Problems

1. Baking Cookies

1)

X_i is an indicator random variable which indicates the Tiago's ruined cookie.
$X_i = \begin{cases} 1 & \text{with probability=1/2} \\ 0 & \text{with probability=1/2} \end{cases}$
$E(X) = E(X_1{+}X_2{+}X_3{+}X_4{+}X_5) = E(X_1){+}E(X_2){+}E(X_3){+}E(X_4){+}E(X_5)(\because linearity\ of\ expectation) = 5E(X_1)(\because symmetry) = \frac{5}{2}$
X: Binomial(5,1/2)
$Var(X) = 5 \cdot 1/2 \cdot (1 - 1/2) = 5/4$
Y_i is an indicator random variable which indicates the Tim's ruined cookie.
Y: Binomial(10,1/5)
$Var(Y) = 10 \cdot 1/5 \cdot (1 - 1/5) = 40/25$
$Var(2X + 2Y) = 4Var(X) + 4Var(Y) = 5 + 160/25 = 285/25$
$\sigma(2X + 2Y) = \sqrt{Var(2X + 2Y)} = \sqrt{285/25}$

2)

$Pr(Y > X \mid X + Y = 2) = Pr(X = 0, Y = 2 \mid X + Y = 2)$($\because$ the only case satisfies theses conditions)
$= \frac{Pr((X=0,Y=2)\cap X+Y=2)}{Pr(X+Y=2)}$($\because$ definition of conditional probability)
$= \frac{Pr(X=0,Y=2)}{Pr(X=0,Y=2)+Pr(X=1,Y=1)+Pr(X=2,Y=0)}$
$= \frac{(1/2)^2\cdot(1/5)^2}{(1/2)^2\cdot(1/5)^2+(1/2)^2\cdot(1/5)\cdot(4/5)+(1/2)^2\cdot(4/5)^2} = 1/21$

3)

$E(X{+}Y) = E(X{+}Y \mid X{+}Y = 0){+}E(X{+}Y \mid X{+}Y = 1){+}E(X{+}Y \mid X{+}Y \geq 2)$($\because$ law of total expectation)
X: Binomial(5,1/2)
$E(X) = 5 \cdot 1/2 = 5/2$
Y: Binomial(10,1/5)
$E(Y) = 10 \cdot 1/5 = 2$
$E(X + Y) = 5/2 + 2 = 9/2$($\because$ linearity of expectation)
$E(X + Y \mid X + Y = 0) = 0$
$E(X + Y \mid X + Y = 1) = 1$
$E(X + Y \mid X + Y \geq 2) = 9/2 - 0 - 1 = 7/2$

2. Recognizing Distributions

1)

Geometric distribution(1/20) is involved. (The parameter p is 1/20 since there are 20 flavors.)
Because of the memory less property, past trials would not affect the expected number of cookies.
X: "The expected number of cookies I have to eat until I get Smoked Popcorn Cookie"
$E(X) = 1/p = \frac{1}{1/20} = 20$

2)

Y: "The expected amount of cookies I have to eat in order to taste all remaining flawors"
Negative binomial distribution(15,1/20) is involved.(The parameter r is 15 since there are 15 remaining flavors. The parameter p is 1/20 since there are 20 total flavors.)
$E(Y) = r \cdot 1/p = 20 \cdot 20 = 400$

3)

Geometric distribution(1/20*1/20=1/400) is involved.(The parameter p is 1/400 since I have to pick consecutive same flavor and there are 20 flavors.)
Z: "the expected number of cookies requested until I eat two consecutive Strawberry Cookies"
$E(Z) = 1/p = \frac{1}{1/400} = 400$

4)

Negative binomial distribution(2,1/20) is involved.(The parameter r is 2 since I wanna eat 2 flavor. The parameter p is 1/20 since there are 20 flavors.)
K: "The expected number of cookies requested until I eat a Chocolate Cookie and a Gingerbread Cookie"
$E(K) = r \cdot 1/p = 2 \cdot 20 = 40$

3. Allergic Reaction

1)

Markov's inequality: $Pr(X \geq a) \leq \frac{E(X)}{a}$
X has a binomial distribution(10,0.4). (The parameter n is 10 since there are 10 cookies which means trials. Tha parameter p is 0.4 because of the problem's condition.)
$E(X) = n \cdot p = 10 \cdot 0.4 = 4$
$Pr(X \geq 6) \leq 4/6 = 2/3$ (Because of the formula of Markov's inequality)

2)

$Pr(X \geq 6) = Pr((X-4) \geq (6-4)) \leq \frac{Var(X)}{(6-4)^2}$
$Var(X) = n \cdot p \cdot (1-p)(\because$ binomial distribution$) = 10 \cdot 0.4 \cdot 0.6 = 10 \cdot (0.24) = 2.4$
$Pr(X \geq 6) \leq \frac{Var(X)}{4} = 2.4/4 = 0.6$

3)

$Pr(X \geq 8) = Pr(X = 8) + Pr(X = 9) + Pr(X = 10)$
$= \binom{10}{8} \cdot (0.4)^8 \cdot (0.6)^2 + \binom{10}{9} \cdot (0.4)^9 \cdot (0.6) + \binom{10}{10}(0.4)^{10} \cdot (0.6)^0(\because$ binomial distribution$)$

4. Accept Cookies?

There is a consistent demand for cookies, averaging three requests every "10 minutes" from the moment the exam commences.
There are 0.3 requests every 1 minute on average which means $\lambda = 0.3 request/1 minute$.

1)

T: "Time from the start of the exam until the first cookie is requested"
T has an exponential distribution($\lambda = 0.3$).
$Pr(T \geq 5)) = e^{-0.3 \cdot 5} = e^{-1.5}$

2)

X: "the number of cookies requested during the first time interval of the exam"
X is a kind of Poisson random variable, so we can use the equation of Poisson RV.
$Pr(X \geq 10) = 1 - Pr(X \leq 9)(\because \text{complement rule}) = 1 - \sum_{k=0}^{9} \frac{(\frac{3}{10})^k \cdot e^{-\frac{3}{10}}}{k!}$

3)

Y: "the number of cookies requested during a 2 hours exam"
Y_i: "The number of requested cookies during the i-th 10 minutes out of 120 minutes"
Y_i is a kind of Poisson random variable, so we can use the equation of Poisson RV.
$E(Y) = \sum_{i=1}^{12} E(Y_i) = \sum_{i=1}^{12} 3 = 36$