CAUSAL INFERENCE WITH ORDINAL OUTCOMES

Density Estimation Based Approach

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MOTIVATION

- Growing emphasis on causal identification in political science
- Outcomes are measured in Ordinal scale
  - Approval ratings (??)
  - Trade policy (???)
- The Problem:
  - The usual causal inference tools are designed to serve cardinal or at least interval outcomes

THE GOAL

- Treatment effect can only be identified up to scale
  - ATE resides in the unknown latent space
  - Consistent normalization may help comparison and interpretation
- Flexible density estimation based estimators
  - Standard parametric approaches rely on strong distributional assumptions, inconsistent
  - Density estimation techniques can help

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PROBLEM SETTING

- Suppose a DGP

$$Y_i^* = f(X_i, \beta) + D_i^\top \tau + \varepsilon_i,$$

- Y_i^* is outcome in unidimensional space that we cannot observe
- D_i denotes a binary treatment
- We only can observe the transformed version of it, $Y_i \in \{1, \ldots, j, \ldots, J\}$

$$Y_i = \begin{cases} 0 & \text{if } \alpha_{-1} < Y_i^* \leq \alpha_0 \\ \vdots & \vdots \\ J & \text{if } \alpha_{J-1} < Y_i^* \leq \alpha_J \end{cases}$$

, where α_j denotes the threshold points for each ordinal category

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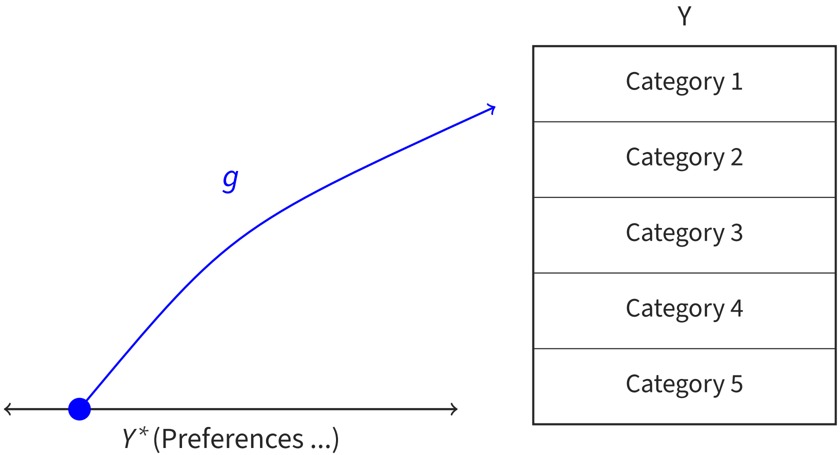

POTENTIAL OUTCOME FRAMEWORK

- Denote the binary treatment status of individual $D_i \in \{0, 1\}$
- Denote the potential outcomes in latent preference scale as $Y_i^*(D_i)$
  - $Y_i^*(1) = \tau + f(\beta, X_i) + \epsilon_i$
  - $Y_i^*(0) = f(\beta, X_i) + \epsilon_i$
- Denote the potential outcomes in observed ordinal scale (POO) as $Y_i(d_i) = g(Y_i^*(d_i)) \in \{1, \ldots, j, \ldots J\}$
- We are interested in the average treatment effect in the latent space (LTE)

$$\begin{aligned} LTE &= \mathbb{E}\left[Y^*(1)\right] - \mathbb{E}\left[Y^*(0)\right] \\ &= \tau \end{aligned}$$

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LTE WITH $Y_i(D_i)$

- If Y_i^* is known, there is not problem
- If g is known and it maps one value of Y_i^* to Y_i, then we may get LTE, but this not true in most settings

$$\mathbb{E}\left[g^{-1}(Y_i(1))\right] - \mathbb{E}\left[g^{-1}(Y_i(0))\right] \neq LTE$$

- Should we give up?

UP TO SCALE IDENTIFICATION

- We can identify LTE up to scale (Theorem 1)
- From the model and data we know:

$$\mathbb{P}\left(Y(D) \leq j\right) = \mathbb{P}\left(Y^* \leq \alpha_j\right)$$

- We can construct MLE with using this probability.
- However, if we scale Y^* side by a constant, $c > 0$, we still get the same probability

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NORMALIZATION

- Up to scale identification is disappointing
- Proper normalization helps interpretation and comparison across models and across samples
- Fixing $Var(\varepsilon_i) = 1$ is one way
- This works as mapping Y_i^* to the space where the variance of the error is a unit / probit space

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COMMON TRANSFORMATION: CARDINALIZATION AND BINARIZATION

- Cardinalization
  - Relies on luck
- Binarization
  - Lose significant efficiency
  - Estimate may be sensitive to how you group the ordinal outcomes

SMALL EXAMPLE

	$Y_i^*(0)$	$Y_i^*(1)$
A	1.37	0.09
B	-0.56	1.71
C	0.36	0.11
D	0.63	2.22
E	0.40	0.14

- The LTE in the latent preference space is: 0.41

SMALL EXAMPLE – CARDINALIZATION

- Suppose two transformation functions: g_a and g_b
- Both are monotone, but with different thresholds (α_j)

	g_a		g_b	
	$Y_{i,g_a}(0)$	$Y_{i,g_a}(1)$	$Y_{i,g_b}(0)$	$Y_{i,g_b}(1)$
A	Agree	Agree	Agree	Disagree
B	Neither	Agree	Disagree	Agree
C	Agree	Agree	Neither	Disagree
D	Agree	Strongly Agree	Neither	Agree
E	Agree	Agree	Neither	Disagree

SMALL EXAMPLE – CARDINALIZATION

- A researcher impose common cardinalization of 1 to 5,
- In case the true transformation is g_a, the estimate is -0.4
- In case the true transformation is g_b, the estimate is 0.2
- Without compelling reason to impose such numeric labels, cardinalization relies on pure luck

ESTIMATION

- There has been tools for ordinal outcomes
- Standard parametric approaches such as Ordered Probit and Ordered Logit
- Two estimators based on density estimation techniques

ORDERED LOGIT AND PROBIT

- Assume that the error term follows a specific distribution (Logistic and Standard Normal)
- And use maximum likelihood to estimate the coefficients
- Identify coefficients up to scale
- Once distributional assumption is violated, inconsistent

ORDERED LOGIT AND PROBIT

- The true log-likelihood is:

$$\ell(\alpha, \beta, \tau) = \sum_{i=1}^{n} \sum_{j=0}^{J} 1_{\{Y_i = j\}} \left\{ \log F\left(\alpha_j - f(X_i, \beta) + D_i^\top \tau\right) \right.$$
$$\left. - \log F\left(\alpha_{j-1} - f(X_i, \beta) + D_i^\top \tau\right) \right\}$$

- F is the CDF of the true error.
- If F is not standard normal or standard logistic, MLE is inconsistent.
- The error is never known, and omitted or unobserved confounder may also distort the error

ALTERNATIVE: ESTIMATE THE F

- Instead of assuming a specific F, we can estimate as $\hat{F}$
- Then the log-likelihood becomes:

$$\hat{\ell}(\alpha, \beta, \tau) = \sum_{i=1}^{n} \sum_{j=0}^{J} 1_{\{Y_i = j\}} \Big\{ \log \hat{F}\big(\alpha_j - f(X_i, \beta) + D_i^\top \tau\big) - \log \hat{F}\big(\alpha_{j-1} - f(X_i, \beta) + D_i^\top \tau\big) \Big\}$$

- I propose to use two density estimation methods: KDE and Normalizing Flows

KERNEL DENSITY ESTIMATION

- Nonparametric method
- Smooth each observation using a kernel (usually Gaussian)

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

- Generally, the moderate bandwidth ensures enough convergence rate for $\sqrt{n}$ consistency for the main estimator (semiparametric efficiency)

NORMALIZING FLOWS

- A flexible class of models that transform complex continuous density to simple base density (usually Gaussian)
- Use the change-of-variable formula

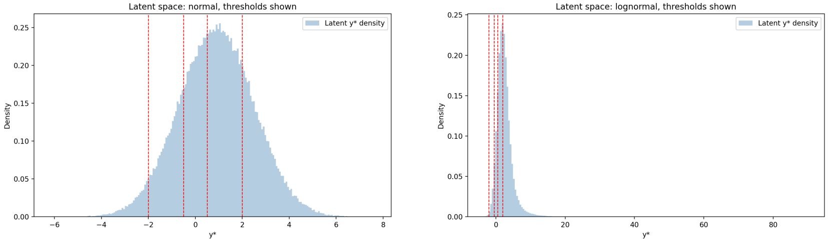
$$f_\theta(h) = f_Z\left(T_\theta^{-1}(h)\right) \left|\det T_\theta^{-1}(h)\right|,$$

- T_θ is a set of *invertible* transformation
- Rational Quadratic Spline Flow
- Since this is fully parametric, for finite θ, MLE ensures $\sqrt{n}$ consistency

SIMULATION

- Ordered Probit Ordered Logit and NF based only
- Errors: Standard Normal and Log Normal distribution
- A binary treatment randomized
- Covariates: One binary and three continuous (following normal distribution)
- Thresholds: $-2, -1, 1, 2.5 \rightarrow 5$ categories
- Sample sizes: 200, 500, 1000, 2000
- 200 Replications

SIMULATION


Latent space: normal, thresholds shown
Latent y* density
Density
y*
0.00
0.05
0.10
0.15
0.20
0.25
−6
−4
−2
0
2
4
6
8
Latent space: lognormal, thresholds shown
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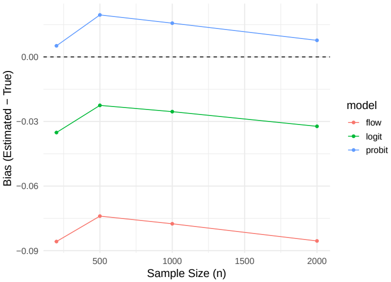
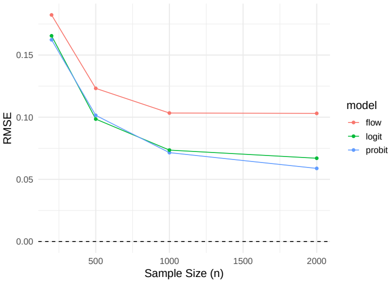
20
40
60
80

SIMULATION – NORMAL CASE

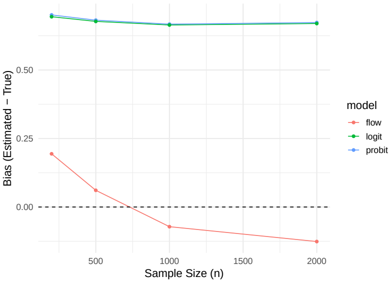
- True treatment size: 1.0

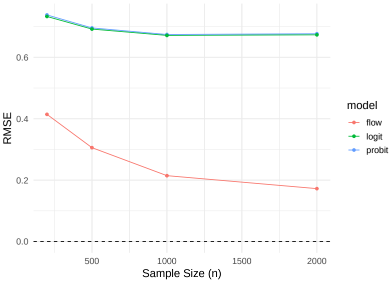




SIMULATION – LOGNORMAL CASE

- True treatment size: 0.46





CONCLUSION

- With ordinal outcomes, treatment effects can only be identified up to scale
- Cardinalization and Binarization have its pitfalls
- Standard parametric approaches relies on strong distributional assumption
- KDE and NF based approaches may provide flexible ways to estimate the true effects
- For KDE, the normalization of coefficients is an issue...
- For NF, Hyperparameter selection is an issue, trying to automatically adjust it while traing