Financial Econometrics
Problem Set 2 Solution

3. Let X be a discrete random variable with the following p.m.f.

$$\boldsymbol{P\{X = x\} = \frac{\lambda^x \exp(-\lambda)}{x!}}$$

Then we say that X is distributed with Poisson Distribution with parameter λ. This is a well-known distribution proposed by a French Mathematician, Poisson (1837), and is used to express the probability of the number of events occurring in a given fixed interval of time, provided that each event happens independently of the others. Note that X can take values 0, 1, 2, …., and all positive integers. For example, X may be the number that you receive a phone call in a day, given that the chance that one person calls you doesn't affect the chance that the others call you.

a. Suppose that you know that your data $X_1, \dots, X_n \sim iid\ Poisson(\lambda)$ but you don't know λ and want to estimate it by MLE. Write down the likelihood function and log-likelihood function of observing the data set.

Since the observed data are drawn iid, the joint probability of observing this dataset, i.e. the likelihood, is

$$L(X;\lambda) = P\{X = X_1\} \cdot P\{X = X_2\} \dots P\{X = X_n\} = \prod_{i=1}^{n} \left\{\frac{\lambda^{X_i} \exp(-\lambda)}{X_i!}\right\}$$

For example, if the dataset is $\{X_1 = 3, X_2 = 7, X_3 = 4\}$, then the likelihood is

$$L(X;\lambda) = \left\{\frac{\lambda^3 \exp(-\lambda)}{3!}\right\} \cdot \left\{\frac{\lambda^7 \exp(-\lambda)}{7!}\right\} \cdot \left\{\frac{\lambda^4 \exp(-\lambda)}{4!}\right\}$$

Log-likelihood is the natural log of the likelihood function. Use the property of log:

$$\log(XY) = \log X + \log Y\,; \quad \log\left(\frac{X}{Y}\right) = \log X - \log Y$$
$$\log \lambda^X = X \log \lambda\,; \quad \log(\exp(-\lambda)) = -\lambda$$

Then, taking log of the likelihood function, we get

$$l(X;\lambda) = \sum_{i=1}^{n} \log\left\{\frac{\lambda^{X_i} \exp(-\lambda)}{X_i!}\right\} = \sum_{i=1}^{n} \{X_i \log \lambda + (-\lambda) - \log(X_i!)\}$$

b. Prove that $\hat{\lambda}_{MLE} = \bar{X}_n$.

The MLE can be found by the first-ordered condition:

$$0 = \frac{\partial}{\partial \lambda} l(X;\lambda) = \sum_{i=1}^{n} \left\{\frac{X_i}{\lambda} - 1\right\} \to \left\{\frac{1}{\lambda}\sum_{i=1}^{n} X_i\right\} - n = 0 \to \lambda = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$$

4. [Simultaneous Equation Revisit] Let the demand curve of a commodity is given by $\boldsymbol{Q = \beta_0 + \beta_1 P + u}$, where Q denotes quantity demanded, P denotes price, and U denotes factors other than price that determine demand. Let the supply curve of the same commodity is given by $\boldsymbol{Q = \gamma_0 + \gamma_1 P + v}$, where Q denotes quantity supply and v denotes factors other price that determine supply. Suppose that u and v both have zero mean, have variance σ_u^2, σ_v^2, and are mutually uncorrelated

a. **An equilibrium price (or market-clearing price) is a price that equates demand and supply of the commodity. Solve for the equilibrium price in terms of $\boldsymbol{\beta_0, \beta_1, \gamma_0, \gamma_1, U, v}$.**

$$\beta_0 + \beta_1 P + u = \gamma_0 + \gamma_1 P + v \rightarrow P = \frac{\gamma_0 - \beta_0}{\beta_1 - \gamma_1} + \frac{1}{\beta_1 - \gamma_1} v - \frac{1}{\beta_1 - \gamma_1} u$$

b. **Solve for an equilibrium quantity in terms of $\boldsymbol{\beta_0, \beta_1, \gamma_0, \gamma_1, U, v}$.**

$$Q = \beta_0 + \beta_1 P + u = \beta_0 + \beta_1 \left(\frac{\gamma_0 - \beta_0}{\beta_1 - \gamma_1} + \frac{1}{\beta_1 - \gamma_1} v - \frac{1}{\beta_1 - \gamma_1} u \right) + u$$

$$= \frac{\beta_0(\beta_1 - \gamma_1)}{\beta_1 - \gamma_1} + \frac{\beta_1(\gamma_0 - \beta_0)}{\beta_1 - \gamma_1} + \frac{\beta_1}{\beta_1 - \gamma_1} v - \frac{\beta_1 - (\beta_1 - \gamma_1)}{\beta_1 - \gamma_1} u$$

$$= \frac{\beta_0 \gamma_1 + \beta_1 \gamma_0}{\beta_1 - \gamma_1} + \frac{\beta_1}{\beta_1 - \gamma_1} v + \frac{\gamma_1}{\beta_1 - \gamma_1} u$$

c. **Find the variance of the equilibrium price and quantity found in parts a) and b).**

Since $cov(u, v) = 0$,

$$Var(P) = Var\left(\frac{\gamma_0 - \beta_0}{\beta_1 - \gamma_1} + \frac{1}{\beta_1 - \gamma_1} v - \frac{1}{\beta_1 - \gamma_1} u \right) = \left(\frac{1}{\beta_1 - \gamma_1} \right)^2 \sigma_v^2 + \left(\frac{1}{\beta_1 - \gamma_1} \right)^2 \sigma_u^2$$

$$Var(Q) = Var\left(\frac{\beta_0 \gamma_1 + \beta_1 \gamma_0}{\beta_1 - \gamma_1} + \frac{\beta_1}{\beta_1 - \gamma_1} v + \frac{\gamma_1}{\beta_1 - \gamma_1} u \right) = \left(\frac{\beta_1}{\beta_1 - \gamma_1} \right)^2 \sigma_v^2 + \left(\frac{\gamma_1}{\beta_1 - \gamma_1} \right)^2 \sigma_u^2$$

d. **Find the covariance between the equilibrium price and quantity found in parts a) and b).**

$$cov(P, Q) = cov\left(\frac{\gamma_0 - \beta_0}{\beta_1 - \gamma_1} + \frac{1}{\beta_1 - \gamma_1} v - \frac{1}{\beta_1 - \gamma_1} u, \frac{\beta_0 \gamma_1 + \beta_1 \gamma_0}{\beta_1 - \gamma_1} + \frac{\beta_1}{\beta_1 - \gamma_1} v + \frac{\gamma_1}{\beta_1 - \gamma_1} u \right)$$

$$= cov\left(\frac{1}{\beta_1 - \gamma_1} v - \frac{1}{\beta_1 - \gamma_1} u, \ \frac{\beta_1}{\beta_1 - \gamma_1} v + \frac{\gamma_1}{\beta_1 - \gamma_1} u \right)$$

$$= \beta_1 \left(\frac{1}{\beta_1 - \gamma_1} \right)^2 \sigma_v^2 + \gamma_1 \left(\frac{1}{\beta_1 - \gamma_1} \right)^2 \sigma_u^2$$

e. **Normally, we only observe equilibrium prices and quantities when collecting for data on actual transactions, as customers and sellers must agree upon the prices. Suppose you use the OLS to estimate the model $\boldsymbol{Q = \alpha_0 + \alpha_1 P + \eta}$. What is the probability limit of the OLS estimator for $\boldsymbol{\alpha_1}$, i.e. $\boldsymbol{plim(\widehat{\alpha}_1)}$, in terms of $\boldsymbol{\beta_0, \beta_1, \gamma_0, \gamma_1, \sigma_U^2, \sigma_v^2}$. Note that this is the best linear approximation of quantity given price (i.e. interpretation 2 of the linear regression model). Is it equal to the effect of price on demand ($\boldsymbol{\beta_1}$)? Is it equal to the effect of price of supply ($\boldsymbol{\gamma_1}$)?**

$$\hat{\alpha}_1^{OLS} = \frac{\widehat{cov(P, Q)}}{\widehat{Var(P)}} \xrightarrow{p} \frac{cov(P, Q)}{Var(P)} = \frac{\left(\frac{1}{\beta_1 - \gamma_1} \right)^2 [\beta_1 \sigma_v^2 + \gamma_1 \sigma_u^2]}{\left(\frac{1}{\beta_1 - \gamma_1} \right)^2 [\sigma_v^2 + \sigma_u^2]} \neq \beta_1 \ or \ \gamma_1$$

Note that demand curve is usually negative sloping, which means β_1 should be expected to be negative. On the contrary, we should expect slope of a supply curve γ_1 to be positive. Thus, $\beta_1 < \gamma_1$, and the OLS estimated relationship between price and quantity:

$$\alpha_1 = \frac{\beta_1 \sigma_v^2 + \gamma_1 \sigma_u^2}{\sigma_v^2 + \sigma_u^2} > \frac{\beta_1 \sigma_v^2 + \beta_1 \sigma_u^2}{\sigma_v^2 + \sigma_u^2} = \beta_1$$

So, the OLS tends to give higher value of the slope of the demand curve. But since $\beta_1 < 0$, this means that the magnitude of the OLS estimates, if negative, should be smaller than $|\beta_1|$. In other words, the actual demand curve slope should be steeper than the OLS estimate, or the OLS may give a positive slope. Similarly, the actual supply curve will be steeper than the OLS estimate, because

$$\alpha = \frac{\beta_1 \sigma_v^2 + \gamma_1 \sigma_u^2}{\sigma_v^2 + \sigma_u^2} < \frac{\gamma_1 \sigma_v^2 + \gamma_1 \sigma_u^2}{\sigma_v^2 + \sigma_u^2} = \gamma_1$$

5. **Suppose that we want to study the effect of policy interest rate, denoted by int, on inflation rate as denoted by inf. Suppose that we use the finite distributed lag model and get the following result.**

Dependent Variable inf				
Regressors	**_con**	$\boldsymbol{int}$	$\boldsymbol{l.int}$	$\boldsymbol{l2.int}$
Coefficient	**1.633**	**0.479**	**-0.157**	**-0.323**
Std. Error	**0.65**	**0.22**	**0.004**	**0.023**

a. **What is the immediate impact propensity of the policy rate on inflation?**
The model is

$$inf_t = \beta_0 + \beta_1 int_t + \beta_2 int_{t-1} + \beta_3 int_{t-2} + u_t$$

The immediate impact propensity is defined as the change that happens in the same period as a temporary policy change. That is, for 1 unit change of int at time τ, the impact propensity is

$$inf_\tau - inf_{\tau-1}$$

To simplify thing without loss of generality, we can let the policy interest rate be zero except for time τ that is raised to 1, and we can ignore the error term. So,

$$inf_\tau = \beta_0 + \beta_1 int_\tau + \beta_2 int_{\tau-1} + \beta_3 int_{\tau-2} = \beta_0 + \beta_1$$
$$inf_{\tau-1} = \beta_0 + \beta_1 int_{\tau-1} + \beta_2 int_{\tau-2} + \beta_3 int_{\tau-3} = \beta_0$$

Therefore, the impact propensity is β_1, which is estimated to be 0.479

b. **Suppose that the change in interest rate is temporary, what are the effect of policy rate change on inflation rate after one, two, and three periods?**
Again for a temporary change, to simplify thing without loss of generality, we can let the policy interest rate be zero all periods except for time τ that is raised to 1:

Period	int_t	int_{t-1}	int_{t-2}
$\tau - 2$	0	0	0
$\tau - 1$	0	0	0
τ	1	0	0
$\tau + 1$	0	1	0
$\tau + 2$	0	0	1

$$inf_{\tau+1} = \beta_0 + \beta_1 int_{\tau+1} + \beta_2 int_\tau + \beta_3 int_{\tau-1} = \beta_0 + \beta_2 \rightarrow \ inf_{\tau+1} - inf_{\tau-1} = \beta_2$$
$$inf_{\tau+2} = \beta_0 + \beta_1 int_{\tau+2} + \beta_2 int_{\tau+1} + \beta_3 int_\tau = \beta_0 + \beta_3 \rightarrow \ inf_{\tau+2} - inf_{\tau-1} = \beta_3$$

Plugging in the value, therefore, the estimated effect after one period is -0.157, and after two period is -0.323

c. **What is the long-run multiplier of the policy rate on inflation?**
The long-run multiplier is defined as the final effect of one unit permanent change, which is $\beta_1 + \beta_2 + \beta_3$ and estimated to be 0.479 – 0.157 – 0.323 = -0.001. Or in words, if the government increases the policy rate by one unit and keep it permanently, inflation rate in the long run will decrease by 0.001 unit.

d. **Suppose that the central bank set interest rate by also looking at contemporaneous inflation rate. Do you think we can use OLS to estimate this model? If not, what should be a proper estimator to use?**
If the central bank set interest rate, int_t, by looking at contemporaneous inflation rate, inf_t, this means that inf_t causes int_t in addition to int_t causes inf_t. In other words, this is a problem of simultaneous equations, which is a kind of endogeneity problem. So, the OLS estimator is inconsistent and cannot be used.

6. **Suppose that we have a sample of working-age Thai people with college degrees, and we want to study factors contributing to the probability that they invest in the Stock Exchange of Thailand. Let SET be a dummy variable taking value 1 if a person invests in SET; inc is the person's annual income in units of thousand Baht; $exper$ is the person's working experience in years; age is the person's age in years; and fin is a dummy variable taking 1 if the person graduated with a finance-related degree. Suppose that we use the Logit model to estimate and get the following result:**

Dependent variable: SET					
Regressors	**Constant**	inc	$exper$	age	fin
$\hat{\beta}$	**-2.74**	**0.0112**	**0.0079**	**-0.0014**	**1.280**
$se(\hat{\beta})$	**(0.12)**	**(0.001)**	**(0.003)**	**(0.028)**	**(0.56)**

a. **What is the probability predicted by the estimated model that a 30 years old person with a Finance degree and 7 years of work experience who earn 10,000 Baht a month would invest in SET?**

From the given information, $age = 30, fin = 1, exper = 7$, and $inc = 120$, because 10,000 Baht a month is equivalent to 120,000 Baht annual income, which is 120 thousand Baht. This is the Logit model; so, the predicted probability that $SET = 1$ is

$$\frac{\exp\{-2.74 + 0.0112 \times 120 + 0.0079 \times 7 - 0.0014 \times 30 + 1.280\}}{1 + \exp\{-2.74 + 0.0112 \times 120 + 0.0079 \times 7 - 0.0014 \times 30 + 1.280\}}$$

$$\Pr\{SET = 1 | inc, exper, age, fin\} = \frac{e^{-0.1027}}{1 + e^{-0.1027}} = 47.43\%$$

b. **For a person ages 25 years old with 2 years of work experience earning 120,000 Baht a year, what is the marginal effect of a finance degree on the probability that he invests in SET?**

From the given information, $age = 25, exper = 2$, and $inc = 120$. Then, if he doesn't have a finance degree,

$$X\hat{\beta} = -2.74 + 0.0112 \times 120 + 0.0079 \times 2 - 0.0014 \times 25 = -1.4152$$

$$\Pr\{SET = 1|inc, exper, age, fin\} = \frac{e^{-1.4152}}{1+e^{-1.4152}} = 19.54\%$$

But if he gets a finance degree, then $X\hat{\beta} = -1.4152 + 1.280 = -0.1352$, and

$$\Pr\{SET = 1|inc, exper, age, fin\} = \frac{e^{-0.1352}}{1+e^{-0.1352}} = 46.63\%$$

Therefore, the effect of a finance degree on the probability is 46.63 – 19.54 = 27.09%

c. **For a person ages 25 years old with a finance degree and 2 years of work experience earning 120,000 Baht a year, what is the effect of income level on the probability that he invests in SET?**

The effect of income level on the probability is

$$\frac{\partial}{\partial inc}\left\{\frac{\exp(X\beta)}{1+\exp(X\beta)}\right\} = \left\{\frac{\exp(X\beta)}{(1+\exp(X\beta))^2}\right\}\cdot\frac{\partial}{\partial inc}X\beta$$
$$= \left\{\frac{\exp(\beta_0+\beta_1 inc+\beta_2 exper+\beta_3 age+\beta_4 fin)}{(1+\exp(\beta_0+\beta_1 inc+\beta_2 exper+\beta_3 age+\beta_4 fin))^2}\right\}\beta_1$$

From the given information, $e = 25, exper = 2, fin = 1, inc = 120$. Then, from part b), $X\hat{\beta}$ = -0.1352. Therefore, the estimated effect is

$$\left\{\frac{\exp(X\hat{\beta})}{\left(1+\exp(X\hat{\beta})\right)^2}\right\}\hat{\beta}_1 = \left\{\frac{e^{-0.1352}}{(1+e^{-0.1352})^2}\right\}\times 0.0112 = 0.0028 = 0.28\%$$

d. **Now suppose that the given result is from Probit instead of Logit. What is the probability predicted by the estimated model that a 30 years old person with a Finance degree and 7 years of work experience who earns 10,000 Baht a month would invest in SET?**

The given information of the sample is the same as in part a); so, $X\hat{\beta} = -0.1027$. For the Probit model, the estimated probability is

$$\Pr\{SET = 1|inc, exper, age, fin\} = \Phi(X\hat{\beta}) = \Phi(-0.1027) = 45.91\%$$

where $\Phi(.)$ is the cdf of the standard Normal distribution.

7. Suppose a researcher has a sample of 200 monthly observations on the stochastic process $\{Y_t\}$ and wants to fit the data with an ARMA model. In order to select the model, he computes the ACF and PACF as follows:

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
ACF	**0.83**	**0.71**	**0.60**	**0.45**	**0.44**	**0.35**	**0.29**	**0.20**	**0.11**
PACF	**0.83**	**0.16**	**-0.09**	**0.05**	**0.04**	**-0.05**	**0.01**	**0.10**	**-0.03**

a **The researcher decides not to use the MA model to fit the data. Based on the given correlogram, why do you think he makes such decision?**

For MA(q), the ACF will be non-zero up to $k = q$, then dropping to zero. We do not observe this pattern in the given correlogram. Instead, we see that the ACF is gradually decreasing like an AR process, and the PACF pattern suggests that it's more like an AR process.

b **Suppose that the researcher suspects that AR(2) may be better. So, he runs AR(2) and get the following result:**

$$\boldsymbol{Y_t = 0.74Y_{t-1} + \widehat{\rho}_2 Y_{t-2} + e_t}$$

What is the numerical value of $\widehat{\boldsymbol{\rho}}_2$ in the above equation?

Since the PACF($k = 2$) is derived by estimating the coefficient of the second lag of the AR(2) model. Then, $\hat{\rho}_2 = 0.16$

c **Conduct a hypothesis testing to see if he would prefer AR(1) or AR(2). State the null and alternative hypotheses, test statistics, distribution of the test statistics, critical value at 5% significance level, p-value, and the conclusion of the test whether AR(1) or AR(2) is preferred.**

First, we may test that PACF(1) is not equal to zero by running the following regression and conducting the following hypothesis test:

$$Y_t = \rho_{11} Y_{t-1} + e_t$$
$$H_0 : \rho_{11} = 0 \quad vs \quad H_1 : \rho_{11} \neq 0$$

If we reject the null hypothesis, it's possible that it's AR(1); so, moving on to run the following regression and testing the following hypothesis:

$$Y_t = \rho_{21} Y_{t-1} + \rho_{22} Y_{t-2} + e_t$$
$$H_0 : \rho_{22} = 0 \quad vs \quad H_1 : \rho_{22} \neq 0$$

If we fail to reject the null hypothesis, then we can conclude for AR(1). If not, AR(2) is better for this dataset. The test statistics (T_n) are $\sqrt{T} \cdot \hat{\rho}_{11}$ and $\sqrt{T} \cdot \hat{\rho}_{22}$ respectively for the first and the second tests above, and both of them are distributed according to the Standard Normal Distribution.

If we have to do just one test, then conduct the second test:

$$T_n = \sqrt{200} \times 0.16 = 2.26 \sim N(0,1)$$

With the 5% significance level for two-sided test, the critical value is 1.96, which is less than our test statistics. Alternatively, using the test-stat to find p-value:

$$2\big(1 - \Phi(2.26)\big) = 2(1 - 0.9881) = 0.0238 < 5\%$$

So, we can reject the null hypothesis that the PACF(2) = 0. Hence, the AR(2) model would be better than the AR(1) here.

8. **A researcher uses a dataset that comprises a sample of 545 full-time working males who have completed their schooling by 1980. This is a balanced panel covering the period from years 1980 to 1987. He does panel data analysis and gets the following results**

Dependent Variable: ln(wage)

Regressors	**Between**	**Fixed Effect**	**Pooled OLS**	**Random Effect**
Constant	**0.490** (0.221)	-	**-0.034** (0.065)	**-0.104** (0.111)
Schooling	**0.095** (0.011)	-	**0.099** (0.005)	**0.101** (0.009)
Experience	**-0.050** (0.050)	**0.116** (0.008)	**0.089** (0.010)	**0.112** (0.008)
Experience2	**0.0051** (0.0032)	**-0.0043** (0.0006)	**-0.0028** (0.0007)	**-0.0041** (0.0006)
Union Member	**0.274** (0.047)	**0.081** (0.019)	**0.180** (0.017)	**0.106** (0.018)
Married	**0.145** (0.041)	**0.045** (0.018)	**0.108** (0.016)	**0.063** (0.017)
Black	**-0.139** (0.049)	-	**-0.144** (0.024)	**-0.144** (0.048)
Hispanic	**0.005** (0.043)	-	**0.016** (0.021)	**0.020** (0.043)
Public Sector	**-0.056** (0.109)	**0.035** (0.039)	**0.004** (0.037)	**0.030** (0.036)

where *Wage* indicate earning in unit of dollars per month; *Schooling* is years of education, *Experience* is working experience in unit of years, *Union member* is a dummy variable taking value 1 if the worker is a member of a union; *Married* is a dummy variable taking value 1 if the worker is married; *Black* is a dummy variable taking value 1 if the worker is African American; *Hispanic* is a dummy variable taking value 1 if the worker is Hispanic; and *Public Sector* is a dummy variable taking value 1 if the worker works in the public sector. Standard errors are displayed in parentheses.

a. **Explain why there are no results for *Schooling, Black*, and *Hispanic* estimated by the Fixed Effect model?**

These variables are time-invariant (i.e. don't change over time). Hence, $X_{it} - \bar{X}_i = 0$ for all i, which is invalid as a regressor.

b. **Suppose somebody questions that married male workers have higher wage than the others. Use the result from the Fixed Effect model, how would you test this hypothesis? State the null hypothesis, alternative hypothesis, test-statistics, distribution of the test statistics, 5% critical value, and the p-value.**

This dataset only contain male workers; so, we just need to test the impact of being married, which is the variable "Married." Let β_4 denote the population coefficient of this variable, then we test

$$H_0: \beta_4 = 0 \ \ vs \ \ H_1: \beta_4 > 0$$

Using the FE estimator:

$$T_n = \frac{\hat{\beta}_4^{FE}}{SE(\hat{\beta}_4^{FE})} = \frac{0.045}{0.018} = \frac{5}{2} = 2.5 \ \sim \ t_{545\times(8-1)-5} \approx N(0,1)$$

Note that the degree of freedom for FE estimator is $n(T-1) - \#regressors$. For the 5% significance level, the one-sided critical value is 1.645. The p-value = $1 - \Phi(2.5) =$

0.0062. Since the t-stat is greater than the critical value, and the p-value is less than the significance level, we can reject the null hypothesis and conclude that married workers enjoyed higher wage

c. Construct a 95% confidence interval for the effect of getting married on wage.

A conventional CI is based on the two-sided critical value, which is 1.96 for 5% significance level:

$$\Pr\left\{|T_n| = \left|\frac{\hat{\beta}_4 - \beta_4}{SE(\hat{\beta}_4)}\right| < 1.96\right\} = 95\%$$

$$\hat{\beta}_4 - 1.96SE(\hat{\beta}_4) < \beta_4 < \hat{\beta}_4 + SE(\hat{\beta}_4)$$

So, the 95% CI is (0.045 – 1.96*0.018, 0.045 + 1.96*0.045) = (0.00972, 0.08028)

d. Suppose that the researcher carries out the Hausman Test and get the test statistic equal to 7.90. State the distribution of the Hausman Test Statistics, the critical value at 5% significance level, whether you can reject the null hypothesis, and your conclusion about Random Effect model.

The Hausman Test is $H_0: \beta^{RE} = \beta^{FE}\ vs\ H_1: \beta^{RE} \neq \beta^{FE}$ where β^{RE} denoted the population coefficients estimated by the Random Effect estimator, and β^{FE} denoted the population coefficients estimated by the Fixed Effect estimator. The test statistics distributes according to Chi-square with degree of freedom equal to the number of the FE estimates, which is 5 in this case. The critical value for χ_5^2 at 5% significance level is 11.1. So, since 7.90 < 11.1, we fail to reject the null and conclude that the underlying assumption of the Random Effect model is fine.

e. Do you think the Between estimator is consistent if you reject the null hypothesis of the Hausman test? Why?

If we reject the null of the Hausman Test, then the underlying assumption of the Random Effect model that the unobserved heterogeneity (α_i) is uncorrelated with the regressors is false. The Between estimator still has this α_i in the error term which is correlated with the regressor $\bar{X}_i$. Thus, it isn't consistent.

f. Do you think the Pooled OLS estimator is consistent if you can reject the null hypothesis of the Hausman test? Why?

Again, since we have $cov(\alpha_i, X_{it}) \neq 0$ if we can reject the Hausman Test, then OLS is inconsistent.

9. Let $Y_t = 0.9Y_{t-1} + e_t + 0.5e_{t-1} + 0.2e_{t-2}$, where $\{e_t\}$ is the White-Noise process. Compute the covariance $cov(Y_t, Y_{t+k})$ for $k = 1, 2, 3, 4$ in terms of variance of Y, and variance of e, i.e. σ_Y^2, σ_e^2 respecitively.

Recall that $cov(e_\tau, e_k) = 0$ and $cov(Y_\tau, e_s) = 0$ for all $k \neq \tau$ and $\tau < s$.

Let $Y_t = \rho Y_{t-1} + e_t + \alpha e_{t-1} + \beta e_{t-2}$. Consider the following covariance

k	Y_{t+k}	Covariance with Y_{t+k}		
		e_{t+1}	e_t	e_{t-1}
-1	$\rho Y_{t-2} + e_{t-1} + \alpha e_{t-2} + \beta e_{t-3}$	0	0	σ_e^2
0	$\rho Y_{t-1} + e_t + \alpha e_{t-1} + \beta e_{t-2}$	0	σ_e^2	$\rho Cov(Y_{t-1}, e_{t-1}) + \alpha\sigma_e^2 = (\rho+\alpha)\sigma_e^2$

Then, $Cov(Y_t, Y_{t+1}) = Cov(Y_t, \rho Y_t + e_{t+1} + \alpha e_t + \beta e_{t-1})$

$$= Cov(Y_t, \rho Y_t) + Cov(Y_t, e_{t+1}) + Cov(Y_t, \alpha e_t) + Cov(Y_t, \beta e_{t-1})$$

$$= \rho\sigma_Y^2 + 0 + \alpha\sigma_e^2 + \beta(\rho+\alpha)\sigma_e^2$$

$$= \rho\sigma_Y^2 + (\alpha + \beta\rho + \beta\alpha)\sigma_e^2$$

Now, find $Cov(Y_t, Y_{t+2})$

$$Cov(Y_t, Y_{t+2}) = Cov(Y_t, \rho Y_{t+1} + e_{t+2} + \alpha e_{t+1} + \beta e_t)$$

$$= \rho Cov(Y_t, Y_{t+1}) + \beta Cov(Y_t, e_t)$$

$$= \rho[\rho\sigma_Y^2 + (\alpha + \beta\rho + \beta\alpha)\sigma_e^2] + \beta\sigma_e^2$$

$$= \rho^2\sigma_Y^2 + (\beta + \alpha\rho + \beta\rho^2 + \beta\alpha\rho)\sigma_e^2$$

Now, find $Cov(Y_t, Y_{t+3})$

$$Cov(Y_t, Y_{t+3}) = Cov(Y_t, \rho Y_{t+2} + e_{t+3} + \alpha e_{t+2} + \beta e_{t+1})$$

$$= \rho Cov(Y_t, Y_{t+2})$$

$$= \rho[\rho^2\sigma_Y^2 + (\beta + \alpha\rho + \beta\rho^2 + \beta\alpha\rho)\sigma_e^2]$$

$$= \rho^3\sigma_Y^2 + \rho(\beta + \alpha\rho + \beta\rho^2 + \beta\alpha\rho)\sigma_e^2$$

Now, find $Cov(Y_t, Y_{t+4})$

$$Cov(Y_t, Y_{t+4}) = Cov(Y_t, \rho Y_{t+3} + e_{t+4} + \alpha e_{t+3} + \beta e_{t+2})$$

$$= \rho Cov(Y_t, Y_{t+3})$$

$$= \rho^4\sigma_Y^2 + \rho^2(\beta + \alpha\rho + \beta\rho^2 + \beta\alpha\rho)\sigma_e^2$$

Keep iterating, we will have that for $k \geq 2$,

$$Cov(Y_t, Y_{t+k}) = \rho^k\sigma_Y^2 + \rho^{k-2}(\beta + \alpha\rho + \beta\rho^2 + \beta\alpha\rho)\sigma_e^2$$

Then, we can plug in $\rho = 0.9, \alpha = 0.5, \beta = 0.2$ to get the numerical values.