Paths in Graphs: Fastest Route

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Higher School of Economics

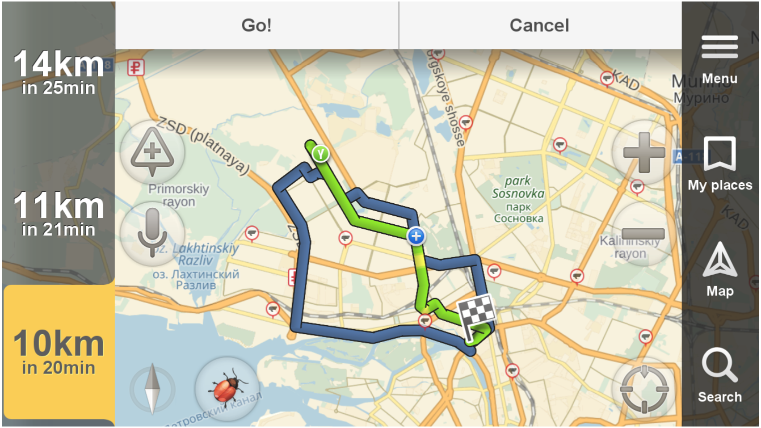
Graph Algorithms
Data Structures and Algorithms

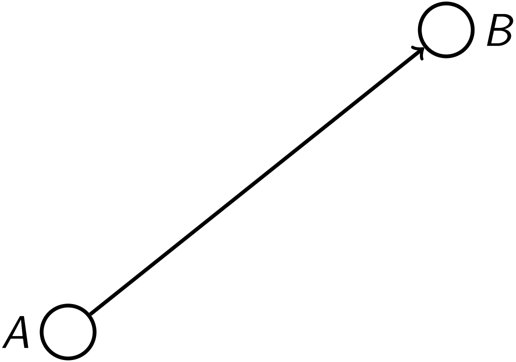
Outline

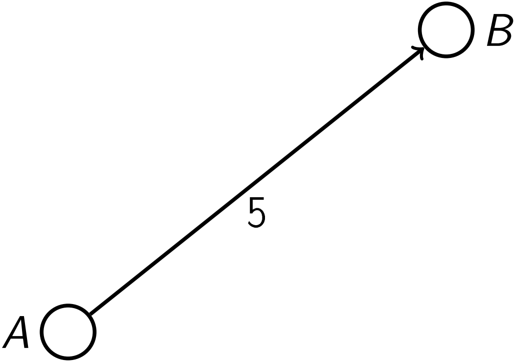
1. Fastest Route
2. Naive Algorithm
3. Dijkstra's Algorithm

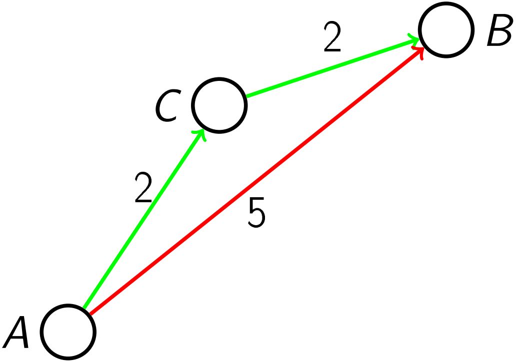
Fastest Route

What is the fastest route to get home from work?

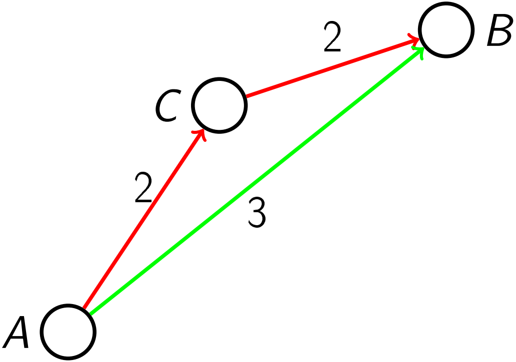



A
B


B
5
A

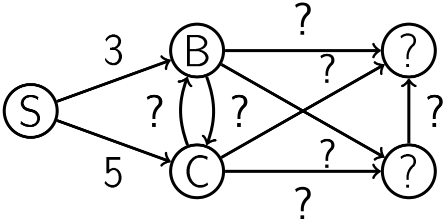


C
B
A
2
2
5


A
B
C
2
2
3

Intuition

- Assume that we stay at S and observe two outgoing edges:



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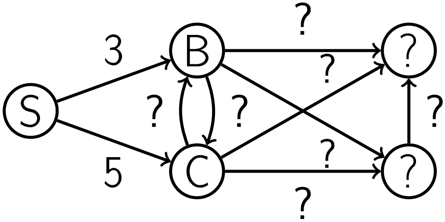

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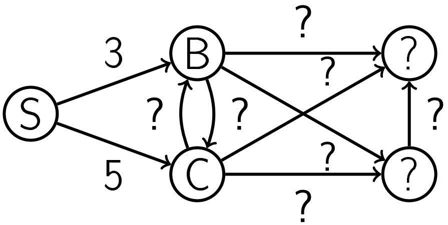

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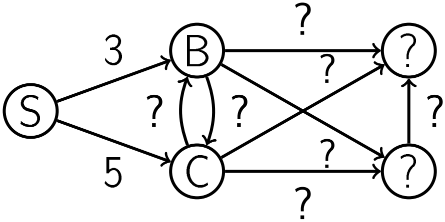

- No, because the weight of the edge (B, C) might be equal to, say, 1.

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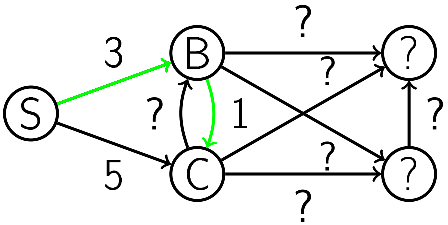
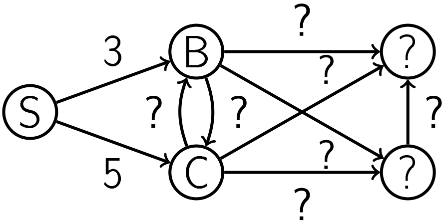


- No, because the weight of the edge (B, C) might be equal to, say, 1.

Intuition

- Can we be sure that the distance from S to B is 3?



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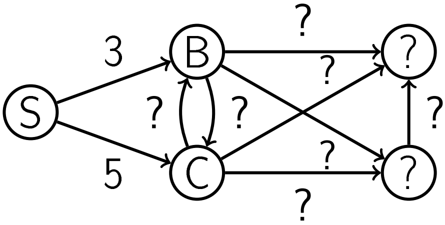

- Yes, because there are no negative weight edges.

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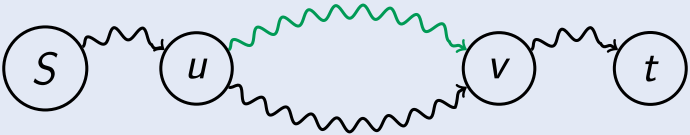
Optimal substructure

Observation

Any subpath of an optimal path is also optimal.

Proof

Consider an optimal path from S to t and two vertices u and v on this path. If there were a shorter path from u to v we would get a shorter path from S to t.



Corollary

If $S \to \ldots \to u \to t$ is a shortest path from S to t, then

$$d(S, t) = d(S, u) + w(u, t)$$

Edge relaxation

- `dist`[v] will be an upper bound on the actual distance from S to v.

Edge relaxation

- `dist`$[v]$ will be an upper bound on the actual distance from S to v.
- The edge relaxation procedure for an edge (u, v) just checks whether going from S to v through u improves the current value of `dist`$[v]$.

$\texttt{Relax}((u, v) \in E)$

```
if  dist[v] > dist[u] + w(u, v):
    dist[v] ← dist[u] + w(u, v)
    prev[v] ← u
```

Naive approach

$\texttt{Naive}(G, S)$

$\texttt{for all } u \in V:$

$\quad \mathit{dist}[u] \leftarrow \infty$

$\quad \mathit{prev}[u] \leftarrow \mathit{nil}$

$\mathit{dist}[S] \leftarrow 0$

$\mathit{do}:$

$\quad \texttt{relax all the edges}$

$\mathit{while} \texttt{ at least one } \mathit{dist} \texttt{ changes}$

Correct distances

Lemma

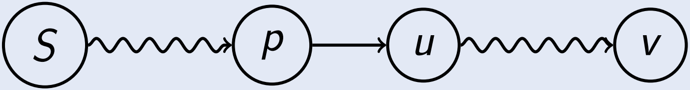
After the call to `Naive` algorithm all the distances are set correctly.

Proof

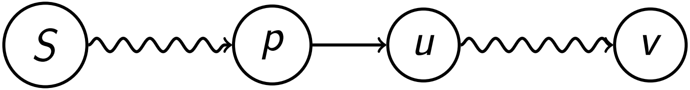
- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\text{dist}[v] > d(S, v)$.

Proof

- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex v such that $\mathrm{dist}[v] > d(S, v)$.
- Consider a shortest path from S to v and let u be the first vertex on this path with the same property. Let p be the vertex right before u.



Proof (continued)



- Then $d(S, p) = \text{dist}[p]$ and hence $d(S, u) = d(S, p) + w(p, u) = \text{dist}[p] + w(p, u)$

Proof (continued)

$$S \rightsquigarrow p \longrightarrow u \rightsquigarrow v$$

- Then $d(S, p) = \text{dist}[p]$ and hence $d(S, u) = d(S, p) + w(p, u) = \text{dist}[p] + w(p, u)$
- $\text{dist}[u] > d(S, u) = \text{dist}[p] + w(p, u) \Rightarrow$ edge (p, u) can be relaxed — a contradiction. $\square$

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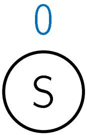

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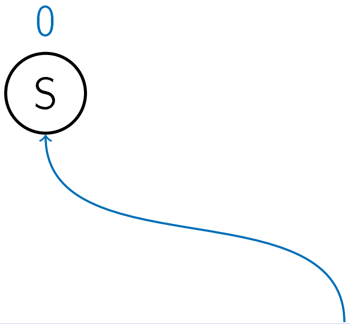
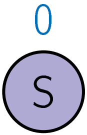


initially, we only know the distance to S

Intuition



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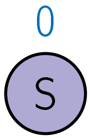
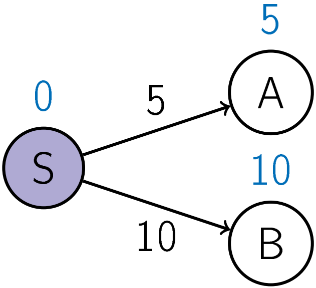


let's relax all the edges from S

Intuition



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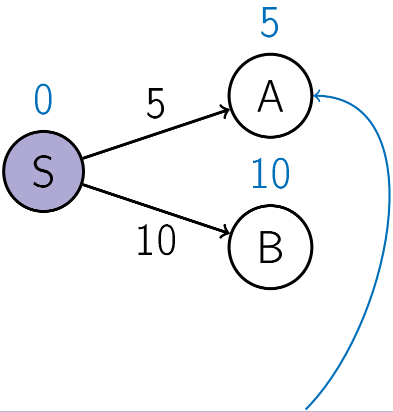
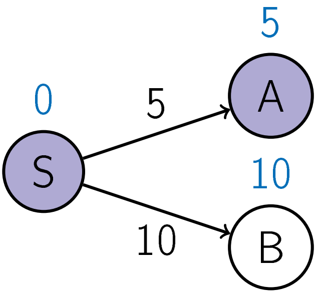


we now know the distance for A

Intuition



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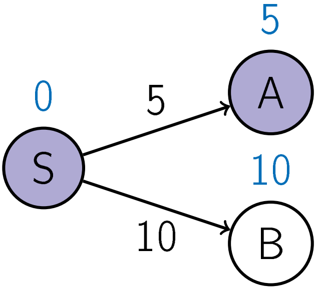

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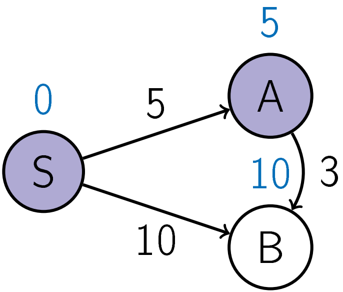

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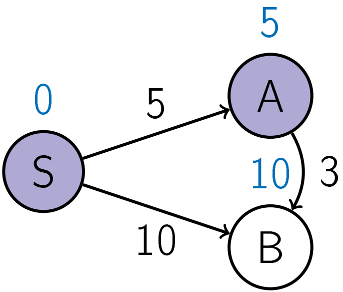

we discover an edge (A, B) of weight 3 that updates dist$[B]$

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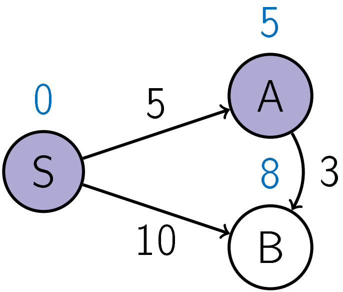

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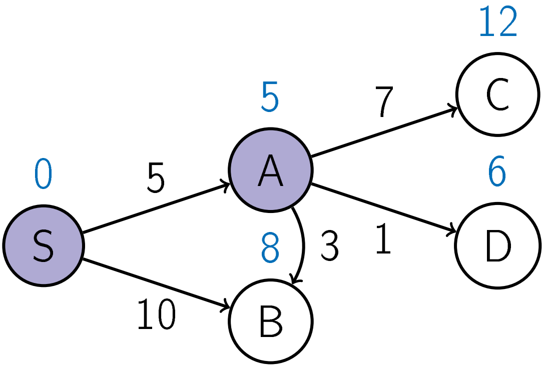
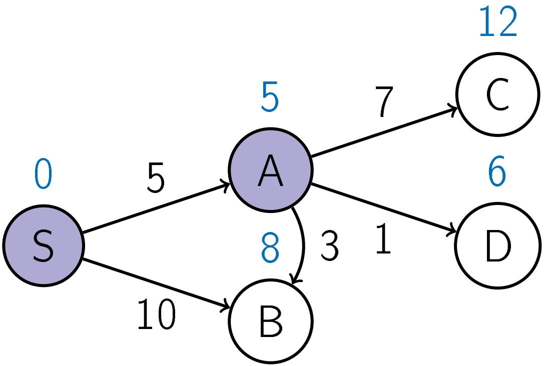


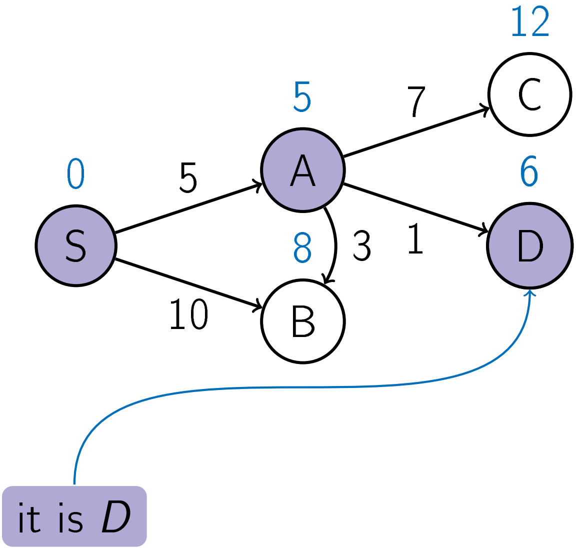
we also discover a few more outgoing edges

Intuition



what is the next vertex for which we already know the correct distance?

Intuition



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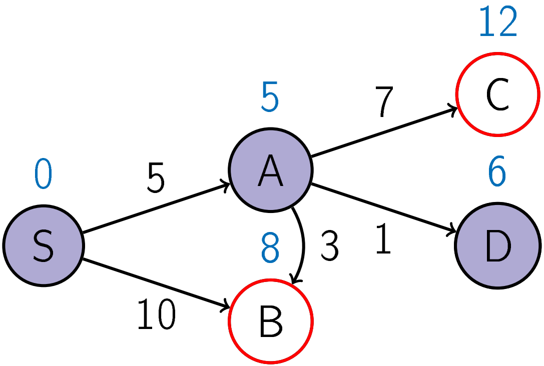

while for B and C it is possible that their `dist` values are larger than actual distances

Main ideas of Dijkstra's Algorithm

- We maintain a set R of vertices for which `dist` is already set correctly ("known region").

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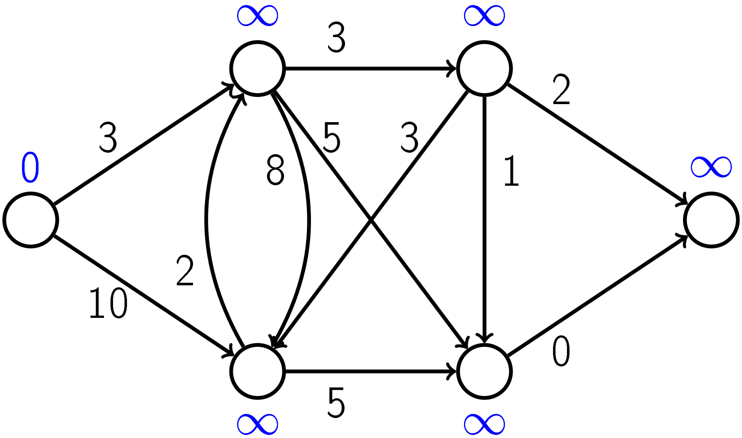
- We maintain a set R of vertices for which `dist` is already set correctly ("known region").
- The first vertex added to R is S.
- On each iteration we take a vertex outside of R with the minimal `dist`-value, add it to R, and relax all its outgoing edges.

Example



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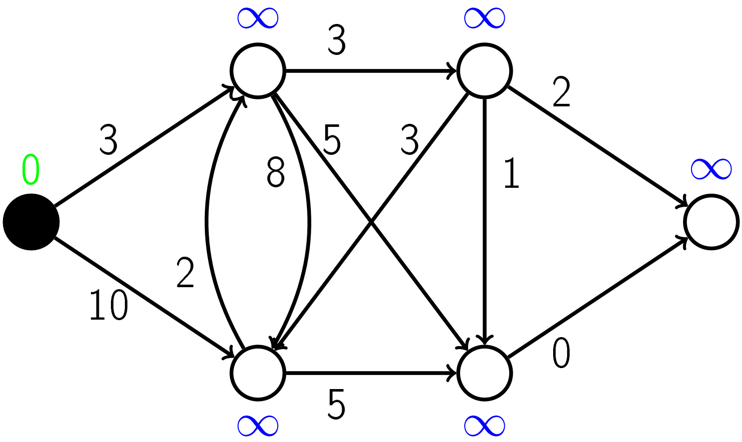

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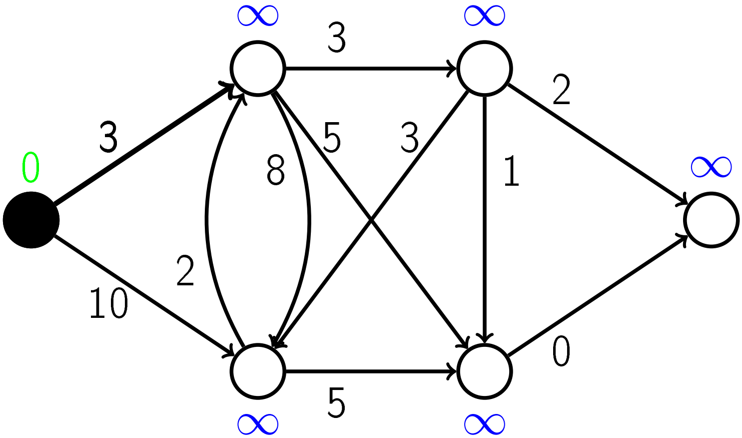

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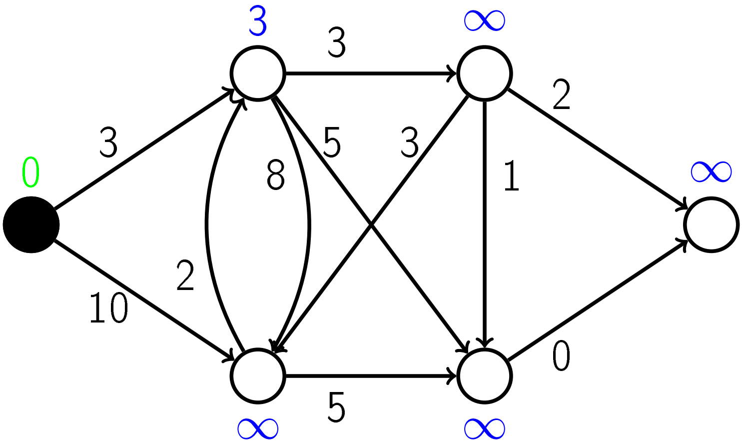

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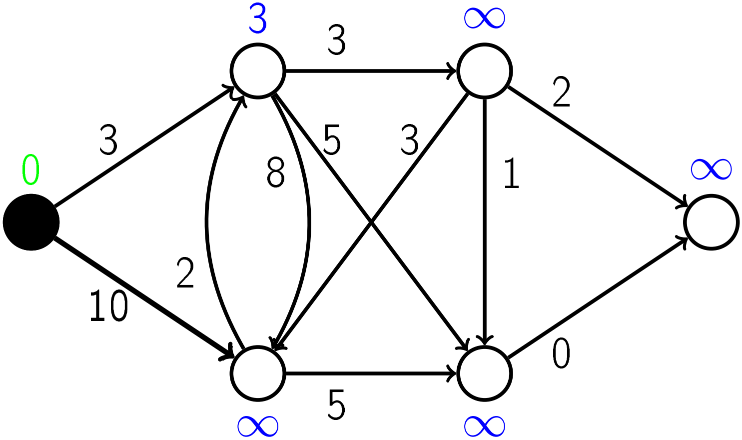

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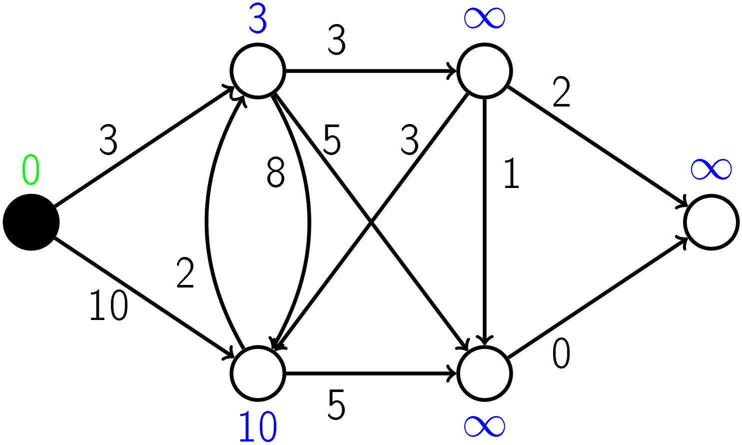

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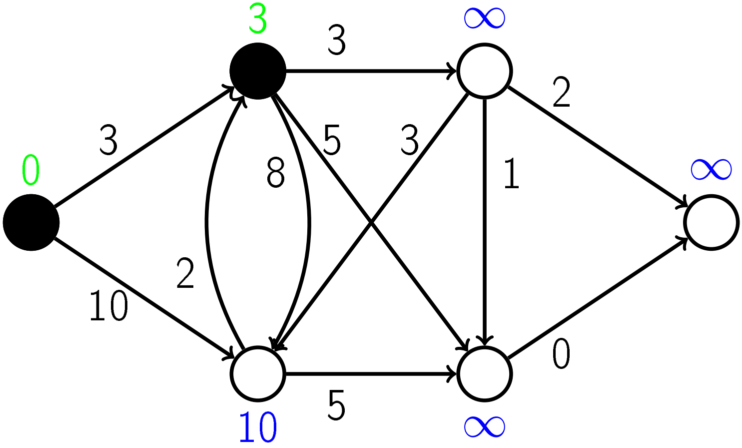

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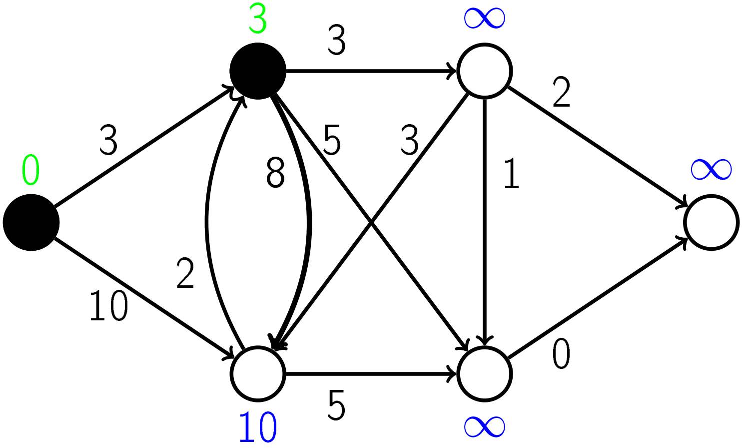

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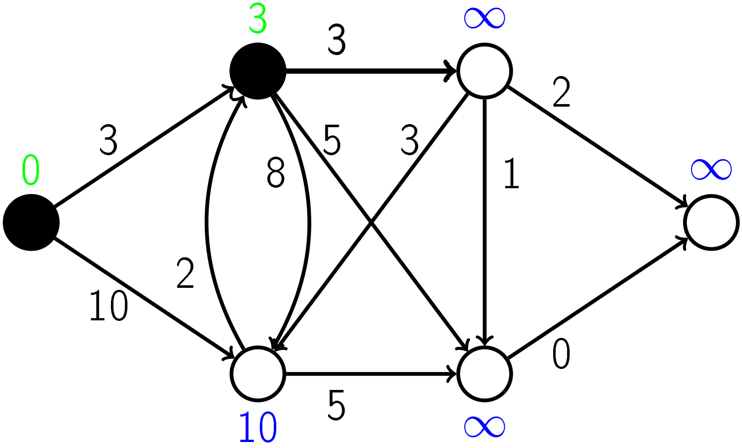

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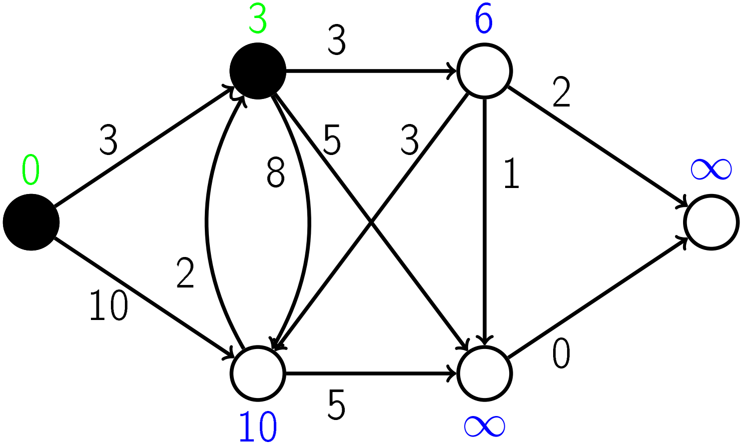

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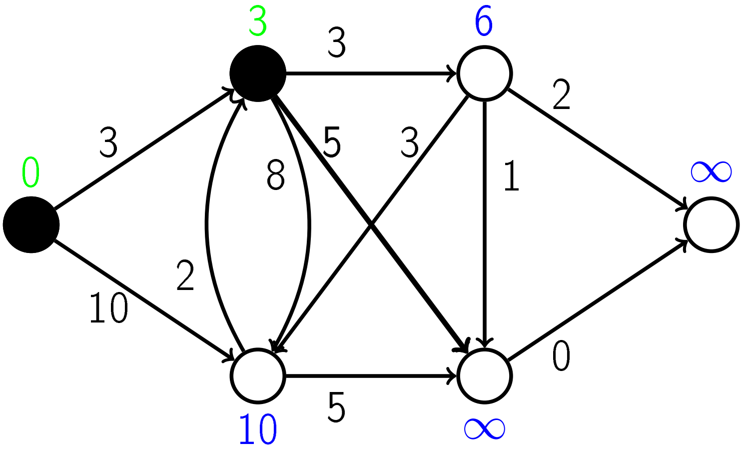

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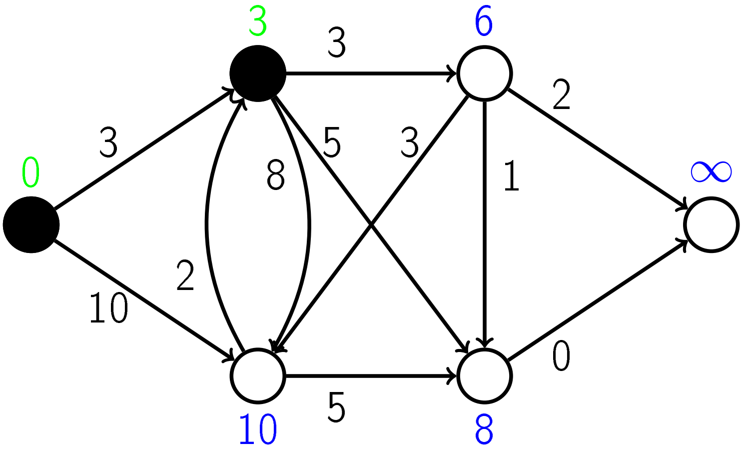

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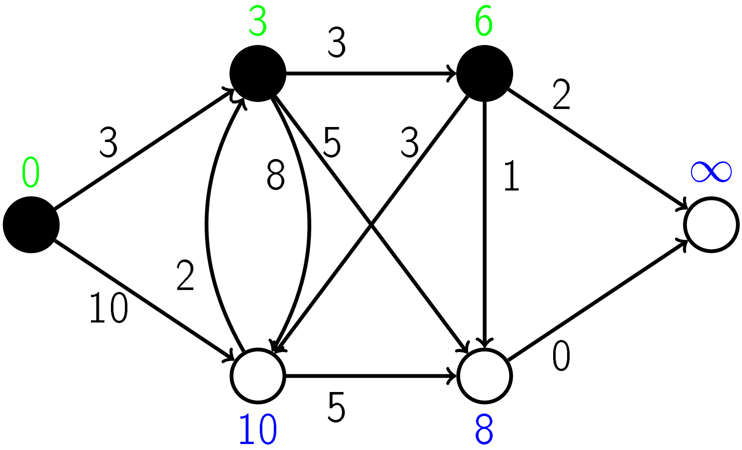

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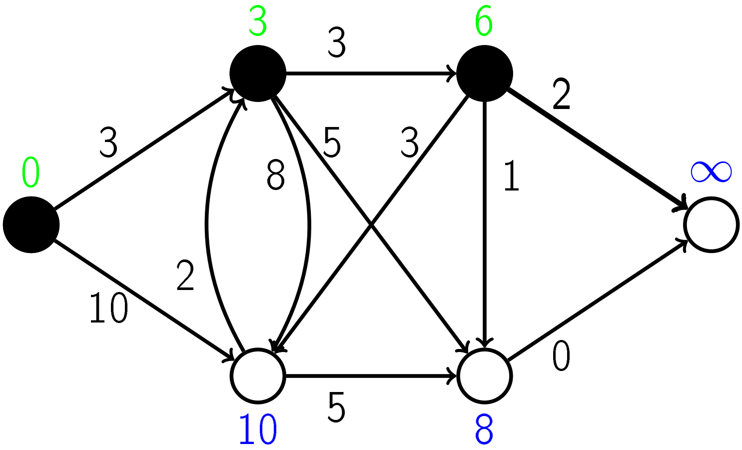

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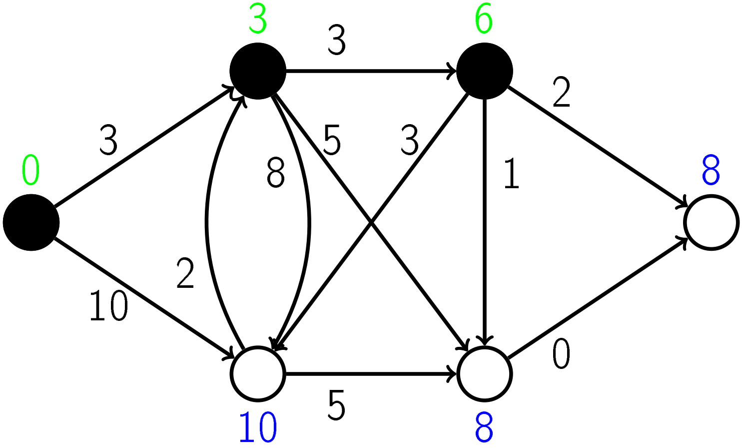

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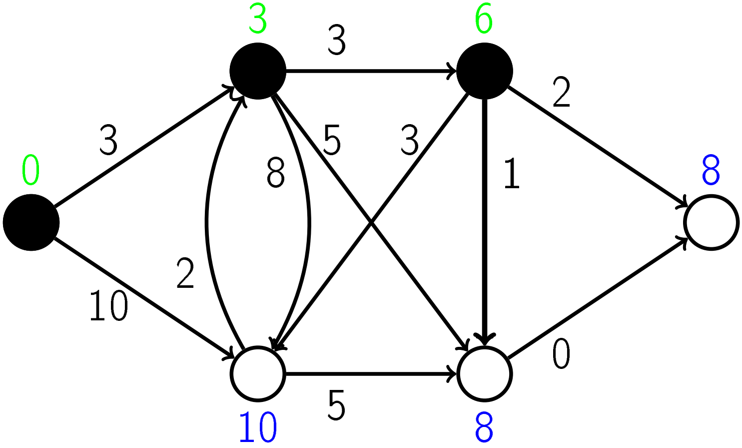

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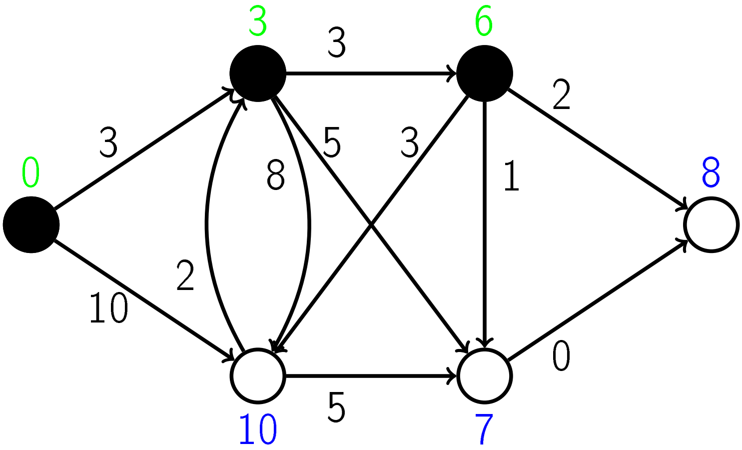

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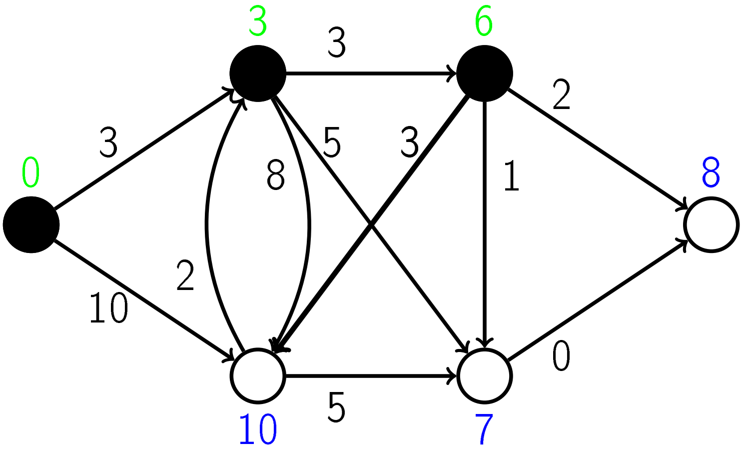

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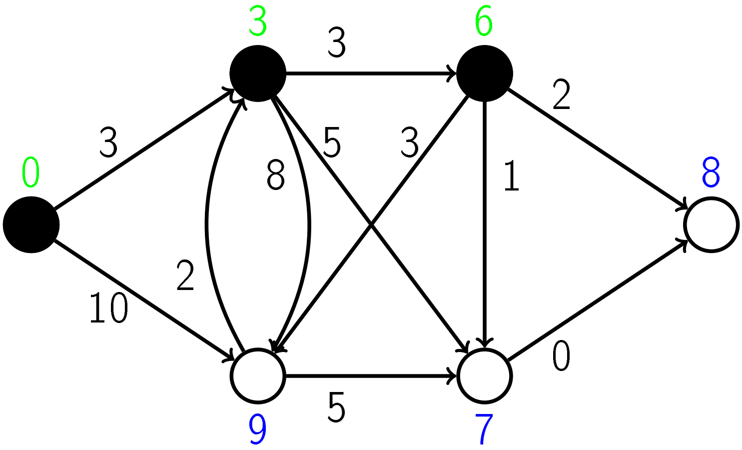

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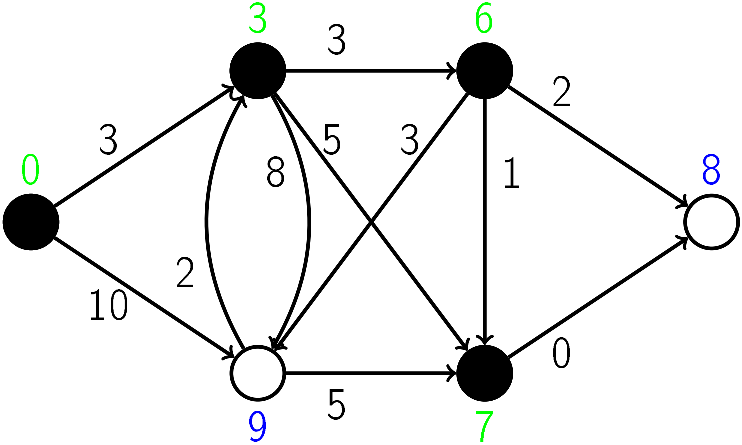

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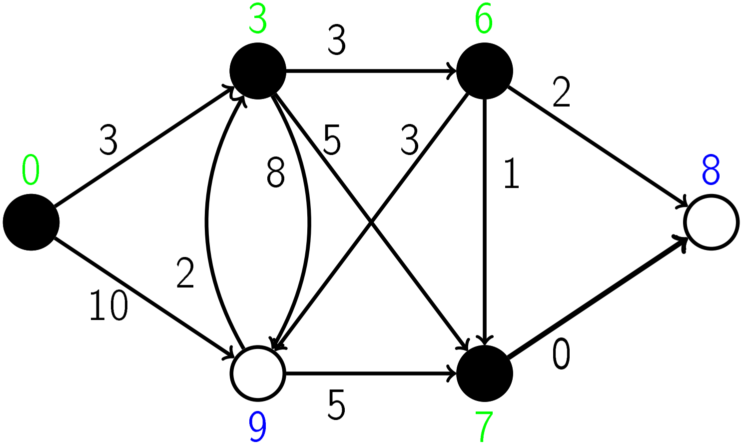

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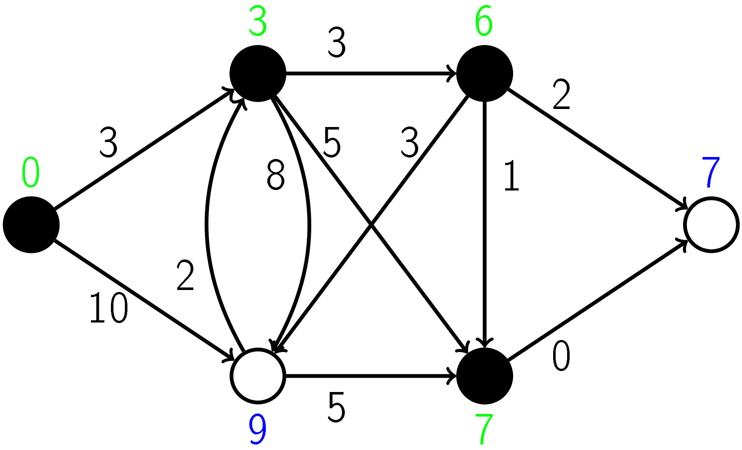

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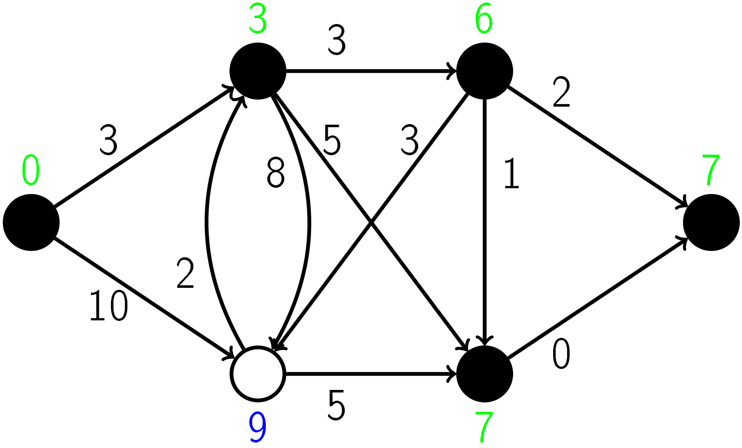

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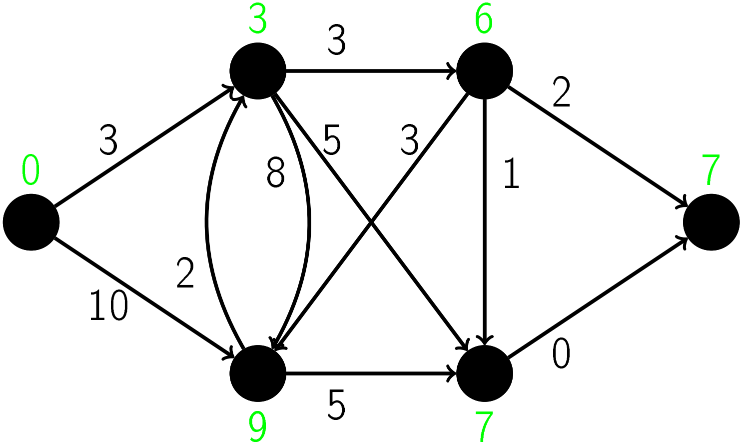

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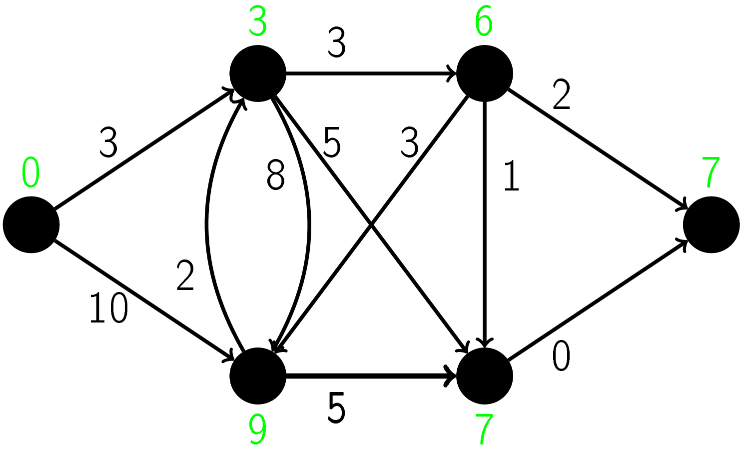

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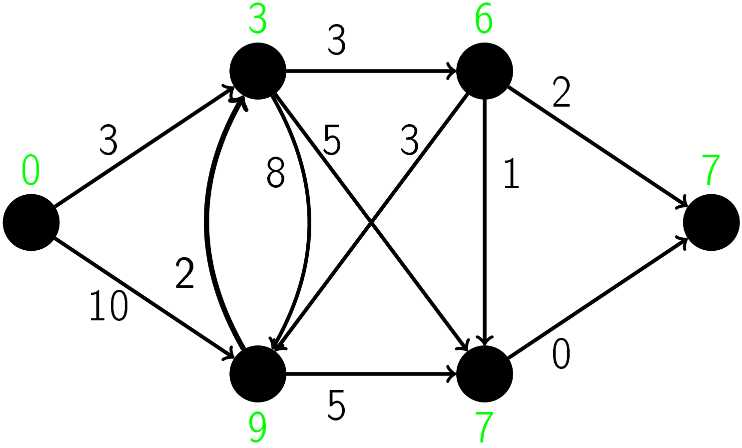

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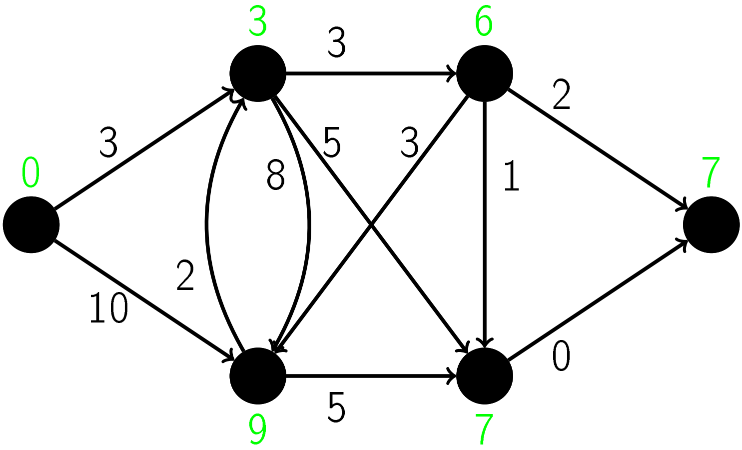

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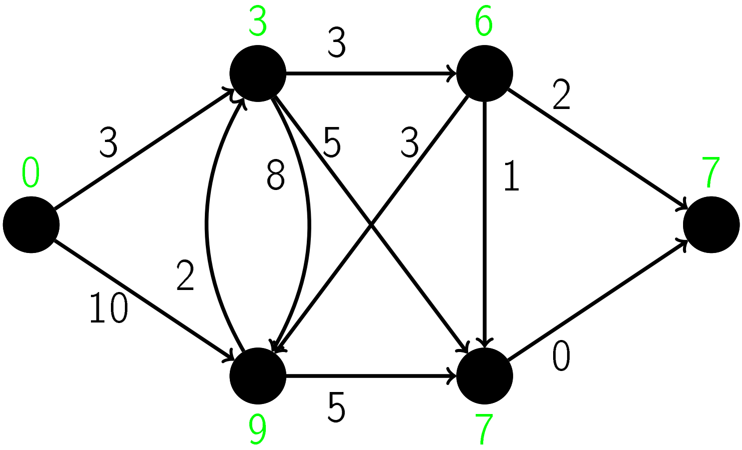


Pseudocode

Dijkstra(G, S)

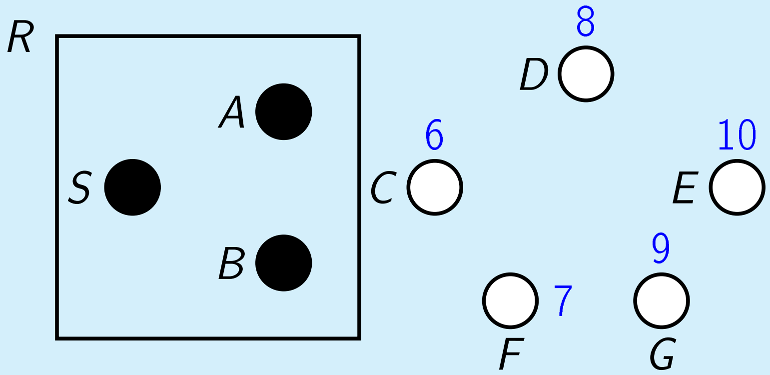
```
for all u ∈ V:
  dist[u] ← ∞, prev[u] ← nil
dist[S] ← 0
H ← MakeQueue(V) {dist-values as keys}
while H is not empty:
  u ← ExtractMin(H)
  for all (u, v) ∈ E:
    if dist[v] > dist[u] + w(u, v):
      dist[v] ← dist[u] + w(u, v)
      prev[v] ← u
      ChangePriority(H, v, dist[v])
```

Correct distances

Lemma

When a node u is selected via `ExtractMin`, $\mathrm{dist}[u] = d(S, u)$.

Proof



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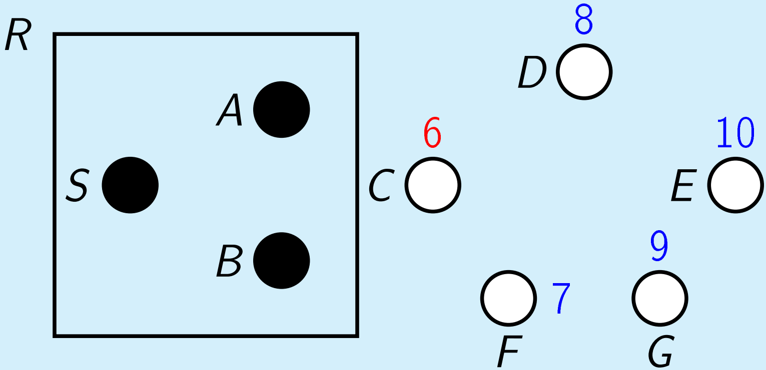
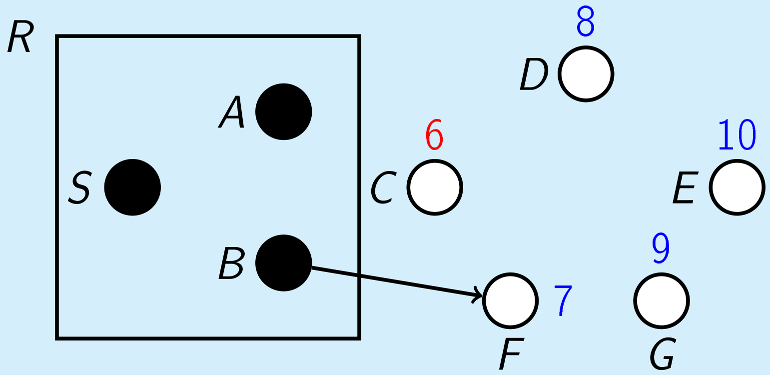
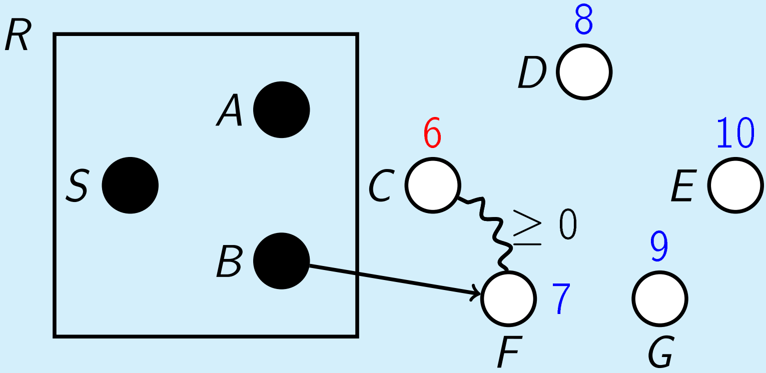


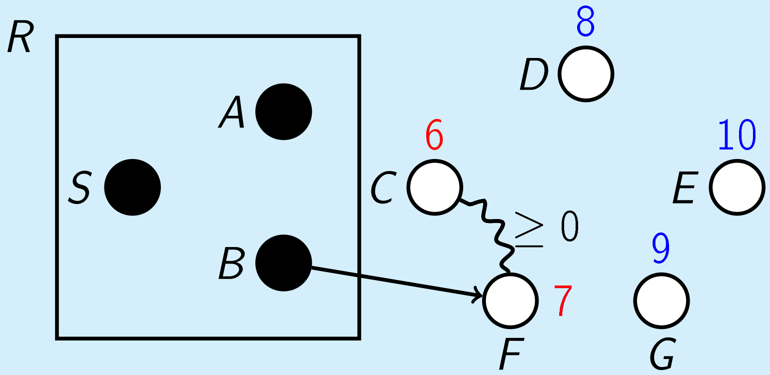
Proof



Proof



Proof



Running time

Total running time:

$$T(\texttt{MakeQueue}) + |V| \cdot T(\texttt{ExtractMin}) + |E| \cdot T(\texttt{ChangePriority})$$

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Priority queue implementations:

- array:

$$O(|V| + |V|^2 + |E|) = O(|V|^2)$$

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Priority queue implementations:

- array:

$$O(|V| + |V|^2 + |E|) = O(|V|^2)$$

- binary heap:
$O(|V| + |V| \log |V| + |E| \log |V|) =$
$O((|V| + |E|) \log |V|)$

Conclusion

- Can find the minimum time to get from work to home
- Can find the fastest route from work to home
- Works for any graph with non-negative edge weights
- Works in $O(|V|^2)$ or $O((|V| + |E|) \log(|V|))$ depending on the implementation