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2604-639

Finance Theories

Probability distribution

↳ PDF

↳ CDF

Topic 3:

Stochastic Dominance

↳ use to derive asset pricing model in the past

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Agenda

1 The Concept of an Efficient Criterion

FSD

2 First Order Stochastic Dominance

SSD

3 Second Order Stochastic Dominance

4 Conclusion

1. THE CONCEPT OF AN EFFICIENT CRITERION

1.1 Introduction

1.2 The Concept of an Efficient Criterion

1.3 Probability Density and Cumulative Distribution Functions

1.1 Introduction

- The last lecture has developed cardinal utility functions (i.e., functions generating cardinal numbers that can be used as utility indexes).
- We have seen that the (maximizing) expected utility framework can be applied to explain how individuals derive the price of a risky investment.

  to make optimal decision

- In this lecture, we will employ the expected utility framework to derive investment criteria, called stochastic dominance, which can be used to classify alternative risky investments into efficient and inefficient sets.

1.1 Introduction

two stocks to choose

$r_F = 10\%$ $r_G = 20\%$

\$100 $\$W_0$ F vs. G

utility depends on return

$U(W_F) = U(W_0(1+r_F))$

$U(W_G) = U(W_0(1+r_G))$

$U(W)$

- In investment analysis, we usually write utility as a function of "the rate of return" from an investment, rather than as a function of "wealth" or \$-return.

utility of return

$$\text{Utility} = U(r)$$

W

- At time 0, an investor invests \$200 on a stock. Three months later, he earns \$3 dividend and sells the stock for \$205.
  - \$-return = (205+3)–200 = \$8 over 3 months
  - %-return = [(205+3)–200]/200 = (208/200)–1 = 0.04 or 4% per 3 months or 16% pa (simple compounding)

1.1 Introduction

- We assume that when choosing among risky investment alternatives, the objective function of an investor is to maximize his/her expected utility which, in turn, is derived from risky returns.

unknown

expected utility that derive from return

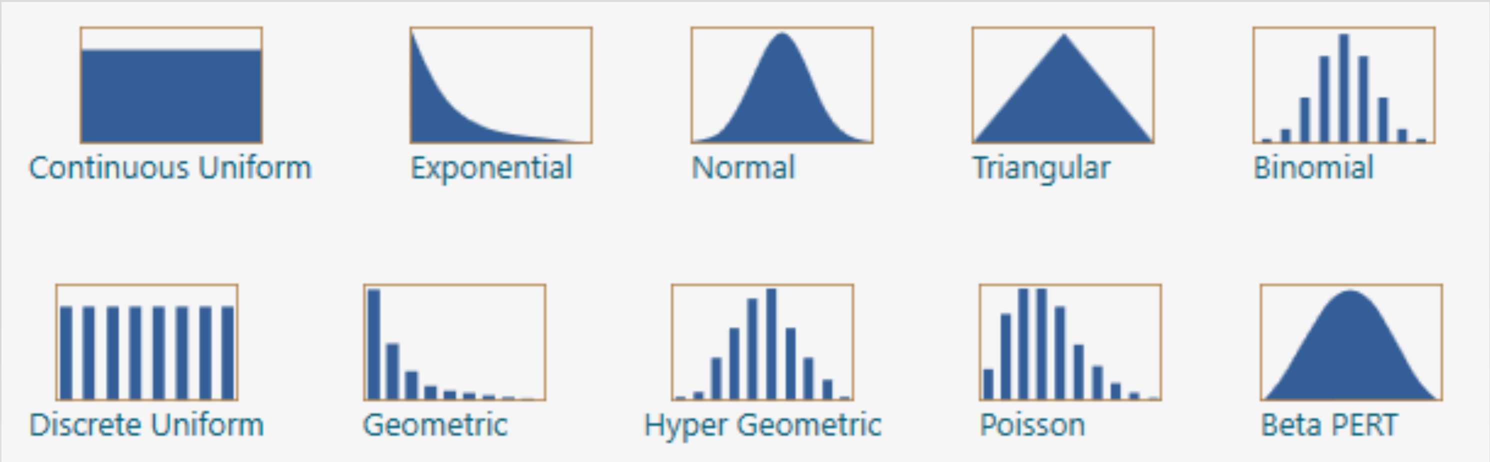
$$\text{Maximize } E[U(r)]$$

- Investment return is a random variable ($\tilde{r}$). It is subject to probability distribution.

we don't know return for sure

1.1 Introduction

- The most general form of stochastic dominance can be applied to any forms of probability distribution of asset return and whether the distribution is discrete or continuous in variable.
- It makes use of all points/moments from the probability distribution of returns (i.e., all information from the probability distribution is used). In addition, only very general forms of investors' utility functions are assumed.



modern model

mean variance criterian

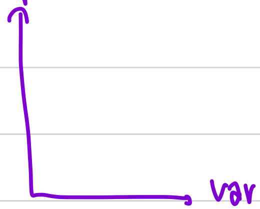
MPT → Markowitz → MVC

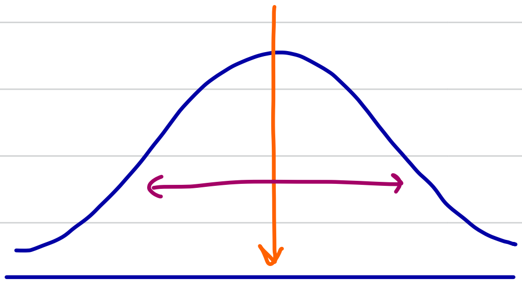
↓can be applied only when

$\tilde{r} \sim$ Normally

expected return

use only 2 moments





Stochastic dominance

↓use all information unlike Markowitz

μ_1 = mean 1st moment

μ_2 = var

μ_3 = skewness

μ_4 = kurtosis

1.2 The Concept of an Efficient Criterion

- Making an investment decision among risky alternatives can be divided into three steps.

  [1] Take all possible investment alternatives into consideration and form the investment feasibility set.

  - This step requires market information on financial assets.

  [2] Classify the feasible investment set into 2 subsets; the efficient investment and inefficient investment subsets.

  - Efficient investments can be identified by using an efficient criterion suitable to a given class of investors.
  - This step requires general descriptions of the investor's preference/taste toward risk and return.

work for all risk averse investors

1.2 The Concept of an Efficient Criterion

[3] The subject selects the optimal investment (the investment option that maximizes his/her expected utility) from the efficient investment set.

- This step requires a detailed descriptions of the investor’s preference/taste toward risk and return.

- An efficient criterion is a decision rule for dividing all potential investment options into two mutually exclusive sets: efficient and inefficient sets.

1.2 The Concept of an Efficient Criterion

- An efficient criterion must be consistent with the expected utility framework.
- That is, given two risky investment alternatives, F and G, if an investment criterion concludes that F is preferred to G, then it must be the case that $E[U(r_F)]$ is higher than $E[U(r_G)]$.

conclusion

$$F \succ G \quad \text{if and only if} \quad E[U(r_F)] > E[U(r_G)]$$

- That is, the criterion used to conclude that $F \succ G$ is both sufficient and necessary condition for $E[U(r_F)] > E[U(r_G)]$.

1.3 Probability Density and Cumulative Distribution

- Definitions

current wealth

- W_0 is the principal invested in a risky asset at time 0.

$W_0(1+r_i)$

- W_1 is the end-of-the-period investment outcome.
- If security or portfolio i is selected, then $W_1 = W_0(1+r_i)$ where r_i is the rate of return on security i.
  - Note: W_0 is a constant while r_i is a random variable.
- When choosing between two investment alternatives; F and G, the investor compares between $W_0(1+r_F)$ and $W_0(1+r_G)$.

compare return from 2 stocks

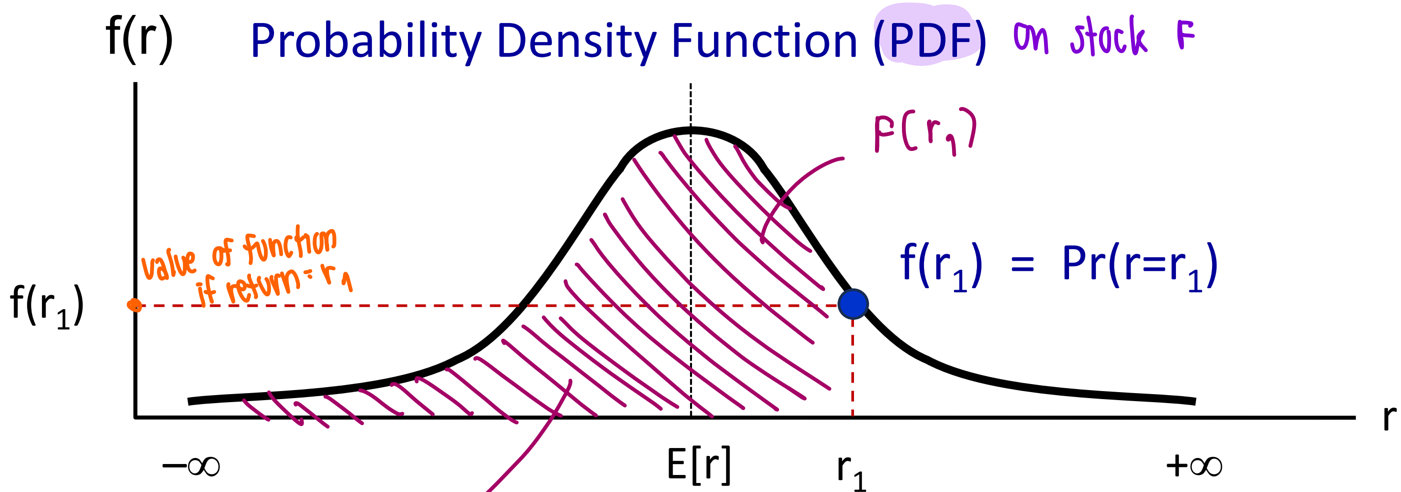
1.3 Probability Density and Cumulative Distribution

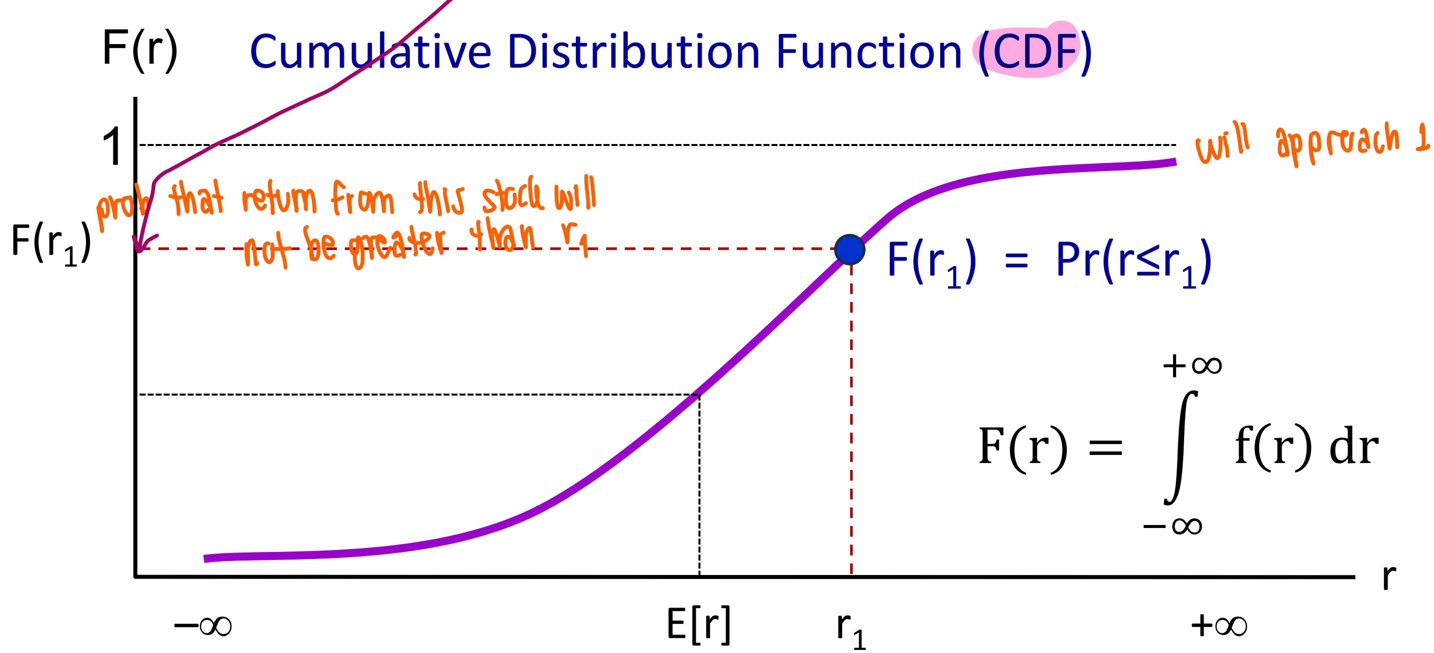
- Since W_0 is the same, the choice becomes r_F vs. r_G. That means, the individual is comparing between;

$$E[U(r_F)] \quad \text{and} \quad E[U(r_G)]$$

- Let
  - $f(r)$ and $g(r)$ = probability density functions, or PDF, of returns on assets F and G, respectively.
  - $F(r)$ and $G(r)$ = corresponding cumulative distribution functions, or CDF.

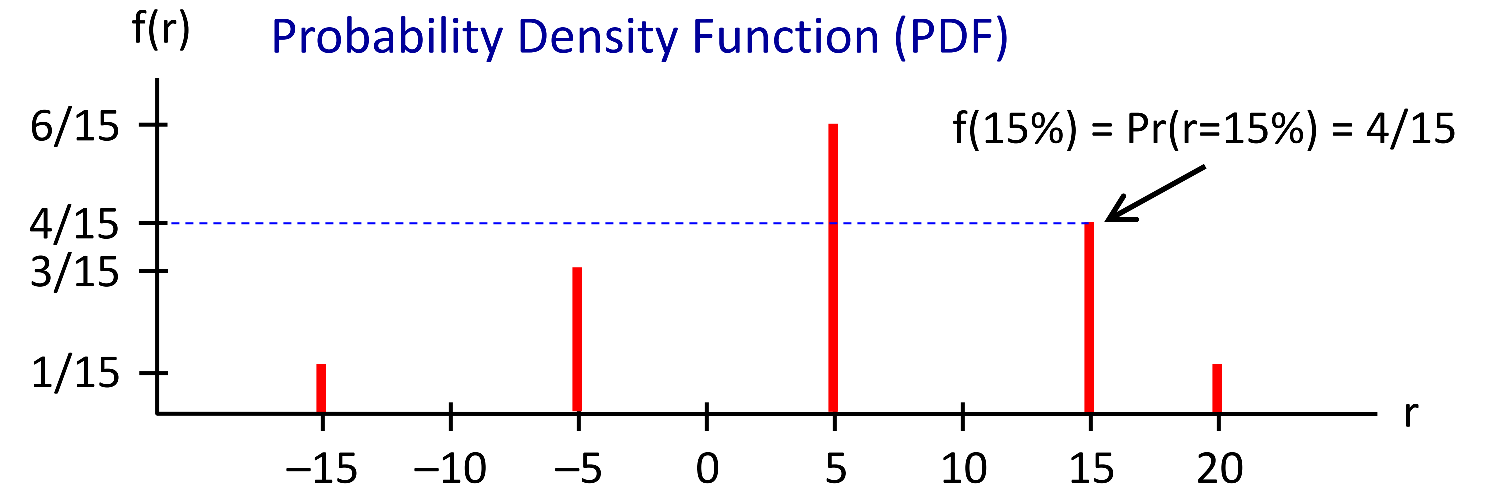
PDF and Corresponding CDF: Continuous Distribution

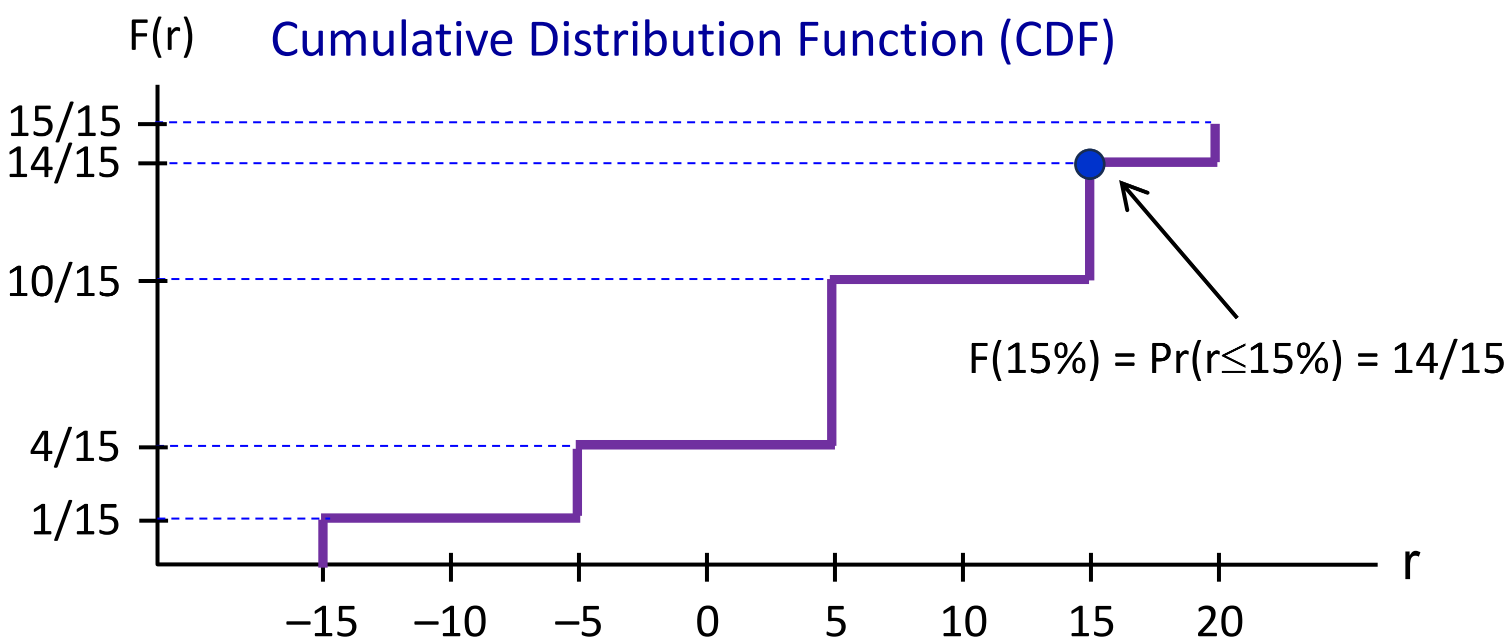




$$F(r) = \int_{-\infty}^{+\infty} f(r)\, dr$$

PDF and Corresponding CDF: Discrete Distribution





1.3 Probability Density and Cumulative Distribution

- Note:

1) $F(r) = \int f(r)\,dr$ and $G(r) = \int g(r)\,dr$

2) $dF(r)/_{dr} = f(r)$ and $dG(r)/_{dr} = g(r)$

3) $E[r_F] = \int r\,f(r)\,dr = \int r\,dF(r)$

$E[r_G] = \int r\,g(r)\,dr = \int r\,dG(r)$

4) $E[U(r_F)] = \int U(r)\,f(r)\,dr = \int U(r)\,dF(r)$

$E[U(r_G)] = \int U(r)\,g(r)\,dr = \int U(r)\,dG(r)$

2. FIRST ORDER STOCHASTIC DOMINANCE

2.1 The FSD Rule

2.2 Interpretation of the FSD

2.3 The FSD and Distribution of Returns

2.1 The FSD Ruls

can be apply to any investors with different risk preference

investors prefer more to less

- **Assumption:** Investors have utilities that are non-decreasing with respect to returns (U'(r) > 0).

- **FSD Rule:** A risky investment F will be preferred over an investment G by the FSD if and only if (iff);

F ≻ G

at all level of return

$$F(r) \le G(r), \forall r$$

cumulative

zero at some point

positive

$G(r) - F(r) \geq 0 \rightarrow F \succ G$

$G(r) - F(r) \leq 0 \rightarrow G \succ F$

and

for some level of return

$$F(r) < G(r), \exists r$$

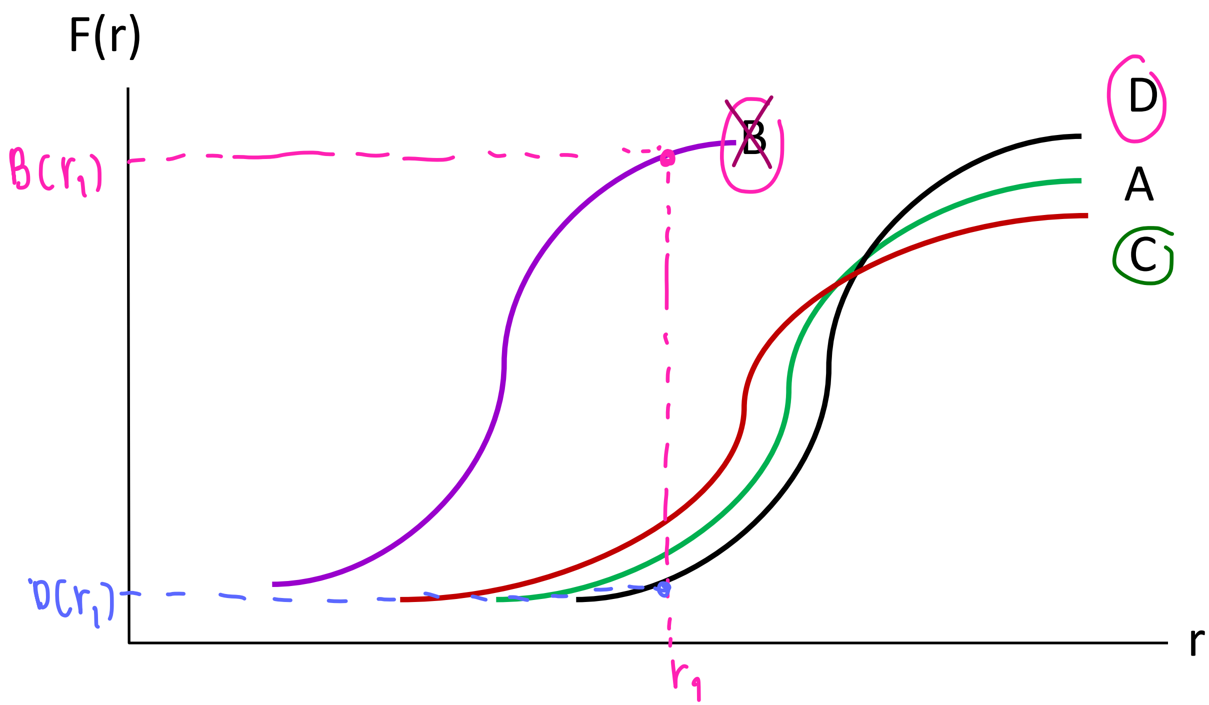
to avoid comparing same distribution

  - That is, F ≻ G when the cumulative distribution of F lies on the right-hand side (RHS) of the cumulative distribution of G. In other words, $G(r) - F(r) \geq 0$.

[1st] Find $B(r_1)$ & $D(r_1)$

whole curve is on left side of D

EX 1: CDF of 4 Securities



$D(r_1)$ vs $C(r_1)$

cannot make conclusion because they cross each other

According to the FSD;

Efficient set: A, C and D

Inefficient set: B

Note: The cumulative distribution of an inefficient investment lies to the LHS of an efficient investment.

EX 2: Comparing 2 Investments

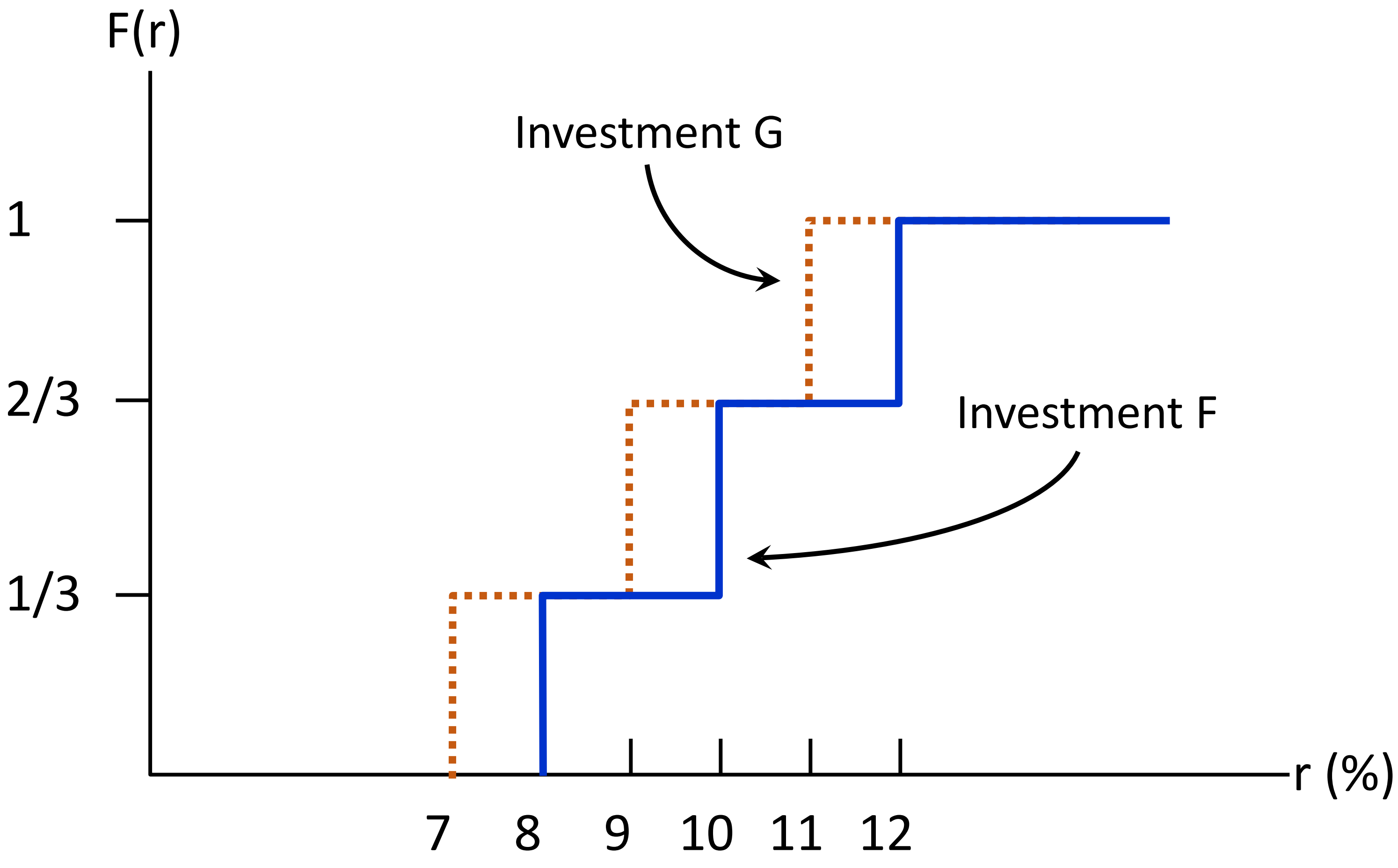
Question

Security F (given PDF)		Security G	
Outcome (%)	Prob.	Outcome (%)	Prob.
12	1/3	11	1/3
10	1/3	9	1/3
8	1/3	7	1/3

เรียง

r (%)	F(r)	G(r)	G(r) – F(r)
7	0	1/3	1/3
8	1/3	ไม่มี 1/3	0
9	ไม่มี 1/3	มี + 1/3 2/3	1/3
10	2/3	2/3	0
11	2/3	1	1/3
12	1	1	0

EX 2: Comparing 2 Investments (cont.)



As $G(r) \geq F(r)$ for $\forall r$ and $G(r) > F(r)$ for $\exists r$, it is said that investment F stochastically dominates investment G. Note that FSD allows the cumulative distribution to touch, but not cross, each other.

prob that A gives -10 → $\frac{2}{8}$

N: 8 stocks → $\frac{8^2 - 8}{2} = 28$

	A	B	C	D
A	x			
B		x		
C			x	
D				x

$\frac{4^2 - 4}{2} = 6$

don't have to double compare

EX 3: Historical Returns of 8 Securities

annual return from each stock

	A	B	C	D	E	F	G	H
2000	-10	-30	60	45	-40	-70	50	10
2001	10	-10	70	45	-40	-20	70	40
2002	10	-30	70	20	-80	-20	70	40
2003	10	-10	70	90	-60	-70	60	60
2004	10	30	70	20	-40	-20	70	60
2005	-10	-30	70	45	-30	40	70	60
2006	10	30	60	90	20	-20	80	60
2007	10	-30	70	45	-30	-70	70	60
E[r]	**5.00**	**-10.00**	**67.50**	**50.00**	**-37.50**	**-31.25**	**67.50**	**48.75**
SD	**9.26**	**26.19**	**4.63**	**26.99**	**28.66**	**37.96**	**8.86**	**18.08**

located on right
$C(r) - A(r) \leq 0 \Rightarrow \therefore C \succsim A$

EX 3: Cumulative Distribution of 8 Securities (cont.)

	A	B	C	D	E	F	G	H
-80	0	0	0	0	1/8	0	0	0
-70	0	0	0	0	1/8	3/8	0	0
-60	0	0	0	0	2/8	3/8	0	0
-50	0	0	0	0	5/8	3/8	0	0
-40	0	0	0	0	7/8	3/8	0	0
-30	0	4/8	0	0	7/8	3/8	0	0
-20	0	4/8	0	0	7/8	7/8	0	0
-10	2/8	6/8	0 −2/8	0	7/8	7/8	0	0
0	2/8	6/8	0 −2/8	0	7/8	7/8	0	0
10	1	6/8	0 −1	0	7/8	7/8	0	1/8
20	1	6/8	0 −1	2/8	1	7/8	0	1/8
30	1	1	0 −1	2/8	1	7/8	0	1/8
40	1	1	0 −1	2/8	1	1	0	3/8
50	1	1	0 −1	6/8	1	1	1/8	3/8
60	1	1	2/8 −6/8	6/8	1	1	2/8	1
70	1	1	1 0	6/8	1	1	7/8	1
80	1	1	1 0	6/8	1	1	1	1
90	1	1	1 0	1	1	1	1	1

EX 3: Identifying the Efficient Investment Set (cont.)

- Cumulative distributions of each pair of securities are compared using the FSD rule.
  - C $\succ$ A. Hence, A is inefficient.
  - C $\succ$ B. Hence, B is inefficient.
  - The final efficient set consists of C, D and G
- If the exact utility function is known, the optimal investment can be identified. (r is in percentage form, i.e., 50% = 50).

EX 1: $U(r) = 100 + 100 \cdot \ln(r)$

$E[U(r_C)] = 521.0$

$E[U(r_D)] = 477.7$

$E[U(r_G)] = 502.4$

$\boxed{C}$

prob	return	$U(r_c)$
0.25	60	$100+100 \cdot \ln(60)$
0.75	70	$200+100 \cdot \ln(70)$

$0.25(509.43) + 0.75(524.48)$

$= 521.0$

EX 1: $U(r) = 10 + (1/100)\, r^2$

$E[U(r_C)] = 55.8$

$E[U(r_D)] = 41.4$

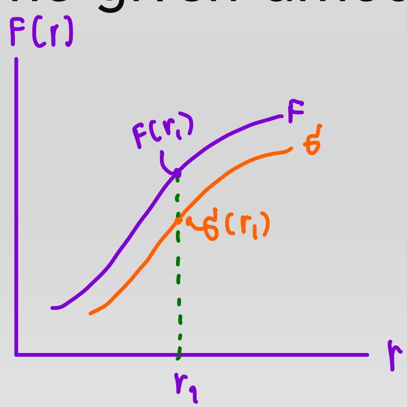
$E[U(r_G)] = 56.3$

2.2 Interpretation of the FSD

- $F \succ G$ iff $F(r) \leq G(r)$ with at least one level of r the strong inequality holds.
- This is equivalent to requiring that the probability of receiving a return lower than some given amount, r_1, will always be smaller for F than for G.

$$F(r) \leq G(r) \leftrightarrow Pr_F(r \leq r_1) \leq Pr_G(r \leq r_1)$$

- In other words, the probability of receiving a return higher than some given amount, r_1, will always be higher for F than for G.



$1-G(r_1) \Rightarrow$ chance that G give return $> r_1$

$1-F(r_1) \Rightarrow$ " —— F " ——

$\Downarrow$

$\therefore G \succ F$

2.2 Interpretation of the FSD

- To prove that the FSD rule leads to the optimal decision, one must prove that the FSD rule ($F(r) < G(r)$ for $\forall r$) is both sufficient and necessary conditions for $E[U(r_F)] > E[U(r_G)]$ (i.e., $F \succ G$).
    - To prove that FSD rule is sufficient (**if**) to conclude that $F \succ G$, one must show that "$F(r) < G(r)$ for $\forall r$" implies $E[U(r_F)] > E[U(r_G)]$.
    - To prove that FSD rule is necessary (**only if**) to conclude that $F \succ G$, one must show that $E[U(r_F)] > E[U(r_G)]$ implies "$F(r) < G(r)$ for $\forall r$".

2.3 The FSD and Distribution of Returns

- It can be shown that

mean value of F   mean value of G

$$G(r) \geq F(r) \text{ implies } E[r_F] \geq E[r_G]$$

- Therefore, a necessary condition (but not sufficient) for investment F to be preferred to G under the FSD is that the expected return of F be no less than that of G.

- However, a similar systematic relationship cannot be established between variance of the returns.

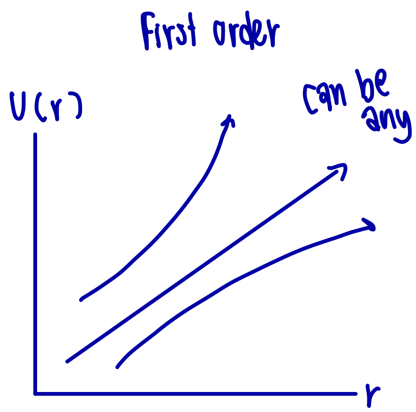
because there's no assumption regarding risk aversion

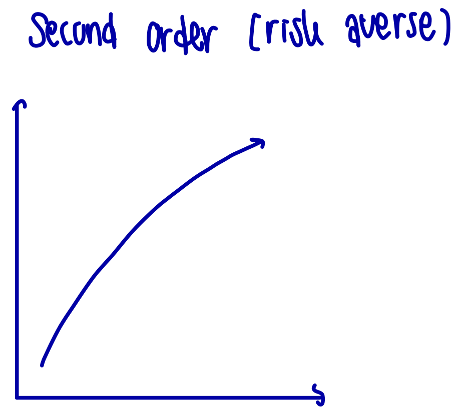
FSD → has low power to differentiate btw efficient & inefficient

3. SECOND ORDER STOCHASTIC DOMINANCE

3.1 The SSD Rule

3.2 The SSD and Distribution of Returns





3.1 The SSD Rule

- **Assumption:** Investors have utilities that are non-decreasing with respect to returns (U'(r) > 0) and are risk *averse (U''(r) < 0).

F ≻ G

- **SSD Rule:** A risky investment F will be preferred over an investment G by the ~~F~~SSD if and only if (iff);

accumulative of difference btw cumulative prob of G & F always non-zero for all returns

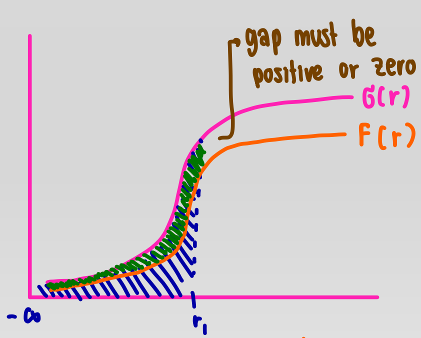
$$\boxed{\int (G(r) - F(r))}\, dr \geq 0, \forall r$$

and

strong inequility for some return

$$\int (G(r) - F(r))\, dr > 0, \exists r$$

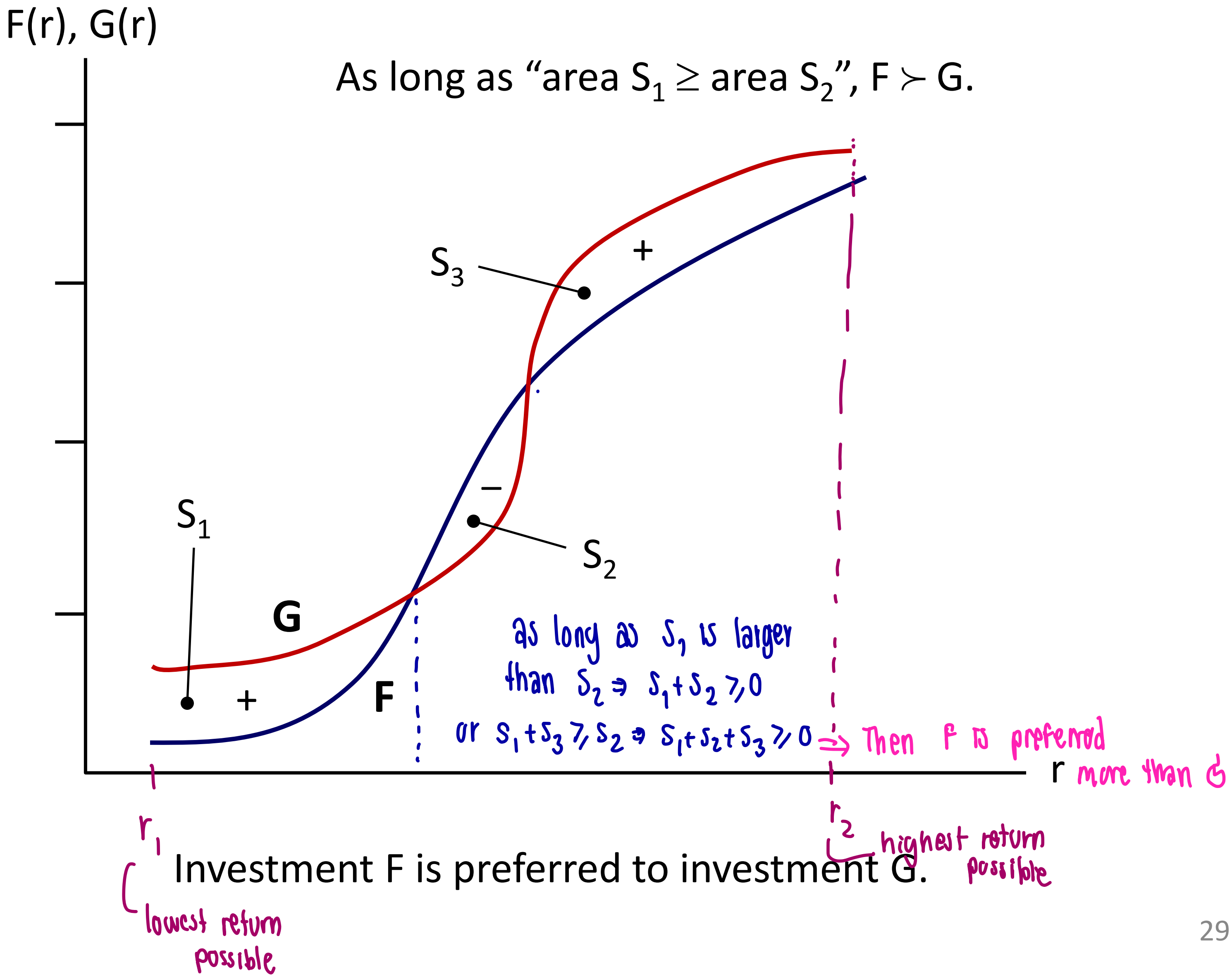
$$\left[\int G(r)\,dr - \int F(r)\,dr\right] \geq 0$$



$$\int_{-\infty}^{r_1} G(r)\,dr - \int_{-\infty}^{r_1} F(r)\,dr$$

SSD and Cumulative Distribution

As long as "area $S_1 \geq$ area S_2", $F \succ G$.

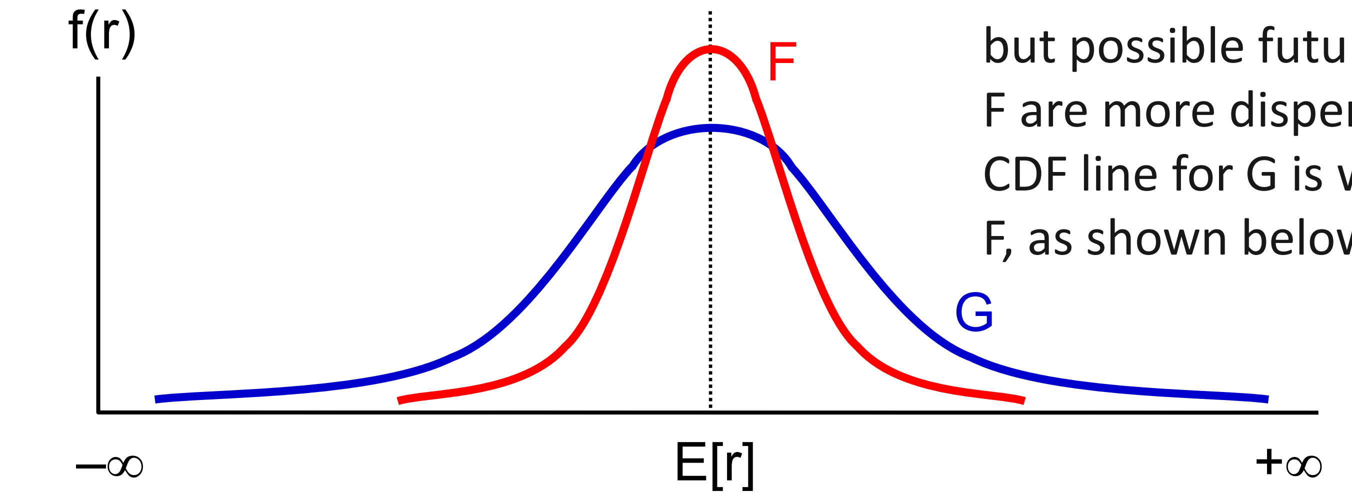


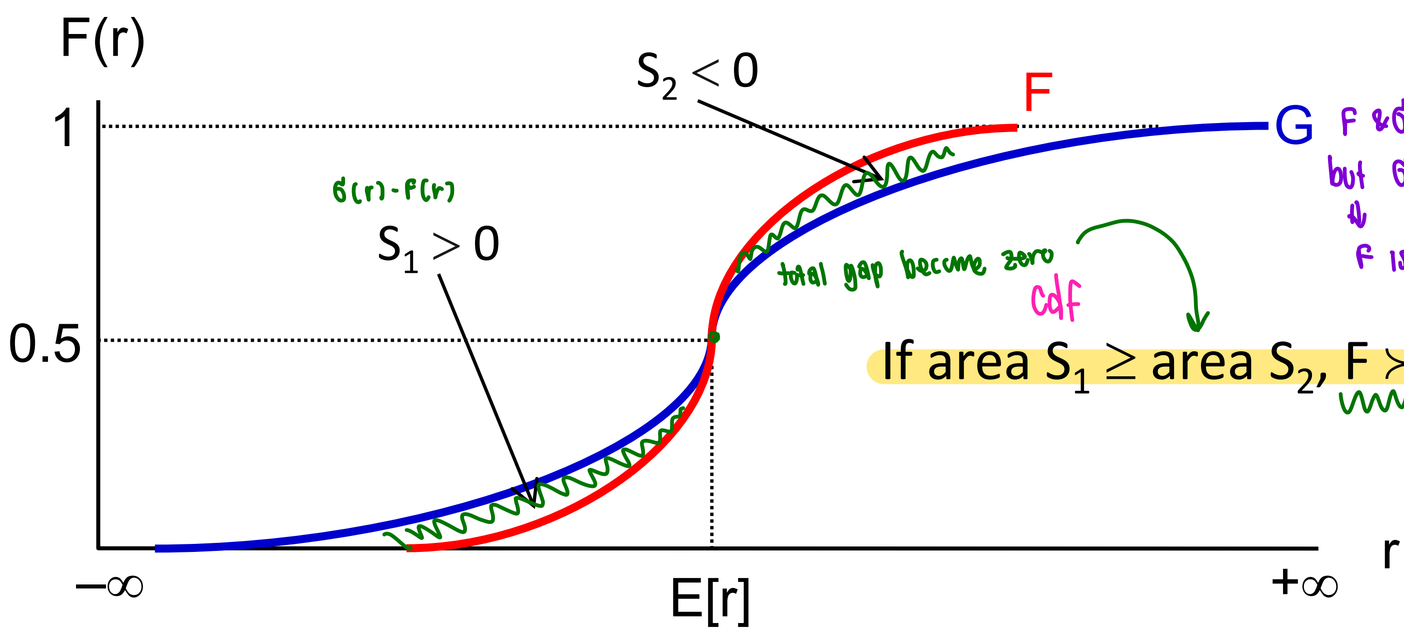
Investment F is preferred to investment G.

SSD: E[r] and Risk

pdf

Assets F and G have the same E[r], but possible future returns on asset F are more disperse than F. Thus, CDF line for G is wider (longer) than F, as shown below.





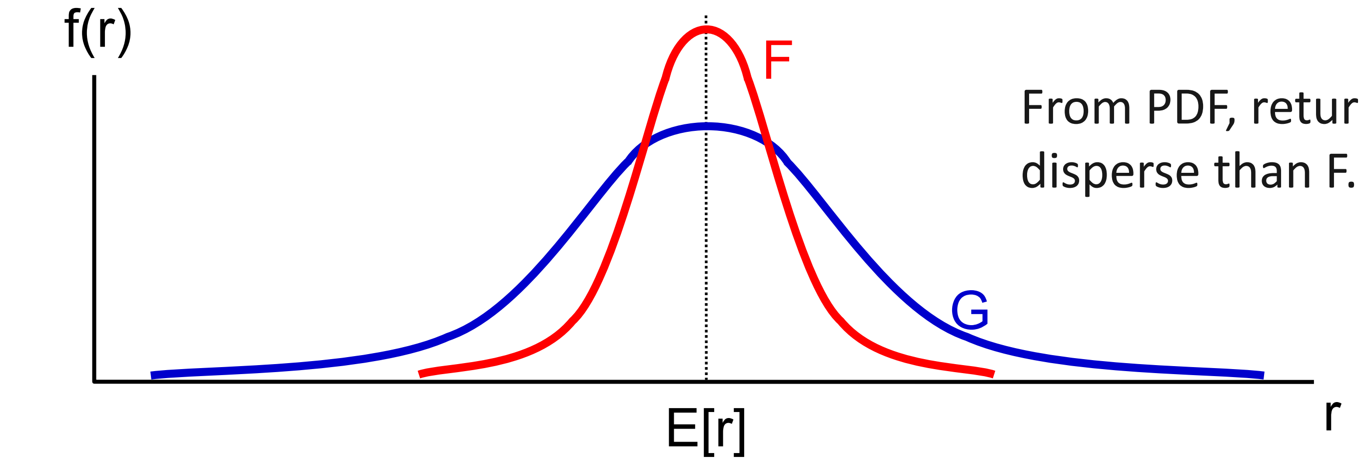
F & G has same expected return but G is riskier.

⇓

F is more preferred for risk averse investors

If area $S_1 \geq$ area S_2, $F \succ G$ under SSD.

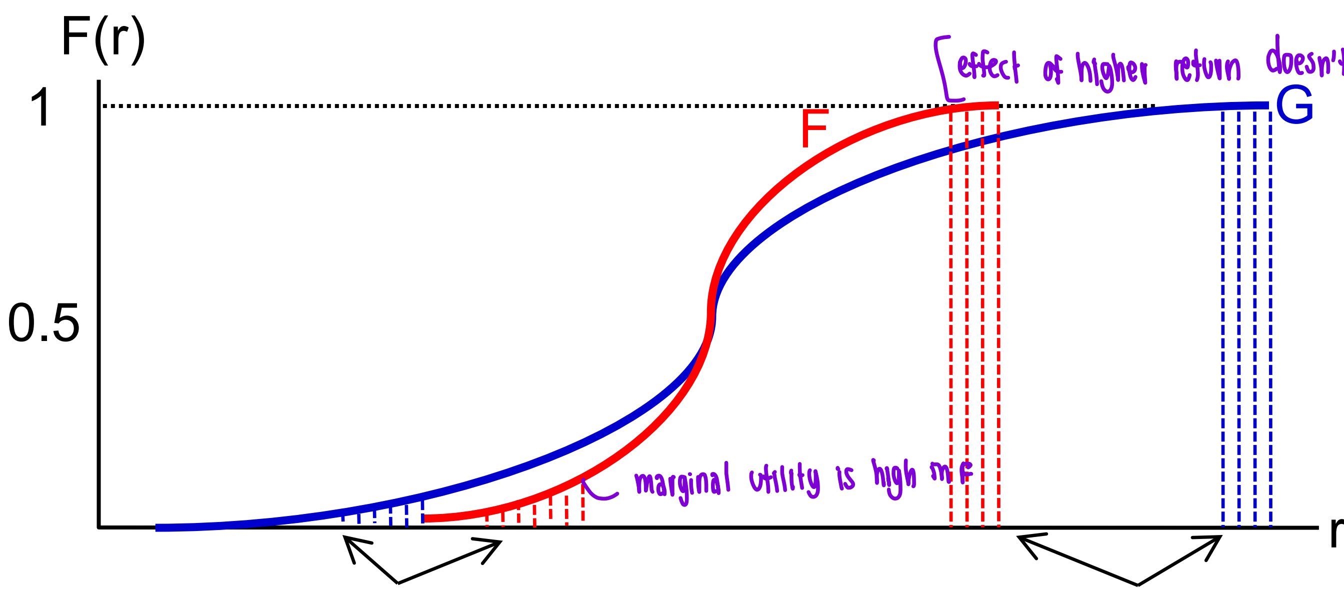
Area under CDF and Dispersion of Returns



From PDF, returns on G is more disperse than F.

even G gives chance to earn higher return but:

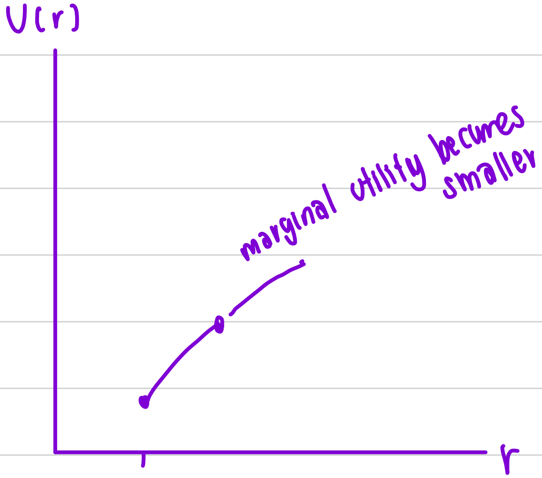
[effect of higher return doesn't effect total utility much due to diminishing marginal utility]



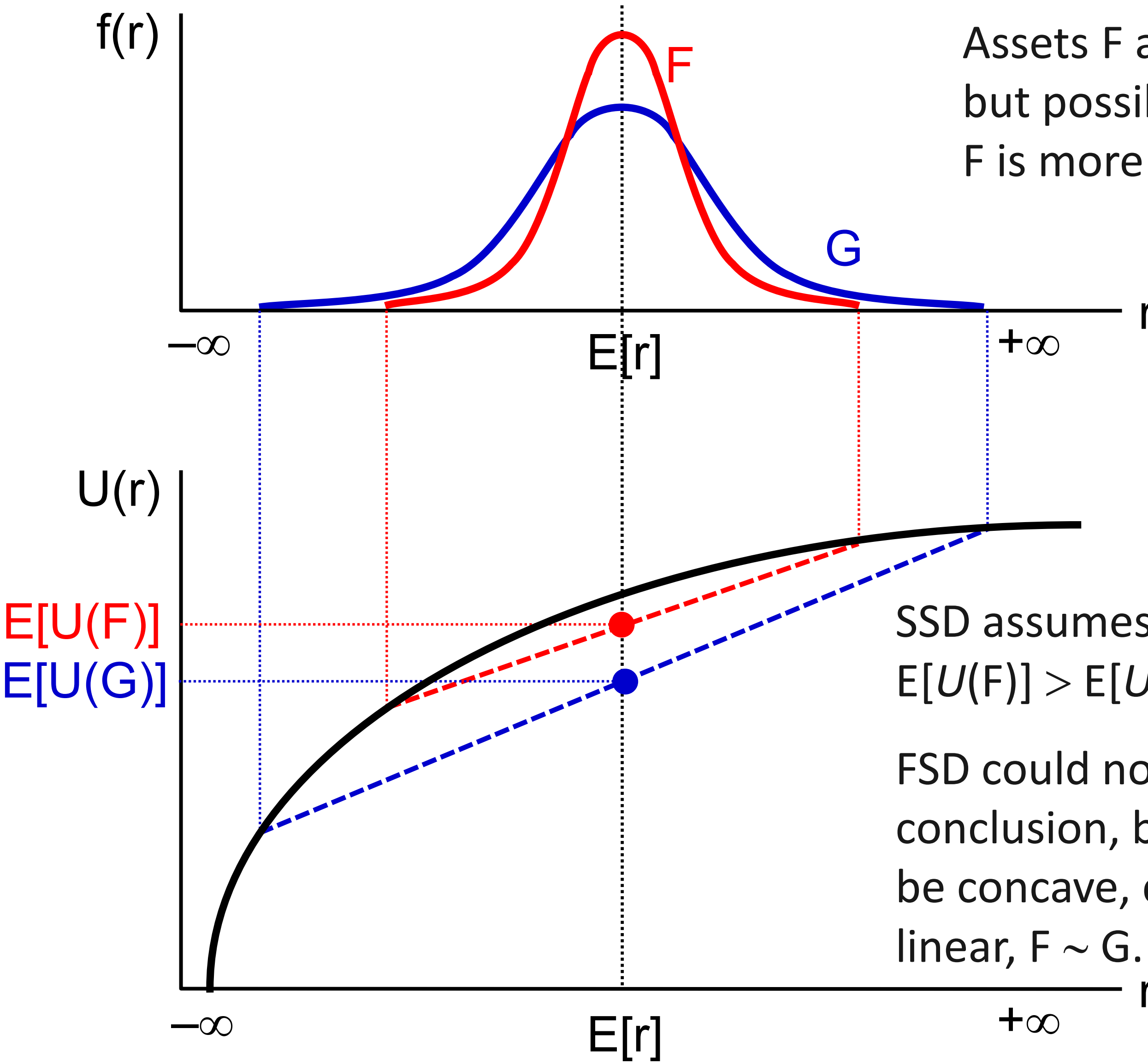
Area under CDF of G is larger than F implying that returns on G is more disperse than F.

Higher returns give higher *U*(r)

With concave U(r), MU becomes smaller as r becomes higher.


U(r)
marginal utility becomes smaller
r

SSD and Utility Function



Assets F and G have the same E[r], but possible future return on asset F is more disperse than F.

SSD assumes risk aversion. Thus, E[U(F)] > E[U(G)] or F $\succ$ G.

FSD could not reach the same conclusion, because it allows U(r) to be concave, convex or linear. If U(r) is linear, F $\sim$ G.

EX 4: Possible Outcomes of 2 Investments

PDF → CDF

A		B	
Outcome	Prob.	Outcome	Prob.
6	1/4	5	1/4
8	1/4	9	1/4
10	1/4	10	1/4
12	1/4	12	1/4

EX 4: Cumulative Distribution of 2 investments (cont.)

accumulate

	Cumulative Distribution		Sum of Cumulative Distribution		either zero or positive
Outcome	A	B	$\int A(r)dr$ A	B	B – A
4	0	0	0	0	0
5	0	1/4	0	1/4	1/4
6	1/4	1/4	1/4	2/4	1/4
7	1/4	1/4	2/4	3/4	1/4
8	2/4	1/4	1	1	0
9	2/4	2/4	1 1/2	1 1/2	0
10	3/4	3/4	2 1/4	2 1/4	0
11	3/4	3/4	3	3	0
12	1	1	4	4	0

$\Downarrow$ comply with SSD $\Downarrow$ $A \succ B$

Note: With discrete distribution, the result is sensitive to the size of incremental return that we assume. For example, if returns are assumed to be 5, 6, 8, 9, 10 or 12% rather than 1% incremental, results could be different.

If we have N stocks = 600

$$\frac{N^2 - N}{2} = \frac{600^2 - 600}{2}$$ comparison

→ not efficient

Stock A & B have infinite combination
portfolio

$\boxed{SD}$

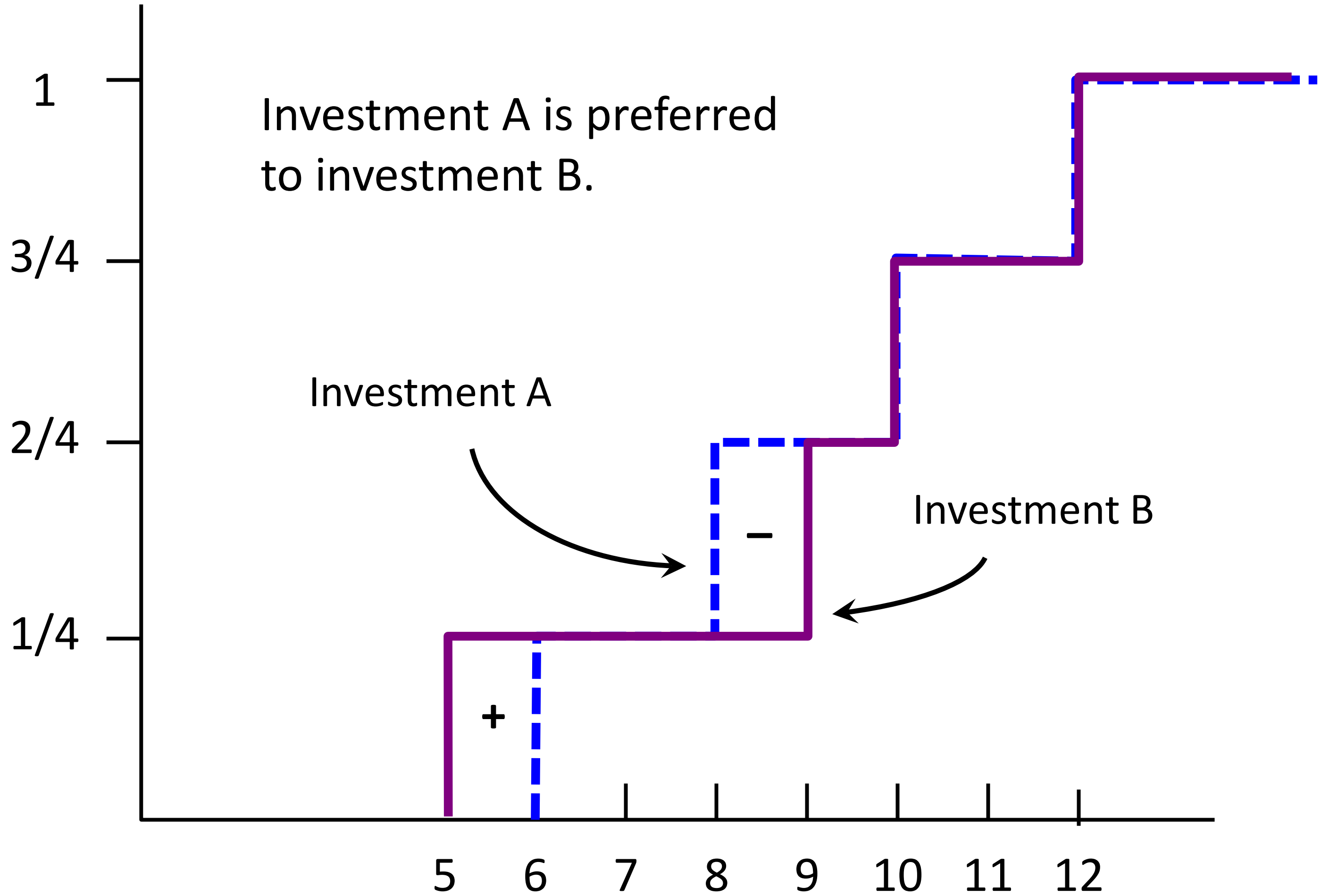
Advantages

- We don't have to set many assumptions for utilities
- general

Drawbacks

- we have to compare pair by pair
- pair of securities to be compaired are infinite

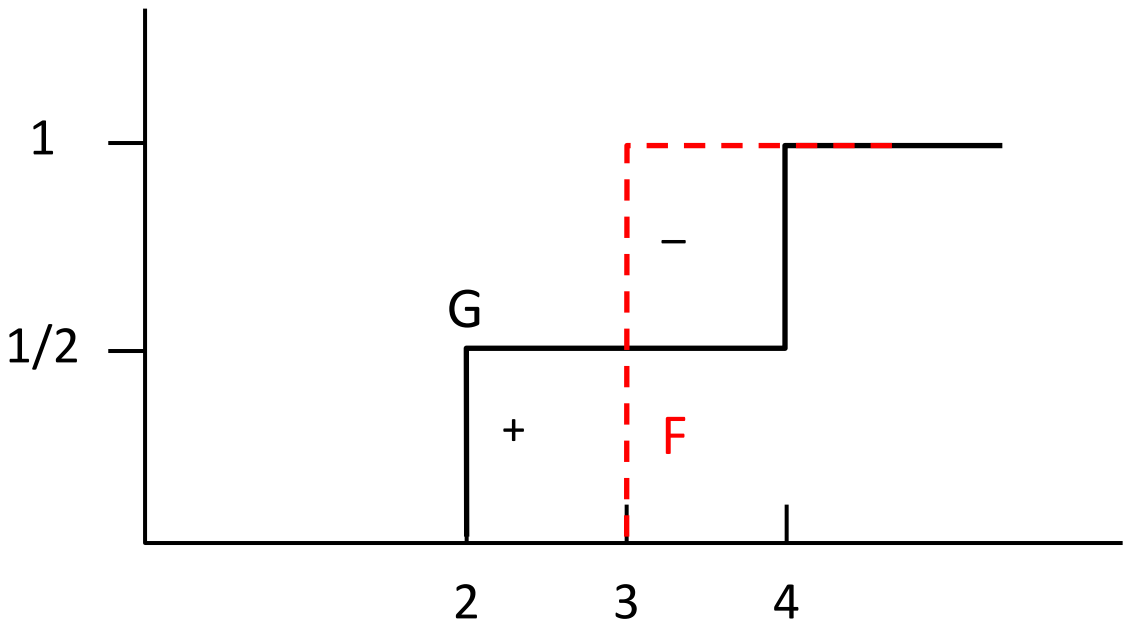
EX 4: Cumulative Distribution of 2 investments (cont.)



Although the two areas (with + and – signs) are of the same size, A $\succ$ B because of the diminishing marginal utility.

EX 5: Comparing 2 Investment Alternatives

G		F	
Outcome	Prob.	Outcome	Prob.
2	1/2	3	1
4	1/2		



Under SSD, $F \succ G$.

Hence, given the same E[r], the SSD will choose an investment with lower Var[r].

3.2 The SSD and Distribution of Returns

- It can be shown that a necessary condition (but not sufficient) for investment F to be preferred to G under the SSD is that the expected return of F be no less than that of G

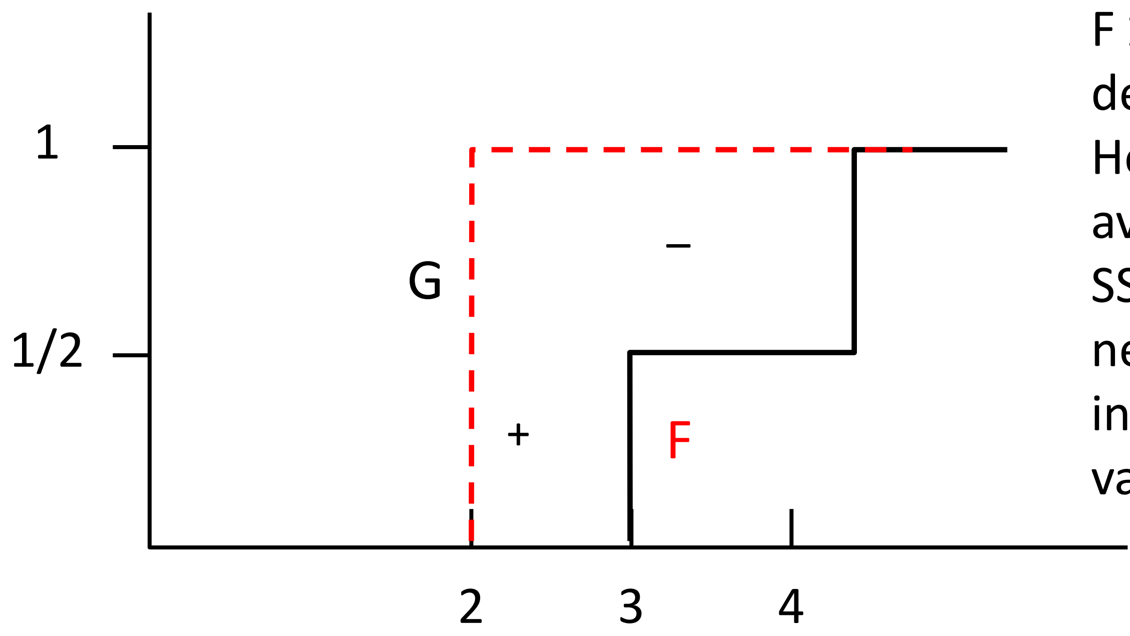
$$\int (G(r)-F(r)\ dr \geq 0 \text{ implies } E[r_F] \geq E[r_G]$$

- Thus, similar to FSD, the mean return plays a critical role in determining the preference order.

- However, despite the underlying assumption of risk aversion, a similar systematic relationship cannot be established between variance of the returns.

EX 6: Comparing 2 Investment Alternatives

G		F	
Outcome	Prob.	Outcome	Prob.
3	1/2	2	1
4	1/2		



$F \succ G$ under the SSD, despite $Var[r_F] > Var[r_G]$. Hence, despite the risk aversion assumption, the SSD efficient set does not necessarily exclude investment having higher variance.

3.2 The SSD and Distribution of Returns

- Given two investment alternatives, F and G, whose expected returns are equal and return distributions are symmetric, the investment with less variance will be preferred to the one with higher variance under the SSD rule.
- For a given universe of investment alternatives, the SSD efficient set is a subset of the FSD efficient set.

4. CONCLUSION

4.1 Conclusion

4.1 Conclusion

- Stochastic dominance is an important result. It is properly founded on the ground of expected utility maximization.
- It takes into consideration every point in the probability distribution of asset returns. It does not ignore certain moments of the distribution.
- The FSD only assumes non-decreasing utility functions while SSD only assumes further that investors are risk averse.
- If we further make an additional assumption regarding the degree of risk aversion, we can derive the Third Order Stochastic Dominance, TSD.

4.1 Conclusion

- Drawbacks of the stochastic dominance approach.
  - It consider the whole distribution of each investment alternative.
  - With two securities, there could be infinite number of investment alternatives. Hence, applying the stochastic dominance becomes very complicated.

Exercises

1. What is the relationship (if any) of the FSD and SSD rules to investment's expected return? What is their relationship to the variance of the distribution of returns?

2. Consider the following five investment options:

A		B		C		D		E	
Pr.	**r**	**Pr.**	**r**	**Pr.**	**r**	**Pr.**	**r**	**Pr.**	**r**
1/2	10	1/3	10	1/2	5	1/2	12	1/4	6
1/2	20	1/3	15	1/4	6	1/2	20	3/4	40
		1/3	30	1/4	8				

- Which options comprise the FSD efficient set?
- Which options comprise the SSD efficient set?

3. The FSD rule is better than SSD criterion since it does not require strong assumptions regarding investors' preferences." Appraise.

Exercises

4. Consider the following two investment options:

F		G	
Pr.	**r**	**Pr.**	**r**
1/4	1	3/16	1/2
1/4	2	3/16	1 ½
1/4	9	4/16	2 ½
1/4	10	3/16	3 ½
		3/16	4 ½

- Which options comprise the FSD efficient set?
- Which options comprise the SSD efficient set?