Problem 1 Error estimate in stochastic simulations

Let $f : (a, b) \to \mathbb{R}$ be a real function defined in the open interval (a, b). The mean value of f is defined as

$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x) dx \tag{1}$$

We can also define the variance, $\sigma^2 = \langle f^2 \rangle - \langle f \rangle^2$, where

$$\langle f^2 \rangle = \frac{1}{b-a} \int_a^b f(x)^2 dx \tag{2}$$

Let's take a = 0 and b = 1, and let $x_i, i = 1 \ldots N$, be a random variable drawn from a uniform distribution in the interval (0, 1). We can define a random variable S as the sum

$$S = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \tag{3}$$

Consider the function

$$f(x) = \frac{1}{(x - x^2)^p}, \quad 0 < p < 1 \tag{4}$$

Despite the singularities at $x = 0$ and $x = 1$, the expectation value is well defined:

$$\langle f \rangle = \frac{\Gamma(1-p)^2}{\Gamma(2-2p)}, \tag{5}$$

where Γ is the gamma function. Note that, for large N, the law of large numbers dictates that most values of S should fall close to the analytical result given by Equation 5.

a) Set $p = 0.1$ and calculate S for a fixed N several times and plot a histogram of the results. Does the distribution look gaussian? Try different values of N. Does the width of the distribution change like $\frac{1}{\sqrt{N}}$?
b) Repeat the analysis and answer the questions above for $p = 0.9$.
c) Why is the behaviour different for $p < 0.5$ and $p > 0.5$? (Hint: Consider the integral in equation 2. Close to $x = 0$ the integrand behaves as $f(x)^2 \sim \frac{1}{x^{2p}}$. The contribution to the integral coming from the interval (x$(x_0, x_1), \quad 0 < x_0 < x_1 \ll 1, x_1)$, can thus be approximated by

$$\int_{x_0}^{x_1} f(x)^2 dx \approx \int_{x_0}^{x_1} \frac{1}{x^2} dx \tag{6}$$

Calculate this integral and let $x_0 \to 0$. How is the answer different for $p < 0.5$ and $p > 0.5$? If the central limit theorem holds, what should be the standard deviation of S (the error estimate) for $p \geq 0.5$?)