrt{1+x}}=(1+x)^{-1/2}=\displaystyle\sum_{k=0}^{n}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!}x^k+o(x^n)$

$$=\sum_{k=0}^{n}(-1)^k\frac{(2k-1)!!}{(2k)!!}x^k+o(x^n)$$

$$\frac{1}{\sqrt{1-x^2}}=\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}x^{2k}+o(x^n),$$

$$\arcsin x=\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\int_0^x t^{2k}dx+\int_0^x o(t^n)dt$$

$$=\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!(2k+1)}x^{2k+1}+o(t^{2n+1}).$$

4.利用Taylor公式求下列极限:

$$(1)\lim_{x\to 0}\frac{1-x^2-e^{-x^2}}{x\sin^3 2x}=\lim_{x\to 0}\frac{1-x^2-\left(1-x^2+\frac{x^4}{2}+o(x^4)\right)}{8x^4}=-\frac{1}{16}.$$

$$(2)\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}=\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}=\lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{x(x+o(x))}=\frac{1}{2}.$$

$$(3)\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{\sin x}=\lim_{x\to 0}\left(\frac{\sin x-x\cos x}{x\sin x}\right)\frac{1}{\sin x}$$

$$=\lim_{x\to 0}\frac{\sin x-x\cos x}{x^3}=\lim_{x\to 0}\frac{(x-\frac{x^3}{6})-x\left(1-\frac{x^2}{2}\right)+o(x^3)}{x^3}=\frac{1}{3}.$$

5.当x较小时,可用$\sin a + x\cos a$近似代替$\sin(a+x)$,其中a为常数,试证其误差不超过$|x|^2/2$.

证$f(x)=\sin(a+x)-(\sin a + x\cos a)$

$f(0)=0,\ f'(x)=\cos(a+x)-\cos a,$

$f''(x)=-\sin(a+x).$

$$f(x)=f(0)+f'(0)x+\frac{f''(c)}{2}x^2=\frac{-\sin(a+c)}{2}x^2,$$

$$|f(x)|=|\sin(a+x)-(\sin a + x\cos a)|\le\frac{x^2}{2}.$$

6.设$0<x\le 1/3$,按公式$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$计算$e^x$的近似值,试证公式误差不超过$8\times10^{-4}$.

证$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{e^{\theta x}}{24}x^4,\left|e^x-\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3\right)\right|=\frac{e^{\theta x}}{24}x^4\le\frac{e^{1/3}}{24}\times\left(\frac{1}{3}\right)^4$

$=.000717\cdots<8\times10^{-4}.$

习题 4.4

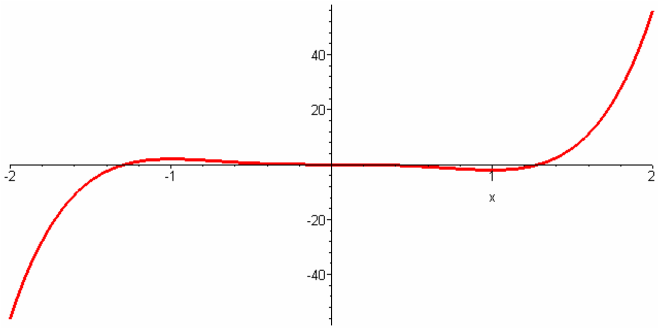
1.求下列函数的单调性区间与极值点:

(1)$y=3x^5-5x^3$.

解$y'=15x^4-15x^2=15x^2(x^2-1)$,

$y'=15x^2(x^2-1)=15x^2(x-1)(x+1)=0, x_1=-1, x_2=0, x_2=1.$

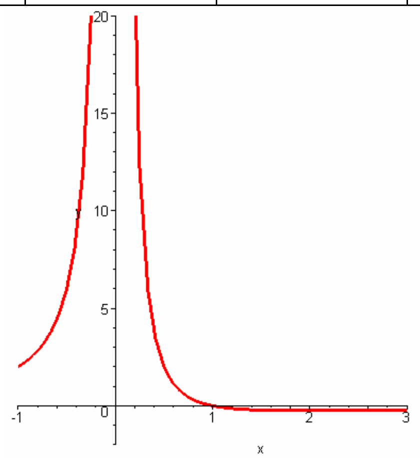
x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,+\infty)$
y'	+	0	–	0	–	0	+
y	↗	极大值	↘	无极值	↘	极小值	↗



(2)$y=\dfrac{1}{x^2}-\dfrac{1}{x}.x\neq 0.$

$y'=-\dfrac{2}{x^3}+\dfrac{1}{x^2}=\dfrac{x-2}{x^3}=0, x_1=2.$

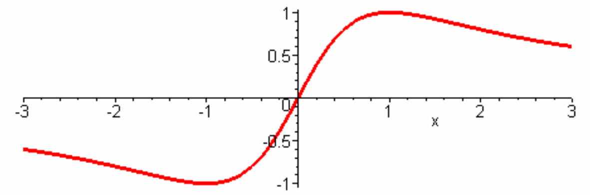
x	$(-\infty,0)$	$(0,2)$	2	$(2,+\infty)$
y'	+	–	0	+
y	↗	↘	极小值	↗



$(3) y=\frac{2x}{1+x^2}, x\in(-\infty,+\infty).$

$y'=2\times\frac{1+x^2-2x^2}{(1+x^2)^2}=2\times\frac{1-x^2}{(1+x^2)^2}=0, x=\pm1.$

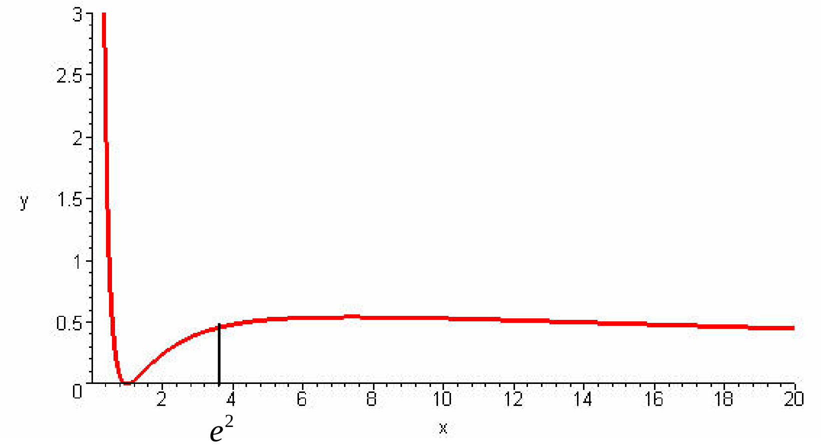
x	$(-\infty,-1)$	-1	$(-1,1)$	1	(1,+ ∞)
y'	–	0	+	0	–
y	↘	极小值−1	↗	极大值 1	↘



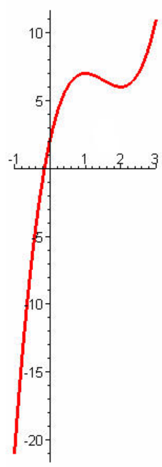
$(4) y=\frac{1}{x}\ln^2 x, x>0.$

$y'=\frac{2(\ln x)(1/x)x-\ln^2 x}{x^2}=\frac{2(\ln x)-\ln^2 x}{x^2}=\frac{\ln x[2-\ln x]}{x^2}=0, x=1, x=e^2.$

x	(0,1)	1	$(1,e^2)$	e^2	$(e^2,+\infty)$
y'	–	0	+	0	–
y	↘	极小值	↗	极大值	↘



2.求函数$f(x)=2x^3-9x^2+12x+2$在区间$[-1,3]$上的最大值与最小值,并指明最大值点与最小值点.



解$f'(x)=6x^2-18x+12=6(x^2-3x+2)$

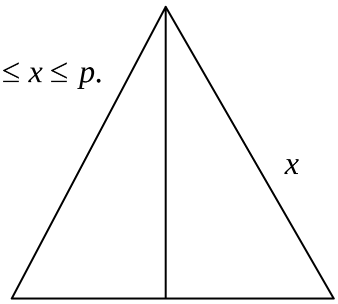
$=6(x-1)(x-2)=0, x=1,2.$

$f(-1)=-21, f(1)=7, f(2)=6, f(3)=11.$

$f(-1)=-21$是最小值, $f(3)=11$是最大值.

3.将周长为$2p$的等腰三角形绕其底边旋转一周,求使所得旋转体体积最大的等腰三角形的底边长度.

解 设腰长为x, 则



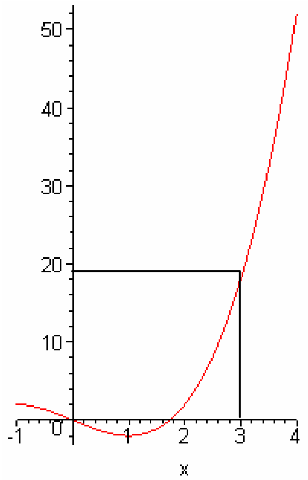
$$V=\frac{2\pi}{3}\left(x^2-(p-x)^2\right)(p-x)=\frac{2\pi}{3}\left(2px-p^2\right)(p-x),\ p/2\le x\le p.$$

$$V'=\frac{2\pi}{3}\left(2p(p-x)-(2px-p^2)\right)=\frac{2\pi}{3}\left(-4px+3p^2\right)=0,$$

$$x_0=\frac{3}{4}p, V(p/2)=V(p)=0. V(\frac{3}{4}p)\text{是最大值.}$$

等腰三角形的底边长度$=\ 2p-\frac{3}{2}p=\frac{1}{2}p.$

4.求出常数l与k的值,使函数$f(x)=x^3+lx^2+kx$在$x=-1$处有极值2,并求出在这样的l与k之下$f(x)$的所有极值点,以及在$[0,3]$上的最小值和最大值.



解 $f'(x)=3x^2+2lx+k, 3-2l+k=0, -1+l-k=2.$

$k=-3, l=0.$

$f(x)=x^3-3x. f'(x)=3x^2-3=3(x-1)(x+1)=0,$

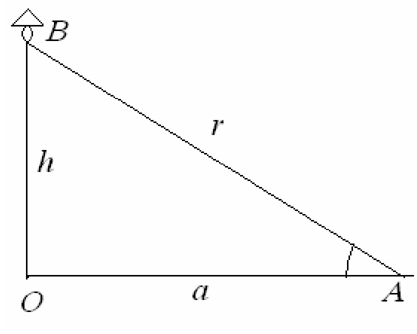
$x=\pm1, f''(x)=6x, f''(\pm1)=\pm6,$

$f(1)$是极小值, $f(-1)$是极大值.

$f(0)=0, f(1)=-2, f(3)=18. f(1)=-2$是最小值,

$f(3)=18$是最大值.

5.设一电灯可以沿垂直线OB移动,OA是一条水平线,长度为a. 问灯距离O点多高时,A点有最大的照度.

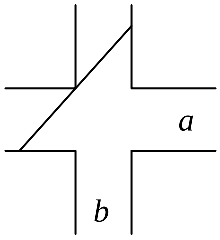


解 $J=K\dfrac{\sin\varphi}{a^2+a^2\tan^2\varphi}=\dfrac{K}{a^2}\sin\varphi\cos^2\varphi, 0\le\varphi\le\dfrac{\pi}{2}.$

$J'=\dfrac{K}{a^2}\left(\cos^3\varphi-2\sin^2\varphi\cos\varphi\right)=0, \tan\varphi_0=\dfrac{1}{\sqrt{2}}.$

$J(0)=J(\pi/2)=0, J(\varphi_0)$是最大值, 这时灯的高度 $h=a\tan\varphi_0=\dfrac{a}{\sqrt{2}}.$

6.若两条宽分别为a及b的河垂直相交,若一船从一河转入另一河,问其最大的长度是多少?



解 设船与一岸夹角为θ,则船长为

$l = a\csc\nu + b\sec\theta, 0 < \theta < \dfrac{\pi}{2}.$

$l' = -a\csc\theta\cot\theta + b\sec\theta\tan\theta = 0, \dfrac{\sec\theta\tan\theta}{\csc\theta\cot\theta} = \dfrac{a}{b},$

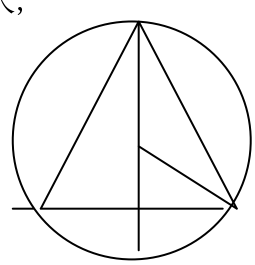
$\tan^3\theta = \dfrac{a}{b}, \tan\theta = \sqrt[3]{\dfrac{a}{b}}, \theta_0 = \arctan\sqrt[3]{\dfrac{a}{b}}.$

$\lim\limits_{\theta\to 0} l(\theta) = +\infty, \lim\limits_{\theta\to\pi/2} l(\theta) = +\infty, l$在$\left(0, \dfrac{\pi}{2}\right)$有最小值，$\theta_0$是最小值点.

此时船长1=a$\sqrt{1+\left(\sqrt[3]{\dfrac{b}{a}},\right)^2} + b\sqrt{1+\left(\sqrt[3]{\dfrac{a}{b}},\right)^2}$

$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + a^{2/3}\sqrt{a^{2/3}+b^{2/3}} = \sqrt{a^{2/3}+b^{2/3}}^3.$

7.在半径为a的球内作一内接圆锥体, 要使锥体体积最大, 问其高及底半径应是多少?



解 设球心到内接圆锥体底的距离为x, 则锥体体积

$V = \dfrac{\pi}{3}(a^2 - x^2)(a + x), 0 \le x \le a.$

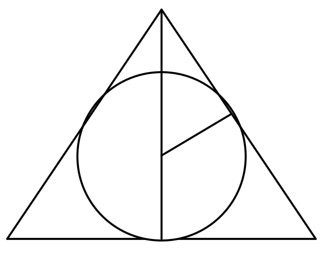
$V' = \dfrac{\pi}{3}\left(-2x(a+x) + a^2 - x^2\right) = \dfrac{\pi}{3}\left(-3x^2 - 2ax + a^2\right)$

$= -\dfrac{\pi}{3}\left(3x^2 + 2ax - a^2\right) = -\dfrac{\pi}{3}\left(3x - a\right)(x + a) = 0, x_0 = \dfrac{a}{3}.$

$V(0) = \dfrac{\pi}{3}a^3, V(a) = 0, V(\dfrac{a}{3}) = \dfrac{\pi}{3}a^3 \times \dfrac{32}{27}. V(\dfrac{a}{3})$为最大值.

底半径＝$\sqrt{a^2 - x_0^2} = \sqrt{a^2 - \left(\dfrac{a}{3}\right)^2} = \dfrac{2\sqrt{2}}{3}a$, 高$h$＝$a + x_0 = a + \dfrac{a}{3} = \dfrac{4a}{3}.$

8.在半径为a的球外作一外切圆锥体, 要问其高及底半径取多少才能使锥体体积最小?



解 设锥的高为$h, \dfrac{r}{h} = \dfrac{a}{\sqrt{(h-a)^2 - a^2}}, r = \dfrac{ah}{\sqrt{(h-a)^2 - a^2}}.$

$V = V(h) = \dfrac{\pi}{3}\dfrac{a^2h^3}{(h-a)^2 - a^2}.$

$V' = \dfrac{\pi a^2}{3}\dfrac{3h^2((h-a)^2 - a^2) - 2h^3(h-a)}{((h-a)^2 - a^2)^2} = \dfrac{\pi a^2}{3}\dfrac{h^2[3((h-a)^2 - a^2) - 2h(h-a)]}{((h-a)^2 - a^2)^2}$

$= \dfrac{\pi a^2}{3}\dfrac{h^2[h^2 - 4ah]}{((h-a)^2 - a^2)^2} = \dfrac{\pi a^2}{3}\dfrac{h^3[h - 4a]}{((h-a)^2 - a^2)^2} = 0, h_0 = 4a.$

当$h<4a$时,$V'<0$,当$h>4a$时,$V'>0$,$V(4a)$为最小值, 此时$r_0=\dfrac{4a^2}{\sqrt{8a^2}}=\sqrt{2}a$.

9.在曲线$y^2=4x$上求出到点$(18,0)$的距离最短的点.

解 $d^2=f(y)=\left(\dfrac{y^2}{4}-18\right)^2+y^2=\left(\dfrac{z}{4}-18\right)^2+z=g(z),0\le z<+\infty(z=y^2)$.

$\lim\limits_{z\to+\infty}g(z)=+\infty$, $g(z)$在$[0,+\infty)$有最小值.

$g'(z)=2\left(\dfrac{z}{4}-18\right)\dfrac{1}{4}+1=\dfrac{z}{8}-8=0, z=64, g(0)=324, g(64)=68<g(0)$,

$g(64)$为最小值. $y=\sqrt{z}=\pm 8, x=\dfrac{y^2}{4}=16$.

曲线$y^2=4x$上到点$(18,0)$的距离最短的点$(16,8),(16,-8)$.

10.试求内接于已知圆锥且有最大体积的正圆柱的高度.

解设已知圆锥的高度为H,底半径为H.设内接正圆柱的底半径为x,则其体积为

$V=\pi x^2(R-x)\dfrac{H}{R},0\le x\le R$.

$V'=\pi\left(2x(R-x)-x^2\right)=\pi\left(2Rx-3x^2\right)=\pi x\left(2R-3x\right)=0, x=0,\dfrac{2}{3}R$.

$V(0)=V(R)=0.V\left(\dfrac{2}{3}R\right)$为最大值.此时内接正圆柱的高度$h=(R-\dfrac{2}{3}R)\dfrac{H}{R}=\dfrac{H}{3}$.

11.试求内接于椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$且其底平行于$x$轴的最大等腰三角形的面积.

解$\begin{cases}x=a\cos t\\ y=b\sin t\end{cases},0\le t\le 2\pi$.

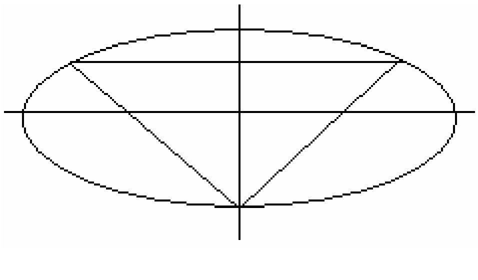
设内接等腰三角形的顶点在$(-b,0)$,而底边上的一个顶点在第一象限.

内接三角形面积$S=ab\cos t(1+\sin t),0\le t\le\dfrac{\pi}{2}$.

$S'=ab[-\sin t(1+\sin t)+\cos^2 t]=ab[1-\sin t-2\sin^2 t](\sin t=z)$

$=-ab(2z^2+z-1)=-ab(2z-1)(z+1)=0, z=\sin t_0=\dfrac{1}{2}$.

$S(0)=ab, S(\dfrac{\pi}{2})=0, S(t_0)=ab\sqrt{1-\dfrac{1}{4}}\left(1+\dfrac{1}{2}\right)=\dfrac{3\sqrt{3}}{4}ab$为最大值.



12.设动点A自平面坐标的原点O开始以速度8m/min沿y轴正向前进,而点B在x轴的正向距离原点50m处,同时沿x轴向原点作匀速运动,速度为6m/min.问何时A与B距离最近?最近的距离是多少?.

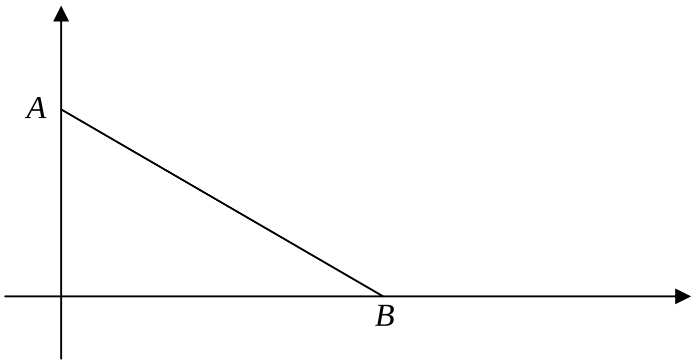
解$s^2 = f(t) = (8t)^2 + (50-6t)^2, t \ge 0.$

$\lim\limits_{t\to+\infty} f(t) = +\infty$, $f(t)$在$t \ge 0$取最小值.

$f'(t) = 128t - 12(50-6t) = 200t - 600 = 0, t_0 = 3.$

$f(0) = 50$, $f(3) = 24^2 + 32^2 = 1600 = d^2, d = 40.$

开始后3分钟达到最近距离40m.



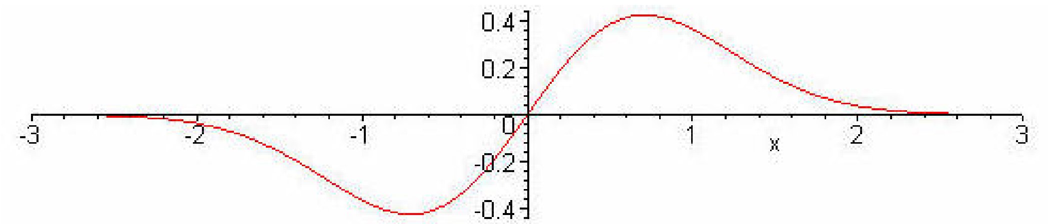
习题 4.5

1.求函数$f(x)=xe^{-x^2}$ 的凸凹性区间及拐点.

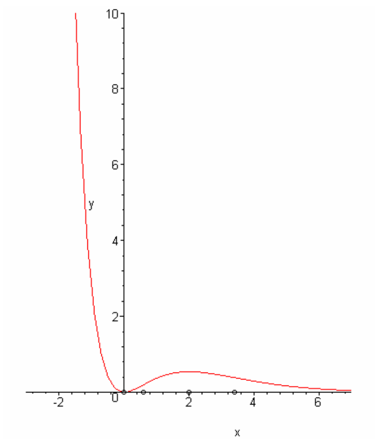
解$f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}\left(1-2x^2\right),\ f''(x)=e^{-x^2}\left(1-2x^2\right)(-2x)-4xe^{-x^2}$

$=e^{-x^2}\left(-6x+4x^3\right)=2xe^{-x^2}\left(-3+2x^2\right)=0,\ x=0,\pm\sqrt{\dfrac{3}{2}}.$

x	$(-\infty,-\sqrt{\frac{3}{2}})$	$-\sqrt{\frac{3}{2}}$	$(-\sqrt{\frac{3}{2}},0)$	0	$(0,-\sqrt{\frac{3}{2}})$	$\sqrt{\frac{3}{2}}$	$(\sqrt{\frac{3}{2}},+\infty)$
f''	–	0	+	0	–	0	+
f	∩	拐点	∪	拐点	∩	拐点	∪



作下列函数的图形：



2. $y=x^2-\dfrac{1}{3}x^3,\ x\in(-\infty,\infty).$

$y'=2x-x^2=x(2-x)=0,\ x=0,2.$

$y''=2-2x=0,\ x=1.$

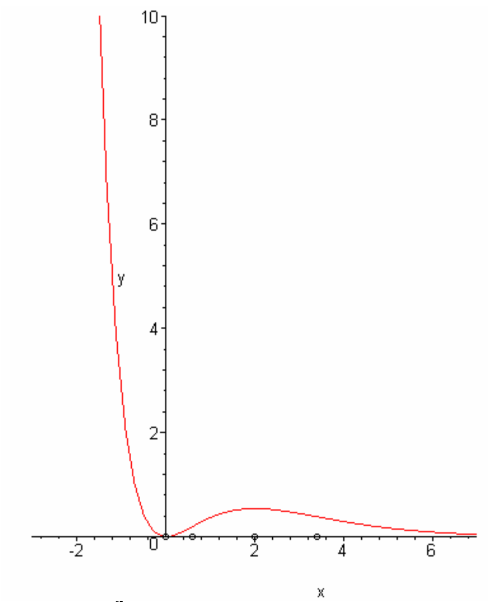
x	$(-\infty,0)$	0	$(0,1)$	1	$(1,2)$	2	$(2,+\infty)$
y'	–	0	+		+	0	–
y''	+		+		–		–
y	↘∪	极小值	∪↗	拐点	↗∩	极大值	↘∩

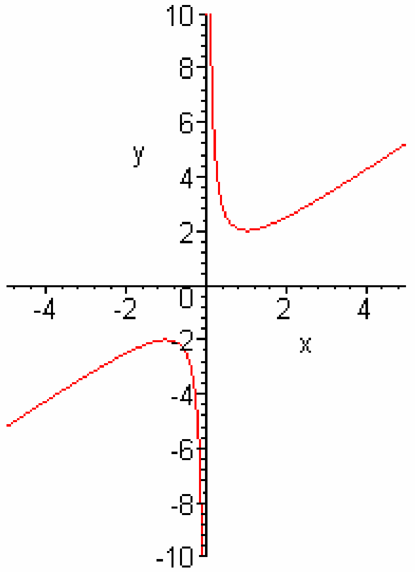
$3.y=x^2e^{-x}, x\in(-\infty,+\infty).y'=2xe^{-x}-x^2e^{-x}=e^{-x}(2x-x^2)=e^{-x}x(2-x)=0, x=0,2;$

$y''=-e^{-x}(2x-x^2)+e^{-x}(2-2x)=e^{-x}(x^2-4x+2)=0,$

$x=2\pm\sqrt{2}.$

x	$(-\infty,0)$	0	$(0,2-\sqrt{2})$	$2-\sqrt{2}$	$(2-\sqrt{2},2)$	2	$(2,2+\sqrt{2})$	$2+\sqrt{2}$	$(2+\sqrt{2},+\infty)$
y'	$-$	0	$+$		$+$	0	$-$		$-$
y''	$+$		$+$	0	$-$		$-$	0	$+$
y	↘∪	极小值	↗∪	拐点	↗∩	极大值	↘∩	拐点	↘∪





$4.y=x+\frac{1}{x}, x\neq 0.$

$y'=1-\frac{1}{x^2}=\frac{x^2-1}{x^2}=0,$

$x=\pm 1;\ y''=\frac{2}{x^3}.$

x	$(-\infty,-1)$	-1	$(-1,0)$	$(0,1)$	1	$(1,+\infty)$
y'	$+$	0	$-$	$-$	0	$+$
y''	$-$		$-$	$+$		$+$
y	↗∩	极大值	↘∩	∪	极小值	↗∪

$$5.\ y=\frac{(x+1)^3}{(x-1)^2},\ x\neq 1.$$

$$y'=\frac{3(x+1)^2(x-1)^2-2(x+1)^3(x-1)}{(x-1)^4}$$

$$=\frac{(x+1)^2(x-1)(3x-3-2x-2)}{(x-1)^4}=\frac{(x+1)^2(x-1)(x-5)}{(x-1)^4}=\frac{(x+1)^2(x-5)}{(x-1)^3},$$

$$y'=0,\ x=-1,5.$$

$$y''=\frac{[2(x+1)(x-5)+(x+1)^2](x-1)^3-3(x+1)^2(x-5)(x-1)^2}{(x-1)^6}$$

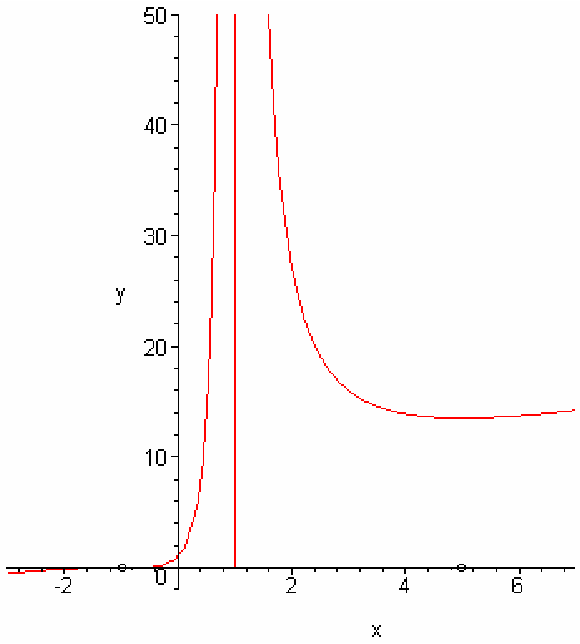
$$=\frac{[2(x+1)(x-5)+(x+1)^2](x-1)-3(x+1)^2(x-5)}{(x-1)^4}$$

$$=\frac{(x+1)\{[2(x-5)+(x+1)](x-1)-3(x+1)(x-5)\}}{(x-1)^4}$$

$$=\frac{(x+1)\{(3x-9)(x-1)-3(x^2-4x-5)\}}{(x-1)^4}$$

$$=\frac{(x+1)\{(3x^2-12x+9)-3(x^2-4x-5)\}}{(x-1)^4}=\frac{24(x+1)}{(x-1)^4}=0,\quad x=-1.$$

x	$(-\infty,-1)$	-1	$(-1,1)$	$(1,5)$	5	$(5,+\infty)$
y'	+	0	+	–	0	+
y''	–	0	+	+		+
y	↗∩	拐点	↗∪	↘∪	极小值	↗∪



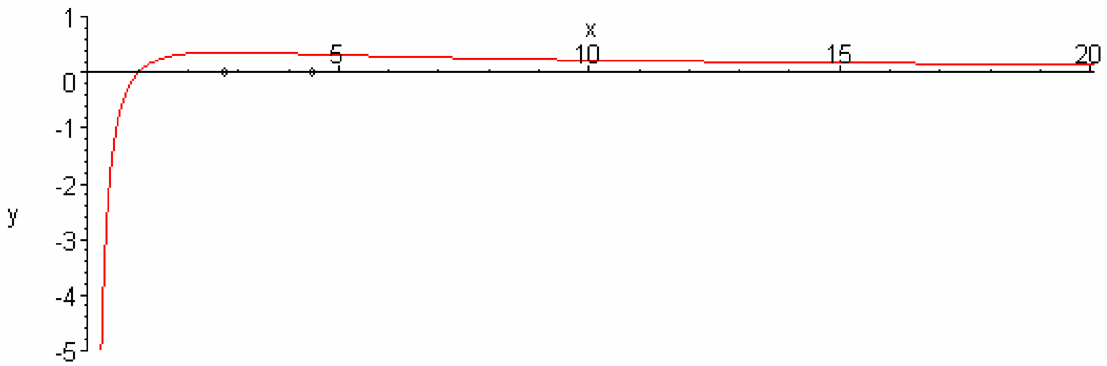
$6. y=\dfrac{\ln x}{x}, x>0.$

$y'=\dfrac{1-\ln x}{x^2}=0, x=e.$

$y''=\dfrac{-\frac{1}{x}\times x^2-2x(1-\ln x)}{x^4}=-\dfrac{1+2(1-\ln x)}{x^3}=-\dfrac{3-2\ln x}{x^3},$

$y''=0, x=e^{3/2}.$

x	$(-\infty,e)$	e	$(e,e^{3/2})$	$e^{3/2}$	$(e^{3/2},+\infty)$
y'	+	0	−		−
y''	−		−	0	+
y	↗∩	极大值	↘∩	拐点	↘∪



7. 设函数$y=f(x)$在(a,b)内有二阶导数$f''(x)$且在(a,b)内向上凸.证明$f''(x)\le 0, x\in(a,b)$.

证$y=f(x)$在(a,b)在(a,b)内向上凸,故对于任意$x_1,x_2\in(a,b), x_1<x_2$,

$f(x_1)\le f(x_2)+f'(x_2)(x_1-x_2), f(x_2)\le f(x_1)+f'(x_1)(x_2-x_1).$

两式相加得

$0\le(f'(x_1)-f'(x_2))(x_2-x_1),$

消去$x_2-x_1>0$得$0\le f'(x_1)-f'(x_2)$,即$f'(x_2)\le f'(x_1)$, $f'(x)$是单调递减函数,故$f''(x)\le 0$,
$x\in(a,b)$.

习题 4.6

1.求下列曲线在指定点的曲率:

(1) $y=3x^3-x+1$在$\left(-\frac{1}{3},\frac{11}{9}\right)$处;

(2) $y=\frac{x^2}{x-1}$在$\left(3,\frac{9}{2}\right)$处;

(3) $x(t)=a(t-\sin t), y(t)=a(1-\cos t)$,其中$a$为常数,在$t=\pi/2$处.

解(1) $y'=9x^2-1, y''=18x, K=\frac{|y''|}{(1+y'^2)^{3/2}}=\frac{|-6|}{(1+0^2)^{3/2}}=6.$

(2) $y=x+1+\frac{1}{x-1}, y'=1-\frac{1}{(x-1)^2}, y''=\frac{2}{(x-1)^3}.K=\frac{\frac{1}{4}}{(1+\frac{9}{16})^{3/2}}=\frac{16}{125}.$

(3) $x'=a(1-\cos t), x''=a\sin t, y'=a\sin t, y''=a\cos t, K=\frac{a^2}{(a^2+a^2)^{3/2}}=\frac{1}{2\sqrt{2}a}.$

2.求曲线$y=2x^2+1$在点(0,1)处的曲率圆方程.

解$y'=4x, y''=4.\alpha=\mathrm{x}_0-\frac{y'(1+y'^2)}{y''}=0, \beta=y_0+\frac{(1+y'^2)}{y''}=1+\frac{1}{4}=\frac{5}{4},$

$K=\frac{|y''|}{(1+y'^2)^{3/2}}=4, R=\frac{1}{4}$,曲率圆方程:$x^2+(y-\frac{5}{4})^2=\left(\frac{1}{4}\right)^2.$

3.问曲线$y=2x^2-4x+3$上哪一点处曲率最大?并对其作几何解释.

解$y'=4x-4, y''=4.K=\frac{|y''|}{(1+y'^2)^{3/2}}=\frac{4}{(1+(4x-4)^2)^{3/2}}$当$x=1$时最大,对应点(1,1)恰是抛物线的顶点.

18.设函数$f(x)$在$(-\infty,+\infty)$内可导,且a,b是方程$f(x)=0$的两个实根.证明方程
$f(x)+f'(x)=0$在(a,b)内至少有一个实根.

证设 $g(x)=e^{x}f(x), g(a)=g(b)=0, g$在 $[a,b]$连续， 在(a,b)可导),.

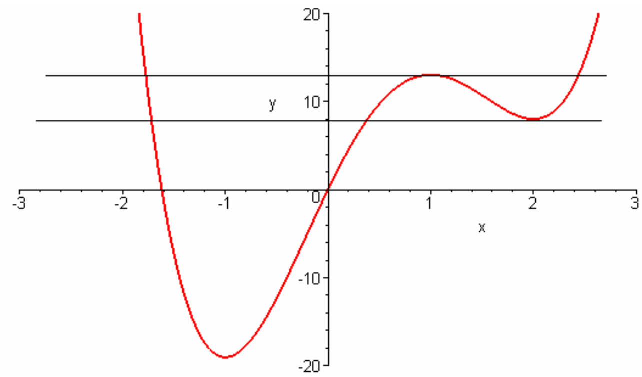
根据Rolle定理， 存在 $c\in(a,b)$, 使得$g'(x)=e^{x}(f(x)+f'(x))=0$,即$f(x)+f'(x)=0$.

19.决定常数A的范围, 使方程$3x^4-8x^3-6x^2+24x+A$有四个不相等的实根.

解$P(x)=3x^4-8x^3-6x^2+24x, P'(x)=12x^3-24x^2-12x+24$
$=12(x^3-2x^2-x+2)=12[x^2(x-2)-(x-2)]=12(x-2)(x^2-1)=12(x-2)(x-1)(x+1)$
$=0$,.

$x_1=-1, x_2=1, x_3=2. P(x_1)=-19, P(1)=13, P(2)=8.$

根据这些数据画图， 由图易知当在区间$(-P(1),-P(2))=(-13,-8)$时
$3x^4-8x^3-6x^2+24x+A$有四个不相等的实根.



20.设$f(x)=1-x+\dfrac{x^2}{2}-\dfrac{x^3}{3}+\cdots+(-1)^n\dfrac{x^n}{n}$.证明:方程$f(x)=0$当$n$为奇数时有一个实根,当$n$为偶数时无实根.

证当$x\le 0$时$f(x)>0$,故f 只有正根,当$n=2k-1$为奇数时, $\lim\limits_{x\to-\infty}f(x)=+\infty$,
$\lim\limits_{x\to+\infty}f(x)=-\infty$,存在$a,b,a<b, f(a)>0, f(b)<0$.

根据连续函数的中间值定理, 存在$x_0\in(a,b)$,使得$f(x_0)=0$.

$f'(x)=-1+x-x^2+\cdots-x^{2k-2}=\dfrac{x^{2k-1}+1}{-x-1}<0(x>0)$,当$x>0$时, f严格单调递减,故实根唯一.

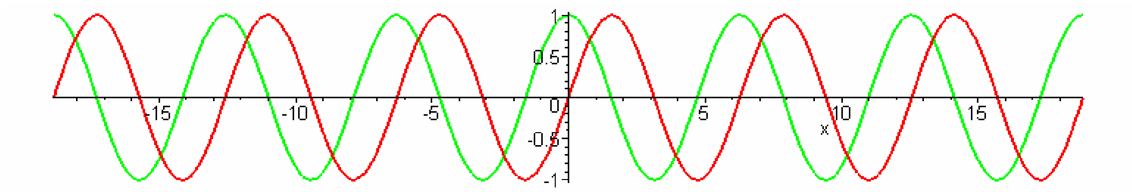
当$n=2k$为偶数时, $f'(x)=-1+x-x^2+\cdots+x^{2k-1}=\dfrac{-x^{2k}+1}{-x-1}=0, x=1$.

$0<x<1, f'(x)<0, x>1, f'(x)>0, f(1)$是$x>0$时的最小值, $f(1)>0$,故当n为偶数时$f(x)$无实根.

21.设函数$u(x)$与$v(x)$以及它们的导函数$u'(x)$与$v'(x)$在区间$[a,b]$上都连续,且$uv'-u'v$在$[a,b]$上恒不等于零.证明$u(x)$在$v(x)$的相邻根之间必有一根,反之也对.即有$u(x)$与$v(x)$的根互相交错地出现.试句举处满足上述条件的$u(x)$与$v(x)$.

证 设x_1,x_2是$u(x)$的在$[a,b]$的两个根, $x_1<x_2$.由于$u'v-uv'\neq 0, v(x_1)\neq 0, v(x_2)\neq 0$.如果$v(x)$在$[x_1,x_2]$上没有根,则$w=\dfrac{u}{v}$在$[a,b]$连续, $w(x_1)=w(x_2)=0$,由Rolle定理,存在$c\in[x_1,x_2]$,使得$w'(c)=\dfrac{u'v-uv'}{v^2}(c)=0$,即$(u'v-uv')(c)=0$,此与$u'v-uv'$恒不等于零的假设矛盾. 故v(x)在$[x_1,x_2]$上有根.

例如$u=\cos(x), v=\sin x, u'v-uv'=-1\neq 0$, $\sin x\cos x$的根交错出现.



22.证明:当$x>0$时函数$f(x)=\dfrac{\arctan x}{\tanh x}$单调'递增,且$\arctan x<\dfrac{\pi}{2}(\tanh x)$.

证$f'(x)=\left(\dfrac{\arctan x}{\tanh x}\right)'=\dfrac{\dfrac{\tanh x}{1+x^2}-\dfrac{\arctan x}{\cosh^2 x}}{\tanh^2 x}=\dfrac{\sinh x\cosh x-(1+x^2)\arctan x}{(1+x^2)\tanh^2 x\cosh^2 x}$

$$=\frac{\frac{1}{2}\sinh 2x-(1+x^2)\arctan x}{(1+x^2)\tanh^2 x\cosh^2 x}=\frac{g(x)}{(1+x^2)\tanh^2 x\cosh^2 x}.$$

$g(0)=0.$

$g'(x)=\cosh 2x-1-2x\arctan x, g'(0)=0,$

$g''(x)=2\sinh 2x-2\arctan x-\dfrac{2x}{1+x^2}, g''(0)=0,$

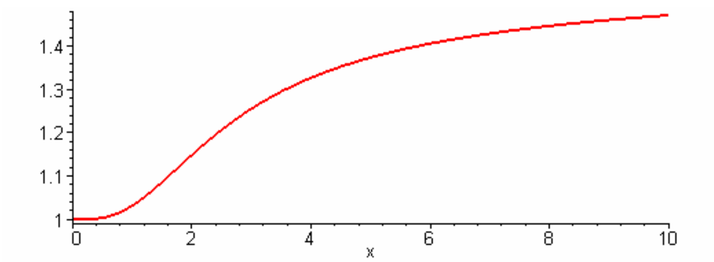
$$g'''(x)=4\cosh 2x-\frac{2}{1+x^2}-2\times\frac{(1+x^2)-2x^2}{(1+x)^2}=4\cosh 2x-\frac{2}{1+x^2}-\frac{2(1-x^2)}{1+x^2}$$

$=4\cosh 2x-\dfrac{4}{1+x^2}+\dfrac{4x^2}{1+x^2}>0$(当$x>0$时$\cosh x>1$),

由Taylor公式, 对于$x>0$有

$g(x)=\dfrac{g(\theta x)}{3!}x^3>0$, $f'(x)>0$, f严格单调递增.

$\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}\dfrac{\arctan x}{\tanh x}=\dfrac{\pi}{2}$,故对于$x>0$有$\dfrac{\arctan x}{\tanh x}<\dfrac{\pi}{2}$.



$23.$证明:当$0<x<\dfrac{\pi}{2}$时有$\dfrac{x}{\sin x}<\dfrac{\tan x}{x}$.

证$f(x)=\sin x\tan x-x^2$,

$f'(x)=\cos x\tan x+\sin x\sec^2 x-2x=\sin x+\sin x\sec^2 x-2x$,

$f''(x)=\cos x+\sec x+2\sin x\sec^2 x\tan x-2=(\cos x+\sec x-2)+2\sin^2 x\sec x-2>0$

$(\cos x+\sec x=\cos x+\dfrac{1}{\cos x}\ge 2, x\in(0,\pi/2))$.

$f(0)=f'(0)=0$,根据Taylor公式,

$f(x)=\dfrac{f''(\theta x)}{2}x^2>0, \sin x\tan x-x^2>0, \dfrac{x}{\sin x}<\dfrac{\tan x}{x}(x\in(0,\pi/2))$.

24.证明下列不等式:

(1)$e^x>1+x, x\ne 0$.

(2)$x-\dfrac{x^2}{2}<\ln(1+x), x>0$.

(3)$x-\dfrac{x^3}{6}<\sin x<x, x>0$.

证 (1)$e^x=1+x+\dfrac{e^{\theta x}}{2}x^2>1+x, x\ne 0$.

(2) $\ln(1+x)=x-\dfrac{1}{(1+\theta x)^2}x^2<x, x>0$.

$\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{1}{3(1+\theta x)^3}x^3>x-\dfrac{x^2}{2}, x>0$.

(3) $f(x)=x-\sin x$, $f(0)=0$, $f'(x)=1-\cos x\ge 0$,仅当$x=2n\pi$时$f'(x)=0$,故当$x>0$时f严格单调递增, $f(x)>f(0)=0, x>0$.

$g(x)=\sin x-\left(x-\dfrac{x^3}{6}\right)$,

$g'(x)=\cos x-\left(1-\dfrac{x^2}{2}\right), g''(x)=-\sin x+x>0, x>0$.$g$当$x>0$时严格单调递增, $g(x)>g(0)=0, x>0$.

25.设$x_n=(1+q)(1+q^2)\cdots(1+q^n)$,其中常数$q\in[0,1)$.证明序列$x_n$有极限.

证x_n单调递增.$\ln x_n=\sum\limits_{i=1}^{n}\ln(1+q^i)<\sum\limits_{i=1}^{n}q^i=\dfrac{q-q^{n+1}}{1-q}<\dfrac{q}{1-q}$,

$x_n=e^{\ln x_n}<e^{\frac{q}{1-q}}$.$x_n$有上界.故$x_n$有极限.

26.求函数$f(x)=\tan x$在$x=\pi/4$处的三阶Taylor多项式, 并由此估计$\tan(50^\circ)$的值.

解$f'(x)=\sec^2 x$, $f''(x)=2\sec^2 x\tan x$, $f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x$.

$f(\dfrac{\pi}{4})=1, f'(\dfrac{\pi}{4})=2, f''(\dfrac{\pi}{4})=4, f'''(\dfrac{\pi}{4})=16$.

$$f(x)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+o\left(\left(x-\frac{\pi}{4}\right)^3\right).$$

$$\tan(50^\circ)=\tan\left(\frac{\pi}{4}+\frac{\pi}{36}\right)\approx 1+2\times\frac{\pi}{36}+2\left(\frac{\pi}{36}\right)^2+\frac{8}{3}\left(\frac{\pi}{36}\right)^3\approx 1.191536480.$$

27.设$0<a<b$,证明$(1+a)\ln(1+a)+(1+b)\ln(1+b)<(1+a+b)\ln(1+a+b)$.

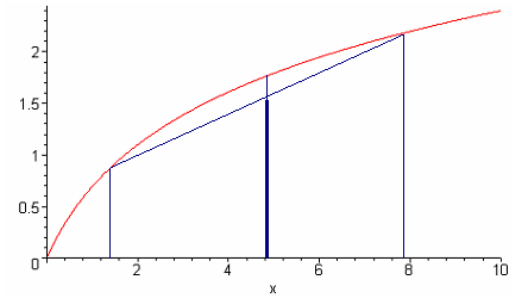
证$f(x)=\ln(1+x)$, $f'(x)=\dfrac{1}{1+x}$, $f''(x)=-\dfrac{1}{(1+x)^2}<0$,

f在$x>0$上凸,

$$\frac{(1+a)}{(1+a+b)}\ln(1+a)+\frac{(1+b)}{(1+a+b)}\ln(1+b)$$

$$<\ln\left(1+\frac{(1+a)a}{(1+a+b)}+\frac{(1+b)b}{(1+a+b)}\right)$$

$$<\ln\left(1+\frac{(1+a+b)a}{(1+a+b)}+\frac{(1+a+b)b}{(1+a+b)}\right)=\ln(1+a+b).$$



28.设有三个常数a,b,c,满足

$a<b<c, a+b+c=2, ab+bc+ca=1$.证明:$0<a<\dfrac{1}{3},\dfrac{1}{3}<b<1,1<c<\dfrac{4}{3}$.

证考虑多项式$f(x)=(x-a)(x-b)(x-c)=x^3-2x^2+x-abc$.

$f'(x)=3x^2-4x+1=(3x-1)(x-1)=0, x_1=\dfrac{1}{3}, x_2=1.$

当$x<\dfrac{1}{3}$或$x>1$时$f'(x)>0$, f严格单调递增,当$\dfrac{1}{3}<x<1$时$f'(x)<0$, f严格单调递减.

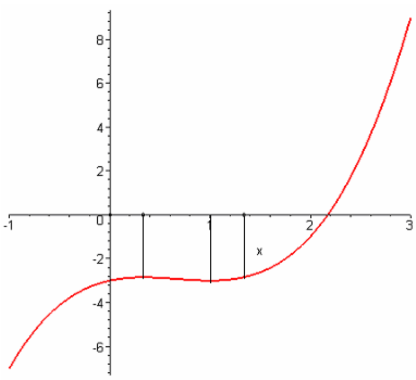
如果$f(0)=f(1)=-abc\geq 0$, f将至多有两个

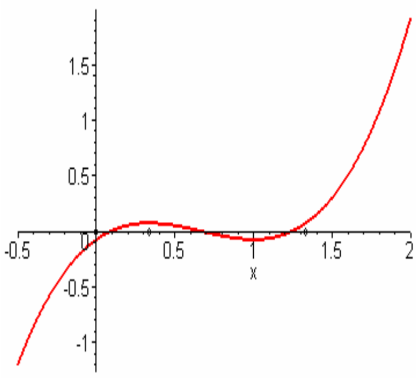
实根.如果$f(\dfrac{1}{3})=f(\dfrac{4}{3})=\dfrac{4}{27}-abc\leq 0$, f也将至多有两个

根(见附图). 而f实际有根a,b,c.故$f(0)=f(1)=-abc<0$,并且$f(\dfrac{1}{3})=f(\dfrac{4}{3})=\dfrac{4}{27}-abc>0$.

考虑到严格单调性,于是f

在$(0,\dfrac{1}{3}),(\dfrac{1}{3},1),(1,\dfrac{4}{3})$各有一实根,正是$a,b,c$,故结论成立.





29.设函数$f(x)$的二阶导数$f''(x)$在$[a,b]$上连续,且对于每一点$x\in[a,b]$, $f''(x)$与$f(x)$同号.证明:若有两点$c,d\in[a,b]$,使$f(c)=f(d)=0$,则$f(x)\equiv 0, x\in[c,d]$.

证由于$f''(x)$与$f(x)$同号,$(f(x)f'(x))'=f'^2(x)+f(x)f''(x)\ge 0, g(x)=f(x)f'(x)$单调,

$g(c)=g(d)=0$,故$f(x)f'(x)\equiv 0, x\in[c,d]$.$(f^2(x))'=2f(x)f'(x)\equiv 0, x\in[c,d]$.

$f^2(x)\equiv C, x\in[c,d]$.$f^2(c)=0$,故$f^2(x)\equiv 0, x\in[c,d]$,即$f(x)\equiv 0, x\in[c,d]$.

30.求多项式$P_3(x)=2x^3-7x^2+13x-9$在$x=1$处的Taylor公式.

解$P_3'(x)=6x^2-14x+13, P_3''(x)=12x-14, P_3'''(x)=12$.

$P_3(1)=-1, P_3'(1)=5, P_3''(1)=-2, P_3'''(1)=12$.

$P_3(x)=-1+5(x-1)-(x-1)^2+2(x-1)^3$.

31.设$P_n(x)$是一个n次多项式.

(1)证明:$P_n(x)$在任一点x_0处的Taylor公式为

$$P_n(x)=P_n(x_0)+P_n'(x_0)+\cdots+\frac{1}{n!}P_n^{(n)}(x_0).$$

(2)若存在一个数a,使$P_n(a)>0, P_n^{(k)}(a)\ge 0(k=1,2,\cdots n)$.证明$P_n(x)$的所有实根都不超过$a$.

证(1)$P_n(x)$是一个n次多项式.

(1)证明:因为$P_n(x)$是一个n次多项式,$P_n^{(n+1)}(x)\equiv 0, x\in(-\infty,+\infty)$.故在任一点$x_0$处,根据带Lagrange余项的Taylor公式

$$P_n(x)=P_n(x_0)+P_n'(x_0)(x-x_0)+\cdots+\frac{1}{n!}P_n^{(n)}(x_0)(x-x_0)^n+\frac{1}{(n+1)!}P_n^{(n+1)}(c)(x-x_0)^{n+1}$$

$$=P_n(x_0)+P_n'(x_0)(x-x_0)+\cdots+\frac{1}{n!}P_n^{(n)}(x_0)(x-x_0)^n.$$

(2)$P_n(x)=P_n(a)+P_n'(a)(x-a)+\cdots+\frac{1}{n!}P_n^{(n)}(a)(x-a)^n\ge P_n(a)>0(x\ge a)$,

故$P_n(x)$的所有实根都小于a.

32.设函数$f(x)$在$(0,+\infty)$上有二阶导数,又知对于一切$x>0$,有

$|f(x)|\le A, |f''(x)|\le B$其中$A,B$为常数.证明:$|f'(x)|\le 2\sqrt{AB}, x\in(0,+\infty)$.

证任意取$x\in(0,+\infty), h>0. f(x+h)=f(x)+f'(x)h+\frac{f''(c)}{2}h^2$,

$f'(x)=\frac{1}{h}(f(+h)-f(x))-\frac{f''(c)}{2}h.$

$|f'(x)|\le\frac{2A}{h}+\frac{B}{2}h(*).$

当$\frac{2A}{h}=\frac{B}{2}h$时$(*)$右端取最小值. 在$(*)$中取$h=2\sqrt{\frac{A}{B}}$,即得$|f'(x)|\le 2\sqrt{AB}$.

第四章总练习题

1.设y=f(x)在$[x_0-h, x_0+h]$(h>0)内可导.证明存在θ, 0<θ<1使得

$$f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

证令g(x)=$f(x_0+x)-f(x_0-x), x\in[0,h]$.$g$(x)在[0, h]内可导,

$$g'(x)=f'(x_0+x)+f'(x_0-x), g(0)=0.$$

根据Lagrange公式,存在$\theta\in(0,1)$使得

$$g(h)-g(0)=g'(\theta h)h,\ 即 f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$$

2.证明:当$x\ge 0$时,等式$\sqrt{x+1}-\sqrt{x}=\dfrac{1}{2\sqrt{x+\theta(x)}}$

中的$\theta(x)$满足$1/4\le\theta(x)\le 1/2$且$\lim\limits_{x\to 0}\theta(x)=1/4, \lim\limits_{x\to+\infty}\theta(x)=1/2$.

证$\sqrt{x+1}-\sqrt{x}=\dfrac{1}{2\sqrt{x+\theta(x)}}, 2\sqrt{x+\theta(x)}=\dfrac{1}{\sqrt{x+1}-\sqrt{x}}=\sqrt{x+1}+\sqrt{x},$

$$4(x+\theta(x))=2x+1+2\sqrt{x(x+1)},$$

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x).$$

$$\theta(x)\ge\frac{1}{4}(1+2\sqrt{x(x)}-2x)=\frac{1}{4},$$

由算术$-$几何平均不等式得

$$\theta(x)=\frac{1}{4}(1+2\sqrt{x(x+1)}-2x)\le\frac{1}{4}(1+(x+x+1)-2x)=\frac{1}{2}.$$

$$\lim_{x\to 0}\theta(x)=\lim_{x\to 0}\frac{1}{4}(1+2\sqrt{x(x+1)}-2x)=\frac{1}{4}.$$

$$\lim_{x\to+\infty}\theta(x)=\lim_{x\to+\infty}\frac{1}{4}(1+2\sqrt{x(x+1)}-2x)$$

$$=\frac{1}{4}\lim_{x\to+\infty}\frac{(1+2\sqrt{x(x+1)}-2x)(1+2\sqrt{x(x+1)}+2x)}{(1+2\sqrt{x(x+1)}+2x)}$$

$$=\frac{1}{4}\lim_{x\to+\infty}\frac{1+4x+4\sqrt{x(x+1)}}{(1+2\sqrt{x(x+1)}+2x)}==\frac{1}{4}\lim_{x\to+\infty}\frac{1/x+4+4\sqrt{1+1/x}}{(1/x+2\sqrt{(1/x+1)}+2)}=\frac{1}{2}.$$

3.设$f(x)=\begin{cases}\dfrac{3-x^2}{2}, 0\le x\le 1\\ \dfrac{1}{x}, \quad 1<x<+\infty\end{cases}$ 求$f(x)$在闭区间$[0,2]$上的微分中值定理的中间值.

解$f'(x)=\begin{cases}-x, 0\le x\le 1\\ -\dfrac{1}{x^2}, \quad 1<x<+\infty\end{cases}.\dfrac{f(2)-f(0)}{2-0}=\dfrac{1/2-3/2}{2}=-\dfrac{1}{2}.$

$-x=-\dfrac{1}{2}, x=\dfrac{1}{2}; -\dfrac{1}{x^2}=-\dfrac{1}{2}, x=\sqrt{2}. f(x)$在闭区间$[0,2]$上的微分中值定理的中间值为$\dfrac{1}{2}$或$\sqrt{2}$.

4.在闭区间$[-1,1]$上Cauchy中值定理对于函数$f(x)=x^2$与$g(x)=x^3$是否成立?并说明理由.

解 由于$g'(x)=3x^2$有零点$0\in(-1,1)$, Cauchy中值定理的条件不满足. 其实其结论也不成立.

因为若$\dfrac{f(1)-f(-1)}{g(1)-g(-1)}=0=\dfrac{f'(c)}{g'(c)}$, $f'(c)=2c=0$, $c=0$,但$g'(0)=0$, $\dfrac{f'(c)}{g'(c)}$无意义.

5.设$f(x)$在$[a,b]$上连续,在(a,b)上有二阶导数,且$f''(x)\neq 0, x\in(a,b)$又$f(a)=f(b)=0$,证明当$x\in(a,b)$时$f(x)\neq 0$.

证一 若存在$c\in(a,b)$, $f(c)=0$,则由Rolle定理,存在$c_1\in(a,c), c_2\in(c,b)$使得$f'(c_1)=f'(c_2)=0$.

对于$f'(x)$在$[c_1,c_2]$应用定理, 存在$\xi\in(c_1,c_2)$,使得$f''(\xi)=0$,此与条件$f''(x)\neq 0$, $x\in(a,b)$矛盾.

证二 由假设, $f''(x)\neq 0, x\in(a,b)$,根据*Darboux*定理, $f''(x)$恒正或恒负.不妨设$f''(x)$恒正,于是f下凸, 曲线严格在连结$(a,f(a))=(a,0)(b,f(b))=(b,0)$的弦下方,故$f(x)<0$, $x\in(a,b)$.

6.设$f(x)$在$[a,b]$上有二阶导数,且$f(a)=f(b)=0$,又存在$c\in(a,b)$使$f(c)>0$.证明:在(a,b)内至少存在一点x_0使$f''(x_0)<0$.

证一 由公式,存在$c_1\in(a,c)$,满足$f'(c_1)=\dfrac{f(c)-f(a)}{c-a}=\dfrac{f(c)}{c-a}>0$,

存在$c_2\in(c,b)$,满足$f'(c_2)=\dfrac{f(b)-f(c)}{b-c}=\dfrac{-f(c)}{c-a}<0$.

对于$f'(x)$在$[c_1,c_2]$应用Lagrange公式, 存在$x_0\in(c_1,c_2)$,使得

$$f''(x_0)=\frac{f'(c_2)-f'(c_1)}{c_2-c_1}<0.$$

证二 若不然, $f''(x)\geq 0, x\in(a,b)$, f在$[a,b]$下凸,曲线在连结$(a,f(a))=(a,0)$ $(b,f(b))=(b,0)$的弦下方,故$f(x)\leq 0, x\in(a,b)$.

7.证明方程$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=\dfrac{a_0}{n+1}+\dfrac{a_1}{n}+\dfrac{a_2}{n-1}+\cdots+\dfrac{a_{n-1}}{2}+\dfrac{a_n}{1}$

在0与1之间有一个根.

证 考虑函数

$$f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\frac{a_2x^{n-1}}{n-1}+\cdots+\frac{a_nx}{1}-\left(\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}\right)x,$$

$$f'(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n-\left(\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}\right)$$

$f(0)=f(1)=0$.由*Rolle*定理,存在$c\in(0,1)$, $f'(c)=0$,即c是

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}$$

在0与1之间的一个根.

8.设函数$f(x)$在有限区间(a,b)内可导,但无界,证明$f'(x)$在(a, b)内也无界.
逆命题是否成立?试举例说明.

证若不然,设f'(x)在(a, b)内有界M, 取定$x_0\in(a,b)$, 则对于任意 $x\in(a,b)$,
根据 Lagrange 公式,$f(x)-f(x_0)=f'(c)(x-x_0)$,
$|f(x)|=|f(x_0)+f'(c)(x-x_0)|\le|f(x_0)|+|f'(c)||(x-x_0)|\le|f(x_0)|+|M|(b-a)$.
逆命题不成立. 例如$\sqrt{x}$在(0,1)内有界,$0<\sqrt{x}<1$,但是$\sqrt{x}'=\dfrac{1}{2\sqrt{x}}$在(0,1)内无界.

9.若函数$f(x)$在区间$[a,b]$上有n个根(一个k重根算作k个根),且存在$f^{(n-1)}(x)$,
证明$f^{(n-1)}(x)$在$[a,b]$至少有一个根.(注意:若$f(x)$可以表示成$f(x)=(x-x_0)^k g(x)$且
$g(x_0)\ne0$,则称x_0为$f(x)$的k重根).

证我们对于n作归纳法证明.函数$f(x)$在区间$[a,b]$上有2个根. 如果x_0是2重根,则
$f(x)=(x-x_0)^2g(x)$且$g(x_0)\ne0$,则$f'(x)=2(x-x_0)g(x)+(x-x_0)^2g'(x)$, $f'(x)$有根x_0.
如果$f(x)$在区间$[a,b]$上有2个不同的根$x_1,x_2,x_1<x_2$,在$[x_1,x_2]$应用Rolle定理,存在
$x_0\in(x_1,x_2)$,使得$f'(x_0)=0$.设结论对于n个根的情况成立.现在假定$f(x)$在区间$[a,b]$
上有$n+1$个根. 如果f有$n+1$重根重根x_0,则
$f(x)=(x-x_0)^{n+1}g(x)$且$g(x_0)\ne0$,则
$f'(x)=(n+1)(x-x_0)^n g(x)+(x-x_0)^{n+1}g'(x)=(x-x_0)^n((n+1)g(x)+(x-x_0)g(x))$,
$(n+1)g(x)+(x-x_0)g(x)=g_1(x),g_1(x_0)=(n+1)g(x_0)\ne0$, $f'(x)$有n重根x_0.
如果如果f有$n+1$个单重根$x_1,\ldots x_{n+1}$,在区间$[x_1,x_2],\ldots,[x_n,x_{n+1}]$上应用Rolle定理,
存在$c_1\in(x_1,x_2),\ldots,c_n\in(x_n,x_{n+1}])$使得$f'(c_1)=\cdots=f'(c_n)=0$, $f'(x)$至少有n个根.
如果f有不同的根$x_1,\ldots,x_k$,重数分别为$n_1,\ldots,n_k,n+1>k>1,\sum_{i=1}^k n_i=n+1$.在$[x_1,x_2]$,
$\ldots,[x_{k-1},x_k]$上应用Rolle定理,存在$c_1\in(x_1,x_2),\ldots,c_{k-1}\in(x_{k-1},x_k)$使得
$f'(c_1)=\cdots=f'(c_{k-1})=0$. $f'(x)$至少有根$k-1+\sum_{i=1}^k(n_i-1)=n$个. 对f'(x)用归纳假设,
$(f'(x))^{(n)}=f^{(n+1)}(x)$至少有一个根.

10.证明:Lerendre多项式$P_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)]^n$在$(-1,1)$内有$n$个根.

证$f(x)=\dfrac{1}{2^n n!}(x^2-1)]^n$, $f(1)=f(-1)=0$,对于f在$[-1.1]$应用Rolle定理，存在
$c_1^1\in(-1,1)$,使得$f'(c_1^1)=0$. $f'(-1)=f'(1)=0$(当$n>1$时),对于f'在（-1，c_1^1）
$(c_1^1,1)$应用Rolle定理，存在
$c_1^2\in$（-1，c_1^1）,$c_2^2\in$（$c_1^1,1$)使得$f'(c_1^2)=f'(c_2^2)=0$.如此下去，$f^{(n-1)}(x)$在
$(-1,1)$有零点$c_1^{n-1},\ldots,c_{n-1}^{n-1}$，$f^{(n-1)}(-1)=f^{(n-1)}(1)=0$，在（$-1,c_1^{n-1}$），($c_1^{n-1},c_2^{n-1}$)，
$\ldots$，(c_{n-1}^{n-1}, 1)应用Rolle定理，得到$x_1,x_2,\ldots,x_n\in(-1,1)$使得$f^{(n)}(x)=P_n(x)=0$.
$P_n(x)$是n次多项式，至多有n个零点，故$P_n(x)$恰有n个零点.

11.设函数f在$(-\infty,+\infty)$内可导,且$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)$.证明:必存在一点$c\in(-\infty,+\infty)$,使得$f'(c)=0$.

证若$f(x)\equiv\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)=A.x\in(-\infty,+\infty)$,取任意一点$c\in(-\infty,+\infty)$,都有$f'(c)=0$.

设存在$f(x_0)\neq A$,不妨设$f(x_0)>A$.根据极限不等式,存在a, b, 满足:$a<b$,
$x_0\in(a,b)$, $f(a)<f(x_0)$, $f(b)<f(x_0)$. f在$[a,b]$连续,必在一点$c\in[a,b]$取最大值. $f(c)\geq f(x_0)>f(a)$, $f(c)\geq f(x_0)>f(b)$, 故$x_0\in(a,b)$, x_0为极大值点,根据Fermat引理, $f'(c)=0$.

12. 设函数$f(x)$在无穷区间$(x_0,+\infty)$可导, 且$\lim\limits_{x\to+\infty} f'(x)=0$, 证明$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=0$.

证由于$\lim\limits_{x\to+\infty} f'(x)=0$, 根据极限定义, 存在正数$x_1>x_0$,使得$x>x_1$时$|f'(x)|<\varepsilon$.

$$\left|\frac{f(x)}{x}\right|=\left|\frac{f(x)-f(x_1)+f(x_1)}{x}\right|=\left|\frac{f'(c)(x-x_1)+f(x_1)}{x}\right|\leq\frac{\varepsilon(x-x_1)+|f(x_1)|}{x}$$

$<\varepsilon+\dfrac{|f(x_1)|}{x}$.为使$\dfrac{|f(x_1)|}{x}<\varepsilon$,只需$x>\dfrac{|f(x_1)|}{\varepsilon}$.令$X=\max\{x_1,\dfrac{|f(x_1)|}{\varepsilon}\}$,

当$x>X$时,必有$\left|\dfrac{f(x)}{x}\right|<2\varepsilon$,故$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=0$.

13.设函数$f(x)$在无穷区间$[a,+\infty)$内连续,且当$x>a$时$f'(x)>l>0$,其中1为常数. 证明:若$f(a)<0$,则在区间$\left(a,a-\dfrac{f(a)}{l}\right)$内方程$f(x)=0$有唯一实根.

证$f(a)<0$,

$$\left(a-\frac{f(a)}{l}\right)=f(a)+f'(c)\left(-\frac{f(a)}{l}\right)>f(a)+l\left(-\frac{f(a)}{l}\right)=0,$$

f在$\left[a,a-\dfrac{f(a)}{l}\right]$连续,由连续怀念书函数的中间值定理,

在区间$\left(a,a-\dfrac{f(a)}{l}\right)$内方程$f(x)=0$至少有一实根.若有两个实根,根据Rolle定理, $f'(x)$将在$\left(a,a-\dfrac{f(a)}{l}\right)$有一零点,这与条件$f'(x)>l>0$矛盾.

14.设函数$f(x)$在$(-\infty,+\infty)$上可导，且$\lim\limits_{x\to\infty} f'(x)=0$.现令$g(x)=f(x+1)-f(x)$,证明$\lim\limits_{x\to\infty} g(x)=0$.

证$\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}(f(x+1)-f(x))=\lim\limits_{x\to\infty} f'(x+\theta)(0<\theta<1)=0$.

15.称函数$f(x)$在$[a,b]$满足Lipschiz条件,若存在常数$L>0$,使对于任意$x_1,x_2\in[a,b]$,都有$|f(x_1)-f(x_2)|\le L|x_1-x_2|$.

(1)若$f'(x)$在$[a,b]$连续,则$f(x)$在$[a,b]$满足Lipschiz条件

(2)(1)中所述事实的逆命题是否成立?

(3)举一个在$[a,b]$上连续但不满足Lipschiz条件的函数.

解(1) $f'(x)$在$[a,b]$连续, 存在常数$L>0$,使得$|f'(x)|\le L.x\in[a,b]$.

根据中值公式, 对于任意$x_1,x_2\in[a,b],x_1<x_2$,存在$c\in[x_1,x_2]$,使得

$|f(x_1)-f(x_2)|=|f'(c)(x_1-x_2)|=|f'(c)|(x_2-x_1)\le L(x_2-x_1)$.

(2)否. $f(x)$在$[a,b]$满足Lipschiz条件, 未必处处可导, 更谈不到$f'(x)$在$[a,b]$连续. 例如, $f(x)=|x|$在 $[-1,1]$满足Lipschiz条件, 但在0不可导.

(3) $f(x)=\sqrt{x}$在$[0,1]$连续,但不满足Lipschiz条件, 因其导函数

$f'(x)=\dfrac{1}{2\sqrt{x}}$在$(0,1]$无界.

16.设$F(x)$在$[a,b]$可导,且其导函数$F'(x)=f(x)$在$[a,b]$上可积,证明

$\int_a^b f(x)dx=F(b)-F(a)$.

证$F(b)-F(a)=\sum_{i=1}^{n}(F(x_i)-F(x_{i-1}))=\sum_{i=1}^{n}F'(\xi_i)(x_i-x_{i-1})$

$\sum_{i=1}^{n}f(\xi_i)(x_i-x_{i-1})\to\int_a^b f(x)dx(\lambda(\Delta)\to 0)$.

$\{x_i\}$为$[a,b]$的分割.

17.设多项式$P(x)-a$与$P(x)-b$的全部根都是单实根,证明对于任意实数$c\in(a,b)$,多项式$P(x)-c$的根也全都是单实根.

证不妨设a=0, b>0, c∈(0, b), $P(x)$是n次多项式, 且首项系数为正.

$P(x)$有单实根$x_1<\cdots<x_n$,则这些根把实轴分为$n+1$个区间,每个区间保持固定正负号,且正负相间. 否则某个根将为极值点, 导数为零, 此与单实根矛盾. 在两个无穷区间保持正号, 且严格单调递增或递减, 在每个有穷区间有一个最值点, 且在其两侧分别递增和递减, 设$n=2k$为偶数, 则$\lim_{x\to\infty}P(x)=+\infty$.设$b>0$且$P(x)=b$有n个单实根$x_1'<\cdots<x_n'$. 必有

$x_1'<x_1,x_2',x_3'\in(x_2,x_3),\cdots,x_{2k-2}',x_{2k-1}'\in(x_{2k-2},x_{2k-1}),x_{2k}'\in(x_{2k},+\infty),P(x_i')=b$.

根据连续函数的中间值定理,对于$c\in(0,b)$,存在$c_1\in(-\infty,x_1),c_2\in(x_2,x_2')$,

$c_3\in(x_3',x_3),c_{2k-2}\in(x_{2k-2},x_{2k-2}'),c_{2k-1}\in(x_{2k-1}',x_{2k-1}+\infty),c_{2k}\in(x_{2k},+\infty)$,

使得$P(c_i)=c.P$为n次多项式,c_i是P(x)=c的所有单实根.

第四章总练习题

1.设y=f(x)在$[x_0-h, x_0+h]$(h>0)内可导.证明存在θ, 0<θ<1使得

$f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$

证令g(x)=$f(x_0+x)-f(x_0-x), x\in[0,h]$.$g$(x)在[0, h]内可导,

$g'(x)=f'(x_0+x)+f'(x_0-x), g(0)=0.$

根据Lagrange公式,存在$\theta\in(0,1)$使得

$g(h)-g(0)=g'(\theta h)h$, 即$f(x_0+h)-f(x_0-h)=[f'(x_0+\theta h)+f'(x_0-\theta h)]h.$

2.证明:当$x\geq 0$时,等式$\sqrt{x+1}-\sqrt{x}=\dfrac{1}{2\sqrt{x+\theta(x)}}$

中的$\theta(x)$满足$1/4\leq\theta(x)\leq 1/2$且$\lim\limits_{x\to 0}\theta(x)=1/4$, $\lim\limits_{x\to+\infty}\theta(x)=1/2.$

证$\sqrt{x+1}-\sqrt{x}=\dfrac{1}{2\sqrt{x+\theta(x)}}$, $2\sqrt{x+\theta(x)}=\dfrac{1}{\sqrt{x+1}-\sqrt{x}}=\sqrt{x+1}+\sqrt{x}$,

$4(x+\theta(x))=2x+1+2\sqrt{x(x+1)}$,

$\theta(x)=\dfrac{1}{4}(1+2\sqrt{x(x+1)}-2x).$

$\theta(x)\geq\dfrac{1}{4}(1+2\sqrt{x(x)}-2x)=\dfrac{1}{4}$,

由算术$-$几何平均不等式得

$\theta(x)=\dfrac{1}{4}(1+2\sqrt{x(x+1)}-2x)\leq\dfrac{1}{4}(1+(x+x+1)-2x)=\dfrac{1}{2}.$

$\lim\limits_{x\to 0}\theta(x)=\lim\limits_{x\to 0}\dfrac{1}{4}(1+2\sqrt{x(x+1)}-2x)=\dfrac{1}{4}.$

$\lim\limits_{x\to+\infty}\theta(x)=\lim\limits_{x\to+\infty}\dfrac{1}{4}(1+2\sqrt{x(x+1)}-2x)$

$=\dfrac{1}{4}\lim\limits_{x\to+\infty}\dfrac{(1+2\sqrt{x(x+1)}-2x)(1+2\sqrt{x(x+1)}+2x)}{(1+2\sqrt{x(x+1)}+2x)}$

$=\dfrac{1}{4}\lim\limits_{x\to+\infty}\dfrac{1+4x+4\sqrt{x(x+1)}}{(1+2\sqrt{x(x+1)}+2x)}==\dfrac{1}{4}\lim\limits_{x\to+\infty}\dfrac{1/x+4+4\sqrt{1+1/x}}{(1/x+2\sqrt{(1/x+1)}+2)}=\dfrac{1}{2}.$

3.设$f(x)=\begin{cases}\dfrac{3-x^2}{2}, 0\leq x\leq 1\\ \dfrac{1}{x}, \quad 1<x<+\infty\end{cases}$求$f(x)$在闭区间$[0,2]$上的微分中值定理的中间值.

解$f'(x)=\begin{cases}-x, 0\leq x\leq 1\\ -\dfrac{1}{x^2}, \quad 1<x<+\infty\end{cases}$.$\dfrac{f(2)-f(0)}{2-0}=\dfrac{1/2-3/2}{2}=-\dfrac{1}{2}.$

$-x=-\dfrac{1}{2}, x=\dfrac{1}{2}; -\dfrac{1}{x^2}=-\dfrac{1}{2}, x=\sqrt{2}$. $f(x)$在闭区间$[0,2]$上的微分中值定理的中间值为$\dfrac{1}{2}$或$\sqrt{2}$.

4.在闭区间$[-1,1]$上Cauchy中值定理对于函数$f(x)=x^2$与$g(x)=x^3$是否成立?并说明理由.

解 由于$g'(x)=3x^2$有零点$0\in(-1,1)$, Cauchy中值定理的条件不满足. 其实其结论也不成立.

因为若$\dfrac{f(1)-f(-1)}{g(1)-g(-1)}=0=\dfrac{f'(c)}{g'(c)}$, $f'(c)=2c=0$, $c=0$, 但$g'(0)=0$, $\dfrac{f'(c)}{g'(c)}$无意义.

5.设$f(x)$在$[a,b]$上连续, 在(a,b)上有二阶导数, 且$f''(x)\neq 0$, $x\in(a,b)$又$f(a)=f(b)=0$, 证明当$x\in(a,b)$时$f(x)\neq 0$.

证一 若存在$c\in(a,b)$, $f(c)=0$, 则由Rolle定理, 存在$c_1\in(a,c)$, $c_2\in(c,b)$使得$f'(c_1)=f'(c_2)=0$.

对于$f'(x)$在$[c_1,c_2]$应用定理, 存在$\xi\in(c_1,c_2)$, 使得$f''(\xi)=0$, 此与条件$f''(x)\neq 0$, $x\in(a,b)$矛盾.

证二 由假设, $f''(x)\neq 0$, $x\in(a,b)$, 根据*Darboux*定理, $f''(x)$恒正或恒负.不妨设$f''(x)$恒正, 于是f下凸, 曲线严格在连结$(a,f(a))=(a,0)(b,f(b))=(b,0)$的弦下方, 故$f(x)<0$, $x\in(a,b)$.

6.设$f(x)$在$[a,b]$上有二阶导数, 且$f(a)=f(b)=0$, 又存在$c\in(a,b)$使$f(c)>0$.证明: 在(a,b)内至少存在一点x_0使$f''(x_0)<0$.

证一 由公式, 存在$c_1\in(a,c)$, 满足$f'(c_1)=\dfrac{f(c)-f(a)}{c-a}=\dfrac{f(c)}{c-a}>0$,

存在$c_2\in(c,b)$, 满足$f'(c_2)=\dfrac{f(b)-f(c)}{b-c}=\dfrac{-f(c)}{c-a}<0$.

对于$f'(x)$在$[c_1,c_2]$应用Lagrange公式, 存在$x_0\in(c_1,c_2)$, 使得

$$f''(x_0)=\frac{f'(c_2)-f'(c_1)}{c_2-c_1}<0.$$

证二 若不然, $f''(x)\geq 0$, $x\in(a,b)$, f在$[a,b]$下凸, 曲线在连结$(a,f(a))=(a,0)$ $(b,f(b))=(b,0)$的弦下方, 故$f(x)\leq 0$, $x\in(a,b)$.

7.证明方程$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=\dfrac{a_0}{n+1}+\dfrac{a_1}{n}+\dfrac{a_2}{n-1}+\cdots+\dfrac{a_{n-1}}{2}+\dfrac{a_n}{1}$ 在0与1之间有一个根.

证 考虑函数

$$f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\frac{a_2x^{n-1}}{n-1}+\cdots+\frac{a_nx}{1}-\left(\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}\right)x,$$

$$f'(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n-\left(\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}\right)$$

$f(0)=f(1)=0$.由*Rolle*定理, 存在$c\in(0,1)$, $f'(c)=0$, 即c是

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+\frac{a_n}{1}$$

在0与1之间的一个根.

8.设函数$f(x)$在有限区间(a,b)内可导,但无界,证明$f'(x)$在(a, b)内也无界.
逆命题是否成立?试举例说明.

证若不然,设f'(x)在(a, b)内有界M, 取定$x_0 \in (a, b)$, 则对于任意 $x \in (a, b)$,
根据 Lagrange 公式,$f(x)-f(x_0)=f'(c)(x-x_0)$,
$|f(x)|=|f(x_0)+f'(c)(x-x_0)|\le|f(x_0)|+|f'(c)||(x-x_0)|\le|f(x_0)|+|M|(b-a)$.
逆命题不成立. 例如$\sqrt{x}$在$(0,1)$内有界,$0<\sqrt{x}<1$,但是$\sqrt{x}'=\dfrac{1}{2\sqrt{x}}$在$(0,1)$内无界.

9.若函数$f(x)$在区间$[a,b]$上有n个根(一个k重根算作k个根),且存在$f^{(n-1)}(x)$,
证明$f^{(n-1)}(x)$在$[a,b]$至少有一个根.(注意:若$f(x)$可以表示成$f(x)=(x-x_0)^k g(x)$且
$g(x_0)\ne 0$,则称x_0为$f(x)$的k重根).

证我们对于n作归纳法证明.函数$f(x)$在区间$[a,b]$上有2个根. 如果x_0是2重根,则
$f(x)=(x-x_0)^2 g(x)$且$g(x_0)\ne 0$,则$f'(x)=2(x-x_0)g(x)+(x-x_0)^2 g'(x)$, $f'(x)$有根x_0.
如果$f(x)$在区间$[a,b]$上有2个不同的根$x_1,x_2,x_1<x_2$,在$[x_1,x_2]$应用Rolle定理,存在
$x_0\in(x_1,x_2)$,使得$f'(x_0)=0$.设结论对于n个根的情况成立.现在假定$f(x)$在区间$[a,b]$
上有$n+1$个根. 如果f有$n+1$重根重根x_0,则
$f(x)=(x-x_0)^{n+1}g(x)$且$g(x_0)\ne 0$,则
$f'(x)=(n+1)(x-x_0)^n g(x)+(x-x_0)^{n+1}g'(x)=(x-x_0)^n((n+1)g(x)+(x-x_0)g(x))$,
$(n+1)g(x)+(x-x_0)g(x)=g_1(x), g_1(x_0)=(n+1)g(x_0)\ne 0$, $f'(x)$有n重根x_0.
如果如果f有$n+1$个单重根$x_1,\ldots x_{n+1}$,在区间$[x_1,x_2],\ldots,[x_n,x_{n+1}]$上应用Rolle定理,
存在$c_1\in(x_1,x_2),\ldots,c_n\in(x_n,x_{n+1}])$使得$f'(c_1)=\cdots=f'(c_n)=0$, $f'(x)$至少有n个根.
如果f有不同的根$x_1,\ldots,x_k$,重数分别为$n_1,\ldots,n_k,n+1>k>1,\sum_{i=1}^{k}n_i=n+1$.在$[x_1,x_2]$,
$\ldots,[x_{k-1},x_k]$上应用Rolle定理,存在$c_1\in(x_1,x_2),\ldots,c_{k-1}\in(x_{k-1},x_k)$使得
$f'(c_1)=\cdots=f'(c_{k-1})=0$. $f'(x)$至少有根$k-1+\sum_{i=1}^{k}(n_i-1)=n$个. 对f'(x)用归纳假设,
$(f'(x))^{(n)}=f^{(n+1)}(x)$至少有一个根.

10.证明:Lerendre多项式$P_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)]^n$在$(-1,1)$内有$n$个根.

证$f(x)=\dfrac{1}{2^n n!}(x^2-1)]^n$, $f(1)=f(-1)=0$,对于f在$[-1.1]$应用Rolle定理，存在
$c_1^1\in(-1,1)$,使得$f'(c_1^1)=0$. $f'(-1)=f'(1)=0$(当$n>1$时),对于f'在（-1，c_1^1）
$(c_1^1,1)$应用Rolle定理，存在
$c_1^2\in$（-1，c_1^1）,$c_2^2\in$（$c_1^1,1$）使得$f'(c_1^2)=f'(c_2^2)=0$.如此下去，$f^{(n-1)}(x)$在
$(-1,1)$有零点$c_1^{n-1},\ldots,c_{n-1}^{n-1}$，$f^{(n-1)}(-1)=f^{(n-1)}(1)=0$，在（$-1,c_1^{n-1}$），（$c_1^{n-1},c_2^{n-1}$），
$\ldots$，（$c_{n-1}^{n-1},1$）应用Rolle定理，得到$x_1,x_2,\ldots,x_n\in(-1,1)$使得$f^{(n)}(x)=P_n(x)=0$.
$P_n(x)$是n次多项式，至多有n个零点，故$P_n(x)$恰有n个零点.

11.设函数f在$(-\infty,+\infty)$内可导,且$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)$.证明:必存在一点$c\in(-\infty,+\infty)$,使得$f'(c)=0$.

证若$f(x)\equiv\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)=A.x\in(-\infty,+\infty)$,取任意一点$c\in(-\infty,+\infty)$,都有$f'(c)=0$.

设存在$f(x_0)\neq A$,不妨设$f(x_0)>A$.根据极限不等式,存在a, b,满足:$a<b$, $x_0\in(a,b)$, $f(a)<f(x_0)$, $f(b)<f(x_0)$. f在$[a,b]$连续,必在一点$c\in[a,b]$取最大值. $f(c)\geq f(x_0)>f(a)$, $f(c)\geq f(x_0)>f(b)$, 故$x_0\in(a,b)$, x_0为极大值点,根据Fermat引理, $f'(c)=0$.

12. 设函数$f(x)$在无穷区间$(x_0,+\infty)$可导,且$\lim\limits_{x\to+\infty} f'(x)=0$,证明$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=0$.

证由于$\lim\limits_{x\to+\infty} f'(x)=0$,根据极限定义,存在正数$x_1>x_0$,使得$x>x_1$时$|f'(x)|<\varepsilon$.

$$\left|\frac{f(x)}{x}\right|=\left|\frac{f(x)-f(x_1)+f(x_1)}{x}\right|=\left|\frac{f'(c)(x-x_1)+f(x_1)}{x}\right|\leq\frac{\varepsilon(x-x_1)+|f(x_1)|}{x}$$

$<\varepsilon+\dfrac{|f(x_1)|}{x}$.为使$\dfrac{|f(x_1)|}{x}<\varepsilon$,只需$x>\dfrac{|f(x_1)|}{\varepsilon}$.令$X=\max\{x_1,\dfrac{|f(x_1)|}{\varepsilon}\}$,

当$x>X$时,必有$\left|\dfrac{f(x)}{x}\right|<2\varepsilon$,故$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=0$.

13.设函数$f(x)$在无穷区间$[a,+\infty)$内连续,且当$x>a$时$f'(x)>l>0$,其中l为常数. 证明:若$f(a)<0$,则在区间$\left(a,a-\dfrac{f(a)}{l}\right)$内方程$f(x)=0$有唯一实根.

证$f(a)<0$,

$$\left(a-\frac{f(a)}{l}\right)=f(a)+f'(c)\left(-\frac{f(a)}{l}\right)>f(a)+l\left(-\frac{f(a)}{l}\right)=0,$$

f在$\left[a,a-\dfrac{f(a)}{l}\right]$连续,由连续怀念书函数的中间值定理,

在区间$\left(a,a-\dfrac{f(a)}{l}\right)$内方程$f(x)=0$至少有一实根.若有两个实根,根据Rolle定理, $f'(x)$将在$\left(a,a-\dfrac{f(a)}{l}\right)$有一零点,这与条件$f'(x)>l>0$矛盾.

14.设函数$f(x)$在$(-\infty,+\infty)$上可导,且$\lim\limits_{x\to\infty} f'(x)=0$.现令$g(x)=f(x+1)-f(x)$,证明$\lim\limits_{x\to\infty} g(x)=0$.

证$\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}(f(x+1)-f(x))=\lim\limits_{x\to\infty} f'(x+\theta)(0<\theta<1)=0$.

15.称函数$f(x)$在$[a,b]$满足Lipschiz条件,若存在常数$L>0$,使对于任意$x_1,x_2\in[a,b]$,都有$|f(x_1)-f(x_2)|\le L|x_1-x_2|$.

(1)若$f'(x)$在$[a,b]$连续,则$f(x)$在$[a,b]$满足Lipschiz条件

(2)(1)中所述事实的逆命题是否成立?

(3)举一个在$[a,b]$上连续但不满足Lipschiz条件的函数.

解(1) $f'(x)$在$[a,b]$连续, 存在常数$L>0$, 使得$|f'(x)|\le L. x\in[a,b]$.

根据中值公式, 对于任意$x_1,x_2\in[a,b], x_1<x_2$, 存在$c\in[x_1,x_2]$, 使得

$|f(x_1)-f(x_2)|=|f'(c)(x_1-x_2)|=|f'(c)|(x_2-x_1)\le L(x_2-x_1)$.

(2)否. $f(x)$在$[a,b]$满足Lipschiz条件, 未必处处可导, 更谈不到$f'(x)$在$[a,b]$连续. 例如, $f(x)=|x|$在 $[-1,1]$满足Lipschiz条件, 但在0不可导.

(3) $f(x)=\sqrt{x}$在$[0,1]$连续,但不满足Lipschiz条件, 因其导函数

$f'(x)=\dfrac{1}{2\sqrt{x}}$在$(0,1]$无界.

16.设$F(x)$在$[a,b]$可导,且其导函数$F'(x)=f(x)$在$[a,b]$上可积,证明

$\int_a^b f(x)dx=F(b)-F(a)$.

证$F(b)-F(a)=\sum_{i=1}^{n}(F(x_i)-F(x_{i-1}))=\sum_{i=1}^{n}F'(\xi_i)(x_i-x_{i-1})$

$\sum_{i=1}^{n}f(\xi_i)(x_i-x_{i-1})\to\int_a^b f(x)dx(\lambda(\Delta)\to 0)$.

$\{x_i\}$为$[a,b]$的分割.

17.设多项式$P(x)-a$与$P(x)-b$的全部根都是单实根,证明对于任意实数$c\in(a,b)$,多项式$P(x)-c$的根也全都是单实根.

证不妨设a=0, b>0, c∈(0, b)，$P(x)$是n次多项式, 且首项系数为正.

$P(x)$有单实根$x_1<\cdots<x_n$,则这些根把实轴分为$n+1$个区间,每个区间保持固定正负号,且正负相间. 否则某个根将为极值点, 导数为零, 此与单实根矛盾. 在两个无穷区间保持正号, 且严格单调递增或递减, 在每个有穷区间有一个最值点, 且在其两侧分别递增和递减, 设$n=2k$为偶数,

则$\lim\limits_{x\to\infty}P(x)=+\infty$.设$b>0$且$P(x)=b$有n个单实根$x_1'<\cdots<x_n'$. 必有

$x_1'<x_1, x_2', x_3'\in(x_2,x_3), \cdots, x_{2k-2}', x_{2k-1}'\in(x_{2k-2},x_{2k-1}), x_{2k}'\in(x_{2k},+\infty), P(x_i')=b$.

根据连续函数的中间值定理,对于$c\in(0,b)$,存在$c_1\in(-\infty,x_1), c_2\in(x_2,x_2')$,

$c_3\in(x_3',x_3), c_{2k-2}\in(x_{2k-2},x_{2k-2}'), c_{2k-1}\in(x_{2k-1}',x_{2k-1}+\infty), c_{2k}\in(x_{2k},+\infty)$,

使得$P(c_i)=c. P$为n次多项式, c_i是P(x)=c的所有单实根.

习题 5.1

1.设$ABCD$为一平行四边形,$\overrightarrow{AB}=\boldsymbol{a},\overrightarrow{AD}=\boldsymbol{b}$.试用$\boldsymbol{a},\boldsymbol{b}$表示$\overrightarrow{AC},\overrightarrow{DB},\overrightarrow{MA}$($M$为平行四边形对角线的交点).

解$\overrightarrow{AC}=\boldsymbol{a}+\boldsymbol{b},\overrightarrow{DB}=\boldsymbol{a}-\boldsymbol{b},\overrightarrow{MA}=-\overrightarrow{AM}=-\frac{1}{2}\overrightarrow{AC}=-\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

2.设M为线段$\overrightarrow{AB}$的中点,O为空间中的任意一点,证明

$\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$.

证$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}=\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$

$=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$.

3.设M为三角形ABC的重心,O为空间中任意一点,

证明$\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$.

证$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AD}=\overrightarrow{OA}+\frac{2}{3}\times\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$

$=\overrightarrow{OA}+\frac{1}{3}(\overrightarrow{AB}+\overrightarrow{AC})$,

$\overrightarrow{OM}=\overrightarrow{OB}+\frac{1}{3}(\overrightarrow{BA}+\overrightarrow{BC}),\overrightarrow{OM}=\overrightarrow{OC}+\frac{1}{3}(\overrightarrow{CA}+\overrightarrow{CB})$.

$3\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC},\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$.

4.设平行四边形$ABCD$的对角线交点为M,O为空间中的

任意一点,证明$\overrightarrow{OM}=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$.

证$\overrightarrow{OM}=OA+AM=OA+\frac{1}{2}(AB+AD)$,

$\overrightarrow{OM}=OB+\frac{1}{2}(BA+AD),\overrightarrow{OM}=OC+\frac{1}{2}(BA+DA)$,

$\overrightarrow{OM}=OD+\frac{1}{2}(AB+DA)$.

$4\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD},\overrightarrow{OM}=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$.

5.对于任意三个向量$\boldsymbol{a},\boldsymbol{b}$与$\boldsymbol{c}$,判断下列各式是否成立？

(1)$(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$;

(2)$(\boldsymbol{a}\cdot\boldsymbol{b})^2=\boldsymbol{a}^2\cdot\boldsymbol{b}^2$;

(3)$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}$.

解(1)不成立.例如: $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{i},\ \boldsymbol{c}=\boldsymbol{j}.(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=\boldsymbol{j},(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}=\boldsymbol{0}$.

(2)不成立.例如: $\boldsymbol{a}=\boldsymbol{i},\ \boldsymbol{b}=\boldsymbol{j},(\boldsymbol{a}\cdot\boldsymbol{b})^2=0,\boldsymbol{a}^2\cdot\boldsymbol{b}^2=1$.

(3)成立,都是$\boldsymbol{a},\boldsymbol{b}$与$\boldsymbol{c}$组成的平行六面体的有向体积.

6.利用向量证明三角形两边中点的连线平行于第三边,
并且等于第三边长度之半.

证 $DE=DA+AE=\frac{1}{2}BA+\frac{1}{2}AC$

$=\frac{1}{2}(BA+AC)=\frac{1}{2}BC.$

7.利用向量证明:

(1)菱形的对角线互相垂直,且平分顶角;(2)勾股弦定理.

证(1)$\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BC}+\overrightarrow{CD})$

$=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BC}-\overrightarrow{CD})=|\overrightarrow{BC}|^2-|\overrightarrow{CD}|^2=0.$

$$\cos\alpha=\frac{AB\cdot AC}{|AB||AC|}\frac{AB\cdot(AB+AD)}{|AB||AC|}=\frac{AB\cdot AB+AB\cdot AD}{|AB||AC|}=\frac{a^2+AB\cdot AD}{|a|AC|},$$

$$\cos\beta=\frac{AD\cdot(AB+AD)}{|AB||AC|}=\frac{AD\cdot AB+AD\cdot AD}{|AB||AC|}=\frac{a^2+AB\cdot AD}{|a|AC|}=\cos\alpha.$$

α与β都是锐角,故$\alpha=\beta$.

(2)$|AC|^2=\overrightarrow{AC}\cdot\overrightarrow{AC}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{AB}+\overrightarrow{BC})$

$=|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2+2\overrightarrow{AB}\cdot\overrightarrow{BC}=|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2$.

8.证明恒等式$(\boldsymbol{a}\times\boldsymbol{b})^2+(\boldsymbol{a}\cdot\boldsymbol{b})^2=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2$.

证$(\boldsymbol{a}\times\boldsymbol{b})^2+(\boldsymbol{a}\cdot\boldsymbol{b})^2=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2\cos^2\alpha+|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2\sin^2\alpha$

$=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2(\cos^2\alpha+\sin^2\alpha)=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2$.

9.试用向量$\overrightarrow{AB}$与$\overrightarrow{AC}$表示三角形ABC的面积.

解ΔABC的面积$=\frac{1}{2}$▱$ABDC$的面积$=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

10.给定向量$\boldsymbol{a}$,记$\boldsymbol{a}\cdot\boldsymbol{a}$为$\boldsymbol{a}^2$,即$\boldsymbol{a}^2=\boldsymbol{a}\cdot\boldsymbol{a}$.现设$\boldsymbol{a},\boldsymbol{b}$为任意向量,证明:

$(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2=2(\boldsymbol{a}^2+\boldsymbol{b}^2)$.

证$(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})+(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})$

$=\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}+2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}-2\boldsymbol{a}\cdot\boldsymbol{b}=2(\boldsymbol{a}^2+\boldsymbol{b}^2)$.

11.对于任意向量$\boldsymbol{a},\boldsymbol{b}$,证明:$(\boldsymbol{a}\times\boldsymbol{b})^2\le\boldsymbol{a}^2\boldsymbol{b}^2$.问:等号成立的充分必要条件是什么?

证$(\boldsymbol{a}\times\boldsymbol{b})^2=|\boldsymbol{a}\times\boldsymbol{b}|^2=(|\boldsymbol{a}||\boldsymbol{b}|\sin\alpha)^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\alpha\le|\boldsymbol{a}|^2|\boldsymbol{b}|^2=\boldsymbol{a}^2\boldsymbol{b}^2$.

等号成立的充分必要条件是$\boldsymbol{a},\boldsymbol{b}$正交**.**

习题 5.2

1.写出点(x,y,z)分别到x轴,y轴,z轴,Oxy平面,Oyz平面以及原点的距离.

解$d_x=\sqrt{y^2+z^2},d_y=\sqrt{x^2+z^2},d_z=\sqrt{x^2+y^2},d_{xy}=|z|,d_{yz}=|x|,d_O=\sqrt{x^2+y^2+z^2}.$

2.已知三点$A=(-1,2,1),B=(3,0,1),C=(2,1,2)$,求$\overrightarrow{AB},\overrightarrow{BA},\overrightarrow{AC},\overrightarrow{BC}$的坐标与模.

解$\overrightarrow{AB}=(3,0,1)-(-1,2,1)=(4,-2,0),|\overrightarrow{AB}|=\sqrt{20}=2\sqrt{5},$

$|\overrightarrow{BA}=-\overrightarrow{AB}=-(4,-2,0)=(-4,2,0)=-2\sqrt{5},$

$\overrightarrow{AC}=(2,1,2)-(-1,2,1)=(3,-1,1),|\overrightarrow{AC}|=\sqrt{11},$

$\overrightarrow{BC}=(2,1,2)-(3,0,1)-=(-1,1,1),|\overrightarrow{BC}|=\sqrt{3}.$

3.$\boldsymbol{a}=(3,-2,2),\boldsymbol{b}=(1,3,2),\boldsymbol{c}=(8,6,-2),$

$3\boldsymbol{a}-2\boldsymbol{b}+\frac{1}{2}\boldsymbol{c}=(9,-6,6)+(-2,-6,-4)+(4,3,-1)=(11,-9,1).$

4.设$\boldsymbol{a}=(2,5,1),\boldsymbol{b}=(1,-2,7)$,分别求出沿$\boldsymbol{a}$和$\boldsymbol{b}$方向的单位向量,并求常数$k$,使$k\boldsymbol{a}+\boldsymbol{b}$与$xy$平面平行.

解$\boldsymbol{a}^\circ=\frac{1}{\sqrt{30}}(2,5,1),\boldsymbol{b}^\circ=\frac{1}{3\sqrt{6}}(1,-2,7).$

$k\boldsymbol{a}+\boldsymbol{b}=(2k,5k,k)+(1,-2,7)=(2k+1,5k,-2,k+7),k+7=0,k=-7.$

5.设A,B两点的坐标分别为(x_1,y_1,z_1)和(x_2,y_2,z_2),求A,B连线中点C的坐标.

解$\overrightarrow{OC}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})=\frac{1}{2}((x_1,y_1,z_1)+(x_2,y_2,z_2))=\frac{1}{2}(x_1+x_2,y_1+y_2,z_1+z_2).$

6.设$\boldsymbol{a}=(1,-2,3),\boldsymbol{b}=(5,2,-1)$,求

(1)$2\boldsymbol{a}\cdot 3\boldsymbol{b}$ (2)$\boldsymbol{a}\cdot\boldsymbol{i}$ (3)$\cos<\boldsymbol{a},\boldsymbol{b}>$.

解(1)$2\boldsymbol{a}\cdot 3\boldsymbol{b}=6\boldsymbol{a}\cdot\boldsymbol{b}=6\times(-2)=-12.$

(2)$\boldsymbol{a}\cdot\boldsymbol{i}=1$.

(3)$\cos<\boldsymbol{a},\boldsymbol{b}>=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\frac{-2}{\sqrt{14}\sqrt{30}}=-\frac{1}{\sqrt{105}},$

7.设$|\boldsymbol{a}|=1,|\boldsymbol{b}|=3,|\boldsymbol{c}|=2,|\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|=\sqrt{17+6\sqrt{3}}$且$\boldsymbol{a}\perp\boldsymbol{c}<\boldsymbol{a},\boldsymbol{b}>=\pi/3$,求$<\boldsymbol{b},\boldsymbol{c}>=?$

解$17+6\sqrt{3}=|\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|^2=(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\cdot(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

$=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+|\boldsymbol{c}|^2+2(\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{c})=$

$=1+9+4+2(3\times\frac{1}{2}+\boldsymbol{b}\cdot\boldsymbol{c}),$

$\boldsymbol{b}\cdot\boldsymbol{c}=3\sqrt{3},\cos<\boldsymbol{b},\boldsymbol{c}>=\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}||\boldsymbol{c}|}=\frac{3\sqrt{3}}{3\times 2}=\frac{\sqrt{3}}{2}.<\boldsymbol{b},\boldsymbol{c}>=\frac{\pi}{6}.$

8.设$|\boldsymbol{a}|=2,|\boldsymbol{b}|=6$,试求常数$k$,使$\boldsymbol{a}+k\boldsymbol{b}\perp\boldsymbol{a}-k\boldsymbol{b}$.

解$(\boldsymbol{a}+k\boldsymbol{b})\cdot(\boldsymbol{a}-k\boldsymbol{b})=|\boldsymbol{a}|^2-k^2|\boldsymbol{b}|^2=4-36k^2=0,k=\pm1/3.$

9.$\boldsymbol{a}=(1,-2,1),\boldsymbol{b}=(1,-1,3),\boldsymbol{c}=(2,5,-3)$

(1)$\boldsymbol{a}\times\boldsymbol{b}$ (2)$\boldsymbol{c}\times\boldsymbol{j}$ (3)$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ (4)$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ (5)$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$.

解(1)$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-2&1\\1&-1&3\end{vmatrix}=(-5,-2,1),$

(2)$\boldsymbol{c}\times\boldsymbol{j}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&5&-3\\0&1&0\end{vmatrix}=(3,0,2).$

(3)$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}1&-2&1\\1&-1&3\\2&5&-3\end{vmatrix}=-23.(4)(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-5&-2&1\\2&5&-3\end{vmatrix}=(1,-13,-21).$

(5)$\boldsymbol{b}\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-1&3\\2&5&-3\end{vmatrix}=(-12,9,7),\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-2&1\\-12&9&7\end{vmatrix}=(-23,-19,-15).$

10.在平行四边形$ABCD$中,$\overrightarrow{AB}=(2,1,0)\overrightarrow{AD}=(0,-1,2)$,求两对角线的夹角$<\overrightarrow{AC},\overrightarrow{BD}>$.

解$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}=(2,1,0)+(0,-1,2)=(2,0,2),$

$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}=(0,-1,2)-(2,1,0)=(-2,-2,2).$

$\cos<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}||\overrightarrow{BD}|}=\dfrac{0}{|\overrightarrow{AC}||\overrightarrow{BD}|}=0,<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\pi}{2}.$

解二$|\overrightarrow{AB}|=|\overrightarrow{AD}|=\sqrt{5}$,平行四边形$ABCD$为菱形,故两对角线的夹角$<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\pi}{2}$.

11.已知三点$A(3,4,1),B(2,3,0),C(3,5,1)$,求三角形$ABC$的面积.

解$\overrightarrow{AB}=(-1,-1,-1)=-(1,1,1),\overrightarrow{AC}=(0,1,0),\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&1&1\\0&1&0\end{vmatrix}=(-1,0,1),$

三角形ABC的面积$=\dfrac{1}{2}\times\sqrt{2}$.

12.证明向量$\boldsymbol{a}=(3,4,5),\boldsymbol{b}=(1,2,2)$和$\boldsymbol{c}=(9,14,16)$是共面的.

证因为$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=\begin{vmatrix}3 & 4 & 5\\ 1 & 2 & 2\\ 9 & 14 & 16\end{vmatrix}=0$,故$\boldsymbol{a},\boldsymbol{b}$和$\boldsymbol{c}$是共面的.

13.已知$|\boldsymbol{a}|=1,|\boldsymbol{b}|=5,\boldsymbol{a}\cdot\boldsymbol{b}=-3$,求$|\boldsymbol{a}\times\boldsymbol{b}|$.

解$\cos<\boldsymbol{a},\boldsymbol{b}>=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\dfrac{-3}{5},\sin<\boldsymbol{a},\boldsymbol{b}>=\dfrac{4}{5},|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin<\boldsymbol{a},\boldsymbol{b}>=1\times5\times\dfrac{4}{5}=4.$

14.设向量$\boldsymbol{a}$的方向余弦$\cos\alpha,\cos\beta,\cos\gamma$,在下列各情况下,指出$\boldsymbol{a}$的方向特征.

(1) $\cos\alpha=0,\cos\beta\neq0,\cos\gamma\neq0$;

(2) $\cos\alpha=\cos\beta=0,\cos\gamma\neq0$;

(3) $\cos\alpha=\cos\beta=\cos\gamma$.

解(1)$\boldsymbol{a}$与x轴垂直.

(2)$\boldsymbol{a}$是沿z轴的的向量.

(3) $\boldsymbol{a}$与三个轴的夹角相等, 都是$\arccos\frac{1}{\sqrt{3}}$或$\pi-\arccos\frac{1}{\sqrt{3}}$.

15.设$|\boldsymbol{a}|=\sqrt{2}$, $\boldsymbol{a}$的三个方向角满足$\alpha=\beta=\frac{1}{2}\gamma$,求$\boldsymbol{a}$的坐标.

解$2\cos^2\alpha+\cos^2 2\alpha=1,2\cos^2\alpha+(2\cos^2\alpha-1)^2=1$.

$\cos^2\alpha=x,2x+(2x-1)^2=1,4x^2-2x+1=1,2x(2x-1)=0,x=0,x=\frac{1}{2}$.

$\cos^2\alpha=0,\alpha=\frac{\pi}{2},\boldsymbol{a}=(0,0,-\sqrt{2})$.

$\cos^2\alpha=\frac{1}{2},\cos\alpha=\pm\frac{1}{\sqrt{2}},\alpha=\frac{\pi}{4},\frac{3\pi}{4}.\boldsymbol{a}=(1,1,0)$.

16.设$\boldsymbol{a},\boldsymbol{b}$为两非零向量,且$(7\boldsymbol{a}-5\boldsymbol{b})\perp(\boldsymbol{a}+3\boldsymbol{b}),(\boldsymbol{a}-4\boldsymbol{b})\perp(7\boldsymbol{a}-2\boldsymbol{b})$,

求$\cos<\boldsymbol{a},\boldsymbol{b}>$.

解$(7\boldsymbol{a}-5\boldsymbol{b})g(\boldsymbol{a}+3\boldsymbol{b})=0,7|\boldsymbol{a}|^2-15|\boldsymbol{b}|^2+16|\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>=0$,

$(\boldsymbol{a}-4\boldsymbol{b})g(7\boldsymbol{a}-2\boldsymbol{b})=0,7|\boldsymbol{a}|^2+8|\boldsymbol{b}|^2-30|\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>=0$.

$$\begin{cases}-15\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}+16\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\cos<\boldsymbol{a},\boldsymbol{b}>=-7,\\ 8\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}-30\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\cos<\boldsymbol{a},\boldsymbol{b}>=-7.\end{cases}$$

$$\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}=\frac{\begin{vmatrix}-7 & 16\\ -7 & -30\end{vmatrix}}{\begin{vmatrix}-15 & 16\\ 8 & -30\end{vmatrix}}=1,\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}=1$$

$$\cos<\boldsymbol{a},\boldsymbol{b}>=\frac{\begin{vmatrix}-15 & -7\\ 8 & -7\end{vmatrix}}{\begin{vmatrix}-15 & 16\\ 8 & -30\end{vmatrix}}=\frac{1}{2}.$$

习题 5.3

1.指出下列平面位置的特点:

(1)$5x-3z+1=0$(2)$x+2y-7z=0$(3)$y+5=0$(4)$2y-9z=0$(5)$x-y-5=0$(6)$x=0$.

解 (1)平行于y轴.(2)过原点.(3)平行于Oxz平面.

(4)过x轴.(5)平行于z轴.(6)Oyz平面.

2.求下列各平面的方程:

(1)平行于y轴且通过点$(1,-5,1)$和$(3,2,-2)$;

(2)平行于Oxz平面且通过点$(5,2,-8)$;

(3)垂直于平面$x-4y+5z=1$且通过点$(-2,7,3)$及$(0,0,0)$;

(4)垂直于Oyz平面且通过点$(5,-4,3)$及$(-2,1,8)$.

解 (1)$\boldsymbol{a}=(0,1,0),\boldsymbol{b}=(2,7,-3),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 1 & 0\\ 2 & 7 & -3\end{vmatrix}=(-3,0,-2).$

$-3(x-1)-2(z-1)=0,3x+2z-5=0.$

(2)$y=2$.

(3)$\boldsymbol{a}=(1,-4,5),\boldsymbol{b}=(-2,7,3),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & -4 & 5\\ -2 & 7 & 3\end{vmatrix}=(-47,-13,-1).$

$47x+13y+z=0$

(4)$\boldsymbol{a}=(1,0,0),\boldsymbol{b}=(-7,5,5),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & 0\\ -7 & 5 & 5\end{vmatrix}=(0,-5,5)=5(0,-1,1).$

$-(y+4)+(z-3)=0,y-z+7=0.$

3.求通过点$A(2,4,8),B(-3,1,5)$及$C(6,-2,7)$的平面方程.

解 $\boldsymbol{a}=(-5,-3,-3),\boldsymbol{b}=(4,-6,-1).$

$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -5 & -3 & -3\\ 4 & -6 & -1\end{vmatrix}=(-15,-17,42),$

$-15(x-2)-17(y-4)+42(z-8)=0,15x+17y-42z+238=0.$

4.设一平面在各坐标轴上的截距都不等于零并相等，且过点$(5,-7,4)$，求此平面的方程.

解：$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1,\frac{5}{a}+\frac{-7}{a}+\frac{4}{a}=1,a=2,x+y+z-2=0.$

5.已知两点$A(2,-1,-2)$及$B(8,7,5)$,求过B且与线段AB垂直的平面.

解 $\boldsymbol{n}=(6,8,7).6(x-8)+8(y-7)+7(z-5)=0,6x+8y+7z-139=0.$

6.求过点$(2,0,-3)$且与$2x-2y+4z+7=0, 3x+y-2z+5=0$垂直的平面方程.

解 $\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 4 \\ 3 & 1 & -2 \end{vmatrix}=(0,16,8)=8(0,2,1). 2y+(z+3)=0, y+z+3=0.$

7.求通过x轴且与平面$9x-4y-2z+3=0$垂直的平面方程.

解 $By+Cz=0, -4B-2C=0$, 取$B=1, C=-2, y-2z=0.$

8.求通过直线$l_1:\begin{cases} x+2z-4=0 \\ 3y-z+8=0 \end{cases}$且与直线$l_2:\begin{cases} x-y-4=0 \\ y-z-6=0 \end{cases}$平行的平面方程.

解 $\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2 \\ 0 & 3 & -1 \end{vmatrix}=(-6,1,3), \boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}=(1,1,1),$

$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -6 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}=(-2,9,-7)$.用$z_0=0$代入$l_1$的方程,得$x_0=4, y_0=-8/3.$

$-2(x-4)+9(y+8/3)-7(z)=0, -2x+9y-7z+32=0.$

9.求直线$l_1:\dfrac{x+3}{3}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$与直线$l_2:\begin{cases} x=3t+8 \\ y=t+1 \\ z=2t+6 \end{cases}$的交点坐标,

并求通过此两直线的平面方程.

解 求两条直线交点坐标:

$\dfrac{3t+8+3}{3}=\dfrac{t+1+1}{2}=\dfrac{2t+6-2}{4}, t+\dfrac{11}{3}=\dfrac{t}{2}+1=\dfrac{t}{2}+1, t=-\dfrac{16}{3},$

$x_0=-8, y_0=-\dfrac{13}{3}, z_0=-\dfrac{14}{3}$,交点$(-8,-\dfrac{13}{3},-\dfrac{14}{3})$.

$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 4 \\ 3 & 1 & 2 \end{vmatrix}=(0,6,-3)=3(0,2,-1). 2(y+1)-(z-2)=0, 2y-z+4=0.$

10.求通过两直线$l_1:\dfrac{x-1}{2}=\dfrac{y+1}{-1}=\dfrac{z+1}{1}$和$l_2:\dfrac{x+2}{-4}=\dfrac{y-2}{2}=\dfrac{z}{-2}$的平面方程.

解 两直线平行. 平面过点$(1,-1,-1)$和$(-2,2,0)$.$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 1 \\ -3 & 3 & 1 \end{vmatrix}=(-4,-5,3).$

$-4(x-1)-5(y+1)+3(z+1)=0, -4x-5y+3z+2=0.$

11.证明两直线$l_1:\dfrac{x-1}{-1}=\dfrac{y}{2}=\dfrac{z+1}{1}$和$l_2:\dfrac{x+2}{0}=\dfrac{y-1}{1}=\dfrac{z-2}{-2}$是异面直线.

证 首先,两直线的方向向量$(-1,2,1)$ 和 $(0,1,-2)$不平行.

$l_2\begin{cases}x=-2\\y=1+t\\z=2-2t\end{cases}$ $\dfrac{-2-1}{-1}=\dfrac{1+t}{2}=\dfrac{-2t+3}{1}, t=5, t=0,$矛盾.故两直线无公共点.

两直线不平行, 又无交点, 故是异面直线.

12. 将下列直线方程化为标准方程及参数方程:

(1)$\begin{cases}2x+y-z+1=0\\3x-y+2z-8=0;\end{cases}(2)\begin{cases}x-3z+5=0\\y-2z+8=0.\end{cases}$

解 (1)$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\2 & 1 & -1\\3 & -1 & 2\end{vmatrix}=(1,-7,-5).$

(1)中令$x_0=0,\begin{cases}y-z+1=0\\-y+2z-8=0;\end{cases}$解之得$y_0=6, z_0=7.$

标准方程$\dfrac{x}{1}=\dfrac{y-6}{-7}=\dfrac{z-7}{-5}.$

参数方程:$\begin{cases}x=t\\y=6-7t\\z=7-5t\end{cases}, -\infty<t<+\infty.$

(2)(1)$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\1 & 0 & -3\\0 & 1 & -2\end{vmatrix}=(3,2,1).$

(2)中令$z_0=0,$直接得$x_0=-5, y_0=-8.$

标准方程$\dfrac{x+5}{3}=\dfrac{y+8}{2}=\dfrac{z}{1}.$

参数方程:$\begin{cases}x=-5+3t\\y=-8+2t\\z=t\end{cases}, -\infty<t<+\infty.$

13.求通过点$(3,2,-5)$及x轴的平面与平面$3x-y-7z+9=0$的交线方程.

解：第一个平面的法向量$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 3 & 2 & -5 \end{vmatrix}=(0,5,2)$,

平面方程$5y+2z=0$.

直线方程$\begin{cases} 5y+2z=0 \\ 3x-y-7z+9=0. \end{cases}$

直线的方向向量$\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 5 & 2 \\ 3 & -1 & -7 \end{vmatrix}=(-33,6,-15)=3(-11,2,-5)$.

$z_0=0,\begin{cases} 5y=0 \\ 3x-y+9=0. \end{cases} y_0=0, x_0=-3.$

直线方程:$\dfrac{x+3}{-11}=\dfrac{y}{2}=\dfrac{z}{-5}$.

14.当D为何值时,直线$\begin{cases} 3x-y+2z-6=0 \\ x+4y-z+D=0 \end{cases}$与$Oz$轴相交?

解:直线$\begin{cases} 3x-y+2z-6=0 \\ x+4y-z+D=0 \end{cases}$与$Oz$轴相交$\Leftrightarrow$存在$(0,0,z_0)$在此直线上,

$\Leftrightarrow \begin{cases} 2z_0-6=0 \\ -z_0+D=0 \end{cases} \Leftrightarrow D=z_0=3.$

15.试求通过直线$l_1:\begin{cases} x-2z-4=0 \\ 3y-z+8=0 \end{cases}$并与直线$l_2:\begin{cases} x-y-4=0 \\ z-y+6=0 \end{cases}$平行的平面方程.

解:l_1的方向向量$\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -2 \\ 0 & 3 & -1 \end{vmatrix}=(6,1,3)$.

l_2的方向向量$\boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}=(-1,-1,-1)=-(1,1,1)$.

平面的法向量$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}=(-2,-3,5)$.

在的方程中令$z_0=0$得$x_0=4, y_0=-\dfrac{8}{3}$.

所求平面方程:$-2(x-4)-3(y+\dfrac{8}{3})+5z=0$,即$2x+3y-5z=0$.

16.求点$(1,2,3)$到直线$\frac{x}{1}=\frac{y-4}{-3}=\frac{z-3}{-2}$的距离.

解 过点$(1,2,3)$垂直于直线的平面:

$(x-1)-3(y-2)-2(z-3)=0.$

直线参数方程:$\begin{cases} x=t \\ y=4-3t. \\ z=3-2t \end{cases}$

代入平面方程得对应交点的参数:

$(t-1)-3(4-3t-2)-2(3-2t-3)=0, t_0=\frac{1}{2},$

直线与平面交点为$(\frac{1}{2},\frac{5}{2},2)$.

所求距离$d=\sqrt{(1-\frac{1}{2})^2+(2-\frac{5}{2})^2+(3-2)^2}=\frac{\sqrt{6}}{2}.$

17.求点$(2,1,3)$到平面$2x-2y+z-3=0$的距离与投影.

解 过点$(2,1,3)$垂直于平面$2x-2y+z-3=0$的直线方程的参数方程:

$\begin{cases} x=2+2t \\ y=1-2t \\ z=3+t \end{cases}, -\infty<t<+\infty.$代入平面方程

$2(2+2t)-2(1-2t)+(3+t)-3=0. t_0=-\frac{2}{9}.$

$x_0=\frac{14}{9}, y_0=\frac{13}{9}, z_0=\frac{25}{9}.$

点$(2,1,3)$在平面$2x-2y+z-3=0$上的投影为$\left(\frac{14}{9},\frac{13}{9},\frac{25}{9}\right)$.

点$(2,1,3)$在平面$2x-2y+z-3=0$的距离为

$\sqrt{(2-\frac{14}{9})^2+(1-\frac{13}{9})^2+(3-\frac{25}{9})^2}=\frac{2}{3}.$

18.求两平行直线$\dfrac{x-1}{1}=\dfrac{y+1}{-2}=\dfrac{z}{3}$与$\dfrac{x}{1}=\dfrac{y+1}{-2}=\dfrac{z-1}{3}$的距离.

解 所求的就是点$(1,-1,0)$到直线$\dfrac{x}{1}=\dfrac{y+1}{-2}=\dfrac{z-1}{3}$的距离.

作法与16题雷同. 过点$(1,-1,0)$垂直于直线$\dfrac{x}{1}=\dfrac{y+1}{-2}=\dfrac{z-1}{3}$的平面:

$(x-1)-2(y+1)+3z=0.$

直线的参数方程$\begin{cases} x=t \\ y=-1-2t \\ z=1+3t \end{cases}$,代入平面方程

$(t-1)-2(-2t)+3(1+3t)=0, t_0=-\dfrac{1}{7}.$

直线与平面交点$(-\dfrac{1}{7},-\dfrac{5}{7},\dfrac{4}{7})$.

所求距离$d=\sqrt{(1+\dfrac{1}{7})^2+(-1+\dfrac{5}{7})^2+(0-\dfrac{4}{7})^2}=2\sqrt{\dfrac{3}{7}}.$

19.求过点$A(2,1,3)$并与直线$l_1:\dfrac{x+1}{3}=\dfrac{y-1}{2}=\dfrac{z}{-1}$垂直且相交的直线方程.

解 过点A垂直于直线l_1的平面方程$3(x-2)+2(y-1)-(z-3)=0.$

直线l_1的参数方程$\begin{cases} x=-1+3t \\ y=1+2t \\ z=-t \end{cases}$

代入平面方程求交点对应的参数t:

$3(-3+3t)+2(2t)-(-t-3)=0, t_0=\dfrac{3}{7}.$

交点$B(\dfrac{2}{7},\dfrac{13}{7},-\dfrac{3}{7})$.

连结点A, B的直线的方向向量

$AB=(\dfrac{2}{7}-2,\dfrac{13}{7}-1,-\dfrac{3}{7}-3)=(-\dfrac{12}{7},\dfrac{6}{7},-\dfrac{24}{7})=-\dfrac{6}{7}(2,-1,4).$

所求直线方程:$\dfrac{x-2}{2}=\dfrac{y-1}{-1}=\dfrac{z-3}{4}.$

20.求两平行平面$3x+6y-2z-7=0$与$3x+6y-2z+14=0$之间的距离.

解 点$A(0,0,-\frac{7}{2})$在第一张平面上.

过A垂直于第二张平面的直线的参数方程:$\begin{cases} x=3t \\ y=6t \\ z=-7/2-2t \end{cases}$

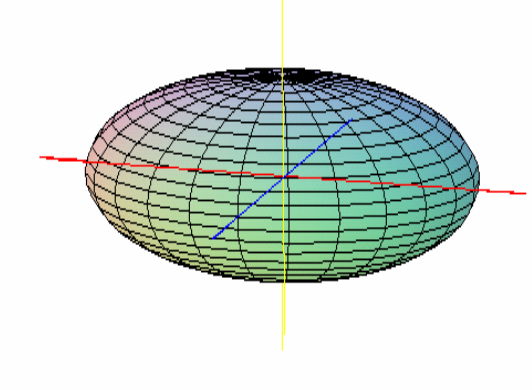
求直线与第二张平面的交点:$3(3t)+6(6t)-2(-7/2-2t)+14=0$,

$t_0=-\frac{3}{7},(-\frac{9}{7},-\frac{18}{7},-\frac{37}{14})$.

所求距离$=\sqrt{(\frac{9}{7})^2+(\frac{18}{7})^2+(\frac{6}{7})^2}=3$.

习题 5.4

1.求椭球面$2x^2+3y^2+4z^2-4x-6y+16z+16=0$的中心的坐标及三个半轴之长度.



解$2x^2+3y^2+4z^2-4x-6y+16z+16=0$,

$$2x^2+3y^2+4z^2-4x-6y+16z+17$$
$$=2(x-1)^2-2+3(y-1)^2-3+4(z+2)^2-16+16$$
$$=2(x-1)^2+3(y-1)^2+4(z+2)^2-5=0.$$
$$\frac{(x-1)^2}{\left(\sqrt{\frac{5}{2}}\right)^2}+\frac{(y-1)^2}{\left(\sqrt{\frac{5}{3}}\right)^2}+\frac{(z+2)^2}{\left(\frac{\sqrt{5}}{2}\right)^2}=1,$$

中心坐标:$(1,1,-2)$,半轴:$\sqrt{\frac{5}{2}},\sqrt{\frac{5}{3}},\frac{\sqrt{5}}{2}$.

2.说出下列曲面的名称,并画出略图:

(1)$8x^2+11y^2+24z^2=1$;椭球面.

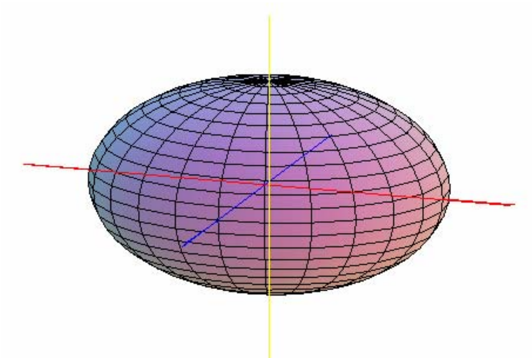
(2)$4x^2-9y^2-14z^2=-25$;单叶双曲面.

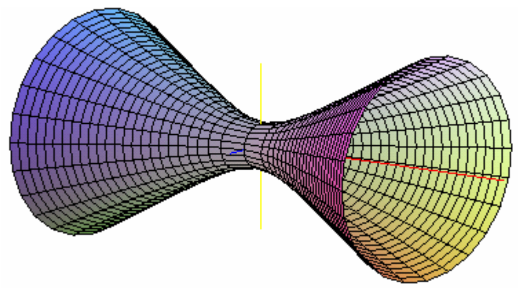
(3)$2x^2+9y^2-16z^2=-9$;双叶双曲面.

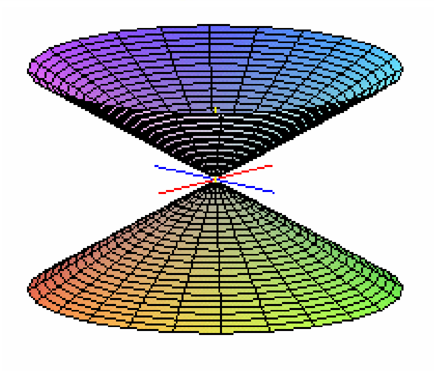
(4)$x^2-y^2=2x$;双曲柱面.

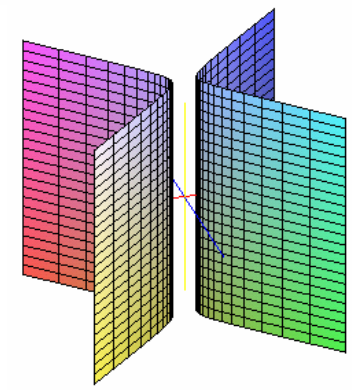
(5)$2y^2+z^2=x$;椭圆抛物面.

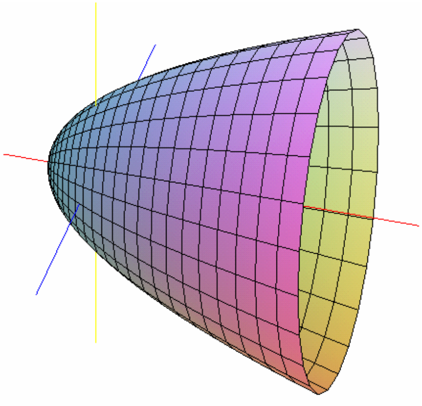
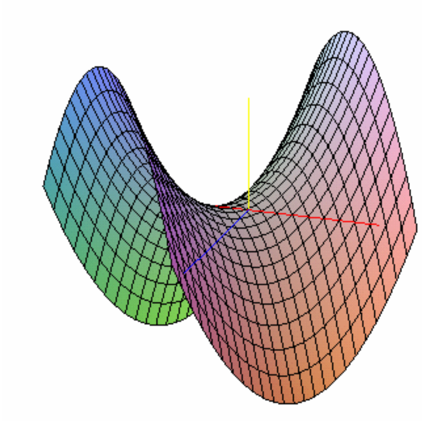
(6)$z=xy$.双曲抛物面.









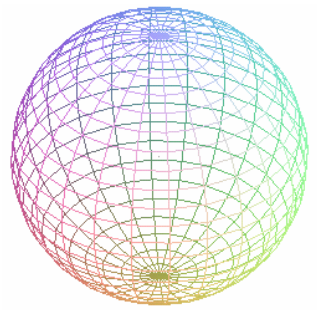
3.求下列曲面的参数方程：

$(1)(x-1)^2+(y+1)^2+(z-3)^2=R^2$;

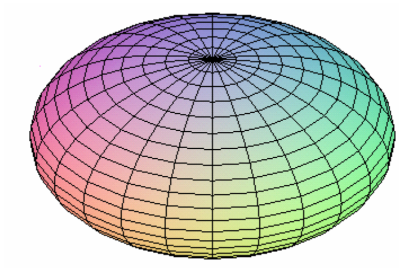
$(2)x^2+\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$;$(3)\dfrac{x^2}{4}+\dfrac{y^2}{9}-\dfrac{z^2}{16}=1$;

$(4)z=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$;$(5)z=\dfrac{z^2}{a^2}+\dfrac{y^2}{b^2}$.

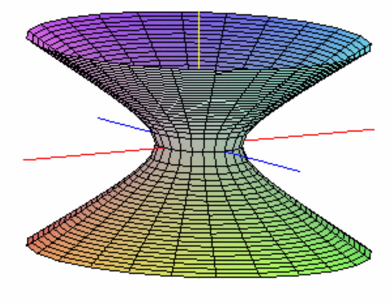
$$解:(1)\begin{cases} x=1+R\sin\varphi\cos\theta \\ y=-1+R\sin\varphi\sin\theta\, 0\le\varphi\le\pi,0\le\theta<2\pi; \\ z=3+R\cos\varphi \end{cases}$$



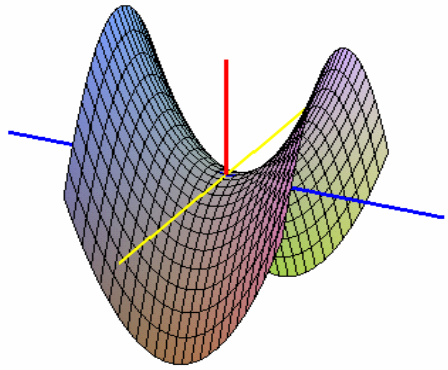
$$(2)\begin{cases} x=\sin\varphi\cos\theta \\ y=3\sin\varphi\sin\theta\, 0\le\varphi\le\pi,0\le\theta<2\pi; \\ z=2\cos\varphi \end{cases}$$



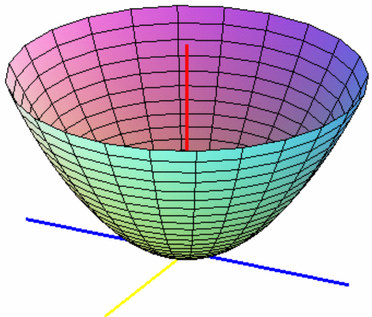
$$(3)\begin{cases} x=\quad 2\,\mathrm{ch}\varphi\cos\theta \\ y=\quad 3\,\mathrm{ch}\,\varphi\sin\theta \quad -\theta<\varphi<+\infty,0\le\theta<2\pi; \\ z=\quad 4\,\mathrm{sh}\varphi \end{cases}$$



$$(5)\begin{cases} x = ar\sec\theta \\ y = br\tan\theta \quad 0 \le r < +\infty, 0 \le \theta \le 2\pi \\ z = r^2 \end{cases}$$



$$(5)\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \quad 0 \le r < +\infty, 0 \le \theta \le 2\pi \\ z = r^2 \end{cases}$$



习题 5.5

1. 求下列曲线在指定点P_0的切线方程和法平面方程：

(1)$x=t, y=t^2, z=t^3, P_0=(1,1,1)$;

(2)曲面$z=x^2$与$y=x$的交线, $P_0=(2,2,4)$;

(3)柱面$x^2+y^2=R^2(R>0)$与平面$z=x+y$的交线$P_0=(R,0,R)$.

解（1）$x'=1, y'=2t, z'=3t^2, \boldsymbol{t}=(1,2,3)$, 切线方程:$\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-1}{3}$,

法平面方程:$(x-1)+2(y-1)+3(z-1)=0, x+2y+3z-6=0$.

(2)$x=x, y=x, z=x^2, x'=1, y'=1, z'=2x, \boldsymbol{t}=(1,1,4)$.切线方程:$\dfrac{x-2}{1}=\dfrac{y-2}{1}=\dfrac{z-4}{4}$,

法平面方程:$(x-2)+(y-2)+4(z-4)=0, x+y+4z-20=0$.

$$(3)\boldsymbol{n}_1=(2x,2y,0)=(2R,0,0), \boldsymbol{n}_1=(1,1,-1), \boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2R & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix}=(0,2R,2R)=2R(0,1,1),$$

切线方程:$\dfrac{x-R}{0}=\dfrac{y}{1}=\dfrac{z-R}{1}$,法平面方程:$y+z-R=0$.

2.求出螺旋线$\begin{cases} x=R\cos t \\ y=R\sin t\,(R>0, b>0, 0\le t\le 2\pi) \\ z=bt \end{cases}$在任意一

点处的切线的

方向余弦，并证明切线与z轴之夹角为常数.

解$(x',y',z')=(-R\sin t, R\cos t, b)$,

$\boldsymbol{t}=(\cos\alpha,\cos\beta,\cos\gamma)=\dfrac{1}{\sqrt{R^2+b^2}}(-R\sin t, R\cos t, b)$,

$\cos<\boldsymbol{t},\boldsymbol{k}>=\dfrac{b}{\sqrt{R^2+b^2}}=$常数.$0<<\boldsymbol{t},\boldsymbol{k}><\pi, <\boldsymbol{t},\boldsymbol{k}>=$常数.

3.设$\boldsymbol{a}=\boldsymbol{a}(t)$与$\boldsymbol{b}=\boldsymbol{b}(t)$是两个可导的向量函数,$\alpha<t<\beta$.证明

$$\frac{d}{dt}\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=\boldsymbol{a}'(t)\cdot\boldsymbol{b}(t)+\boldsymbol{a}(t)\cdot\boldsymbol{b}'(t).$$

证设$\boldsymbol{a}(t)=(a_1(t),a_2(t),a_3(t)),\boldsymbol{b}(t)=(b_1(t),b_2(t),b_3(t)),$

$\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=a_1(t)b_1(t)+a_2(t)b_2(t)+a_3(t)b_3(t),$

$$\frac{d}{dt}\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=\frac{d}{dt}[a_1(t)b_1(t)+a_2(t)b_2(t)+a_3(t)b_3(t)]$$

$$=a_1'(t)b_1(t)+a_1(t)b_1'(t)+a_2'(t)b_2(t)+a_2(t)b_2'(t)+a_3'(t)b_3(t)+a_3(t)b_3'(t)$$

$$=[a_1'(t)b_1(t)+a_2'(t)b_2(t)+a_3'(t)b_3(t)]+[a_1(t)b_1'(t)+a_2(t)b_2'(t)+a_3(t)b_3'(t)]$$

$$=\boldsymbol{a}'(t)\cdot\boldsymbol{b}(t)+\boldsymbol{a}(t)\cdot\boldsymbol{b}'(t).$$

4.设$\boldsymbol{r}=\boldsymbol{r}(t)(\alpha<t<\beta)$是一条光滑曲线,切$|\boldsymbol{r}(t)|=C$(常数).证明$\boldsymbol{r}(t)$与切线垂直,即$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0.$

证$\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)=C^2,\dfrac{d}{dt}\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)=\dfrac{d}{dt}C^2,\boldsymbol{r}'(t)\cdot\boldsymbol{r}(t)+\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0,2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0,$

$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0.$

第五章总练习题

1. 设$\boldsymbol{a},\boldsymbol{b}$ 为两个非零向量,指出下列等式成立的充分必要条件:

(1) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}-\boldsymbol{b}|$;(2) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}|-|\boldsymbol{b}|$;(3)$\boldsymbol{a}+\boldsymbol{b}$与$\boldsymbol{a}-\boldsymbol{b}$共线.

解(1) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}-\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}+\boldsymbol{b}|^2=|\boldsymbol{a}-\boldsymbol{b}|^2 \Leftrightarrow |\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}$

$\Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=0 \Leftrightarrow \boldsymbol{a},\boldsymbol{b}$正交.

(2) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}|-|\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}+\boldsymbol{b}|^2=(|\boldsymbol{a}|-|\boldsymbol{b}|)^2 \Leftrightarrow |\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b} \Leftrightarrow$

$|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>$

$=-|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow \cos<\boldsymbol{a},\boldsymbol{b}>=-1 \Leftrightarrow \boldsymbol{a},\boldsymbol{b}$共线且方向相反.

(3)$\boldsymbol{a}+\boldsymbol{b}$与$\boldsymbol{a}-\boldsymbol{b}$共线$\Leftrightarrow (\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b})=0 \Leftrightarrow \boldsymbol{b}\times\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}=0 \Leftrightarrow \boldsymbol{a}\times\boldsymbol{b}=0$

$\Leftrightarrow \boldsymbol{a},\boldsymbol{b}$共线.

2.设$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$为非零向量,判断下列等式是否成立:

(1)$(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$;(2)$(\boldsymbol{a}\cdot\boldsymbol{b})^2=\boldsymbol{a}^2\boldsymbol{b}^2$;(3)$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.

解(1)不成立. 例如:$(\boldsymbol{i}\cdot\boldsymbol{i})\boldsymbol{j}=\boldsymbol{j}\neq\boldsymbol{i}(\boldsymbol{i}\cdot\boldsymbol{j})=\boldsymbol{0}$.

(2)不成立.例如:$(\boldsymbol{i}\cdot\boldsymbol{j})^2=0\neq\boldsymbol{i}^2\boldsymbol{j}^2=1$.

(3)成立.$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$和$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$都是$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$的有向体积,且定向相同.

3.设$\boldsymbol{a},\boldsymbol{b}$为非零向量, 且$7\boldsymbol{a}-5\boldsymbol{b}$与$\boldsymbol{a}+3\boldsymbol{b}$正交, 与$\boldsymbol{a}-4\boldsymbol{b}$与$7\boldsymbol{a}-2\boldsymbol{b}$正交, 求$\boldsymbol{a}^2-\boldsymbol{b}^2$.

解 $(7\boldsymbol{a}-5\boldsymbol{b})\cdot(\boldsymbol{a}+3\boldsymbol{b})=0,(\boldsymbol{a}-4\boldsymbol{b})\cdot(7\boldsymbol{a}-2\boldsymbol{b})=0.$

$$\begin{cases}7\boldsymbol{a}^2-15\boldsymbol{b}^2+16\boldsymbol{a}\cdot\boldsymbol{b}=0 & (1)\\ 7\boldsymbol{a}^2+8\boldsymbol{b}^2-30\boldsymbol{a}\cdot\boldsymbol{b}=0 & (2)\end{cases}$$

$(1)\times15+(2)\times8$

$161(\boldsymbol{a}^2-\boldsymbol{b}^2)=0,\boldsymbol{a}^2-\boldsymbol{b}^2=0.$

4.利用向量运算,证明下列几何命题:射影定理.考虑直角三角形ABC,其中$\angle A$为直角,AD是斜边上的高,则$\overline{AD}^2=\overline{BD}\cdot\overline{CD},\overline{AB}^2=\overline{BD}\cdot\overline{BC},\overline{AC}^2=\overline{CD}\cdot\overline{CB}.$

证$\overrightarrow{AB}=\overrightarrow{AD}+\overrightarrow{DB},\overrightarrow{AC}=\overrightarrow{AD}+\overrightarrow{DC},$

$0=\overrightarrow{AB}\cdot\overrightarrow{AC}=(\overrightarrow{AD}+\overrightarrow{DB})\cdot(\overrightarrow{AD}+\overrightarrow{DC})=\overline{AD}^2+\overrightarrow{AD}\cdot\overrightarrow{DC}+\overrightarrow{DB}\cdot\overrightarrow{AD}+\overrightarrow{DB}\cdot\overrightarrow{DC}$

$=\overline{AD}^2+\overrightarrow{DB}\cdot\overrightarrow{DC},\overline{AD}^2=-\overrightarrow{DB}\cdot\overrightarrow{DC}=\overrightarrow{BD}\cdot\overrightarrow{DC}=\overline{BD}\times\overline{DC}(\overrightarrow{BD},\overrightarrow{DC}$同向).

$\overline{AB}^2=\overline{AD}^2+\overline{BD}^2=\overline{BD}\cdot\overline{CD}+\overline{BD}^2=\overline{BD}(\overline{CD}+\overline{BD})=\overline{BD}\cdot\overline{BC}).$

$\overline{AC}^2=\overline{AD}^2+\overline{CD}^2=\overline{BD}\cdot\overline{CD}+\overline{CD}^2=\overline{CD}(\overline{BD}+\overline{CD})=\overline{CD}\cdot\overline{BC}.$

5.已知三点A,B,C的坐标分别为$(1,0,0),(1,1,0),(1,1,1)$.若$ACDBD$是一平行四边形，求点D的坐标.

解$A=(1,0,0),B=(1,1,0),C=(1,1,1).\overrightarrow{AC}=(0,1,1),\overrightarrow{AB}=(0,1,0),\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{AC}=(0,2,1),$
$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=(1,0,0)+(0,2,1)=(1,2,1)$.点$D$的坐标$(1,2,1)$.

6.设$\boldsymbol{a},\boldsymbol{b}$为非零向量, 证明$(\boldsymbol{a}\times\boldsymbol{b})^2=\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2$.

证$(\boldsymbol{a}\times\boldsymbol{b})^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2<\boldsymbol{a},\boldsymbol{b}>=|\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2<\boldsymbol{a},\boldsymbol{b}>)$

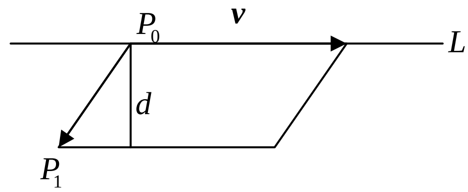
$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2<\boldsymbol{a},\boldsymbol{b}>=\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2$.

7.设有两直线$L_1:\dfrac{x-1}{-1}=\dfrac{y}{2}=\dfrac{z+1}{1},L_2:\dfrac{x+2}{0}=\dfrac{y-1}{1}=\dfrac{z-2}{-2}$,求平行于$L_1,L_2$且与它们等距的平面方程.

解$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-1&2&1\\0&1&-2\end{vmatrix}=(-5,-2,-1)$,所求平面过点$A=(-1/2,1/2,1/2)$,

所求平面:$-5(x+1/2)-2(y-1/2)-(z-1/2)=0,5x+2y+z+1=0$.

8.设直线L通过点P_0且其方向向量为$\boldsymbol{v}$,证明L外一点P_1到L的距离d可表为$d=\dfrac{|\overrightarrow{P_0P_1}\times\boldsymbol{v}|}{|\boldsymbol{v}|}$.



证平行四边形P_0P_1AB的面积
$=d\times|\boldsymbol{v}|=||\overrightarrow{P_0P_1}\times\boldsymbol{v}||$.

9.设两直线L_1,L_2分别通过点P_1P_2,且它们的方向向量为$\boldsymbol{v}_1,\boldsymbol{v}_2$.证明$L_1$与$L_2$共面的充分必要条件为$\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)=0$.

证L_1与L_2共面$\Leftrightarrow\overrightarrow{P_1P_2},\boldsymbol{v}_1,\boldsymbol{v}_2$共面$\Leftrightarrow\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)=0$.

10.设两直线L_1,L_2分别通过点P_1,P_2,且它们的方向向量为$\boldsymbol{v}_1,\boldsymbol{v}_2$.$L_1$与$L_2$之间的距离定义为$d=\min\limits_{\substack{Q_1\in L_1\\Q_2\in L}}|\overrightarrow{Q_1Q_2}|$证明:(1)当$L_1$与$L_2$平行时,它们之间的距离可表示为$d=\dfrac{\overrightarrow{P_1P_2}\times\boldsymbol{v}_1}{|\boldsymbol{v}_1|}$

(2)当L_1与L_2为异面直线时, 它们之间的距离可表示为$d=\dfrac{|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)|}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}$.

证(1)当L_1与L_2平行时, 它们之间的距离为L_1上任意一点到L_2的距离,由第8题,

$d=\dfrac{\overrightarrow{P_1P_2}\times\boldsymbol{v}_1}{|\boldsymbol{v}_1|}$.

(2)$\dfrac{\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}=\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)^\circ$是$\overrightarrow{P_1P_2}$在$L_1$与$L_2$的公垂线方向的单位向量上的投影,

故其长度$|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)^\circ|=\dfrac{|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)|}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}$是异面直线$L_1$与$L_2$之间的距离.

11.设直线L的方程为$L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$

证明:(1)对于任意两个不全为零的常数λ_1,λ_2, 方程

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0$

表示一个通过直线L的平面;

(2)任意给定一个通过直线L的平面π,必存在两个不全为零的实数λ_1,λ_2, 使平面π的方程为$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

证(1)向量(A_1,B_1,C_1)与(A_2,B_2,C_2)不共线,故对于两个不全为零的常数λ_1,λ_2,

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0$的主系数

$\lambda_1(A_1,B_1,C_1)+\lambda_2(A_2,B_2,C_2)\neq(0,0,0)$,是一个平面的方程, 并且 L上点的坐标

(x,y,z)满足$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$,故满足

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

(2)设平面π通过直线L,其方程为

$A(x-x_0)+B(y-y_0)+C(z-z_0)=Ax+By+Cz+D=0.$

(x_0,y_0,z_0)在L上.三个向量(A,B,C) (A_1,B_1,C_1)与(A_2,B_2,C_2)均垂直于L的方向向量,故共面,又(A_1,B_1,C_1)与(A_2,B_2,C_2)都是非零向量,故存在两个不全为零的常数λ_1,λ_2,使得

$(A,B,C)=\lambda_1(A_1,B_1,C_1)+\lambda_2(A_2,B_2,C_2).$

$D=-Ax_0-By_0-Cz_0=-(\lambda_1A_1+\lambda_2A_2)x_0-(\lambda_1B_1+\lambda_2B_2)y_0-(\lambda_1C_1+\lambda_2C_2)z_0$

$=-\lambda_1(A_1x_0+B_1y_0+C_1z_0)-\lambda_2(A_2x_0+B_2y_0+C_2z_0)=\lambda_1D_1+\lambda_2D_2.$

故π表示为$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

12.试求通过直线$L_1:\begin{cases}x-2z-4=0\\3y-z+8=0\end{cases}$且与直线$L_2:x-1=y+1=z-3$平行的平面方程.

解根据11题的结论,所求平面方程有形式

$\lambda_1(x-2z-4)+\lambda_2(3y-z+8)=0,\lambda_1x+3\lambda_2y+(-2\lambda_1-\lambda_2)z-4\lambda_1+8\lambda_2=0.$

由于平面与L_2平行,$(\lambda_1,3\lambda_2,-2\lambda_1-\lambda_2)g(1,1,1)=0,\lambda_1+3\lambda_2-2\lambda_1-\lambda_2=0,-\lambda_1+2\lambda_2=0.$

令$\lambda_1=2,\lambda_2=1$,得所求平面方程$2x+3y-5z=0.$

13.已知曲面S的方程为$S:x^2+\dfrac{y^2}{4}+\dfrac{z^2}{2}=1$,平面$\pi$的方程为$\pi:2x+y+2z+6=0.$

(1)求曲面S的平行于π的切平面方程;

(2)在曲面S上求到平面π距离为最短及最长的点,并求最短及最长的距离.

解 (1)S的法向量$(2x,\dfrac{y}{2},z).4x+\dfrac{y}{2}+2z=0.$

$2x(X-x)+\dfrac{y}{2}(Y-y)+z(Z-z)=0$

13.已知曲面S的方程为$S: x^2+\frac{y^2}{4}+\frac{z^2}{2}=1$,平面$\pi$的方程为$\pi: 2x+y+2z+6=0$.

(1)求曲面S的平行于π的切平面方程;

(2)在曲面S上求到平面π距离为最短及最长的点,并求最短及最长的距离.

解 (1)S上的点记为(x,y,z).S的法向量$(2x,\frac{y}{2},z)$.

切平面与π平行, 则法向量对应坐标成比例:$\frac{2x}{2}=\frac{y/2}{1}=\frac{z}{2}$.$z=2x, y=z$.

与曲面方程联立:$x^2+x^2+2x^2=1, x=\pm\frac{1}{2}, y=\pm 1, z=\pm 1$.

切平面方程:$2x(X-x)+\frac{y}{2}(Y-y)+z(Z-z)=0$,

利用曲面方程得$2xX+\frac{y}{2}Y+zZ=2$.$\pm X\pm\frac{1}{2}Y\pm Z=2$.

平面π过点$A=(-3,0,0)$.$\overrightarrow{P_1A}=$

点$P_1=(\frac{1}{2},1,1)$,$\overrightarrow{P_1A}=(-\frac{7}{2},-1,-1)$,

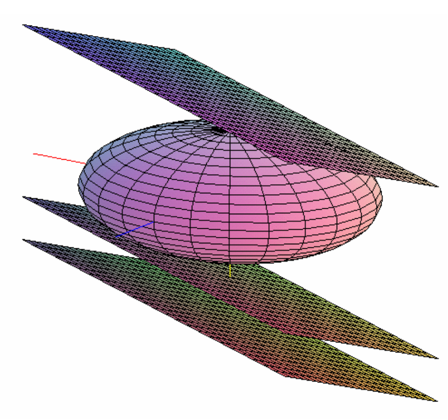
P_1到平面π的距离$d_1=\frac{\left|\overrightarrow{P_1A}\cdot\boldsymbol{n}\right|}{|\boldsymbol{n}|}=\frac{\left|(-\frac{7}{2},-1,-1)\cdot(2,1,2)\right|}{3}=\frac{10}{3}$.

点$P_2=(-\frac{1}{2},-1,-1)$,$\overrightarrow{P_2A}=(-\frac{5}{2},1,1)$,

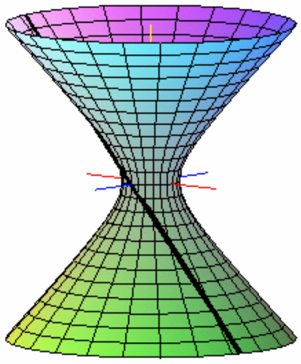
P_2到平面π的距离$d_2=\frac{\left|\overrightarrow{P_2A}\cdot\boldsymbol{n}\right|}{|\boldsymbol{n}|}=\frac{\left|(-\frac{5}{2},1,1)\cdot(2,1,2)\right|}{3}=\frac{2}{3}$.

在曲面S上到平面π距离为最短及最长的点分别是$(-\frac{1}{2},-1,-1)$和$(\frac{1}{2},1,1)$,

并求最短及最长的距离分别是$\frac{2}{3}$和$\frac{10}{3}$.



14.直线$\frac{x}{1}=\frac{y-1}{0}=\frac{z}{1}$绕z轴旋转一周,求所得旋转曲面的方程.

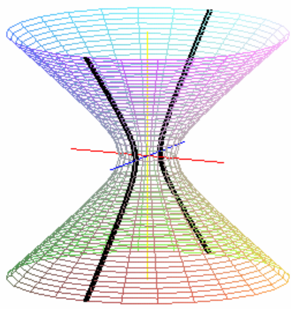


解直线参数方程$\begin{cases}x=z\\y=1\\z=z\end{cases}-\infty<z<+\infty.$

直线$\frac{x}{1}=\frac{y-1}{0}=\frac{z}{1}$绕z轴旋转,对于固定的z,旋转曲面上的点组成一个圆,其半径为$\sqrt{1+z^2}$,

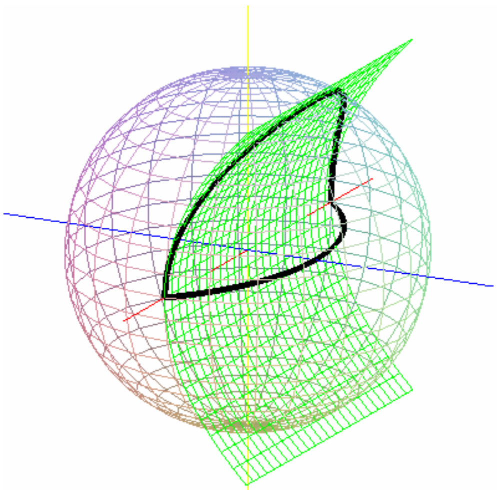
故旋转曲面的方程$\begin{cases}x=\sqrt{1+z^2}\cos\theta\\y=\sqrt{1+z^2}\sin\theta\\z=z\end{cases}-\infty<z<+\infty,0\le\theta\le 2\pi.$

15.求双曲线$\begin{cases}\frac{y^2}{b^2}-\frac{z^2}{c^2}=1(b,c>0),\\x=0\end{cases}$绕z轴旋转一周所得曲面的方程.



解$\frac{x^2+y^2}{b^2}-\frac{z^2}{c^2}=1.$

16.求曲线$\begin{cases}x^2+y^2+z^2=1\\z^2=2y\end{cases}$在$Oxy$平面上的投影曲线的方程.



解$x^2+y^2+2y=1,x^2+(y+1)^2=2.$

习题 6.1

1.确定下列函数的定义域并且画出定义域的的图形:

$(1) z=(x^2+y^2-2x)^{1/2}+\ln(4-x^2-y^2); x^2+y^2-2x\geq 0, x^2-y^2<4.$

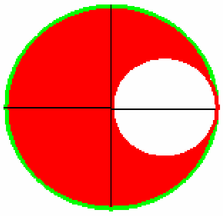
$(2) z=(x^2-y^2)^{-1}; x^2\neq y^2.$

$(3) z=\ln(y-x^2)+\ln(1-y); y-x^2>0, y<1.$

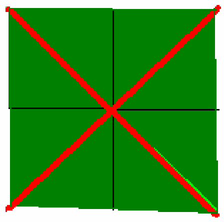
$(4) z=\arcsin\dfrac{x}{a}+\arccos\dfrac{y}{b}(a>0,b>0); |x|\leq a, |y|\leq b.$

$(5) z=\sqrt{1-x^2-y^2}+\ln(x+y); x^2+y^2\leq 1, x+y>0.$

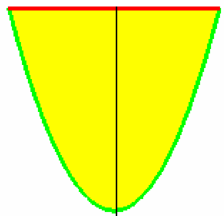
$(6) z=\arcsin(x^2+y^2)+\sqrt{xy}. x^2+y^2\leq 1, xy\geq 0.$



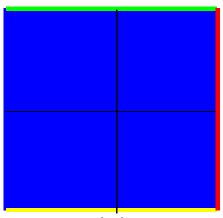
1(1)



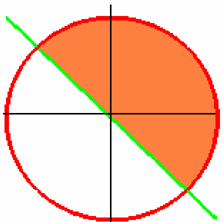
1(2)



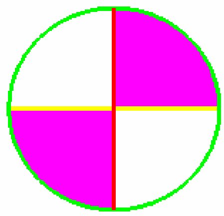
1(3)



1(4)



1(5)



1(6)

2.指出下列集合中哪些集合在中是开集, 哪些是区域?哪些是有界区域?哪些是有界闭区域?

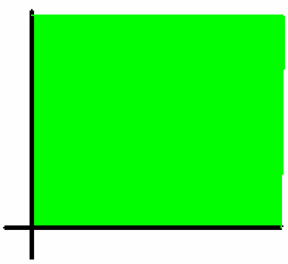
$(1) E_1=\{(x,y)\,|\,x>0, y>0\}$;开集, 区域.

$(2) E_2=\{(x,y)\,\|\,x|<1, |y-1|<2\}$;开集, 区域, 有界区域.

$(3) E_3=\{(x,y)\,|\,y\geq x^2, x\geq y^2\}$;有界闭区域.

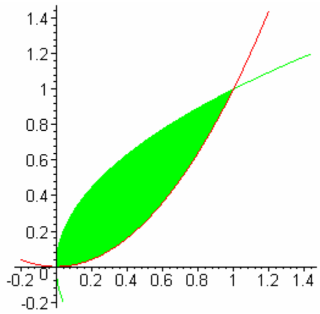
$(4) E_4=\{(x,y)\,|\,y\neq\sin\dfrac{1}{x}\text{且}x\neq 0\}$. 开集, 边界点集合

$\partial E_4=\{(x,\sin\dfrac{1}{x})\,|\,x\neq 0\}\cup\{(0,y)\,|\,-1\leq y\leq 1\}.$

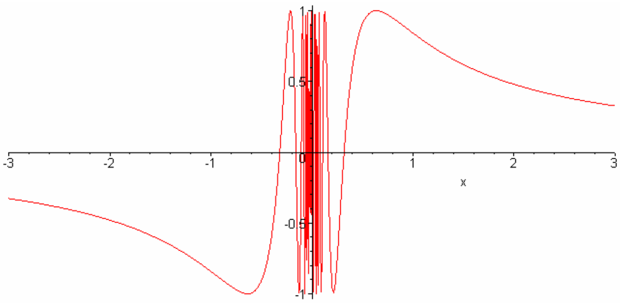


2(1)

2(2)



2(3)



2(4)

3.设$E\subset \mathbf{R}^n$,∂E为E的边界点集合.试证明$\overline{E}=E\cup\partial E$是一个闭集.

证设 $P_0\notin\overline{E}$,则 $P_0\notin E$且 $P_0\notin\partial E$.于是存在$r>0$,使得$U_r(P_0)$不含E的点,从而不含∂E的点.否则,存在$Q\in U_r(P_0)\cap\partial E$,$Q$作为$E$的边界点,存在 $U_\rho(Q)\subseteq U_r(P_0)$,$U_\rho(Q)$含$E$的点,于是$U_r(P_0)$含$E$的点,矛盾.因此,$U_r(P_0)$不含$E\cup\partial E=\overline{E}$的点,$P_0$不是$\overline{E}$的的边界点.这表明$\overline{E}$的边界点全属于$\overline{E}$.故$\overline{E}$是闭集合.

4.像在 $\mathbf{R}^2$中一样,我们把$\mathbf{R}^n$中的点$(x_1,\cdots,x_n)$同时也视作一个向量,并定义两个向量$\alpha=(x_1,\cdots,x_n)$及$\beta=(y_1,\cdots,y_n)$的加法运算

$\alpha+\beta=(x_1+y_1,\cdots,x_n+y_n)$

及数乘运算

$\lambda\alpha=(\lambda x_1,\cdots,\lambda x_n),\forall\lambda\in\mathbf{R}.$

此外=,我们也可以定义两个向量之内积

$\alpha\cdot\beta=x_1y_1+\cdots+x_ny_n$, 并规定

$\sqrt{\alpha\cdot\alpha}=|\alpha|$作为向量的模. 试证明

(1)$|\alpha\cdot\beta|\le|\alpha||\beta|,\forall\alpha,\beta\in\mathbf{R}^n$;

(2)$|\alpha-\beta|\le|\alpha-\gamma|+|\gamma-\beta|,\forall\alpha,\beta,\gamma\in\mathbf{R}^n$;

(3)将点$P(x_1,\cdots,x_n)$及$Q(y_1,\cdots,y_n)$分别看成向量α及β,则有P到Q的距离$d(P,Q)=|\alpha-\beta|$.由此,可由(2)中之不等式导出三角不等式.

证(1)$\beta=0$ 时结论显然成立.设$\beta\neq 0$.考虑二次函数

$|\alpha+\lambda\beta|^2=|\beta|^2\lambda^2+2\alpha\cdot\beta\lambda+|\alpha|^2\ge 0,\forall\lambda\in\mathbf{R}.$

其判别式$|\alpha\cdot\beta|^2-|\alpha|^2|\beta|^2\le 0,|\alpha\cdot\beta|\le|\alpha||\beta|$.

(2)$|\alpha+\beta|^2=|\alpha|^2+|\beta|^2+2\alpha\cdot\beta\le|\alpha|^2+|\beta|^2+2|\alpha||\beta|=(|\alpha|+|\beta|)^2$,

$|\alpha+\beta|\le|\alpha|+|\beta|$.

$|\alpha-\beta|=|(\alpha-\gamma)-(\beta-\gamma)|\le|(\alpha-\gamma)|+|\beta-\gamma|=|\alpha-\gamma|+|\gamma-\beta|$.

(3)$P=\alpha,Q=\beta,R=\gamma,d(P,R)=|\alpha-\gamma|\le|\alpha-\beta|+|\beta-\gamma|=d(P,Q)+d(Q,R)$.

习题 6.2

1.求下列极限:

(1) $\lim\limits_{(x,y)\to(0,0)}\dfrac{3x^2-y^2+5}{x^2+y^2+2}=\dfrac{5}{2}.$

(2) $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin(x^2+y^2)}{x^2+y^2}=\lim\limits_{u\to0}\dfrac{\sin u}{u}=1.$

(3) $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}=0$(有界变量乘无穷小量得无穷小量).

(4) $\lim\limits_{(x,y)\to(0,1)}\dfrac{x^3+(y-1)^3}{x^2+(y-1)^2}=\lim\limits_{(x,y)\to(0,1)}\dfrac{x^3}{x^2+(y-1)^2}+\lim\limits_{(x,y)\to(0,1)}\dfrac{(y-1)^3}{x^2+(y-1)^2}$

$=\lim\limits_{(x,y)\to(0,1)}x\dfrac{x^2}{x^2+(y-1)^2}+\lim\limits_{(x,y)\to(0,1)}(y-1)\dfrac{(y-1)^2}{x^2+(y-1)^2}=0+0=0$

$\left(0\le\dfrac{x^2}{x^2+(y-1)^2}\le1,0\le\dfrac{(y-1)^2}{x^2+(y-1)^2}\le1\right).$

(5) $\lim\limits_{(x,y)\to(1,1)}\dfrac{xy-y-2x+2}{x-1}=\lim\limits_{(x,y)\to(1,1)}\dfrac{y(x-1)-2(x-1)}{x-1}=\lim\limits_{(x,y)\to(1,1)}(y-2)=-1.$

(6) $\lim\limits_{(x,y,z)\to(1-2,0)}\ln\sqrt{x^2+y^2+z^2}=\ln\sqrt{5}.$

2.证明:当$(x,y)\to(0,0)$时下列函数无极限:

(1) $f(x,y)=\dfrac{x^4-y^2}{x^4+y^2}$.由于

$\lim\limits_{\substack{(x,y)\to(0,0)\\y=x^2}}\dfrac{x^4-y^2}{x^4+y^2}=0,\ \lim\limits_{\substack{(x,y)\to(0,0)\\y=x}}\dfrac{x^4-y^2}{x^4+y^2}=\lim\limits_{x\to0}\dfrac{x^4-x^2}{x^4+x^2}=\lim\limits_{x\to0}\dfrac{x^2-1}{x^2+1}=-1\ne0,$

故当$(x,y)\to(0,0)$时上述函数无极限.

(2) $f(x,y)=\begin{cases}\dfrac{x+y}{x-y}, y\ne x,\\ 0, \quad y=x.\end{cases}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\y=x}}f(x,y)=0,\ \lim\limits_{\substack{(x,y)\to(0,0)\\y=2x}}f(x,y)=\lim\limits_{x\to0}\dfrac{3x}{-x}=-3\ne0.$

3.讨论当$(x,y)\to(0,0)$时下列函数是否有极限,若有极限,求出其值:

(1) $f(x,y)=(x+2y)\ln(x^2+y^2)=x\ln(x^2+y^2)+2y\ln(x^2+y^2),$

$|x\ln(x^2+y^2)|\le2\sqrt{|x|^2+|y|^2}\,|\ln\sqrt{|x|^2+|y|^2}|\to0((x,y)\to(0,0)),$

$\lim\limits_{(x,y)\to(0,0)}x\ln(x^2+y^2)=0.$类似有$\lim\limits_{(x,y)\to(0,0)}2y\ln(x^2+y^2)=0.$

故$\lim\limits_{(x,y)\to(0,0)}f(x,y)=0.$

(2)

(2) $f(x,y)=\dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$.$1-\cos(x^2+y^2)\sim\dfrac{1}{2}(x^2+y^2)^2$.

只需讨论$\dfrac{(x^2+y^2)^2}{(x^2+y^2)x^2y^2}=\dfrac{x^2+y^2}{x^2y^2}=\dfrac{1}{y^2}+\dfrac{1}{x^2}$极限存在与否.

$$\lim_{(x,y)\to(0,0)}\left(\frac{1}{y^2}+\frac{1}{x^2}\right)=+\infty,\lim_{(x,y)\to(0,0)}f(x,y)=+\infty.$$

不存在有限极限.

(3) $f(x,y)=(x^2+y^2)^{x^2y^2}=e^{x^2y^2\ln(x^2+y^2)}$,

$$|x^2y^2\ln(x^2+y^2)|=x^2y^2|\ln(x^2+y^2)|\le\frac{1}{2}(x^2+y^2)|\ln(x^2+y^2)|\to 0$$

$((x,y)\to(0,0))$. $\lim\limits_{(x,y)\to(0,0)}x^2y^2\ln(x^2+y^2)=0$,

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}e^{x^2y^2\ln(x^2+y^2)}=e^{\lim\limits_{(x,y)\to(0,0)}x^2y^2\ln(x^2+y^2)}=e^0=1.$$

(4) $f(x,y)=\dfrac{P_n(x,y)}{\rho^{n-1}}$, $n\ge 1$,其中$\rho=\sqrt{x^2+y^2}$, $P_n(x,y)$为n次齐次多项式.

$$0\le\alpha\le n,\left|\frac{x^\alpha y^{n-\alpha}}{\rho^{n-1}}\right|=\frac{|x|^\alpha|y|^{n-\alpha}}{\rho^{n-1}}\le\frac{\rho^\alpha\rho^{n-\alpha}}{\rho^{n-1}}=\rho\to 0(\rho\to 0).$$

故极限存在,并且等于零.

4.求下列函数的累次极限$\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)$及$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)$:

(1) $f(x,y)=\dfrac{|x|-|y|}{|x|+|y|}$.

$$\lim_{x\to0}\lim_{y\to0}\frac{|x|-|y|}{|x|+|y|}=\lim_{x\to0}\frac{|x|}{|x|}=1,\lim_{y\to0}\lim_{x\to0}\frac{|x|-|y|}{|x|+|y|}=\lim_{y\to0}\frac{-|y|}{|y|}=-1.$$

(2) $f(x,y)=\dfrac{y^3+\sin x^2}{x^2+y^2}$.

$$\lim_{x\to0}\lim_{y\to0}\frac{y^3+\sin x^2}{x^2+y^2}=\lim_{x\to0}\frac{\sin x^2}{x^2}=1,$$

$$\lim_{y\to0}\lim_{x\to0}\frac{y^3+\sin x^2}{x^2+y^2}=\lim_{y\to0}\frac{y^3}{y^2}=0.$$

(3) $f(x,y)=(1+x)^{\frac{y}{x}}(x\ne0)$, $f(0,y)=1$.

$$\lim_{x\to0}\lim_{y\to0}(1+x)^{\frac{y}{x}}=\lim_{x\to0}1=1,$$

$$\lim_{y\to0}\lim_{x\to0}(1+x)^{\frac{y}{x}}=\lim_{y\to0}e^y=1.$$

习题 6.3

1.下列函数在哪些点连续?

$(1) z=\dfrac{1}{x^2+y^2}.(x,y)\neq(0,0).$

$(2) z=\dfrac{1}{\sin x}+\dfrac{1}{\cos y}.x\neq m\pi, y\neq\dfrac{\pi}{2}(2n+1), m,n\in\mathbf{Z}.$

$(3) z=\dfrac{y^2+x}{y^2-2x}.y^2-2x\neq 0.$

2.设$\overline{D}$是平面Oxy上的有界闭区域,$P_0(x_{0,}y_0)$是D的外点.证明:在$\overline{D}$内一定存在与P_0距离最长的点,也存在与P_0距离最近的点.

证 考虑函数$f(x,y)=d((x,y),(x_{0,}y_0))=\sqrt{(x-x_0)^2+(y-y_0)^2}, (x,y)\in\overline{D}.$

f在$\overline{D}$上连续,$\overline{D}$是平面Oxy上的有界闭区域. 根据有界闭区域上连续函数的最值定理, 存在$P_1,P_2\in\overline{D}$,使得$\mathrm{f}(\mathrm{P}_2)\leq)\,\mathrm{f}(\mathrm{P})\leq\mathrm{f}(\mathrm{P}_1)$, P_1,P_2分别是$\overline{D}$内与P_0距离最长和最近的点.

3.设函数$f(x,y)$在区域D内连续,又点$(x_i,y_i)\in D(i=1,2,\cdots,n)$.证明:在$D$内存在点$(\xi,\eta)$,使

$$f(\xi,\eta)=\frac{1}{n}[f(x_1,y_1)+\cdots+f(x_n,y_n)].$$

证 写出连结点$(x_i,y_i)\in D(i=1,2,\cdots,n)$的折线方程:

$$L:\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}\alpha\leq t\leq\beta.L\subset D.$$

设$(\varphi(t_i),\psi(t_i))=(x_i,y_i),(i=1,2,\cdots,n).$

考虑一元函数$g(t)=f(\varphi(t),\psi(t))\in C([\alpha,\beta]).$

根据一元函数的介值定理, 存在$\tau\in[\alpha,\beta]$,使得

$$g(\tau)=\frac{1}{n}[g(t_1)+\cdots+g(t_n)],$$

令$(\varphi(\tau),\psi(\tau))=(\xi,\eta)$,则有

$$f(\xi,\eta)=\frac{1}{n}[f(x_1,y_1)+\cdots+f(x_n,y_n)].$$

4.已知二元函数 $f(x,y)$在(x_0,y_0)处连续,证明函数$u=f(x,y_0)$在x_0处连续.

证 $\forall\varepsilon>0$, $\exists\delta>0$, 使得当$|x-x_0|<\delta,|y-y_0|<\delta$时,$|f(x,y)-f(x_0,y_0)|<\varepsilon.$

特别有, 当$|x-x_0|<\delta$时,$|f(x,y_0)-f(x_0,y_0)|<\varepsilon$, 即$u=f(x,y_0)$在$x_0$处连续.

5.将区间套原理推广到$\mathbf{R}^2$中,也即证明下列命题:

设$R_n=\{(x,y)\,|\,a_n\leq x\leq b_n, c_n\leq y\leq d_n\}$,其中$0<b_n-a_n\to 0$且$0<d_n-c_n\to 0$, $R_{n+1}\subseteq R_n,\forall n=1,2,\cdots$.则存在唯一的一个点$(\xi,\eta)\in R_n,\forall n=1,2,\cdots$.

证

6.举出一个例子说明一个二元函数$u=f(x,y)$在$D=\{(x,y)\mid x^2+y^2<1\}$连续，但它在$D$中是无界的.

解$f(x,y)=\dfrac{1}{1-(x^2+y^2)},(x,y)\in D$在$D$连续，但它在$D$中是无界的.

7.设$z=f(P)$在区域D中连续,且D内有两点P_1与P_2.证明:对于任意η,
$f(P_1)\le\eta\le f(P_2)$,在D内存在一点P_0,使得$f(P_0)=\eta$.

证由于D是连通的,存在折线

$$L:\begin{cases}x=x(t)\\y=y(t)\end{cases},\alpha\le t\le\beta.,$$

$L\subset D,(x(\alpha),y(\alpha))=P_1,(x(\alpha),y(\alpha))=P_1,(x(\beta),y(\beta))=P_2.$

$\varphi(t)=f(x(t),y(t))$在$[\alpha,\beta]$连续,并且$\varphi(\alpha)\le\eta\le\varphi(\beta)$.

根据连续函数的介值定理,存在$t_0\in[\alpha,\beta]$,使得$\varphi(t_0)=\eta$.

记$(x(t_0),y(t_0))=P_0$,则有$f(P_0)=\eta$.

习题 6.4

1.求下列函数的一阶偏导数：

(1)$z=\ln(x+\sqrt{x^2+y^2})$.

$$\frac{\partial z}{\partial x}=\frac{1+\dfrac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{1}{\sqrt{x^2+y^2}},$$

$$\frac{\partial z}{\partial y}=\frac{\dfrac{y}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}.$$

(2)$z=\dfrac{x}{\sqrt{x^2+y^2}}$.

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}-\frac{x^2}{\sqrt{x^2+y^2}^3}=\frac{y^2}{\sqrt{x^2+y^2}^3},$$

$$\frac{\partial z}{\partial y}=-\frac{xy}{\sqrt{x^2+y^2}^3}.$$

(3)$z=x^{x^y}, \ln z=x^y\ln x$.

$$\frac{1}{z}=yx^{y-1}\ln x+x^{y-1},\frac{\partial z}{\partial x}=z(yx^{y-1}\ln x+x^{y-1}),$$

$$\frac{1}{z}\frac{\partial z}{\partial y}=x^y\ln x\ln x,\frac{\partial z}{\partial y}=z(x^y\ln^2 x).$$

(4)$z=\dfrac{xy}{x-y}$.

$$\frac{\partial z}{\partial x}=y\left(\frac{x-y-x}{(x-y)^2}\right)=\frac{-y^2}{(x-y)^2},$$

$$\frac{\partial z}{\partial y}=x\left(\frac{x-y+y}{(x-y)^2}\right)=\frac{x^2}{(x-y)^2}.$$

(5)$z=\arcsin(x\sqrt{y})$.

$$\frac{\partial z}{\partial x}=\frac{\sqrt{y}}{\sqrt{1-x^2y}},\frac{\partial z}{\partial y}=\frac{x}{2\sqrt{y}\sqrt{1-x^2y}}.$$

(6)$z=xe^{-xy}$.

$$\frac{\partial z}{\partial x}=e^{-xy}+xe^{-xy}(-y)=e^{-xy}(1-xy),\frac{\partial z}{\partial y}=-x^2e^{-xy}.$$

(7)$u=\dfrac{y}{x}+\dfrac{z}{y}-\dfrac{x}{z}$.

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}-\frac{1}{z},\frac{\partial u}{\partial y}=\frac{1}{x}-\frac{z}{y^2},\frac{\partial u}{\partial z}=\frac{1}{y}+\frac{x}{z^2}.$$

$(8) u=(xy)^z.$

$$\frac{\partial u}{\partial x}=yz(xy)^{z-1},\frac{\partial u}{\partial y}=xz(xy)^{z-1},\frac{\partial u}{\partial z}=(xy)^z.\ln(xy)$$

2求下列函数在指定点的偏导数:

$(1) z=\dfrac{x\arccos(y-1)-(y-1)\cos x}{1+\sin x+\sin(y-1)}$,求$\left.\dfrac{\partial z}{\partial x}\right|_{(0,1)}$ 及$\left.\dfrac{\partial z}{\partial y}\right|_{(0,1)}$.

$$\left.\frac{\partial z}{\partial x}\right|_{(0,1)}=\frac{d}{dx}\left.\frac{x}{1+\sin x}\right|_{x=0}=\frac{d}{dx}\left.\frac{1+\sin x-x\cos x}{(1+\sin x)^2}\right|_{x=0}=1,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,1)}=\frac{d}{dy}\left.\frac{-(y-1)}{1+\sin(y-1)}\right|_{y=1}=\frac{d}{dy}\left.\frac{-(1+\sin(y-1))+(y-1)\cos(y-1)}{(1+\sin(y-1))^2}\right|_{y=1}=-1.$$

$(2) z=\dfrac{2y}{y+\cos x}$,求$\left.\dfrac{\partial z}{\partial x}\right|_{(\frac{\pi}{2},1)}$ 及$\left.\dfrac{\partial z}{\partial y}\right|_{(\frac{\pi}{2},1)}$.

$$\frac{\partial z}{\partial x}=\frac{2y\sin x}{(y+\cos x)^2},\left.\frac{\partial z}{\partial y}\right|=2\times\frac{y+\cos x-y}{(y+\cos x)^2}=\frac{2\cos x}{(y+\cos x)^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{2},1)}=2,\left.\frac{\partial z}{\partial y}\right|_{(\frac{\pi}{2},1)}=0.$$

$(3) f(x,y,z)=\ln(xy+z)$,求$f_x(2,1,0),\ f_y(2,1,0),\ f_z(2,1,0)$.

$$f_x(x,y,z)=\frac{y}{xy+z},\ f_y(x,y,z)=\frac{x}{xy+z},\ f_z(x,y,z)=\frac{1}{xy+z}.$$

$$f_x(2,1,0)=\frac{1}{2},\ f_y(2,1,0)=1,\ f_z(x,y,z)=\frac{1}{2}.$$

3.证明函数$f(x,y)=\begin{cases}\dfrac{x^2+y^2}{|x|+|y|},(x,y)\neq(0,0),\\ 0, \qquad (x,y)=(0,0)\end{cases}$

在$(0,0)$连续,但是$f_x(0,0)$不存在.

证$|f(x,y)|=\dfrac{x^2+y^2}{|x|+|y|}\leq|x|+|y|\to 0((x,y)\to(0,0))$,

$f(x,y)\to f(0,0)=0((x,y)\to(0,0))$,

$f(x,y)$在$(0,0)$连续.

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\Delta x^2}{|\Delta x|}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{|\Delta x|}$$ 不存在.

4.设$z=\sqrt{x}\sin\frac{y}{x}$, 证明$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{z}{2}$.

证为齐1/2次函数,根据关于齐次函数微分的一个定理,立得结论.

直接计算如下.

$$\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x}}\sin\frac{y}{x}+\sqrt{x}\cos\frac{y}{x}\left(-\frac{y}{x^2}\right),\frac{\partial z}{\partial y}=\sqrt{x}\cos\frac{y}{x}\left(\frac{1}{x}\right),$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\sqrt{x}}{2}\sin\frac{y}{x}-\frac{y}{\sqrt{x}}\cos\frac{y}{x}+\frac{y}{\sqrt{x}}\cos\frac{y}{x}=\frac{\sqrt{x}}{2}\sin\frac{y}{x}=\frac{z}{2}.$$

5.求下列函数的二阶混合偏导数f_{xy}：

(1) $f(x,y)=\ln(2x+3y)$.

$$f_x=\frac{2}{2x+3y},\ f_{xy}=\frac{-6}{(2x+3y)^2}.$$

(2) $f(x,y)=y\sin x+e^x$.

$$f_x=y\cos x+e^x,\ f_{xy}=\cos x.$$

(3) $f(x,y)=x+xy^2+4x^3-\ln(x^2+1)$.

$$f_x=1+y^2+12x^2-\frac{2x}{x^2+1},\ f_{xy}=2y.$$

(4) $f(x,y)=x\ln(xy)=x\ln x+x\ln y$.

$$f_x=\ln y+\ln x+1,\ f_{xy}=\frac{1}{y}.$$

6.设$u=e^{-3y}\cos 3x$,证明u满足平面Laplace方程$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

证Q $\dfrac{\partial u}{\partial x}=-3e^{-3y}\sin 3x,\dfrac{\partial^2 u}{\partial x^2}=-9e^{-3y}\cos 3x,$

$$\frac{\partial u}{\partial y}=-3e^{-3y}\cos 3x,\frac{\partial^2 u}{\partial y^2}=9e^{-3y}\cos 3x,$$

$$\therefore \Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

7.证明函数$u(x,t)=e^{x+ct}+4\cos(3x+3ct)$满足波动方程$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$.

证$\dfrac{\partial u}{\partial t}=ce^{x+ct}-12c\sin(3x+3ct),\dfrac{\partial^2 u}{\partial t^2}=c^2e^{x+ct}-36c^2\cos(3x+3ct),$

$$\frac{\partial u}{\partial x}=e^{x+ct}-12\sin(3x+3ct),\frac{\partial^2 u}{\partial x^2}=e^{x+ct}-36\cos(3x+3ct),$$

故$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$.

8.设$u=u(x,y)$及$v=v(x,y)$在D内又连续的二阶偏导数,且满足方程组$\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$.证明$u$及$v$在$D$内满足平面Laplace方程$\Delta u=\Delta v=0$,其中$\Delta=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

证$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\dfrac{\partial v}{\partial y}=\dfrac{\partial^2 v}{\partial x\partial y},\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{\partial}{\partial y}\dfrac{\partial v}{\partial x}=-\dfrac{\partial^2 v}{\partial y\partial x}=-\dfrac{\partial^2 v}{\partial x\partial y}$($\dfrac{\partial^2 v}{\partial y\partial x}$和$\dfrac{\partial^2 v}{\partial x\partial y}$连续),

故$\Delta u=0$.类似证$\Delta v=0$.

9.已知函数$z(x,y)$满足$\dfrac{\partial z}{\partial x}=-\sin y+\dfrac{1}{1-xy}$以及$z(0,y)=2\sin y+y^2$. 试求z的表达式.

解$z=\int\left(-\sin y+\dfrac{1}{1-xy}\right)dx=-x\sin y-\dfrac{1}{y}\ln(1-xy)+C,$

$z(0,y)=C=2\sin y+y^2, z(x,y)=-x\sin y-\dfrac{1}{y}\ln|1-xy|+2\sin y+y^2$

$=(2-x)\sin y+y^2-\dfrac{1}{y}\ln|1-xy|.$

10.求下列函数的全微分:

(1)$z=e^{y/x}$.

$dz=e^{y/x}d\dfrac{y}{x}=e^{y/x}\dfrac{xdy-ydx}{x^2}.$

(2)$z=\dfrac{x+y}{x-y}.dz=\dfrac{(dx+dy)(x-y)-(x+y)(dx-dy)}{(x-y)^2}=\dfrac{(-2y)dx+(2x)dy)}{(x-y)^2}.$

(3)$z=\arctan\dfrac{y}{x}+\arctan\dfrac{x}{y}=\arctan\dfrac{y}{x}+\operatorname{arccot}\dfrac{y}{x}=\dfrac{\pi}{2}, dz=0.$

(4)$u=\sqrt{x^2+y^2+z^2}, du=\dfrac{d(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}=\dfrac{2(xd(x+ydy+zdz)}{\sqrt{x^2+y^2+z^2}}.$

11.已知函数$z(x,y)$的全微分

$dz=(4x^3+10xy^3-3y^4)dx+(15x^2y^2-12xy^3+5y^4)dy$,求$f(x,y)$的表达式.

解$\dfrac{\partial z}{\partial x}=4x^3+10xy^3-3y^4,\dfrac{\partial z}{\partial y}=15x^2y^2-12xy^3+5y^4.$

$z=\int(4x^3+10xy^3-3y^4)dx=x^4+5x^2y^3-3xy^4+C(y),$

$\dfrac{\partial z}{\partial y}=15x^2y^2-12xy^3+C'(y)=15x^2y^2-12xy^3+5y^4,$

$C'(y)=5y^4,C(y)=y^5+C.f(x,y)=x^4+5x^2y^3-3xy^4+y^5+C.$

12.已知函数$z=f(x,y)$的全微分

$dz=\left(x-\dfrac{y}{x^2+y^2}\right)dx+(y+\dfrac{x}{x^2+y^2})dy$,求$z(x,y)$的表达式.

解$dz=\left(x-\dfrac{y}{x^2+y^2}\right)dx+(y+\dfrac{x}{x^2+y^2})dy$

$=xdx+ydy+\dfrac{xdy-ydx}{x^2+y^2}$

$=\dfrac{1}{2}d(x^2+y^2)+\dfrac{\dfrac{xdy-ydx}{x^2}}{1+\left|\dfrac{y}{x}\right|^2}=\dfrac{1}{2}d(x^2+y^2)+\dfrac{d\dfrac{y}{x}}{1+\left|\dfrac{y}{x}\right|^2}=\dfrac{1}{2}d(x^2+y^2)+d\arctan\dfrac{y}{x}$

$=d\left(\dfrac{1}{2}(x^2+y^2)+\arctan\dfrac{y}{x}\right).$

$z=\dfrac{1}{2}(x^2+y^2)+\arctan\dfrac{y}{x}+C.$

13.$z=f(x,y)D:\{(x-x_0)^2+(y-y_0)^2<R^2\}\dfrac{\partial f}{\partial x}=0,\dfrac{\partial f}{\partial y}=0$.证明：$f(x,y)$

在区域上恒等于常数.

证$\forall(x,y)\in D,$

$f(x,y)-f(x_0,y_0)=[f(x,y)-f(x_0,y)]+[f(x_0,y)-f(x_0,y_0)]$

$=f_x(\xi,y)(x-x_0)+f_y(x_0,\eta)(y-y_0)=0.f(x,y)=f(x_0,y_0),(x,y)\in D.$

14.证明:函数$f(x,y)=\sqrt{|xy|}$在点$(0,0)$处连续,$f_x(0,0),f_y(0,0)$存在,但$f(x,y)$在$(0,0)$处不可微.

证$|f(x,y)|=\sqrt{|xy|}\to 0=f(0,0)((x,y)\to(0,0)),f(x,y)=\sqrt{|xy|}$在点$(0,0)$处连续.

$f_x(0,0)=0,f_y(0,0)=0$.若$f(x,y)$在$(0,0)$处可微，将有

$f(x,y)=o(\sqrt{x^2+y^2})(\sqrt{x^2+y^2}\to 0)$,特别应有

$f(x,x)=|x|=o(\sqrt{2}|x|)(x\to 0),$

但此式显然不成立.

15.设$P(x,y)dx+Q(x,y)dy$在区域D中是某个函数$u(x,y)$之全微分,且$P,Q\in C^1(D)$.
证明$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

证 由假设$du=P(x,y)dx+Q(x,y)dy.\frac{\partial u}{\partial x}=P,\frac{\partial u}{\partial y}=Q.$

$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial y\partial x},\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x\partial y},$

由$P,Q\in C^1(D)$得$\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}\in C(D)$,故$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$,即$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

16.设函数$f(x,y)=\begin{cases}\frac{(x^2-y^2)xy}{x^2+y^2},(x,y)\neq(0,0)\\ 0 \qquad (x,y)=(0,0).\end{cases}$

(1)计算$f_x(0,y)(y\neq 0)$;

(2)根据偏导数定义证明$f_x(0,0)=0$;

(3)在上述结果的基础上证明$f_{xy}(0,0)=-1$;

(4)重复上述步骤于$f_y(x,0)$,并证明$f_{yx}(0,0)=1$.

证(1)设$y\neq 0$,则$f_x(x,y)=\frac{[2x^2y+(x^2-y^2)y](x^2+y^2)-2x(x^2-y^2)xy}{(x^2+y^2)^2}$,

$f_x(0,y)=\frac{-y^5}{y^4}=-y.$

(2) $f(x,0)=0,\ f_x(0,0)=0.$

(3) $f_{xy}(0,0)=(-y)'|_{y=0}=-1.$

(4)设$x\neq 0$,则$f_y(x,y)=\frac{[-2xy^2+(x^2-y^2)x](x^2+y^2)-2y(x^2-y^2)xy}{(x^2+y^2)^2}$,

$f_y(x,0)=x.f(0,y)=0,\ f_y(0,0)=0.f_{yx}(0,0)=x'|_{x=0}=1.$

17.设$z=x\ln(xy)$,求$\frac{\partial^3 z}{\partial x^3}\frac{\partial^3 z}{\partial x\partial y^2}$.

解 $\frac{\partial z}{\partial x}=\ln(xy)+x\frac{y}{xy}=\ln(xy)+1,\frac{\partial^2 z}{\partial x^2}=\frac{1}{x},\frac{\partial^3 z}{\partial x^3}=-\frac{1}{x^2},$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{y},\frac{\partial^3 z}{\partial x\partial y^2}=-\frac{1}{y^2}.$

习题 6.5

在下面的习题中,出现的函数$f(u,v)$或$F(u)$一律假定有连续的一阶偏导数或导数.

1.求下列复合函数的偏导数或导数:

(1)$z=\sqrt{u^2+v^2}$,其中$u=xy,v=y^2$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{uy}{\sqrt{u^2+v^2}}=\frac{xy^2}{\sqrt{x^2y^2+y^4}}=\frac{x|y|}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{ux}{\sqrt{u^2+v^2}}+\frac{2vy}{\sqrt{u^2+v^2}}$$

$$=\frac{x^2y}{\sqrt{x^2y^2+y^4}}+\frac{2y^3}{\sqrt{x^2y^2+y^4}}=\frac{x^2\operatorname{sgn}y}{\sqrt{x^2+y^2}}+\frac{2y^2\operatorname{sgn}y}{\sqrt{x^2+y^2}}=\frac{(x^2+2y^2)\operatorname{sgn}y}{\sqrt{x^2+y^2}}.$$

(2)$z=\dfrac{u^2}{v}$,其中$u=ye^x,v=x\ln y$.

$$\frac{\partial z}{\partial x}=\frac{2u}{v}\times ye^x-\frac{u^2}{v^2}\ln y=\frac{2ye^x}{x\ln y}\times ye^x-\frac{(ye^x)^2}{(x\ln y)^2}\ln y=\frac{2y^2e^{2x}}{x\ln y}-\frac{(ye^x)^2}{x^2\ln y}=\frac{(2x-1)y^2e^{2x}}{x^2\ln y}$$

$$\frac{\partial z}{\partial y}=\frac{2ue^x}{v}-\frac{u^2x}{v^2y}=\frac{2ye^{2x}}{x\ln y}-\frac{y^2e^{2x}x}{x^2(\ln^2y)y}=\frac{ye^{2x}(2\ln y-1)}{x(\ln^2y)}.$$

(3)$z=f(u,v)$,其中$u=\sqrt{xy},v=x+y$.

$$\frac{\partial z}{\partial x}=f_u(u,v)\frac{y}{2\sqrt{xy}}+f_v(u,v),\frac{\partial z}{\partial y}=f_u(u,v)\frac{x}{2\sqrt{xy}}+f_v(u,v).$$

(4)$z=f\left(xy,\dfrac{x}{y}\right).\dfrac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot\dfrac{1}{y},\dfrac{\partial z}{\partial y}=f_1'\cdot x-f_2'\cdot\dfrac{x}{y^2}.$

(5)$z=f(x^2-y^2,e^{xy}),\dfrac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot e^{xy}y,\dfrac{\partial z}{\partial y}=f_1'\cdot(-2y)+f_2'\cdot e^{xy}x.$

2.设$u=f(x+y+z,x^2+y^2+z^2)$,求$\Delta u=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}+\dfrac{\partial^2u}{\partial z^2}$.

$$\frac{\partial u}{\partial x}=f_1'+2xf_2',$$

$$\frac{\partial^2u}{\partial x^2}=f_{11}''+2xf_{12}''+2f_2'+2x(f_{21}''+2xf_{22}'')=f_{11}''+4xf_{12}''+4x^2f_{22}''+2f_2',$$

$$\frac{\partial^2u}{\partial y^2}=f''+4yf_{12}''+4y^2f_{22}''+2f_2',\frac{\partial^2u}{\partial z^2}=f_{11}''+4zf_{12}''+4z^2f_{22}''+2f_2',$$

$$\Delta u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=3f_{11}''+4(x+y+z)f_{12}''+4(x^2+y^2+z^2)f_{22}''+6f_2'.$$

4.设$z=x^nf(\dfrac{y}{x^2})$其中函数f可微.证明z满足下列方程

$$x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial x}=nz.$$

证$\dfrac{\partial z}{\partial x}=nx^{n-1}f(\dfrac{y}{x^2})+x^n f'(\dfrac{y}{x^2})\left(-\dfrac{2y}{x^3}\right),\dfrac{\partial z}{\partial y}=x^n f'(\dfrac{y}{x^2})\left(\dfrac{1}{x^2}\right).$

$x\dfrac{\partial z}{\partial x}+2y\dfrac{\partial z}{\partial x}=x\left(nx^{n-1}f(\dfrac{y}{x^2})+x^n f'(\dfrac{y}{x^2})\left(-\dfrac{2y}{x^3}\right)\right)+2yx^n f'(\dfrac{y}{x^2})\left(\dfrac{1}{x^2}\right)$

$=nx^n f(\dfrac{y}{x^2})=nz.$

5.设$z=\dfrac{y}{F(x^2-y^2)}$,试证明$\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}.$

证$\dfrac{\partial z}{\partial x}=-\dfrac{2xyF'(x^2-y^2)}{(F(x^2-y^2))^2},\dfrac{\partial z}{\partial x}=\dfrac{F(x^2-y^2)+2y^2F'(x^2-y^2)}{(F(x^2-y^2))^2}.$

$\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}=-\dfrac{2yF'(x^2-y^2)}{(F(x^2-y^2))^2}+\dfrac{F(x^2-y^2)+2y^2F'(x^2-y^2)}{y(F(x^2-y^2))^2}$

$=\dfrac{1}{yF(x^2-y^2)}=\dfrac{1}{y^2}g\dfrac{y}{F(x^2-y^2)}=\dfrac{z}{y^2}.$

6.设函数$u(x,y)$有二阶连续偏导数且满足Laplace方程$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

证明,作变量替换$x=e^s\cos t, y=e^s\sin t$后,u依然满足关于s,t的Laplace方程

$\dfrac{\partial^2 u}{\partial s^2}+\dfrac{\partial^2 u}{\partial t^2}=0.$

证$\dfrac{\partial u}{\partial s}=\dfrac{\partial u}{\partial x}e^s\cos t+\dfrac{\partial u}{\partial y}e^s\sin t,$

$\dfrac{\partial^2 u}{\partial s^2}=e^s\cos t\left(\dfrac{\partial^2 u}{\partial x^2}e^s\cos t+\dfrac{\partial^2 u}{\partial x\partial y}e^s\sin t\right)+\dfrac{\partial u}{\partial x}e^s\cos t+e^s\sin t\left(\dfrac{\partial^2 u}{\partial x\partial y}e^s\cos t+\dfrac{\partial^2 u}{\partial y^2}e^s\sin t\right)$

$+e^s\sin t\dfrac{\partial u}{\partial y},$

$\dfrac{\partial u}{\partial t}=-\dfrac{\partial u}{\partial x}e^s\sin t+\dfrac{\partial u}{\partial y}e^s\cos t,$

$\dfrac{\partial^2 u}{\partial t^2}=-e^s\sin t\left(-\dfrac{\partial^2 u}{\partial x^2}e^s\sin t+\dfrac{\partial^2 u}{\partial x\partial y}e^s\cos t\right)-\dfrac{\partial u}{\partial x}e^s\cos t+e^s\cos t\left(-\dfrac{\partial^2 u}{\partial x\partial y}e^s\sin t+\dfrac{\partial^2 u}{\partial y^2}e^s\cos t\right)$

$-e^s\sin t\dfrac{\partial u}{\partial y}.$

$\dfrac{\partial^2 u}{\partial s^2}+\dfrac{\partial^2 u}{\partial t^2}=e^s\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

7.验证下列各式:

(1)$u=F(x^2+y^2)$则,$y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=0;$

(2)$u=F(x-ct)$,c为常数,则$\dfrac{\partial u}{\partial t}+c\dfrac{\partial u}{\partial x}=0.$

7.验证下列各式:

(1)$u=F(x^2+y^2)$则,$y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=0;$

(2)$u=F(x-ct)$,c为常数,则$\dfrac{\partial u}{\partial t}+c\dfrac{\partial u}{\partial x}=0.$

证(1)$\dfrac{\partial u}{\partial x}=F'(x^2+y^2)2x,\dfrac{\partial u}{\partial y}=F'(x^2+y^2)2y,$

$y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=yF'(x^2+y^2)2x-xF'(x^2+y^2)2y=0.$

(2)$\dfrac{\partial u}{\partial t}=F'(x-ct)(-c),\dfrac{\partial u}{\partial x}=F'(x-ct),$

$\dfrac{\partial u}{\partial t}+c\dfrac{\partial u}{\partial x}=F'(x-ct)(-c)+cF'(x-ct)=0.$

8.若$f(x,y,z)$满足关系式$f(tx,ty,tz)=t^n f(x,y,z)$,其中t为任意实数,则称f为n次齐次函数.证明,任意一个可微的n次齐次函数满足下列方程

$xf_x+yf_y+zf_z=nz.$

证$f(tx,ty,tz)=t^n f(x,y,z)$,对t求导,

$f_1'(tx,ty,tz)x+f_2'(tx,ty,tz)y+f_3'(tx,ty,tz)z=nt^{n-1}f(x,y,z),$

令$t=1$得$xf_x+yf_y+zf_z=nz.$

9.设$z=f(x,y)$在一个平面区域D中有定义.假定D有这样的性质,对于其中任意一点(x_0,y_0),区域D与直线$y=y_0$之交是一个区间.又设$z=f(x,y)$在区域D内有连续的一阶偏导数,若$f(x,y)$对x的偏导数恒为零,也即$\dfrac{\partial f(x,y)}{\partial x}=0,\forall(x,y)\in D.$

证明:$f(x,y)$可以表示成y的函数,也即存在一个函数$F(y)$,使得

$f(x,y)=F(y),\forall(x,y)\in D.$

证设$(x,y)\in D,(x_0,y)\in D,x_0<x.$ 由Lagrange中值公式,

$f(x,y)-f(x_0,y)=\dfrac{\partial f(\xi,y)}{\partial x}(x-x_0)=0.$

即$f(x,y)$的值不依赖x,只依赖y,其值记为$F(y)$,则有$f(x,y)=F(y),\forall(x,y)\in D.$

10.设$z=f(x,y)$在全平面上有定义,且有连续的一阶偏导数,满足方程

$xf_x(x,y)+yf_y(x,y)=0$.证明:存在一个函数$F(\theta)$,

使得$f(r\cos\theta,r\sin\theta)=F(\theta).$

证$x=r\cos\theta,y=r\sin\theta.\dfrac{\partial z}{\partial r}=\dfrac{\partial z}{\partial x}(r\cos\theta)+\dfrac{\partial z}{\partial y}(r\sin\theta)$

$=\dfrac{\partial z}{\partial x}(x)+\dfrac{\partial z}{\partial y}(y)=0.$

由上题,存在一个函数$G(r)$,使得$f(r\cos\theta,r\sin\theta)=F(\theta).$

11.设$z=f(x,y)$在全平面上有定义,且有连续的一阶偏导数,满足方程

$yf_x(x,y)-xf_y(x,y)=0$.证明:存在一个函数$G(r)$,

使得$f(r\cos\theta,r\sin\theta)=G(r)$.

证$x=r\cos\theta, y=r\sin\theta$. $\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$

$=\frac{\partial z}{\partial x}(-y)+\frac{\partial z}{\partial y}(x)=0.$

由9题,存在一个函数$G(r)$,使得$f(r\cos\theta,r\sin\theta)=G(r)$.

习题 6.6

1.求函数$f(x,y)=x^2-xy+y^2$在点$P_0(2+\sqrt{3},1+2\sqrt{3})$处沿极角为$\theta$的方向$l$的方向导数. 并问$\theta$取何值时,对应的方向导数(1)达到最大值;(2)达到最小值;(3)等于0.

解(1)$\nabla f(x,y)=(2x-y,-x+2y),\nabla f(2+\sqrt{3},1+2\sqrt{3})=(3,3\sqrt{3})=3(1,\sqrt{3})$.

$\frac{\partial f}{\partial l}=6(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta)=6(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta)=6\cos(\frac{\pi}{3}-\theta)$.

$\theta=\frac{\pi}{3}$.(2)$\theta=\frac{4\pi}{3}$.(3)$\theta=\frac{5\pi}{6},\frac{11\pi}{6}$.

2.求函数$f(x,y)=x^3-3x^2y+3xy^2+2$在点$P_0(3,1)$处沿从P_0到$P(6,5)$方向的方向导数.

解$\nabla f(x,y)=(3x^2-6xy+3y^2,-3x^2+6xy)$,

$\nabla f(3,1)=(12,-9)=3(4,-3)=3\sqrt{5}\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}}\right)$.

$l=(6,5)-(3,1)=(3,4)=5\left(\frac{3}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right),\frac{\partial f}{\partial l}(3,1)=3\sqrt{5}\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}}\right)g\left(\frac{3}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)=0$.

3.求函数$f(x,y)=\ln(x+y)$在点$(1,2)$沿抛物线$y=2x^2$在该点的切线方向的方向导数.

解$y'=4x$,切线斜率k=4, 方向$l=\pm(\frac{1}{\sqrt{17}},\frac{4}{\sqrt{17}})$. $\nabla f(x,y)=(\frac{1}{x+y},\frac{1}{x+y})$,

$\nabla f(1,2)=(\frac{1}{3},\frac{1}{3})$. $\frac{\partial f}{\partial l}=\pm\left(\frac{1}{3\sqrt{17}}+\frac{4}{3\sqrt{17}}\right)=\pm\frac{5}{3\sqrt{17}}$.

4.求函数$u(x,y,z)=xy+yz+zx$在点$P_0(2,1,3)$沿着与各坐标轴构成等角的方向的方向导数.

解设方向l与各坐标轴构成等角$\alpha,3\cos^2\alpha=1,\cos\alpha=\pm\frac{1}{\sqrt{3}}$.

$\nabla u(x,y,z)=(y+z,x+z,x+y),\nabla u(2,1,3)=(4,5,3)$.

$\frac{\partial u}{\partial l}=\pm\frac{12}{\sqrt{3}}=\pm4\sqrt{3}$.

5.求$z=f(x,y)=x^2+2xy+y^2$在点$(1,2)$处的梯度.

解$\nabla f(x,y)=(2x+2y,2x+2y)=2(x+y.x+y)$,

$\nabla f(1,2)==2(3,3)=6(1,1)$.

6.求$z=f(x,y)=\arctan\frac{y}{x}$在点$(x_0,y_0)$的梯度,并求沿向量$(x_0,y_0)$的方向导数.

$$\text{解}\nabla f(x,y)=\left(\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2},\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right)=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\right),$$

$$\nabla f(x_0,y_0)=\left(-\frac{y_0}{{x_0}^2+{y_0}^2},\frac{x_0}{{x_0}^2+{y_0}^2}\right),$$

$$\frac{\partial f}{\partial l}(x_0,y_0)=-\frac{y_0}{{x_0}^2+{y_0}^2}\frac{x_0}{\sqrt{{x_0}^2+{y_0}^2}}+\frac{x_0}{{x_0}^2+{y_0}^2}\frac{y_0}{\sqrt{{x_0}^2+{y_0}^2}}=0.$$

7.求函数$z=f(x,y)=\ln\dfrac{y}{x}$分别在点$A\left(\dfrac{1}{3},\dfrac{1}{10}\right)$及点$B\left(1,\dfrac{1}{6}\right)$处的两个梯度之间的夹角余弦.

解 $z=\ln|y|-\ln|x|.\nabla f(x,y)=\left(-\dfrac{1}{x},\dfrac{1}{y}\right),\nabla f\left(\dfrac{1}{3},\dfrac{1}{10}\right)=(-3,10),$

$\nabla f\left(1,\dfrac{1}{6}\right)=(-1,6).$

$$<\nabla f\left(\frac{1}{3},\frac{1}{10}\right),\nabla f\left(1,\frac{1}{6}\right)>=\frac{(-3,10)g(-1,6)}{\sqrt{109}\sqrt{37}}=\frac{63}{\sqrt{109}\sqrt{37}}.$$

8.求函数$f(x,y)=x(x-2y)+x^2y^2$在点$(1,1)$处沿方向$(\cos\alpha,\cos\beta)$的方向导数,并求出最大的与最小的方向导数,它们各沿什么方向?

解$\nabla f(x,y)=(2x-2y+2xy^2,-2x+2x^2y),\nabla f(1,1)=(2,0).$

$\dfrac{\partial f}{\partial l}(1,1)=2\cos\alpha.$

最大的与方向导数$:2,$最小的方向导数$:-2,$分别沿方向x轴方向和负x轴方向.

9.证明函数$f(x,y)=\dfrac{y}{x^2}$在椭圆周$x^2+2y^2=1$上任一点处沿椭圆周法方向的方向导数等于0.

证$\nabla f(x,y)=\left(-\dfrac{2y}{x^3},\dfrac{1}{x^2}\right)$.椭圆周法方向n(x, y)=(2x, 4y).

$$方向导数=\frac{1}{|n|}\left(-\frac{2y}{x^3}\times 2x+\frac{1}{x^2}\times 4y\right)=\frac{2(-2xy+2xy)}{|n|x^3}=0.$$

习题 6.7

1.求函数$f(x,y)=xy-y$在点(1,1)的二阶Taylor多项式.

解$f(x,y)=xy-y=(x-1+1)(y-1+1)-(y-1)-1$

$=(x-1)+(x-1)(y-1).$

2.在点(0,0)的邻域内,将下列函数按带Peano型余项展开成Taylor公式(到二阶):

$$(1) f(x,y)=\frac{\cos x}{\cos y}=\frac{1-\frac{x^2}{2}+o(x^2)}{1-\frac{y^2}{2}+o(y^2)}=\left(1-\frac{x^2}{2}+o(x^2)\right)\left(1+\frac{y^2}{2}+o(y^2)\right)$$

$$=1-\frac{x^2}{2}+\frac{y^2}{2}+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(2)\ f(x,y)=\ln(1+x+y)=x+y-\frac{1}{2}(x+y)^2+o(x^2+y^2)$$

$$=x+y-\frac{1}{2}(x^2+2xy+y^2)+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(3)\ f(x,y)=\sqrt{1-x^2-y^2}=1-\frac{1}{2}(x^2-y^2)+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(4)\ f(x,y)=\sin(x^2+y^2)=x^2+y^2 o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

3.在点(0,0)的邻域内,将函数$f(x,y)=\ln(1+x+y)$按Lagrange余项展开成Taylor公式(到一阶).

解 $\ln(1+x)=x-\frac{1}{2(1+\theta x)^2}x^2.$

$$\ln(1+x+y)=x+y-\frac{1}{2(1+\theta x+\theta y)^2}(x+y)^2.$$

4.利用Taylor公式证明:当$|x|,|y|,|z|$充分小时,有近似公式

$\cos(x+y+z)-\cos x\cos y\cos z\approx-(xy+yx+zx).$

证由于$\cos(x+y+z)-\cos x\cos y\cos z$

$$=1-\frac{1}{2}(x+y+z)^2+o(\rho^2)-\left(1-\frac{x^2}{2}+o(\rho^2)\right)\left(1-\frac{y^2}{2}+o(\rho^2)\right)\left(1-\frac{z^2}{2}+o(\rho^2)\right)$$

$=-(xy+yx+zx)+o(\rho^2)(\rho\to 0).$

故当$|x|,|y|,|z|$充分小时,有近似公式

$\cos(x+y+z)-\cos x\cos y\cos z\approx-(xy+yx+zx).$

5.设D是单位圆,即$D=\{(x,y)\,|\,x^2+y^2<1\}$,又设函数$f(x,y)$在$D$内有连续的偏导数且满足$xf_x(x,y)+yf_y(x,y)=0,(x,y)\in D$.证明: $f(x,y)$在D内是一常数.

证$f(x,y)-f(0,0)=f_x(\theta x,\theta y)x+f_y(\theta x,\theta y)y$

$$=\frac{1}{\theta}[f_x(\theta x,\theta y)\theta x+f_y(\theta x,\theta y)\theta y]=0.$$

$f(x,y)=f(0,0),(x,y)\in D.$

习题 6.8

在本节习题中所涉及的函数f或F都是有连续一阶偏导数的函数.

1.求由下列方程确定的隐函数$z=z(x,y)$的所有一阶偏导数:

(1)$x^3z+z^3x-2yz=0.$

$$3x^2z+x^3\frac{\partial z}{\partial x}+3z^2\frac{\partial z}{\partial x}x+z^3-2y\frac{\partial z}{\partial x}=0,$$

$$x^3\frac{\partial z}{\partial y}+3z^2\frac{\partial z}{\partial y}x-2z-2y\frac{\partial z}{\partial y}=0.$$

$$=-\frac{3x^2z+z^3}{x^3+3xz^2-2y},\frac{\partial z}{\partial x}=\frac{2z}{x^3+3xz^2-2y}.$$

(2)$yz-\ln z=x+y.$

$$y\frac{\partial z}{\partial x}-\frac{1}{z}\frac{\partial z}{\partial x}=1,z+y\frac{\partial z}{\partial y}-\frac{1}{z}\frac{\partial z}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=\frac{1}{y-\frac{1}{z}}=\frac{z}{yz-1},\frac{\partial z}{\partial y}=\frac{1-z}{y-\frac{1}{z}}=\frac{z-z^2}{yz-1}.$$

(3)$x+z-\varepsilon\sin z=y(0<\varepsilon<1).$

$$1+(1-\varepsilon\cos z)\frac{\partial z}{\partial x}=0,(1-\varepsilon\cos z)\frac{\partial z}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=-\frac{1}{1-\varepsilon\cos z},\frac{\partial z}{\partial y}=\frac{1}{1-\varepsilon\cos z}.$$

(4)$z^x=y^z.$

$$z^x\ln z+xz^{x-1}\frac{\partial z}{\partial x}=y^z\ln y\frac{\partial z}{\partial x},xz^{x-1}\frac{\partial z}{\partial y}=zy^{z-1}+y^z\ln y\frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial x}=-\frac{z^x\ln z}{xz^{x-1}-y^z\ln y}=-\frac{z^x\ln z}{xz^{x-1}-z^x\ln y}=-\frac{z\ln z}{x-z\ln y},$$

$$\frac{\partial z}{\partial y}=\frac{zy^{z-1}}{xz^{x-1}-y^z\ln y}=\frac{zy^z}{xyz^{x-1}-y^zy\ln y}=\frac{zz^x}{xyz^{x-1}-z^xy\ln y}=\frac{z^2}{xy-zy\ln y}.$$

(5)$x\cos y+y\cos z+z\cos x=1.$

$$\cos y-z\sin x+(-y\sin z+\cos x)\frac{\partial z}{\partial x}=0,\frac{\partial z}{\partial x}=\frac{\cos y+z\sin x}{y\sin x-\cos x}.$$

$$-x\sin y+\cos z+(-y\sin z+\cos x)\frac{\partial z}{\partial y}=0,\frac{\partial z}{\partial y}=\frac{x\sin y-\cos z}{\cos x-y\sin z}.$$

2.设由方程$f(xy^2,x+y)=0$确定隐函数为$y=y(x)$,求$\frac{dy}{dx}$.

解$f_1'(xy^2,x+y)(y^2+2xyy')+f_2'(xy^2,x+y)(1+y')=0,$

$$\frac{dy}{dx}=-\frac{y^2f_1'(xy^2,x+y)+f_2'(xy^2,x+y)}{2xyf_1'(xy^2,x+y)+f_2'(xy^2,x+y)}.$$

3.设$z+\cos xy=e^z$,求$\dfrac{\partial z}{\partial x}$及$\dfrac{\partial^2 z}{\partial x^2}$.

解$z+\cos xy-e^z=0.-y\sin xy+(1-e^z)\dfrac{\partial z}{\partial x}=0,\dfrac{\partial z}{\partial x}=\dfrac{y\sin xy}{1-e^z}$.

$$\frac{\partial^2 z}{\partial x^2}=\frac{y^2(1-e^z)\cos xy-y\sin xy(-e^z\dfrac{\partial z}{\partial x})}{(1-e^z)^2}$$

$$=\frac{y^2(1-e^z)\cos xy-y\sin xy(-e^z\dfrac{y\sin xy}{1-e^z})}{(1-e^z)^2}$$

$$=y^2\frac{(1-e^z)^2\cos xy+e^z\sin^2 xy}{(1-e^z)^3}.$$

4.设$F(x,x+y,x+y+z)=0$,求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解$F_1'+F_2'+F_3'(1+\dfrac{\partial z}{\partial x})=0,\dfrac{\partial z}{\partial x}=-\dfrac{F_1'+F_2'+F_3'}{F_3'}$.

$F_2'+F_3'(1+\dfrac{\partial z}{\partial y})=0,\dfrac{\partial z}{\partial y}=-\dfrac{F_2'+F_3'}{F_3'}$.

另解$\mathrm{d}F(x,x+y,x+y+z)=0$,

$F_1'dx+F_2'(dx+dy)+F_3'(dx+dy+dz)=0$,

$dz=-\dfrac{F_1'+F_2'+F_3'}{F_3'}dx-\dfrac{F_2'+F_3'}{F_3'}dy,\dfrac{\partial z}{\partial x}=-\dfrac{F_1'+F_2'+F_3'}{F_3'},\dfrac{\partial z}{\partial y}=-\dfrac{F_2'+F_3'}{F_3'}$.

5.设$z=z(x,y)$是方程$F(x,y,z)=0$确定的隐函数,利用一阶微分形式的不变型,

证明$dz=-\dfrac{F_x}{F_z}dx-\dfrac{F_y}{F_z}dy(F_z\neq 0)$,并求

$F(x^2+y^2+z^2,xy-z^2)=0$确定的隐函数$z=z(x,y)$的8微分dz.

证$dF(x,y,z)=F_xdx+F_ydy+F_zdz=0,dz=-\dfrac{F_x}{F_z}dx-\dfrac{F_y}{F_z}dy(F_z\neq 0)$.

记$dF(x^2+y^2+z^2,xy-z^2)=0$.

$F_1'(2xdx+2ydy+2zdz)+F_2'(ydx+xdy-2zdz)=0$,

$(2xF_1'+yF_2')dx+(2yF_1'+xF_2')dy+(2zF_1'-2zF_2')dz=0$,

$$dz=\frac{(2xF_1'+yF_2')dx+(2yF_1'+xF_2')dy}{2z(F_2'-F_1')}.$$

6.证明球坐标变换的Jacobi行列式$J=r^2\sin\varphi$.

证$\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$

证$dx=\sin\varphi\cos\theta dr+r\cos\varphi\cos\theta d\varphi-r\sin\varphi\sin\theta d\theta,$

$dy=\sin\varphi\sin\theta dr+r\cos\varphi\sin\theta d\varphi+r\sin\varphi\cos\theta d\theta,$

$dz=\cos\varphi dr-r\sin\varphi d\varphi.$

$$J=\begin{vmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta\\ \cos\varphi & -r\sin\varphi & 0\end{vmatrix}$$

$=r^2\cos^2\varphi\sin\varphi+r^2\sin^3\varphi=r^2\sin\varphi.$

7.设由$x=u+v,y=u^2+v^2$，$z=u^3+v^3$确定函数$z=z(x,y)$，求当

$x=0,y=u=\dfrac{1}{2},v=-\dfrac{1}{2}$时,$\dfrac{\partial z}{\partial x}$与$\dfrac{\partial z}{\partial y}$的值.

解$\mathrm{dz}=3\mathrm{u}^2du+3v^2dv.$

$$\begin{cases}du+dv=dx\\ 2udu+2vdv=dy\end{cases}du=\frac{2vdx-dy}{2v-2u},dv=\frac{dy-2udx}{2v-2u}.$$

$x=0,y=u=\dfrac{1}{2},v=-\dfrac{1}{2}$

$$du=\frac{-dx-dy}{-2},dv=\frac{dy-dx}{-2},$$

$$dz=\frac{3}{4}\times\frac{1}{2}(dx+dy)+\frac{3}{4}\times\frac{1}{2}(dx-dy)=\frac{3}{4}dx,$$

$$\frac{\partial z}{\partial x}=\frac{3}{4},\frac{\partial z}{\partial y}=0.$$

8.设$\begin{cases} xu+yv=0, \\ uv-xy=5. \end{cases}$

求当$x=1, y=-1, u=v=2$时$\dfrac{\partial^2 u}{\partial x^2}$与$\dfrac{\partial^2 u}{\partial x \partial y}$的值.

解$\begin{cases} udx+xdu+vdy+ydv=0, \\ vdu+udv-ydx-xdy=0. \end{cases}$

$$\begin{cases} xdu+ydv=-udx-vdy, \\ vdu+udv=ydx+xdy. \end{cases}$$

$$du=\frac{u(-udx-vdy)-y(ydx+xdy)}{xu-yv}=\frac{(-u^2-y^2)dx+(-uv-xy)dy}{xu-yv}$$

$$\frac{\partial u}{\partial x}=-\frac{u^2+y^2}{xu-yv}, \frac{\partial u}{\partial y}=-\frac{uv+xy}{xu-yv}.$$

$$dv=\frac{x(ydx+xdy)+v(udx+vdy)}{xu-yv}=\frac{(xy+uv)dx+(x^2+v^2)dy}{xu-yv},$$

$$\frac{\partial v}{\partial x}=\frac{(xy+uv)}{xu-yv}.$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial}{\partial \mathrm{x}}\frac{u^2+y^2}{xu-yv}=-\frac{\left(2u\dfrac{\partial u}{\partial x}\right)(xu-yv)-(u^2+y^2)\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)}{(xu-yv)^2}.$$

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial}{\partial x}\frac{\partial u}{\partial y}=-\frac{\partial}{\partial x}\frac{uv+xy}{xu-yv}$$

$$=-\frac{\left(\dfrac{\partial u}{\partial x}v+\dfrac{\partial v}{\partial x}u+y\right)(xu-yv)-\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)(uv+xy)}{(xu-yv)^2}.$$

当$x=1, y=-1, u=v=2$时,

$$\frac{\partial u}{\partial x}=-\frac{u^2+y^2}{xu-yv}=-\frac{5}{4}, \frac{\partial v}{\partial x}=\frac{(xy+uv)}{xu-yv}=\frac{3}{4}$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\left(2u\dfrac{\partial u}{\partial x}\right)(xu-yv)-(u^2+y^2)\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)}{(xu-yv)^2}$$

$$=-\frac{4\times(-5)-5\times(-\frac{5}{4}+\frac{3}{4})}{16}=\frac{55}{32}.$$

$$\frac{\partial^2 u}{\partial x \partial y}=--\frac{\left(\dfrac{\partial u}{\partial x}v+\dfrac{\partial v}{\partial x}u+y\right)(xu-yv)-\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)(uv+xy)}{(xu-yv)^2}=\frac{25}{32}.$$

9.设$x^2+y^2=\frac{1}{2}z^2, x+y+z=2$,求当$x=1, y=-1, z=2$时$\frac{dx}{dz}$与$\frac{dy}{dz}$的值.

解$\begin{cases}2xdx+2ydy=zdz\\ dx+dy+dz=0\end{cases}$

$$\begin{cases}2xdx+2ydy=zdz\\ dx+dy=-dz\end{cases}$$

$$dx=\frac{z+2y}{2-2y}dz, dy=\frac{-2x-z}{2-2y}dz,$$

$$\frac{dx}{dz}=\frac{z+2y}{2-2y}, \frac{dy}{dz}=\frac{-2x-z}{2-2y}.$$

当$x=1, y=-1, z=2$时

$$\frac{dx}{dz}=\frac{z+2y}{2-2y}=\frac{0}{4}=0, \frac{dy}{dz}=\frac{-2x-z}{2-2y}=\frac{-4}{4}=-1.$$

10.设$x=\cos\varphi\cos\theta, y=\cos\varphi\sin\theta, z=\sin\varphi$,求$\frac{\partial z}{\partial x}$.

解$\begin{cases}-\sin\varphi\cos\theta d\varphi-\cos\varphi\sin\theta d\theta=dx\\ -\sin\varphi\sin\theta d\varphi+\cos\varphi\cos\theta d\theta=dy\\ \cos\varphi d\varphi=dz\end{cases}$

由前两个方程解出

$$d\varphi=\frac{\begin{vmatrix}dx & -\cos\varphi\sin\theta\\ dy & \cos\varphi\cos\theta\end{vmatrix}}{\begin{vmatrix}-\sin\varphi\cos\theta & -\cos\varphi\sin\theta\\ -\sin\varphi\sin\theta & \cos\varphi\cos\theta\end{vmatrix}}=-\frac{\cos\varphi\cos\theta dx+\cos\varphi\sin\theta dy}{\sin\varphi\cos\varphi}$$

$$=-\frac{\cos\theta}{\sin\varphi}dx-\frac{\sin\theta}{\sin\varphi}dy,$$

$$dz=\cos\varphi d\varphi=-\frac{\cos\varphi\cos\theta}{\sin\varphi}dx-\frac{\cos\varphi\sin\theta}{\sin\varphi}dy$$

$$\frac{\partial z}{\partial x}=-\frac{\cos\varphi\cos\theta}{\sin\varphi}=-\frac{x}{z}.$$

另解$x^2+y^2+z^2=1, 2xdx+2zdz=0,$

$$\frac{\partial z}{\partial x}=-\frac{x}{z}.$$

再解$z=\pm\sqrt{1-x^2-y^2-z^2}$,

$$\frac{\partial z}{\partial x}=\frac{-x}{\pm\sqrt{1-x^2-y^2-z^2}}=-\frac{x}{z}.$$

11.设$u=u(x,y)$及$v=v(x,y)$有连续一阶偏导数, 又设$x=x(\xi,\eta)$及$y=y(\xi,\eta)$也有连续一阶偏导数, 且使复合函数$u=u(x(\xi,\eta),y(\xi,\eta))$及$v=v(x(\xi,\eta),y(\xi,\eta))$有定义. 证明

$$\frac{D(u,v)}{D(\xi,\eta)}=\frac{D(u,v)}{D(x,y)}\frac{D(x,y)}{D(\xi,\eta)}.$$

证 $\dfrac{\partial u}{\partial \xi}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \xi}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \xi},\dfrac{\partial u}{\partial \eta}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \eta}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \eta},$

$$\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi},\frac{\partial v}{\partial \eta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta},$$

$$\frac{D(u,v)}{D(\xi,\eta)}=\begin{vmatrix}\dfrac{\partial u}{\partial \xi} & \dfrac{\partial u}{\partial \eta}\\ \dfrac{\partial v}{\partial \xi} & \dfrac{\partial v}{\partial \eta}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \xi}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \xi} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \eta}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \eta}\\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial \xi}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial \xi} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial \eta}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial \eta}\end{vmatrix}$$

$$\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta}\\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\end{vmatrix}=\frac{D(u,v)}{D(x,y)}\frac{D(x,y)}{D(\xi,\eta)}.$$

习题 6.9

1.求下列函数的极值:

(1)$z=x^2(x-1)^2+y^2$.

$\frac{\partial z}{\partial x}=2x(x-1)^2+2(x-1)x^2$

$=x(x-1)(2x-2+2x)=x(x-1)(4x-2)=0,$

$x=0,\frac{1}{2},1.$

$\frac{\partial z}{\partial y}=2y=0, y=0.$

三个稳定点$(0,0),(\frac{1}{2},0),(1,0).2x(x-1)^2+2(x-1)x^2$

$A=\frac{\partial^2 z}{\partial x^2}=2(x-1)^2+4x(x-1)+2x^2+4x(x-1)=2(x-1)^2+8x(x-1)+2x^2,$

$C=\frac{\partial^2 z}{\partial y^2}=2, B=\frac{\partial^2 z}{\partial x\partial y}=0.$

$(0,0), A=2>0, B=0, C=2, AC-B^2=4,$极小值点，极小值$z(0,0)=1.$

$(\frac{1}{2},0), A=-1, B=0, C=2, AC-B^2=-2,$非极小值点.

$(1,0), A=2, B=0, C=2, AC-B^2=4>0,$极小值点.极小值$z(1,0)=0.$

(2)$z=2xy-5x^2-2y^2+4x+4y-1.$

$\frac{\partial z}{\partial x}=2y-10x+4=2(y-5x+2),$

$\frac{\partial z}{\partial y}=2x-4y+4=2(x-2y+2).$

$\begin{cases}-5x+y=-2\\ x-2y=-2\end{cases} x=\frac{2}{3}, y=\frac{4}{3}.$

稳定点$(\frac{2}{3},\frac{4}{3}).$

$A=\frac{\partial^2 z}{\partial x^2}=-10<0, C=\frac{\partial^2 z}{\partial y^2}=-4, B=\frac{\partial^2 z}{\partial x\partial y}=2.$

$AC-B^2=36>0,(\frac{2}{3},\frac{4}{3})$极大值点.

极大值$=z(\frac{2}{3},\frac{4}{3})=3.$

$(3) z=6x^2-2x^3+3y^2+6xy+1.$

$$\frac{\partial z}{\partial x}=12x-6x^2+6y=6(2x-x^2+y)$$

$$\frac{\partial z}{\partial y}=6y+6x=6(x+y)$$

$$\begin{cases}2x-x^2+y=0\\x+y=0\end{cases} x=0,1,\text{相应地}y=0,-1.$$

稳定点$(0,0),(1,-1)$.

在点$(0,0)$, $A=\dfrac{\partial^2 z}{\partial x^2}=12-12x=12>0, C=\dfrac{\partial^2 z}{\partial y^2}=6, B=\dfrac{\partial^2 z}{\partial x\partial y}=6.$

$AC-B^2=66>0,(0,0)$极小值点, 极小值z(0, 0)=1.

在点$(1,-1)$， $A=\dfrac{\partial^2 z}{\partial x^2}=12-12x=0, C=\dfrac{\partial^2 z}{\partial y^2}=6, B=\dfrac{\partial^2 z}{\partial x\partial y}=6.$

$AC-B^2=-36<0.z$不取极值.

$(4) z=4xy-x^4-y^4+5.$

$$\frac{\partial z}{\partial x}=4y-4x^3=4(y-x^3),$$

$$\frac{\partial z}{\partial y}=4x-4y^3=4(x-y^3).$$

$$\begin{cases}y-x^3=0\\x-y^3=0\end{cases} x=0,\pm1,\text{相应地}y=0,\pm1.\text{稳定点}(0,0),(1,1),(-1,-1).$$

在点$(0,0)$, $A=\dfrac{\partial^2 z}{\partial x^2}=-12x^2=0, C=\dfrac{\partial^2 z}{\partial y^2}=-12y^2=0, B=\dfrac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=-16<0,(0,0)$不是极值点.

在点$(1,1)$， $A=\dfrac{\partial^2 z}{\partial x^2}=-12<0, C=\dfrac{\partial^2 z}{\partial y^2}=-12, B=\dfrac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=128>0.z$取极大值$z(1,1)=7$.

在点$(-1,-1)$， $A=\dfrac{\partial^2 z}{\partial x^2}=-12<0, C=\dfrac{\partial^2 z}{\partial y^2}=-12, B=\dfrac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=128>0.z$取极大值$z(-1,-1)=7$.

$(5) z = x^3 y^2 (6 - x - y)(x > 0, y > 0).$

$$\frac{\partial z}{\partial x} = 3x^2 y^2 (6 - x - y) - x^3 y^2 = x^2 y^2 (18 - 3x - 3y - x) = x^2 y^2 (18 - 4x - 3y)$$

$$\frac{\partial z}{\partial y} = 2x^3 y(6 - x - y) - x^3 y^2 = x^3 y(12 - 2x - 2y - y) = x^3 y(12 - 2x - 3y).$$

$\begin{cases} 4x + 3y = 18 \\ 2x + 3y = 12 \end{cases}$在$\{(x, y) \mid x > 0, y > 0\}$的稳定点$(x, y) = (3, 2).$

在稳定点$(3, 2), A = \dfrac{\partial^2 z}{\partial x^2} = 2xy^2 (18 - 4x - 3y) - 4x^2 y^2 = -144,$

$$C = \frac{\partial^2 z}{\partial y^2} = x^3 (12 - 2x - 3y) - 3x^3 y = -162,$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = 2x^2 y(18 - 4x - 3y) - 3x^2 y^2 = -108.$$

$AC - B^2 = 144 \times 162 - 108^2 = 11664 > 0, (3, 2)$极大值点, 极大值$z(3, 2) = 108.$

2.确定下列函数在所给条件下的最大值及最小值:

(1)$z=x^2+y^2$,当$\frac{x}{2}+\frac{y}{3}=1$时.

解由于$\sqrt{x^2+y^2}\to+\infty$时，$z\to+\infty$,又$z=x^2+y^2$是连续函数，故在平面$\frac{x}{2}+\frac{y}{3}=1$

上取极小值. $z=$代入法.$z=x^2+(3(1-\frac{x}{2}))^2=x^2+\frac{9}{4}(2-x)^2=\frac{13}{4}x^2-9x+9$

$=\frac{1}{4}(13x^2-36x+36)=f(x),x\in(-\infty,+\infty)$.

$f'(x)=\frac{1}{4}(26x-36)=0,x_0=\frac{18}{13},y_0=3(1-\frac{9}{13})=\frac{12}{13}$.

$f''(x)=\frac{13}{2}>0$.$\frac{18}{13}$是唯一极值点,且是极小值点, 故是最小值点.

最小值$f(\frac{18}{13})=\frac{36}{13}$.

对二次函数f用配方法当然得到同一结果.

再解Lagrange乘子法. 考虑Lagrange函数

$F(x,y,\lambda)=x^2+y^2+\lambda\left(\frac{x}{2}+\frac{y}{3}-1\right)$.

$$\begin{cases}2x+\frac{\lambda}{2}=0,\\ 2y+\frac{\lambda}{3}=0, \quad x=-\frac{\lambda}{4},y=-\frac{\lambda}{6},-\frac{\lambda}{8}-\frac{\lambda}{18}-1=0,\lambda_0=-\frac{72}{13}.\\ \frac{x}{2}+\frac{y}{3}-1=0.\end{cases}$$

得到满足条件的唯一点$x_0=\frac{18}{13},y_0=\frac{12}{13}$.$z(x_0,y_0)$是最小值.

3.在某一行星表面要安装一个无线电望远镜,为了减少干扰,要将望远镜装在磁场最弱的位置.设该行星为一球体,半径为6个单位.若以球心为坐标原点建立坐标系$Oxyz$,则行星表面上点(x,y,z)处的磁场强度为$H(x,y,z)=6x-y^2+xz+60$.问,应将望远镜安装在何处?

解球面方程:$x^2+y^2+z^2=36$.$F(x,y,z,\lambda)=H(x,y,z)+\lambda(x^2+y^2+z^2-36)$.

$$\begin{cases}\frac{\partial H}{\partial x}=6+z+2\lambda x=0 & (1)\\ \frac{\partial H}{\partial y}=-2y+2\lambda y=2y(\lambda-1)=0 & (2)\\ \frac{\partial H}{\partial z}=x+2\lambda z=0 & (3)\\ x^2+y^2+z^2=36 & (4)\end{cases}$$

由(2), $y=0$或$\lambda=1$.

设$y=0$,则有$\begin{cases}6+z+2\lambda x=0\\ x+2\lambda z=0\\ x^2+y^2+z^2=36\end{cases}$

解之得$(\pm5,0,3),(0.0,-6)$,相应H值为105,15和60.

设$\lambda=1$,则

$$\begin{cases}6+z+2x=0\\ x+2z=0\\ x^2+y^2+z^2=36\end{cases}$$

解之得$(-4,\pm4,2)$,相应H值为12.各条件极值比较得 $(x,y,z)=(-4,\pm4,2)$时

H取最小值12.

4.已知三角形的周长为$2p$,问怎样的三角形绕自己的一边旋转所得的体积最大?

解 设三角形底边上的高为x,垂足分底边的长度为y,z.设三角形饶底边旋转,

旋转体体积

$$V=\frac{\pi}{3}x^2(y+z),y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}=2p,x\geq0,y\geq0,z\geq0.$$

V在有界闭集上取最大值.

$$L(x,y,z,\lambda)=x^2(y+z)+\lambda(y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}-2p),$$

$$\begin{cases}2x(y+z)+\lambda\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{x}{\sqrt{x^2+z^2}}\right)=0,(1)\\ x^2+\lambda\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)=0, \qquad (2)\\ x^2+\lambda\left(1+\frac{z}{\sqrt{x^2+z^2}}\right)=0. \qquad (3)\\ y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}-2p=0. \quad (4)\end{cases}$$

$$(2)-(3)\Rightarrow\lambda(\frac{y}{\sqrt{x^2+y^2}}-\frac{z}{\sqrt{x^2+z^2}})=0.$$

若$\lambda=0$,将有$x=0$,不可能. 故$\frac{y}{\sqrt{x^2+y^2}}-\frac{z}{\sqrt{x^2+z^2}}=0$.

由于$y>0,z>0$,易得$y=z$.

$$\begin{cases}2xy+\frac{\lambda x}{\sqrt{x^2+y^2}}=0,\\ x^2+\lambda\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)=0,\\ y+\sqrt{x^2+y^2}=p.\end{cases}\begin{cases}2y+\frac{\lambda}{\sqrt{x^2+y^2}}=0,\\ x^2+\lambda\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)=0,\\ y+\sqrt{x^2+y^2}=p.\end{cases}$$

解之得$y=z=\dfrac{p}{4}$,底边长$=\dfrac{p}{2}$,两腰长$=\dfrac{1}{2}(2p-\dfrac{p}{2})=\dfrac{3p}{4}$.

5.在两平面有$y+2z=12$及$x+y=6$的交线上求到原点距离最近的点.

解$u=x^2+y^2+z^2$,

$$z=6-\frac{y}{2},x=6-y,u=(6-y)^2+y^2+\left(6-\frac{y}{2}\right)^2=\frac{9}{4}y^2-18y+72.$$

$z'=\dfrac{9}{2}y-18=0,z''=\dfrac{9}{2}.y_0=4$是唯一极值点,且是极小值点,故是最小值点.

$x_0=2,z_0=4$.所求的点为$(2,4,4)$.

6.求椭球面$x^2+y^2+\dfrac{z^2}{4}=1$与平面$x+y+z=0$的交线上到坐标原点的最大距离与最小距离.

解$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2+\dfrac{z^2}{4}-1)+\mu(x+y+z)$.

$$(*)\begin{cases}L_x=2x+2\lambda x+\mu=0,\\ L_y=2y+2\lambda y+\mu=0,\\ L_z=2z+\dfrac{1}{2}\lambda z+\mu=0,\\ x^2+y^2+\dfrac{z^2}{4}=1,\\ x+y+z=0.\end{cases}$$

由前三个方程得

$$(**)\begin{cases}2x(1+\lambda)=2z+\dfrac{1}{2}\lambda z,\\ 2y(1+\lambda)=2z+\dfrac{1}{2}\lambda z.\end{cases}$$

下面分两种情况求解.

(1)$\lambda=-1$.由方程组(**)得$z=0$,再由(*)的后两个方程得$(\dfrac{1}{\sqrt{2}},-\dfrac{1}{\sqrt{2}},0)$,

$(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}},0)$.这两点与原点距离为1.

(2)$\lambda\neq-1$..由方程组(**)得$x=y$,再由(*)的后两个方程得$(\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},-\dfrac{2}{\sqrt{3}})$,

$(-\dfrac{1}{\sqrt{3}},-\dfrac{1}{\sqrt{3}},\dfrac{2}{\sqrt{3}})$.这两点与原点距离为2.

在$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$和$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$有最小距离1,在$(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{3}})$和$(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}})$有最大距离2.

7.在已知圆锥体内做一内接长方体,长方体的底面在圆锥体的底面上, 求使体积最大的那个长方体的边长.

解 设圆锥体高为H,底半径为R.取其底面为xy平面,底面中心为坐标原点.设内接长方体底面边长为 2x, 2y, 高为z, 则长方体体积

$V=4xyz, x\geq 0, y\geq 0, z\geq 0$ 满足圆锥面方程 $(H-z)^2=\frac{H^2}{R^2}(x^2+y^2)$.

$$L(x,y,z,\lambda)=xyz+\lambda\left((H-z)^2-\frac{H^2}{R^2}(x^2+y^2)\right).$$

$$(*)\begin{cases} L_x=yz-2\lambda\frac{H^2}{R^2}x=0, \\ L_y=xz-2\lambda\frac{H^2}{R^2}y=0, \\ L_z=xy-2\lambda(H-z)=0, \\ (H-z)^2=\frac{H^2}{R^2}(x^2+y^2). \end{cases}$$

由(*)的前两个方程易得$x=y$.由(*)的前三个方程易得$x^2=y^2=\frac{R^2}{H^2}z(H-z)$.

再与第四个方程联立得$(H-z)^2=2z(H-z), z=\frac{H}{3}, x=y=\frac{\sqrt{2}}{3}R$.

8.当n个正数$x_1,\cdots,x_n$的和等于常数l时,求它们的乘积的最大值.并证明:n个正数$a_1,\cdots,a_n$的几何平均值不超过算术平均值,即$\sqrt[n]{a_1\cdots a_n}\le\dfrac{a_1+\cdots+a_n}{n}$.

解$f(x_1,\cdots,x_n)=x_1\cdots x_n, x_1+\cdots+x_n=l.$

$F(x_1,\cdots,x_n,\lambda)=x_1\cdots x_n+\lambda(x_1+\cdots+x_n-l),$

$$\begin{cases}F_{x_1}=x_2x_3\cdots x_n+\lambda=0\\F_{x_2}=x_1x_3\cdots x_n+\lambda=0\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\F_{x_n}=x_1x_2\cdots x_{n-1}+\lambda=0.\end{cases}$$

$$\begin{cases}x_1x_2x_3\cdots x_n+\lambda x_1=0\\x_1x_2x_3\cdots x_n+\lambda x_2=0\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\x_1x_2\cdots x_{n-1}x_n+\lambda x_n=0.\end{cases}\quad \lambda x_1=\lambda x_2\cdots=\lambda x_n.$$

若$\lambda=0$,将有$x_1x_2\cdots x_{n-1}x_n=0$,不会是最大值.若$\lambda\ne0$,则有$x_1=x_2\cdots=x_n=\dfrac{l}{n}$.

$$x_1x_2\cdots x_{n-1}x_n=\left(\frac{l}{n}\right)^n,\sqrt[n]{x_1x_2\cdots x_{n-1}x_n}=\left(\frac{l}{n}\right)=\frac{x_1+\cdots+x_n}{n}.$$

10.求函数$f(x,y)=\dfrac{1}{2}(x^n+y^n)(n>1$是常数$,x\ge0,y\ge0,)$在条件

$x+y=A(A>0)$下的最小值,并由此证明

$$\frac{1}{2}(x^n+y^n)\ge\left(\frac{x+y}{2}\right)^n(x>0,y>0).$$

9.在椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$上哪些点处,其切线与坐标轴构成的三角形面积最大?

解 $\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$,切线斜率:$y'=-\frac{b^2x}{a^2y}$,切线的点$(X,Y)$满足方程:

$Y-y=-\frac{b^2x}{a^2y}(X-x).Y_0=0,X_0=x+\frac{a^2y^2}{b^2x}.X_1=0,Y_1=y+\frac{b^2x^2}{a^2y}.$

三角形面积$f(x,y)=\left(x+\frac{a^2y^2}{b^2x}\right)\left(y+\frac{b^2x^2}{a^2y}\right)=\frac{a^2b^2}{xy}$,$(x,y)$满足

$x>0,y>0,\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

由于$x\to 0\,y\to 0$时$f(x,y)\to+\infty$,故f在所述条件下取极小值.

令$L(x,y,\lambda)=\frac{a^2b^2}{xy}+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$,

$$\begin{cases}L_x=-\frac{a^2b^2}{x^2y}+2\frac{\lambda x}{a^2}=0,\\ L_y=-\frac{a^2b^2}{xy^2}+2\frac{\lambda y}{b^2}=0,\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.\end{cases}\begin{cases}-\frac{a^2b^2}{x^2y^2}+2\frac{\lambda x}{a^2y}=0\\ -\frac{a^2b^2}{x^2y^2}+2\frac{\lambda y}{b^2x}=0\end{cases}\frac{\lambda x}{a^2y}=\frac{\lambda y}{b^2x}$$

易见$\lambda\neq 0$,故$\frac{x}{a^2y}=\frac{y}{b^2x},\frac{y}{x}=\frac{b}{a},y=\frac{b}{a}x$,代入椭圆方程得

$\frac{x^2}{a^2}+\frac{x^2}{a^2}=1,x=\frac{a}{\sqrt{2}},y=\frac{b}{\sqrt{2}}.$

在第一象限,$(x,y)=(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})$时,该点切线与坐标轴构成的三角形面积

最小. 由对称性,$(-\frac{a}{\sqrt{2}},-\frac{b}{\sqrt{2}}),(\frac{a}{\sqrt{2}},-\frac{b}{\sqrt{2}}),(-\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})$也满足要求.

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习题 6.10

在指定的各点求曲面的切平面：

$(1)\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1(a>0,b>0,c>0)$,在$\left(0,\dfrac{b}{\sqrt{2}},\dfrac{c}{\sqrt{2}}\right)$点.

$\boldsymbol{n}=\left(\dfrac{2x}{a^2},\dfrac{2y}{b^2},\dfrac{2z}{c^2}\right)=\left(0,\dfrac{\sqrt{2}}{b},\dfrac{\sqrt{2}}{c}\right)$,

$\dfrac{\sqrt{2}}{b}(x-\dfrac{b}{\sqrt{2}})+\dfrac{\sqrt{2}}{c}(x-\dfrac{c}{\sqrt{2}})=0$,

$\dfrac{\sqrt{2}}{b}x+\dfrac{\sqrt{2}}{c}y-2=0$.

$(2)z=x^2-y^2,(2,1,3).x^2-y^2-z=0$

$\boldsymbol{n}=(2x,-2y,-1)=(4,-2,-1)$,

$4(x-2)-2(y-1)-(z-3)=0,4x-2y-3=0$.

$(3)x=\cosh\rho\cos\theta,y=\cosh\rho\sin\theta,z=\rho(\rho>0,0\le\theta\le 2\pi),\rho=1,\theta=\dfrac{\pi}{2}$.

$$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \sinh\rho\cos\theta & \sinh\rho\sin\theta & 1\\ -\cosh\rho\sin\theta & \cosh\rho\cos\theta & 0\end{vmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & \sinh 1 & 1\\ -\cosh 1 & 0 & 0\end{vmatrix}=(0,-\cosh 1,\sinh 1\cosh 1),$$

$(0,\cosh 1,1)$,

$-\cosh 1(y-\cosh 1)+\sinh 1\cosh 1(z-1)=0$.

$(4)e^z-2z+xy=3,(2,1,0)$

$\boldsymbol{n}=(y,x,e^z-2)=(1,2,-1)$,

$(x-2)+2(y-1)-z=0,x+2y-z-4=0$.

2.试证明曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$上任一点的切平面在各坐标轴上截距之和等于$a$.

证$\boldsymbol{n}=\left(\dfrac{1}{2\sqrt{x}},\dfrac{1}{2\sqrt{y}},\dfrac{1}{2\sqrt{z}}\right)$,

$\dfrac{1}{\sqrt{x}}(X-x)+\dfrac{1}{\sqrt{y}}(Y-y)+\dfrac{1}{\sqrt{z}}(Z-z)=0$,

$\dfrac{1}{\sqrt{x}}X+\dfrac{1}{\sqrt{y}}Y+\dfrac{1}{\sqrt{z}}Z-\sqrt{a}=0$,

x轴上截距$X_0=\sqrt{x}\sqrt{a},Y_0=\sqrt{y}\sqrt{a},Z_0=\sqrt{z}\sqrt{a}$,

$X_0+Y_0+Z_0=\sqrt{x}\sqrt{a}+\sqrt{y}\sqrt{a},+\sqrt{z}\sqrt{a}=(\sqrt{x}+\sqrt{y},+\sqrt{z})\sqrt{a}=\sqrt{a}\sqrt{a}=a$.