

Spanning Trees: Efficient Algorithms

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Russian Academy of Sciences

Graph Algorithms
Data Structures and Algorithms

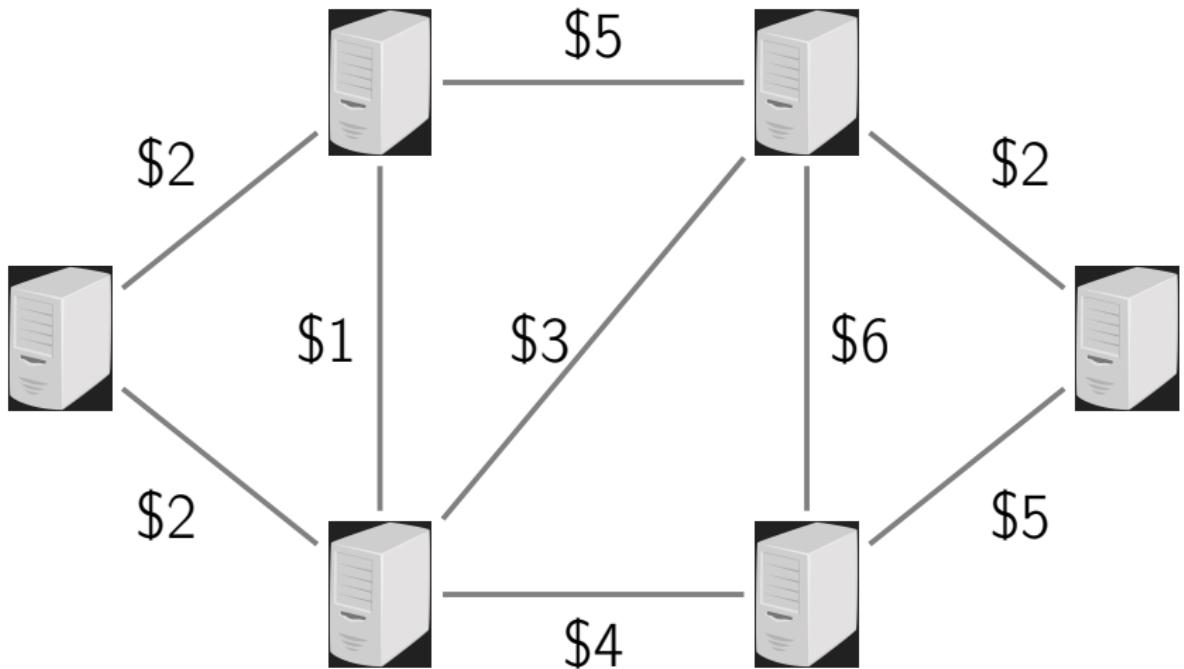
Outline

- ① Building a Network
- ② Greedy Algorithms
- ③ Cut Property
- ④ Kruskal's Algorithm
- ⑤ Prim's Algorithm

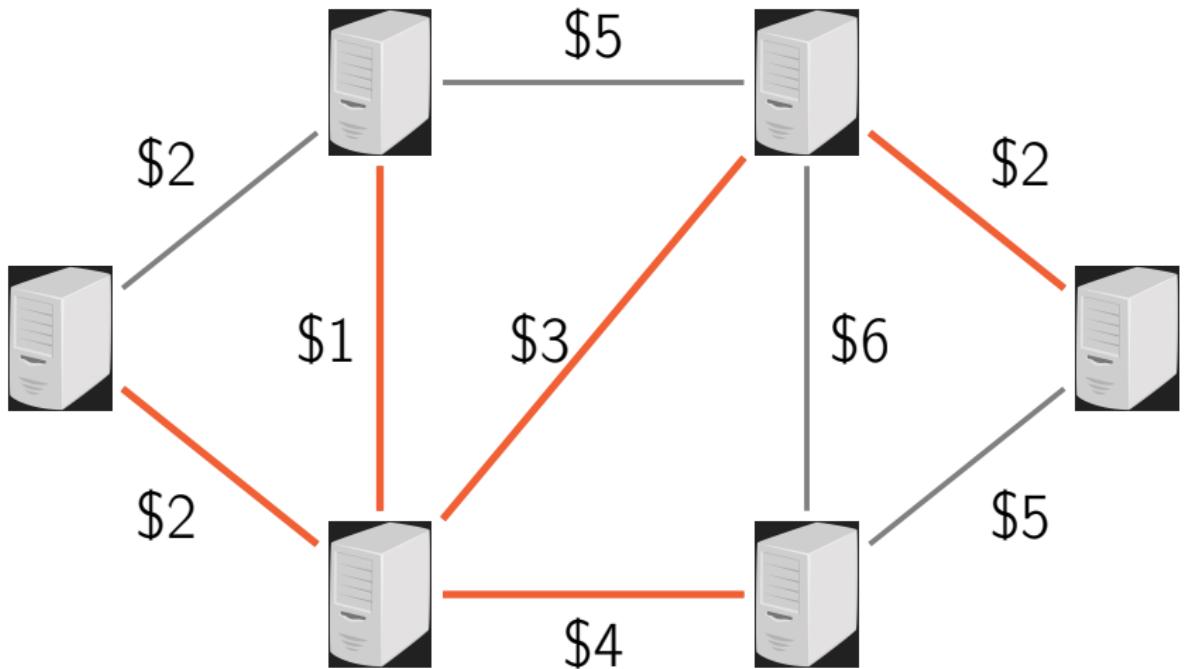
Connecting Computers by Wires



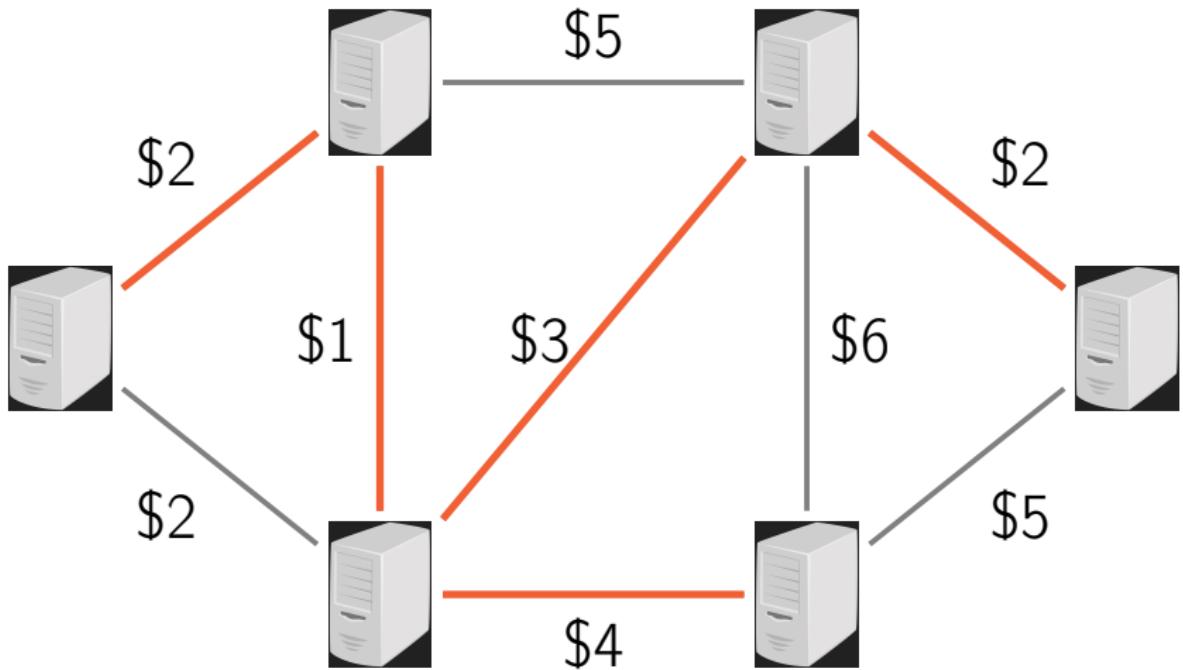
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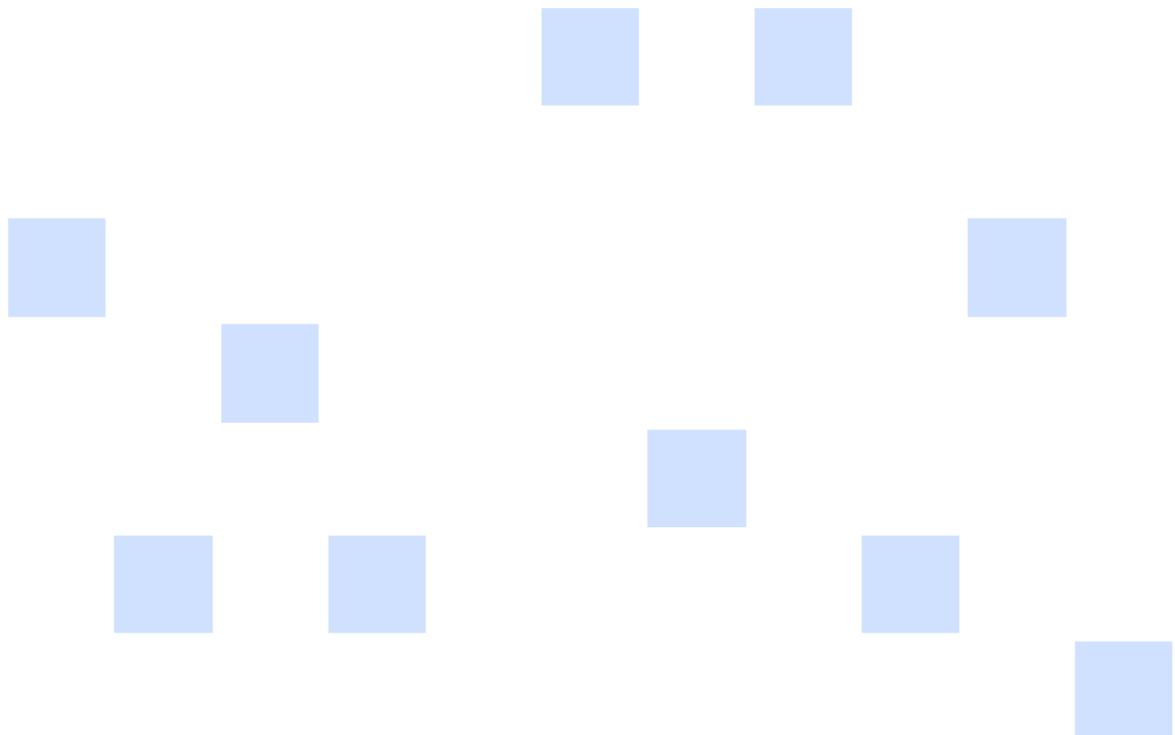
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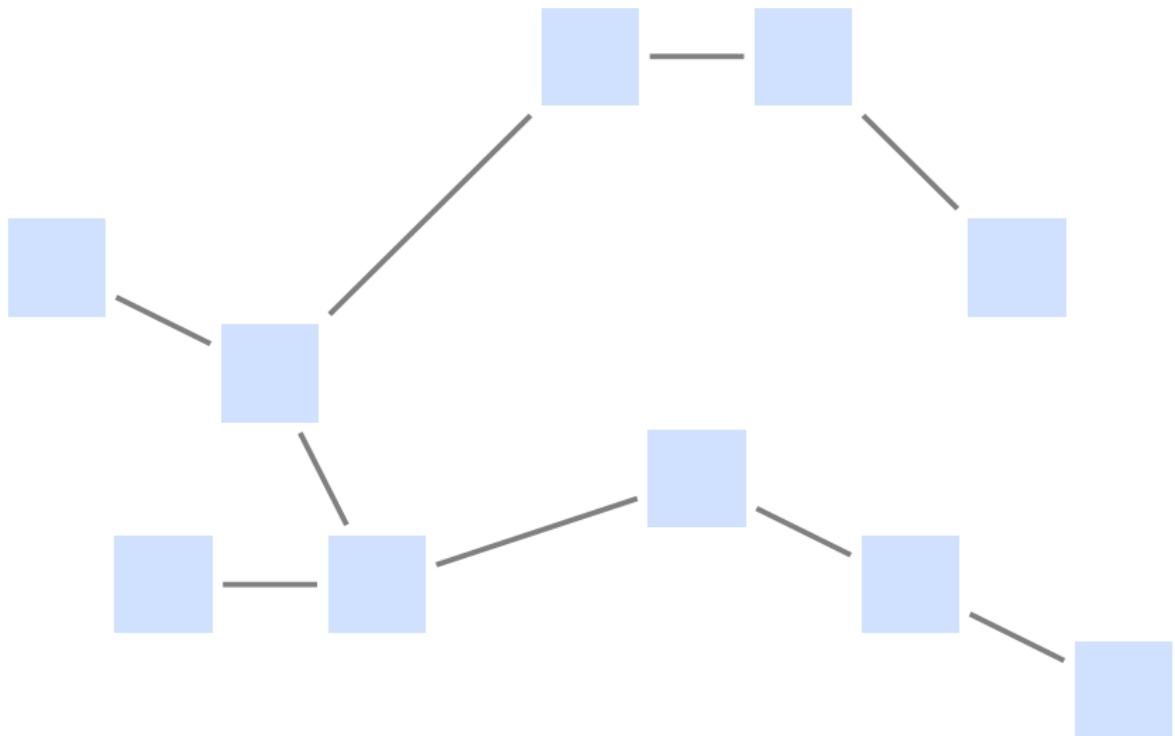
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Building Roads



Building Roads



Minimum spanning tree (MST)

Input: A connected, undirected graph $G = (V, E)$ with positive edge weights.

Output: A subset of edges $E' \subseteq E$ of minimum total weight such that the graph (V, E') is connected.

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Remark

The set E' always forms a tree.

Properties of Trees

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- A tree on n vertices has $n - 1$ edges.
- Any connected undirected graph $G(V, E)$ with $|E| = |V| - 1$ is a tree.
- An undirected graph is a tree iff there is a unique path between any pair of its vertices.

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This lesson

Two efficient greedy algorithms for the minimum spanning tree problem.

Kruskal's algorithm

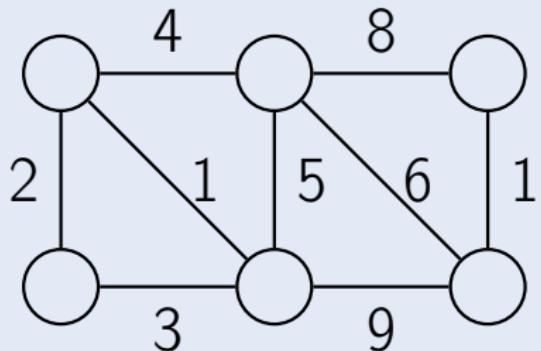
repeatedly add the next lightest edge if this doesn't produce a cycle

Prim's algorithm

repeatedly attach a new vertex to the current tree by a lightest edge

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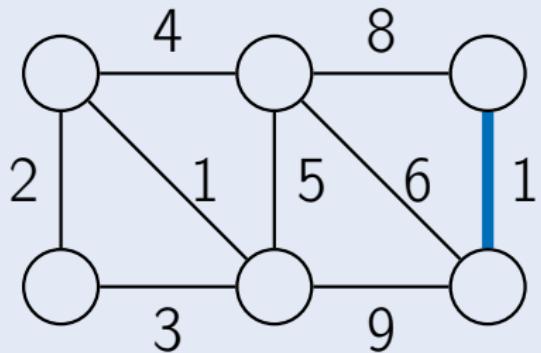


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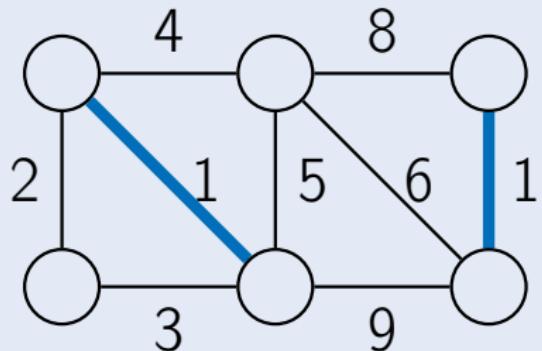


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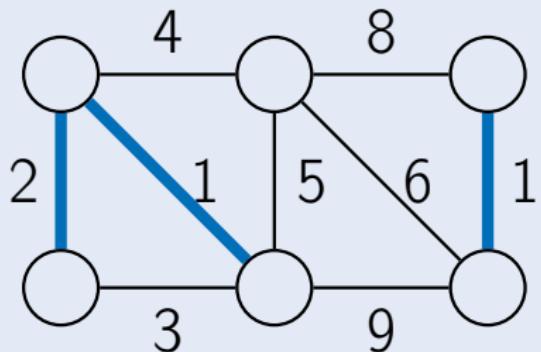


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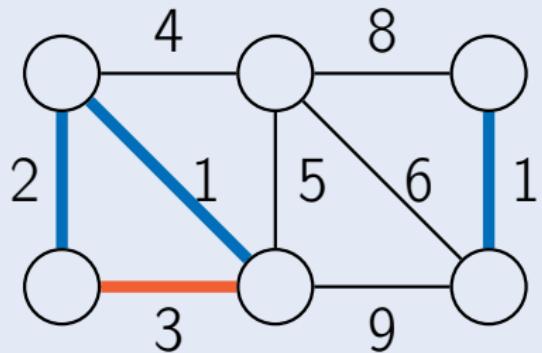


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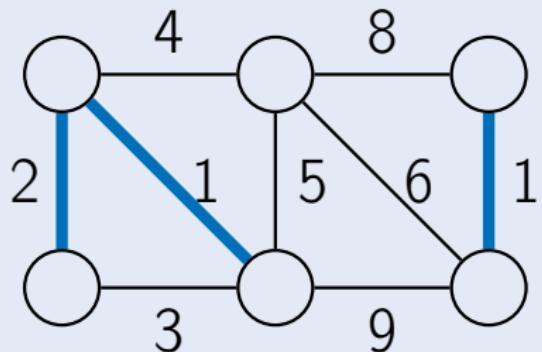


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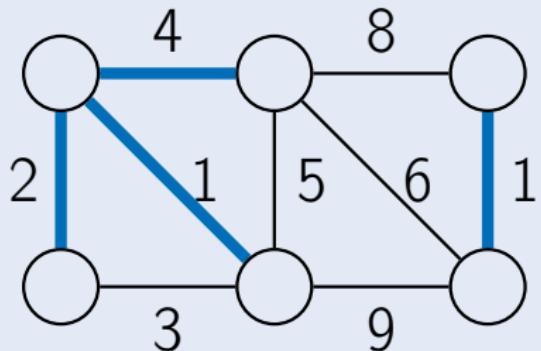


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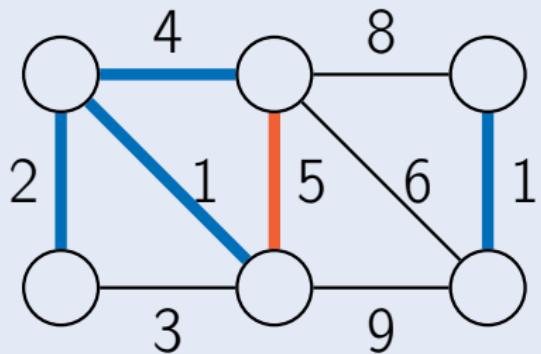


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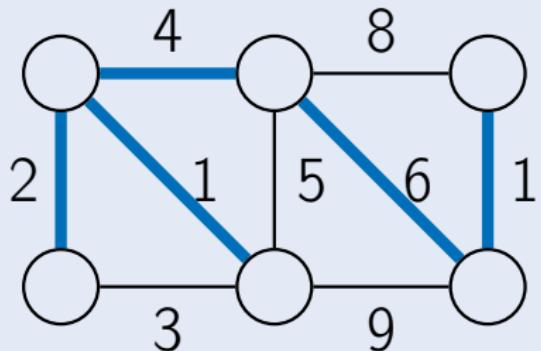


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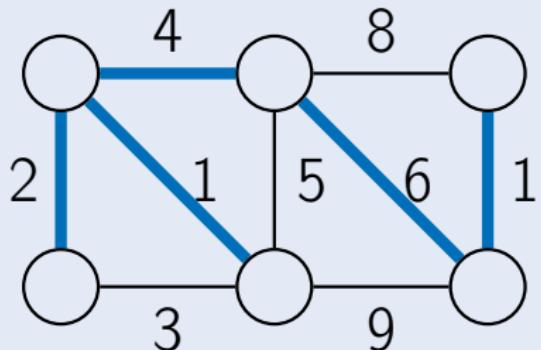


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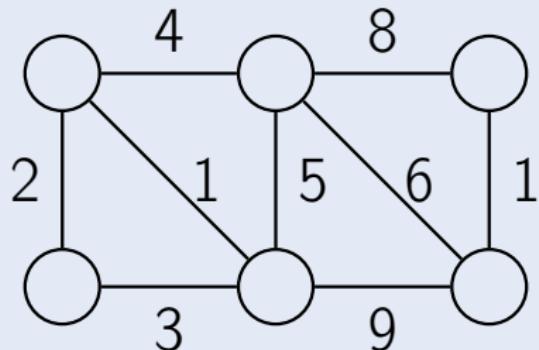
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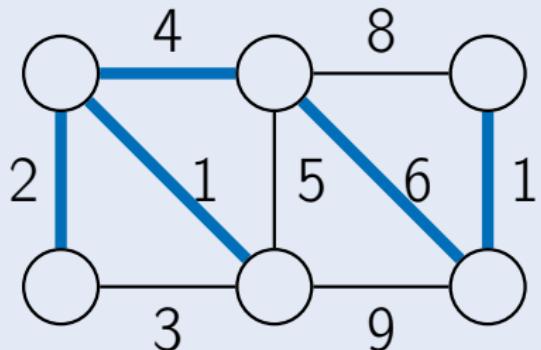
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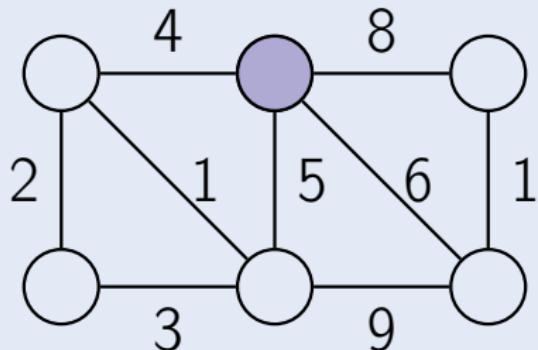
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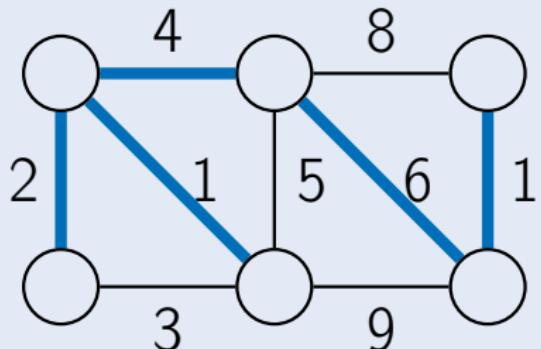
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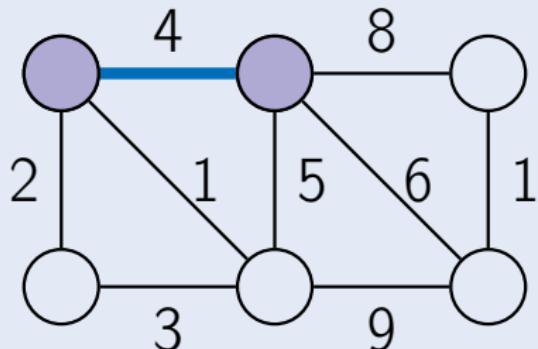
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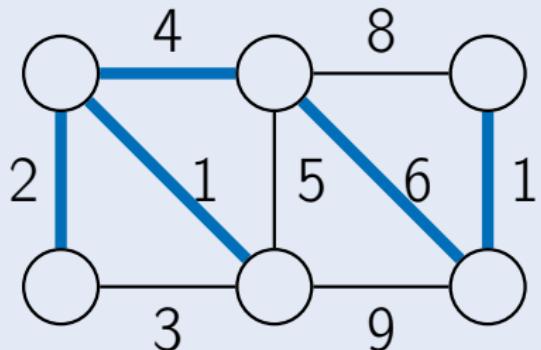
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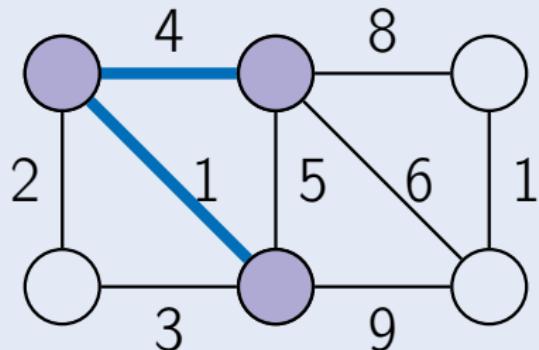
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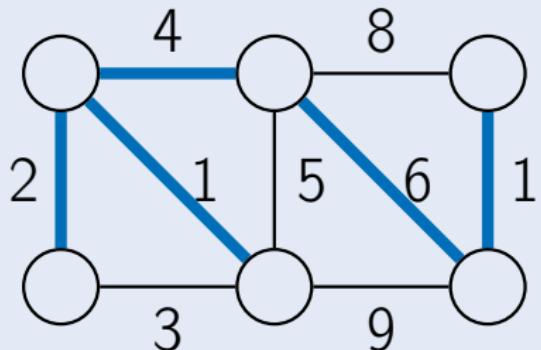
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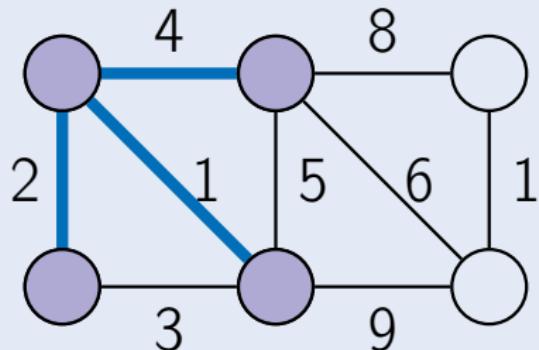
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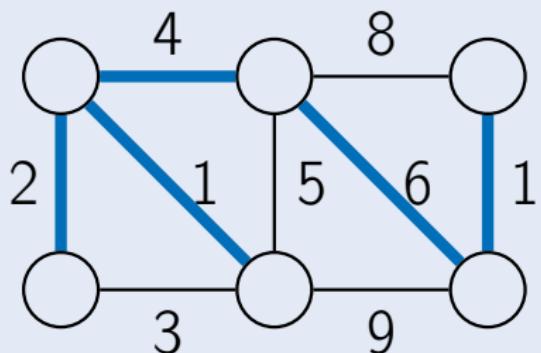
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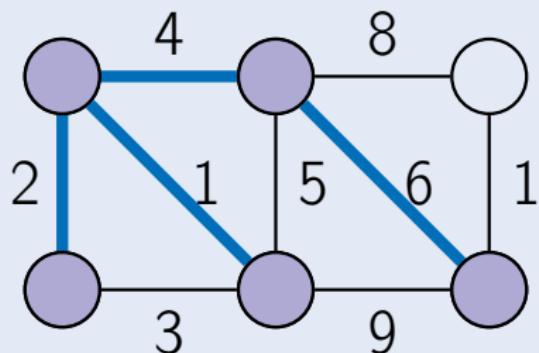
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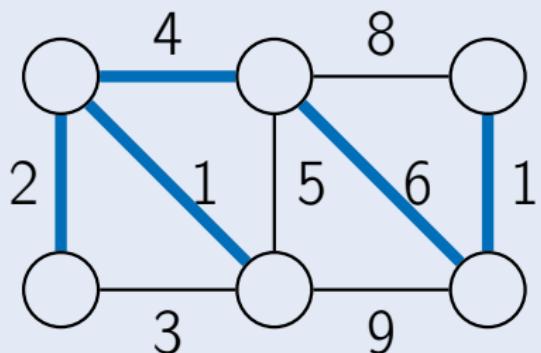
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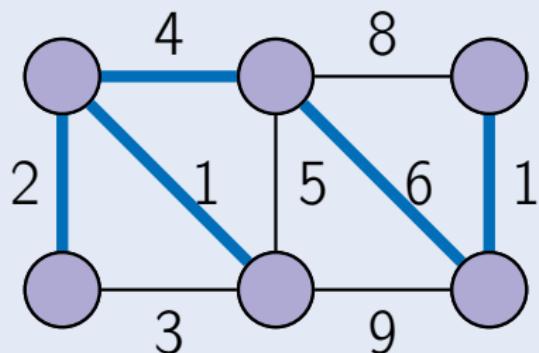
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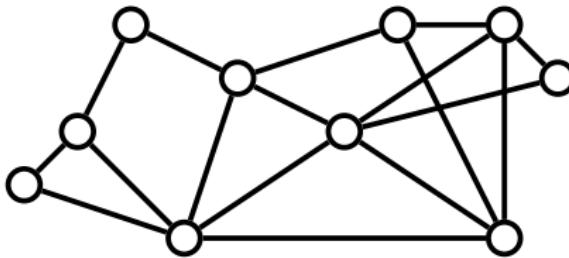
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Cut property

Let $X \subseteq E$ be a part of a MST of $G(V, E)$, $S \subseteq V$ be such that no edge of X crosses between S and $V - S$, and $e \in E$ be a lightest edge across this partition. Then $X + \{e\}$ is a part of some MST.

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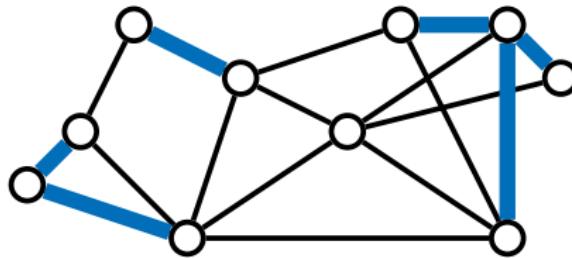
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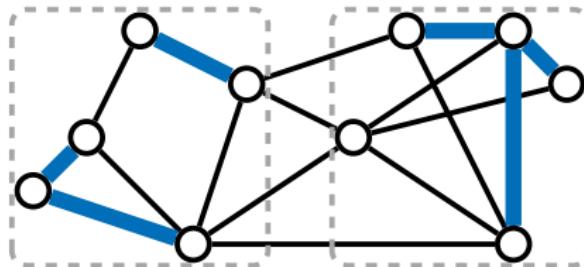
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subset $X \subseteq E$ of some MST

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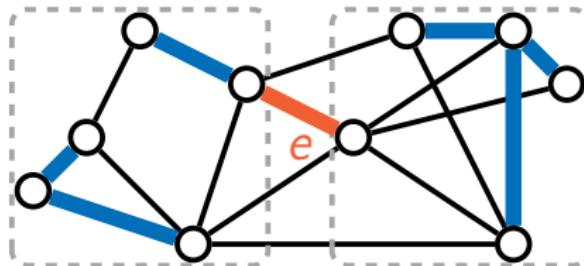
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partition of V into S and $V - S$

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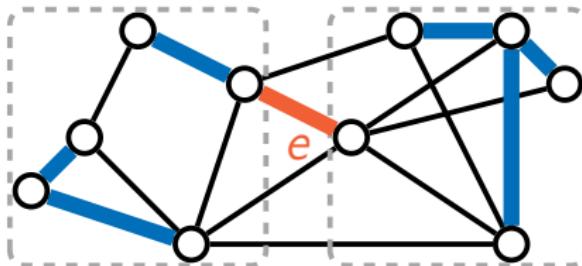
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lightest edge e between S and $V - S$

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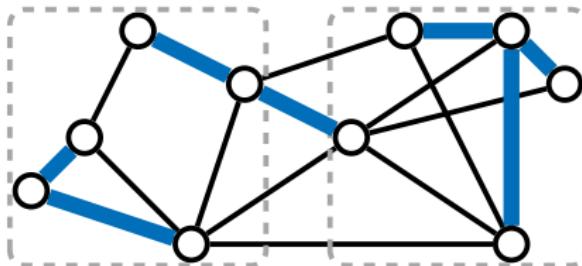
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cut property states that $X + \{e\}$ is also a part of some MST

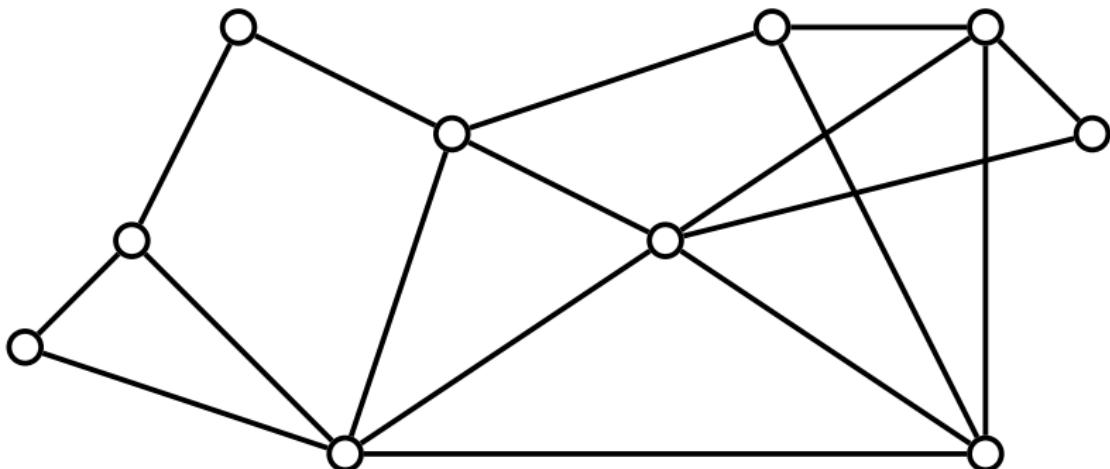
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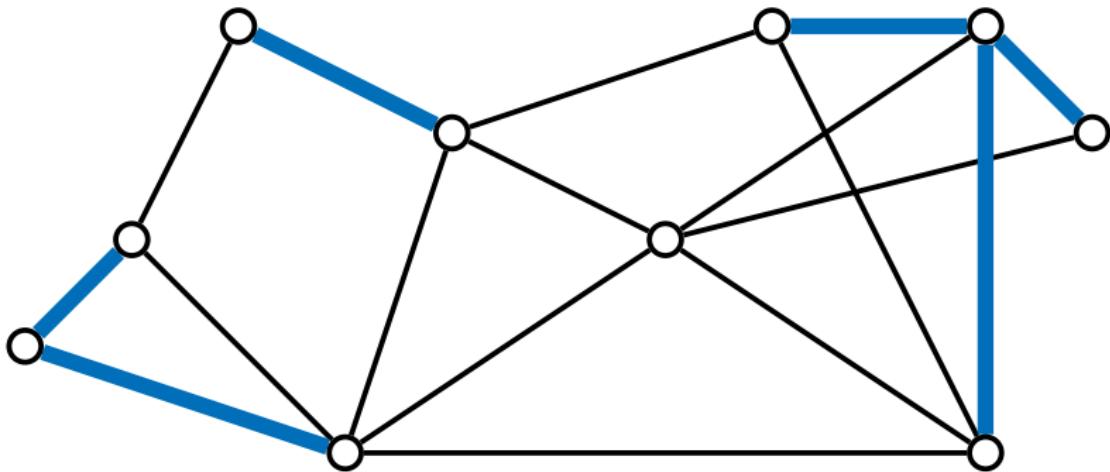
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Proof



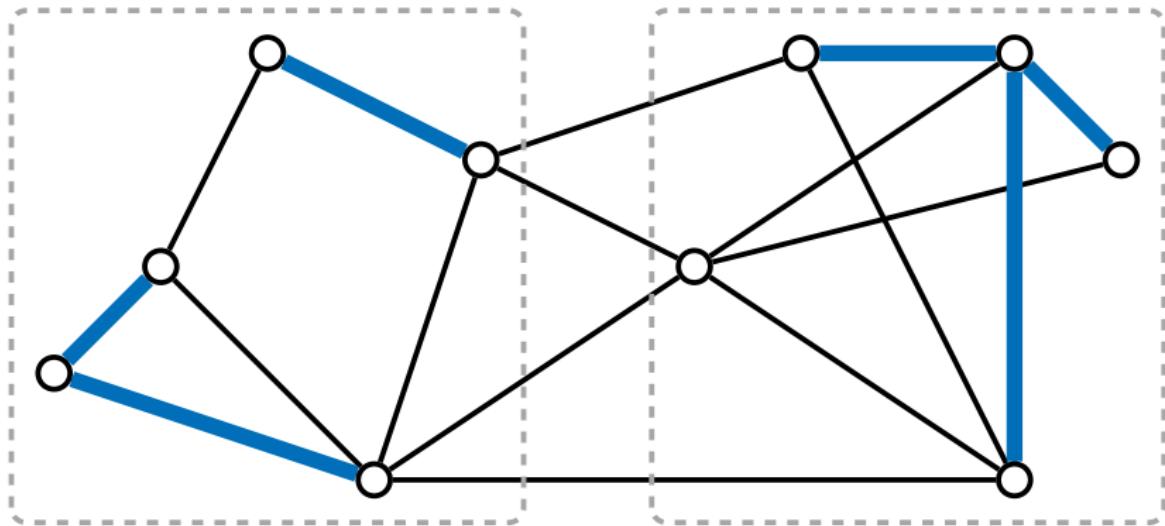
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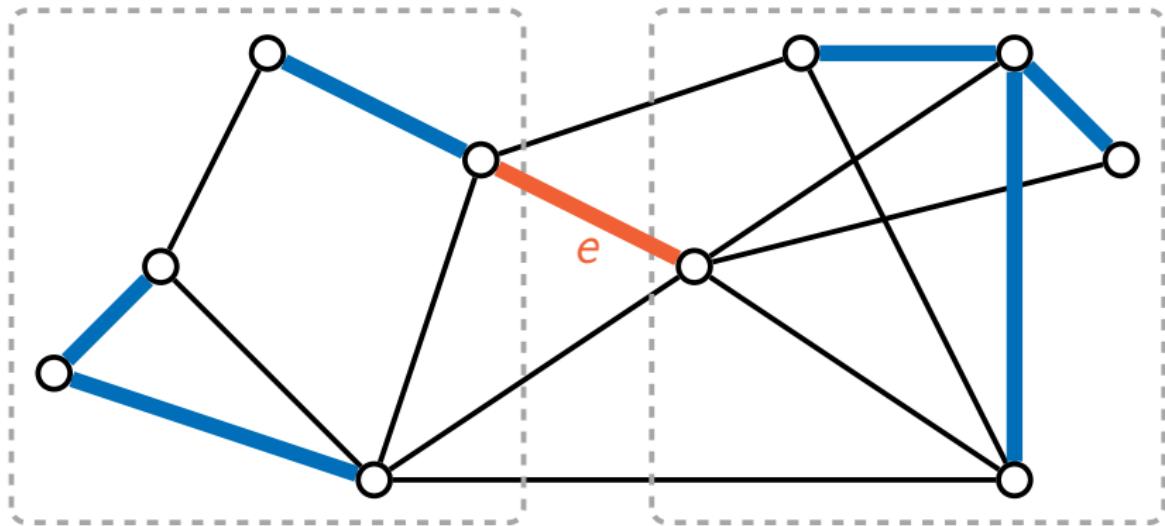
subset $X \subseteq E$ of some MST T

Proof



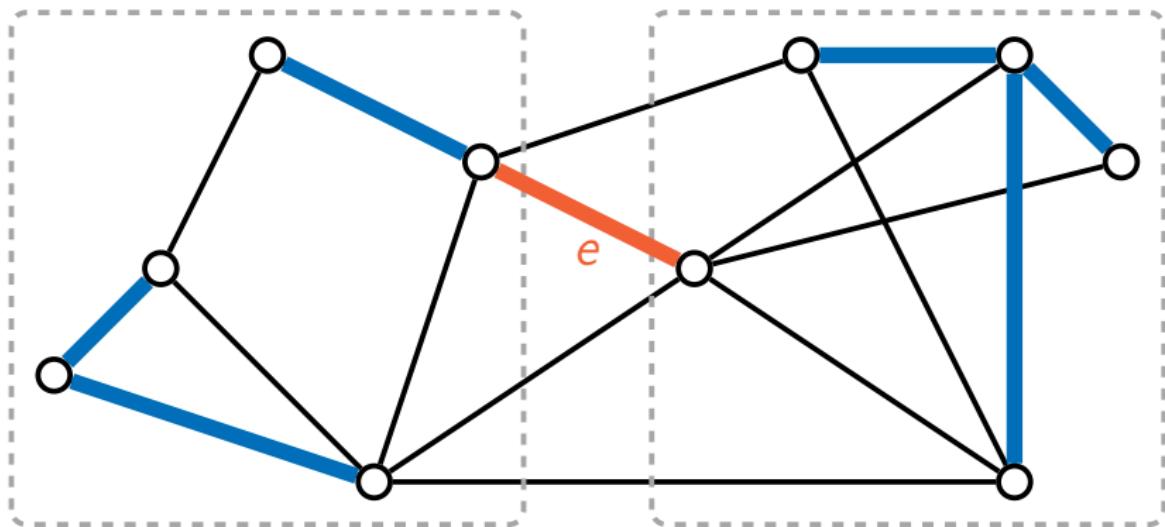
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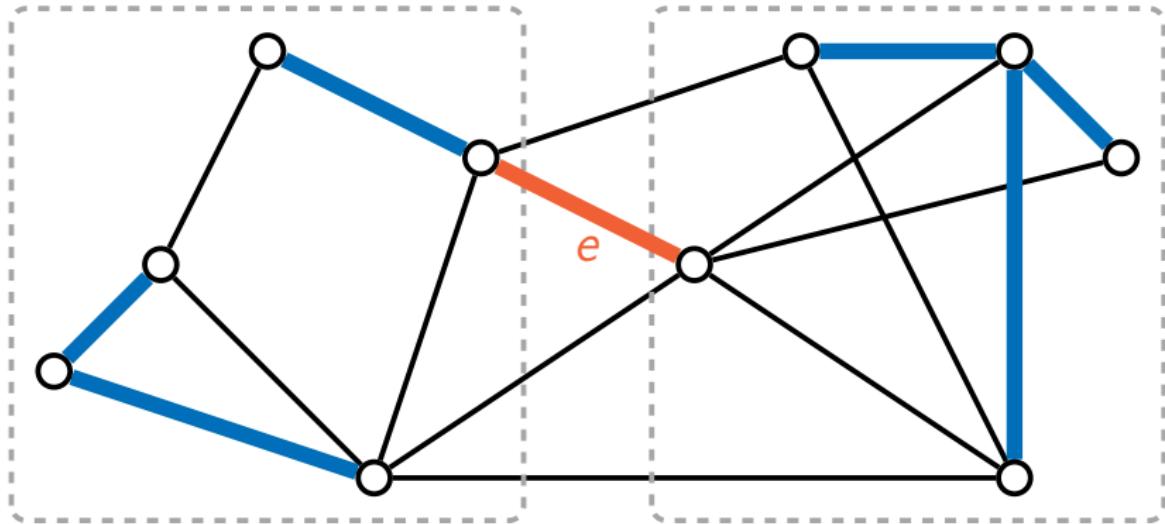
lightest edge e between S and $V - S$

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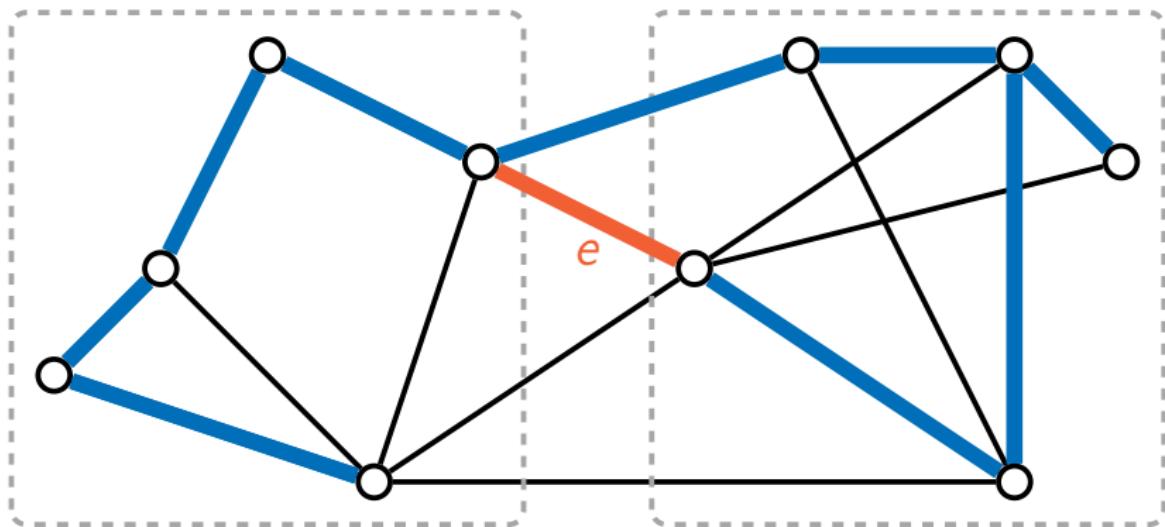
we know that X is a part of some MST T and
need to show that $X + \{e\}$ is also a part of a
(possibly different) MST

Proof



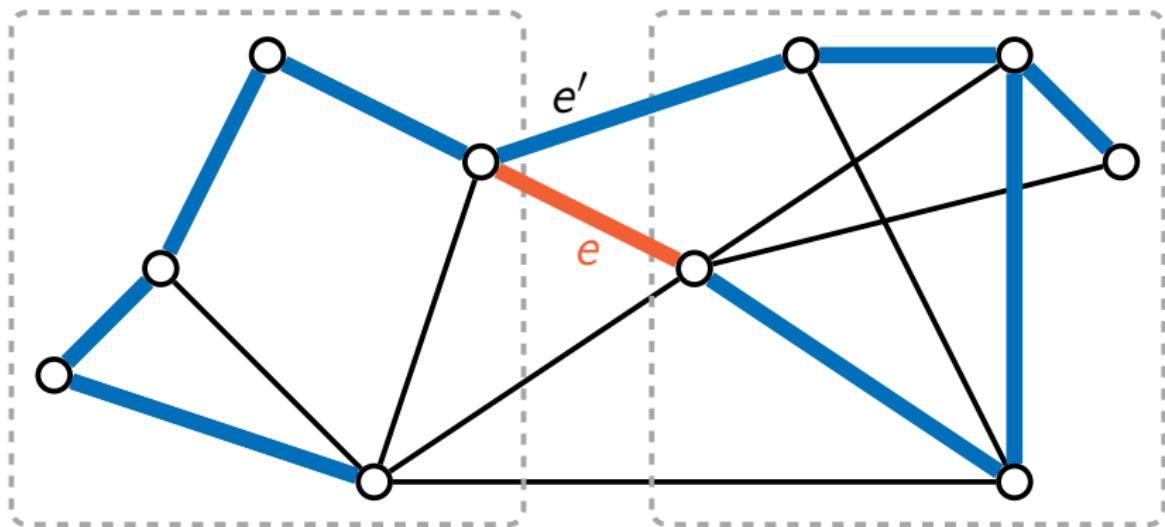
if $e \in T$ then there is nothing to prove; so
assume that $e \notin T$

Proof



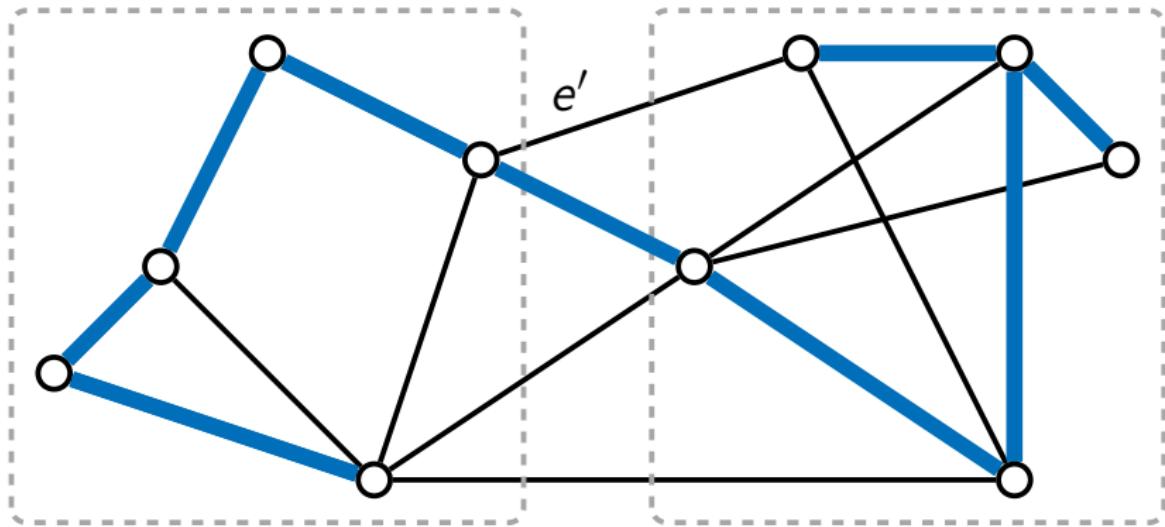
consider the tree T

Proof



adding e to T creates a cycle; let e' be an edge of this cycle that crosses S and $V - S$

Proof



then $T' = T - \{e'\} + \{e\}$ is an MST containing $X + \{e\}$: it is a tree, and $w(T') \leq w(T)$ since $w(e) \leq w(e')$

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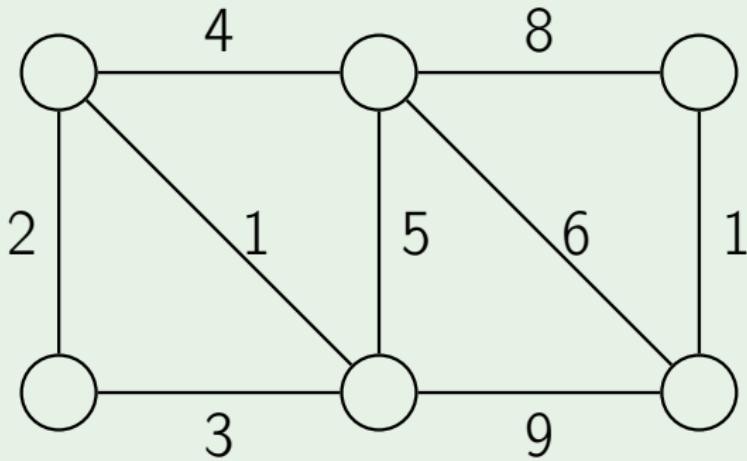
Kruskal's Algorithm

- Algorithm: repeatedly add to X the next lightest edge e that doesn't produce a cycle
- At any point of time, the set X is a forest, that is, a collection of trees
- The next edge e connects two different trees—say, T_1 and T_2
- The edge e is the lightest between T_1 and $V - T_1$, hence adding e is safe

Implementation Details

- use disjoint sets data structure
- initially, each vertex lies in a separate set
- each set is the set of vertices of a connected component
- to check whether the current edge $\{u, v\}$ produces a cycle, we check whether u and v belong to the same set

Example



Kruskal(G)

for all $u \in V$:

 MakeSet(v)

$X \leftarrow$ empty set

sort the edges E by weight

for all $\{u, v\} \in E$ in non-decreasing
weight order:

 if Find(u) \neq Find(v):

 add $\{u, v\}$ to X

 Union(u, v)

return X

Running Time

- Sorting edges:

$$\begin{aligned}O(|E| \log |E|) &= O(|E| \log |V|^2) = \\O(2|E| \log |V|) &= O(|E| \log |V|)\end{aligned}$$

- Processing edges:

$$\begin{aligned}2|E| \cdot T(\text{Find}) + |V| \cdot T(\text{Union}) &= \\O((|E|+|V|) \log |V|) &= O(|E| \log |V|)\end{aligned}$$

- Total running time: $O(|E| \log |V|)$

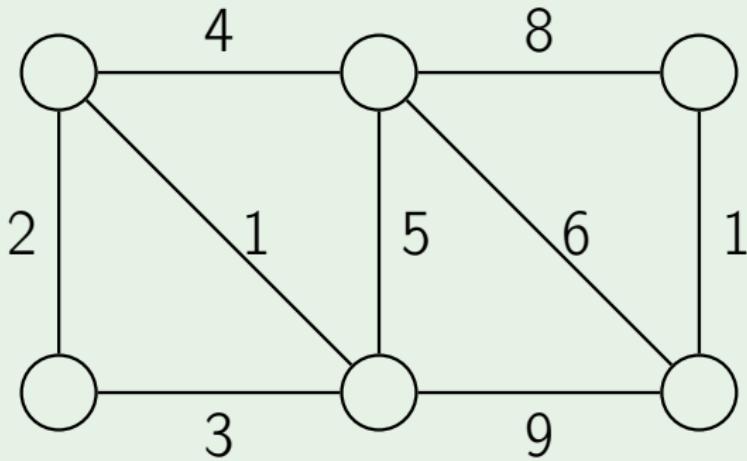
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Prim's Algorithm

- X is always a subtree, grows by one edge at each iteration
- we add a lightest edge between a vertex of the tree and a vertex not in the tree
- very similar to Dijkstra's algorithm

Example



Prim's Algorithm

Prim(G)

for all $u \in V$:

$cost[u] \leftarrow \infty$, $parent[u] \leftarrow nil$

pick any initial vertex u_0

$cost[u_0] \leftarrow 0$

$PrioQ \leftarrow \text{MakeQueue}(V)$ {priority is cost}

while $PrioQ$ is not empty:

$v \leftarrow \text{ExtractMin}(PrioQ)$

for all $\{v, z\} \in E$:

if $z \in PrioQ$ and $cost[z] > w(v, z)$:

$cost[z] \leftarrow w(v, z)$, $parent[z] \leftarrow v$

$\text{ChangePriority}(PrioQ, z, cost[z])$

Running Time

- the running time is

$$|V| \cdot T(\text{ExtractMin}) + |E| \cdot T(\text{ChangePriority})$$

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- for array-based implementation, the running time is $O(|V|^2)$

- for binary heap-based implementation, the running time is

$$O((|V| + |E|) \log |V|) = O(|E| \log |V|)$$

Summary

Kruskal: repeatedly add the next lightest edge if this doesn't produce a cycle; use disjoint sets to check whether the current edge joins two vertices from different components

Prim: repeatedly attach a new vertex to the current tree by a lightest edge; use priority queue to quickly find the next lightest edge