Kalman Filter

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Contents

In this note I will be focusing on the state space model of type:

$$x_t = Ax_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q), \tag{1}$$
$$y_t = Cx_t + \delta_t, \delta_t \sim N(0, R). \tag{2}$$

Forward Filtering

```
import numpy as np;
def kfilter(y, num, x0, v0, A, C, Q, R):
    x_f = np.zeros(num); v_f = np.zeros(num);
    x_p = np.zeros(num); v_p = np.zeros(num);
    x_f[-1] = x0; v_f[-1] = v0;
    for i in range(num):
        x_p[i] = A*x_f[i-1]
        v_p[i] = A*v_f[i-1]*A + Q
        K = v_p[i]*C/(C*v_p[i]*C+R)
        x_f[i] = x_p[i] + K*(y[i]-C*x_p[i])
        v_f[i] = v_p[i] - K*C*v_p[i]
    return {'x_f':x_f,'v_f':v_f,'x_p':x_p,'v_p':v_p, 'K_T':K}
```

Kalman filtering computes $\mu_t, \Sigma_t, \mu_{t|t-1}, \Sigma_{t|t-1}$:

$$\begin{aligned}
\mu_t &= \mathrm{E}[x_t|y_{1:t}] \\
\Sigma_t &= \mathrm{E}[(x_t - \mu_t)(x_t - \mu_t)'|y_{1:t}] \\
\mu_{t|t-1} &= \mathrm{E}[x_t|y_{1:t-1}] \\
\Sigma_{t|t-1} &= \mathrm{E}[(x_t - \mu_{t|t-1})(x_t - \mu_{t|t-1})'|y_{1:t-1}].
\end{aligned}$$

These variables are denoted by `x_f, v_f, x_p, v_p` in function `kfilter` above. Procedure for estimating these values are summarized in two following steps: *prediction* and *measurement* step.

Prediction

At time step t, assume that we have computed μ_{t-1}, Σ_{t-1}, and that the distribution is normally distributed

$$x_{t-1}|y_{1:t-1} \sim N(\mu_{t-1}, \Sigma_{t-1}). \tag{3}$$

Prediction of the state variable x_t in the next step is given by:

$$\begin{aligned}
x_t|y_{t-1} &\sim N(\mu_{t|t-1}, \Sigma_{t|t-1}). \\
\mu_{t|t-1} &= \mathrm{E}[x_t|y_{1:t-1}], \\
&= \mathrm{E}[Ax_{t-1} + \epsilon_t|y_{1:t-1}], \\
&= A\mathrm{E}[x_{t-1}|y_{1:t-1}], && (\epsilon_t \text{ has mean zero}) \\
&= A\mu_{t-1}. \\
\Sigma_{t|t-1} &= \mathrm{Var}[x_t|y_{1:t-1}] \\
&= \mathrm{Var}[Ax_{t-1} + \epsilon_t|y_{1:t-1}], \\
&= \mathrm{Var}[Ax_{t-1}|y_{1:t-1}] + \mathrm{Var}[\epsilon_t|y_{1:t-1}], && (\mathrm{Cov}(x_{t-1}, \epsilon_t) = 0) \\
&= A\Sigma_{t-1}A' + Q.
\end{aligned}$$

Measurement

In prediction step, we have certain belief of how x_t is distributed given past observation $y_1, \ldots, y_{t-1}$. This can be quantitatively represented by $\mathrm{P}(x_t|y_{1:t-1})$, which is called *prior distribution* in the language of bayesian framework. In measurement step, a newly observed data point y_t is obtained first. The *likelihood* is according to Equaiton 2:

$$\mathrm{P}(y_t|x_t, y_{1:t-1}) = \frac{1}{|2\pi R|^{-\frac{1}{2}}} e^{-\frac{1}{2}(y_t - Cx_t)'R^{-1}(y_t - Cx_t)} \tag{4}$$

Conceptually, one can update distribution using bayes rule:

$$\mathrm{P}(x_t|y_t, D^{t-1}) \propto \mathrm{P}(y_t|x_t, y_{1:t-1}, D^{t-1}), \mathrm{P}(x_t|D^{t-1}), \qquad (D^{t-1} := y_{1:t-1})$$

in order to obtain *posterior* $\mathrm{P}(x_t|y_{1:t})$.

Kalman filtering deals with special case of such updating rules by assuming all distribution to be normal. For this reason, two parameters μ_t, Σ_t which is a representative for posterior distribution at time t can be computed analytically as:

$$\begin{aligned}
\mu_t &= \mu_{t|t-1} + K_t r_t \\
\Sigma_t &= (\mathbb{I} - K_t C_t)\, \Sigma_{t|t-1},
\end{aligned}$$

where,

$$\begin{aligned}
r_t &= y_t - \mathrm{E}[y_t|y_{1:t-1}], \\
&= y_t - \mathrm{E}[Cx_t + \delta_t|y_{1:t-1}], \\
&= y_t - C\mu_{t|t-1}, \\
K_t &= \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C' + R)^{-1}.
\end{aligned}$$

Backward Smoothing

```
def ksmooth(y, num, x_f, v_f, x_p, v_p, A, vvT):
    x_s = np.zeros(num); v_s = np.zeros(num); J = np.zeros(num-1); P = np.zeros(num)
    x_s[num-1] = x_f[num-1]; v_s[num-1] = v_f[num-1]
    P[num-1] =  v_s[num-1] + x_s[num-1]*x_s[num-1]
    for i in range(1,num)[::-1]:
        J[i-1] = v_f[i-1]*A/v_p[i]
        x_s[i-1] = x_f[i-1] + J[i-1]*(x_s[i]-x_p[i])
        v_s[i-1] = v_f[i-1] + J[i-1]*(v_s[i]-v_p[i])*J[i-1]
        P[i-1] =  v_s[i-1] + x_s[i-1]*x_s[i-1]
    PP = np.zeros(num-1); vv = np.zeros(num-1)
    vv[num-2] = vvT; PP[num-2] = vv[num-2] + x_s[num-1]*x_s[num-2]
    for i in range(2,num)[::-1]:
        vv[i-2] = v_f[i-1]*J[i-2] + J[i-1]*(vv[i-1]-A*v_f[i-1])*J[i-2]
        PP[i-2] = vv[i-2] + x_s[i-1]*x_s[i-2]
    return {'x_s':x_s,'v_s':v_s,'J':J,'P':P,'PP':PP}
```

By alternating between prediction and measurement steps, kalman filtering computes the distribution of $x_t|y_{1:t}$ forwards in time. Once computation at final time point $t = T$ is finished, we can work backwards by using *kalman smoothing* algorithm. The goal of this algorithm is to compute distribution $\mathrm{P}(z_t|y_{1:T})$ by also incorporating future data $y_{t+1:T}$ in additional to $y_{1:t}$. Specifically, the backwards smoothing algorithm computes $\mu_{t|T}, \Sigma_{t|T}$ for which the following holds true:

$$z_t|y_{1:T} \sim N(\mu_{t|T}, \Sigma_{t|T})$$

.

Conditional Distribution for MVN

Before going into the derivation details of this procedure, let recall an elementary result from multivariate normal distribution. Let $X = (X_1, X_2)'$ be jointly distributed normal random variable with

$$\mathrm{E}[X] = \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, \mathrm{Var}[X] = \begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}$$

.

Then,

$$X_1|X_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}). \tag{5}$$

Also, it is not hard to work out the joint distribution:

$$(x_t, x_{t+1})|y_{1:t} \sim N\left(\begin{bmatrix}\mu_t \\ \mu_{t+1|t}\end{bmatrix}, \begin{bmatrix}\Sigma_t & \Sigma_t A' \\ A\Sigma_t & \Sigma_{t+1|t}\end{bmatrix}\right).$$

Then, applying (5) to obtain:

$$\begin{aligned} x_t|x_{t+1}, y_{1:t} &\sim N(\mu_t + \Sigma_t A'\Sigma_{t+1|t}^{-1}(x_{t+1} - \mu_{t+1|t}), \Sigma_t - \Sigma_t A'\Sigma_{t+1|t}^{-1}A\Sigma_t), \\ &\sim N(\mu_t + J_t(x_{t+1} - \mu_{t+1|t}), \Sigma_t - J_t\Sigma_{t+1|t}J_t'), \\ J_t &:= \Sigma_t A'\Sigma_{t+1|t}^{-1}. \end{aligned} \tag{6}$$

Calculating $\mu_{t|T}$

$$
\begin{aligned}
\mu_{t|T} &= \mathrm{E}[x_t|y_{1:T}] && (7)\\
&= \mathrm{E}[\mathrm{E}[x_t|x_{t+1}, y_{1:T}]|y_{1:T}] && (8)\\
&= \mathrm{E}[\mathrm{E}[x_t|x_{t+1}, y_{1:t}]|y_{1:T}] && (9)\\
&= \mathrm{E}[\mu_t + J_t(x_{t+1} - \mu_{t+1|t})|y_{1:T}] && (10)\\
&= \mu_t + J_t(\mathrm{E}[x_{t+1}|y_{1:T}] - \mu_{t+1|t}) && (11)\\
&= \mu_t + J_t(\mu_{t+1|T} - \mu_{t+1|t}). && (12)
\end{aligned}
$$

We reach from (7) to (8) by using *law of iterated expectation*. Transition to line 9 involves property called *conditional independent* which true by state space model assumption. One intuitive way to understand why $y_{1:T}$ changes to $y_{1:t}$ is that x_{t+1} already contains all the information about dependency between x_t and $y_{t+1:T}$, so $y_{t+1:T}$ is nolonger affecting probability of x_t once x_{t+1} is given.

By applying (6), we obtain (10)

$\Sigma_{t|T}$ can be derived similarly using law of iterated variance.

Local level model

I apply above three method: prediction, measurement (or filtering) and smoothing to local level model to see how well they can track this time serie:

$$
\begin{aligned}
x_t &= x_{t-1} + \epsilon_t, \epsilon_t \sim N(0,1)\\
y_t &= x_t + \delta_t, \delta_t \sim N(0,1)
\end{aligned}
$$

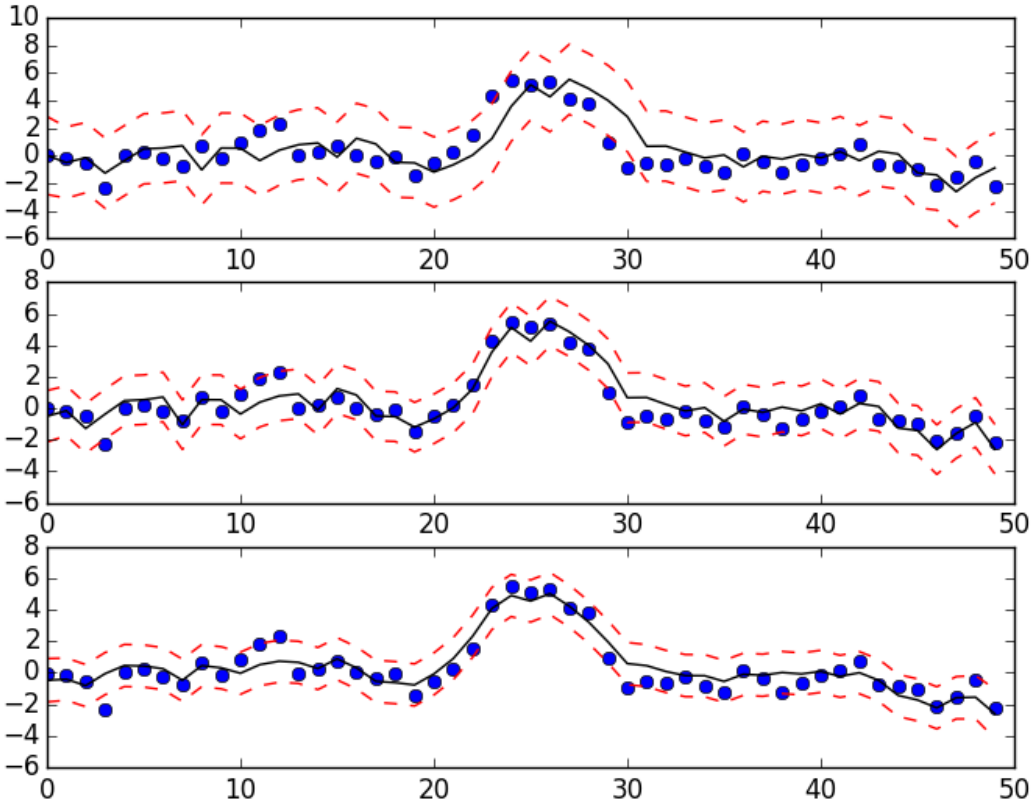
As expected, kalman filtering `x_f` track the observation better than `x_p` from prediction step. `x_s` looks smooth the most and has lowest variance `v_s` among three method.

```
import matplotlib.pyplot as plt
num = 50; x0 = 0; v0 = 1; A=1; C=1; Q=1; R=1;
w = np.random.normal(x0,Q,num+1); v = np.random.normal(0,R,num);
mu = np.cumsum(w);
y = mu[1:] + v;
t = np.arange(num)

ans1 = kfilter(y, num, x0, v0, A, C, Q, R)
x_f = ans1['x_f']; v_f = ans1['v_f']; x_p = ans1['x_p']; v_p = ans1['v_p']; K_T = ans1['K_T'];
ans2 = ksmooth(y, num, x_f, v_f, x_p, v_p, A, (1-K_T*C)*A*v_f[num-2]);

plt.figure(2)
plt.subplot(311)
plt.plot(t,mu[1:],'bo',t,ans1['x_p'],'k')
plt.plot(t,ans1['x_p']+2*np.sqrt(ans1['v_p']),'--r');
plt.plot(t,ans1['x_p']-2*np.sqrt(ans1['v_p']),'--r');
plt.subplot(312)
plt.plot(t,mu[1:],'bo',t,ans1['x_f'], 'k')
plt.plot(t,ans1['x_f']+2*np.sqrt(ans1['v_f']),'--r');
plt.plot(t,ans1['x_f']-2*np.sqrt(ans1['v_f']),'--r');
plt.subplot(313)
```

```
plt.plot(t,mu[1:],'bo',t,ans2['x_s'], 'k')
plt.plot(t,ans2['x_s']+2*np.sqrt(ans2['v_s']),'--r');
plt.plot(t,ans2['x_s']-2*np.sqrt(ans2['v_s']),'--r');
```



Parameter Learning

```
def klearn(y, num, x0, v0, A, C, Q, R):
    while True:
        flt = kfilter(y, num, x0, v0, A, C, Q, R)
        x_f = flt['x_f']; v_f = flt['v_f']
        x_p = flt['x_p']; v_p = flt['v_p']; K_T = flt['K_T']
        sm = ksmooth(y, num, x_f, v_f, x_p, v_p, A, (1-K_T*C)*A*v_f[num-2])
        x_s = sm['x_s']; P = sm['P']; PP = sm['PP']
        C = np.sum(y*x_s) / sum(P)
        Q = 1 # eps is normalized
        R = np.sum(y*y-C*x_s*y) / num
        A = np.sum(PP) / np.sum(P[:-1])
        x0 = x_s[0]
        v0 = P[0] - x_s[0]*x_s[0]
        yield x0, v0, A, C, Q, R
```

In this section, I write an EM algorithm for estimation of $\theta = (\mu_0, \Sigma_0, A, C, Q, R)$. One remark is that because the state variable x_t cannot be observed. We can always scales it up and down and have another valid model. For example, we could say that our model is:

$$
\begin{aligned}
x_t &= Ax_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q) \\
y_t &= Cx_t + \delta_t, \delta_t \sim N(0, R),
\end{aligned}
$$

or equally valid, it is

$$
\begin{aligned}
x_t &= Ax_{t-1} + \epsilon_t, \epsilon_t \sim N(0, 2Q) \\
y_t &= \frac{C}{2}x_t + \delta_t, \delta_t \sim N(0, R),
\end{aligned}
$$

which is obtained by replacing x_t with $2x_t$. For this reason, it is safe to let the state equation variance remain constant with $Q = 1$. In other word, ϵ_t is normalized so it follow standard normal distribution.

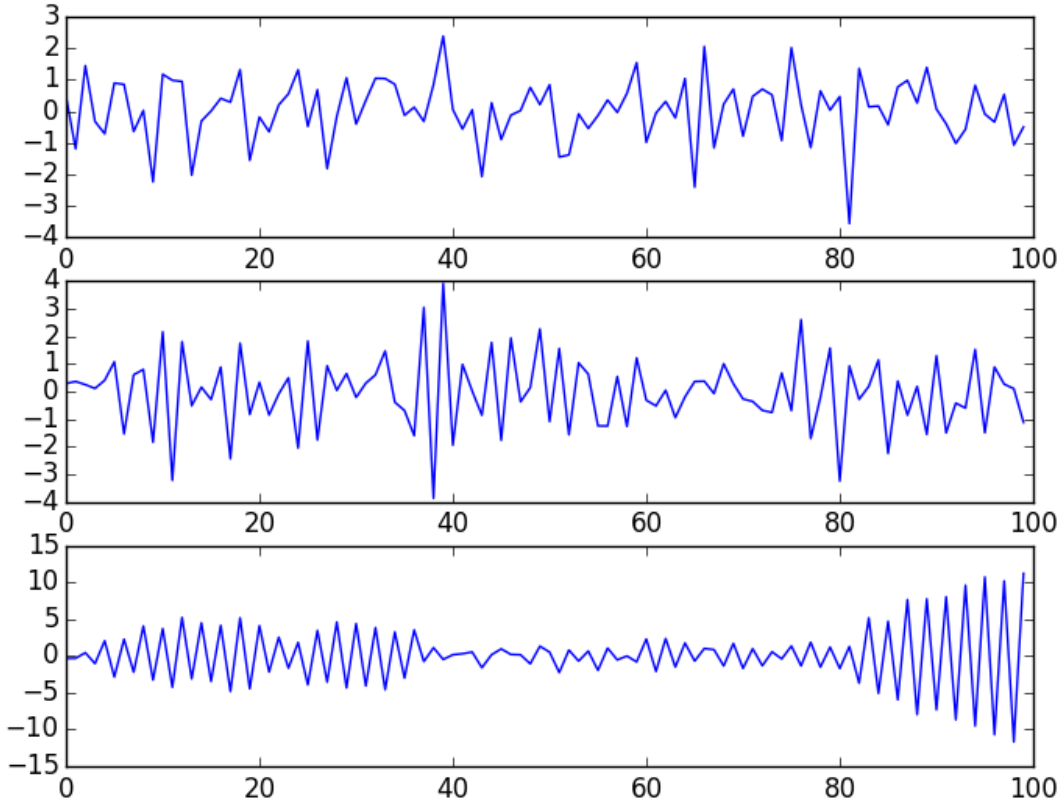
AR(1) Model with Observation Noise.

In this section I generate AR(1) model with observation noise:

$$
\begin{aligned}
x_t &= \phi x_{t-1} + \epsilon_t, \epsilon_t \sim N(0, Q) \\
y_t &= x_t + \delta_t, \delta_t \sim N(0, R).
\end{aligned}
$$

My goal is to see how well `klearn` function can recover the values of ϕ, when we try to estimate it from noisy y_t. Below are three auto-regressive series generated with three different auto-regressive coefficient $\phi = -0.01, -0.7, -0.99$.

```
from statsmodels.tsa.arima_process import arma_generate_sample, ArmaProcess
plt.figure(3)
series = [];
phiList = [-0.01,-0.7,-0.99]
np.random.seed(1234);
for i, phi in enumerate(phiList):
    arparams = np.array([1, -phi])
    maparams = np.array([1])
    arma_t = ArmaProcess(arparams, maparams)
    series.append(arma_t.generate_sample(nsample=100,scale=1))
    plt.subplot(311+i)
    plt.plot(series[i])
plt.show()
```



I then add noise to those three generated series by before estimating model parameters. The magnitude of noise varies from as low as $\sigma = 0.01$ up to around $\sigma = 1.5$. It can be seen below that model with $\phi = 0.01$ is the hardest to estimated with accuracy. As ϕ is estimated to be `phi1=-0.328` even when the magnitude of noise is low (`noise=0.11`). At this noise level, estimated $\hat{\phi}$ in the second and third is close to its true value as can be seen from `phi2=-0.691, phi3=-1.011`, which quite close to $\phi = -0.7, -0.99$.

As the level of noise is grows higher, estimated ϕ for the second model starts to deviate more from $\phi = -0.7$. This can be contrast to the estimation for third model which is quite robust.

```
estIndexbyNoise = []
nList = np.arange(0.01,1.5,0.1)
initpars = np.array([0,1,1,1,1,1]) # initial values of x0, v0, A, C, Q, R
np.random.seed(1234);
for i, noise in enumerate(nList):
    est = []
    for j in range(3):
        learn = klearn(series[j]+np.random.normal(0,noise,100), 100, *initpars)
        for k in range(500):
            learn.next()
        _, _, phi, _, _, _ = learn.next()
        est.append(phi)
    estIndexbyNoise.append(est)
    if i%2 != 0:
        print 'noise={0}  phi1={1:6.3f}, phi2={2:6.3f}, phi3={3:6.3f} \n'.format(noise, *est)
```

```
## noise=0.11  phi1=-0.328, phi2=-0.691, phi3=-1.011
## noise=0.31  phi1=-0.336, phi2=-0.654, phi3=-1.003
## noise=0.51  phi1=-0.477, phi2=-0.611, phi3=-1.005
## noise=0.71  phi1=-0.282, phi2=-0.599, phi3=-1.022
## noise=0.91  phi1=-0.200, phi2=-0.579, phi3=-1.018
## noise=1.11  phi1=-0.033, phi2=-0.525, phi3=-1.025
## noise=1.31  phi1=-0.366, phi2=-0.445, phi3=-1.011
```