Notes on Ihmsen et al. "Implicit Incompressible SPH"

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Feb 2017

1 Motivation

$$\begin{aligned}\frac{D\rho}{Dt} &= -\rho\nabla\cdot\mathbf{v} && (1)\\ \frac{\rho_f(t+\Delta t)-\rho_f(t)}{\Delta t} &= \sum_{f_f} m_{f_f}\mathbf{v}_{ff_f}(t+\Delta t)\nabla W_{ff_f}(t)\\ &+\sum_{f_b} m_{f_b}\mathbf{v}_{ff_b}(t+\Delta t)\nabla W_{ff_b}(t) && (2)\end{aligned}$$

with $\mathbf{v}_{ff_f}=\mathbf{v}_f-\mathbf{v}_{f_f}$ and $\mathbf{v}_{ff_b}=\mathbf{v}_f-\mathbf{v}_{f_b}$. f_f are fluid neighbors of sample f. f_b are boundary neighbors of sample f. The density at the next timestep should be the rest density $\rho_f(t+\Delta t)=\rho_f^0$. The velocities at the next timestep are computed as $\mathbf{v}(t+\Delta t)=\mathbf{v}(t)+\Delta t\mathbf{a}^n+\Delta t\mathbf{a}^p$. $\mathbf{a}^n$ contains non-pressure accelerations, $\mathbf{a}^p$ is the acceleration due to pressure. Timestep t is omitted in the following equations:

$$\begin{aligned}\frac{\rho_f^0-\rho_f}{\Delta t} &= \sum_{f_f} m_{f_f}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n+\Delta t\mathbf{a}_f^p-\mathbf{v}_{f_f}-\Delta t\mathbf{a}_{f_f}^n-\Delta t\mathbf{a}_{f_f}^p\right)\nabla W_{ff_f}\\ &+\sum_{f_b} m_{f_b}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n+\Delta t\mathbf{a}_f^p-\mathbf{v}_{f_b}(t+\Delta t)\right)\nabla W_{ff_b} && (3)\end{aligned}$$

Separating knowns from unknowns:

$$\begin{aligned}\frac{\rho_f^0-\rho_f}{\Delta t} &= \sum_{f_f} m_{f_f}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_f}-\Delta t\mathbf{a}_{f_f}^n\right)\nabla W_{ff_f}\\ &+\sum_{f_b} m_{f_b}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_b}(t+\Delta t)\right)\nabla W_{ff_b}\\ &+\sum_{f_f} m_{f_f}\left(\Delta t\mathbf{a}_f^p-\Delta t\mathbf{a}_{f_f}^p\right)\nabla W_{ff_f}+\sum_{f_b} m_{f_b}\Delta t\mathbf{a}_f^p\nabla W_{ff_b} && (4)\end{aligned}$$

$$\underbrace{\rho_f^0-\rho_f}_{\text{current density error}} \underbrace{-\Delta t\sum_{f_f} m_{f_f}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_f}-\Delta t\mathbf{a}_{f_f}^n\right)\nabla W_{ff_f}}_{\text{density error from fluid velocity divergence}}$$

$$\underbrace{-\Delta t\sum_{f_b} m_{f_b}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_b}(t+\Delta t)\right)\nabla W_{ff_b}}_{\text{density error from boundary velocity divergence}}$$

$$=\Delta t^2\sum_{f_f} m_{f_f}\left(\mathbf{a}_f^p-\mathbf{a}_{f_f}^p\right)\nabla W_{ff_f}+\Delta t^2\sum_{f_b} m_{f_b}\mathbf{a}_f^p\nabla W_{ff_b} \quad (5)$$

The left-hand side of Eq. 5 is referred to as source term s_f:

$$s_f := \rho_f^0-\rho_f-\Delta t\sum_{f_f} m_{f_f}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_f}-\Delta t\mathbf{a}_{f_f}^n\right)\nabla W_{ff_f}$$

$$-\Delta t\sum_{f_b} m_{f_b}\left(\mathbf{v}_f+\Delta t\mathbf{a}_f^n-\mathbf{v}_{f_b}(t+\Delta t)\right)\nabla W_{ff_b} \quad (6)$$

resulting in

$$s_f=\underbrace{\Delta t^2\sum_{f_f} m_{f_f}\left(\mathbf{a}_f^p-\mathbf{a}_{f_f}^p\right)\nabla W_{ff_f}+\Delta t^2\sum_{f_b} m_{f_b}\mathbf{a}_f^p\nabla W_{ff_b}}_{(\mathbf{Ap})_f} \quad (7)$$

The right-hand side of Eq. 7 is referred to as $(\mathbf{Ap})_f$ with $\mathbf{A}$ being a system matrix that approximates the Laplacian and $\mathbf{p}$ being the vector of all unknown pressure values p_f.

Eq. 7 requires the computation of pressure accelerations $\mathbf{a}_f^p$. Pressure accelerations are only considered at fluid particles. We make the simplifying assumption that there are no pressure accelerations at boundary particles. Fluid pressure is mirrored to adjacent boundary samples, e.g. Akinci et al. 2012.

$$\mathbf{a}_f^p=-\sum_{f_f} m_{f_f}\left(\frac{p_f}{(\rho_f^0)^2}+\frac{p_{f_f}}{(\rho_{f_f}^0)^2}\right)\nabla W_{ff_f}-\gamma\sum_{f_b} m_{f_b}\left(2\frac{p_f}{(\rho_f^0)^2}\right)\nabla W_{ff_b} \quad (8)$$

The value γ is typically set to 0.7.

Solving $\mathbf{Ap}=\mathbf{s}$ can require the computation of diagonal elements $\mathbf{A}_{ff}$ of matrix $\mathbf{A}$. Therefore, Eq. 8 can be rewritten as

$$\mathbf{a}_f^p=-\sum_{f_f} m_{f_f}\frac{p_f}{(\rho_f^0)^2}\nabla W_{ff_f}-\sum_{f_f} m_{f_f}\frac{p_{f_f}}{(\rho_{f_f}^0)^2}\nabla W_{ff_f}-\gamma\sum_{f_b} m_{f_b}2\frac{p_f}{(\rho_f^0)^2}\nabla W_{ff_b} \quad (9)$$

$$\mathbf{a}_f^p=p_f\cdot\underbrace{\left(-\sum_{f_f}\frac{m_{f_f}}{(\rho_f^0)^2}\nabla W_{ff_f}-2\gamma\sum_{f_b}\frac{m_{f_b}}{(\rho_f^0)^2}\nabla W_{ff_b}\right)}_{\mathbf{c}_f}-\sum_{f_f} m_{f_f}\frac{p_{f_f}}{(\rho_{f_f}^0)^2}\nabla W_{ff_f} \quad (10)$$

The term $\sum_{f_f} m_{f_f} \frac{p_{f_f}}{(\rho^0_{f_f})^2} \nabla W_{ff_f}$ can be split into $\sum_{f_f \neq f} m_{f_f} \frac{p_{f_f}}{(\rho^0_{f_f})^2} \nabla W_{ff_f} + m_f \frac{p_f}{(\rho^0_f)^2} \nabla W_{ff}$. Due to $\nabla W_{ff} = \mathbf{0}$, there is no additional contribution to the coefficient of p_f. Therefore,

$$\mathbf{a}^p_f = p_f \cdot \mathbf{c}_f - \sum_{f_f \neq f} m_{f_f} \frac{p_{f_f}}{(\rho^0_{f_f})^2} \nabla W_{ff_f}. \tag{11}$$

Combining Eq. 11 and Eq. 7, we get

$$\begin{aligned}
s_f =& \Delta t^2 \sum_{f_f} m_{f_f} \left(p_f \cdot \mathbf{c}_f - \sum_{f_f \neq f} m_{f_f} \frac{p_{f_f}}{(\rho^0_{f_f})^2} \nabla W_{ff_f} \right) \nabla W_{ff_f} \\
&+ \Delta t^2 \sum_{f_f} m_{f_f} \left(-p_{f_f} \cdot \mathbf{c}_{f_f} + \sum_{f_{f_f} \neq f_f} m_{f_{f_f}} \frac{p_{f_{f_f}}}{(\rho^0_{f_{f_f}})^2} \nabla W_{f_f f_{f_f}} \right) \nabla W_{ff_f} \\
&+ \Delta t^2 \sum_{f_b} m_{f_b} \left(p_f \cdot \mathbf{c}_f - \sum_{f_f \neq f} m_{f_f} \frac{p_{f_f}}{(\rho^0_{f_f})^2} \nabla W_{ff_f} \right) \nabla W_{ff_b}
\end{aligned} \tag{12}$$

Now, the main diagonal value $\mathbf{A}_{ff}$ can be computed as

$$\begin{aligned}
\mathbf{A}_{ff} =& \Delta t^2 \sum_{f_f} \left(m_{f_f} \mathbf{c}_f \cdot \nabla W_{ff_f} \right) \\
&+ \Delta t^2 \sum_{f_f} \left(m_{f_f} \left(\frac{m_f}{(\rho^0_f)^2} \nabla W_{f_f f} \right) \cdot \nabla W_{ff_f} \right) \\
&+ \Delta t^2 \sum_{f_b} \left(m_{f_b} \mathbf{c}_f \cdot \nabla W_{ff_b} \right)
\end{aligned} \tag{13}$$

Comment: The coefficient extraction in lines one and three of Eq. 12 ist straightforward. Inside the main sum, we have the term $p_f \cdot \mathbf{c}_f$ that results in the coefficient $\mathbf{c}_f$. We further have the term $\sum_{f_f \neq f} \ldots$. This term explicitly excludes f. So, there is no contribution to $\mathbf{A}_{ff}$. In line two of Eq. 12, we have the term $\sum_{f_{f_f} \neq f_f} \ldots$. This term contributes to the coefficient for $f_{f_f} = f$, i.e. $\frac{m_f}{(\rho^0_f)^2} \nabla W_{f_f f}$. The term $-p_{f_f} \cdot \mathbf{c}_{f_f}$ does not contribute to $\mathbf{A}_{ff}$ as $\nabla W_{ff_f} = 0$ for $f_f = f$.

$$
\begin{aligned}
\mathbf{A}_{ff} =& \Delta t^2 \sum_{f_f} m_{f_f} \left(-\sum_{f_f} \frac{m_{f_f}}{(\rho_f^0)^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{(\rho_f^0)^2} \nabla W_{f f_b} \right) \cdot \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_f} m_{f_f} \left(\frac{m_f}{(\rho_f^0)^2} \nabla W_{f_f f} \right) \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_b} m_{f_b} \left(-\sum_{f_f} \frac{m_{f_f}}{(\rho_f^0)^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{(\rho_f^0)^2} \nabla W_{f f_b} \right) \cdot \nabla W_{f f_b}
\end{aligned} \tag{14}
$$

2 Algorithm

2.1 Initialization

- Compute density: $\rho_f = \sum_{f_f} m_{f_f} W_{f f_f} + \sum_{f_b} m_{f_b} W_{f f_b}$
- Compute non-pressure accelerations and predicted velocity: $\mathbf{v}_f^* = \mathbf{v}_f + \Delta t \mathbf{a}_f^n$
- Compute source term, i.e. the predicted density error that has to be corrected:

$$
\begin{aligned}
s_f =& \rho_f^0 - \rho_f - \Delta t \sum_{f_f} m_{f_f} \left(\mathbf{v}_f^* - \mathbf{v}_{f_f}^* \right) \nabla W_{f f_f} \\
&- \Delta t \sum_{f_b} m_{f_b} \left(\mathbf{v}_f^* - \mathbf{v}_{f_b}(t + \Delta t) \right) \nabla W_{f f_b}
\end{aligned} \tag{15}
$$

- Compute diagonal element:

$$
\begin{aligned}
\mathbf{A}_{ff} =& \Delta t^2 \sum_{f_f} m_{f_f} \left(-\sum_{f_f} \frac{m_{f_f}}{(\rho_f^0)^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{(\rho_f^0)^2} \nabla W_{f f_b} \right) \cdot \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_f} m_{f_f} \left(\frac{m_f}{(\rho_f^0)^2} \nabla W_{f_f f} \right) \nabla W_{f f_f} \\
&+ \Delta t^2 \sum_{f_b} m_{f_b} \left(-\sum_{f_f} \frac{m_{f_f}}{(\rho_f^0)^2} \nabla W_{f f_f} - 2\gamma \sum_{f_b} \frac{m_{f_b}}{(\rho_f^0)^2} \nabla W_{f f_b} \right) \cdot \nabla W_{f f_b}
\end{aligned} \tag{16}
$$

- Set: $p_f^0 = 0$

2.2 Iteration l:

2.2.1 First loop:

- Compute pressure acceleration:
  $(\mathbf{a}_f^p)^l = -\sum_{f_f} m_{f_f} \left(\frac{p_f^l}{(\rho_f^0)^2} + \frac{p_{f_f}^l}{(\rho_{f_f}^0)^2} \right) \nabla W_{ff_f} - \gamma \sum_{f_b} m_{f_b} 2 \frac{p_f^l}{(\rho_f^0)^2} \nabla W_{ff_b}$

2.2.2 Second loop:

- Compute the divergence of the velocity change $\Delta t \mathbf{a}_f^p$ due to the pressure acceleration:
  $(\mathbf{Ap})_f^l = \Delta t^2 \sum_{f_f} m_{f_f} \left((\mathbf{a}_f^p)^l - (\mathbf{a}_{f_f}^p)^l \right) \nabla W_{ff_f} + \Delta t^2 \sum_{f_b} m_{f_b} (\mathbf{a}_f^p)^l \nabla W_{ff_b}$
- Update pressure if $\mathbf{A}_{ff} \neq 0$: $p_f^{l+1} = \max \left(p_f^l + \omega \frac{s_f - (\mathbf{Ap})_f^l}{\mathbf{A}_{ff}}, 0 \right)$
  If a particle has no neighbors, then $\mathbf{A}_{ff} = 0$ and pressure should not be updated.
- Compute predicted density error per particle: $(\rho_f^{\text{error}})^l = (\mathbf{Ap})_f^l - s_f$