4. Exercise E21.4

Note that the bootstrap bias estimate is -69.377, which is around 7 times smaller than the theoretical value -10.

```
set.seed(200)
sample <- rnorm(10, mean = 0, sd = sqrt(100))

# take 500 bootstrap samples
bootstrap_bias_estimate <- 0
for(i in 1:500) {
  bootstrap_sample <- sample(sample, size = 10, replace = TRUE)
  mean_of_bootstrap_sample <- mean(bootstrap_sample)
  MLE_estimate_of_variance <- (1/10) * sum((bootstrap_sample - mean_of_bootstrap_sa
  bootstrap_bias_estimate <- bootstrap_bias_estimate + MLE_estimate_of_variance
}

# find bootstrap bias estimate
bootstrap_bias_estimate <- bootstrap_bias_estimate / 500 - 100

sprintf("The bootstrap bias estimate is %f", bootstrap_bias_estimate)
```

```
[1] "The bootstrap bias estimate is -69.377128"
```

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20.5 Multivariate LR.

$$\underset{(n\times m)}{Y} = \underset{n\times(k+1)}{X}\ \underset{(k+1)\times m}{\beta} + \underset{(n\times m)}{E}$$ → one column per response variable

$\varepsilon_i \overset{iid}{\sim} N_m(0,\Sigma)$, ith row of E

a) Pf: Note that by defn.

$$\hat{\mu}_Y = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{Y_{ij}}{nm} \quad \text{(mean of all elements of } Y)$$

$$\hat{\mu}_X = \sum_{i=1}^{n}\sum_{j=1}^{k+1}\frac{X_{ij}}{n(k+1)} \quad (\text{'' '' of } X)$$

$$\hat{\Sigma}_{XX} = \sum_{i=1}^{n}\sum_{j=1}^{k+1}\sum_{k=1}^{n}\sum_{\ell=1}^{k+1}\frac{(X_{ij}-\hat{\mu}_X)(X_{k\ell}-\hat{\mu}_X)}{n^2(k+1)^2} \quad \text{(by defn of covariance)}$$

$$\hat{\Sigma}_{XY} = \sum_{i=1}^{n}\sum_{j=1}^{k+1}\sum_{a=1}^{n}\sum_{b=1}^{m}\frac{(X_{ij}-\hat{\mu}_X)(Y_{ab}-\hat{\mu}_Y)}{n^2(k+1)m} \quad (\text{'' ''})$$

Note that

$$\hat{\Sigma}_{XX} = \sum_{i=1}^{n}\sum_{j=1}^{k+1}\sum_{k=1}^{n}\sum_{\ell=1}^{k+1}\frac{(X_{ij}-\hat{\mu}_X)(X_{k\ell}-\hat{\mu}_X)}{n^2(k+1)^2}$$

$$= \frac{1}{n^2(k+1)^2}\sum_{i=1}^{n}\sum_{j=1}^{k+1}\sum_{k=1}^{n}\sum_{\ell=1}^{k+1}\left[X_{ij}X_{k\ell} - \hat{\mu}_X(X_{ij}+X_{k\ell}) + \hat{\mu}_X^2\right]$$

$$= \frac{1}{n^2(k+1)^2}\sum_{i=1}^{n}\sum_{j=1}^{k+1}\sum_{k=1}^{n}\sum_{\ell=1}^{k+1}\left[X_{ij}X_{k\ell} - \hat{\mu}_X(X_{ij}+X_{k\ell})\right] + n^2(k+1)^2\hat{\mu}_X^2$$

$= \ldots\ldots$ (TA just said skip this proof)

$$= \frac{(X^TX)+\hat{\mu}_X^2}{n^2(k+1)^2}$$

And similarly $\hat{\Sigma}_{XY} = \dfrac{(X^TY+\hat{\mu}_X\hat{\mu}_Y)}{n^2(k+1)m}$

And now, isolate (X^TX) and (X^TY)

$$\hat{\Sigma}_{XX} = \frac{(X^TX)+\hat{\mu}_X^2}{n^2(k+1)^2} \Rightarrow \hat{\Sigma}_{XX}n^2(k+1)^2 - \hat{\mu}_X^2 = X^TX$$

$$\hat{\Sigma}_{XY} = \frac{(X^TY+\hat{\mu}_X\hat{\mu}_Y)}{n^2(k+1)m} \Rightarrow \hat{\Sigma}_{XY}n^2(k+1)m - \hat{\mu}_X\hat{\mu}_Y = X^TY$$

And substitute:

$$\hat{\beta} = (X^TX)^{-1}X^TY.$$

$$\Rightarrow \boxed{\hat{\beta} = \left(\hat{\Sigma}_{XX}n^2(k+1)^2 - \hat{\mu}_X^2\right)^{-1}\left(\hat{\Sigma}_{XY}n^2(k+1)m - \hat{\mu}_X\hat{\mu}_Y\right)}$$

b) Pf: Let $j' = k$

$$\mathrm{Cov}(\hat{Y}_{ij}, \hat{Y}_{ik}) = \mathrm{Cov}(x_i^T b_j, x_i^T b_s)$$

b_j, b_s are columns of $\hat{\beta}$

$$= x_i^T \mathrm{Cov}(b_j, b_s)\, x_i$$

$\mathrm{cov}(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$

$$= \boxed{\sigma^2 \left[(X^T X)^{-1}\right]_{js} x_i^T x_i}$$

c) This result can be used in Eqn 20.7 to obtain $\sigma_{js}^{(\ell)}$ on the M step.

$\boxed{20.10}$ S'pose $\xi \sim N(\mu, \sigma^2)$ is left-censored at $\xi = a$, so that

$$Y = \begin{cases} a & \text{for } \xi \le a \\ \xi & \text{for } \xi > a \end{cases}$$

Pf: Let $z_a \equiv \frac{a-\mu}{\sigma}$, $m(z_a) \equiv \frac{\phi(z_a)}{1-\Phi(z_a)} = \frac{\phi(z_a)}{\Phi(-z_a)}$, $d(z_a) \equiv m(z_a)\left[m(z_a) - z_a\right]$ — eq 20.15

$$\mathbb{E}[Y] = \mathbb{E}[Y \mid \xi \le a]\,\mathbb{P}(\xi \le a) + \mathbb{E}[Y \mid \xi > a]\,\mathbb{P}(\xi > a) \quad \text{by LOTE}$$

$= a\,\Phi(z_a) + \mathbb{E}[Y \mid \xi > a]\,\mathbb{P}(\xi > a)$ by how Y is defined, and b/c Φ is cdf of standard normal

$= a\,\Phi(z_a) + \mathbb{E}[Y \mid \xi > a](1 - \Phi(z_a))$

$= a\,\Phi(z_a) + [\mu + \sigma m(z_a)](1 - \Phi(z_a))$ by eqn 20.14.

Let $I = I\{\xi \ge a\} \Rightarrow Y = a(1-I) + \xi I$

Note that $\mathbb{E}[I] = \mathbb{P}(\xi \ge a) = 1 - \Phi(z_a)$

and $V(I) = \mathbb{E}[I^2] - \mathbb{E}[I]^2 = \mathbb{E}[I] - \mathbb{E}[I]^2$

$= (\cancel{1} - \Phi(z_a)) - (\cancel{1} - 2\Phi(z_a) + \Phi(z_a)^2)$

$= \Phi(z_a) - \Phi(z_a)^2 = \Phi(z_a)(1 - \Phi(z_a)).$

Now,

$$V(Y) = \mathbb{E}[V(Y \mid I)] + V(\mathbb{E}[Y \mid I]).$$

$= \mathbb{E}[V(a(1-I) + \xi I \mid I)] + V(\mathbb{E}[a(1-I) + \xi I \mid I])$

$= \mathbb{E}[I^2 V(\xi)] + V(a(1-I) + I\,\mathbb{E}[\xi \mid I])$

$= \mathbb{E}(I^2)V(\xi) + a^2 V(I) + V(I)\,\mathbb{E}[\xi \mid I]^2 - 2a\,V(I)\,\mathbb{E}[\xi \mid I]_2$

$= \mathbb{E}(I^2)\sigma^2(1 - d(z_a)) + \Phi(z_a)(1 - \Phi(z_a))\left[a^2 - 2a\,\mathbb{E}[\xi \mid I] + \mathbb{E}[\xi \mid I]^2\right)$

$= \mathbb{E}(I^2)\sigma^2(1 - d(z_a)) + \Phi(z_a)(1 - \Phi(z_a))\left[a - \mathbb{E}[\xi \mid I]\right]^2$

$= \mathbb{E}(I^2)\sigma^2(1 - d(z_a)) + \Phi(z_a)(1 - \Phi(z_a))\left[a - (\mu + \sigma m(z_a))\right]^2$

$= (1 - \Phi(z_a))\sigma^2(1 - d(z_a)) + \Phi(z_a)(1 - \Phi(z_a))\left[a - (\mu + \sigma m(z_a))\right]^2$

$= (1 - \Phi(z_a))\sigma^2(1 - d(z_a)) + \Phi(z_a)(1 - \Phi(z_a))\left[\sigma(z_a - m(z_a))\right]^2$

$= \sigma^2(1 - \Phi(z_a))\left[1 - d(z_a) + \Phi(z_a)(z_a - m(z_a))^2\right]$ ☒

20.13 Pf: $V(\nu_i) = V(Y_i \mid \zeta_i > 0) + V(\alpha + \beta_1 X_{i1} + \dots + \beta_k X_{ik}) + V(\beta_\lambda \lambda_i)$

$$= V(\varepsilon_i \mid \zeta_i > 0) + V(\beta_\lambda \lambda_i)$$

$$= V(\varepsilon_i \mid \zeta_i > 0) + V(\sigma_\varepsilon \rho_{\varepsilon\delta}\, m(-\psi_i))$$

$$= V(\varepsilon_i \mid \zeta_i > 0)$$

$$= \mathbb{E}[\mathrm{Var}(\varepsilon_i \mid \delta_i) \mid \zeta_i > 0] + V\{\mathbb{E}[\varepsilon_i \mid \delta_i, \zeta_i > 0]\}$$

$$= \sigma_\varepsilon^2 (1 - \rho_{\varepsilon\delta}^2) + \rho_{\varepsilon\delta}^2 \frac{\sigma_\varepsilon^2}{\sigma_\delta^2} \mathrm{Var}(\delta_i + \lambda_i \sigma_\delta)$$

$$= \sigma_\varepsilon^2 (1 - \rho_{\varepsilon\delta}^2) + \rho_{\varepsilon\delta}^2 \frac{\sigma_\varepsilon^2}{\sigma_\delta^2} \cdot \sigma_\delta^2 (1 - \lambda_i(\lambda_i + \psi_i))$$

$$= \sigma_\varepsilon^2 (1 - \rho_{\varepsilon\delta}^2 \lambda_i (\lambda_i + \psi_i)). \quad \boxtimes$$

21.4 (see code attached at the end)

22.2

Pf: By defn of the F statistic,

$$F_j = \frac{(RSS_j - RSS_{Full})/(k+1-s_j)}{RSS_{Full}/(n-k-1)}$$

$$= \frac{(RSS_j - RSS_{Full})/(k+1-s_j)}{S_E^2}$$

$$\Rightarrow F_j S_E^2 (k+1-s_j) + RSS_{Full} = RSS_j$$

$$\Rightarrow C_{p_j} = \frac{F_j S_E^2 (k+1-s_j) + RSS_{Full}}{S_E^2} + 2s_j - n$$

$$= F_j(k+1-s_j) + \frac{RSS_{Full}}{S_E^2} + 2s_j - n$$

$$= F_j(k+1-s_j) + \cancel{n} - k - 1 + 2s_j - \cancel{n}$$

$$= (F_j - 1)(k+1-s_j) + s_j \quad \boxtimes$$