Surface Integral over a Cylinder

We are asked to compute the surface integral:

$$\iint_\Sigma \frac{1}{x^2 + y^2 + z^2} \, dS$$

where Σ is the closed surface bounded by the cylinder $x^2 + y^2 = R^2$, between the planes $z = 0$ and $z = H$. Thus, the surface Σ consists of three parts:

- The lateral surface of the cylinder.
- The top disk at $z = H$.
- The bottom disk at $z = 0$.

We perform the computation in cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

1. Lateral Surface

On the side of the cylinder, we fix $r = R$, and let $\theta \in [0, 2\pi]$, $z \in [0, H]$. Then:

$$x^2 + y^2 + z^2 = R^2 + z^2, \quad dS = R \, d\theta \, dz$$

Therefore, the integral over the side is:

$$\iint_{\text{side}} \frac{1}{x^2 + y^2 + z^2} \, dS = \int_0^{2\pi} \int_0^H \frac{1}{R^2 + z^2} R \, dz \, d\theta = R \int_0^{2\pi} d\theta \int_0^H \frac{1}{R^2 + z^2} \, dz$$

Evaluate the inner integral:

$$\int_0^H \frac{1}{R^2 + z^2} \, dz = \frac{1}{R} \tan^{-1}\left(\frac{H}{R}\right)$$

So:

$$\iint_{\text{side}} = R \cdot 2\pi \cdot \frac{1}{R} \tan^{-1}\left(\frac{H}{R}\right) = 2\pi \tan^{-1}\left(\frac{H}{R}\right)$$

2. Top Disk

At the top, $z = H$, $r \in [0, R]$, $\theta \in [0, 2\pi]$, and:

$$x^2 + y^2 + z^2 = r^2 + H^2, \quad dS = r\,dr\,d\theta$$

Then:

$$\iint_{\text{top}} \frac{1}{x^2 + y^2 + z^2}\,dS = \int_0^{2\pi} \int_0^R \frac{1}{r^2 + H^2} r\,dr\,d\theta$$

Inner integral:

$$\int_0^R \frac{r}{r^2 + H^2}\,dr = \frac{1}{2} \ln\left(\frac{R^2 + H^2}{H^2}\right)$$

So:

$$\iint_{\text{top}} = 2\pi \cdot \frac{1}{2} \ln\left(\frac{R^2 + H^2}{H^2}\right) = \pi \ln\left(1 + \frac{R^2}{H^2}\right)$$

3. Bottom Disk

At the bottom, $z = 0$, so $x^2 + y^2 + z^2 = r^2$, and:

$$\iint_{\text{bottom}} \frac{1}{r^2} \cdot r\,dr\,d\theta = \int_0^{2\pi} \int_0^R \frac{1}{r} dr\,d\theta$$

This integral diverges at $r = 0$, so:

$$\boxed{\text{The surface integral diverges due to the singularity at } (0, 0, 0).}$$

Final Conclusion

The total surface integral is:

$$\iint_\Sigma \frac{1}{x^2 + y^2 + z^2}\,dS = \boxed{\text{Divergent}}$$