Problems

1. Dice

a)

$R_X = \{1, 2, 3, 4, 5, 6\}$
$R_Y = \{1, 2, 3, 4, 5, 6\}$

$$Pr(X = x) = \begin{cases} 1/6 & \text{if } x = 1 \\ 1/6 & \text{if } x = 2 \\ 1/6 & \text{if } x = 3 \\ 1/6 & \text{if } x = 4 \\ 1/6 & \text{if } x = 5 \\ 1/6 & \text{if } x = 6 \end{cases}$$

$$Pr(Y = y) = \begin{cases} 1/6 & \text{if } y = 1 \\ 1/6 & \text{if } y = 2 \\ 1/6 & \text{if } y = 3 \\ 1/6 & \text{if } y = 4 \\ 1/6 & \text{if } y = 5 \\ 1/6 & \text{if } y = 6 \end{cases}$$

b)

$Pr(X = 2, Y = 5) = Pr(X = 2 \; and \; Y = 5) = Pr(X = 2 \, \cap \, Y = 5) = Pr(X = 2) \cdot Pr(Y = 5) = 1/6 \cdot 1/6 = 1/36$
We can convert $Pr(X = 2 \, \cap \, Y = 5)$ to $Pr(X = 2) \cdot Pr(Y = 5)$ using product rule since X and Y are independent.

c)

$Pr(X > 3 \mid Y < 4) = Pr(X > 3) = 1/6 \;\; (\because definition \;\; of \;\; independent)$

d)

$R_Z = \{2,3,4,5,6,7,8,9,10,11,12\}$

$$Pr(Z = z) = \begin{cases} 1/36 & \text{if } x = 2 \ (\because (x,y) \in \{(1,1)\}) \\ 2/36 & \text{if } x = 3 \ (\because (x,y) \in \{(1,2),(2,1)\}) \\ 3/36 & \text{if } x = 4 \ (\because (x,y) \in \{(1,3),(2,2),(3,1)\}) \\ 4/36 & \text{if } x = 5 \ (\because (x,y) \in \{(1,4),(2,3),(3,2),(4,1)\}) \\ 5/36 & \text{if } x = 6 \ (\because (x,y) \in \{(1,5),(2,4),(3,3),(4,2),(5,1)\}) \\ 6/36 & \text{if } x = 7 \ (\because (x,y) \in \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) \\ 5/36 & \text{if } x = 8 \ (\because (x,y) \in \{(2,6),(3,5),(4,4),(5,3),(6,2)\}) \\ 4/36 & \text{if } x = 9 \ (\because (x,y) \in \{(3,6),(4,5),(5,4),(6,3)\}) \\ 3/36 & \text{if } x = 10 \ (\because (x,y) \in \{(4,6),(5,5),(6,4)\}) \\ 2/36 & \text{if } x = 11 \ (\because (x,y) \in \{(5,6),(6,5)\}) \\ 1/36 & \text{if } x = 12 \ (\because (x,y) \in \{(6,6)\}) \end{cases}$$

e)

$Pr(X = 4 \mid Z = 6) = \frac{Pr(X=4 \cap Z=6)}{Pr(Z=6)}(\because \textit{definition of conditional probability}) = \frac{Pr(X=4 \cap Y=2)}{Pr(Z=6)}(\because Z = X + Y) = \frac{Pr(X=4) \cdot Pr(Y=2)}{Pr(Z=6)}(\because \textit{product rule}) = \frac{1/6 \cdot 1/6}{5/36} = 1/5$

2. Independent Random Variables

$$Pr(X = x) = \begin{cases} 1/3 & \text{if } x = 1 \\ 2/3 & otherwise \end{cases}$$
$$Pr(Y = y) = \begin{cases} 1/4 & \text{if } y = -3 \\ a/4 & \text{if } y = 2 \end{cases}$$

a)

$a = 3$ ($\because$ *Probability Axiom, Normalization*)

b)

$E(3X + 2Y) = 3E(X) + 2E(Y)(\because linearity\ of\ expectation)$
$= 3(1 \cdot 1/3) + 2((-3) \cdot 1/4 + 3/4 \cdot 2)(\because definition\ of\ expectation)$
$= 1 + 2 \cdot 5/4 = 1 + 5/2 = 7/2$

c)

$E(X^2) = \sum_{x \in R} x^2 P_X(x) = 1 \cdot P_X(1) = 1 \cdot 1/3 = 1/3(\because Law\ of\ The\ Unconscious\ Statistician)$
$E(Y^2) = \sum_{y \in R} y^2 P_Y(y) = 3^2 \cdot 1/4 + 2^2 \cdot 3/4 = 9/4 + 12/4 = 21/4(\because Law\ of\ The\ Unconscious\ Statistician)$
$E(2X^2 - Y^2) = 2E(X^2) - E(Y^2)(\because linearity\ expectation) = 1/3 - 21/4 = -61/12$

d)

$E(XY) = E(X) \cdot E(Y)(\because Product\ of\ independent\ RVs\ since\ they\ are\ independent.) = (1 \cdot 1/3) \cdot ((-3) \cdot 1/4 + 2 \cdot 3/4) = 1/3 \cdot 3/4 = 1/4$

e)

$E(X^2Y) = E(X^2) \cdot E(Y)(\because Product\ of\ independent\ RVs\ since\ they\ are\ independent.) = 1/3 \cdot 3/4 = 1/4$

3. More Socks

a)

$1(\because for\ the\ 1st\ draw,\ any\ color\ is\ okay.) \times 1/19(\because it\ should\ be\ matched\ with\ the\ 1st\ one.) = 1/19$

b)

X="number of distinct colored socks you select"

$$Pr(X=x) = \begin{cases} \frac{\binom{10}{2}\cdot\binom{8}{1}\cdot 2}{\binom{20}{5}} & \text{if } x=3 \\ \frac{\binom{10}{1}\cdot\binom{9}{3}\cdot 2^3}{\binom{20}{5}} & \text{if } x=4 \\ \frac{\binom{10}{5}\cdot 2^5}{\binom{20}{5}} & \text{if } x=5 \end{cases}$$

$E(X) = \sum_{x\in R} x \cdot P_X(x) = 720/15504 \cdot 3 + 6720/15504 \cdot 4 + 8064/15504 \cdot 5 = 4.47\cdots$

c)

Y="number of matching pairs if you randomly select 10 pairs, one pair at a time" $E(Y) = E(Y \mid Y=0) \cdot Pr(Y=0) + E(Y \mid Y \geq 1) \cdot Pr(Y \geq 1)(\because Law\ of\ Total\ Expectation) = 0 \cdot (1-1/19)^{10}(\because answer\ of\ (a)) + (E(Y)+1)(\because linearity\ of\ expectation) \cdot (1-(1-1/19)^{10}) = 0 \cdot (0.58\cdots) + (E(Y)+1)(0.42\cdots) = 0.42 \cdot E(Y) + 0.42$

$0.58 \cdot E(Y) = 0.42,\ E(Y) = 42/58 = 0.72\cdots$

d)

When the machine eats four socks, two pairs which have matching colors, the probability is $\frac{\binom{10}{2}}{\binom{20}{4}} = 45/4845$.

When the machine eats four socks - two socks are a pair which have the same color and other two socks have different colors, the probability is $\frac{\binom{10}{3}\cdot 2\cdot 2}{\binom{20}{4}} = 480/4845$.

When the machine eats four socks - all four socks have different colors, the probability is $\frac{\binom{10}{4}\cdot 2\cdot 2\cdot 2\cdot 2}{\binom{20}{4}} = 3360/4845$.

Z="the number of incomplete pairs "

$E(Z) = 45/4845 \cdot 2 + 480/4845 \cdot 3 + 3360/4845 \cdot 4 = 3.08\cdots$

Z'="the number of complete pairs that make it out alive"

$E(Z') = 10 - E(Z) = 6.92\cdots$

4. Boston Weather

a)

$Pr(rain-Sunday) = Pr(rain-Sunday \mid rain-Saturday) + Pr(rain-Sunday \mid not\ rain-Saturday) (\because Law\ of\ Total\ Probability) = 0.6 + 0.3 = 0.9$

b)

$Pr(Sunny - Saturday \mid rain - Sunday) = Pr(not\ rain - Saturday \mid rain - Sunday)$
$= \frac{Pr(rain-Sunday|not\ rain-Saturday) \cdot Pr(not\ rain-Saturday)}{Pr(rain-Sunday)} (\because Bayes's\ rule) = (0.3 \cdot 0.6)/0.9 = 0.2$

c)

X="amount of rain on Saturday"
$E(X) = E(X \mid rain - Saturday) \cdot Pr(rain - Saturday) + E(X \mid not\ rain - Saturday) \cdot Pr(not\ rain - Saturday)(\because Law\ of\ Total\ Expectation) = 2 \cdot 0.4 + 0 \cdot 0.6 = 0.8$

d)

Y="amount of rain on Sunday"
$E(Y) = E(Y \mid rain-Sunday) \cdot Pr(rain-Sunday) + E(Y \mid not\ rain-Sunday) \cdot Pr(not\ rain-Sunday)(\because Law\ of\ Total\ Expectation) = 3 \cdot 0.9 + 0 \cdot 0.1 = 2.7$

e)

$E(X+Y) = E(X) + E(Y)(regardless\ of\ indepedence) = 0.8 + 2.7 = 3.5$

5. Bland cookies

a)

X="number of cookies one person can get"
$E(X) = E(X = 0) \cdot Pr(X = 0) + E(X \mid X \geq 1) \cdot Pr(X \geq 1)(\because \textit{Law of Total Expectation}) = 0 \cdot 2/3 + (E(X) + 1) \cdot 1/3$
$E(X) \cdot 1/3 + 1/3 = E(X),\ E(X) = 1/2$

I have considered an additional solution for the problem (a).
$E(X) = \binom{100}{1} \cdot 1/300 + \binom{100}{2} \cdot 1/300^2 \cdot 2 + \cdots + \binom{100}{100} \cdot 1/300^{100} \cdot 100$
$= \sum_{n=1}^{100} (\frac{\binom{100}{n}}{300^n} \cdot n)$

b)

One person can get 1/2 cookie which means there are 200 expected people who get at least one cookie. Thus, the expected number of students who do not receive any cookies is 300-200=100.

c)

Y="the number of students who receive at least two cookies"
$E(Y \mid X \geq 2) = E(Y = 300) - E(Y \mid X = 0) - E(Y \mid X = 1)(\because \textit{Law of Total Expectation}) = 300 - 100 - E(Y \mid X = 1)$
Sorry, but I don't know how to solve the problem from here.