**證明$\boldsymbol{f'(x) = f(x)[1 - f(x)]}$**

將方程整理為：

$$
\begin{aligned}
&f'(x) = f(x)[1 - f(x)] \\
&\Leftrightarrow \frac{df}{dx} = f(x)[1 - f(x)] \\
&\Leftrightarrow \frac{1}{f(x)[1 - f(x)]} \times df = dx \\
&\Leftrightarrow \left(\frac{1}{f(x)} + \frac{1}{1 - f(x)}\right) df = dx \ (\because \text{部分分式分解})
\end{aligned}
$$

對兩邊積分：

$$
\begin{aligned}
&\int \frac{1}{f(x)} df + \int \frac{1}{1 - f(x)} df = \int dx \\
&\Leftrightarrow ln|f(x)| - ln|1 - f(x)| = x + C \\
&\Leftrightarrow ln\left|\frac{f(x)}{1 - f(x)}\right| = x + C \\
&\Leftrightarrow \frac{f(x)}{1 - f(x)} = Ce^x
\end{aligned}
$$

解出$f(x)$：

$$
\begin{aligned}
&\frac{f(x)}{1 - f(x)} = Ce^x \\
&\Leftrightarrow f(x) = Ce^x\big(1 - f(x)\big) \\
&\Leftrightarrow f(x) = Ce^x - Ce^x f(x) \\
&\Leftrightarrow f(x) + Ce^x f(x) = Ce^x \\
&\Leftrightarrow (1 + Ce^x) f(x) = Ce^x \\
&\Leftrightarrow f(x) = \frac{Ce^x}{1 + Ce^x}
\end{aligned}
$$

驗證解：

對$f(x) = Ce^x/(1 + Ce^x)$求導

$$f'(x) = \frac{Ce^x \times (1 + Ce^x) - Ce^x \times Ce^x}{(1 + Ce^x)^2} = \frac{Ce^x}{(1 + Ce^x)^2}$$

計算$f(x)[1 - f(x)]$

$$f(x)[1 - f(x)] = \frac{Ce^x}{1 + Ce^x} \times \left(1 - \frac{Ce^x}{1 + Ce^x}\right) = \frac{Ce^x}{1 + Ce^x} \times \frac{1}{1 + Ce^x} = \frac{Ce^x}{(1 + Ce^x)^2}$$

$$\therefore f'(x) = f(x)[1 - f(x)]$$

$\blacksquare$