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SD is part of this

Efficiency criterian

# 2604-639

# Finance Theories

## Topic 4:

mean-variance criterian

## The MV Analysis

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# Agenda

1 Introduction

2 The Mean-Variance Criterion (MVC)

- prefer more to less
- risk aversion
- quadratic utility function /asset returns must be have normal distribution

3 Portfolio Diversification

4 The Efficient Portfolios

# 1. INTRODUCTION

## 1.1 Moments of Return Distribution and the Expected Utility

# 1.1 Introduction

- When making an investment decision on risky assets, an investor confronts with the probability distribution of returns on each investment alternative.
- Let $r$ = investment return (a random variable)

  $(1+r)$ = terminal wealth (a random variable) (scaling initial investment to \$1)

  $U(1+r)$ = utility derived from investment return
- Although the exact form of $U(\cdot)$ is not known, we can apply the Taylor's series expansion around the fixed point "$1+E[r]$".

  used to derive degree of risk aversion

# 1.1 Introduction

What drive expected utility from risky return

$$E\left[U(1+r)\right] = U(1+E[r]) + \frac{U'(1+E[r])}{1!}(1+r-(1+E[r])) \quad = 0$$

$$+ \frac{U''(1+E[r])}{2!}(1+r-(1+E[r]))^2 \quad \rightarrow \sigma^2$$

$$+ \frac{U'''(1+E[r])}{3!}(1+r-(1+E[r]))^3 \quad \mu_3$$

$$+ \frac{U^4(1+E[r])}{4!}(1+r-(1+E[r]))^4 \quad \mu_4$$

$$+ \ldots$$

- Take expectation on both sides of the equation.

# 1.1 Introduction

all moments of return distribution affect expected utility

always positive

$$E[U(1+r)] = U(1+E[r]) + \left(\frac{U''(1+E[r])}{2!}\right)\sigma^2$$

$$+ \frac{U'''(1+E[r])}{3!} \cdot \mu^3$$

$$+ \frac{U^4(1+E[r])}{4!} \cdot \mu^4 + \ldots$$

in SD we make use all points on distribution

↳ use all moments

If we ignore $\mu_3$ & $\mu_4$ insignificant effect on utility

⇓

only mean & variance of return distribution matter

- $\sigma^2$ is the variance (second moment) of the distribution of return defined as $E[(r\text{–}E[r])^2]$.

$E[r] \xrightarrow{+} E[U(1+r)]$

assume more prefer to less

$\sigma_r^2 \xrightarrow{-} E[U(1+r)]$

$U''(r) < 0$

assume risk aversion

- $\mu^3$ and $\mu^4$ are skewness (3rd moment) and kurtosis (4th moment) of the distribution of return defined as $E[(r\text{–}E[r])^3]$ and $E[(r\text{–}E[r])^4]$, respectively.

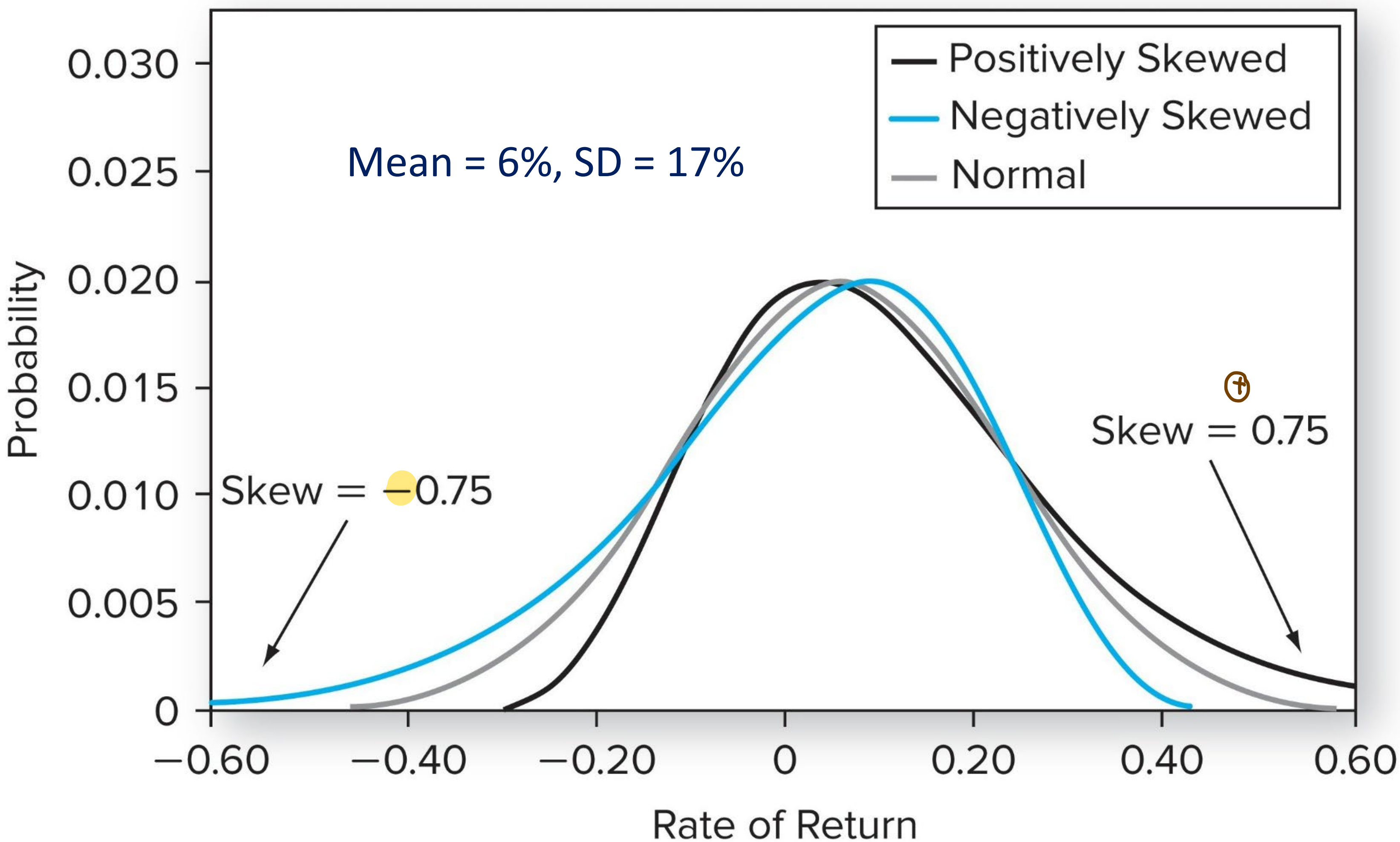
# 1.1 Introduction

- Although E[U(r)] depends on all the moments of the probability distribution of returns, many studies find that only the first few moments have significant effects on E[U(r)] of an average investor.
- It is also generally found that
  - $U'(1+r) > 0$ meaning E[r] is considered as desirable.
  - $U''(1+r) < 0$ meaning $\sigma^2$ is undesirable.
  - $U'''(1+r) > 0$ meaning positive skewness ($\mu^3 > 0$) is desirable while negative skewness ($\mu^3 < 0$) is undesirable.

has high prob to get high returns ⊕ skewness → expected utility

only difference is that one has ⊕ skewness
another has ⊖ skewness

# Symmetric and Skewed Distributions




Source: Bodie, Kane and Marcus (2021)

## 1.1 Introduction

- An efficiency criterion is a decision rule for dividing all potential investments into two mutually exclusive sets: an efficient and inefficient sets.
  - The efficient set contains all desirable investments for a particular class of investors.
  - All individuals belonging to the class being analyzed will make their final choice from the efficient set.
- The mean-variance criterion (MVC) focuses only on the mean and variance of return distribution, as opposed to the whole distribution used in stochastic dominance.

# 2. THE MEAN VARIANCE CRITERION

2.1 Basic Principles

2.2 Applying the MVC

2.3 Assumptions Underlying the MVC

2.4 The Graphical Representation of the MVC

# 2.1 Basic Principle

- MVC states that: An investment F dominates (is preferred to) an investment G by MVC if and only if

$$E[r_F] \geq E[r_G]$$

and

$$\sigma_F^2 \leq \sigma_G^2$$

In one of the relation

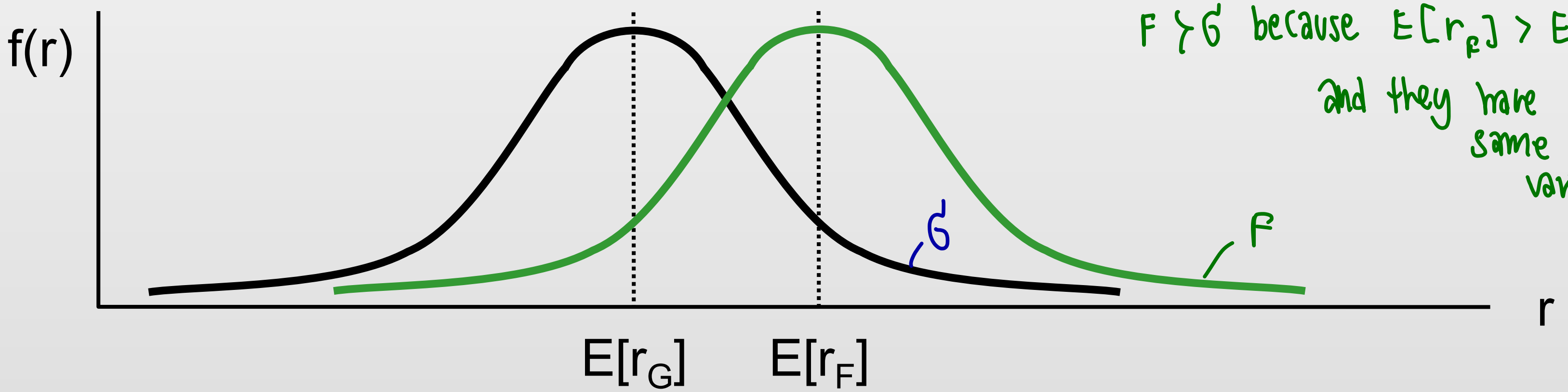
On the condition that at least one strong inequality holds.

to avoid comparing same stock

- The advantage of the MVC is that an investor can confine himself on the first two distribution moments of each investment alternative being considered.

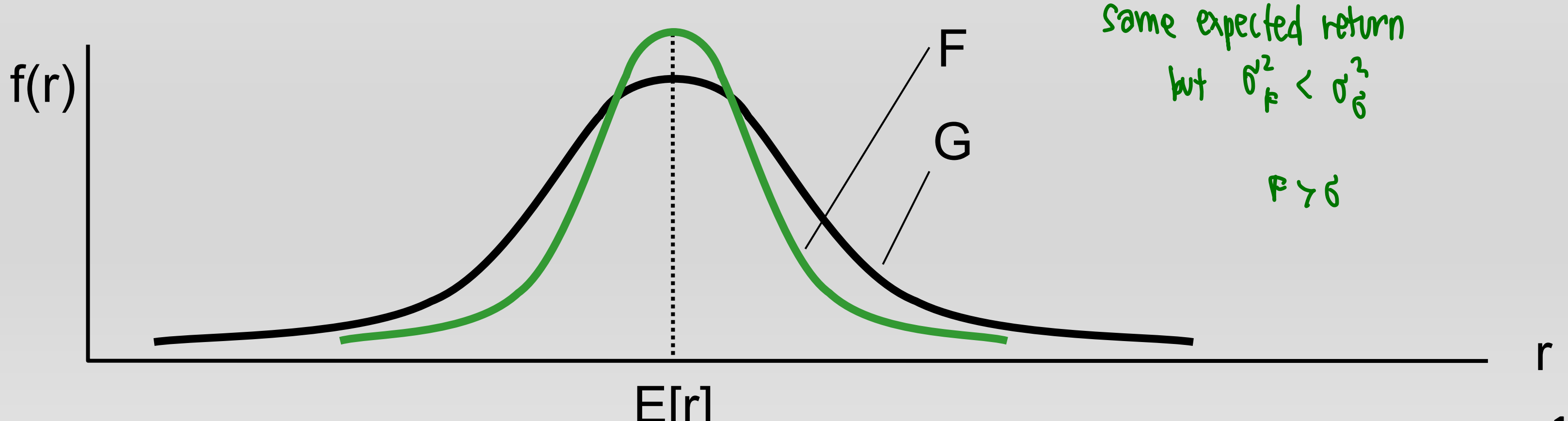
# 2.2 Applying the MVC

1) F and G have the same $\sigma$, which security is MV efficient?




$F \succ G$ because $E[r_F] > E[r_G]$ and they have same variance

2) F and G have the same E[r], which security is MV efficient?




same expected return but $\sigma_F^2 < \sigma_G^2$

$F \succ G$

# 2.2 Applying the MVC

- EX: Consider 5 investment alternatives.

| | A | B | C | D<br>inconclusive to A | E<br>Inconclusive to A |
|---|---|---|---|---|---|
| E[r] | 10 | 8 | 9 | 11 | 12 |
| $\sigma^2$ | 10 | 11 | 10 | 12 | 11 |

- Based on the MVC, A dominates B and C while E dominates D. Thus, B, C and D are relegated to the inefficient set.
- The MVC efficient set consists of A and E. When making investment decision, individuals only need to consider A and E and ignore the rest.

Have to impose more assumption → to use MVC

# 2.3 Assumptions Underlying the MVC

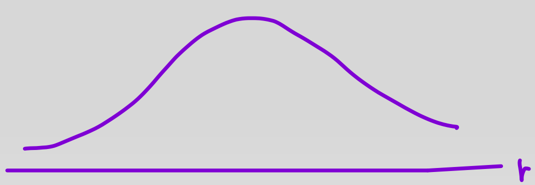
- The MVC is based on certain underlying assumptions regarding investor's tastes. Hence, it is considered as a relevant decision rule for a particular class of investors.

can use one of these two to justify MVC as an approach to maximize your expected utility

- The MVC provides relevant decision rule in two cases;
  - When investors utility functions are quadratic function
  - When asset returns are normally distributed

imply increasing absolute degree of risk aversion
⇕ not make sense!
as we become wealthier
we tend to invest more in risky asset

MVC leads to maximize expected utility




if we know mean & variance
⇓
we can know distribution
⇓
MVC

Skip

# Quadratic Utility Function

- Investors are assumed to be risk averse with the following utility function

quadratic function

$$U(W) \quad = \quad a + bW + cW^2$$

where W is the final wealth and $W = W_0(1+r)$

a, b and c are a constant

$b > 0$ and $c < 0$

Note, r and W are random variables.

- Assuming $U'(W) = b + 2cW > 0$ and $U''(W) = 2c < 0$.

# Quadratic Utility Function

- The expected utility is given by

$$E[U(W)] = a + bE[W] + cE[W^2]$$

- Since $\sigma_W^2 = E[(W-E[W])^2] = E[W^2 - 2W \cdot E[W] + E[W]^2]$

$$= E[W^2] - E[W]^2$$

- The expected utility can be written as

$$E[U(W)] = a + bE[W] + cE[W]^2 + \sigma_W^2$$

- From, $W = W_0(1+r)$

$$E[W] = W_0(1+E[r])$$

$$\sigma_W^2 = Var[W_0(1+r)] = W_0^2\sigma_r^2$$

# Quadratic Utility Function

- Take the 1st order derivative of $E[U(W)]$ with respect to $E[r]$ and $\sigma_r^2$ by applying the chain rule.

  [1] $\partial E[U(W)]/\partial E[r] = (\partial E[U(W)]/\partial E[W])(\partial E[W]/\partial E[r])$

  [2] $\partial E[U(W)]/\partial \sigma_r^2 = (\partial E[U(W)]/\partial \sigma_W^2)(\partial \sigma_W^2/\partial \sigma_r^2)$

- If we restrict $c < 0$ and $b > -2cW$, then;

$$\partial E[U(W)]/\partial E[r] = (b + 2cE[W])(W_0) > 0$$

$$\partial E[U(W)]/\partial \sigma_r^2 = (c)(W^2) < 0$$

- We can conclude that $E[U(W)]$ is an increasing function of $E[r]$ and a decreasing function of $\sigma_r^2$.

# Quadratic Utility Function

- EX: An investor has a quadratic utility function with the coefficients: a = 1, b = 100 and c = −0.1. He has initial investment of $W_0$ = \$100 and is considering 2 investments.

| | A | B |
|---|---|---|
| **E[r]** | 1.00% | 2.00% |
| $\sigma_r$ | 2.00% | 1.00% |

- $E[U(W_A)] = 1 + 100[101] - 0.1[101]^2 - 0.1[100^2 \times 0.02^2] = 9{,}080.5$
- $E[U(W_B)] = 1 + 100[102] - 0.1[102]^2 - 0.1[100^2 \times 0.01^2] = 9{,}160.5$
- $E[U(W_B)] > E[U(W_A)]$, which is consistent with the MVC.

# Normal Distribution of Asset Returns

- Instead of restricting to an unreasonable form of the utility function, we can alternatively justify the MVC for a wide class of utility functions by imposing the restriction that the distribution of investment returns is normal.

$$r \sim N(E[r], \sigma_r)$$

- Investment return can be written in a standard normal variable;

$$z = \frac{r - E[r]}{\sigma_r} \sim N(0, 1)$$

- Therefore,

$$r = E[r] + z \cdot \sigma_r$$

# Normal Distribution of Asset Returns

- From the utility function:

$$U(W) \;=\; U(W_0(1+r)).$$

- We can now rewrite the utility function as:

$$U(W) \;=\; U(W_0(1+E[r]+z\cdot\sigma_r))$$

# Normal Distribution of Asset Returns

- The expected utility of a risky asset is derived from

$$E[U(W)] = \int U(W)\, f(W)\, dW$$

$$= \int U(W_0(1+E[r]+z\cdot\sigma_r))\, f(z)\, dz$$

affect

- As f(z) is standard normal distribution and we consider utility of a person, f(z) and $W_0$ is the same across different assets.
- The two factors causing different expected utilities across assets are;
  - The asset's expected return, E[r]
  - The standard deviation of future returns, $\sigma_r$

Skip end

# Normal Distribution of Asset Returns

- If we assume $U'(W) > 0$ and $U''(W) < 0$, then;

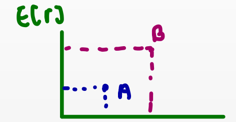
$$\partial E[U(W)]/\partial E[r] > 0$$

$$\partial E[U(W)]/\partial \sigma_r < 0$$

- Thus, by assuming that 1) asset returns are normally distributed and 2) investors prefer more to less and are risk averse, it can be concluded that investors are only concerned with E[r] and $\sigma_r$ when making investment decision.
- Furthermore, the expected utility from risky investment is an increasing function of E[r] and a decreasing function of $\sigma_r$.

More detail is in another file

MVC
A → $E[r_A]$, $\sigma_A^2$
B → $E[r_B]$, $\sigma_B^2$
need 2 factors to rank



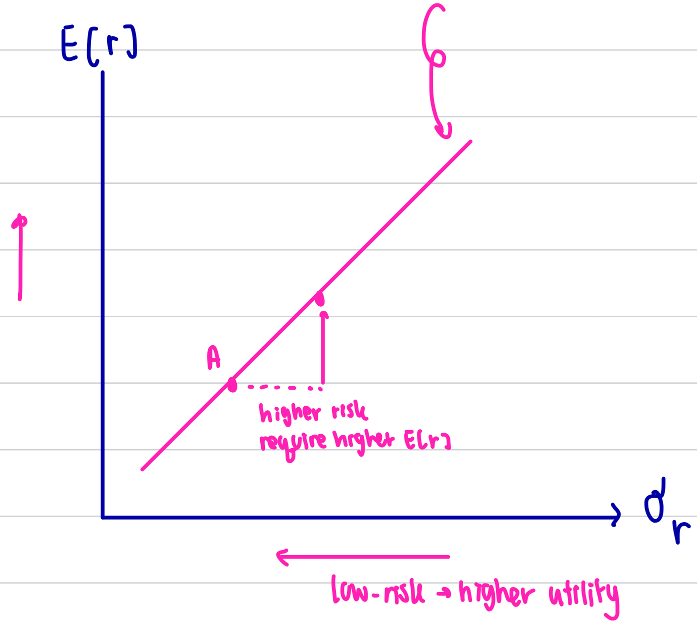

can present analysis on 2D graph
use S.D. instead of $\sigma^2$

# 2.4 The Graphical Representation of the MVC

2D graph

- As the MVC is two-dimensional, the efficiency analysis using this criterion readily lends itself to graphical presentation.
- It is convention to use the x-axis to represent $\sigma_r$ and the y-axis to represent E[r].
- An investment opportunity (or feasibility) set is presented by plotting individual assets and portfolios attainable by the investor on the E[r]-$\sigma_r$ plane.
- The taste of the individual is presented by IC map on the same E[r]-$\sigma_r$ plane.

Bring in risk-return preference



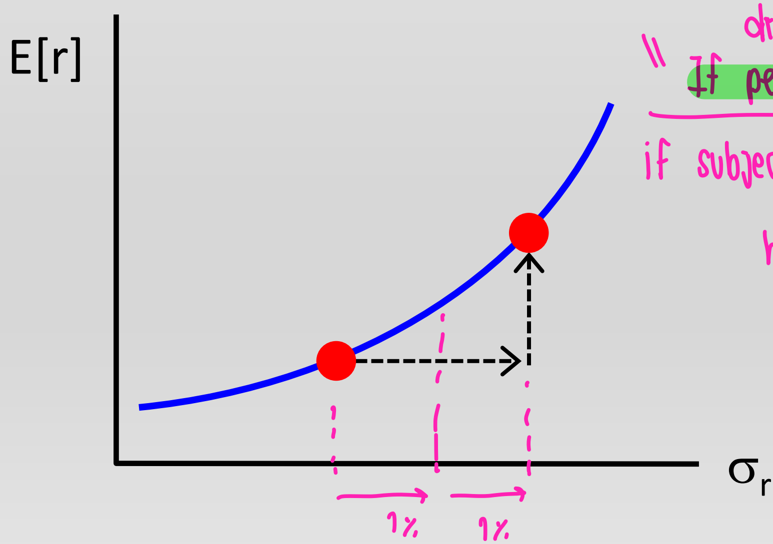
# 2.4 The Graphical Representation of the MVC

- It can be shown that by holding the level of utility constant (see accompany document);

$$\partial E[r]/\partial\sigma_r \quad > \quad 0$$

$$\partial(\partial E[r]/\partial\sigma_r)/\partial\sigma_r \;=\; \partial^2 E[r]/(\partial\sigma_r)^2 \quad > \quad 0$$

- This implies that an IC on the E[r]-$\sigma_r$ plane has positive slope and a convex shape.




because more $\sigma_r$ damage utility

diminishing marginal utility

// If person is risk averse //

if subject take 1 more unit of risk

↓

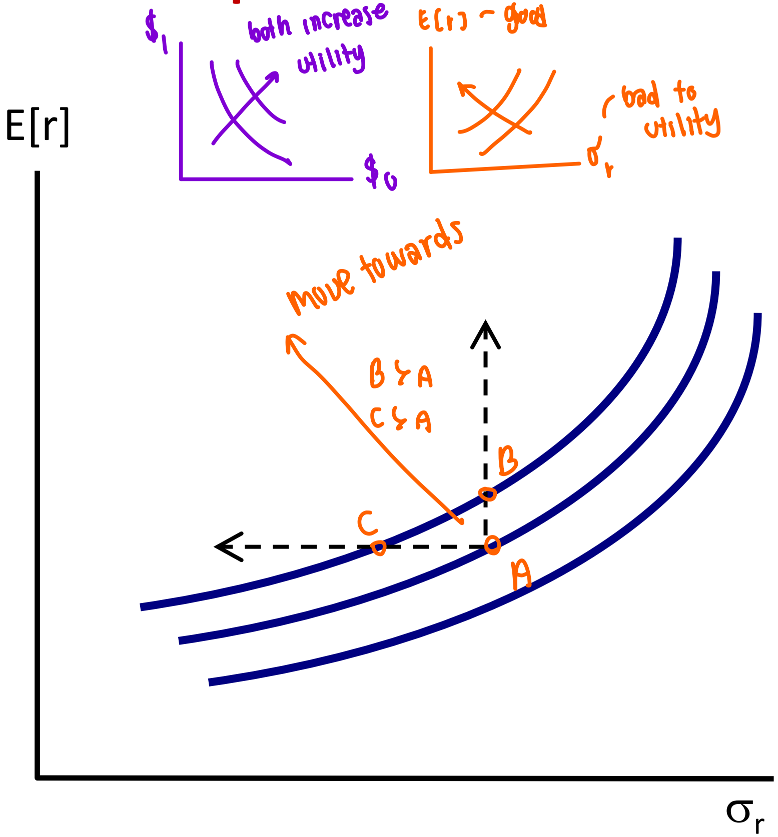
he needs more compensation with expected return

⇓

graph is convex

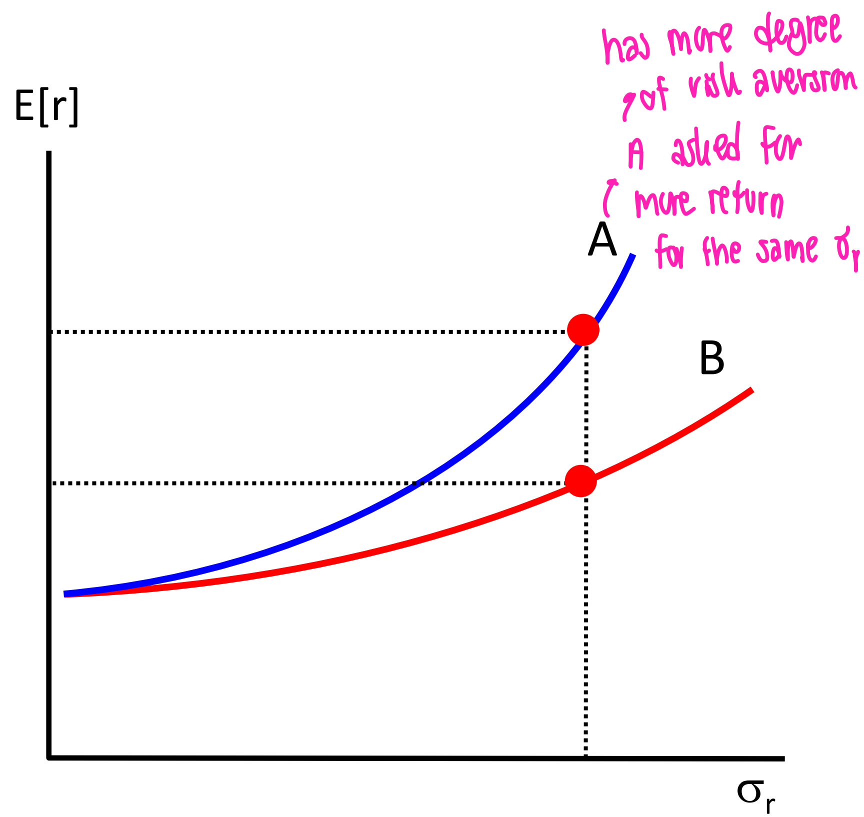
Slope tells degree of risk aversion

## IC Map of an Individual




An upper-left IC delivers higher level of total utility

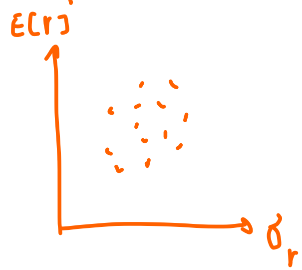
## Degree of Risk Aversion




Steeper IC reflects higher degree of risk aversion

# 3. PORTFOLIO DIVERSIFICATION

How to represent investment opportunity set on plane




3.1 Portfolio Mathematic

3.2 Diversification Benefits

# 3.1 Portfolio Mathematic

- Let $r_X$ and $r_Y$ to be future returns on securities X and Y. As their values are not known today, they are random variables.

| Measures | Symbol | Historical Return | Scenario Analysis |
|---|---|---|---|
| Expected rate of return | $E[r_X]$ | $\bar{r_X} = \frac{1}{T}\sum_{t=1}^{T} r_{X,t}$ | $\sum_{i=1}^{N} p_i r_{X,i}$ |
| Variance of rate of return $\sigma_{i,i} = \sigma_i^2$ | $\sigma_X^2$ | $\frac{1}{T}\sum_{t=1}^{T}(r_{X,t} - \bar{r_X})^2$ deviation from mean | $\sum_{i=1}^{N} p_i(r_{X,i} - E[r_X])^2$ |
| Standard Deviation of return | $\sigma_X$ | $\sqrt{\sigma^2}$ | $\sqrt{\sigma^2}$ |
| Covariance $\sigma_{i,j}$ between $r_X$ and $r_Y$ only tell whether two stocks have +/- or relationship | $\sigma_{X,Y}$ | same equation $\frac{1}{T}\sum_{t=1}^{T}(r_{X,t} - \bar{r_X})(r_{Y,t} - \bar{r_Y})$ | $\sum_{i=1}^{N} p_i(r_{X,i} - \bar{r_X})(r_{Y,i} - \bar{r_Y})$ |
| Correlation bound [-1, 1] between $r_X$ and $r_Y$ can tell magnitude strength of relation | $\rho_{X,Y}$ | $\frac{\sigma_{X,Y}}{\sigma_X \cdot \sigma_Y}$ | $\frac{\sigma_{X,Y}}{\sigma_X \cdot \sigma_Y}$ |

# 3.1 Portfolio Mathematic

- **Portfolio of 2 securities**

$w_1, w_2$ depend on investor's decision

others from market information

weight on stock 1

don't know

$$E[r_P] = E[\underbrace{w_1 r_1 + w_2 r_2}_{}]$$

constant → we know at t=0

- Take expectation and variance both sides:

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2} \Rightarrow \sigma_{1,2} = \sigma_1 \sigma_2 \rho_{1,2}$$

$$E[r_P] = w_1 E[r_1] + w_2 E[r_2]$$

constant

$$Var[r_p] = Var[w_1 r_1 + w_2 r_2] = Var[w_1 r_1] + Var[w_2 r_2] + 2 w_1 w_2 \sigma_{1,2}$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2}$$

Note: $E[aX+bY] = aE[X] + bE[Y]$

$Var[aX+bY] = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{1,2}$

$$[w_1 \;\; w_2 \;\; w_3]_{1\times 2} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{2,1} & \sigma_2^2 & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{array}{ccc} w_1 & w_2 & w_3 \end{array}$$

$$= [w_1\sigma_1^2 + w_2\sigma_{2,1} + w_3\sigma_{3,1} \quad w_1\sigma_{1,2} + w_2\sigma_2^2 + w_3\sigma_{3,2} \quad w_1\sigma_{1,3} + w_2\sigma_{2,3} + w_3\sigma_3^2] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3}$$

# EX: Calculating Portfolio E[r] and SD[r]

unit : % per month

| | E[r] | σ |
|---|---|---|
| Motorola | 1.75% | 9.73% is riskier & give more E[r] |
| GM | 1.08% | 6.23% |
| Correlation (ρ) | 0.37 on average, the two returns tend to move together but not that strong (closer to 0) | |

The table shows average monthly returns, standard deviations and correlation coefficient of returns on two securities, Motorola and GM.

Calculate E[r] and SD[r] on a portfolio consists of \$700 and \$400 worth of Motorola and GM shares?

$W_{Mot} = 7/11$

$W_{GM} = 4/11$

$$E[r_P] = (700/1100)\times 0.0175 + (400/1100)\times 0.0108$$

$$= 0.01506 \text{ or } 1.506\% \text{ per month}$$

start with variance then square root it

$$\sigma_P = [(7/11)^2 \cdot (0.0973^2) + (4/11)^2 \cdot (0.0623^2) + 2(7/11)(4/11)(0.0973)(0.0623)(0.37)]^{1/2}$$

$$= 0.07338 \text{ or } 7.338\% \text{ per month}$$

How risk & return change

# EX: Portfolio E[r] and SD[r] as Weights Change

E[r_p] per 1% of σ_p or per 1 unit of risk

each row is different portfolio

| | $w_{Mot}$ | $w_{GM}$ | $E[r_P]$ | $\sigma_P^2$ | $\sigma_P$ | $E[r_P]/\sigma_P$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1.08 | 38.8 | 6.23 | 0.173 |
| 2 | 0.25 | 0.75 | 1.25 | 36.2 | 6.01 | 0.208 |
| 3 | 0.50 | 0.5 | 1.42 | 44.6 | 6.68 | 0.213 |
| 4 | 0.75 | 0.25 | 1.58 | 64.1 | 8.00 | 0.198 |
| 5 | 1 | 0 | 1.75 | 94.6 | 9.73 | 0.180 |
| 6 | 1.25 | -0.25 | 1.92 | 136.3 | 11.67 | 0.165 |

diversify

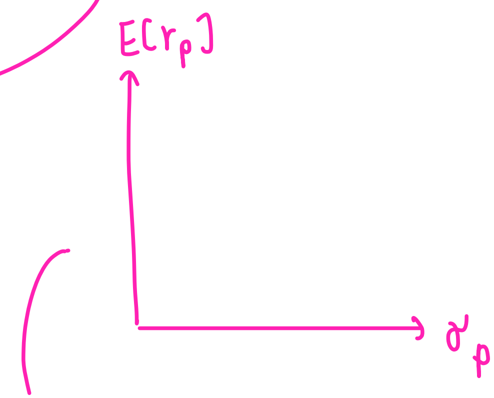
trade-off return

reduce

the high the better

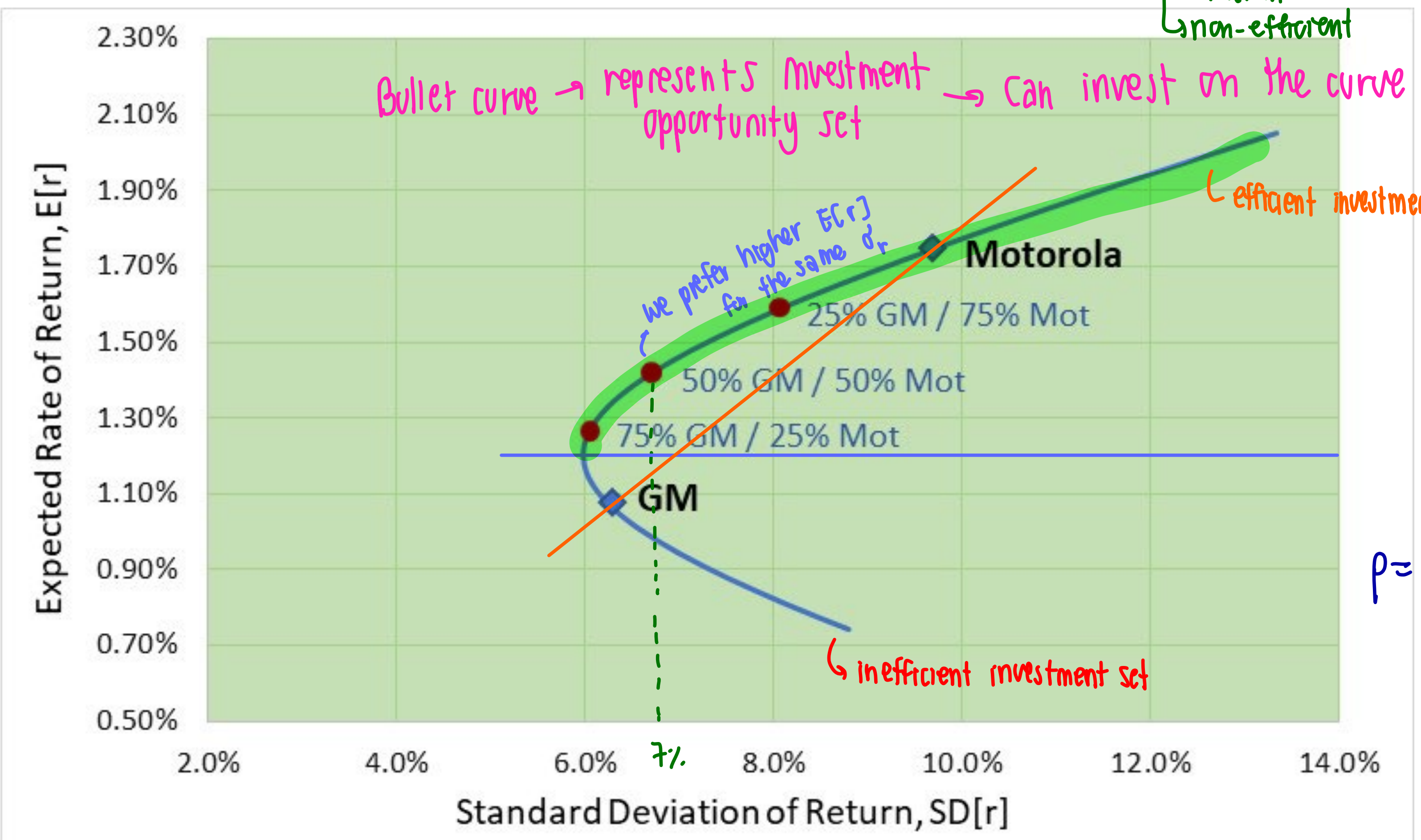
diversification → you can increase the return per unit of risk <free lunch in finance>

short sell consists of
1) borrow stocks
2) sell them
→ has money > 100



# The Investment Opportunity Set and the Efficient Investment Set: The Case of 2 Assets

MVC → divide into two subsets
↳ efficient
↳ non-efficient



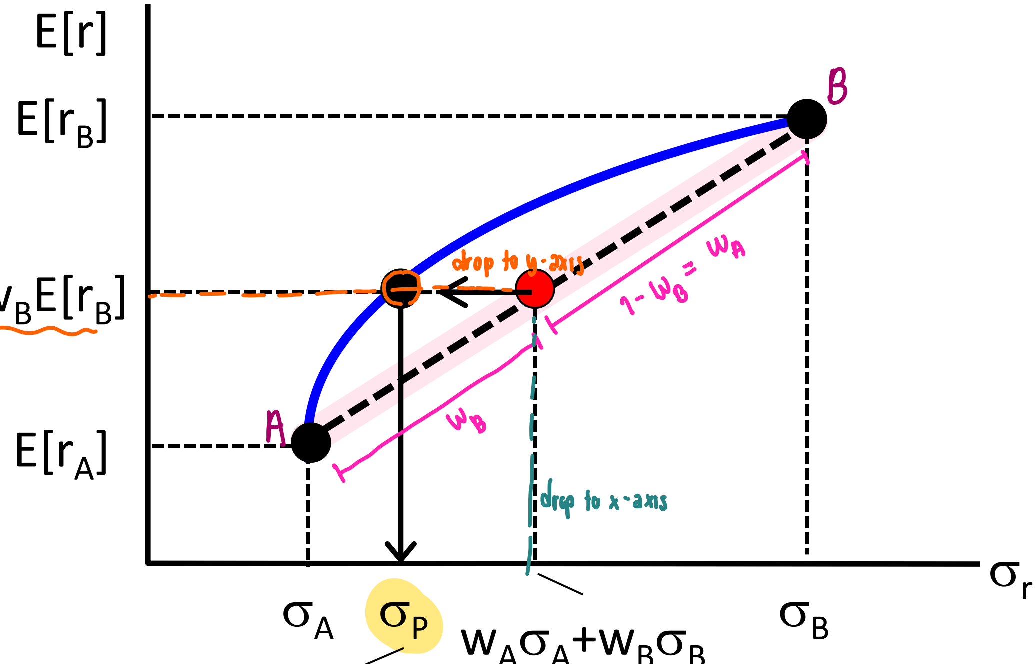

$\rho = 0.37$

efficient frontier : gives the highest E[r] at specific risk

# The Shape of the Bullet Curve

portfolio return

$E[r_P] = w_A E[r_A] + w_B E[r_B]$




↳ weighted average of two s.d.

$\sigma_P < w_A\sigma_A + w_B\sigma_B$ if $\rho_{A,B} < 1$

$(x+y)^2 = x^2 + y^2 + 2xy$ ; $x = w_A\sigma_A$, $y = w_B\sigma_B$

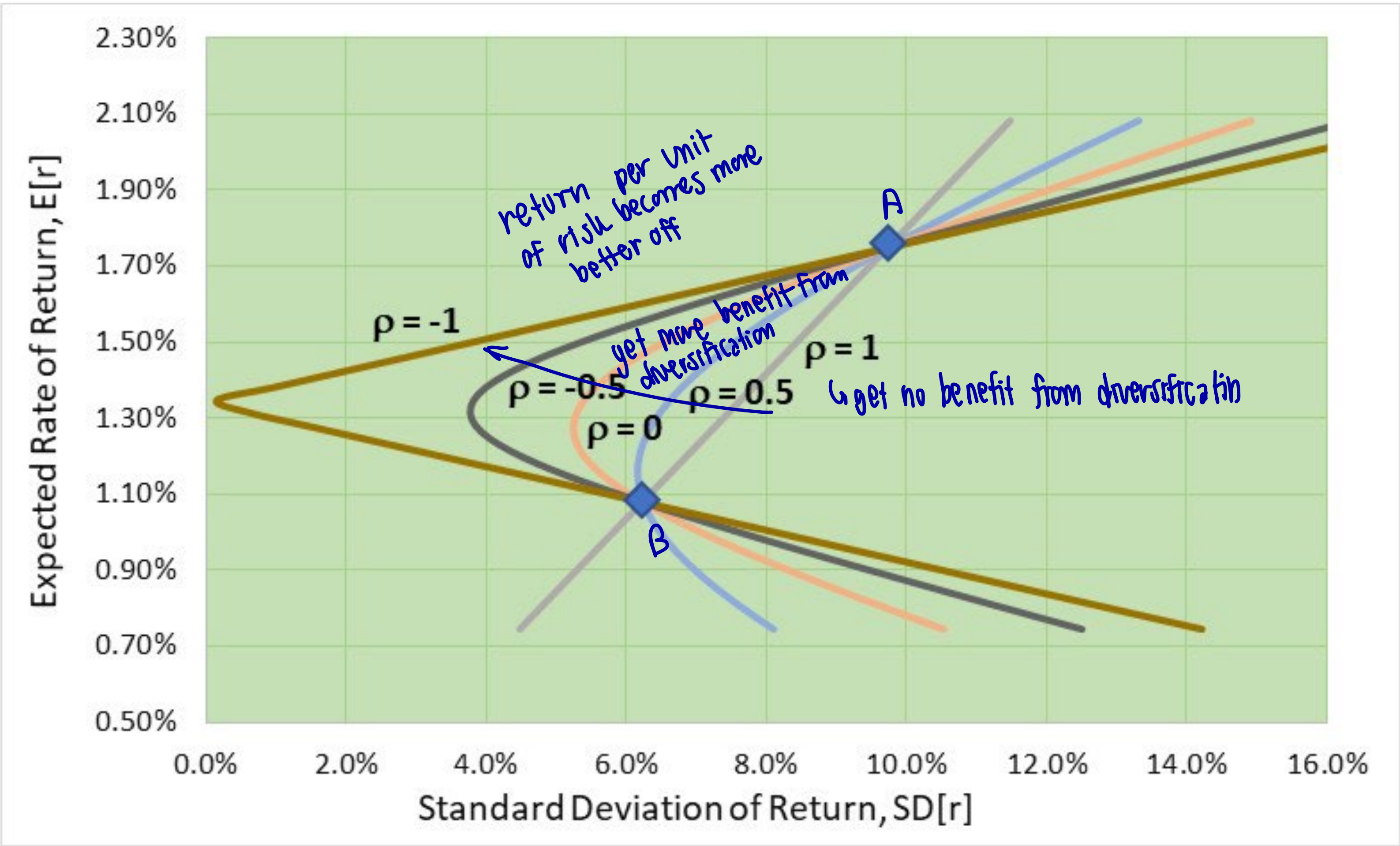
When $\rho_{A,B} = 1$, $\boxed{\sigma_P = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B)^{1/2}} = ((w_A\sigma_A + w_B\sigma_B)^2)^{1/2}$
$= w_A\sigma_A + w_B\sigma_B$

On average $-1 < \rho_{A,B} < 1$. Hence, for average assets A and B; $\sigma_P < w_A\sigma_A + w_B\sigma_B$

$\sigma_p$ must be located on left hand side

We prefer more curvy curve

# Correlation and Diversification Benefits



# 3.1 Portfolio Mathematic

Matrix operation on Excel / Matlab

$\sum_{i=1}^{N} w_i = 1$

- **Portfolio of N securities**

$$r_P = w_1 r_1 + w_2 r_2 + \dots + w_N r_N$$

- Take expectation and variance both sides:

$$E[r_P] = w_1 E[r_1] + w_2 E[r_2] + \dots + w_N E[r_N]$$

$$= \sum_{i=1}^{N} w_i E[r_i]$$

add 1 stock to N=10 portfolio
add 1 variance
& add 10 covariances

# 3.1 Portfolio Mathematic

need to know var of each stock & covariance of each pair → as N become bigger

$$\sigma_P^2 = (w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \ldots + w_N^2\sigma_N^2) +$$

e.g.

2 direction of covariance ($\sigma_{1,2}$ & $\sigma_{2,1}$)

$$2(w_1w_2\sigma_1\sigma_2\rho_{1,2} + w_1w_3\sigma_1\sigma_3\rho_{1,3} + \ldots$$

ww

$$+ w_{N-1}w_N\sigma_{N-1}\sigma_N\rho_{N-1,N})$$

$$= \sum_{i=1}^{N} w_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ \boxed{j\neq i}}}^{N} w_iw_j\sigma_{i,j}$$

$w_i w_i \sigma_{i,i}$

double counting

must not be same stock! (variance)

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} w_iw_j\sigma_{i,j}$$

# 3.1 Portfolio Mathematic

- In matrix form.
  - $E[r_P] = \mathbf{w}^T\mathbf{r}$.
  - $\sigma_P^2 = \mathbf{w}^T\mathbf{V}\mathbf{w}$

$$\ell = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{N\times 1}$$

square & symmetric matrix

"variance - covariance matrix" $[\Sigma]$ not summation

$$\underset{\text{column vector of weight}}{\mathbf{w}} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}_{N\times 1} \qquad \underset{\text{expected return on each stock}}{\mathbf{r}} = \begin{pmatrix} E[r_1] \\ E[r_2] \\ \vdots \\ E[r_N] \end{pmatrix}_{N\times 1} \qquad \mathbf{V} = \begin{pmatrix} \underset{\text{var}}{\sigma_{1,1}} & \underset{\text{covariance}}{\sigma_{1,2}} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,N} \\ \vdots & \vdots & & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_{N,N} \end{pmatrix}_{N\times N}$$

* Portfolio return in matrix form

$$\underset{\underline{1 \times N}}{w^T} \cdot \underset{\underline{N \times 1}}{r} = [w_1 \quad w_2] \begin{bmatrix} E[r_1] \\ E[r_2] \end{bmatrix}$$

$$= w_1 E[r_1] + w_2 E[r_2]$$

$$= E[r_p]$$

$$\underset{(1\times N)(N \times N)(N \times 1)}{w^T \cdot V + w} = [w_1 \; w_2] \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} w_1\sigma_{1,1} + w_2\sigma_{2,1} & w_1\sigma_{1,2} + w_2\sigma_{2,2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= w_1^2 \sigma_2^2 + \underbrace{w_1 w_2 \sigma_{2,1} + w_1 w_2 \sigma_{1,2}}_{2} + w_2^2 \sigma_2^2$$

Var[$r_p$] = mmult(mmult(transpose(weight column), var-cov matrix), weight column)

multiplication of matrix

row vector

E[$r_p$] = mmult(transpose(weight column), expected return)

⇒ ctrl + shift + enter

# 3.1 Portfolio Mathematic

- Three stocks, A, B and C, have the following E[r], SD[r] and correlations. Note returns and SD are in % pa unit.

%²

| | E[r] |
|---|---|
| A | 6.25 |
| B | 7.50 |
| C | 3.50 |

| | A | B | C |
|---|---|---|---|
| A | 40.69 | 37.38 | -27.38 |
| B | 37.38 | 63.27 | -17.25 |
| C | -27.38 | -17.25 | 20.75 |

- Calculate E[r] and SD[r] of an equally weighted portfolio.

| | w |
|---|---|
| A | 1/3 |
| B | 1/3 |
| C | 1/3 |

- $E[r_P] = \mathbf{w}^T\mathbf{r} = 5.75\%$ pa
- $\sigma_P^2 = \mathbf{w}^T\mathbf{V}\mathbf{w} = 12.25$
- $\sigma_P = (12.25)^{0.5} = 3.50\%$ pa

# 3.2 Diversification Benefits

- If the market is efficient, prices of stocks with good or bad prospects will already reflect the good or bad news.
- For an average investors with average information, it would be impossible to predict the next price movement.
- Picking a next good stock becomes a random process.
- What would happen to the risk of an average 1-stock portfolio as more randomly selected stocks were added?
- $\sigma_P$ would decrease because the added stocks would not be perfectly correlated, but $r_P$ would remain relatively constant.

# 3.2 Diversification Benefits

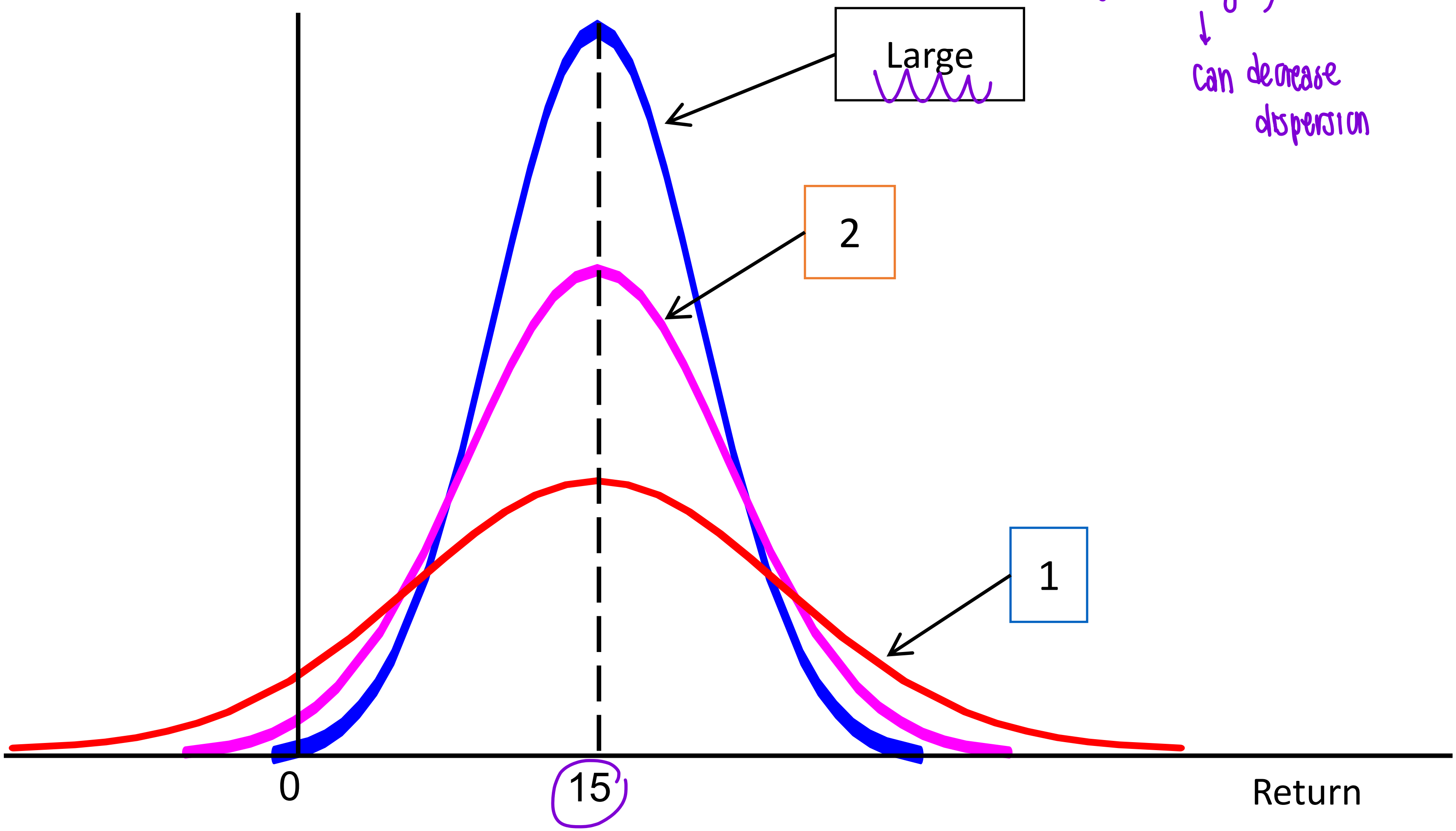
- For an N-asset portfolio, $\sigma_P$ consists of;
  - N variance terms, and
  - $N(N-1)/2$ covariance terms.
- As N gets larger, the covariance terms dominate the variance terms in $\sigma_P$.

Random selection

# Distribution of Portfolio Returns as N Increases

by diversifying → don't have to sacrify expected return

↓ can decrease dispersion



# Power of Diversification

- Consider a diversified portfolio with equal weights ($w_i$=1/N). # of stocks in market

$$\sigma_P^2 = \sum_{i=1}^{N} \overbrace{w_i^2}^{\text{constant}} \sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} \overbrace{w_i w_j}^{\frac{1}{N}\cdot\frac{1}{N}} \sigma_{i,j}$$

variance of portfolio

$$= \frac{1}{N}\cdot\frac{1}{N}\sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N}\cdot\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{i,j}$$

equally weighted $w_i = \frac{1}{N}$

$w_i^2 = \frac{1}{N}\cdot\frac{1}{N}$

- Define the average variance and average covariance of securities;

$$\overline{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^2 \quad \text{and} \quad \overline{\sigma_{i,j}} = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{i,j}$$

systematic risk

- Substitute $\overline{\sigma^2}$ and $\overline{\sigma_{i,j}}$ to the $\sigma_P^2$ equation.

can't diversify

covariance risk

total risk

$$\boldsymbol{\sigma_P^2 = \frac{1}{N}\overline{\sigma^2} + \frac{N-1}{N}\overline{\sigma_{i,j}}}$$

$\overline{\sigma_{i,j}}$ becomes more important

as we increase N → $\frac{N-1}{N}$ approach to 1

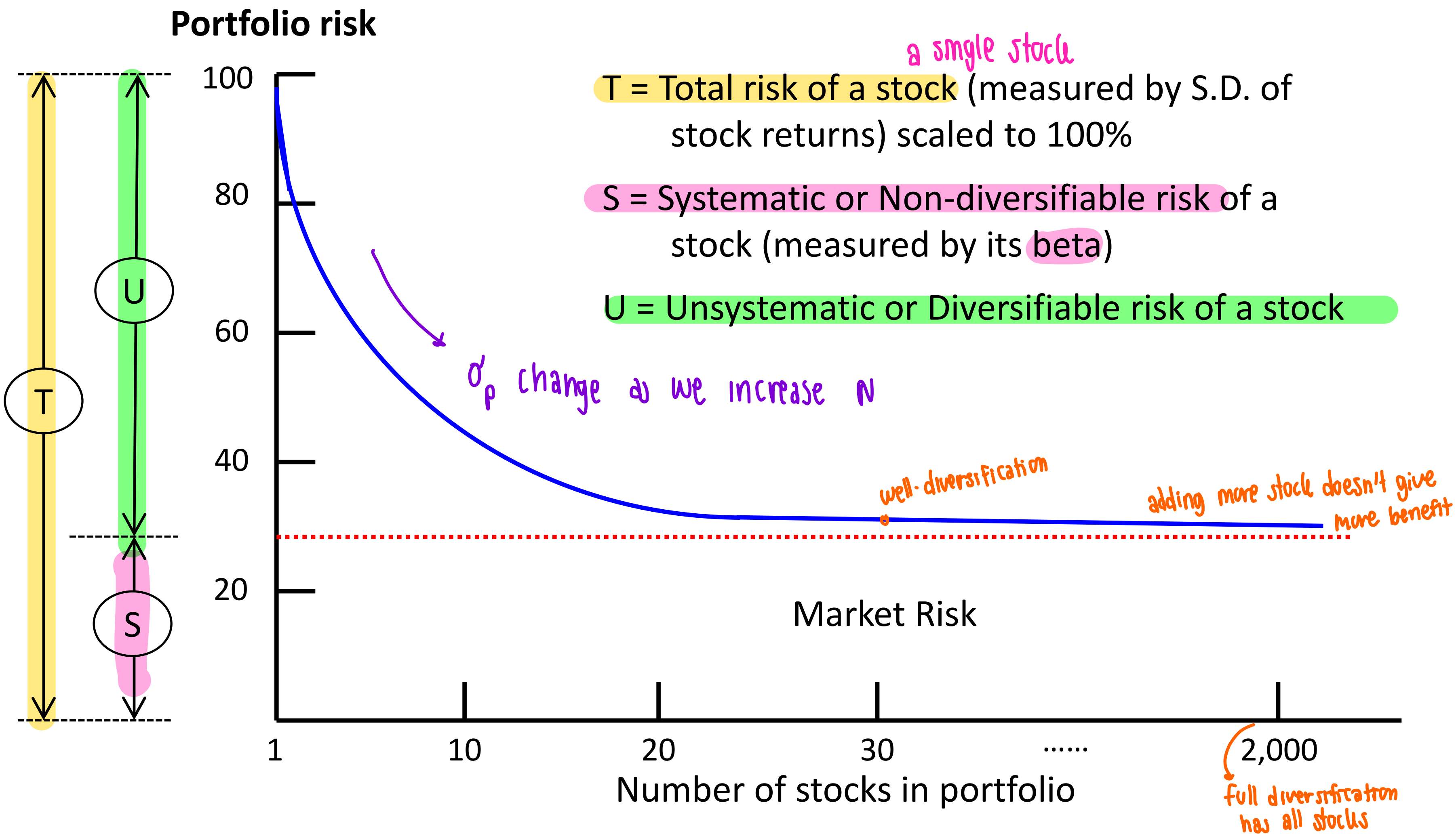
as N become larger → $\overline{\sigma^2}$ has less influence

- The total risk of a portfolio ($\sigma_P{}^2$) consist of the risk of each security (1st term) plus the covariance risk (2nd term).

# Power of Diversification (cont.)

- As $N \rightarrow \infty$, the 1st term$\rightarrow$ 0 and the 2nd term$\rightarrow \overline{\sigma_{i,j}}$.
- The risk that is specific to each security is diversifiable, while the covariance risk is not.
- The non-diversifiable risk of a diversified portfolio depends on the covariance of the returns of the component securities, which in turn, is a function of the systematic factors in the economy.

# Portfolio SD as N Increases




Since investors could not diversify away **<u>the systematic risk</u>** without incurring additional cost, they will **<u>require return for bearing this risk</u>**. Hence the cost of equity capital must reflect the systematic risk (beta) of the stock issued by the firm.

# 3.2 Diversification Benefits

- Market risk is that part of a security's stand-alone risk that cannot be eliminated by diversification.

e.g hold another stock with low correlation to the stock e.g. APPLE & competitor stock

- Diversifiable risk is that part of a security's stand-alone risk that can be eliminated by diversification. These risks are either firm-specific or industry-specific.
- As more stocks are added, each new stock has a smaller risk-reducing impact on the portfolio.
- By forming a well-diversified portfolio, investors can eliminate a significant part of the riskiness of owning a single stock.
- By forming a fully-diversified portfolio, investors can eliminate all unsystematic risk of the portfolio.

# 4. THE EFFICIENT PORTFOLIOS

4.1 The Efficient Portfolios without Risk-free Asset

4.2 The Efficient Portfolios with Risk-free Asset

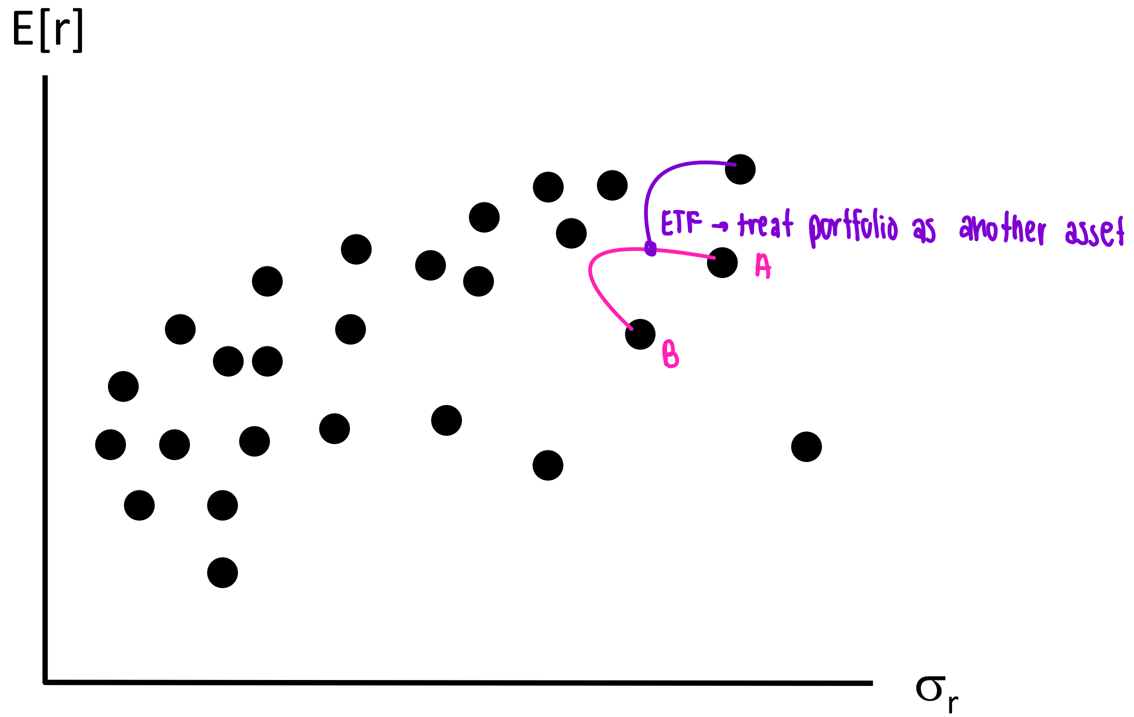
4.3 The Separation Property of Portfolio Construction

# 4.1 The Efficient Portfolio without RF Asset

- The next slide shows various investment options faced by an individual.
- Although, he only wants to pick one optimal investment, it could be more convenient if he could identify the efficient investment set first.
- Under the MVC, an efficient investment is the one that;
  - Minimize $\sigma_r$ for a given level of E[r], or
  - Maximize E[r] for a given level of $\sigma_r$.

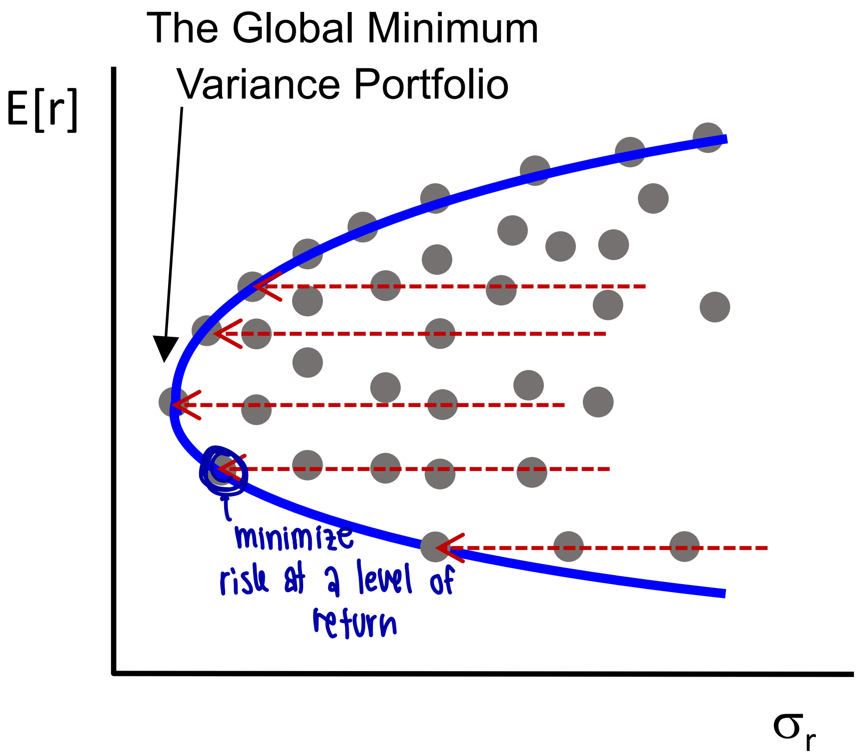
# Investment Opportunity Set with N Risky Assets

MVC



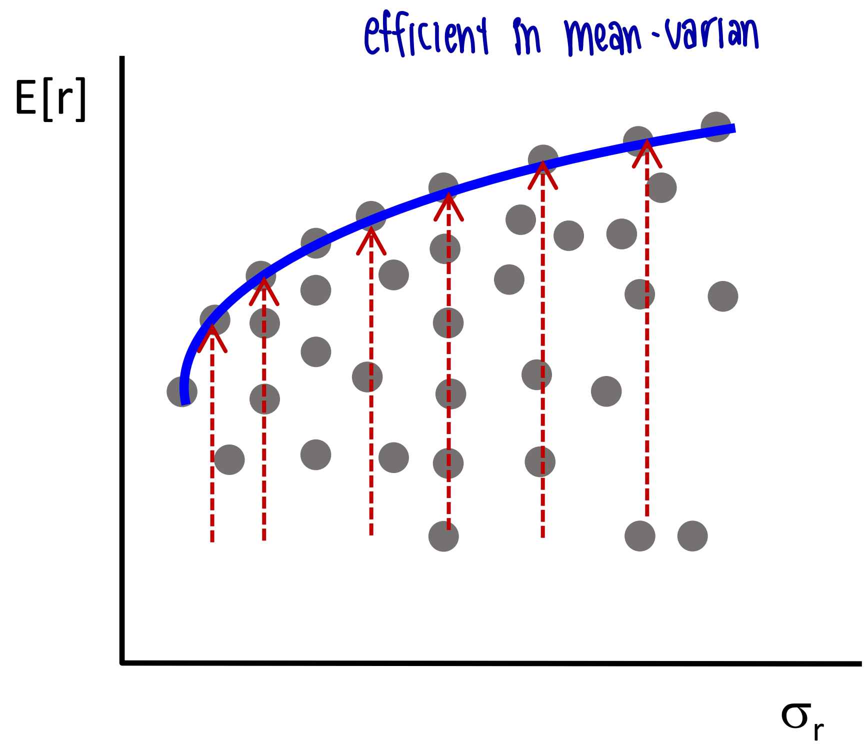

Each point represents an investment, which could be an individual assets or a portfolio of assets.

# Obtaining The MV Efficient Set




**Minimum Variance Boundary**

obj function : given a level of return → minimize $\sigma_p$




**The Efficient Frontier**

→ Given a level of $\sigma_p$ → then find the portfolio that maximize expected return $E[r_p]$

INVESTMENTS <Kane & Marcus>
textbook ⇒ index part tells how to use solver

# 4.1 The Efficient Portfolio without RF Asset

- Assume there are N individual securities, the objective function for finding the efficient portfolio is;

use excel to find $w_1$ & $w_2$ that maximize $E[r_p]$

$$\text{Max.} \quad E[r_P] = \sum_{i=1}^{N} w_i E[r_i]$$

$$\text{s.t.} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j} = \sigma_P^2$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0 \quad \forall\, i$$

↳ if we're not allowed to short sell

$$\text{Max.} \quad E[r_P] = \mathbf{w^T r}$$

$$\text{s.t.} \quad \mathbf{w^T V w} = \sigma_P^2$$

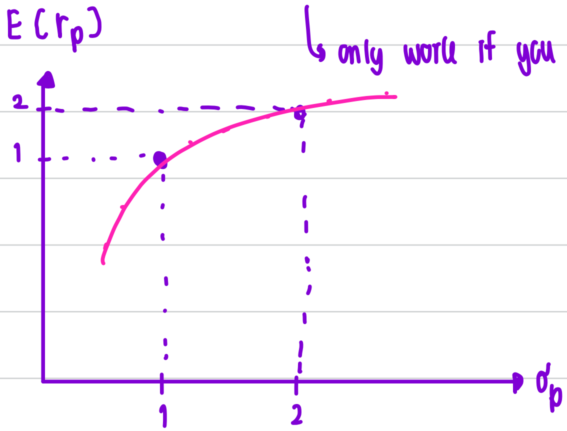
$$\mathbf{w^T 1} = 1$$

$$w_i \geq 0 \quad \forall\, i$$

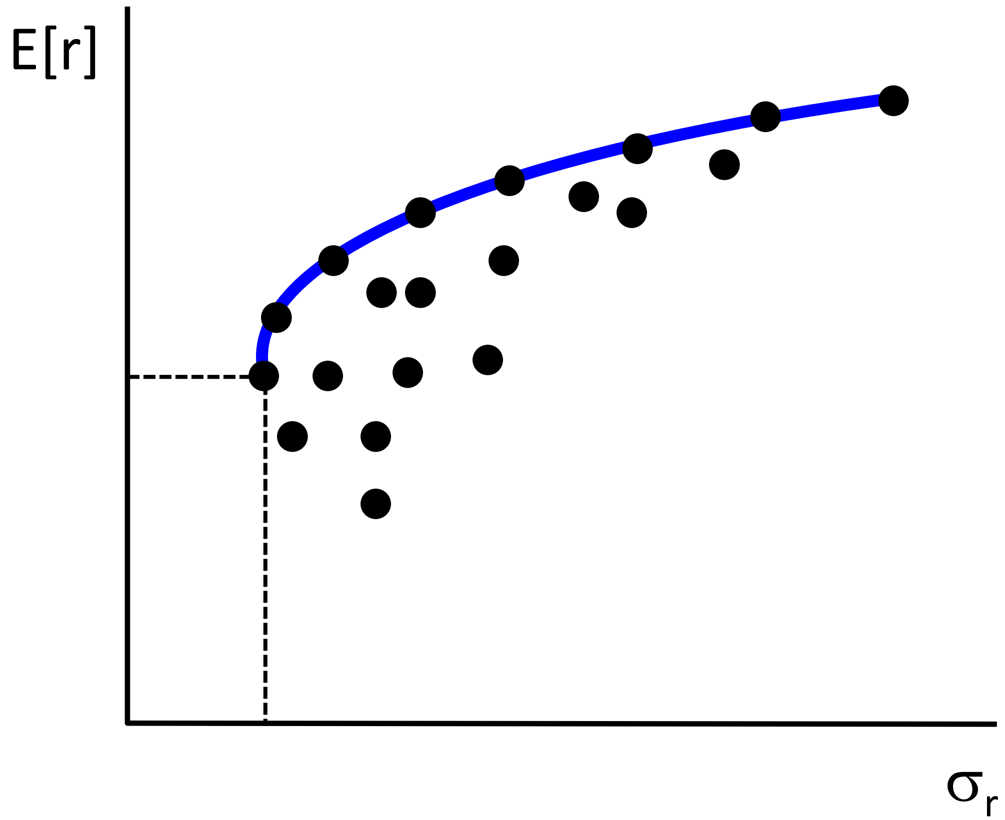
Note: Assume there is no short sale. Solve for $w_i$'s. **1**=vector of 1's.

combination of mean-varian efficient ports also efficient

↳ only work if you allow short-sell !!!

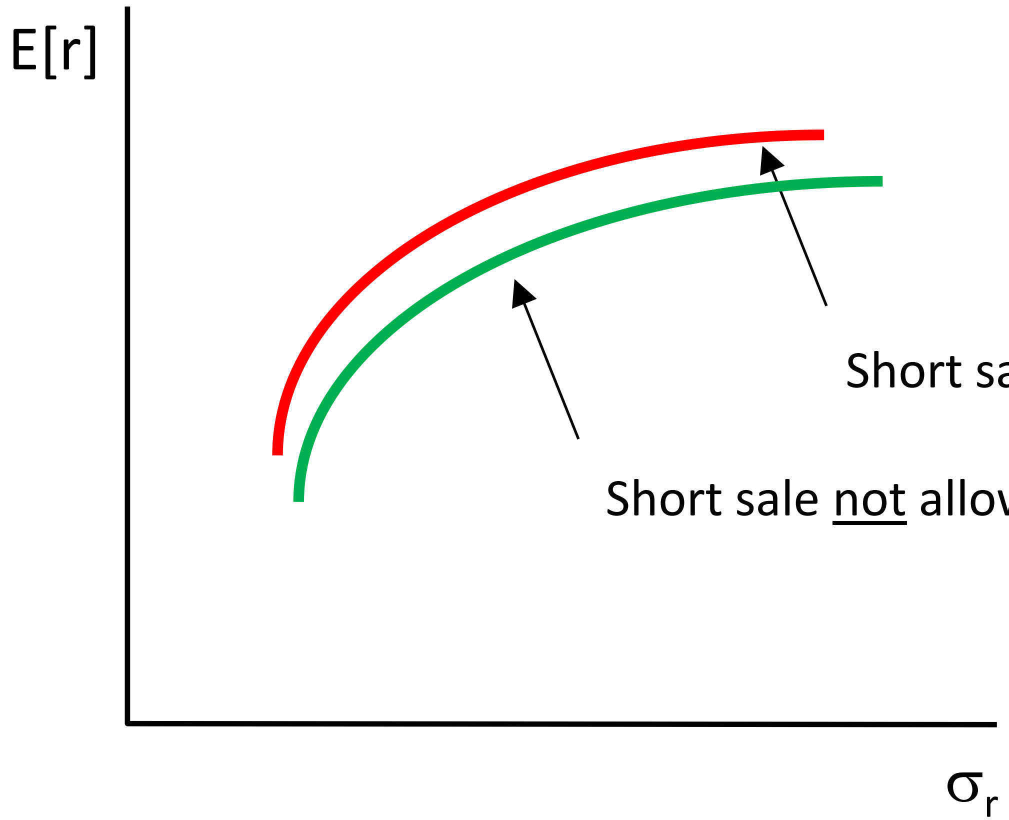


# The Efficient Frontier of Risk Assets




The efficient frontier represent the optimal investment opportunities.

## The Efficient Frontier with vs. without Short Sale Restriction




; less mean-varian effiosent than short sale allowed

# Preparing the Inputs for Optimization Procedure

[1] Collect stock prices information.

drawback ; price split — price ↓ but wealth is the same

- Select stocks into the universe of investable assets all stocks in the market

each stock can obtain 3 types of price series — already capture capitalization overtime — both capital gain/loss & dividend return

- Choose price series: Price, Price Index or Total Return Index

actual price observed in market — but only reflect capital gain, loss not dividend return

time gap btw price

- Choose a price (sampling) interval: Daily, Weekly, Monthly, etc.
- Choose a sampling period e.g. 5-yr of daily data ⇒ should use monthly data for long sampling period

[2] Convert prices into rate of returns

- Discrete return $((P_{i,t+1}/P_{i,t})-1)$ vs Continuous return $(\ln(P_{i,t+1}/P_{i,t}))$

[3] Calculate Ave[$r_i$], $\sigma_i^2$ and $\sigma_{i,j}$ for all individual stocks

mean return for 60 prices

- Arithmetic vs Geometric Average

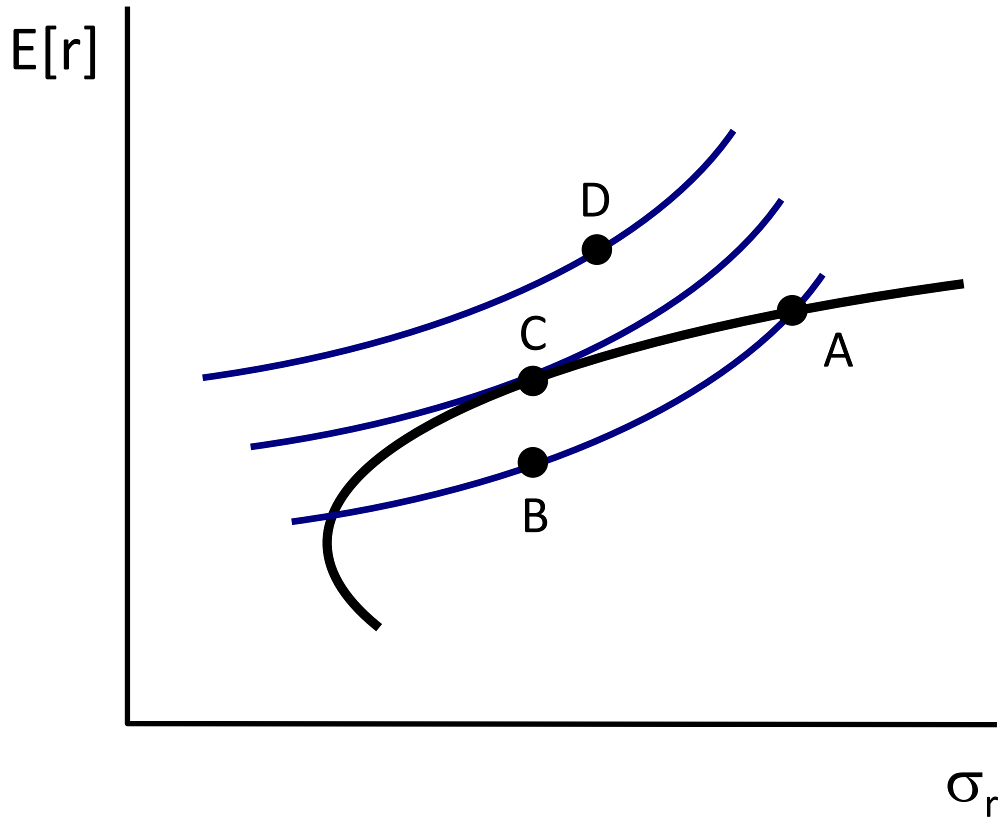
unbiased estimator for mean return

- If monthly returns are used, find arithmetic average and SD of monthly return first, then annualized as follows:
  - Annualized rate of return = $(1+\text{Ave.}[r_i])^{12}-1$
  - Annualized SD = $(12^{0.5})\times$(SD of Monthly Return)

add-in analysis toolpax ⇒ variance - covariance matrix

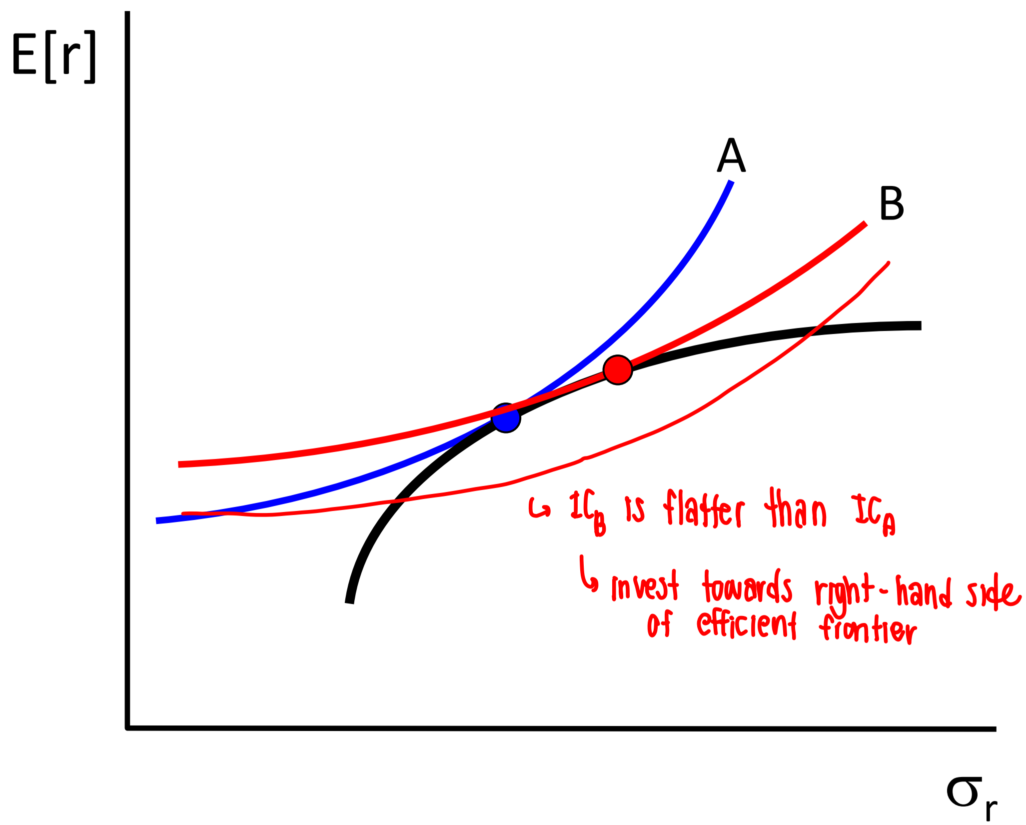
# Finding the Optimal Investment Decision

Need investor preference!

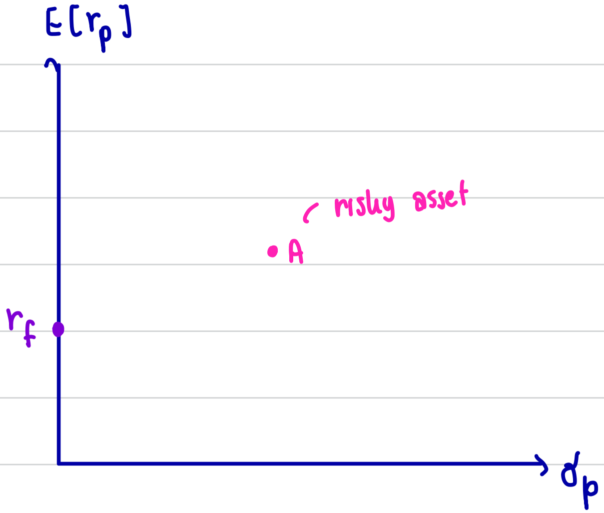



The optimal investment decision is on the efficient frontier where the slope of the efficient frontier is equal to the slope of the IC. In the graph, investments A and B are suboptimal, C is the optimal choice, while D is unattainable.

# Degree of Risk Aversion and the Investment Decision




As the degree of risk aversion reduces, the optimal portfolio moves along the efficient frontier from left to right.



E[r_p]
risky asset
A
r_f
σ_p

# 4.2 The Efficient Portfolio with RF Asset

- We can extend the analysis by allowing an investor to consider all risky assets in the market plus the risk-free (RF) asset.
- The decision on how much to invest in risky assets and how much in the risk-free asset is called capital allocation.
- A government bond can be used to represent risk-free asset. The yield or YTM on the bond represent risk-free rate, $r_F$.
- A portfolio consisting of both the risk-free asset and risky assets is call a complete portfolio (C).

$$\underbrace{F + p}_{\text{complete portfolio}}$$

F: risk-free asset

p: risky part

choose weight of stocks to p
: security selection

# 4.2 The Efficient Portfolio with RF Asset

- Let, y = portfolio weight on a portfolio of risky assets (P) in the complete portfolio (C).
  - $r_C = yr_P + (1-y)r_F$

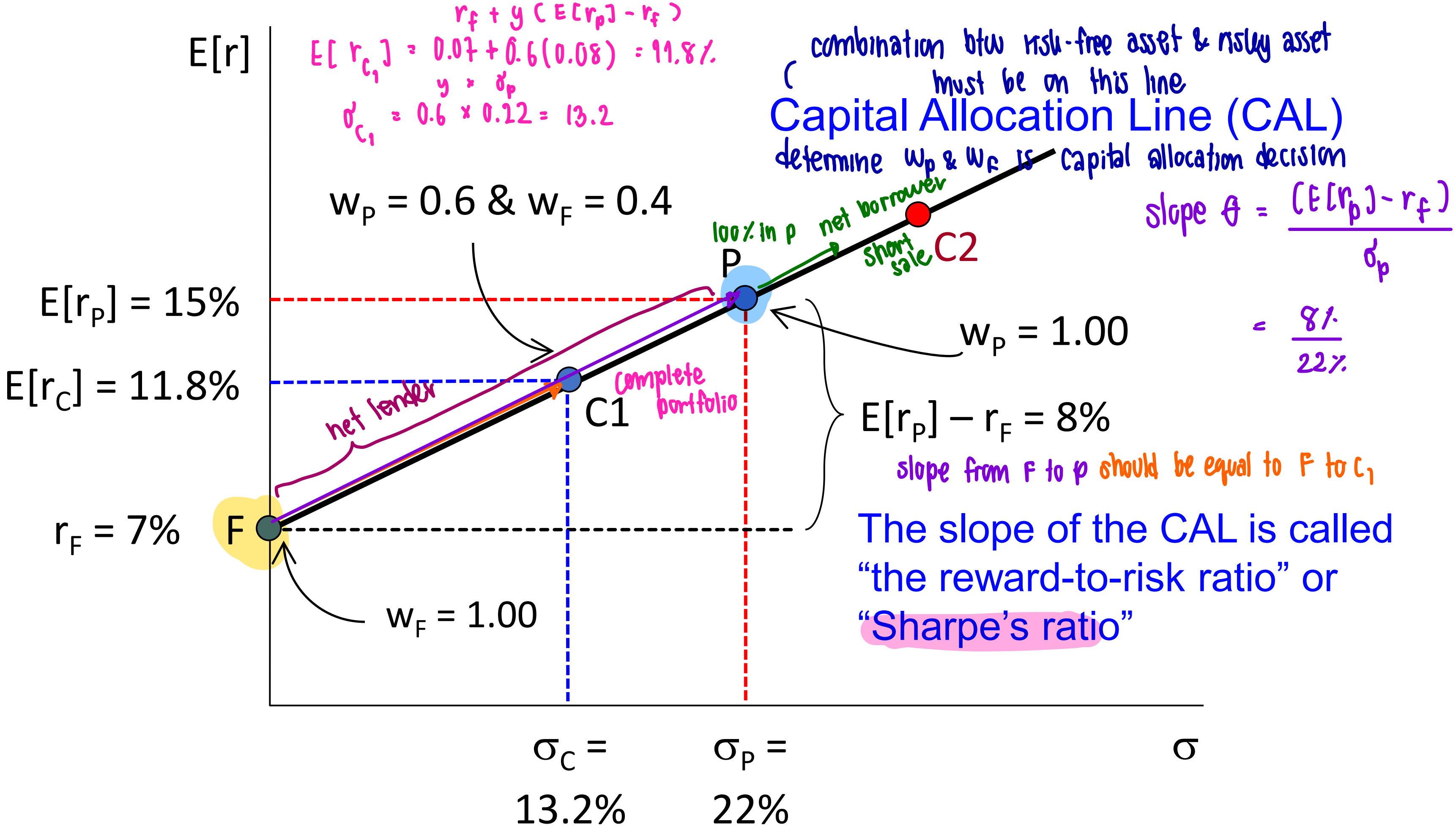
    E( )
  - $E[r_C] = yE[r_P] + (1-y)r_F \qquad = r_F + y\overbrace{(E[r_P]-r_F)}^{\text{risk premium}}$
  - $\sigma_C = \overbrace{(y^2\sigma_P^2 + (1-y)^2\underbrace{\sigma_F^2}_{\text{must be zero}} + 2y(1-y)\underbrace{\sigma_{P,F}}_{\text{must be zero}})}^{\text{variance of portfolio of two assets}}{}^{1/2}$

    $= [y^2\sigma_P^2]^{1/2} \qquad = y\sigma_P$
  - $y = \sigma_C/\sigma_P$
  - $E[r_C] = r_F + (\sigma_C/\sigma_P)\ (E[r_P]-r_F)$

    $= r_F + ((E[r_P]-r_F)/\sigma_P)\cdot\sigma_C$

# The Investment Opportunity Set with Risky and RF Assets

Combine the risk-free asset ($r_F$=7%) with a risky portfolio P ($E[r_P]$=15% and $\sigma_P$=22%). Using equations from the previous slide to calculate $E[r_{C1}]$ and $\sigma_{C1}$. The slopes of the line F-P and F-C1 are the same, $(E[r_P]–r_F)/\sigma_P = 0.36$.

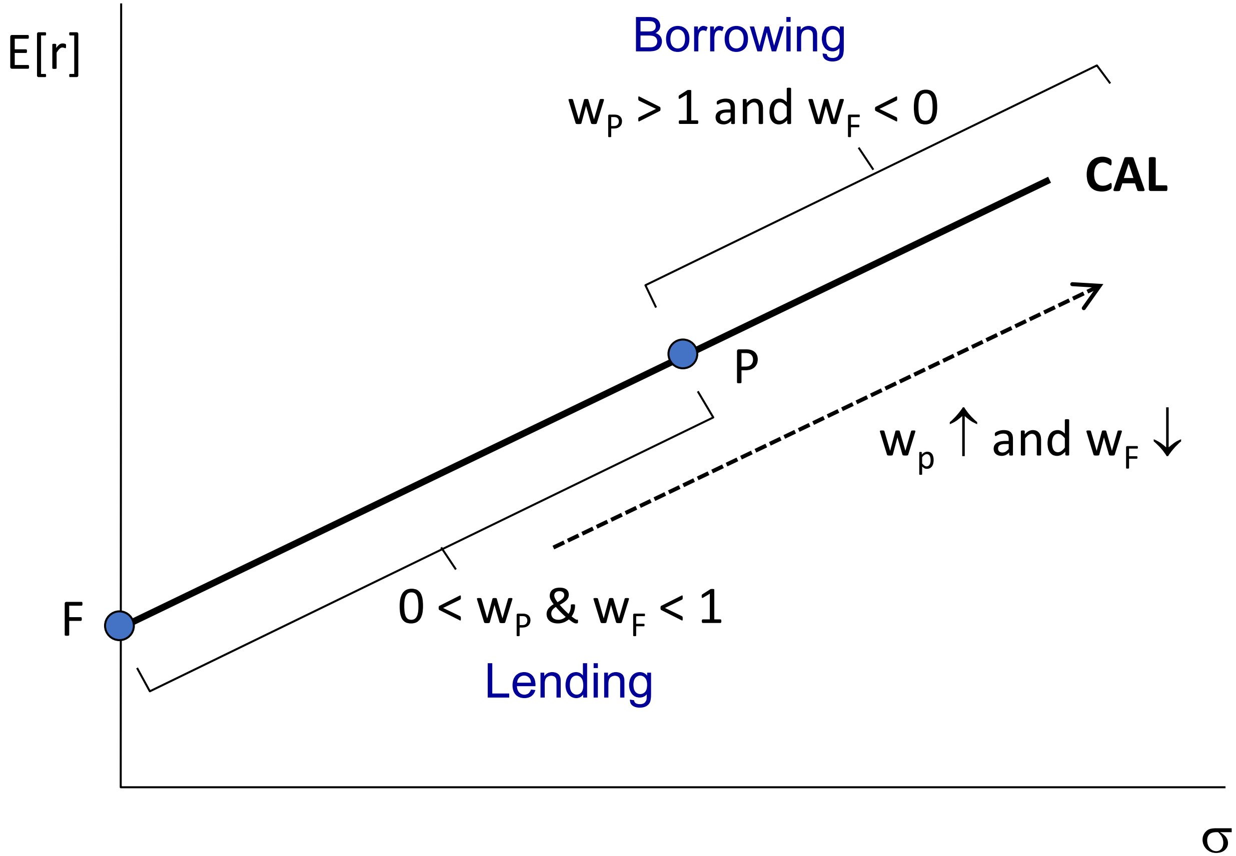



The slope of the CAL is called "the reward-to-risk ratio" or "Sharpe's ratio"

# 4.2 The Efficient Portfolio with RF Asset

- To move the complete portfolio beyond point P, i.e., $E[r_C] > E[r_P]$, the investor must borrow at the risk-free rate (i.e., sell bond) and invest the proceed plus his money in portfolio P.
- An investor starts with \$300 and borrows \$120. He invests all the fund in portfolio P. His complete portfolio is a levered portfolio.
  - $y = w_P = 420/300 = 1.40$
  - $(1-y) = w_F = -0.40$ (a short position = borrowing)
  - $E[r_C] = 0.07 + (1.4 \times 8) = 18.2\%$
  - $\sigma_C = 1.4 \times 22 = 30.8\%$
  - The slope of the line connects F and C = (18.2-7)/30.8 = 0.36

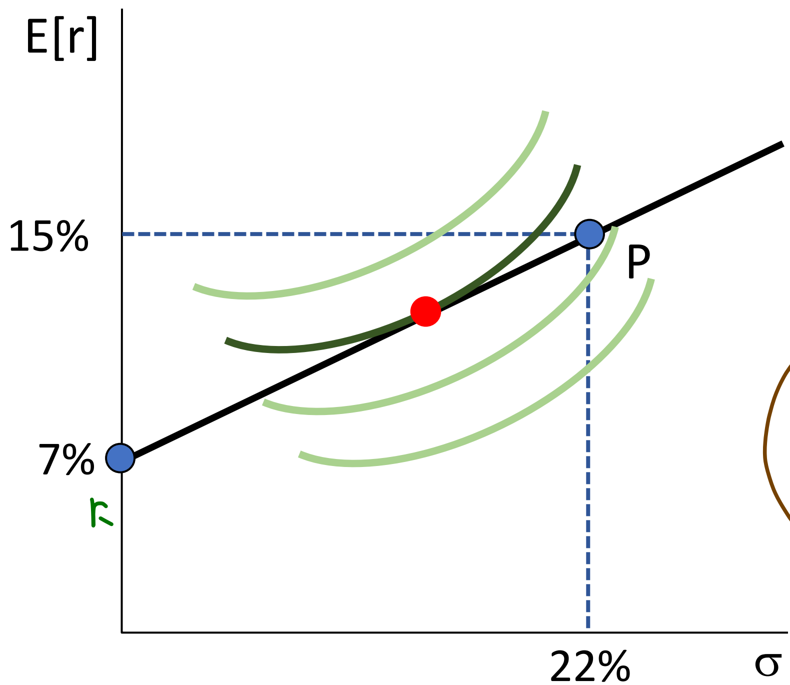
# The Investment Opportunity Set with Risky and RF Assets




Capital allocation decision does not change the reward-to-variability ratio

# The Optimal Complete Portfolio

$C \to \max[U(r)]$




If the utility function is known, we can solve for the optimal weights for the complete portfolio. Assume a quadratic utility function,

→ we can find the best complete portfolio

$$E[U(r_C)] = E[r_C] - (1/2)(A)\sigma_C^2$$

where A is a risk aversion factor.

$E[U(r_C)]$ can be rewritten as,

$$E[U(r_C)] = \boxed{r_F + y(E[r_P]-r_F)} - (1/2)(A)y^2\sigma_P^2$$

We want to find y that max $E[U(r_C)]$. This is done by taking the first order derivative of $E[U(r_C)]$ wrt. y and set the derivative to zero. The optimal y (y*) is found to be;

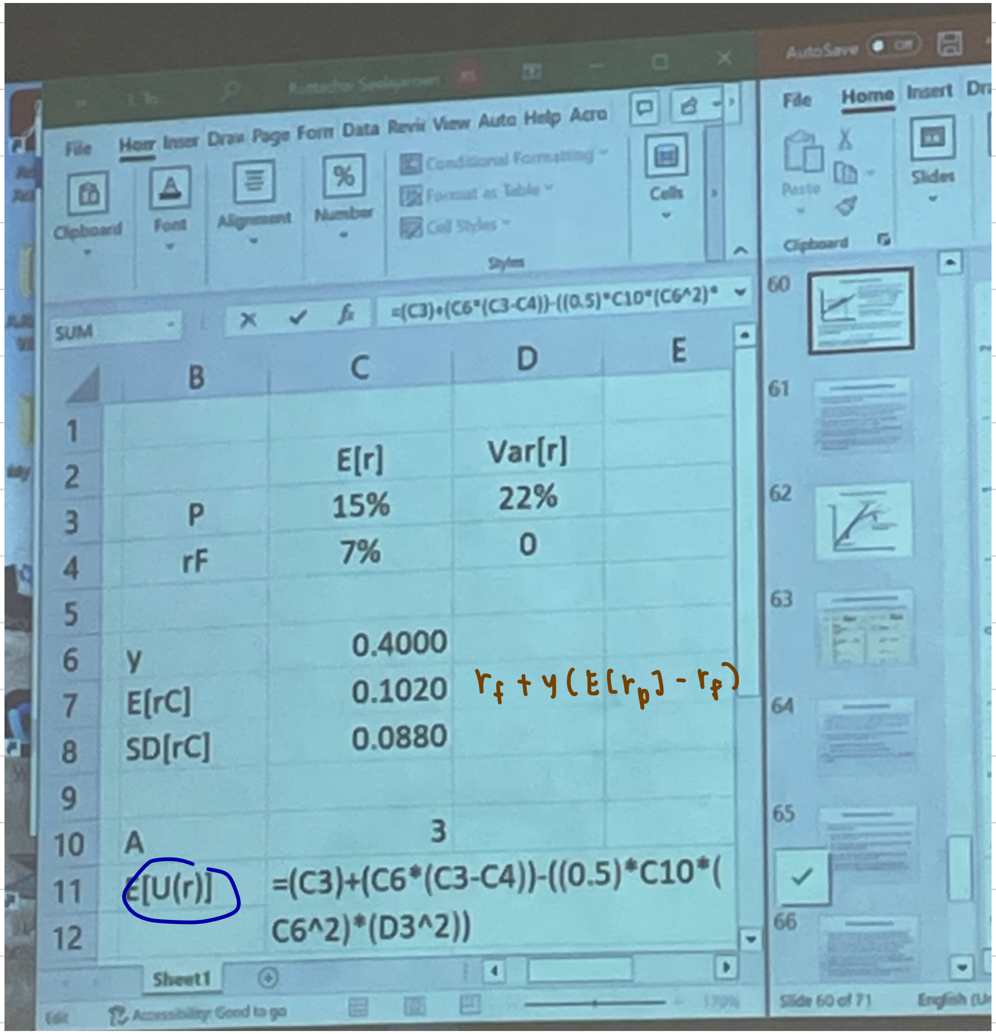
$$y^* = (E[r_P]-r_F) \,/\, A\sigma_P^2$$

For example, if A=3, y*=0.55. The optimal complete portfolio consists of 55% in portfolio P and 45% in risk-free asset.

Solver

↳ maximize expected utility $E[U(r)]$

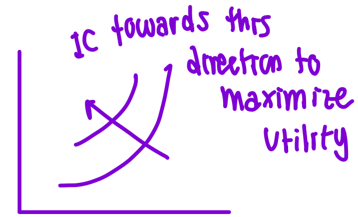
by changing $y$ (weight along CAL)



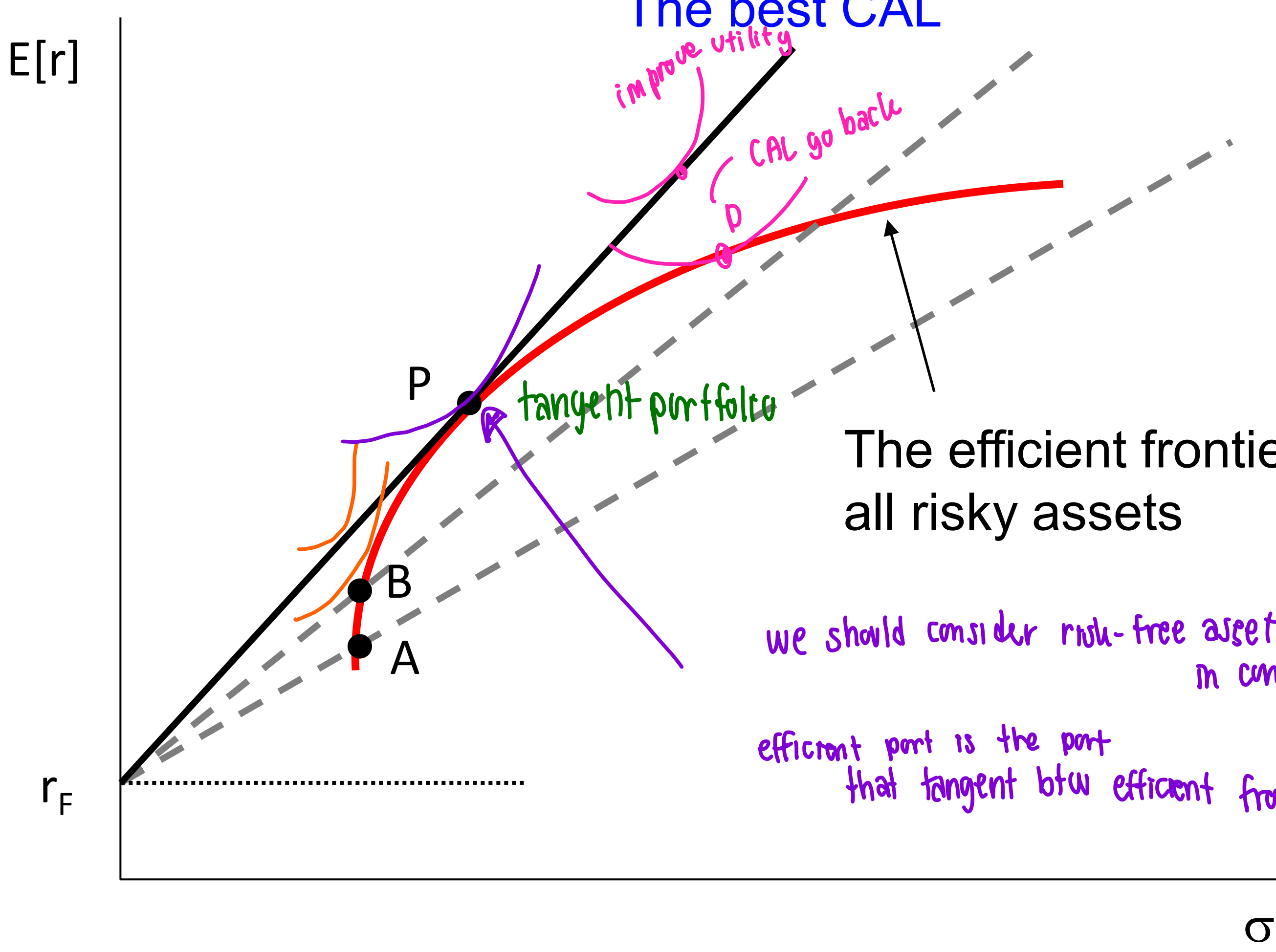
# 4.2 The Efficient Portfolio with RF Asset

- Start with the efficient frontier of risky assets, we pick a risky portfolio and combine with RF.
- The line connecting a risky portfolio with the risk-free asset is called the capital allocation line (CAL). It represents various combinations between the risky portfolio and the RF.
- The optimal risky portfolio is the one that results in the steepest CAL (or Max $SR_P = (E[r_P]-r_F)/\sigma_P$).
- It turns out that the optimal risky portfolio is independent from the investors' degree of risk aversion.
- The optimal risky portfolio can be identified using market information only. No information on the investor's risk aversion is needed.

# Finding the Optimal Risky Portfolio




highest slope! → highest sharp ratio




we should consider risk-free asset in complete port

efficient port is the port that tangent btw efficient frontier and CAL

SIM : $E[r_p] = E[\alpha_p] + \beta_p \cdot E[R_M] + \cancel{E[\varepsilon_p]} \to 0$ by defination

$\sum w_i \alpha_i + (\sum w_i \beta_i) E[R_M]$

$\sigma_p^2 = \underbrace{\beta_p^2 \sigma_M^2}_{\text{systematic}} + \underbrace{\sigma_{\varepsilon p}^2}_{\text{unsystematic}}$

$(\sum w_i \beta_i)^2$ $\quad$ $\cancel{\sum w_i^2 \sigma_{\varepsilon i}^2} \to 0$

# 4.2 The Efficient Portfolio with RF Asset

Look for weight

- The objective function for finding the optimal risky portfolio;

sharpe ratio

$$\text{Max.} \quad SR_P = \frac{E[r_P] - r_F}{\sigma_P}$$

$$\text{s.t.} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j} = \sigma_P^2$$

$$\sum_{i=1}^{N} w_i E[r_i] = E[r_P]$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0 \quad \forall\, i$$

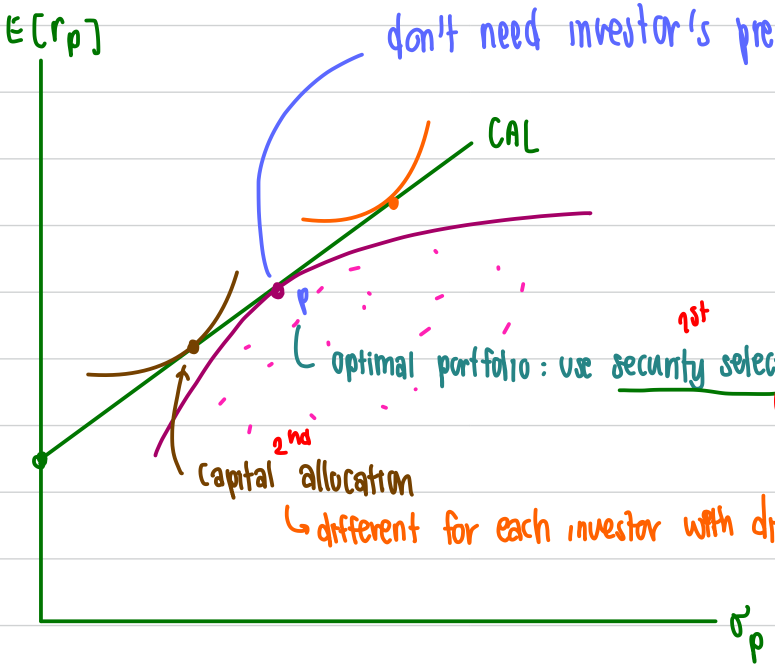
$$\text{Max.} \quad SR_P = \frac{E[r_P] - r_F}{\sigma_P}$$

$$\text{s.t.} \quad \mathbf{w^T V w} = \sigma_P^2$$

$$\mathbf{w^T r} = E[r_P]$$

$$\mathbf{w^T 1} = 1$$

$$w_i \geq 0 \quad \forall\, i$$




don't need investor's preference to find port p

↳ optimal portfolio : use security selection (1st)

→ can let 3rd person do it e.g. fund manager

↳ independ from risk preference

capital allocation (2nd)

↳ different for each investor with different risk preference

# 4.3 The Separation Property

- The independence between the optimal risky portfolio and the degree of risk aversion of an investor allows us to obtain the result called the separation property of investment decision (or the two funds separation theory).
- This result implies that the portfolio choice problem can be separated into two independent tasks.
  - Determination of optimal risky portfolio → security selection
  - Determination of optimal capital allocation
- This result is also called Tobin's Separation Theory, after Tobin (1958).

# Portfolio Construction Process

- The construction of the complete portfolio can be divided into 2 independent steps.
  - Capital Allocation Decision: Allocate funds between risky and risk-free assets
  - Security Selection Decision: Select risky securities to form the optimal risky portfolio.
- The first step requires the knowledge of the investor's risk aversion.
- The second step, however, can be delegated to a fund manager, since only market information is required in this step.

# 4.3 The Separation Property

market portfolio

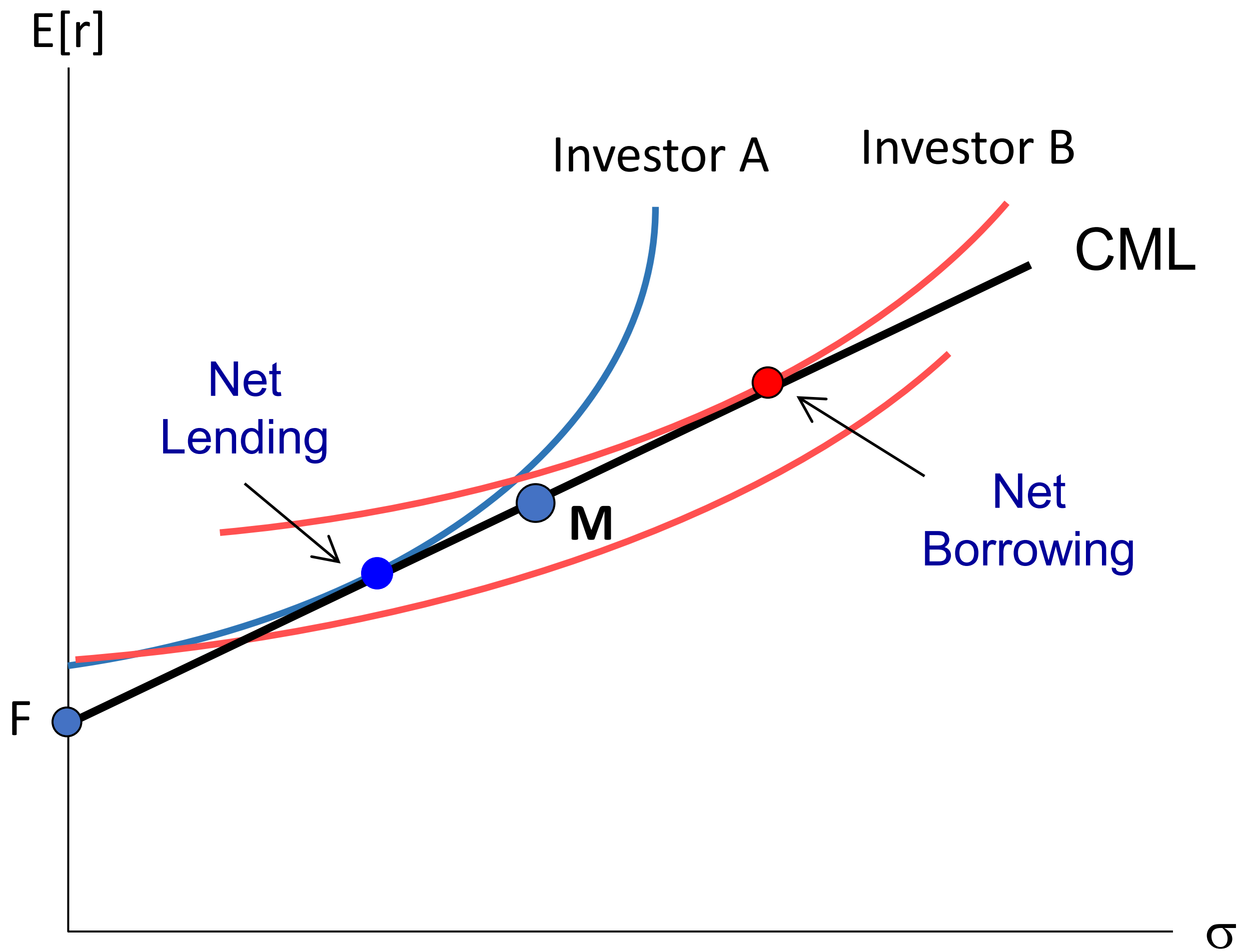
efficient frontier, port P, must be the same across individuals

everybody has same expected return and risk towards same security

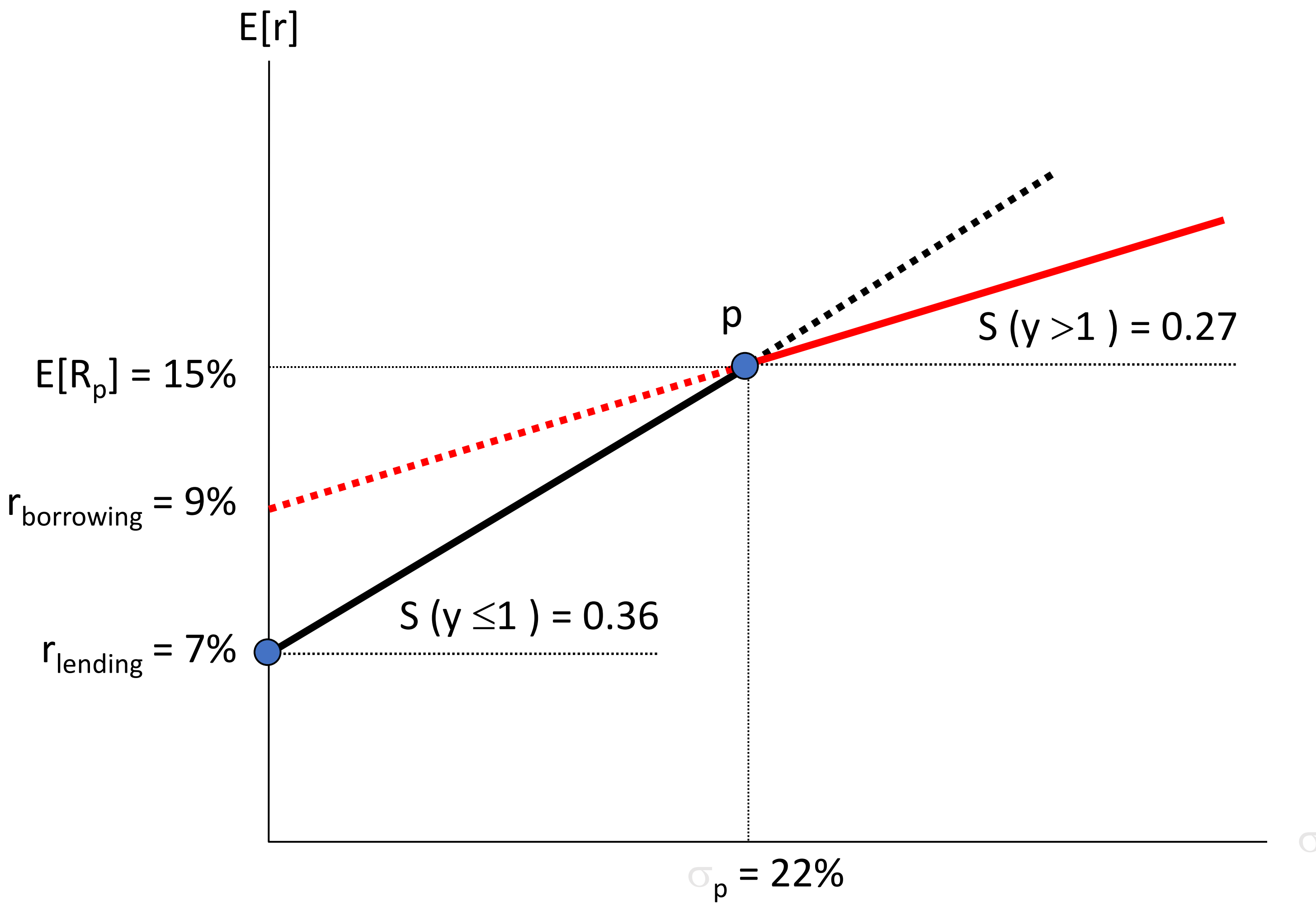
- If we further assume that all investors have homogenous expectation (and perfect capital market), then all investors will have the same efficient frontier of risky assets. They will choose to invest in the same risky portfolio P. They are different only in their capital allocation decisions.
- In this case P represent the market portfolio (M) which consists of all risky assets in the market weighted by relative market capitalization of the assets.
- Under this condition, P becomes the market portfolio. The CAL is called the Capital Market Line (CML).

will break down if mkt is not perfect → lending int rate ≠ borrowing int rate

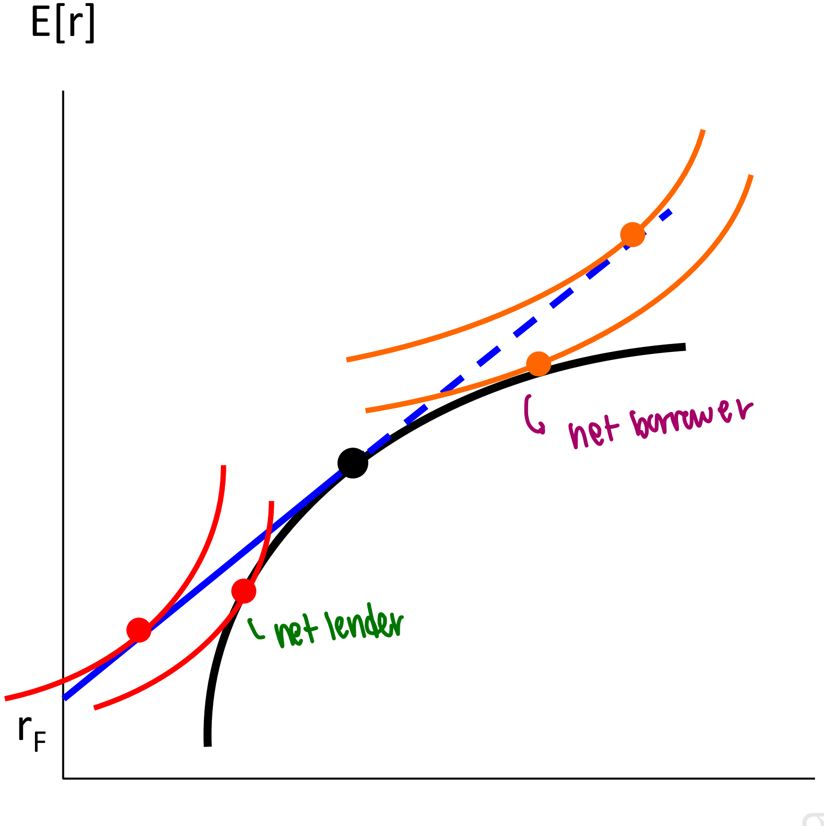
# Degree of Risk Aversion and Optimal Allocation

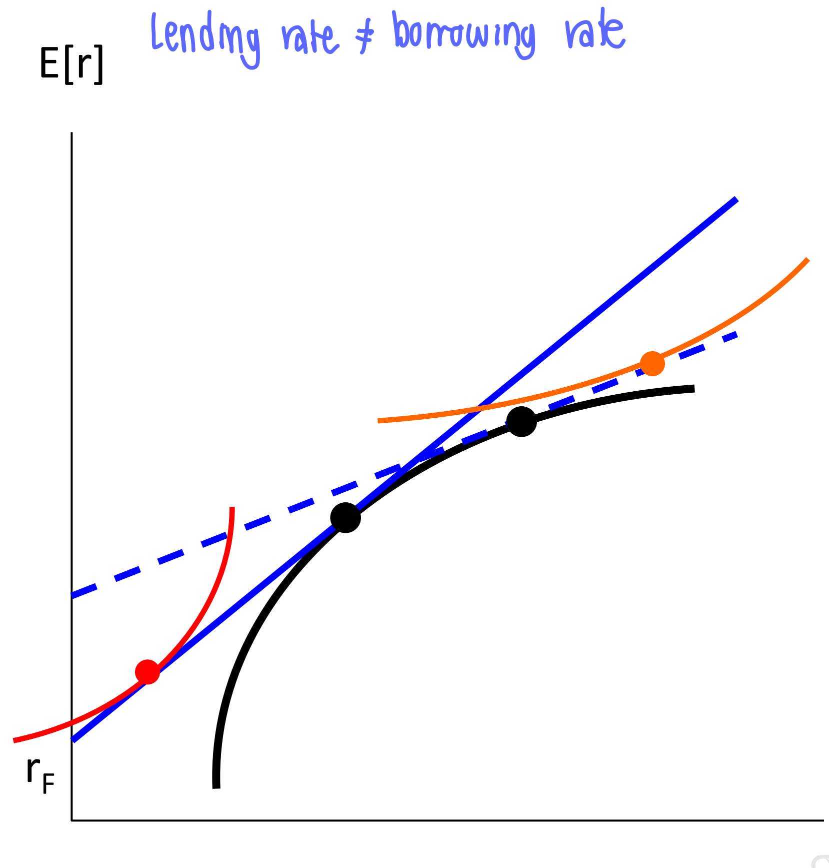


# Different in Lending and Borrowing Interest Rates



# Break-Down of the Separation Property



# Exercises

1. Discuss implications of Tobin's separation to the investment management industry.

2. Below is the information on expected return (in decimal points pa.) and variance-covariances among three securities. Find 3 efficient portfolios whose expected returns are 25%, 30% and 35% pa. Then, find 3 efficient portfolios whose SD's are 20%, 30% and 40% pa.

$$r = \begin{pmatrix} 0.20 \\ 0.30 \\ 0.40 \end{pmatrix}_{3\times 1} \qquad V = \begin{pmatrix} 0.0625 & 0.0700 & 0.1050 \\ & 0.1225 & 0.0840 \\ & & 0.3600 \end{pmatrix}_{3\times 3}$$

3. Now assume there is a risk-free asset with $r_F = 0.08$ pa. Find the efficient portfolio whose expected return is 30% pa. Compare with the portfolio in question 2 above.

# Exercises

4. The accompany data file contains monthly TRI of 10 national market indexes between 2001-2015. Choose 5 stock market indexes. Assume today is 29 May 2015, use the historical information from the latest 5 years to construct the efficient frontier of these 5 risky investments. Do not forget to annualize E[r] and SD[r].

5. Continue from Question 4, now assume the risk-free interest rate is 7% pa., construct the best CAL.