

# Chapter 2

## 2

### 2.1

I think that by "arc", the author is referring to path. (Spivak uses arc to refer to one-one path.) In that case, this is easy.

## 3

### 3.1

The center of a group  $G$  is the set of all elements  $z$  such that  $z \cdot g = g \cdot z$ .

If two path classes gives rises to the same isomorphism if and only if  $\gamma'\gamma^{-1}$  belongs to the center of the group  $\pi(X, x)$ .

### 3.2

We claim the following: this is equivalent to:

Fix a point  $x$ ; if for all values of  $y$ , and all path class  $\gamma$  and  $\gamma'$  connecting  $x$  and  $y$ ,  $\gamma'\gamma^{-1}$  be an element of the center of the group.

Basically this is an extension of the result from problem 3.1. The only difference is that we can simply state the equivalence for a fixed  $x$  and this would be equivalent for any point in  $X$  too.

### 3.3

Suppose  $f \sim g$ , then  $f \cdot \bar{g} = g \cdot \bar{g} = \mathcal{E}_x \in \pi(X, x)$ .

Suppose  $f \cdot \bar{g} = \mathcal{E}_x$ , then  $f \cdot \bar{g}g = \mathcal{E}_x \cdot g \implies f \cdot \mathcal{E}_y = g \implies f \sim g$ .

## 4

### TODO 4.1

### 4.2

Recall that  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{x \times x | x \in X\}$  is closed.

Given two maps  $f, g: X \rightarrow Y$ , then the set  $A = \{x \in X | f(x) = g(x)\}$  is closed if  $Y$  is Hausdorff. This can be proved by making use of the above observation, since  $A = (f \times g)^{-1}(\Delta)$ .

Let  $f: X \rightarrow X$  be a retraction mapping from  $X$  onto  $Z$  and  $i: X \rightarrow X$  be the identity map. Then the points where both these map are equal is clearly the set  $Z$  which will become closed.

### TODO 4.3

#### 4.4

Suppose  $A$  be a retract of  $X$  and  $r: X \rightarrow A$  be the corresponding retraction. Then define a function  $\phi: X \times Y \rightarrow A \times Y$  by  $\phi(x, y) = (r(x), y)$ . This function is clearly continuous and thus forms a retraction mapping from  $X \times Y$  onto  $A \times Y$ .

Suppose that there is a retraction  $\phi: X \times Y \rightarrow A \times Y$ . Define a retraction  $r: X \times A$  by first picking a point  $y_0 \in Y$  and by making  $r(x) = \pi_1 \circ \phi(x, y_0)$ . Once again one can verify that this mapping is a retraction.

#### 4.5

This is easy. Compose the two retraction mappings to get the required retraction.

#### 4.6

Without loss of generality, one can assume that the  $(n - 1)$  dimensional sphere  $S^{n-1}$  is the unit-sphere centered at the origin and the point  $x_0$  is the origin.

Let  $X = \mathbb{R}^n - \{0\}$ , then define the function  $\phi: X \times I \rightarrow X$  by

$$\phi(x, t) = (1 - t)x + t \cdot \frac{x}{\|x\|}.$$

One can verify that this is indeed the required deformation retract.

#### 4.7

The center circle.

### TODO 4.8

### TODO 4.9

#### 4.10

Suppose that  $f_0$  and  $f_1$  be any two path connecting  $x$  and  $y$ . Consider the path from  $x$  to  $x$  given by  $\gamma = f_0 \cdot f_1^{-1}$ .

Since the space is a deformation retract to a single point, the fundamental group  $\pi(X, x)$  is isomorphic to the trivial group  $\{e\}$ , i.e.,  $\gamma$  is identity. This shows that  $f_0$  and  $f_1$  are equivalent.

## TODO 4.11

## 5

### 5.1

Given a loop  $f: I \rightarrow X$  based on  $x_0$ , we need to prove that this belongs to the trivial class.

Suppose  $f(I)$  is entirely contained in one of  $U_i$ , since  $U_i$  is simply connected, the fact that  $f$  belongs to trivial class is obvious. We assume that  $f(I)$  is not completely contained in a single  $U_i$ .

Define a class of open sets  $V_i = f^{-1}(U_i)$ . Since  $[0, 1]$  is compact and since  $V_i$  is an open cover of  $[0, 1]$ , there exists a real number,  $\varepsilon$  (the Lebesgue number corresponding to the open cover), such that any set whose diameter is less than  $\varepsilon$  is completely contained in one of  $U_i$ .

Thus we can find a sequence  $0 = t_1 < t_2 < \dots < t_n = 1$  such that the following conditions are satisfied for all  $i$ :

- (a).  $f([t_i, t_{i+1}]) \subset U_j$  for all some  $j$ ;
- (b).  $f([t_i, t_{i+1}])$  and  $f([t_{i+1}, t_{i+2}])$  are not entirely contained in same  $U_j$  for some  $j$ .

This can be constructed in the following fashion: pick  $t_i$  such that the condition on diameter is satisfied. This guarantees the first condition. If for consecutive intervals lies completely inside a single  $U_i$ , simply merge the intervals to produce a larger one; repeat this procedure until the second condition is satisfied.

This basically implies that for every value of  $i$ ,  $f(t_i)$  belongs to the intersection of  $U_j$  and  $U_k$  for some value of  $j$  and  $k$ .

Let us denote the path component corresponding to  $f$  as  $\beta$  and the ones that correspond to the restriction of  $f$  on the sub intervals as  $\beta_i$ . It is easy to see that

$$\beta = \beta_1 \cdot \beta_2 \cdot \dots.$$

Since  $f(t_i)$  will always belong to the intersection, we can think of a path  $\gamma_i$  that connects  $f(t_i)$  and the point  $x_0$ . Define the loop classes  $\delta_i$  in the following manner:

$$\delta_1 = \beta_1 \gamma_1,$$

$$\delta_i = \gamma_{i-1}^{-1} \beta_i \gamma_i \text{ for } 1 < i < n,$$

$$\delta_n = \gamma_{n-1}^{-1} \beta_n. \text{ (if } n \text{ is finite.)}$$

It is easy to see that

$$\beta = \delta_1 \cdot \delta_2 \cdot \dots.$$

Observe that each  $\delta_i$  is a loop at  $x_0$  and lying entirely inside  $U_i$ , i.e., trivial. We have shown that  $f$  can be decomposed as loops of trivial classes, i.e.,  $f$  belongs to the trivial loop class. ■

### Special cases

- (a). Let  $X$  be a topological space and  $U_1, U_2$  be two open sets that cover  $X$  that has at least one point in common. Suppose these open sets are simply connected and if  $U_1 \cap U_2$  is an arc wise connected space, then  $X$  is simply connected.
- (b). Let  $\{U_i\}$  be a nonempty open cover of  $X$  such that  $U_i \subset U_j$  whenever  $i < j$ . Suppose each of  $U_i$  are arc connected and simply connected. Then the space  $X$  is simply connected.

## 5.2

We make use of the Stereographic projection of  $S^n$ . Let  $p$  and  $-p$  represent two antipodal points of  $S^n$ , then there exists two homeomorphisms  $f: S^n - \{p\} \rightarrow \mathbb{R}^n$   $g: S^n - \{-p\} \rightarrow \mathbb{R}^n$ . This shows that the open subsets  $S^n - \{p\}$  and  $S^n - \{-p\}$ , which has a arc connected intersection whenever  $n \geq 2$  (this is not true for  $n = 1$ ), are also simply connected. From previous exercise, we see that  $S^n$  is simply connected.

## 5.3

$R^2 - \{0\}$  is not simply connected while  $R^n - \{0\}$  is simply connected for  $n > 2$ .

## 5.4

A homeomorphism cannot map a point of  $S^1$  to a point in the interior of  $E^2$ . This is because, there is no neighborhood of a point on the boundary that is homeomorphic  $\mathbb{R}^2$  while there are neighborhoods of  $E^2$  that are homeomorphic to  $\mathbb{R}^2$ .

Alternatively, one can see that deleted neighborhoods of a point on the boundary are simply connected while this is not the case with a point on the interior of  $D^2$ .

# 7

## 7.1

A torus is  $S^1 \times S^1$ , thus  $\pi(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$ .

## 7.2

Consider the retract  $r: S^1 \times S^1 \rightarrow S^1 \times \{p\}$  defined by  $r(x, y) = (x, p)$ . This is clearly, a continuous map.

Recall that, if  $A$  is a deformation retract of  $X$ , then the inclusion map induces an isomorphism of  $\pi(A, a)$  onto  $\pi(X, a)$  for any  $a \in A$ . Observe that the group  $\mathbb{Z} \times \mathbb{Z}$  is not isomorphic to  $\mathbb{Z}$  (the first one is cyclic.)

## Chapter 3

(Skipped through most of the exercises.)

### 3

#### 3.1

Let  $F'$  denote the subgroup of  $F$  that is generated by elements of  $\varphi(S)$ . Given an Abelian group  $A$ , there is a homomorphism  $g$  that makes the diagram in the textbook commutative (lazy to draw it here.)

Firstly we shall show that  $F'$  shall also satisfy the hypothesis required for free group over  $S$ . This can be seen from the fact that the map  $g|_{F'}$  is a homomorphism between  $F'$  and  $A$  which also makes the diagram (in the book) commutative. From Proposition 3.1,  $F'$  and  $F$  are isomorphic.

Since  $F$  is Abelian, here is a unique homomorphism between  $F'$  and  $F$  that makes the diagram commutative. Clearly identity is the unique homomorphism, thus  $F' = F$ .

#### 3.2

We proceed similarly Corollary 3.5. If  $F$  and  $F'$  are isomorphic, then  $F/F^n$  and  $F'/(F')^n$  should also be isomorphic. But the former is finite, while the latter is infinite (the latter can be thought of as the weak product of infinitely infinitely many  $\mathbb{Z}_n$ , which is clearly an infinite group.)

#### 3.3

Reference: Baer's Result: The infinite product of integers has no basis.

An example is the group  $\mathbb{Q}$  under addition. This is clearly an Abelian group. Assume that this is free and let  $S$  be a basis for  $\mathbb{Q}$ .

Let  $1/1$  be represented as  $1/1 = n_1 s_{\alpha_1} + \cdots + n_k s_{\alpha_k}$  where  $n_1, \dots, n_k$  are integers and WLOG  $n_1 \neq 0$ .

Let  $n$  be an integer that does not divide  $n_1$ . The number  $1/n$  has the representation  $1/n = m_1 s_{\beta_1} + \cdots + m_j s_{\beta_j}$ . This implies that  $nm_1 s_{\beta_1} + \cdots + nm_j s_{\beta_j} = n_1 s_{\alpha_1} + \cdots + n_k s_{\alpha_k}$ , which is a contradiction.

#### 3.4

The split is  $\mathbb{Z}_{36} \times \mathbb{Z}_2$ .

### TODO 3.5

## 4

### TODO 4.1

Suppose  $x_1$  and  $x_2$  are elements belonging to two different groups. Then  $x_1x_2$  and  $x_2x_1$  are elements that are not equal, since  $x_1x_2x_1x_2 \neq 1$ . These elements are of infinite order. To show that the center contains only identity, given an arbitrary element  $y$ , let  $y_1 \in G_{y_1}$  denote the first element of the word  $y$  and  $y_2$  denote an element other than unity in and not contained in  $G_{y_1}$ . Clearly  $yy_2 \neq y_2y$ .

(TODO: Need to improve the reasons.)

## 5

### 5.1

This follows from Exercise 4.1 (basically says that the free product of a collection of groups, each containing more than one element is not Abelian.)

### 5.1

Again, this follows from Exercise 4.1

### TODO 5.3

### 5.4

If  $S$  is finite and  $S'$  be an infinite set, if  $F$  and  $F'$  are isomorphic, then so are  $F/[F, F]$  and  $F'/[F', F']$ . But  $F/[F, F]$  and  $F'/[F', F']$  are free Abelian groups with "generating sets" of different cardinality and hence non-isomorphic.

### 5.5