Problems

1. Tim is getting married and needs your help planning his wedding!

a) Tim wants to send out 3 kinds of invitations depending on the recipient: one for friends, one for family, and one for coworkers. He has 5 different colors of paper stock he can pick from. How many different ways are there to choose which paper will be used for which kind of invitations (where repeats are allowed)?

Tim can make different invitations using same color papers. As a result, there are $5 \times 5 \times 5 = 125$ options that Tim can choose in terms of which paper will be used.

b) The same question as in part a, but now Tim doesn't want to have two different kinds of invitation on the same color paper.

Every invitation's color should be different. First, we can choose 3 colors out of 5 colors.($\binom{5}{3}$) Second, we have to choose how to allocate 3 colors chosen to 3 kinds of recipients.($\binom{5}{3} \times 3! = 60$)

c) There is going to be a ice cream bar with 4 flavors of ice cream and 8 different toppings. If a guest wants 2 scoops (either the same or different flavors) and 3 different toppings how many choices does she have?

First, the guest should repeating choosing 1 flavor out of 4 flavors twice because he / she wants 2 scoops. ($4 \times 4 = 16$) Second, the guest should choose three different toppings out of 8 toppings. ($\binom{8}{3}$). Thus, the answer for the question is $4 \times 4 \times \binom{8}{3} = 896$.

d) Guests will be seated at round tables. If there are 8 guests per table, how many different ways can they be arranged? Note that two seating arrangements that are the same but rotated should not be counted separately.

8 seats in the initial condition(no one is assigned to seats) are same because we do not count rotation. When we assign one person to one seat, then rest seats are going to be different. Thus, the answer is 7!.

e) Lastly, Tim wants everyone to line up to take a photo at the end of the night. If there are $n \geq 2$ people, how many different ways can they be arranged, given that Tim and his spouse want to stand next to one another in the photo?

Let's consider Tim and his spouse as an individual member. And there are $(n-1)!$ cases to line up. However, Tim and Spouse can switch their positions. Therefore, the final result is $(n-1)! \times 2$.

2. Sets review

a) Find the number of sets A that satisfy $\{4,5,6\} \subseteq A \subseteq \{1,2,3,4,5,6\}$.

4, 5, 6 should be included in the set A. In addition, 1, 2, 3 may or may not be included in the set A. Thus, there are $2^3 = 8$ sets A that satisfy the condition.

b) Use set notation to describe the set depicted on this Venn diagram.

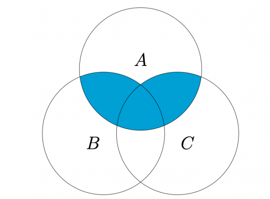

Figure 1: Venn diagram for problem b

The answer is $(B \cup C) \cap A = (A \cap B) \cup (A \cap C)$.

c) Give a Venn diagram for $A \setminus (B \cap C)$.

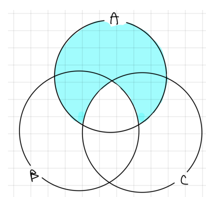

Figure 2: Venn diagram for problem c

3. Proof techniques review

a) Prove by induction

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}(\frac{n(n+1)}{2}) \tag{1}$$

We need to perform inductive reasoning separately for the cases when n is odd and when n is even.

First, we are going to cover when n is odd.

Base case: Prove that the equation is true when $n = 1$. The left hand side is $1^2 = 1$, while the right hand side is $(-1)^0 \times (\frac{1(1+1)}{2}) = 1$. Thus, the base case holds.

Inductions steps: Assume that (1) is true for $n = 2k - 1$. (k is an integer that satisfies $k \geq 1$.) Now, we show that it is true for $n = 2k + 1$.

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{2k-3}(2k-1)^2 = (-1)^{2k-2}\frac{(2k-1)2k}{2} = k(2k-1) \tag{2}$$

$$\begin{aligned} &1^2 - 2^2 + 3^2 - \cdots + (-1)^{2k-1}(2k)^2 + (-1)^{2k}(2k+1)^2 \\ &= k(2k-1) - 4k^2 + 4k^2 + 4k + 1 = 2k^2 + 3k + 1 = (k+1)(2k+1) \\ &= (-1)^{2k}(\frac{(2k+1)(2k+2)}{2}) \end{aligned} \tag{3}$$

Thus, It is proved that (1) is true when n is odd number. And next, we are going to prove (1) when n is even number.

Base case: Prove that the equation is true when $n = 2$. The left hand side is $1^2 - 2^2 = -3$, while the right hand side is $(-1)^1 \times (\frac{2(2+1)}{2}) = -3$. Thus, the base case holds.

Inductions steps: Assume that (1) is true for $n = 2k - 2$. (k is an integer that satisfies $k > 1$.) Now, we show that it is true for $n = 2k$.

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{2k-3}(2k-2)^2 = (-1)^{2k-3}\frac{(2k-2)(2k-1)}{2} = -(k-1)(2k-1) \tag{4}$$

$$\begin{aligned} &1^2 - 2^2 + 3^2 - \cdots + (-1)^{2k-2}(2k-1)^2 + (-1)^{2k-1}(2k)^2 \\ &= -(k-1)(2k-1) + (2k-1)^2 - (2k)^2 = -k(2k+1) \\ &= (-1)^{2k-1}(\frac{(2k)(2k+1)}{2}) \end{aligned} \tag{5}$$

Now, it is proved that (1) holds true regardless of whether n is odd or even.

b) Direct proof

According to the result from (a), when n is odd (n=2k+1),
$\left|1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2\right| = (k+1)(2k+1)$
It indicates that the LHS of the above equation should be composite. In addition, when n is even (n=2k),
$\left|1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2\right| = (k)(2k+1)$
It also indicates that the LHS of the above equation should be composite.

c) Proof by Contradiction

Assume that $x < 0$. Then, we can conclude $\sqrt{x^2} = -x$. And the result of the given equation is going to be 0. ($\sqrt{x^2} + x = 0$)
And when we assume that $x = 0$, it also indicates $\sqrt{x^2} + x = 0$.
Because of these reasons, the necessary condition of x for the $\sqrt{x^2} + x > 0$ is $x > 0$.

4. Sample Space

a) What is Pr({1,2})?

{1} and {2} are disjoint. Thus, Pr({1,2}) = Pr({1})+Pr({2})=1/2+1/3=5/6.

b) List all events A such that Pr(A) = 1/2.

Answer: {1}, {2,3}
The Probability of {1} is given. (Pr({1}) = 1/2) Additionally, the probability of {2,3} can be calculated by the same way covered in 4-(a). (Pr({2,3}) = Pr({2})+Pr({3})=1/3+1/6=1/2)

5. Probability and sets

a) Suppose that an employee arrives late 10% of the time, leaves early 20% of the time, and both arrives late and leaves early 5% of the time. What is the probability that on a given day the employee will arrive late or leave early (or both)?

Pr({arrive late}) = 1/10
Pr({leave early}) = 1/5
Pr({arrive late & leave early}) = 1/20
Pr({arrive late, leave early} = 1/10 + 1/5 - 1/20 = 1/4

b) Suppose your team has a 40% chance of winning or tying today's game and has a 30% chance of winning today's game. What is the probability that today's game will be a tie?

Winning and tying are disjoint.
Pr({winning, tying}) = 4/10
Pr({winning})= 3/10
Pr({tying}) = 1/10

6. Playing with cards

a) What is the probability that at least 10 cards go by before the first jack?

The probability that 0 card go by before the first jack: $\frac{4}{52} = a$
The probability that 1 card go by before the first jack: $\frac{48}{52} \times \frac{4}{51} = b$
The probability that 2 cards go by before the first jack: $\frac{48}{52} \times \frac{47}{51} \times \frac{4}{50} = c$
The probability that 3 cards go by before the first jack: $\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{4}{49} = d$
...
The probability that 9 cards go by before the first jack: $\frac{48 \times 47 \times 46 \times \cdots \times 40}{52 \times 51 \times\ 50 \times \cdots \times 44} \times \frac{4}{43} = j$
The probability that at least 10 cards go by before the first jack: $1 - a - b - c - \cdots - j$

b) What is the probability that at least 10 cards go by before the first jack, but every time you draw a card, you put it back in the deck and shuffle it again?

The probability that 0 card go by before the first jack: $\frac{4}{52} = a$
The probability that 1 card go by before the first jack: $\frac{48}{52} \times \frac{4}{52} = b$
The probability that 2 cards go by before the first jack: $\frac{48}{52} \times \frac{48}{52} \times \frac{4}{52} = c$
The probability that 3 cards go by before the first jack: $\frac{48}{52} \times \frac{48}{52} \times \frac{48}{52} \times \frac{4}{52} = d$
...

The probability that 9 cards go by before the first jack: $\frac{48\times48\times48\times\cdots\times48}{52\times52\times\ 52\times\cdots\times52} \times \frac{4}{52} = j$
The probability that at least 10 cards go by before the first jack: $1 - a - b - c - \cdots - j$

7. Tree diagram

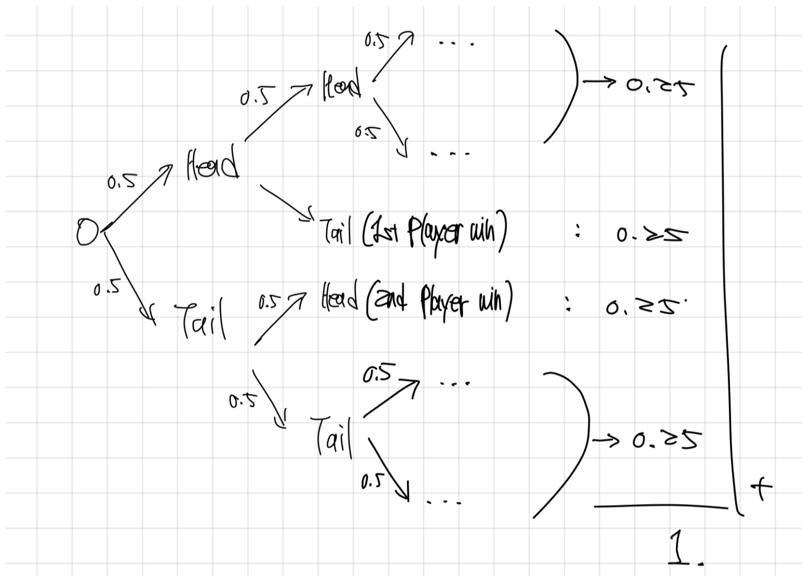

Figure 3: Tree diagram for problem 7

When coin is flipped twice, the probability that neither player wins is 0.5. (0.25+0.25)
When coin is flipped three times, the probability that neither player wins is same as when coin is flipped twice.
When coin is flipped 4 times, the probability that neither player wins is 0.25 $(1 - 0.5 - 0.5 \times 0.5)$
...
When coin is flipped n times, the probability that neither player wins is going to be 0.
(This is because $1-0.5-0.5\times0.5-0.5\times0.5\times0.5-\cdots-(0.5)^k = 1-\frac{0.5(1-(0.5)^k)}{1-0.5} = (0.5)^k$, $\lim_{n\to\infty}(0.5)^k = 0$)