2. ① 沿 $y=x$ 设 $z=t+it$ $(0\le t\le 1)$

$dz=(1+i)\,dt$

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^1(t^2+it)(1+i)\,dt$$

$$=(1+i)\int_0^1(t^2+it)\,dt$$

$$=(1+i)\cdot\frac{1}{3}t^3+\frac{1}{2}it^2\Big|_0^1$$

$$=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)$$

$$=-\frac{1}{6}+\frac{5}{6}i$$

~~② 沿 $y=x^2$ 设 $z=t^2+it^2$ $(0\le t\le 1)$~~

~~$dz=(1+i)\cdot 2t\cdot dt$~~

~~$=2(1+i)\cdot\frac{1}{4}t^4+\frac{1}{4}it^4\Big|_0^1$~~

~~$=2(1+i)\cdot\left(\frac{1}{4}+\frac{i}{4}\right)$~~

~~$=(1+i)\left(\frac{1}{2}+\frac{i}{2}\right)$~~

~~$=i$~~

② 沿 $y=x^2$. 则对 $z=t+it^2$ $(0\le t\le 1)$ $dz=(1+2it)\,dt$

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^t(t^2+it^2)(1+2it)\,dt$$

$$=(1+i)\int_0^1(t^2+2it^3)\,dt$$

$$=(1+i)\cdot\frac{1}{3}t^3+\frac{i}{2}t^4\Big|_0^1$$

$$=(1+i)\cdot\left(\frac{1}{3}+\frac{i}{2}\right)$$

$$=-\frac{1}{6}+\frac{5}{6}i$$

4.

由于 $\bar{z}=\frac{1}{z}$

所以 $\oint_C \bar{z}\,dz = \oint_C \frac{1}{z}\,dz$

柯西积分公式 $\oint_C \frac{\varphi(z)}{z-z_0}\,dz = 2\pi i\,\varphi(z_0)$

取 $z_0=0$, $\varphi(z)=1$. C为正向单位圆周 $|z|=1$

$\therefore \oint_C \frac{1}{z}\,dz = 2\pi i$ 故 $\oint_C \bar{z}\,dz = 2\pi i$

5. (1) 由题(4)知 $\oint_C \frac{1}{z}\,dz = 2\pi i$

由于 $|z|=2$, $\therefore z\cdot\bar{z}=|z|^2=4$. $\therefore \bar{z}=\frac{4}{z}$

$\oint_C \frac{\bar{z}}{|z|}\,dz = \oint_C \frac{\frac{4}{z}}{2}\,dz = \oint_C \frac{2}{z}\,dz = 4\pi i$

(2) 由于 $|z|=4$. $z\cdot\bar{z}=|z|^2=16$ $\therefore \bar{z}=\frac{16}{z}$

$\oint_C \frac{\bar{z}}{|z|}\,dz = \oint_C \frac{\frac{16}{z}}{4}\,dz = \oint_C \frac{4}{z}\,dz = 8\pi i$.

6. (2) $\oint_C \frac{dz}{z^2+2z+4}$ 被积函数有两个奇点 $z_1=-1+\sqrt{3}i$ $z_2=-1-\sqrt{3}i$

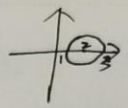
都在 $|z|=1$ 之外. 所以利用柯西一古萨定理知积分为0.

(4) $\oint_C \frac{dz}{z-\frac{1}{2}} = 2\pi i$ 利用柯西积分公式可知.

(6) $\oint_C \frac{dz}{(z-\frac{i}{2})(z+2)} = \oint_C \frac{\frac{1}{z+2}}{z-\frac{i}{2}}\,dz = 2\pi i\cdot\frac{1}{z+2}\Big|_{z=\frac{i}{2}}$

$= \frac{2\pi i}{\frac{i}{2}+2} = \frac{4\pi i}{4+i}$

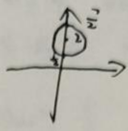
7. (1) $\oint_C \frac{e^z}{z-2}\,dz$. $C: |z-2|=1$



奇点 $z=2$ 在C内.

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\cdot e^z\Big|_{z=2} = 2\pi i\cdot e^2$$

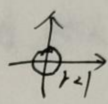
(3). $\oint_C \frac{e^{iz}dz}{z^2+1}$ $C: |z-2i|=\frac{3}{2}$.



奇点 i在C内, $-i$不在C内.

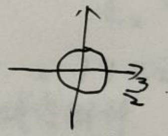
$$\therefore \oint_C \frac{e^{iz}dz}{z^2+1} = \oint_C \frac{\frac{e^{iz}}{z+i}}{z-i}\,dz = 2\pi i\cdot \frac{e^{iz}}{z+i}\Big|_{z=i} = \pi e^{-1} = \frac{\pi}{e}$$

(5) $\oint_C \frac{dz}{(z^2-1)(z^3-1)}$ $C: |z|=r<1$.



奇点 $z=\pm1$ $z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ 都在C之外. 且 C: $|z|=r$ 上及内部解析. $\therefore$ 积分值为0.

(7). $\oint_C \frac{dz}{(z^2+1)(z^2+4)}$ $C: |z|=\frac{3}{2}$.



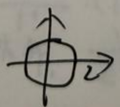
奇点 $\pm i$在C内, $\pm 2i$在C外

$$\therefore \oint_C \frac{dz}{(z^2+1)(z^2+4)} = \oint_C \frac{\frac{1\cdot dz}{z^2+4}}{z^2+1} = \frac{1}{2i}\left[\oint_C \frac{\frac{1}{z^2+4}}{z-i}dz - \oint_C \frac{\frac{1}{z^2+4}}{z+i}dz\right]$$

$$= \frac{1}{2i}\left[2\pi i\ \frac{1}{z^2+4}\Big|_{z=i} - 2\pi i\ \frac{1}{z^2+4}\Big|_{z=-i}\right]$$

$$= 0.$$

(9). $\oint_C \frac{\sin z\,dz}{(z-\frac{\pi}{2})^2}$ $C: |z|=2$



奇点 $z=\frac{\pi}{2}$在C内. 由高阶导数的柯西积分公式得

$$\oint_C \frac{\sin z\,dz}{(z-\frac{\pi}{2})^2} = \frac{2\pi i}{1!}\cdot(\sin z)'\Big|_{z=\frac{\pi}{2}} = 0.$$

8. (2) $\int_{\frac{\pi}{6}i}^{0} \operatorname{ch}3z\,dz = \frac{1}{3}\operatorname{sh}3z \Big|_{\frac{\pi}{6}i}^{0} = \frac{1}{3}\left[0 - \operatorname{sh}\frac{\pi}{2}i\right] = -\frac{i}{3}$

(4) $\int_0^1 z\sin z\,dz = -\int_0^1 z\,d\cos z = -z\cos z\Big|_0^1 + \int_0^1 \cos z\,dz$

$= -\cos 1 + \sin z\Big|_0^1 = \sin 1 - \cos 1$

(6). $\int_1^i \frac{1+\operatorname{tg} z}{\cos^2 z}\,dz = \int_1^i 1+\operatorname{tg} z \; d\operatorname{tg} z = \tan z + \frac{1}{2}\tan^2 z\Big|_1^i$

$= \left(\tan i + \frac{1}{2}\tan^2 i\right) - \left(\tan 1 + \frac{1}{2}\tan^2 1\right)$

$= i\cdot\operatorname{th}1 - \tan 1 - \frac{1}{2}\tan^2 1 - \frac{1}{2}\operatorname{th}^2 1$

9.(1) 两奇点 $z=-1$ 与 $z=-2i$ 都在 C 内. 所以以 $z=-1$ 和 $z=-2i$ 为圆心、$\frac{1}{4}$ 为半径作两个圆周 C_1 与 C_2.

由复合闭路定理得 $\oint_C \left(\frac{4}{z+1}+\frac{3}{z+2i}\right)dz$

$= \oint_{C_1}\left(\frac{4}{z+1}+\frac{3}{z+2i}\right)dz + \oint_{C_2}\left(\frac{4}{z+1}+\frac{3}{z+2i}\right)dz$

$= 4\cdot 2\pi i + 3\cdot 2\pi i = 14\pi i$

(3). 被积函数在 $C=C_1+C_2$ $2\le|z|\le 3$ 的区域内解析.

$\therefore \oint_{C=C_1+C_2} \frac{\cos z}{z^3}\,dz = 0$.

(5) 若 $|a|<1$, 则 $\oint_C \frac{e^z}{(z-a)^3}\,dz = \frac{2\pi i}{2!}(e^z)''\Big|_{z=a} = \pi e^a\cdot i$

若 $|a|>1$. 被积函数在 C 内解析. $\oint_C \frac{e^z}{(z-a)^3}\,dz = 0$.

10. 当 C 包含原点时, $\oint_C \frac{1}{z^2}\,dz = \frac{2\pi i}{1!}\cdot(1)'\Big|_{z=0} = 0$

当 C 不包含原点时, 即原点在 C 外部, $\frac{1}{z^2}$ 在 C 内解析. $\oint_C \frac{1}{z^2}\,dz = 0$.

12. $\int_0^z \frac{1}{1+\xi^2}d\xi$ 设 $z = 1\cdot e^{i\theta}$

$$\int_0^z \frac{1}{1+\xi^2}d\xi = \int_{C_1}\frac{1}{1+\xi^2}d\xi + \int_{C_2}\frac{1}{1+\xi^2}d\xi.$$

$$= \int_0^1 \frac{1}{1+x^2}dx + \int_0^{\theta_0}\frac{1}{1+e^{2i\theta}}de^{i\theta}$$

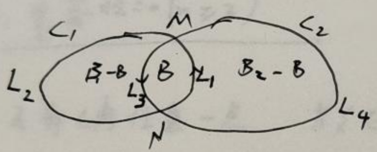
$$= \arctan x\Big|_0^1 + i\int_0^{\theta_0}\frac{e^{i\theta}}{1+e^{2i\theta}}d\theta$$

$$= \arctan 1 + i\int_0^{\theta_0}\frac{1}{e^{-i\theta}+e^{i\theta}}d\theta$$

$$= \frac{\pi}{4} + i\int_0^{\theta_0}\frac{d\theta}{2\cos\theta}$$

$$\mathrm{Re}\left[\int_0^z \frac{1}{1+\xi^2}d\xi\right] = \frac{\pi}{4}.$$

13.



$C_1 = L_1 + L_2$ $C_2 = L_3 + L_4$.

$f(z)$ 在 B_1-B 及边界 $L_2+L_3^-$ 上解析，在 B_2-B 及其边界 $L_4+L_1^-$ 上解析.

$$\oint_{L_2+L_3^-} f(z)dz = \int_{L_2} f(z)dz - \int_{L_3} f(z)dz = 0.$$

$$\oint_{L_4+L_1^-} f(z)dz = \int_{L_4} f(z)dz - \int_{L_1} f(z)dz = 0.$$

所以 $\oint_{C_1} f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$

$\cancel{\oint_{C_2} f(z)dz}$ $= \int_{L_4} f(z)dz + \int_{L_3} f(z)dz = \oint_{C_2} f(z)dz.$

15. 当z_0在C_1内时.

$$\frac{1}{2\pi i}\left[\oint_{C_1}\frac{z^2dz}{z-z_0}+\oint_{C_2}\frac{\sin z\,dz}{z-z_0}\right]=\frac{1}{2\pi i}\cdot\left[2\pi i\cdot z^2\big|_{z=z_0}+0\right]$$
$$=z_0^2.$$

当z_0在C_2内时.

$$原式=\frac{1}{2\pi i}\left[0+2\pi i\cdot\sin z\big|_{z=z_0}\right]=\sin z_0$$

16. ~~不对~~ 不必需. 如$\oint_C\frac{1}{z^2}dz$. $\frac{1}{z^2}$在$0<|z|<1$内解析,且沿任何圆周$C:|z|=r$, $0<r<1$的积分等于0.
但它在$z=0$处不解析.

事实上, $\oint_C\frac{1}{z^n}dz=0\ (n\geq 2)$

17. 设z为C内任意一点. 当$\zeta\in C$时, $f(\zeta)=g(\zeta)$

$$f(z)=\frac{1}{2\pi i}\oint_C\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\oint_C\frac{g(\zeta)}{\zeta-z}d\zeta=g(z)$$

∴得证.

19. 是. 由于$f(z)$在B内处处解析, 所以$f'(z)$在B内也处处解析.
~~所以~~ ∴$\frac{f'(z)}{f(z)}$在B内处处解析. ∴积分为0.

21. 若z_0在闭曲线C之外, $\frac{f'(z)}{z-z_0}$与$\frac{f(z)}{(z-z_0)^2}$都在C内处处解析.
积分都为0. ∴ $\oint_C\frac{f'(z)}{z-z_0}dz=\oint_C\frac{f(z)}{(z-z_0)^2}dz$

若z_0在闭曲线C内, $\oint_C\frac{f(z)\,dz}{(z-z_0)^2}=\frac{2\pi i}{1!}\left(f^{(0)}(z)\right)'\Big|_{z=z_0}=2\pi i\ f'(z_0)$

$$\oint_C\frac{f'(z)}{z-z_0}dz=2\pi i\ f'(z_0)$$

等式仍然成立.