AMDP: Pricing and Hedging of Discrete Arithmetic Average Asian Options

GROUP 3

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1 Introduction

The goal of this paper is to analyze the pricing and hedging of Discrete Arithmetic Average Asian Options (DAA). Although, the payoff of this type of option is simple, there are no closed form formulas for either its price or its delta. A famous trader once stated: options that are trivial to price are difficult to hedge; options that are difficult to price are trivial to hedge[2]. In DAA options we find both problems. The structure of the report is as follows.

In section 2, we analyze different methodologies for pricing DAA options. In addition, we introduce other types of fixed strike Asian Options whose price we compare to those of DAA options. Finally, we perform an analysis of how the price of a DAA option changes with respect to the spot price and the volatility.

In section 3, we explore alternatives for calculating the delta of a DAA option. In the process we introduce the delta of other types of Asian Options, including the delta of a Continuous Arithmetic Asian Option by Finite Difference methods. Again, we present a comparison of the deltas of the different options, and analyze how the deltas changes with changes in the spot price and volatility.

In the final section, we study the effectiveness of different hedging strategies. Each of these hedging strategies is based on calculating a hedging portfolio using a delta from those explored in section 3.

2 Pricing of Asian Options

In this section, we discuss different methodologies for pricing four types of fixed strike Asian options, namely, discrete arithmetic average, discrete geometric average, continuous arithmetic average, and continuous geometric average. In particular, we analyze different methodologies for pricing both continuously and discretely monitored arithmetic Asian options, for which there are no general closed form solutions. We start the section by describing the different options and the pricing methodologies and we conclude it with some numerical results.

2.1 Geometric Average Asian Options

There are closed form solutions for pricing both Continuous and Discrete Geometric Asian Options. For the sake of brevity we do not discuss the formulas here and redirect the interested reader to the AMDP lectures notes. [1]

2.2 Continuous Arithmetic Average

By using the technique of change of numeraire and a series of transformations one can arrive at a PDE for the price of a Continuous Arithmetic Average (CAA) Asian call option:

$$u_t + r(q_t - z)u_z + \frac{1}{2}\sigma^2(q_t - z)^2 u_{zz} = 0$$

Where $q_t = 1 - \frac{t}{T}$, and with boundary conditions $u(T,z) = z^+$, $u(t,z) \to 0$ as $z \to -\infty$, $u(t,z) - z \to 0$ as $z \to \infty$.

We implemented a finite different method solver to solve this PDE and get the prices of CAA options.

Since the final condition is specified, it is convenient to change the direction of time by defining $\tau = T - t$, and approximate the derivative in τ by backward difference. i.e. $u_\tau = \frac{u_n^{\tau+\delta_t} - u_n^\tau}{\delta_\tau}$

The first and second derivatives in z are approximated by central differences, $u_z \approx \frac{u_{n-1}^{\tau+\delta_t} - u_{n+1}^{\tau+\delta_t}}{2\delta_z}$, $u_{zz} \approx \frac{u_{n-1}^{\tau+\delta_t} - 2u_n^{\tau+\delta_t} - u_{n+1}^{\tau+\delta_t}}{\delta_z^2}$.

Using these results one can express the discretized PDE as,

$$\mathrm{u}^{\tau+\delta_\tau} = (\mathrm{I} - \delta_\tau(\mathrm{A_z} + \mathrm{A_{zz}}))^{-1}(\mathrm{u}^\tau + \delta_\tau(\mathrm{b_z} + \mathrm{b_{zz}}))$$

Where u^τ is a vector of option prices for different levels of S at time τ; $\mathrm{A_z}$ and $\mathrm{A_{zz}}$ are banded matrices approximating the derivatives; and $\mathrm{b_z}$ and $\mathrm{b_{zz}}$ are used to compensate the approximations at the boundary grid points. The implicit scheme we used is unconditionally stable [4] and by iterating the matrix formula above one can find u at $\tau = T$, which is the call price at time 0.

In the case of CAA options there is also a closed form formula, but this formula is applicable only under certain parameter conditions so we decided to use the more general method instead.

2.3 Discrete Arithmetic Average

In the case of the discrete arithmetic average we explore three pricing methodologies: Monte Carlo methods, Lévy approximation and Inverse Gaussian.

2.3.1 Monte Carlo Methods

Monte Carlo (MC) method prices the discrete arithmetic average Asian option by simulating a large number of sample paths of the underlying, then generating the corresponding payoffs at each simulation, and finally approximating the expectation of the discounted payoffs by taking the average. It provides a consistent way of pricing by Law of Large Number (LLN).

MC method introduces pricing errors. Although the error converges to zero as number of simulations increase, the rate of convergence is slow. Suppose we generate N independent sample paths $(S^{(i)}(t_1), \ldots, S^{(i)}(t_n))$, where $i = 1, \ldots, N$. Denoting $H(T) = h(S(t_1), \cdots, S(t_n))$ as the payoff, then the price is $H(0) = \mathbb{E}_\mathbb{Q}[e^{-rT}H(T)] = \mathbb{E}_\mathbb{Q}[e^{-rT}h(S(t_1), \cdots, S(t_n))]$, while the price approximated by N times of simulation is $H_N(0) = \frac{1}{N}\sum_{i=1}^N e^{-rT}h(S^{(i)}(t_1), \cdots, S^{(i)}(t_n))$. By LLN, $H(0) = \lim_{N\to\infty} H_N(0)$.

If denoting $H^{(i)}(T) = h(S^{(i)}(t_1), \cdots, S^{(i)}(t_n)$ for $i = 1, \ldots, N$, then an unbiased estimator of the standard error of $H_N(0)$ is $\hat{\sigma}_N = \sqrt{\frac{1}{N}}\sqrt{\frac{1}{N-1}\sum_{i=1}^N (e^{-rT}H^{(i)}(T) - H_N(0))^2}$. Again by LLN, we observe:

$$\hat{\sigma}_N = \frac{e^{-rT}}{\sqrt{N}}\sqrt{\frac{1}{N-1}\sum_{i=1}^N (H^{(i)}(T))^2 - (\frac{1}{N}\sum_{i=1}^N H^{(i)}(T))^2} \to \frac{e^{-rT}}{\sqrt{N}}\sqrt{Var(H(T))}$$

which indicates that $\hat{\sigma}_N$ converges to zero with the rate of $O(\sqrt{N})$.

To effectively improve the accuracy of the MC method, Kemna and Vorst [8] employ the corresponding geometric option as a control variate. Suppose $G(T) = g(S(t_1), \cdots, S(t_n))$ to be the payoff of the geometric option, and its price be $G(0) = \mathbb{E}_\mathbb{Q}[e^{-rT}G(T)]$. Then we can rewrite the price of the discrete arithmetic option as

$$H(0) = e^{-rT}\mathbb{E}_\mathbb{Q}[H(T) - G(T)] + G(0)$$

Since $H(T)$ and $G(T)$ are quite close, the variance of their difference will be significantly reduced.

We implemented both the standard MC method and that incorporating a control variate in pricing the arithmetic type Asian option. Table (1) exhibits one typical simulation result.

	Standard MC		MC with Control Variate	
	Price	Std Errors	Price	Std Errors
10,000	10.4519	0.174349	10.8014	0.015186
100,000	10.6737	0.054920	10.8040	0.004716
1,000,000	10.8280	0.017605	10.8009	0.001474

$K = 100$, $S_0 = 100$, $r = 0.05$, $\sigma = 0.4$, $T = 1$, $m = 12$

Table 1: Monte Carlo Pricing Analysis

2.3.2 Lévy Approximation

According to Edmond Levy's paper in 1992 [5] , we may introduce a M_t term at time t: $M_t = A_T - A_t\frac{n_t}{n}$ where t is the n_tth averaging time point. This term could be considered as the unrealized part of A_T after time t as opposed to $A_t\frac{n_t}{n}$ which consists of prices at past averaging time points. Levy's idea is to approximate the first and second moment of M_t with $Z = \exp(a + bX)$ where X is a standard normal.

$$\mathbb{E}(Z^i) = \mathbb{E}(M_t^i),\ i = 1, 2$$

Solving a, b gives the following expressions

$$a = 2\ln\left(\mathbb{E}^{\mathbb{Q}}(M_t)\right) - \tfrac{1}{2}\ln\left(\mathbb{E}^{\mathbb{Q}}(M_t^2)\right), \quad b^2 = \ln\left(\mathbb{E}^{\mathbb{Q}}(A_T^2)\right) - 2\ln\left(\mathbb{E}^{\mathbb{Q}}(A_T)\right)$$

The underlying asset S_t follows a GBM with drift r under the risk neutral measure $\mathbb{Q}$. We now find the first and second moment of M_t

$$\mathbb{E}^{\mathbb{Q}}(M_t) = \frac{1}{n}\sum_{i=n_t+1}^{n}\mathbb{E}^{\mathbb{Q}}(S_{t_i}) = \frac{S_t}{n}\sum_{i=1}^{n-n_t}\exp(rt_i)$$

$$\mathbb{E}^{\mathbb{Q}}(M_t^2) = \frac{1}{n^2}\sum_{i=n_t+1}^{n}\sum_{j=n_t+1}^{n}\mathbb{E}^{\mathbb{Q}}(S_{t_i}S_{t_j}) = \frac{S_t^2}{n^2}\sum_{j=1}^{n-n_t}\sum_{i=1}^{n-n_t}\exp\left(r(t_i+t_j) + \sigma^2(t_i \wedge t_j)\right)$$

Now using Black-Scholes to find the price of the Asian option

$$e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left((A_T - K)_+\right) \approx e^{-r(T-t)}\left(e^{a+\frac{b^2}{2}}\Phi(d_1) - (K - A_t\frac{n_t}{n})\Phi(d_2)\right), \tag{1}$$

$$d_1 = \frac{\frac{1}{2}\ln\mathbb{E}^{\mathbb{Q}}(M_t^2) - \ln(K - A_t\frac{n_t}{n})}{b}, \qquad d_2 = d_1 - b$$

It is noticeable that if at time t $<$ T the option is already "in the money", then $(K - A_t\frac{n_t}{n})$ would be negative and thus $d1$ would be invalid since it takes logarithm on a negative number. According to a paper by Lars Nielsen [6],$N(d_2)$ could be understood as the risk-adjusted probability of exercise. Therefore when the option is already "in the money", we have $N(d_1) >= N(d_2) = 1$ as in our formula.

2.3.3 Inverse Gaussian

Similar to Lévy approximation, Michel Jacques proposed in his paper [7] to approximate M_t with an inverse Gaussian distribution. The distribution parameters could as well be expressed as the first two moments:

$$\rho = \mathbb{E}^{\mathbb{Q}}(M_t), \beta = \frac{\mathbb{E}^{\mathbb{Q}}(A_T^2) - \mathbb{E}^{\mathbb{Q}}(M_t)}{\mathbb{E}^{\mathbb{Q}}(M_t)}$$

The pricing formula is therefore obtained similarly as Levy approximation case:

$$e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left((A_T - K)_+\right) \approx e^{-r(T-t)}\left((\rho - K)\Phi(d_1) - (\rho + K)e^{2\rho/\beta}\Phi(d_2)\right), \tag{2}$$

$$d_1 = \frac{\rho - K'}{\sqrt{\beta K'}},\ d_2 = -d_1,\ K' = K - A_t\frac{n_t}{n}$$

As also mentioned in Jacques' paper, for some values of volatility and interval length, the $e^{2\rho/\beta}$ term or the $\Phi(d_2)$ term would blow up the computer system under use. This problem has also occurred in this project, for instance when the interval length equals to $\frac{1}{12}$, the approximation is not valid for volatility below 0.35.

2.4 Numerical Results

In this section we start by showing the prices of different type of options under different methodologies under what we will call the standard parameters throughout the paper: Maturity (T): 1 year. Strike (K): 100. S_0: 100. Number of averaging periods (m): 12. r: 0.05. Volatility (σ): 0.4. Number of Monte Carlo simulations (n): 100,000.

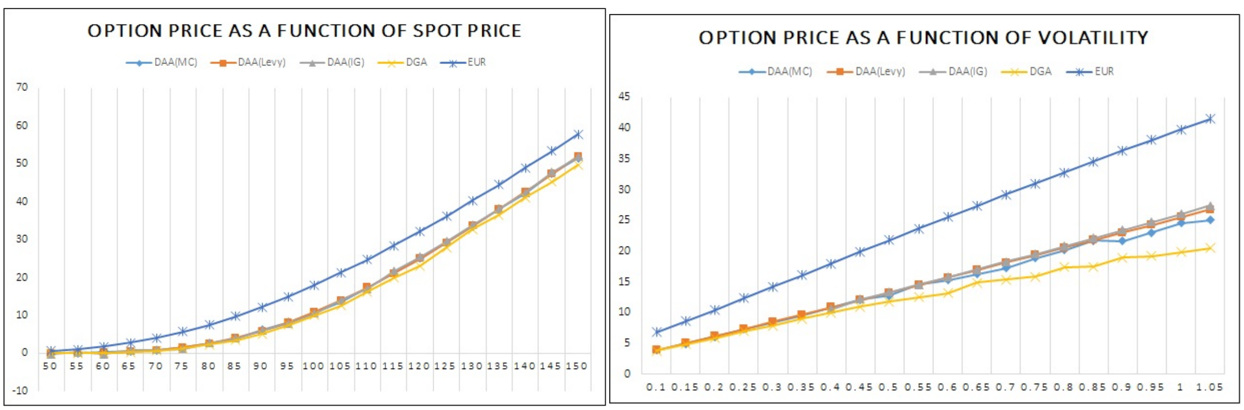
	Disc.Arith.	Disc.Geom.	Cont.Arith.	Cont.Geom.	European Call
Closed form	–	10.029		9.36503	18.0230
Monte Carlo Simulation (Std. errors)	10.7808 (0.1760)	10.0358 (0.1637)	10.3061 (0.1647)	9.3456 (0.1514)	–
Levy Approximation	10.8892	–	–	–	–
Inverse Gaussian	10.5930	–	–	–	–
Finite Difference Method		–	10.4077	–	–

Note: We used MC to evaluate continuous type Asians by setting $m = 100$

Table 2: Options prices under the standard parameters

By comparing the price obtained through Monte Carlo simulation with the price obtained through closed form formulas when these are available we can see that the accuracy of the Monte Carlo Method is quite high.

Next, we analyze how the value of a DAA option changes with respect to changes in the value of the spot price and the volatility, while the remaining parameters remained fixed to the standard values.



Note that in the case of volatility we have only included values for $\sigma > 0.35$ in the case of Inverse Gaussian

Figure 1: CAA price as a function of spot price and volatility

From the graph, we observe that the three different pricing methods for DAA options yield almost the same price for all spot prices. We also observe that the price of a DGA option is slightly lower than that of a DAA option; and that the price of an European Call Option is higher than those of a DAA option and a DGA option for all spot prices. A straight forward application of Jensen's inequality shows that an arithmetic average is larger than a geometric average. Thus, it makes sense that a DAA option has a higher price than a DGA option with the same parameters. On the other hand, one of the reasons why fixed strike Asian options are cheaper than European options is that the volatility is reduced due to the averaging. Finally, we observe that the price of the options increase as the spot price increases.

In the second graph, we observe that a change in volatility has more effect on the price of European options than on the price of Asian options. This is again due to the averaging effect. In addition, we also see that a change in volatility has more effect on DAA options as compared to DGA options . Moreover, it is evident from the graph, that the option prices increase when the volatility increases. Finally, we observe that the three different pricing methods for DAA give approximately the same price when the volatility is between 0.35 and 0.85.

3 Delta calculation for Asian Options

The delta of an option measures the sensitivity of the price of the option with respect to a change in value of the underlying. It is a key measure since it is needed for the hedging of options. In this section we analyze how the delta of different Asian Options can be calculated with emphasis on DAA options.

3.1 Geometric Average

By straightforward differentiation of the closed form formula for the price of Geometric Average Asian Options we arrive at an expression for the delta.

In the discrete case, the only variable in that depends on the asset price at time t_k is $\widetilde{S}(t_k)$:

$$\widetilde{S}(t_k) = [S(t_1)\cdots S(t_k)]^{\frac{1}{n}} = S(t_k)^{\frac{n-k+1}{n}} [S(t_1)\cdots S(t_{k-1})]^{\frac{1}{n}}$$

At time t_k, past asset prices $S(t_1),\ldots,S(t_{k-1})$ are known and can be treated as constants. The following equations illustrate how to obtain the formula for the delta of a Discrete Geometric Average Asian option.

$$\frac{\partial\widetilde{S}(t_k)}{\partial S(t_k)} = \left(\frac{n-k+1}{n}\right) S(t_k)^{\frac{-k+1}{n}} [S(t_1)\cdots S(t_{k-1})]^{\frac{1}{n}}$$

$$\frac{\partial C(S(t_k),t)}{\partial\widetilde{S}(t_k)} = e^{-r(T-t_k)+\mu_A+\frac{1}{2}\sigma_A^2}\Phi(d_1)$$

$$\frac{\partial C(S(t_k),t)}{\partial S(t_k)} = \frac{\partial C(S(t_k),t)}{\partial\widetilde{S}(t_k)}\frac{\partial\widetilde{S}(t_k)}{\partial S(t_k)}$$

Proceeding in a similar way for the continuous case, we find the delta of a Continuous Average Geometric Asian option to be:

$$\begin{aligned}
\Delta &= exp(-rT)\Bigg(\exp(\bar{\mu}+\bar{\sigma}^2/2)\Phi(d_1) + \frac{\exp(\bar{\mu}+\bar{\sigma}^2/2)\phi(d_1)}{\bar{\sigma}} - \frac{X\phi(d_2)}{\bar{\sigma}S_0}\Bigg)\\
d_2 &= \big(\ln S_0 + \bar{\mu} - \ln X\big)/\bar{\sigma} \quad d_1 = d_2 + \bar{\sigma}\\
\bar{\mu} &= (r-\sigma^2)T/2, \quad \bar{\sigma}^2 = \sigma^2 T/3
\end{aligned}$$

3.2 Continuous Arithmetic Average

The delta is a byproduct of the finite difference method explored in section (2.2) . One can compute, $\mathrm{u}_z^\tau = \mathrm{A_z}\mathrm{u}^\tau + \mathrm{b_z}$; and therefore $\partial_{S_0}C = u^T(0, 1-\frac{K}{S_0}) + \frac{K}{S_0}u_z^T(0, 1-\frac{K}{S_0})$, where the super-script of u indicates the time to maturity.

3.3 Discrete Arithmetic Average

To calculate the delta of Discrete Arithmetic Average Asian Options, we will focus again on three methodologies: Lévy Approximation, Path-wise method and Inverse Gaussian.

3.3.1 Lévy Approximation

From the pricing formula (1) in section (2.3.2), taking the derivative with respect to S_t we get the following formula for delta:

$$\Delta \approx e^{-r(T-t)}\big(\frac{1}{S_t}e^{a+\frac{b^2}{2}}\Phi(d_1) + \frac{1}{n}\Phi(d_2)\big)$$

where d_1 and d_2 are the same as in (1).

3.3.2 Path-wise method

As when calculating prices through Monte Carlo methods, simulation is used to obtain the Delta of a DAA option through the Path-wise method.

$$Y = e^{-r(T-t_k)} \left(\bar{S}_T - K\right)^+$$

$$\frac{\partial C(S(t_k),t)}{\partial S(t_k)} = E\left[\frac{dY}{dS(t_k)}\Big|\mathcal{F}_{t_k}\right]$$

$$E\Big[e^{-r(T-t_k)}1\{\bar{S} > K\}\Big(\frac{1}{m}\sum_{i\geq k}\frac{S(t_i)}{S(t_k)}\Big)\Big|\mathcal{F}_{t_k}\Big] \tag{3}$$

At the time t_k (3), asset prices $S(t_1)\ldots S(t_k)$ are known and we simulate asset prices for $t_{k+1},\ldots,t_n = T$, terminal time of the option. We compute the delta by averaging expression inside expectation as appeared in equation (3).

3.3.3 Inverse Gaussian

From the pricing formula (2.3.3) in section(2.3.3), taking the derivative with respect to S_t we get the following formula for delta:

$$\Delta \approx e^{-r(T-t)}\big((\frac{\rho}{S_t} + \frac{1}{n})\Phi(d_1) + (\frac{\rho}{S_t} - \frac{1}{n})e^{2\rho/\beta}\Phi(d_2)\big)$$

where d_1 and d_2 are the same as in (2).

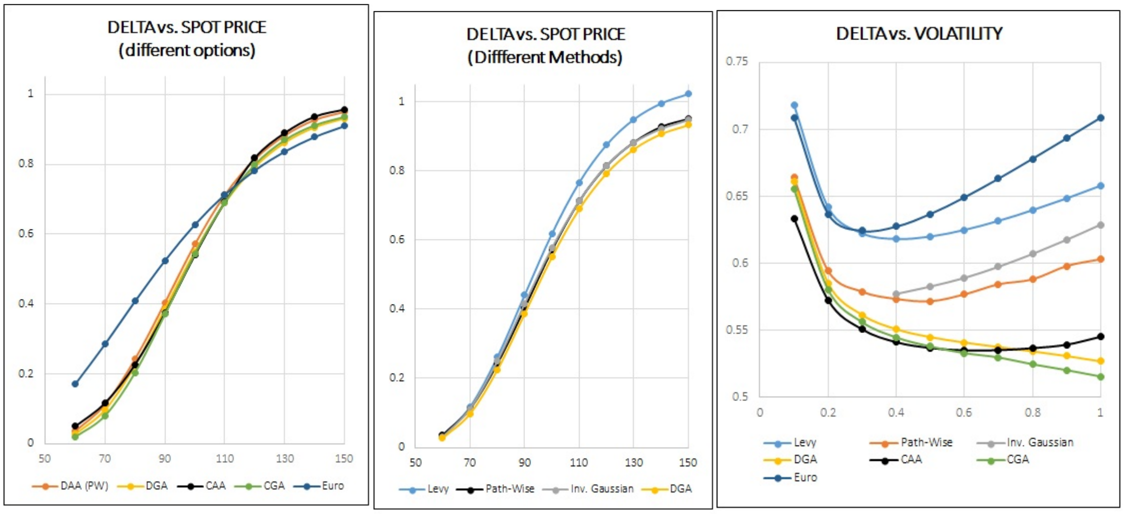
3.4 Numerical results

As we did in the corresponding section for pricing, we start by showing the deltas of different type of options under different methodologies under the standard parameters:

	Disc.Arith.	Disc.Geom.	Cont.Arith.	Cont.Geom.	European Call
Closed form		0.550786		0.544634	0.627409
Path-Wise	0.57777				–
Levy Approximation	0.61818				–
Inverse Gaussian	0.577182				–
Finite Difference Method		–	0.541094	–	–

Table 3: Option Deltas under the standard parameters

Next, we compare the delta of a DAA option with that of other type of options. This will be important as in the next section we will use the delta of a DGA option as a possible hedging strategy.



Note that in the case of volatility we have only included values for $\sigma > 0.35$ in the case of Inverse Gaussian

Figure 2: Deltas as a function of spot and volatility

We observe that the delta of a DAA option is slightly higher than that of a DGA option for all spot prices. One can argue, that this is due to the relationship between the arithmetic and the geometric average that we wrote about in section (2.4). A trader could hedge a DAA option using the closed form solution of the delta of a DGA option because the delta of both kind of options are very similar across all spot prices. Overall, we see that the delta increases when the spot price increases. Furthermore, we observe that the rate of increase of the delta of Asian Options is greater than that of European Options. This indicates that the Gamma of Asian Options is higher, which increases the error under discrete time hedging. Also, we observe that the higher the spot price the bigger the difference between deltas calculated using different methodologies, being the Lévy approximation method the ones that deviates the most.

Regarding delta as a function of volatility we observe that as volatility increases the gap between the different methodologies widens. We could classify the methods into three distinct families of curves in terms of its sensitivity to changes in σ, as reflected by the slop of the curve.: (1) European, (2) Levy, Path-wise, Inv. Gauss. for $\sigma > 0.35$. (3) CAA, DGA and CGA.

When $\sigma < 0.2$, however, the Lévy approximation is in fact closer to the delta of an European option to that of a DAA Asian Option; and the other two families are very close to each other.

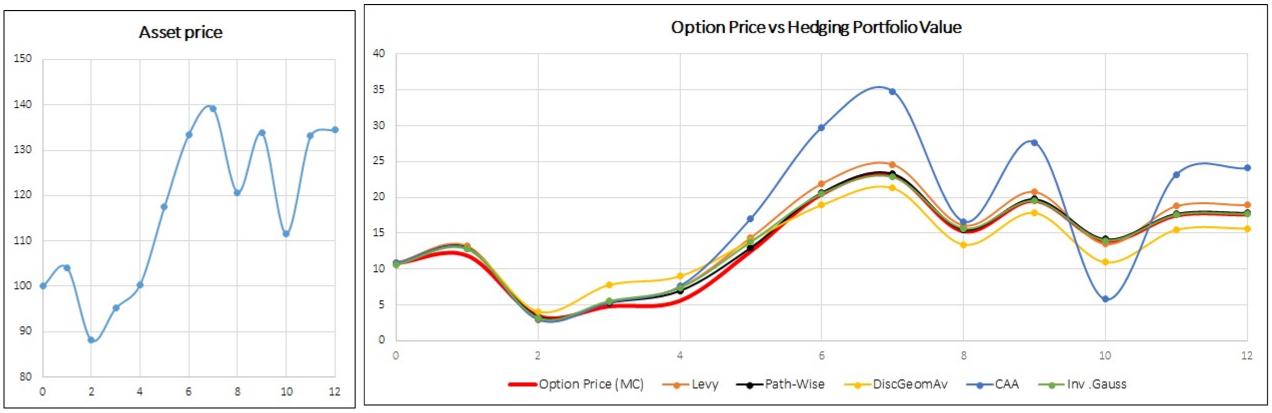
All these observations are consistent with the remarks that we make about the hedging accuracy under different strategies in the next section.

4 Hedging of Discrete Arithmetic Average Asian Call Options

In this section we explore the effectiveness of different methodologies for hedging a Discrete Arithmetic Average (DAA) Asian Call option. The hedging procedure is the same in all hedging strategies. We set up a self-financing portfolio by purchasing delta units of the underlying asset, and borrowing/lending to/from the bank the difference between the cost of such purchase and the money obtained from selling the option. The total cost of setting up the strategy is therefore 0. Then, at each subsequent step, we adjust the number of units of the underlying we hold to the new delta, and again we finance such operation by borrowing/lending to/from the bank account.

Each strategy differs in the way the delta is calculated. We analyzed five different alternatives: Levy Approximation, Path-Wise methods, Inverse Gaussian, delta of a Discrete Geometric Average Asian Call Option (DGA) of the same characteristics, and delta of a CAA option calculated through Finite Difference Methods. Also, for the sake of comparison we calculate the hedging accuracy of a European Call Option, using the closed form solution for delta.

To illustrate this methodology we plot a representative sample path, along with the corresponding option values, and hedging portfolios values along this path. Notice that the difference between the latter two is the hedging error.



T: 1.0. K: 100. S_0: 100. Number of averaging periods: 12. r: 0.05. σ: 0.4. Number of Monte Carlo simulations: 100,000.

Figure 3: Sample path hedging analysis

In the graph we observe how the hedging portfolio calculated both through the Inverse Gaussian approximation and through the path-wise method closely follow the option price; whereas the hedging portfolio calculated using the CAA option delta performs poorly. We will comment further in (4.1).

There are different statistical quantities that can be used for describing the effectiveness of a hedge. Ideally, one would use the percentage error. However, this is not possible for options ending out of the money, since the denominator would be zero. Doing it only for options that end in the money would result in a biased analysis since we have observed the hedging to be better when the option ends in the money.

In this report, we will focus on the average and the standard deviation of the error and of the absolute error, and the 95% VaR of each hedging strategy. The absolute error gives a proxy of the real accuracy of the method. On the other hand, the average error gives an estimate of the money that could be made/lost by pursuing a given hedging strategy and is therefore of interest to a trader. If this quantity is positive, since the initial cost of setting up the hedging portfolio was 0, the hedge would result in a profit. Finally, the 95% VaR can be used to determine how risky a hedging strategy is. In the preparation of the paper other statistics were produced and analyzed and can be generated using the C++ implementation following the instruction on the menu, or can be supplied by request.

We present here a summary of the numerical results obtained. Unless stated other ways we used the following parameters. Number of simulated sample paths per scenario 1000.

		Levy	Path-Wise	Inv. Gauss	DGA	CAA	European
Std	*Error*	−0.12 (3.71)	−0.15 (2.96)	−0.32 (3.47)	−0.19 (4.97)	−0.83 (18.17)	0.05 (3.87)
	Abs. error	3.04 (2.13)	2.29 (1.87)	2.86 (1.99)	3.84 (3.15)	12.17 (13.50)	2.98 (2.47)
	VaR 95%	6.18	5.64	5.85	8.51	21.06	5.99
σ : 0.1	*Error*	−0.04 (1.16)	−0.02 (0.70)		−0.04 (1.24)	−0.15 (4.44)	0.03 (0.89)
	Abs. error	0.91 (0.72)	0.53 (0.45)	*n/a*	0.97 (0.78)	3.43 (2.83)	0.67 (0.58)
	VaR 95%	2.17	1.26		2.28	6.04	1.57
σ : 0.2	*Error*	−0.07 (2.04)	−0.06 (1.47)		−0.08 (2.51)	−0.32 (8.56)	0.08 (1.88)
	Abs. error	1.64733 (1.21)	1.14 (0.93)	*n/a*	1.97 (1.57)	6.27 (5.85)	1.46 (1.19)
	VaR 95%	3.72	2.789		4.49	10.76	3.18
σ : 0.6	*Error*	−0.10 (5.45)	−0.25 (4.45)	0.23 (5.04)	−0.26 (7.66)	1.29 (94.53)	−0.04 (5.89)
	Abs. error	4.45 (3.14)	3.41 (2.87)	4.10 (2.94)	5.76 (5.07)	21.16 (92.14)	4.42 (3.89)
	VaR 95%	8.64	8.41	7.75	13.27	34.92	9.11
r : 0.02 σ : 0.2	*Error*	−0.09 (1.90)	−0.07(1.48)		−0.09 (2.50)	−0.34 (7.83)	0.04 (1.91)
	Abs. error	1.54 (1.10)	1.15(0.93)	*n/a*	1.96 (1.56)	5.58 (5.51)	1.49 (1.21)
	VaR 95%	3.32	2.8546		4.61	9.69	2.97
r : 0.1	*Error*	−0.06 (3.88)	−0.12 (2.95404)	−0.02 (3.69)	−0.17 (4.99)	−0.85 (19.86)	0.11 (3.81)
	Abs. error	3.17 (2.23)	2.30 (1.86)	3.04 (2.09)	3.89 (3.13)	13.66 (14.44)	2.96 (2.40)
	VaR 95%	6.80	5.48	5.83	8.75	23.54	5.94
r : 0.02	*Error*	−0.15 (3.60)	−0.17 (2.96)	0.61 (3.33)	−0.19 (4.95)	−0.81 (17.15)	0.01 (3.90)
	Abs. error	2.94 (2.09)	2.29 (1.88)	2.73 (1.99)	3.81 (3.17)	11.28 (12.95)	2.96 (2.53)
	VaR 95%	5.94	5.71	4.69	8.62	20.02	6.13

K : 75	*Error*	0.03 (2.62)	−0.01 (1.36)	−0.09(1.79)	−0.12 (3.45)	−0.47 (22.87)	0.09 (2.46)
	Abs. error	1.97 (1.72)	0.93 (0.99)	1.31(1.22)	2.66 (2.20)	15.26 (17.04)	1.83 (1.66)
	VaR 95%	4.37	2.38	2.90	6.40	27.47	4.39
K : 90	*Error*	−0.06 (3.28)	−0.07 (2.50)	−0.08 (2.96)	−0.16 (4.45)	−0.83 (19.27)	0.15 (3.37)
	Abs. error	2.59 (2.02)	1.91 (1.61)	2.34 (1.81)	3.46 (2.80)	13.41 (13.87)	2.62 (2.13)
	VaR 95%	5.88	4.53	4.89	7.92	24.09	5.63
K : 110	*Error*	−0.26 (3.90)	−0.230 (3.16)	−0.17 (3.65)	−0.16 (5.35)	−0.91 (16.88)	−0.05 (4.28)
	Abs. error	3.17 (2.28)	2.35 (2.12)	2.94 (2.17)	3.93 (3.63)	10.76 (13.04)	3.19 (2.85)
	VaR 95%	6.19	5.59	5.75	9.77	19.44	6.91
K : 125	*Error*	−0.33 (3.79)	−0.24 (2.80)	−0.13 (3.53)	−0.06 (4.97)	−0.91 (15.13)	−0.15 (4.56)
	Abs. error	2.97 (2.38)	1.99 (1.99)	2.71 (2.27)	3.10 (3.88)	8.72 (12.40)	3.22 (3.23)
	VaR 95%	5.86	5.45	5.31	8.66	16.0448	7.33
m : 3	*Error*	−0.13 (9.70)	0.05 (6.001)	0.16 (6.17)	−0.26 (9.42)	−0.17 (14.69)	−0.13 (7.17)
	Abs. error	7.56 (6.01)	4.64 (3.81)	4.86 (3.80)	7.65 (5.50)	10.15 (10.62)	5.41 (4.71)
	VaR 95%	15.60	11.28	10.87	16.51	20.24	11.87
m : 6	*Error*	0.30 (5.74)	0.21 (4.18)	0.38 (4.55)	0.27 (7.15)	0.07 (17.55)	0.13 (5.21)
	Abs. error	4.62 (3.43)	3.35 (2.52)	3.70 (2.68)	5.61 (4.43)	11.7374(13.0515)	4.06 (3.27)
	VaR 95%	9.01	7.30	7.40	11.81	20.4253	8.30
m : 18	*Error*	0.19 (3.10)	0.03 (2.31)	1.88 (3.03)	0.05 (4.10)	−0.16 (19.54)	−0.03 (3.08)
	Abs. error	2.52 (1.82)	1.81 (1.45)	2.86 (2.13)	3.18 (2.60)	13.09 (14.51)	2.34 (2.00)
	VaR 95%	4.95	4.07	2.73	7.34	−25.39	5.18
m : 24	*Error*	0.18 (2.75)	−0.10(1.97)	−2.05 (2.78)	−0.01 (3.55)	0.01 (20.40)	−0.17 (2.89)
	Abs. error	2.25 (1.59)	1.52 (1.27)	2.87 (1.93)	2.71 (2.29)	13.10(15.64)	2.18 (1.90)
	VaR 95%	4.23	3.57	6.37	6.21	−22.11	5.57
m : 52	*Error*	0.14 (2.38)	0.08(1.31)		0.11 (2.52)	−0.53 (19.06)	0.21 (1.95)
	Abs. error	1.97 (1.35)	1.04(0.81)	n/a	1.94 (1.61)	13.00 (13.96)	1.49 (1.27)
	VaR 95%	3.36	2.06		4.20	22.83	3.03
T : 0.5	*Error*	0.14 (2.53)	0.07 (1.95)		0.16 (3.51)	−0.09 (12.58)	0.01 (2.78)
	Abs. error	2.07 (1.46)	1.54 (1.19)	n/a	2.76 (2.17)	8.40 (9.37)	2.12(1.80)
	VaR 95%	4.14	3.49		6.05	14.71	4.79
T : 2	*Error*	−0.06 (5.27)	−0.22 (4.19)	0.23 (4.94)	−0.25 (7.21)	−0.89 (35.06)	0.04 (5.52)
	Abs. error	4.36 (2.97)	3.24 (2.67)	4.06 (2.83)	5.50 (4.66)	19.62 (29.06)	4.21 (3.56)
	VaR 95%	8.72	8.00	7.61	12.97	34.07	8.52
T : 5	*Error*	0.69 (8.60)	−0.33 (6.69)	1.23 (8.04)	−0.19 (13.90)	−73 (2036)	−0.02 (8.87)
	Abs. error	7.18 (4.77)	5.13 (4.31)	6.70 (4.61)	9.75 (9.91)	108 (2034)	6.65 (5.88)
	VaR 95%	13.00	12.74	11.17	27.18	78	14.04

Quantities in units of currency (uoc)

Table 4: Hedging Numerical Analysis

4.1 Comparison among strategies

First of all, we observe that hedging using the delta from a CAA option is a terrible strategy. This is because the delta of a CAA option is very different than that of a DAA. For example, as the option approaches maturity the delta of a CAA tends to 1, while the delta of a DAA option tends to 0. Therefore, we will no longer consider this strategy in the rest of the analysis. However, it is remarkable that the hedging error using the delta of a DGA option is not extremely higher than in the other cases. This reflects that, contrary to the case of a CAA option, the delta of a DGA option follows a similar pattern as that of a DAA option as we saw in the previous section.

We note that in most scenarios the average error is very close to 0 for all strategies, and that the average absolute error usually lies between 2 and 3. However, when focusing on the standard deviation and VaR we see that the Path-Wise method strategy outperforms the other strategies in most of the scenarios. Moreover, the standard deviation of both the hedging error and the absolute hedging error using the Path-Wise strategy; along with the VaR, is comparable to the same error statistics of the hedging on an Europan Call Option. From this we conclude that the difficulty of hedging DAA options arises not from the lack of a closed form formula, but from the fact that the rebalancing of the portfolio is only done at the averaging time-steps. European Call Options do not have this difficulty and its hedging error can be made very small, not taking into account transaction costs, by increasing the frequency of rebalancing.

We would rank the hedging strategies as follows: (1) Path-wise, (2) IG, only for volatility higher than 0.35, (3) Levy, (4) DGA.

4.2 Influence of the parameters

Volatility: Increasing or decreasing the volatility has little effect on the average error; however, it strongly increases/decreases all other error measures, including the standard deviation of the error. Furthermore, we have observed that for volatilities lower than 0.35, we obtained numerical errors when using the Inverse Gaussian strategy. This is only observed in certain scenarios since we chose a volatility of 0.4 as standard.

Interest rates: We hypothesized that higher interest rates would lead to lower hedging error since it affects the probability of the option ending in the money, and we have seen that the hedge is better in this case. However, the numerical experiments show that interest rates in the range of $0.02 - 0.1$ have little influence in the hedging error.

Strike price: The lower the strike the higher the probability of the option ending in the money; thus, as previously noted, the lower the hedging error.

Number of averaging periods: As expected the higher the frequency of rebalancing, the lower the hedging error. Since in our study the frequency of rebalancing is linked to the number of averaging periods, we observe that those DAA options with a higher number of averaging periods are easier to hedge. However, in the case of 3 averaging periods, we observed a remarkably low average error, though with a high standard deviation. For this scenario we rerun another 2.000 paths and confirmed the results.

Maturity: Since we varied maturity without adapting correspondingly the number of averaging periods, the effect of decreasing/increasing maturity was the same as increasing/decreasing the number of averaging periods.

It is interesting to observe that, as expected VaR and the standard deviation of the error are closely related. We observe that the higher the standard deviation of the error/absolute error the higher the VaR. On the other hand, we note that the Inverse Gaussian Method exhibits an abnormal behavior for different set of parameters..

5 Conclusion

In this report, we have seen that even though there are no closed form formulas for calculating the price and delta of a DAA option, there are effective methods for doing so. Furthermore, we have seen that hedging a DAA option using some of the explored strategies result in a hedging error no bigger than the error resulting from hedging a European Call option. In fact, under 52 averaging periods hedging a DAA option through the path-wise strategy outperformed the hedging of an European Call option. We conclude, that the difficulty in hedging DAA options stems from the fact that we only rebalance our portfolio at the averaging periods; an extension of this project would analyze hedging strategies that do this kind of rebalancing. Another improvement would incorporate gamma hedging into the analysis, since the fact that Asian Options have big high gammas, exacerbates the problem. On the other hand, we have observed inconsistent behaviors of the Inverse Gaussian methodology for $\sigma < 0.35$, and under some other combination of parameters. Maybe a more efficient implementation or under a different software could overcome this difficulty.

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