# Problems

## 1. Probability and Set Operations

### a)

$|\Omega| = 6$, $B \cup C = \{1, 2, 4, 5, 6\}$, $A \cap (B \cup C) = \{1, 2\}$, $|A \cap (B \cup C)| = 2$
$Pr(A \cap (B \cup C)) = \frac{|A \cap (B \cup C|}{|\Omega|} = \frac{2}{6} = \frac{1}{3}$

### b)

$\bar{A} = \{3, 4, 5, 6\}, \bar{B} = \{2, 3, 4\}, \bar{A} \cap \bar{B} = 3, 4, \left|\bar{A} \cap \bar{B}\right| = 2$
$Pr(\bar{A} \cap \bar{B}) = \frac{\left|\bar{A} \cap \bar{B}\right|}{|\Omega|} = \frac{2}{6} = \frac{1}{3}$

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## 2. Tree diagram

### a)

If 1st song is from a band, the probability that the 2nd song is from a band is $2/3 \times 1/4 = 1/10$. In addition, if 1st song is not from a band, the probability that the 2nd song is from a band is $3/5 \times 2/4 = 3/10$. Thus, the total probability is $1/10 + 3/10 = 4/10 = 2/5$.

### b)

3/5(1st song is from a single artist)×2/5(2nd song is from a single artist) $= 6/25$

## 3. True or False

**Fill the bubbles next to the true statements.**

○ The sum of probabilities of all possible events in a probability space is always less than 1.
○ The complement of an event in a probability space is the event that consists of all outcomes that are in the original event.
● The sample space in a probability space represents the set of all possible outcomes of a random experiment.
○ The union of two events in a probability space is the event that consists of all outcomes that belong to both events.
○ If two events are disjoints in a probability space, their intersection is always equal to the union of the two events.
○ If two events are disjoint in a probability space, the probability of their intersection is always equal to the sum of their individual probabilities.

## 4. PDF and CDF practice

### a)

$\int_1^\infty k(e^{-2x} + \frac{1}{x^2})dx = \left[k((-\frac{1}{2}e^{-2x}) - \frac{1}{3}x^{-3})\right]_1^\infty = k(-(-\frac{1}{2}e^{-2} - \frac{1}{3})) = 1$
$k = \frac{1}{\frac{1}{2}e^{-2}+\frac{1}{3}}, k = \frac{6}{3e^{-2}+2}$

### b)

$$F_X(a) = \begin{cases} 0 & \text{for } a < 1 \\ \int_1^a k(e^{-2x} + \frac{1}{x^2})dx & \text{for } a \geq 1 \end{cases}$$

### c)

We can find the PDF by taking the derivative.
$$f_Y(y) = \begin{cases} \frac{2}{9}y & \text{for } 0 \leq y \leq 3 \\ 0 & \text{for } otherwise \end{cases}$$

### d)

$F_Z(z) = Pr(Z \leq z) = Pr(Y^{1/2} \leq z) = Pr(Y \leq z^2)$
By taking $y = z^2$, we can find the CDF of Z.
$$F_Z(z) = \begin{cases} 0 & \text{for } z \in (-\infty, 0) \\ \frac{1}{9}z^4 & \text{for } z \in \left[0, \sqrt{3}\right] \\ 1 & \text{for } otherwise \end{cases}$$

## 5. Game of Dice

### a)

$\Omega = \{(G, W) \mid G \in \{1, 2, 9, 10\}, W \in \{3, 4, 5, 12\}\}$

| (G,W) | C(1,3) | C(1,4) | C(1,5) | C(1,12) | (2,3) | (2,4) | (2,5) | (2,12) | (9,3) | (9,4) | (9,5) | (9,12) | C(9,3) | C(10,4) | C(10,5) | C(10,12) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr((G,W)) | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 |
| winner | W | W | W | W | W | W | W | W | G | G | G | W | G | G | G | W |

Figure 1: Probability function for problem a

$$Pr(G = g, W = w) = \begin{cases} Pr(g > w) = Pr(\{(9,3), (9,4), (9,5), (10,3), (10,4), (10,5)\}) = 6/16 = 3/8 \\ Pr(w > g) = 10/16 = 5/8 \end{cases}$$

### b)

As we can in the solution 5-(a), the answer is 3/8.

### c)

$Pr(g > w \mid g \in \{1, 9\}) = \frac{Pr(g>w \cap g\in\{1,9\})}{Pr(g\in\{1,9\})} = \frac{Pr(g=9>w)}{Pr(g\in\{1,9\})} = \frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{8}$
(When g=1, g cannot satisfy g¿w.)

### d)

$Pr(g > w \mid g \in \{2, 10\} \; and \; w \in 3, 4, 5) = \frac{Pr(g=10>w\in\{3,4,5\})}{Pr(g\in\{2,10\}\cap w\in\{3,4,5\})} = \frac{\frac{1}{4}\times\frac{3}{4}}{\frac{2}{4}\times\frac{3}{4}} = \frac{1}{2}$

**6. Probabilistic Monty**

**If the producers want a player who swaps only to win half the time given that the constant had the opportunity to swap, what should x be?**

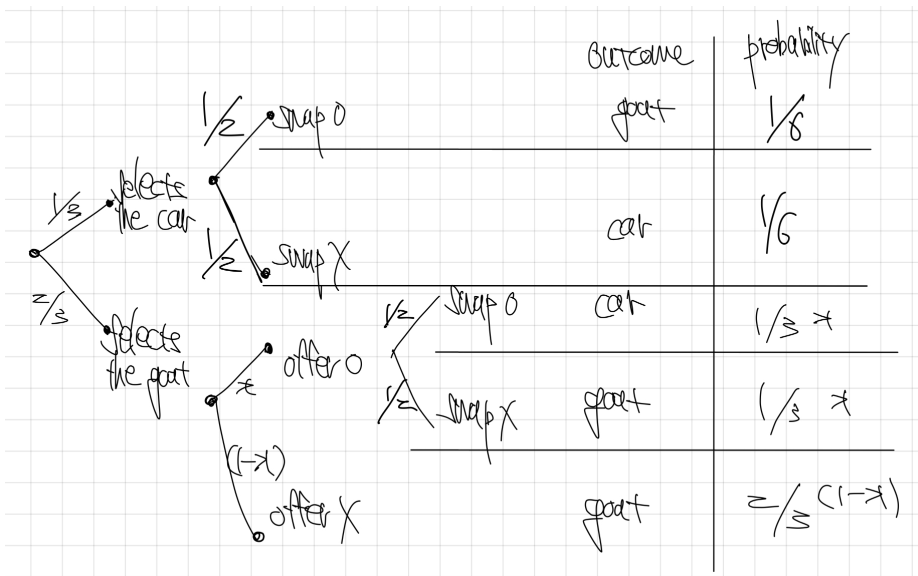


Figure 2: Tree diagram for problem 6

When the player get a opportunity to swap, if he selects the door with a car, he can win the game without the swap. Additionally, if he selects the door with a goat, he can win the game when selects to swap. Since the probability that he win the game without a swap is 1/6, 1/3x which is the probability that he win the game using the swap should be equivalent to 1/6. Thus, the answer is 1/2.

## 7. Continuous Probability

### a)

Tim can reuse a broken string if the break is within 0.5 dm of the saddle.
A = "Tim can reuse a broken string"
$Pr(A) = \frac{0.5}{18} = \frac{1}{36}$

### b)

B = "Two strings are broken and Tim must buy only one new string"
B means two events that one string is broken within 0.5 dm of the saddle and another string is not.
We can calculate the probability of B by calculating the probability of intersection of the two events.
$Pr(B) = \frac{1}{36} \times \frac{35}{36} = \frac{35}{1296}$

### c)

C = "Tim can reuse a string", D = "Tim cannot reuse a string"
Pr(X=0) = Pr(C) = 1/36, Pr(X=1) = Pr(D) = 35/36

$$F_X(a) = \begin{cases} 0 & \text{for } a < 0 \\ \frac{1}{36} & \text{for } 0 \leq a < 1 \\ 1(= \frac{1}{36} + \frac{35}{36}) & \text{for } otherwise \end{cases}$$

## 8. Geometric method

**What is the probability that a ball will drop right through the sound hold instead of hitting the rosette and bouncing away?**

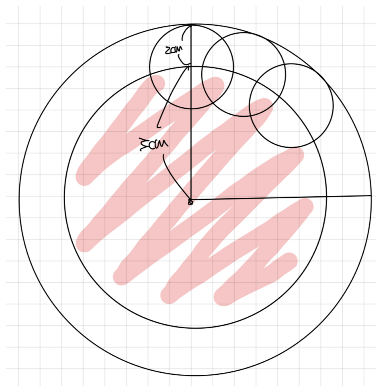


Figure 3: figure for problem 8

A = "ping pong ball drops right through the sound hole"
Except for the pink colored area, the ball cannot pass through the hall completely. This is because the ball collides with the edge of the hall.
$Pr(A) = \frac{4\pi \times 3^2}{4\pi \times 5^2} = \frac{9}{25}$