Chapter 1

2

2.1

I think that by "arc", the author is referring to path. (Spivak uses arc to refer to one-one path.) In that case, this is easy.

3

3.1

The center of a group G is the set of all elements z such that $z \cdot g = g \cdot z$.

If two path classes gives rises to the same isomorphism if and only if $\gamma' \gamma^{-1}$ belongs to the center of the group $\pi(X, x)$.

3.2

We claim the following: this is equivalent to:

Fix a point x; if for all values of y, and all path class γ and γ' connecting x and y, $\gamma' \gamma^{-1}$ be an element of the center of the group.

Basically this is an extension of the result from problem 3.1. The only difference is that we can simply state the equivalence for a fixed x and this would be equivalent for any point in X too.

3.3

Suppose $f \sim g$, then $f \cdot \bar{g} = g \cdot \bar{g} = \mathcal{E}_x \in \pi(X, x)$.

Suppose $f \cdot \bar{g} = \mathcal{E}_x$, then $f \cdot \bar{g}g = \mathcal{E}_x \cdot g \implies f \cdot \mathcal{E}_y = g \implies f \sim g$.

4

TODO 4.1

4.2

Recall that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x | x \in X\}$ is closed.

Given two maps $f, g \colon X \to Y$, then the set $A = \{x \in X | f(x) = g(x)\}$ is closed if Y is Hausdorff. This can be proved by making use of the above observation, since $A = (f \times g)^{-1}(\Delta)$.

Let $f: X \to X$ be a retraction mapping from X onto Z and $i: X \to X$ be the identity map. Then the points where both these map are equal is clearly the set Z which will become closed.

TODO 4.3

4.4

Suppose A be a retract of X and $r: X \to A$ be the corresponding retraction. Then define a function $\phi: X \times Y \to A \times Y$ by $\phi(x,y) = (r(x),y)$. This function is clearly continuous and thus forms a retraction mapping from $X \times Y$ onto $A \times Y$.

Suppose that there is a retraction $\phi: X \times Y \to A \times Y$. Define a retraction $r: X \times A$ by first picking a point $y_0 \in Y$ and by making $r(x) = \pi_1 \circ \phi(x, y_0)$. Once again one can verify that this mapping is a retraction.

4.5

This is easy. Compose the two retraction mappings to get the required retraction.

4.6

Without loss of generality, one can assume that the $(n-1)$ dimensional sphere S^{n-1} is the unit-sphere centered at the origin and the point x_0 is the origin.

Let $X = \mathbb{R}^n - \{0\}$, then define the function $\phi: X \times I \to X$ by

$\phi(x,t) = (1-t)x + t \cdot \frac{x}{\|x\|}$.

One can verify that this is indeed the required deformation retract.

4.7

The center circle.

TODO 4.8

TODO 4.9

4.10

Suppose that f_0 and f_1 be any two path connecting x and y. Consider the path from x to x given by $\gamma = f_0 \cdot f_1^{-1}.$

Since the space is a deformation retract to a single point, the fundamental group $\pi(X,x)$ is isomorphic to the trivial group $\{e\}$, i.e., γ is identity. This shows that f_0 and f_1 are equivalent.

TODO 4.11

5

5.1

Given a loop $f \colon I \to X$ based on x_0, we need to prove that this belongs to the trivial class.

Suppose $f(I)$ is entirely contained in one of U_i, since U_i is simply connected, the fact that f belongs to trivial class is obvious. We assume that $f(I)$ is contained is not completed contained in a single U_i.

Define a class of open sets $V_i = f^{-1}(U_i)$. Since $[0, 1]$ is compact and since V_i is an open cover of $[0, 1]$, there exists a real number, ε (the Lebesgue number corresponding to the open cover), such that any set whose diameter is less than ε is completely contained in one of U_i.

Thus we can find a sequence $0 = t_1 < t_2 < \cdots < t_n = 1$ such that the following conditions are satisfied for all i:

  (a). $f([t_i, t_{i+1}]) \subset U_j$ for all some j;

  (b). $f([t_i, t_{i+1}])$ and $f([t_{i+1}, t_{i+2}])$ are not entirely contained in same U_j for some j.

This can be constructed in the following fashion: pick t_i such that the condition on diameter is satisfied. This guarantees the first condition. If for consecutive intervals lies completely inside a single U_i, simply merge the intervals to produce a larger one; repeat this procedure until the second condition is satisfied.

This basically implies that for every value of i, $f(t_i)$ belongs to the intersection of U_j and U_k for some value of j and k.

Let us denote the path component corresponding to f as β and the ones that correspond to the restriction of f on the sub intervals as β_i. It is easy to see that

$\beta = \beta_1 \cdot \beta_2 \cdots$.

Since $f(t_i)$ will always belong to the intersection, we can think of a path γ_i that connects $f(t_i)$ and the point x_0. Define the loop classes δ_i in the following manner:

$\delta_1 = \beta_1 \gamma_1$,

$\delta_i = \gamma_{i-1}^{-1} \beta_i \gamma^i$ for $1 < i < n$,

$\delta_n = \gamma_{n-1}^{-1} \beta_n$. (if n is finite.)

It is easy to see that

$\beta = \delta_1 \cdot \delta_2 \cdots$.

Observe that each δ_i is a loop at x_0 and lying entirely inside U_i, i.e., trivial. We have shown that f can be decomposed as loops of trivial classes, i.e., f belongs to the trivial loop class. ∎

Special cases

(a). Let X be a topological space and U_1, U_2 be two open sets that cover X that has at least one point in common. Suppose these open sets are simply connected and if $U_1 \cap U_2$ is an arc wise connected space, then X is simply connected.

(b). Let $\{U_i\}$ be a nonempty open cover of X such that $U_i \subset U_j$ whenever $i < j$. Suppose each of U_i are arc connected and simply connected. Then the space X is simply connected.

5.2

We make use of the Stereographic projection of S^n. Let p and $-p$ represent two antipodal points of S^n, then there exists two homeomorphisms $f\colon S^n - \{p\} \to \mathbb{R}^n$ $g\colon S^n - \{-p\} \to \mathbb{R}^n$. This shows that the open subsets $S^n - \{p\}$ and $S^n - \{-p\}$, which has a arc connected intersection whenever $n \geqslant 2$ (this is not true for $n = 1$), are also simply connected. From previous exercise, we see that S^n is simply connected.

5.3

$\mathbb{R}^2 - \{0\}$ is not simply connected while $\mathbb{R}^n - \{0\}$ is simply connected for $n > 2$.

5.4

A homeomorphism cannot map a point of S^1 to a point in the interior of E^2. This is because, there is no neighborhood of a point on the boundary that is homeomorphic $\mathbb{R}^2$ while there are neighborhoods of E^2 that are homeomorphic to $\mathbb{R}^2$.

Alternatively, one can see that deleted neighborhoods of a point on the boundary are simply connected while this is not the case with a point on the interior of D^2.

7

7.1

A torus is $S^1 \times S^1$, thus $\pi(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.