theories try to answer questions regarding allocations of financial resources (corporate financial decisions)

Finance Theory Midterm

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I Investment decisions under certainty

1

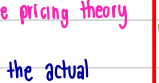
- 2 types of financial theories ↗ including: demand theory, CAPM, arbitrage pricing theory

$Q_D = f(P^-)$

① Positive theory: try to explain relation among economic variable
↳ To verify this, we test whether its hypotheses are consistent with the actual behavior in the market (empirical test) → use actual behavior in mkt & test whether it's consistent with the hypothesis

② Normative theory: try to give recommendation (involving maximization) on how thing should be done ↳ including stochastic dominance, MV criterion, SIM

- Corporation: clear separate of ownership and control + limited liability (shareholder, management)



- Common features of any investment: time and risk

2 Economic theory of choice (utility theory)

how individuals allocate limited resources btw C_0 & C_1

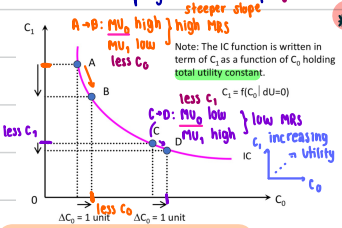
- Investment (PV⊖principal, FV⊕principal & return): inv can only take place if there's saving. (inv defer current consumption to future consumption; saving (income > consumption))
- Assume two time period (t_0 & t_1) and perfect certainty (know outcome of investment for sure)
- 3 main elements:
  1) Object of choice: level of current (C_0) & future consumption (C_1)
  2) Opportunity set: All possible combination of C_0 & C_1 constrained by subject's budget or income
  3) Tastes or preferences of subject towards C_0 & C_1

basket of consumption ($C_0 + C_1 = 100$) → ($60, $40), ($50, $50), ... ($C_0$, C_1) Y_0 & Y_1

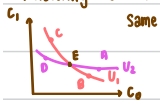
* Axiom of choice: utility theory's assumptions regarding individuals' behavior
  1) Comparability / Completeness: subject has ability to rank his preference / identify >, <, ~
  2) Transitivity / consistency: subject is consistent in his preference rankings: if $x > y$ and $y > z$, then $x > z$
  3) Non-satiation of wants: subject always prefers more to less: if $x = (10, 5)$ and $y = (10, 4)$, then $y \succ x$
  4) Decreasing marginal utility: as more consumption takes place, the subject's satisfaction increases at a decreasing rate. ↳ Diminishing marginal utility (MU) ↗ additional unit of consumption

* Total utility: level of satisfaction obtained from consumption; if utility (x) > utility (y), $x \succ y$ | $U = (C_0 \times C_1)^{0.5}$, Cobb-Douglas utility function
  (C_0, C_1) → utility function $U(C_0, C_1)$ → utility index (used for preference ranking)
  - Cardinal utility: magnitude of diff in utility index is meaniful • Ordinal utility: rank matter
  - Total utility curve: upward sloping (non-satiation) & concave shape (diminishing MU), slope $MU_0 = \frac{\partial U}{\partial C_0} = U'(C_0)$

* Indifference curves: are constructed from total utility curve (C_0-U and C_1-U)
  ↳ each basket on the curve has same utility and IC of the same subject are parallel to each other
  (C_0, C_1) ↳ if you increase C_0, you have to sacrify C_1 to have same utility
  ↳ downward sloping & convex slope } diminishing MU: as we increase C_0, we're willing to less sacrify C_1

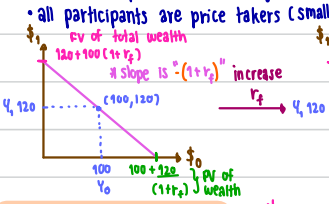


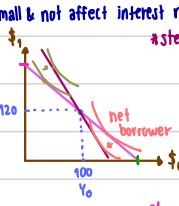
* Marginal rate of substitution (MRS) = $-\frac{\partial C_1}{\partial C_0}\Big|_{dU=0}$ = Slope of IC
  ↳ unit of C_1 that must be given up per unit of C_0 added to maintain a constant level of utility, violate axiom ②
  ! Intersecting IC's of a subject implies inconsistent preferences



Same IC: C~E, D~E } C~D → but C≻D as it has higher C_0 & C_1 → inconsistent!!

* Consumption opportunity set: all possible baskets of consumption (restricted by subject's wealth or budget)
  ↳ Assumptions: current (Y_0) and future (Y_1) incomes are known, existing of capital market.
  ↳ !! Perfect capital market: • equal & costless access to info (no info assymmetry) (frictionless)
  • all participants are price takers (small & not affect interest rate), no transaction fee & taxes

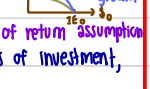




* steeper slope as r_f↑
  ↳ bad for net borrower → utility ↓ (prefers C_0)
  ↳ good for net lender (prefer C_1) utility↑
* Slope of budget line will be the same across individuals (diff in location of line)

"Capital market opportunity line"
Slope = $\frac{\partial W_1}{\partial W_0} = -(1+r)$

* Optimal consumption choice: 1st find budget line Y_0, Y_1 2nd choose basket where IC tangents budget line
  - From non-satiation of wants, subject will only consider baskets of consumption that lie on budget line only. → baskets above line are unattainable & under line are feasible but inefficient.
  ↳ Optimal consumption is where MRS = (1+r) or IC is tangent to budget line.



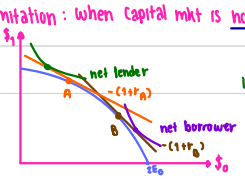
3 Optimal investment - consumption decision:

* Production possibility curve (PPC): possible levels of outputs from different levels of investment, holding available resources and technology constant → assume diminishing MRT (concave shape reflects diminishing marginal rate of return assumption)
  Marginal rate of transformation (MRT): slope of PPC btw current resources & future products (now→now)
  MRT = $\frac{IE(1+IRR)}{IE}$ = (1+IRR) || invest in project that give highest IRR first (descending order)

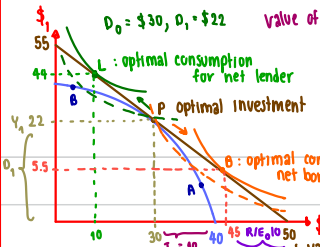
* Optimal investment - consumption decision without capital mkt: MRS = MRT = $(1+IRR_i)$
  ↳ If $C_0 < IE_0$ (initial endowment) → can use excess resource to invest in real production → increase C_1
  $IE_0 - C_0$ = investment I_0 → invest until MRS = MRT (1+IRR)
  ↳ transform to future income
  • If MRS < MRT: return on investment > subject's demand ↳ invest more to keep climbing up PPC

* Optimal investment - consumption decision with capital mkt
  ↳ With capital market, individual can choose whether to invest his resources in real production vs. capital market (lending). ↗ invest until MRT = (1+r)
  1st invest first to maximize his wealth (optimal investment decision) → allows him to maximize the consumption opportunity set
  2nd bring in consumption preference (IC) → chooses where MRS = (1+r) ⇒ optimal consumption decision
  • MRS > 1+r → borrow to increase utility ↗ can make use of capital market

* The case of a corporation: ↗ no info about owner's consumption preference is required
  ↳ According to Fisher Separation Theorem, investment decisions can be made separately and independently from consumption decisions. Thus, to maximized shareholders' utility, the managers should maximize firm's value (W_0) by investing in all projects whose IRR ≥ r. (market value criterion)
  ↳ Limitation: when capital mkt is not perfect, Fisher's Separation Theorem breaks down. ↗ transaction cost, info asymmetry & agency problem



A: deposit, invest more < B: borrowing (steeper), invest less & ask for more dividend
↳ They choose different investment decision



Value of firm is maximized

* Optimal investment is to invest $I_0 = \$10$, $1+IRR_i$
  ↳ keep investing in project until MRT = $1 + r_f$
* value added to shareholder = 50 - 40 = $10 → invest in all IRR ≥ r

: Firm should invest I_0 in real production to reach optimal investment at basket P, where MRT = $1 + r_f$, to maximize firm's value.

price
value of firm = $30 + \frac{22}{(1+10\%)} = 50$ (cost of borrowing r_f)

* At A, MRT > $1 + r_f$ → firm should borrow (low cost) to invest in real production until MRT = $1 + r_f$
  ↳ Increase opp for investors (Fisher's seperation)
* At B, MRT < $1 + r_f$ → cost of borrowing is greater than return from investment
  firm should stop investing beyonds P

* Net lender (optimal consumption at L) holding constant utility willing to sacrifice C_1
  ↳ P is not optimal consumption for net lender as MRS < 1+r. As net leader decrease his current consumption (C_0) and lend in capital mkt more until MRS = $1 + r_f$ (invest excess dividend)
  ↳ increase his utility
* Net borrower (optimal consumption at B)
  ↳ At P, MRS > 1+r. This means, the amount of C_1 that he is willing to sacrifice to increase C_0 by holding total utility constant is greater than amount of C_1 that he has to sacrifice if he were to borrow the money to increase his C_0. By moving downward along capital opportunity line from P → B (increase C_0, reduce C_1), his total utility will increase.

! increase in r causes capital mkt line to be steeper. The new optimal inv decision now move along the old PPC toward the right hand side, implying lower inv in real production. As for net lender, his IC will shift toward upper-right hand direction, implying higher utility. As for net borrower, 〃 lower-left hand direction, lower utility.
Higher r → firm has less projects with IRR ≥ r — higher r gives him more $ return in future (net lender)

II Investment Decisions under Uncertainty

don't know inv outcome for sure ↗ management can make investment decision without shareholder preference info (max. firm value) * mkt value rule

1 Investment risk ↗ investment and consumption decision are independent to each other.

1.1. Fisher's Separation with uncertainty: keep perfect capital market assumption but allow uncertainty
Maximize mkt value of firm to satisfy all shareholders
↳ if the riskiness of firm's investment is not suitable to specific risk-return preference of shdr, then they can undo corporate decision by selling the security & rebalance their portfolios. (in perfect capital mkt: no rebalancing cost)

* Risk → know all possible outcome + prob = distribution of outcome (investment)
* uncertainty: may know all possible outcome, but don't know prob.
  e.g. we know prob & all possible outcomes of toss a fair dice → obtain uniform distribution } uncertainty
  based on certain assumption → there's risk on risky model
  ↳ 2 main approaches to derive prob density of returns: ① Subjective: more forward looking, but diff analyst ② Historical (actual distribution) → not forward looking, used in this subject. (diff outcome)

2 Utility analysis given uncertainty: ↗ optimal decision is to invest in security with highest E(R)

2.1. Maximum expected return criterion ↳ max E(R) ⇒ max expected future monetary outcome of future wealth $E(W_1) = W_0(1 + E(r_i))$ → not objective!

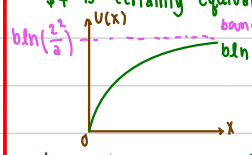
* Most people will not pay up to intrinsic value for the investment. ($IV_0 < E(X)$)

2.2. St. Petersburg Paradox: n = time when 1st head appears; X_n: payoff if H; appears in n^{th} toss; $E(X) = \sum P_n X_n$
- Game of chance: a coin is tossed until 1st head appears. → payoff = $\$2^n$ → $E(X) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n 2^n = \infty$
* But individuals will pay at finite amount to enter the game even Expected payoff is ↑

2.3. Bernoulli's solution: amount of money an individual is willing to pay for the game depends on the game's expected utility not its expected money return. (risk aversion)
↳ individuals don't assign the same value per dollar to all payoffs. ↗ (diminishing marginal utility)
assign satisfaction value to each dollar of payoff → dollar for lower round as it is less risky
e.g. each dollar to be received from 10th flip is viewed as less valuable than each dollar from 3rd flip.
$U(X) = b\ln\left(\frac{X}{a}\right)$ ↳ X is size of money price = 2^n
- increasing function of monetary outcome (prefer more to less) → marginal utility is ⊕ → $\frac{dU}{dX}$ or $U(X) > 0$
- As x increases, U(X) increases at a decreasing rate → diminishing MU → $\frac{d^2U}{dx^2}$ or $U''(x) < 0$
* want to find X_0, such that $U[X_0] = E[U(X)]$ (we need to convert payoff X_n to utility $U(X_n)$)

$$E[U(X)] = \sum_{n=1}^{\infty} P_n U(X_n) = \sum_{n=1}^{\infty}\frac{1}{2^n}\left[b\ln\frac{2^n}{a}\right] = \left[\sum\frac{n}{2^n}\right]b\ln 2 - \left[\sum\frac{1}{2^n}\right]b\ln 2 = 2b\ln 2 - b\ln a$$
$$= b\ln 2^2 - b\ln a = b\ln\left(\frac{2^2}{a}\right) = U(2^2) = U(4) = E[U(X)]$$

* an individual whose taste conforms to Bernoulli's utility function will pay $4 at most
  ↳ he is indifferent btw a perfectly certain promise of a gain of $4 & chance to play game.
  ↳ $4 is certainty equivalent or CE of risky game



As monetary outcome (X) approaches infinity the total utility (U(X)) reaches a finite number.
- Discrete dist: $E[U(X)] = \sum_{i=1}^{n} P_i U(X_i)$
- Continuous dist: $\int_{i=1}^{n} U(X_i) f(X_i)\,dX$

∴ Given risk aversion, expected utility approach can be used to show how individuals derive price of a risky asset

2.4. Modern theory of utility: Von Neumann & Morgenstern (VNM): maximizing E(U) is always optimal solution under certain assumptions (axioms of choice)

2.5. Axiom of choice under uncertainty; x, y, z are elementary mutually exclusive outcomes (final payoff; only 1 outcome can appear once at time)
① Comparability (completeness): has ability to rank preference towards all possible baskets of consumption
② Transitivity (consistency): consistent in preference rankings; $x \succsim y$ & $y \succsim z \rightarrow x \succsim z$
③ Non-satiation of wants (prefer more to less): always prefers a certain outcome with higher payoff
④ Independence: attitudes toward particular outcome aren't affected when combine (everything has its price) | inde: same prob compare only diff outcome
e.g. Camry ≺ Accord ⇒ [0.9 × Vios, 0.6 * Camry] ≺ [0.9 × Vios, 0.6 * Accord]
⑤ Certainty equivalent: Every game of chance have value (CE) ← individual indifferent btw game of chance
↳ If $x \succ y \succ z \rightarrow y \sim [\alpha x, (1-\alpha) z]$ ↳ subject can always give price (btw best & worst outcome) & CE for game of chance; $SCC \succ PTT \rightarrow P_{SCC} > P_{PTT}$
⑥ Ranking: prob assigned to each outcome if diff → we prefer game with more chance to win.

Ranking: same outcome, diff prob (cannot compare directly as the possible outcomes & prob aren't equal → form bridge)

EX1 AIS (100, $0, 0.5) PTT (120, $0, 0.5) } use ranking same outcome investors only
- Independence: same prob, compare only outcome
- Non-satiation of wants & Inde: PTT ≻ AIS
- Inde & Non-satia & CE: $P_{PTT} > P_{AIS}$ } compare prob of each possible outcome

EX3 AIS (100, $0, 0.5) SCC (120, $0, 0.7) PTT (120, $0, 0.5) } From all mentioned & transitivity → SCC ≻ AIS & CE: $P_{SCC} > P_{AIS}$
- Inde & Non-satia: PTT ≻ AIS
- Ranking & non-satia: SCC ≻ PTT

Ex4 Given: Stock B with payoff = (0.6 × \$60, 0.4 × \$30) find rational price range.

(S1) From CE, there's exists a unique α, y ~ [α(x), (1−α)z]. From non-satia \$60 > 30 ∴ value y (or P_B) must line within (\$30, 60) such that P_B ~ [0.6 × \$60, 0.4 × \$30]
price range is \$30 < P_B < 60

(S2) Form bridging securities
A = (\$60, 30, 0) → certain \$30 → P_A = \$30
C = (\$60, 30, 1) → certain \$60 → P_C = \$60
From non-satia & ranking, C ≻ B & B ≻ A
∴ P_A < P_B < P_C or \$30 < P_B < \$60

(S3) • If P_B = 60, buyer will at best breakeven and has chance to make loss. If P_B > 60, buyer will make sure loss. P_B < \$60
• If P_B = 30, (short) seller will at best breakeven or loss, P_B < 30, seller loss for sure → P_B > \$30 ∴ 30 < P_B < 60

3 Attitude toward risk

Max E[U(X)] → optimal decision

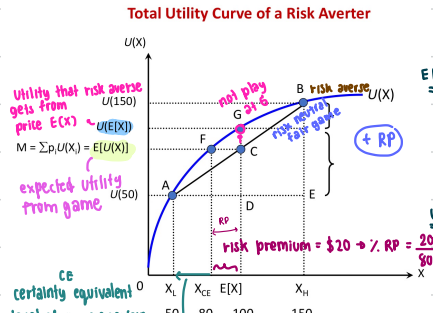
Risk neutral: E[U(X)] = U[E(X)]

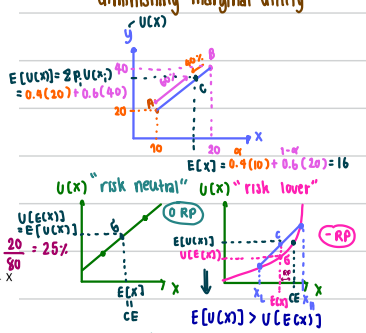
U'(X) > 0 prefer more to less
U''(X) < 0 risk aversion

▸ Fair game: price of participation = expected monetary price E[X] ⇒ E[r] = 0

① Risk neutral: feel indifference btw certain price and E[X] ⇒ E[r] = 0 X_{CE} ∈ $(X_L, E[X])$

② Risk averse: reject fair game because E[U(X)] < U(E(X)] ⇒ total utility function is upward sloping and concave
Speculator E[r] > 0 — expected utility derived from game (ไม่เล่น) — utility from price paid
↓ diminishing marginal utility





CE: certainty equivalent level of price (certain payoff) that investor willing to pay to participate fair game

X_{CE} move to left as subject become more risk averse to have more gap RP

X_{CE} of risk averse ∈ $(X_L, E[X])$ ไม่เกินกึ่ง E[X]
∈ $(X_L, E[X])$ ∈ $(E[X], X_H)$

X_{CE} ∈ $(X_L, E[X])$

★ Risk averter: CE < E[X], Risk lover: CE > E[X], Risk neutral: CE = E[X]

③ Risk lover: always take fair game → E[U(X)] > U[E(X)] willing to pay premium to take risky inv
Gambler E[r] < 0 is rational

★ Portfolio selection with partial information: ① find utility of each investment's payoff ② find expected utility (Step)

Pr	W_A (given individual utility function U(W) = 100 + 2W^0.5)	
1/2	95 ; U(X₁) = 100 + 2(95)^0.5 = 119.5	E[U(X₁)] = 120
1/2	105 ; U(X₂) = 100 + 2(105)^0.5 = 120.5	

E[W_A] = 100

③ compare this with other investments (B & C..) choose

★ make choice btw 2 stocks instead of 3 is easier for investor to

E[X] ← don't have to know utility function under risk averse

investment that gives same expected payoff but has closer range is more preferred.

3 Stochastic Dominance (SD)

(An investment criteria → used to classify alternative risky investments into efficient and inefficient sets.)

F (inv return) is random variable & subject to probability distribution

• Utility depends on return U(r) → maximize E[U(r)] — whether it is discrete or continuous

• SD can be applied to any form of probability distribution of asset return, and make use of all points / moments from prob dist of returns (all info is used) & assume only general forms of investors utility functions

★ Efficient criterian:

[1] Take all possible inv alt into consideration & form inv feasibility set → requires mkt info on fin assets (work for all risk averse investors)

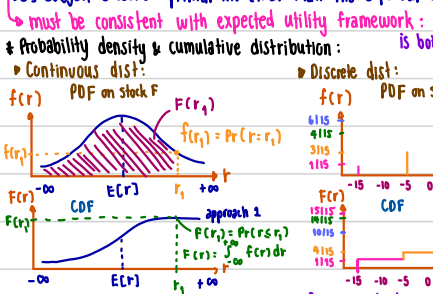
[2] Classify feasible investment into efficient & inefficient investment subsets → requires investor's preference toward risk & return

[3] Subject selects optimal inv (that max his expected utility) from eff inv set

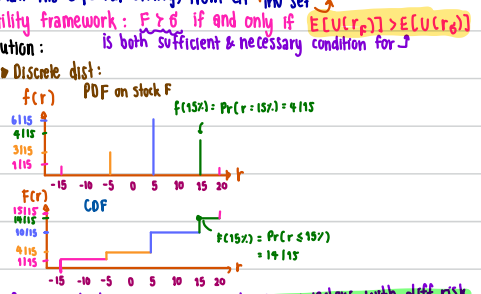
→ must be consistent with expected utility framework: F ≻ G if and only if E[U(r_F)] > E[U(r_G)] is both sufficient & necessary condition for

★ Probability density & cumulative distribution:

▸ Continuous dist:



▸ Discrete dist:



★ First order SD (FSD) → investors prefer more to less → can be apply to any investors with diff risk preference

• Assumption: investors have utilities that are non-decreasing with respect to returns (U'(r) ≥ 0)

• Rule: F ≻ G iff F(r) ≤ G(r), ∀r and F(r) < G(r), ∃r (CDF) — for some level of return to avoid comparing same distribution

F ≻ G when cumulative dist of F lies on right-hand side of cumu dist of G → G(r) − F(r) ≥ 0

Ex1 F(r) — 1st Find G(r₁) > A(r₁) — whole curve is on LHS of A

cannot make conclusion A vs. C because they cross each other

∴ B inefficient, A & C efficient

Ex2

F outcome	(%) Prob.	G outcome	Prob.
12	1/3	11	1/3
10	1/3	9	1/3
8	1/3	7	1/3

r	F(r)	G(r)	G(r) − F(r)
7	0	1/3	1/3
8	1/3	1/3	1/3
9	1/3	2/3	1/3
10	2/3	2/3	0
11	2/3	1	1/3
12	1	1	0

(cumu 0)

∴ F ≻ G ★ FSD allows cumu dist to touch but not cross

Ex3 Given historical returns

of compare = $\frac{N^2 - N}{2}$; N = # of stocks

	A	B	A	B	A(r) − B(r) ≥ 0
2000	-10	60	0	0	0
2001	10	70	-10	2/3	0
2002	10	70	10	1	0
2003	10	70	10	1	1
2004	10	70	10	1	1
2005	-10	70	60	1	2/3
2006	10	70	70	1	1
2007	10	70			
E[r]	5.00	67.5			
SD	9.26	4.63			

(CD)

∴ B ≻ A

A is inefficient

E[U(r_B)] = 0.25(100 + 100 ln(60)) + 0.75(100 + 100 ln(70)) = 521

U(r) = 100 + 100 ln(r)

★ Interpretation of FSD:

F ≻ G iff F(r) ≤ G(r) with at least one level of r the strong inequality holds ↔ $Pr_F(r \le r_1) \le Pr_G(r \le r_1)$

prob of receiving a return lower than given amount r_1 will always be smaller for F than for G

FSD rule (F(r) < G(r) for ∀r) is both sufficient and necessary conditions for E[U(r_F)] > E[U(r_G)] → F ≻ G

• G(r) ≥ F(r) implies E[r_F] ≥ E[r_G] → Thus, a necessary condition (but not sufficient) for investment F to be preferred to G under FSD is that the expected return of F be no less than that of G.

• However, a similar systematic relationship cannot be established btw variance of returns because there's no assumption regarding risk aversion.

∴ FSD has low power to differentiate btw efficient & inefficient

★ Second order SD (SSD) ★ give same E[r] but diff dispersion of outcomes — Same E[r], F less disperse E[U(F)] > E[U(G)] — prefer inv with less dispersion

U'(r) > 0 — U''(r) < 0

• Assumption: investors have utilities that are non-decreasing with respect to returns & risk averse

• SSD rule: F ≻ G iff ∫(G(r) − F(r)) dr ≥ 0, ∀r and ∫(G(r) − F(r)) dr > 0, ∃r — strong inequality for some return

accumulative sum of diff btw cumu prob of G − F always non-zero for all returns

Ex1 F(r), G(r) — as long as S_1 is larger than S_2 → $S_1 + S_2 \ge 0$; then, F ≻ G under SSD
or $S_1 + S_2 \ge S_3$ → $S_1 + S_2 + S_3 \ge 0$

lowest possible r — highest

• SSD assumes risk aversion
↳ E[U(r_F)] > E[U(G)] or F ≻ G

riskier F ≻ G

• Same E[r] but return on G is more disperse than F

• MU is higher in F ; even G gives chance to earn higher return, but effect of higher return doesn't effect total utility much due to diminishing MU.

• with Concave U(r), MU becomes smaller as r↑

↳ FSD: cannot reach same conclusion → as it allows U(r) concave, convex

if $S_1 \ge S_2$, F ≻ G under SSD

IV The MV analysis → oppose to stochastic dominance that use whole distribution

1.1 Intro

▸ What drive expected utility from risky return? (derived from investment return (r), which is random variable)

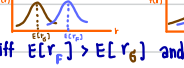
By applying Taylor's series expansion:

$$E[U(1+r)] = U(1+E[r]) + \frac{U''(1+E[r])}{2!}\sigma^2 + \frac{U'''(1+E[r])}{3!}\mu^3 + \frac{U^4(1+E[r])}{4!}\mu^4 + \dots$$

All moments of return distribution affect expected utility. However, with mean-variance criterion (MVC), we focus on mean and variance of return distribution only as any moment higher than 2 (skewness and kurtosis) have insignificant effect on expected utility.

★ • Assume prefer more to less ⇒ U'(1+r) > 0 ⇒ E[r] is desirable ⇒ increase expected utility
• Assume risk aversion ⇒ U''(1+r) < 0 ⇒ σ² is undesirable ⇒ decrease expected utility

เน้นๆ • Positive skewness (μ^3 > 0) is desirable → higher probability to get high return → increase expected U
เน้นๆ Negative skewness (μ^3 < 0) is undesirable → we don't know whether μ^3 is ⊕ or ⊖ → MVC assume μ^3 doesn't matter.



F ≻ G same E[r] on condition that at least one strong in equality holds
↳ avoid comparing same stock

2. Mean-Variance Criterion

2.1. Basic principal: investment F ≻ G iff $E[r_F] > E[r_G]$ and $\sigma_F^2 \le \sigma_G^2$

• Advantage of MVC: investor can confine himself on the first two dist. moments of each investment alternative being considered

2.2

	A	B*	C*	D*	E
E[r]	10	8	9	11	12
σ²	10	11	10	12	11

A ≻ B and C, E ≻ D → B, C, D relegated to inefficient set

↳ MVC efficient set consists of A & E

when making inv decision, individuals only need to consider A & E

2.3 Assumptions

• MVC is based on underlying assumptions regarding investor's tastes, relevant decision

★ can use one of two rules to justify MVC as an approach to maximize expected utility ★

① Investors utility functions are quadratic function: assume risk averse

$W = W_0(1+r)$ (final wealth) — U(W) = a + bW + cW² ; assuming U'(W) = b + 2cW > 0 (+ mean), U''(W) = 2c < 0 (− variance)

$E[U(W)] = a + bE[W] + cE[W^2]$; $\sigma_W^2 = E[W^2] - (E[W])^2$

$= a + bE[W] + cE[W]^2 + c\sigma_W^2$

and from $E[W] = W_0[1 + E[r]]$ → $\sigma_W^2 = Var[W_0(1+r)] = W_0^2\sigma_r^2$

1st order with respect to E[r], σ_r^2:

$\partial E[U(W)]/\partial E[r] = (b + 2cE[W])(W_0) > 0$

$\partial E[U(W)]/\partial \sigma_r^2 = (c)(W^2) < 0$

E[U(W)] is an increasing function of E[r] and decreasing function of σ_r^2.

Ex: a = 1, b = 100, c = −0.1, initial inv W_0 = \$100

	A	B
E[r]	1%	2%
σ_r	2%	1%

$E[U(W_A)] = 1 + 100[101] - 0.1[101]^2 - 0.1(100^2 \cdot 2\%^2) = 9{,}080.5$

$E[U(W_B)] = 1 + 100[102] - 0.1[102]^2 - 0.1(100^2 \cdot 1\%^2) = 9{,}160.5$

∴ B ≻ A as $E[U(W_B)] > E[U(W_A)]$ which is consistent with MVC

★ Quadratic function implies increasing absolute degree of risk adversion. This is not make sense because, as we become wealthier, we tend to invest more in risky assets

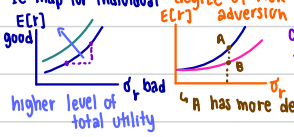
② Asset returns are normally distributed ↲ normally use this

$E[U(W)] = \int U(W_0(1 + E[r] + z \cdot \sigma_r))\, f(z)\, dz$; f(z) & W_0 is the same across diff assets

asset's expected return ; SD of future r ; standard normal dist.

2.4. Graphical Representation of MVC (E[r] − σ_p plane)

IC map for individual — damage his utility



↳ A has more degree of risk adversion (slope)

As we assume risk aversion, if subject takes 1 more unit of risk, he needs higher expected return as compensation. This implies diminishing marginal utility → graph is convex

3. Portfolio diversification:

3.1. Portfolio mathematic:

• $E[r_x] = \frac{1}{T}\sum_{t=1}^{T} r_{x,t}$ or $\sum_{i=1}^{N} P_i r_{x,i}$

• $\sigma_x^2 = \frac{1}{T}\sum_{t=1}^{T}(r_{x,t} - \bar{r}_x)^2$ or $\sum_{i=1}^{N} P_i(r_{x,i} - E[r_x])^2$

• $\sigma_{x,y} = \frac{1}{T}\sum_{t=1}^{T}(r_{x,t} - \bar{r}_x)(r_{y,t} - \bar{r}_y)$ or $\sum_{i=1}^{N} P_i(r_{x,i} - E[r_x])(r_{y,i} - E[r_y])$

• $\rho_{x,y} = \frac{\sigma_{x,y}}{\sigma_x \cdot \sigma_y}$ } correlation, on average, two returns tend to move together?

▸ Portfolio of 2 securities

• $E[r_p] = w_1 E[r_1] + w_2 E[r_2]$

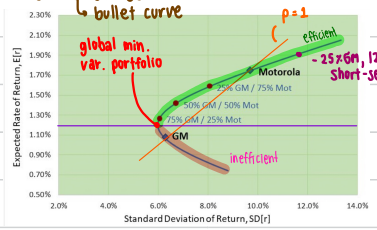
• $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2}$

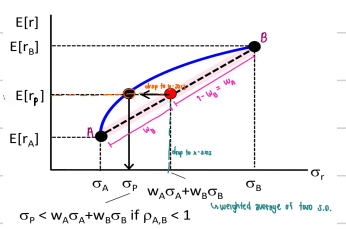
★ $w_1 = \frac{\text{market value of asset 1 in port}}{\text{mkt value of asset 1 + 2}}$

★ Trade-off return: risk ↓, E[r_p] ↓

★ Diversification: increase $E[r_p]/\sigma_p$ return per unit of risk (free lunch in finance)

★ Inv opportunity set & efficient inv set (2 assets): MVC divides inv oppo set into efficient & inefficient set





$\sigma_P < W_A\sigma_A + W_B\sigma_B$ if $\rho_{A,B} < 1$

No benefit from diversification: $\rho_{A,B} = 1$ → $\sigma_p = W_A\sigma_A + W_B\sigma_B$

On average $-1 < \rho_{A,B} < 1$ → $\sigma_p < W_A\sigma_A + W_B\sigma_B$ → we prefer lower ρ (more curvy curve) get more benefit from diversification
↳ return per unit of risk becomes more better off

▸ Portfolio with many risky assets:

$E[r_p] = \sum_{i=1}^{N} w_i E[r_i] = w^T r$ (1×N, N×1) ; $\sum_{i=1}^{N} w = 1$ $[w_1\ w_2 \dots]\begin{bmatrix} E[r_1] \\ \vdots \end{bmatrix}$

$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j} = w^T V w$ (1×N)(N×N)(N×1) $[w_1\ w_2 \dots]\begin{bmatrix} \sigma_1^2 & \cdots \\ \sigma_{21} & \sigma_2^2 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \end{bmatrix}$

★ use mmult in Excel

variance-covariance matrix (square & symmetric)

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can decrease dispersion

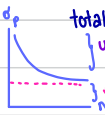
3.2. Diversification benefit: • Uninformed investors, add 1 stock to port will decrease σ_p because stocks would not be perfectly correlated, but r_p would remain relatively constant.

↳ N-asset port has N variance terms & $\frac{N(N-1)}{2}$ covariance terms

total risk $\sigma_p^2 = \frac{1}{N}\bar{\sigma}^2 + \frac{N-1}{N}\bar{\sigma}_{i,j}$ → $N\uparrow \rightarrow \frac{N-1}{N}$ approaches 1 → $\bar{\sigma}_{i,j}$ becomes more important

$N\uparrow \rightarrow \bar{\sigma}^2$ (unsystematic risk has less influence) — firm-specific or industry

we cannot diversify systematic risk — measured by beta — market risk



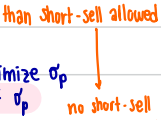
more stocks are added → each new stock has smaller risk-reducing impact on portfolio

4. Efficient portfolios

less MV efficient than short-sell allowed

4.1 Efficient portfolio without RF asset

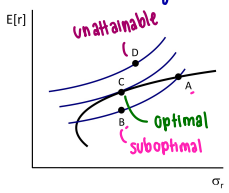
MV efficient set: 1st find global minimum variance port → set obj. minimize σ_p
2nd find the ports that maximize $E[r_p]$ given level of σ_p
↳ set obj. max. $E[r_p]$ solve for weights s.t. $\sum_{i=1}^{N} w = 1$, $w_i \geq 0 \ \forall i$ — no short-sell

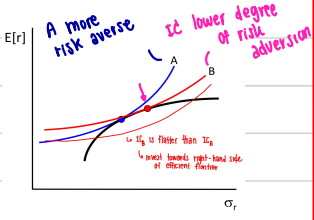


• Arithmetic average ⟨unbiased estimator for average return⟩
• annualized $E[r_p] = (1 + \text{Ave.}[r_j])^{12} - 1$ (monthly), SD $= (12^{0.5}) \times$ (SD of monthly return)

Optimal inv decision: need investor preference; optimal inv decision is on efficient frontier where the slope of EF is equal to slope of IC.

As degree of risk aversion reduces, the optimal port moves from left → right





govt bond

4.2. Efficient portfolio with RF asset

; r_f = yield or YTM on bond = risk-free rate

▸ Capital allocation: decision on how much to invest in risky assets and how much in risk-free asset.

▸ Complete portfolio (C): portfolio consisting of both RF asset and risky assets

• $r_c = y r_p + (1-y) r_f$; y is weight on risky port (P)

• $E[r_c] = r_f + y(E[r_p] - r_f)$ — risk premium

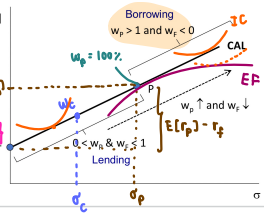
• $\sigma_c = y\sigma_p$; $\sigma_f^2 = 0$ and $\sigma_{p,f} = 0$

• $y = \frac{\sigma_c}{\sigma_p}$

• $E[r_c] = r_f + \frac{\sigma_c(E[r_p] - r_f)}{\sigma_p}$

▸ Capital allocation line (CAL): represents various combination btw the risky portfolio and risk-free asset.
↳ Sharpe's ratio: slope of CAL, reward-to-risk ratio

Sharpe's ratio $= \frac{E[r_p] - r_f}{\sigma_p}$



* To move complete port beyond point P $\{E[r_c] > E[r_p]\}$ investor must borrow (short position / sell bond) to invest more in portfolio P — levered

* Capital allocation decision doesn't change reward-to-variability ratio

2 different for each investor with different risk reference. (ORP: can let 3rd person (fund manager) do it to select security)

* Optimal complete portfolio: assume a quadratic utility function — is the portfolio where CAL is tangent to the efficient frontier. maximize utility

$\max E[U(r_c)] = r_f + y(E[r_p] - r_f) - \left(\frac{1}{2}\right)(A) y^2 \sigma_p^2$ — risk aversion factor

1st order wrt y: $y^* = (E[r_p] - r_f)/A\sigma_p^2$

* Optimal risky portfolio: is independent from investors' degree of risk aversion and can be identified using mkt info only. → max Sharpe's ratio (steepest CAL)

4.3 The Separation Property of investment decision / two funds separation / Tobin's separation theory

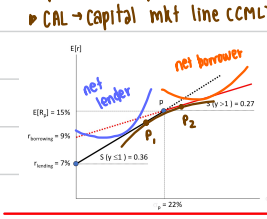
independence btw optimal risky portfolio and degree of risk aversion of investor → determine two independent tasks: ① Optimal risky portfolio (security selection) ② Optimal capital allocation (identify weight using preference)

↳ Top-down approach: capital allocation → asset class allocation → security selection — if not, will break down

* assume all investors have homogenous expectation (& perfect capital mkt), all investors will have the same efficient frontier of risky assets (same port P but diff in capital allocation decision

▸ port P → mkt port will all risky assets in mkt weighted by relative mkt cap of assets.

▸ CAL → capital mkt line (CML)



* Drawbacks of Markowitz model: requires full variance-covariance matrix as the important input. → time consuming
① Large number of estimation must be performed.
↳ tendency to obtain inconsistent estimates of inputs
↳ comes from covariance: use historical data will not cause this
* but using forecasted data might cause this
② No division of labors: 1 team analyst must know all securities and need to do all estimations to avoid inconsistent.
③ Strong implication of passive portfolio management.
↳ implies that holding mkt port is optimal → we want active

V Single index model (SIM)

• assumes that security returns are driven by single factor model

$r_i = \alpha_i + \beta_i \cdot F + \varepsilon_i$

return on security i — constant part of r_i — sensitivity of r_i to a change in value of F — common factor driving returns of all securities — residual term random part of r_i (unexplained by F)

F change 1% → r_i change by β_i. e.g. GDP, inflation, SET index

• based on premise that covariances among securities' returns arise from a common economic factor that affect all firms

* $E[\varepsilon_i] = 0$ → unbiased estimator
$Var[\varepsilon_i] = \sigma_{\varepsilon_i}^2$ → homoskedasticity ↳ for regression
$cov[\varepsilon_i, F] = 0$ → endogeneity
$cov[\varepsilon_i, \varepsilon_j] = 0$ → ทำให้แยกกันทำแต่ละ Industry ได้

$r_i = E[r_i] + \beta_i \cdot (F - E[F]) + \varepsilon_i$

expected return — deviation from $E[r_i]$ due to surprise in value of common factor (F) e.g. broad economy — unanticipated surprise — deviation from $E[r_i]$ due to surprise in firm-/industry-specific events

Single index market model: $r_i = \alpha_i + \beta_i r_M + \varepsilon_i$

r_M = return on mkt portfolio

* **Single Index (market) model** $[r_i = \alpha_i + \beta_i \cdot r_M + \varepsilon_i]$ — แยกกันนำได้ allow division of labor

• $Var[r_i] = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon_i}^2$ — common factor generates correlation across securities → $cov(r_i, r_j) = \beta_i \beta_j \sigma_M^2$ — cov btw any pair of securities is determine by β
Systematic risk — firm-specific unsystematic risk

$cov(r_i, r_M) = \beta_i \cdot Var[r_M] \rightarrow \beta_i = \frac{cov(r_i, r_M)}{Var[r_M]}$ — เทียบ return ขึ้น 1 ตัวกับ return mark $\beta_M = 1$

$corr(r_i, r_j) = \beta_i \beta_j \sigma_M^2 / \sigma_i \sigma_j = corr(r_i, r_M) \cdot corr(r_j, r_M)$

* **Estimation of market model:** SIM is linear → use univariate linear regression

Security characteristic line (SCL): intercept — coefficient of excess return on mkt index — model fit in well

▸ Academic version: $(r_{i,t} - r_{F,t}) = \alpha_i + \beta_i(r_{M,t} - r_{F,t}) + \varepsilon_{i,t}$ —①
(risk-free) — abnormal return — residual term unsystematic risk

use this instead of raw return because we should expect a risky asset to earn return in excess of r_f return

$\alpha_i > 0 \rightarrow w_i \uparrow$
$\alpha_i < 0 \rightarrow w_i \downarrow$

High R-square: mkt return can explain stock return _ %
low $\varepsilon_{i,t}$: low unsystematic risk

▸ Industry version: use raw return of stock i and market

$r_{i,t} = a_i + b_i \cdot r_{M,t} + u_{i,t}$ —②

$r_{i,t} = \alpha_i + (1 - \beta_i) \cdot r_{F,t} + b_i \cdot r_{M,t} + \varepsilon_{i,t}$ —③

Take expectation of ① $E[r_i] - r_F = \alpha_i + \beta_i(E[r_M] - r_F)$ — constant → ①&② yield consistent slope term

"Jensen's Alpha" — expected risk premium — expected risk premium compensates for systematic risk of security

α_i is non-mkt return or abnormal return;
• bare no risk: $Var[\alpha_i] = 0$
• $\alpha_i > 0$: $E[r_i]$ is more than compensates for its systematic risk → in demand
↳ competition in capital mkt will drive α_i down to 0
• $\alpha_i = 0$: not good!! fund manager cannot do anything

* To form SCL ⟨time-series regression⟩
• need historical return on stock, on mkt index, risk-free rates
• calculate excess returns $r_{M,t} - r_{F,t}$ & $r_{i,t} - r_{F,t}$

* **SIM & Portfolio diversification:** use excess return form ⟨$R = r - r_F$⟩

▸ Portfolio's return: $R_p = \alpha_p + \beta_p \cdot R_M + \varepsilon_p$

$R_p = \sum w_i \cdot R_i = \sum w_i \cdot (\alpha_i + \beta_i \cdot R_M + \varepsilon_i)$

$E[R_p] = \underbrace{\sum w_i \cdot \alpha_i}_{\alpha_p} + E(R_M) \cdot \underbrace{\sum w_i \beta_i}_{\beta_p}$ $\quad \varepsilon_p = \sum w_i \varepsilon_i$

assume r_f is constant: $Var[R_M] = Var[r_m - r_f]$, $i = Var[r_m]$

▸ Portfolio's var: $Var[R_p] = (\sum w_i \cdot \beta_i)^2 \sigma_M^2 + (\sum w_i^2) \sigma_{\varepsilon_i}^2 = \beta_p^2 \sigma_M^2 + \sigma_{\varepsilon_p}^2$

$w_i = \frac{1}{N}$

assume equally-weight portfolio: $Var[\varepsilon_p] = \left(\frac{1}{N}\right)^2 \sum Var(\varepsilon_i) = \frac{1}{N} ave(\sigma_{\varepsilon_i}^2) \rightarrow 0$ (as $N \rightarrow \infty$) — SIM assumes $cov(\varepsilon_i, \varepsilon_j) = 0$

↳ as more and more securities included in portfolio → unsystematic risk $\sigma_{\varepsilon_p}^2 \rightarrow 0$ → well-diversified port is only exposed to systematic risk ⟨$\sigma_{\varepsilon_p}^2 \downarrow R^2 \uparrow$⟩

* Max. $SR_p = \frac{E[R_p]}{\sigma_p}$ ⇒ optimal risky portfolio

SR_p increases as $\alpha_p = \sum_{i=1}^{N} w_i \alpha_i > 0$ — will not be fully diversified as it moves away from mkt index → will carry certain risk level of unsystematic risk $\sigma_{\varepsilon_p}^2$ (idiosyncratic)

Information ratio is used to measure how good active manager is — info ratio of stock i $= \frac{\alpha_i}{\text{unsystematic risk} \rightarrow \sigma_{\varepsilon_i}}$ ↑ better

* **SIM vs. Markowitz model:** $\alpha_i, \beta_i, Var[\varepsilon_i], E[r_M]$ & $Var[r_M]$ (1 regression 1 set) — $E[r_i]$ & $Var[r_i]$, $cov(r_i, r_j)$ — depends on how reliable estimated variables are

① SIM use less number of estimated variables ($3N + 2$ vs. $2N + \frac{(N^2 - N)}{2}$)
② Efficient frontier SIM inferior to Markowitz → no guarantee Markowitz outperform SIM in out-of sample setting
③ SIM assumes $cov(\varepsilon_i, \varepsilon_j) = 0$ → SIM allows division of labor & decentralization of macro & security analysis.
overweight securities $\alpha_i > 0$, under-weight securities $\alpha_i < 0 \Rightarrow \sum w_i \alpha_i > 0$
④ SIM can be extended to optimize port with active management → but port will not be fully diversified

VI Capital Asset Pricing Model (CAPM): $E[r_i]$ form in equilibrium in relation to risk;

• CAPM & APT are positive theory (do empirical test) — or utility function is quadratic

* ASSUMPTIONS *
① Individuals' behavior
• All investors are rational MV optimizers (assets returns are normal dist) — no reinvestment risk
• " ——— " have identical single period int. horizon (inv decision is made once today)
• " ——— " have homogeneous expectations regarding $E[r_i]$'s & σ_i's of all assets — same eff frontier, same mkt port
↳ consistent with assumption that all relevant info is publicly available
② Capital mkt structure:
• All assets are divisible & publicly held ↔ lot size regulation & human capital cannot be traded
• There exists a risk-free asset such that investors can borrow/lend unlimited amount at r_f — only govt can borrow
• capital mkt is perfect (frictionless): no transaction cost & taxes, equal & cost less info
↳ no restriction short sales, investors're price takers (mkt price is fair price)

(CML)
* Capital mkt line: mentioned assumptions → imply every investors face same CAL & optimal risky portfolio
• In equilibrium: - asset price must be adjusted until there's no excess demand & supply on financial assets.
- total wealth invested on risk-free asset = 0 (total borrowing = total lending)
- All existing risky assets must be held in optimal risky port, assets're held according to mkt value weights.

* Market portfolio (M): optimal risky port that consists all risky assets traded in the market.

* CML: CAL that connects r_F and mkt portfolio — MVC → CML → CAPM (mkt port)

Ex all investors have same tangent portfolio → $W_T = (0.25, 0.5, 0.25)$ (A, B, C)

Investor	RF	A	B	C	Total
X	100	100	200	100	500
Y	200	200	400	200	1000
Z	−300	450	900	450	1500
Total	0	750	1500	750	3,000

∴ In equilibrium, total dollars holding in each risky asset must be equal to asset's mkt capitalization.
↳ tangent port is market port (same weights)
Total investment in RF asset is $0.

∴ All investors hold the same risky port and are diff only in capital allocations — forming opt risky port independent of investor's risk aversion

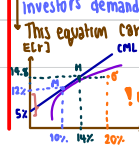
Two-fund separation: port construction consists 2 independent steps: 1) Security selection 2) cap allocation

• Passive security selection strategy is MV efficient. (drawback) ↳ invest 98 mkt port

* CML equation: $E[r_p] = r_f + \left(\frac{E[r_M] - r_f}{\sigma_M}\right)\sigma_p$ — slope — doesn't show how expected return on any port & any individual security (whether it's MV eff or not) is formed in equilibrium.
equilibrium condition
relation btw $E[r_p]$ & σ_p of MV eff port
↳ slope depends on RP_M, RP_M depends on average risk aversion of all mkt participants
Sharpe ratio of mkt → slope $= \frac{E[r_M] - r_f}{\sigma_M}$ (aggregate risk aversion)

• When there's crisis, to take risk, investors demand high expected return → stock price ↓ as mkt become more risk averse (steeper CML)

This equation can overestimate $E[r_p]$ — H not G



From CML equation → can be used on MV eff port only
$E[r_M] = 0.05 + (0.07)(0.14) = 14.8\%$
! $E[r_G] = 0.05 + (0.07)(0.1)(0.2) = 19\%$

* Derivation of CAPM (simplified)
security i's contribution to the risk of the market port is measured by $w_i \times cov(r_i, r_M)$
• marginal contribution of asset i to RP of mkt port: $w_i(E[r_i] - r_f)$ (in equilibrium)
∴ Marginal reward-to-risk ratio for inv in asset i $= \frac{w_i(E[r_i] - r_f)}{w_i \sigma_{i,M}} = \frac{E[r_i] - r_f}{\sigma_{i,M}} = \frac{E[r_i] - r_f}{\sigma_M^2}$

- In equilibrium: sharpe ratio of each stock is equal

$$\frac{E[r_i]-r_F}{\sigma_{i,M}} = \frac{E[r_M]-r_F}{\sigma_M^2} \rightarrow E[r_i]-r_F = \frac{\sigma_{i,M}}{\sigma_M^2}(E[r_M]-r_F) \rightarrow E[r_i] = r_F + \beta_i(E[r_M]-r_F)$$

cov, var

$\sigma_{i,M} = \rho_{i,M}\sigma_i\sigma_M$

CAPM: Can be applied to any stock; not assume asset i is efficient

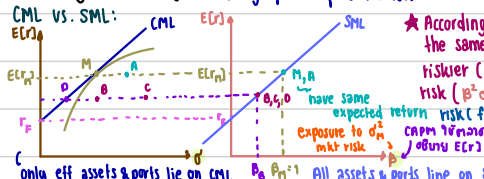
- r_F: compensation for time value of money
- market port has beta of 1

compensation for systematic risk — risk level of mkt price per security i 1 unit of risk

"total risk in CAPM" investors concern only systematic risk

★ Security market line (SML): graphical presentation of CAPM → in equilibrium, all assets & ports must line on SML.

CML vs. SML:



only eff assets & ports lie on CML — All assets & ports line on SML

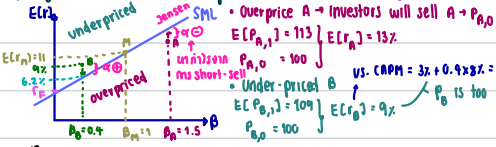
have same expected return; exposure to σ_M^2 mkt risk

★ According to CAPM, investors demand same β_i the same expected return from B&D, but B is riskier (higher σ) because we care only systematic risk ($\beta^2\sigma_M^2$) !!! → we can eliminate unsystematic risk (firm-specific risk) through diversification

CAPM ใช้งานจริง ต้องรู้ E[r]

no cost, no transaction cost → don't charge; must give fair price ← premium in model; not include this because E[r] will be too high & price of stock will be too low & no diversified → no excess supply; equilibrium → no pressure on price

★ Adjustment of E[r] to equilibrium E[r]



★ $E[r] = (E[P_1+D_1]/P_0) - 1$

- Overprice A → investors will sell A → $P_{A,0}$ ↓ → $E[r_A]$↑ → $E[r_A]$ is on SML

$E[r_{A,1}] = 113$, $P_{A,0} = 100$ → $E[r_A] = 13\%$

ทำไมถึงมีการ short-sell

- Under-priced B: $E[P_{B,1}] = 109$, $P_{B,0} = 100$ → $E[r_B] = 9\%$

vs. CAPM = 3% + 0.9×3% = 6.2%

P_B is too low → induce investors to buy B → $P_{B,0}$↑ → $E[r_B]$↓ → $E[r_B]$ is on SML

- Mkt port also change but impact of small stock A, B don't have much effect

★ Properties of CAPM:

① Beta measures the extent to which returns on security & market move together; $\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$

② Beta can be viewed as a measure of systematic risk of an asset.
↳ CAPM shows that investors only demand compensation for systematic risk of assets.

③ $\beta_P = \sum w_i\beta_i$; $w_i = \frac{\text{value of security i in port}}{\text{total value of port}}$

★ Application of CAPM: CFO → investment & financial decision

① Project evaluation: Firm can use E[r_i] as cost of equity → $NPV_0 = \sum_{t=0}^{T} \frac{E[\text{free cash flow}_t]}{(1+k_E)^t}$ (to shareholders)

Risk-adjusted cost of capital: $k_E = r_F + (E[r_M] - r_F)\times\beta_i$ (equity beta of project)

② Fund performance: SML provides benchmark for evaluating inv performance of fund manager
↳ portfolio of skillful fund manager has positive alpha ($\alpha_P > 0$)

active port managers (charge more fee) have specific info leading returns on certain securities are going to be different from fair return (security's alpha α_i)

→ to estimate realized α_i, use time series regression

$r_{P,t} - r_{F,t} = \alpha_P + \beta_P\cdot(r_{M,t} - r_{F,t}) + \varepsilon_{P,t}$

$\alpha_P = \sum w_i\alpha_i$; $\beta_P = \sum w_i\beta_i$ — alpha of a passive portfolio is expected to be zero.

★ Empirical test of CAPM: test whether the prediction of theory is supported by actual (observed data)

- The Ex-post Form of CAPM:
CAPM is expectation model (it predicts E[r_i]) — before event → cannot observe E[r_i]

ex-ante model: ใช้ข้อมูลอดีตเพื่อ predict อนาคต (E[r_i] ใช้ได้)
ex-post model: ใช้ข้อมูลที่เกิดขึ้นแล้วมาทดสอบ (observe ไม่ได้)

- To justify the test of ex-ante model using historical or observed data, we must make one of these 2 assumptions ① Investors have rational expectation (in average, what investors expect is true (unbiased))
② Security returns follow mkt model (we can transform from ex-ante → ex-post return)

↳ $r_{i,t} = E[r_{i,t}] + \beta_i\cdot\delta_{M,t} + \varepsilon_{i,t}$ } mkt model, empirical model based on actual data; surprise of return from mkt portfolio

where $\delta_{M,t} = r_{M,t} - E[r_{M,t}]$ and $E[\delta_{M,t}] = 0$

CAPM

$(r_{i,t} - r_{F,t}) = (r_{M,t} - r_{F,t})\beta_i + \varepsilon_{i,t}$ } ex-post form or testable form of CAPM

Test CAPM from return of single stock has a lot of error ⇓ thus, should test CAPM from portfolio to minimize error

★ Test of CAPM: 2 steps

1st Pass: Time Series Regression (SCL) 50 times (from SIM) — need proxies for r_F (yield govt bond), mkt return (value & equally-weighted index)

$(r_{P,t} - r_{F,t}) = a_P + b_P\cdot(r_{M,t} - r_{F,t}) + e_{P,t}$ [1]

CAPM: $E[r_i] = f(\beta_i)$

$\bar{r}_i = a + b\beta_i + e$ ← linear function of SML

should be $\bar{r}_i - r_F$ to match; $H_0: b = r_M - r_F$ risk premium

2nd pass: Cross-Sectional Regression (SML) — CAPM

$(r_P - r_F) = \gamma_0 + \gamma_1\cdot b_P + u_P$ — run 1 time (average) [2]

There're 500 stocks → each port has 10 stocks; need to run 50 portfolios (smaller firm riskier!!)

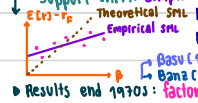
Then, compare to CAPM: risk premium

$E[r_P] - r_F = \beta_P(E[r_M] - r_F)$

$(\bar{r}_P - r_F) = \gamma_0 + \gamma_1 b_P + \gamma_2\sigma_{eP}^2 + \gamma_3 DIV_P + \gamma_4 SIZE_P + u_P$

- Predictions made by CAPM to be tested in [2] — compare two factors: F-test
  - $\gamma_0 = 0$ and γ_1 = average ($r_{M,t} - r_{F,t}$) (risk premium) → $H_0: \gamma_1$ = average ($r_{M,t} - r_{F,t}$)
  - Beta should be the only factor explaining variation in rate of return across assets. — Other terms should have no explanation power. — only systematic risk matter
  - Relationship should be linear in beta → $(\bar{r}_P - r_F) = \gamma_0 + \gamma_1 b_P + \gamma_2 b_P^2 + u_P$

▸ Results prior to Fama and French (1992)
- γ_0 is significantly diff from 0 and γ_1 is less than RP_M — E[r_i]'s aren't affected by nonsystematic risk.
- Support CAPM: Simple linear model fits the data best, β_P dominates σ_{eP}^2 as measure of risk



High-beta " ——— " lower " ——— "
- Basu (1977): Low P/E portfolios earn higher returns than predicted by CAPM
- Banz (1981): Smaller firms tend to have higher returns than bigger firms

▸ Results end 1970s: factors other than β_P are successful in explaining portfolio returns not captured by beta.
+ Value stock (low P/E): tends to outperform growth stock in average even hold same beta mkt constant.
+ Growth stock (high P/E)

▸ Roll's Critique: Roll (1977) argues that CAPM is not testible because only legitimate test of CAPM is whether the mkt portfolio is MV efficient. — reject → just proxy isn't MV efficient
mkt portfolio should contain all risky assets, but we only use proxy to test CAPM. ①
tradable & non-tradable; mkt port is not observable → CAPM cannot be tested/verified

② Size & BVE/MVE Can explain cross-sectional variance better than β

Book-to-mkt ratio (BTM): value stock (high), growth stock (low) } expect HML (high minus low) to be positive because value stock is riskier than growth stock → In long run, value stock will give higher return (higher risk) — Most value firm will experiance drop in BV

★ Fama and French model (1992)

Two-Pass Sort by Size & Beta Average Monthly Return
↳ rank all stocks from low to high beta → sort them further by mkt cap
∴ From the result, beta increases, but return doesn't keep increasing (CAPMX), size can reflect return more
↳ Smaller firms give higher return as they make investors exposed to higher risk than big firms — better support capital's govt can endure crisis

★ Three-factor model: $E[r_i] - r_F = b_i\cdot(E[r_M] - r_F) + s_i\cdot E[SMB] + h_i\cdot E[HML]$ — expected return on port of long high BTM stocks and short low BTM stocks

FF observe that average returns of small stocks and high BTM stocks are higher than predicted by SML — small minus big; small firms give higher return regardless their beta — high minus low: long on r_{high} (BTM, value stock), short on r_{low} (BTM, growth stock)
∴ Size & BTM may be proxies for systematic risk not captured by market beta. — long on r_{small}, short on $r_{big\ firm}$ (big firm)

★ Conclusion:
2 Reasons CAPM fails:
① Bad Research Design: e.g. use daily return in CAPMX (LR); use fragment of mkt index as proxy (S&P 500) X ← improving mkt proxies or statistical methods
② CAPM is invalid: relies on some very restrictive assumptions ← relax some assumptions of CAPM & derive extension versions of CAPM; ← derive new asset pricing theory (APT)

Stephen Ross

VII Arbitrage Pricing Theory (APT): also equilibrium theory explaining how E[r] of a risky asset is formed from CAPM (risk-return dominance, everyone hold mkt port) — large no. of investors required; diff argument; only actions by small no. of arbitrageur with large; APT (no arbitrage argument, not require existence of mkt port)

★ Arbitrage: an investment strategy that requires no net initial inv & results in risk-less profit
↳ Arbitrage profit is risk-less or abnormal profit; doesn't lose any money to invest but get return for sure

$P_{0,A}$: \$10 — 0.5 Econ boom: \$14; 0.5 Econ bust: \$8
$P_{0,B}$: \$10 — 0.5 Econ boom: \$15; 0.5 Econ bust: \$9

short A and get \$10 to invest in B → economic boom: sell B at \$15 and buy A at \$14 to return → profit = \$1; economic bust: sell B at \$9 and buy A at \$8 " " — risk-less profit
zero initial investment

portfolio weight: $w_P = \sum w_i = w_A + w_B = (-10/10) + (10/10) = 0$ — excess demand for B and supply for A → $P_{0,B}$↑, $P_{0,A}$↓

Equilibrium condition: arbitrage opportunity creates excess demand & supply → causes security prices to change.
↳ Mkt equilibrium is reached when there's no arbitrage opportunity → APT uses this as condition of capital mkt equilibrium

★ APT: assumes asset return is from factor model, no assumption about mkt port

▸ Assumption: theory never tell what is K!

① Security return generated by a K factor model

$r_i = \alpha_i + \beta_{i,1}F_1 + \ldots + \beta_{i,K}F_K + \varepsilon_i$ } $r_i = \alpha_i + \beta_{i,1}F_1 + \varepsilon_i$ ①

$E[r_i] = \alpha_i + \beta_{i,1}E[F_1]$ ②

$r_i = r_i - E[r_i]$ ①−②: $r_i = E[r_i] + \beta_{i,1}(F_1 - E[F_1]) + \varepsilon_i$

Surprise from common factor; surprise from firm-specific — cause real return to deviate from expected return

★ $r_i = E[r_i] + \beta_{i,1}f_1 + \ldots + \beta_{i,K}f_K + \varepsilon_i$ — $E[\varepsilon_i]=0$, $cov(\varepsilon_i, F_k)=0$, $cov(\varepsilon_i, \varepsilon_j)=0$; $cov(F_k, F_{k+1})=0$

- $f_k = F_k - E[F_k]$; surprise
- $\beta_{i,K}$: sensitivity (or exposure) of return on i to value of common factor K (factor loading)

ε_i: return on i that is due to firm-specific surprise (unexplained by K factors)

★ Factor model explained:

EX1 common factor (F) is GDP growth, $\beta_{i,Y} = 1.2$, $E[r_i] = 10\%$, $r_F = 2\%$, concensus on GDP growth (Y) = 4%.

$r_i = 0.1 + 1.2f_Y + \varepsilon_i$ or $(r_i - r_F) = 0.08 + 1.2f_Y + \varepsilon_i$ → if $f_Y \neq 0$, there's surprise

★ f_Y is actual minus expected GDP growth

EX2 two-factor model (Y = GDP growth, IR = change in interest rate)

$r_i = E[r_i] + \beta_{i,Y}\cdot f_Y + \beta_{i,IR}\cdot f_{IR} + \varepsilon_i$; for regulated electric-power utility firm, operating CF are insensitive to changes in business cycle → $\beta_{i,Y} < 1$

② There're large number of securities such that well-diversified portfolios could be constructed with negligible idiosyncratic risks → unsystematic or firm-specific risk — no exposure to any risk

③ A diversified portfolio, P, that isn't opposed to any factor risk ($\beta_{P,1} = \ldots = \beta_{P,K} = 0$) must offer r_F — zero sensitivity
↳ $E[r_P] = r_F$

④ There always exist portfolios that're exposed to risk of only single factor, K.

$r_P = E[r_P] + \beta_{P,K}f_K + \varepsilon_P$ } portfolio track movement of GDP perfectly

can be used to represent return of factor K

⑤ A portfolio, P_K, that has unitary risk of factor K, $\beta_{P,K} = 1$, offers a risk premium associated with the factor risk → $E[r_{P_K}] = E[r_K]$ | Portfolio P_K is factor portfolio and $E[r_{P_K}] - r_F$ reflects risk premium on factor K

★ Derivation of APT: To derive APT, we assume that asset returns are generated by a two-factor model

$r_i = E[r_i] + \beta_{i,1}f_1 + \beta_{i,2}f_2 + \varepsilon_i$

↳ Investors can construct a zero-beta (0 systematic risk) zero-investment portfolio

$\sum_i w_i = 0$, $\beta_{P,1} = \sum_i w_i\beta_{i,1} = 0$, $\beta_{P,2} = \sum_i w_i\beta_{i,2} = 0$ where i = 1 to N — large enough to eliminate unsystem

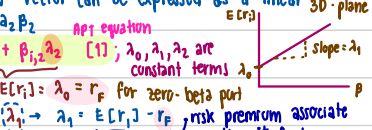
ทำไม short & long i don't use money and create port with no unsystematic risk → arbitrage port → E[r_i] = 0

$r_P = \sum_i w_i r_i = \sum_i w_i E[r_i] + f_1\sum_i w_i\beta_{i,1} + f_2\sum_i w_i\beta_{i,2} + \sum_i w_i\varepsilon_i$ (≈ 0 if N large enough)

$r_P = \sum_i w_i E[r_i]$ and $E[r_P] = \sum_i w_i E[r_i] = 0$ in equilibrium for arbitrage portfolio

- Linear algebra: if the fact that a vector (w) is orthogonal to N-1 vectors (1, β_1 and β_2) implies that it is orthogonal to N^{th} vector (E[r]), then N^{th} vector can be expressed as a linear combination of N-1 vectors. → $E[r_i] = 1 + a_1\beta_1 + a_2\beta_2$

[ ] [ ] = 0 (scalar; normal port)



$E[r_i] = \lambda_0 + \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2$ [1] (APT equation); $\lambda_0, \lambda_1, \lambda_2$ are constant terms

- $\sum_i w_i = 1$, $\sum_i w_i\beta_{i,1} = 0$, $\sum_i w_i\beta_{i,2} = 0$: $E[r_P] = \sum_i w_i E[r_i] = \lambda_0 = r_F$ for zero-beta port
- $\sum_i w_i = 1$, $\sum_i w_i\beta_{i,1} = 1$, $\sum_i w_i\beta_{i,2} = 0$ & $E[r_1] = \lambda_0 + \lambda_1$ → $\lambda_1 = E[r_1] - r_F$ risk premium associate with 1st factor
- $\sum_i w_i = 1$, $\sum_i w_i\beta_{i,1} = 0$, $\sum_i w_i\beta_{i,2} = 1$: $E[r_1] = \lambda_0 + \lambda_2$ → $\lambda_2 = E[r_1] - r_F$ RP associated with 2nd factor

★ Rewrite APT [1] as → $E[r_i] = r_F + \beta_{i,1}(E[r_1] - r_F) + \beta_{i,2}(E[r_2] - r_F)$

★ Logic of APT:

EX1 Assume return is generated by two-factor model: $r_i = E[r_i] + 1.8f_Y - 0.2f_\pi + u_i$ (premium associated with GDP; firm-specific factor)
↳ Given $\beta_{i,Y} = 1.8$, $\beta_{i,\pi} = -0.2$ → view $\beta_{i,K}$ as mimick port weight — procyclical; can hedge inflation

According to APT, asset i's expected return is → $E[r_i] = r_F + \underbrace{1.8E[r_Y]}_{E[r_Y]-r_F} - \underbrace{0.2E[r_\pi]}_{E[r_\pi]-r_F}$ (premium required on asset for bearing GDP and π risk)

↳ If $r_F = 4\%$, $E[r_Y] = 6\%$, $E[r_\pi] = -1\%$ → $E[r_i] = 4 + 1.8(6) - 0.2(-1) = 15\%$

★ Note: $E[r_i] = r_F + 1.8(E[r_Y] - r_F) - 0.2(E[r_\pi] - r_F)$
$= (1 - 1.8 + 0.2)r_F + 1.8E[r_Y] - 0.2E[r_\pi]$
$= -0.6r_F + 1.8E[r_Y] - 0.2E[r_\pi] = 15\%$ ($\sum w_i = 1$)

If they're not equal, can do arbitrage — expected risk premium on GDP and π is 1 relative to GDP and 0 relative to inflation

Then create: Port P replicate stock i → we short sell P: Short GDP 180%, long π 20% & r_F 60% → left 100% to buy stock i (same risk with i; has same systematic risk as asset i)

$\beta_{i,K}$ are viewed as weight

$E[r_P] = -0.6(4) + 1.8(10) - 0.2(3) = 15\%$

If current mkt price of i implies E[r_i] = 15.7%, which is higher than APT → short sell mimick port and buy i — This port: zero initial inv, pay 0.3% for sure, no factor risk & idiosyncratic risk

not equilibrium: E[r_i] will go down until E[r_i] = E[r_P]

★ But APT is approximation → we cannot eliminate idiosyncratic risk 100%

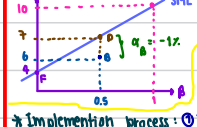
EX2 Port B: $\beta_B = 0.5$, $E[r_B] = 6\%$, $E[r_M] = 10\%$, $r_F = 4\%$, $E[r_M] = 10\% - 4\% = 6\%$ — overvalue

Replicating portfolio B by forming port O (50% r_F & 50% mkt port)

$\beta_O = (0.5\times0) + (0.5\times1) = 0.5$ ($\beta_M = 1$ is factor) and $E[r_O] = (0.5\times4) + (0.5\times10) = 7\% > E[r_B]$ at 6%

→ Short sell port B & long port O: $E[r_{ARB}] = 0.07 - 0.06 = 0.01$, $\beta_{ARB} = 0.5 - 0.5 = 0$ (risk-less)

★ Disequilibrium & adjustment: — price of B will start to drop and $E[r_B]$↑ as everyone do the same



SML is expressed as; $E[r_i] - r_F = \beta_i(E[r_M] - r_F)$ (Portfolio B's alpha)

$E[r_B] - r_F = 6 - 4 = 2 = -1 + 0.5(E[r_M] - r_F)$ — ④ doesn't spell out what syst risk are (CAPM use expected return on mkt as common factor)

★ Implementation of APT: ⑤ What're common factors

★ Drawbacks of APT: ① K = ? & never tell how many factors should be put in model
Like CAPM, APT can be applied to measure fund performance & equity cost of capital

★ Implementation process: ① Identify factors, ② Estimate factor risk premia, ③ Estimate factor loadings (betas) of assets ($\beta_{i,K}$) — difficult to draw academic meaning
should be macro in nature as it affects all firms; risk premium on factor ($E[r_K] - r_F$) — yield curve → difficult to interprete in return

★ 3 approaches to construct APT factors: ① Statistic-based factors (use what's in mkt, derive factor from stock return — use principal components analysis or factor analysis)
② Macroeconomic-based factors e.g. GDP curve, credit-spread
③ Characteristic-based factors: • Char of firms: size and MTB, Char of stock return: momentum, etc.

↳ Fama and French (1996): three-factor model — FF run two-pass regressions to test three-factor

$E[r_i] - r_F = \beta_i\cdot(E[r_M] - r_F) + s_i\cdot E[SMB] + h_i\cdot E[HML]$ → Three factor model doesn't suffer from model miss specification (a = 0), all factors are statistically significant and R^2 are higher than SML

E[r] on port with long small stocks & short big stocks; E[r] on port with long high BTM stocks & short low BTM

★ Two-way sorting procedure (ไม่ใช่แค่ง่ายๆ)

Book-to-mkt ratio	small	medium	big
High	S/H	M/H	B/H
Medium	S/M	M/M	B/M
Low	S/L	M/L	B/L

▸ Construct SMB: 1. Sort stocks into 3 groups based on BTM from high to low
2. In each BTM-sorted group, Sort stocks into 3 group based on mkt cap small to big.
3. SMB_t = difference in returns on ports (S/H + S/M + S/L) and (B/H + B/M + B/L)
▸ Construct HML: Start by sorting stock based on size from small to big
↳ HML_t = difference in returns on ports (S/H + M/H + B/H) and (S/L + M/L + B/L)

★ From CAPM, we calculate β by → $(r_{i,t} - r_{F,t}) = \alpha_i + \beta_i(r_{M,t} - r_{F,t}) + s_i SMB_t + h_i HML_t$

EX1 PTT is large stock → ⊖ SMB

EX2 Amazon (growth stock) → correlated to $r_{B,t}$ rather than $r_{S,t}$ (go together)
↳ regression coefficient for SMB is ⊖, HML is ⊖ too so it

CAPM ignores this →
- undervalue big firm bcs too high cost of equity
- overvalue small firm bcs too low "