Chapter 2

2

2.1

I think that by "arc", the author is referring to path. (Spivak uses arc to refer to one-one path.) In that case, this is easy.

3

3.1

The center of a group G is the set of all elements z such2 that $z \cdot g = g \cdot z$.

If two path classes gives rises to the same isomorphism if and only if $\gamma'\gamma^{-1}$ belongs to the center of the group $\pi(X, x)$.

3.2

We claim the following: this is equivalent to:

Fix a point x; if for all values of y, and all path class γ and γ' connecting x and y, $\gamma'\gamma^{-1}$ be an element of the center of the group.

Basically this is an extension of the result from problem 3.1. The only difference is that we can simply state the equivalence for a fixed x and this would be equivalent for any point in X too.

3.3

Suppose $f \sim g$, then $f \cdot \bar{g} = g \cdot \bar{g} = \mathscr{E}_x \in \pi(X, x)$.

Suppose $f \cdot \bar{g} = \mathscr{E}_x$, then $f \cdot \bar{g}g = \mathscr{E}_x \cdot g \implies f \cdot \mathscr{E}_y = g \implies f \sim g$.

4

TODO 4.1

4.2

Recall that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x | x \in X\}$ is closed.

Given two maps $f, g \colon X \to Y$, then the set $A = \{x \in X | f(x) = g(x)\}$ is closed if Y is Hausdorff. This can be proved by making use of the above observation, since $A = (f \times g)^{-1}(\Delta)$.

Let $f\colon X \to X$ be a retraction mapping from X onto Z and $i\colon X \to X$ be the identity map. Then the points where both these map are equal is clearly the set Z which will become closed.

4.3

We need to show that $\pi(X) = \operatorname{Im} i_* \times \ker r_*$. Since $\operatorname{Im} i_* \approx \pi(A)/\ker i_*$ and $\ker r_* \approx \pi(X)/\operatorname{Im} r_*$, it is enough to show that $\pi(X) \approx \pi(A) \times \pi(X/A)$.

The last result follows from the fact that $\pi(A)$ is a normal subgroup of $\pi(X)$ and observing that the map $(n, g) \mapsto ng$ is an isomorphism between $N \times (G/N)$ and G (here G is an arbitrary group and N is a normal subgroup of G.)

4.4

Suppose A be a retract of X and $r\colon X \to A$ be the corresponding retraction. Then define a function $\phi\colon X \times Y \to A \times Y$ by $\phi(x, y) = (r(x), y)$. This function is clearly continuous and thus forms a retraction mapping from $X \times Y$ onto $A \times Y$.

Suppose that there is a retraction $\phi\colon X \times Y \to A \times Y$. Define a retraction $r\colon X \times A$ by first picking a point $y_0 \in Y$ and by making $r(x) = \pi_1 \circ \phi(x, y_0)$. Once again one can verify that this mapping is a retraction.

4.5

This is easy. Compose the two retraction mappings to get the required retraction.

4.6

Without loss of generality, one can assume that the $(n-1)$ dimensional sphere S^{n-1} is the unit-sphere centered at the origin and the point x_0 is the origin.

Let $X = \mathbb{R}^n - \{0\}$, then define the function $\phi\colon X \times I \to X$ by

$\phi(x, t) = (1-t)x + t \cdot \frac{x}{\|x\|}$.

One can verify that this is indeed the required deformation retract.

4.7

The center circle.

TODO 4.8

TODO 4.9

4.10

Suppose that f_0 and f_1 be any two path connecting x and y. Consider the path from x to x given by $\gamma = f_0 \cdot f_1^{-1}$.

Since the space is a deformation retract to a single point, the fundamental group $\pi(X, x)$ is isomorphic to the trivial group $\{e\}$, i.e., γ is identity. This shows that f_0 and f_1 are equivalent.

TODO 4.11

5

5.1

Given a loop $f\colon I \to X$ based on x_0, we need to prove that this belongs to the trivial class.

Suppose $f(I)$ is entirely contained in one of U_i, since U_i is simply connected, the fact that f belongs to trivial class is obvious. We assume that $f(I)$ is contained is not completed contained in a single U_i.

Define a class of open sets $V_i = f^{-1}(U_i)$. Since $[0, 1]$ is compact and since V_i is an open cover of $[0, 1]$, there exists a real number, ε (the Lebesgue number corresponding to the open cover), such that any set whose diameter is less than ε is completely contained in one of U_i.

Thus we can find a sequence $0 = t_1 < t_2 < \cdots < t_n = 1$ such that the following conditions are satisfied for all i:

(a). $f([t_i, t_{i+1}]) \subset U_j$ for all some j;

(b). $f([t_i, t_{i+1}])$ and $f([t_{i+1}, t_{i+2}])$ are not entirely contained in same U_j for some j.

This can be constructed in the following fashion: pick t_i such that the condition on diameter is satisfied. This guarantees the first condition. If for consecutive intervals lies completely inside a single U_i, simply merge the intervals to produce a larger one; repeat this procedure until the second condition is satisfied.

This basically implies that for every value of i, $f(t_i)$ belongs to the intersection of U_j and U_k for some value of j and k.

Let us denote the path component corresponding to f as β and the ones that correspond to the restriction of f on the sub intervals as β_i. It is easy to see that

$\beta = \beta_1 \cdot \beta_2 \cdot \cdots$.

Since $f(t_i)$ will always belong to the intersection, we can think of a path γ_i that connects $f(t_i)$ and the point x_0. Define the loop classes δ_i in the following manner:

$\delta_1 = \beta_1 \gamma_1$,

$\delta_i = \gamma_{i-1}^{-1} \beta_i \gamma^i$ for $1 < i < n$,

$\delta_n = \gamma_{n-1}^{-1} \beta_n$. (if n is finite.)

It is easy to see that

$\beta = \delta_1 \cdot \delta_2 \cdots$.

Observe that each δ_i is a loop at x_0 and lying entirely inside U_i, i.e., trivial. We have shown that f can be decomposed as loops of trivial classes, i.e., f belongs to the trivial loop class. ■

Special cases

(a). Let X be a topological space and U_1, U_2 be two open sets that cover X that has at least one point in common. Suppose these open sets are simply connected and if $U_1 \cap U_2$ is an arc wise connected space, then X is simply connected.

(b). Let $\{U_i\}$ be a nonempty open cover of X such that $U_i \subset U_j$ whenever $i < j$. Suppose each of U_i are arc connected and simply connected. Then the space X is simply connected.

5.2

We make use of the Stereographic projection of S^n. Let p and $-p$ represent two antipodal points of S^n, then there exists two homeomorphisms $f\colon S^n - \{p\} \to \mathbb{R}^n$ $g\colon S^n - \{-p\} \to \mathbb{R}^n$. This shows that the open subsets $S^n - \{p\}$ and $S^n - \{-p\}$, which has a arc connected intersection whenever $n \geq 2$ (this is not true for $n = 1$), are also simply connected. From previous exercise, we see that S^n is simply connected.

5.3

$R^2 - \{0\}$ is not simply connected while $R^n - \{0\}$ is simply connected for $n > 2$.

5.4

A homeomorphism cannot map a point of S^1 to a point in the interior of E^2. This is because, there is no neighborhood of a point on the boundary that is homeomorphic $\mathbb{R}^2$ while there are neighborhoods of E^2 that are homeomorphic to $\mathbb{R}^2$.

Alternatively, one can see that deleted neighborhoods of a point on the boundary are simply connected while this is not the case with a point on the interior of D^2.

7

7.1

A torus is $S^1 \times S^1$, thus $\pi(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

7.2

Consider the retract $r\colon S^1 \times S^1 \to S^1 \times \{p\}$ defined by $r(x, y) = (x, p)$. This is clearly, a continuous map.

Recall that, if A is a deformation retract of X, then the inclusion map induces an isomorphism of $\pi(A, a)$ onto $\pi(X, a)$ for any $a \in A$. Observe that the group $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$ (the first one is cyclic.)

Chapter 3

3

3.1

Let F' denote the subgroup of F that is generated by elements of $\varphi(S)$. Given an Abelian group A, there is a homomorphism g that makes the diagram in the textbook commutative (lazy to draw it here.)

Firstly we shall show that F' shall also satisfy the hypothesis required for free group over S. This can be seen from the fact that the mep $g|_{F'}$ is a homomorphism between F' and A which also makes the diagram (in the book) commutative. From Proposition 3.1, F' and F are isomorphic.

Since F is Abelian, here is a unique homomorphism between F' and F that makes the diagram commutative. Clearly identity is the unique homomorphism, thus $F' = F$.

3.2

We proceed similarly Corollary 3.5. If F and F' are isomorphic, then F/F^n and $F'/(F')^n$ should also be isomorphic. But the former is finite, while the latter is infinite (the latter can be thought of as the weak product of infinitely many $\mathbb{Z}_n$, which is clearly an infinite group.)

3.3

Reference: Baer's Result: The infinite product of integers has no basis.

An example is the group $\mathbb{Q}$ under addition. This is clearly an Abelian group. Assume that this is free and let S be a basis for $\mathbb{Q}$.

Let $1/1$ be represented as $1/1 = n_1 s_{\alpha_1} + \cdots + n_k s_{\alpha_k}$ where $n_1, \cdots, n_k$ are integers and WLOG $n_1 \neq 0$.

Let n be an integer that does not divide n_1. The number $1/n$ has the representation $1/n = m_1 s_{\beta_1} + \cdots + m_j s_{\beta_j}$. This implies that $nm_1 s_{\beta_1} + \cdots + nm_j s_{\beta_j} = n_1 s_{\alpha_1} + \cdots + n_k s_{\alpha_k}$, which is a contradiction.

The reference paper proves that the direct product of countable number of $\mathbb{Z}$ does not have a basis, and hence cannot be a free Abelian group.

3.4

The split is $\mathbb{Z}_{36} \times \mathbb{Z}_2$.

TODO 3.5

4

4.1

Suppose x_1 and x_2 are elements belonging to two different groups. Then x_1x_2 and x_2x_1 are elements that are not equal, since $x_1x_2x_1x_2 \neq 1$; thus the free product results in a non Abelian group.

Notice that $x_1x_2x_1x_2 \cdots x_1x_2$ (n times) is a reduced word. If this is equal to the identity element, then from uniqueness of the representation of elements of the direct product, one can see that all of them has to be equal to identity, which is a contradiction. This proves that the element x_1x_2 is of infinite order.

To show that the center contains only identity, given an arbitrary element y, let $y_1 \in G_{y_1}$ denote the first element of the word y and y_2 denote an element other than unity in and not contained in G_{y_1}. Clearly $yy_2 \neq y_2y$ (using the uniqueness of the reduced word representation.)

4.2

Let G denote the free product of $\{G_i\}$ and G' denote the free product of $\{G'_i\}$. Let φ_i denote the inclusion homomorphism between G_i and G and φ'_i denote the inclusion homomorphism between G'_i and G'.

Since φ_i is also a homomorphism between G'_i and G, there exists a unique homomorphism $k\colon G' \to G$ such that $k \circ \varphi'_i = \varphi_i$. It is enough to show that the kernel of k is trivial. This follows from the fact that φ_i is a monomorphism for all values of i.

4.3

Since $\varphi'_i \circ f_i\colon G_i \to G'$ is a homomorphism, we observe that there is a unique homomorphism $f\colon G \to G'$ such that $f \circ \varphi_i = \varphi'_i \circ f_i$. This fact makes the diagram (given in the problem) commutative.

Suppose each f_i be a monomorphism, then f is also a monomorphism. Similarly, one can show that f is an epimorphism if each f_i is an epimorphism (making use of the fact that every element of the free product is equal to a finite product of elements of the corresponding family.)

4.4

5

5.1

This follows from Exercise 4.1 (basically says that the free product of a collection of groups, each containing more than one element is not Abelian.)

5.2

Again, this follows from Exercise 4.1

TODO 5.3

5.4

If S if finite and S' be an infinite set, if F and F' are isomorphic, then so are $F/[F,F]$ and $F'/[F',F']$. But $F/[F,F]$ and $F'/[F',F']$ are free Abelian groups with "generating sets" of different cardinality and hence non-isomorphic.

5.5

Chapter 4

3

3.1

First observe that no two V_i and V_j is a proper subset of the other. Thus the only possibility of $\varphi_{\lambda\mu}$ is the inclusion homomorphism between W and V_i. Let us denote this by $\varphi_i\colon \pi(W) \to \pi(V_i)$.

The exercise immediately follows from theorem 2.2.

3.2

For every $i \in I$, let a_i be a point in A_i that is not equal to x_0. Let B_i denote the set $A_i - \{a_i\}$; this set is open relative to A_i. Let us denote $A_i \cap_{j\neq i} B_i$ by C_i; C_i is open in X and such that it is deformation retract to A_i. Let W be the set $\cap B_i$; W is open in X.

Observe that $X = W \cap_i A_i$. This exercise now follows from exercise 3.1.

(Refer to Example 1.21 in Hatcher.)

3.3

Shrinking wedge of circles is an example!

Refer to example 1.25 in Hatcher.

3.4

The space Y can be deformed into a countable wedge sum of circles. The fundamental group of this space is the free group on a countable space.

3.5

If we remove an 8-curve from a torus, we get an open set in Torus, which can be deformed to a point. From exercise 3.1, it is easy to see that the fundamental group of X the Free product of $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$.

4

TODO 4.1

(a). Since φ_2 is an isomorphism and onto; we have a homomorphism $\varphi \colon \pi(V) \to \pi(U) = \varphi_1 \circ \varphi_2^{-1}$. Apply the hypothesis of Theorem 2.1 to $H = \pi(U)$ and maps $\mathrm{id}, \varphi, \varphi_1$, we see that there is a unique homomorphism $\sigma \colon \pi(X) \to \pi(U)$ such that $\sigma \circ \psi_1 = \mathrm{id}$. The one-one-ness of ψ_1 is immediate.

Recall that $\pi(X)$ is generated by $\psi_1(\pi(U))$ and $\psi_2(\pi(V)) = \psi_1 \circ \varphi_1 \circ \varphi_2^{-1}(\pi(V))$. Thus φ_1 has to be an onto mapping and hence an onto isomorphism!

(b).

Chapter 5

2

TODO 2.1

(a) $\implies$ (d)

For every point x and every neighborhood U of x, from (a), the arc components of U are open. Let V be the arc component of U that contains x. It is seen that this neighborhood V satisfies the required conditions for (d).

2.2

I think that one can demonstrate that if f is a local Homeomorphism, then, for every subset A of X, $f|_A$ is also a local Homeomorphism.

2.3

Suppose $A = f^{-1}(\{y\})$ be an infinite set. Corresponding to each point $x \in A$, there is an open set U_x such that f restricted over U_x is a homeomorphism. Notice that U_x cannot have points in A other than x. For points x' of X that are not in A, consider $U_{x'}$ to be its neighborhood such that f can be restricted to a homeomorphism. Again this neighborhood cannot contain more than two points of A. Now $\{U_x\} \cap \{U_{x'}\}$ forms a covering of X. If it has finite sub-cover, we have an obvious contradiction.

Suppose Y is a connected Hausdorff space, let A denote the set of all y such that $f^{-1}(\{y\})$ is non-empty. Since A is non-empty, to prove the $A = Y$, it is enough to show that A is both open and closed.

Observe that f is an open mapping, open-ness of A follows from this. Note that $A = f(X)$, since X is compact A is also compact subset of Y, the closed-ness of A follows from the fact that the space Y is Hausdorff.

TODO 2.4

6

6.1

Consider the universal cover (X, id) of X and the cover $(\tilde{X}, p)$, let φ be the homomorphism between X and $\tilde{X}$, i.e., $p \circ \varphi = \mathrm{id}$. The onto-ness of p is immediate.

6.2

(a). S^1 has the fundamental group $\mathbb{Z}$, the subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$, and since the group is Abelian, the conjugacy class of a subgroup is the singleton set the subgroup.

For $n = 0$, consider the covering space $(\mathbb{R}, t \mapsto (\cos t, \sin t))$. For other values of n, consider the cover $(S^1, \theta \mapsto n\theta)$. It is easily seen that these covering spaces determines all the covering spaces of S^1 upto isomorphism.

(b). The fundamental group of the projective plane is $\mathbb{Z}_2 = \{0, 1\}$. There are only trivial subgroups. $(S^2, p \mapsto [p])$, and $(\mathbb{P}^2, [p] \mapsto [p])$ determines all covering spaces upto isomorphism.

(c).

6.5

7

7.2

(a). Example 2.2. The covering space S^1 where S^1 wraps around S^1, n times. The group of Automorphisms will be $\mathbb{Z}_n$.

(b). Example 2.4.

8

8.1

Let \$G=\\$\varphi_1, \cdots, \varphi_{\mathrm{n}}$\\.\$ Let $y \in Y$, define $y_i = \varphi_i(y)$. For each y_i, let V_i be a neighborhood of y_i that is disjoint from all other y_j. Define $U_i = \varphi_i(V_i)$, then $\cap U_i$ is a neighborhood of y such that $\varphi_i(U)$ is disjoint (pair-wise.)

8.2