Solutions to *Introduction to Commutative Algebra* by Atiyah and Macdonald

Aditya Gudibanda

August 24, 2017

Chapter 1

1.1

Problem

Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution

Note the identity $\left(\sum_{i=0}^{k} x_i\right)(1-x) = 1 - x^{k+1}$. Since x is nilpotent, $x^l = 0$ for some $l \geq 1$. Thus, if we set $k = l-1$ in the identity above, we see that $1-x$ has a multiplicative inverse, and is therefore a unit.

To prove the original statement, given that x is a nilpotent unit, it is clear that $-x$ is nilpotent as well, and by the logic above, $1-(-x) = 1+x$ is a unit, as desired.

Suppose u is an arbitrary unit. Since x is nilpotent, $u^{-1}x$ is nilpotent as well, so by the above, $u^{-1}x+1$ is a unit. Since the product of units is a unit, $u(u^{-1}x+1) = x+u$ is a unit, as desired.

1.2

Problem

Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x]$ $\Leftrightarrow$ a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.
iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$.
iv) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution

TODO

1.3

Problem

Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution

TODO

1.4

Problem

In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution

TODO

1.5

Problem

Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A. Show that
i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?

iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

iv) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution

TODO

1.6

Problem

A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

Solution

TODO

1.7

Problem

Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution

Let $\mathfrak{a}$ be a prime ideal. Then $A/\mathfrak{a}$ is an integral domain. Suppose that $x \in A$. Then $x^n = x$ for some $n > 1$. Let $\bar{x}$ be the image of x under the projection to the quotient by $\mathfrak{a}$. Then $\bar{x}^n - \bar{x} = \bar{x}(\bar{x}^{n-1} - 1) = 0$. Since $A/\mathfrak{a}$ is an integral domain, this implies that either $\bar{x} = 0$ or $\bar{x}^{n-1} = 1$. If

$\bar{x}^{n-1} = 1$, then $\bar{x} \cdot \bar{x}^{n-2} = 1$, so $\bar{x}$ has an inverse. This implies that for every $\bar{x} \in A/\mathfrak{a}$, either $\bar{x} = 0$ or $\bar{x}$ has an inverse. Thus, $A/\mathfrak{a}$ is a field, so $\mathfrak{a}$ is maximal, as desired.

1.8

Problem

Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Solution

TODO

1.9

Problem

Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals.

Solution

$\Leftarrow$: Suppose $\mathfrak{a}$ is an intersection of prime ideals. Clearly, $\mathfrak{a} \subset r(\mathfrak{a})$. Suppose $x \notin \mathfrak{a}$. Then there is some prime ideal $\mathfrak{p}$ which x is not in, but in which $\mathfrak{a}$ is contained. Thus, no power of x will ever be in $\mathfrak{p}$, so no power of x will ever be in $\mathfrak{a}$. Thus, $\mathfrak{a} = r(\mathfrak{a})$, as desired.

$\Rightarrow$: Simply apply Proposition 1.14 - the radical of an ideal $\mathfrak{a}$, in this case $\mathfrak{a}$, is the intersection of the prime ideals which contain $\mathfrak{a}$.

1.10

Problem

Let A be a ring, $\mathcal{R}$ its nilradical. Show that the following are equivalent:

i) A has exactly one prime ideal

ii) every element of A is either a unit or nilpotent
iii) $A/\mathcal{R}$ is a field

Solution

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i $\Rightarrow$ iii: Suppose A has exactly one prime ideal. Let $\mathfrak{m}$ be the maximal ideal of A. Then this must be the unique prime ideal, so it is also the nilradical $\mathcal{R}$. Thus, $\mathcal{R}$ is maximal, so $A/\mathcal{R}$ is a field.

iii $\Rightarrow$ ii: Suppose $A/\mathcal{R}$ is a field. Then R is maximal, so all the elements in $A - \mathcal{R}$ are units. All the elements within $\mathcal{R}$ are obviously nilpotent. Thus, every element of A is either a unit or nilpotent, as desired.

ii $\Rightarrow$ i: Let $\mathfrak{p}$ be a prime ideal. Clearly it must contain all the nilpotent units and can contain none of the units. Thus, every prime ideal must contain exactly all of the nilpotents. Thus, A has exactly one prime ideal.

1.11

Problem

A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that
i) $2x = 0$ for all $x \in A$
ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements
iii) every finitely generated ideal in A is principal.

Solution

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i) $(x+1)^2 = x^2 + 2x + 1 = x + 2x + 1 = x + 1 \Rightarrow 2x = 0$, as desired.

ii) Let $\mathfrak{p}$ be a prime ideal. Then every element in $A/\mathfrak{p}$ satisfies $\bar{x}^2 = \bar{x}$. Since $A/\mathfrak{p}$ is an integral domain, this means that $\bar{x} = 0$ or $\bar{x} = 1$. Thus, $A/\mathfrak{p}$ has only two elements and is hence a field, so $\mathfrak{p}$ is maximal, as desired.

iii) It suffices to be able to reduce the ideal (a, b) and show that it is equivalent to another ideal (c). To this end, let $c = ab + a + b$. It is clear that $c \in (a, b)$. Next, $ca = a^2b + a^2 + ab = ab + a + ab = a$. Next, $cb = ab^2 + ab + b^2 = ab + ab + b = b$. Thus, $(a, b) = (c)$. So given any finite set of generators, we can reduce these to one generator, by induction, by reducing the number of generators one by one.

1.12

Problem

A local ring contains no idempotent $\neq 0, 1$.

Solution

Let $x \in A$, the local ring. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose x is idempotent. Then in $A/\mathfrak{m}$, $\bar{x}^2 = \bar{x} \Rightarrow \bar{x}(\bar{x}-1) = 0$. Since $A/\mathfrak{m}$ is a field, this implies that either $\bar{x} = 0$ or $\bar{x} = 1$.

If $\bar{x} = 1$, then x is a unit, so if $x^2 = x$, then $x(x-1) = 0$, so x is either 0 or 1. 0 is not a unit, so $x = 1$.

If $\bar{x} = 0$, then $x \in \mathfrak{m}$, so $x - 1$ is a unit. Since $x(x-1) = 0$, this means that $x = 0$.

Thus, x is either 0 or 1, as desired.

1.13

Problem

Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K. Let A be the polynomial ring over K generated by indeterminates x_f, one for each $f \in \Sigma$. Let $\mathfrak{a}$ be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let $\mathfrak{m}$ be the maximal ideal of A containing $\mathfrak{a}$, and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K, obtaining a field K_2, and so on. Let $L = \cup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let $\bar{K}$ be the set of all elements of L which are algebraic over K. Then $\bar{K}$ is an algebraic closure of K.

Solution

TODO

1.14

Problem

In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors of A is a union of prime ideals.

Solution

TODO

1.15

Problem

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X, V(1) = \emptyset$.

iii) If $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V(\cup_{i \in I} E_i) = \cap_{i \in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

Solution

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i) If $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{a} \subset \mathfrak{p}$, so $E \subset \mathfrak{p}$. Thus, $\mathfrak{p} \in V(E)$. So $V(\mathfrak{a}) \subset V(E)$. For the other direction, if $\mathfrak{p} \in V(E)$, then $E \subset \mathfrak{p}$. So $\mathfrak{p}$ must contain all the elements in the ideal generated by E, which is $\mathfrak{a}$. Thus, $\mathfrak{a} \subset \mathfrak{p}$, so

$\mathfrak{p} \in V(\mathfrak{a})$. Thus, $V(E) = V(\mathfrak{a})$. Next, let $\mathfrak{p} \in V(\mathfrak{a})$. Let $x \in r(\mathfrak{a}) - \mathfrak{a}$, with $n \in \mathbb{N}$ being the minimal exponent such that $x^n \in \mathfrak{a}$. Clearly, $n > 1$. Then consider the fact that $x \cdot x^{n-1} \in \mathfrak{a}$. By the definition of n, both x and x^{n-1} are not in $\mathfrak{a}$. But $x^n \in \mathfrak{a} \Rightarrow x^n \in \mathfrak{p}$, so either x or x^{n-1} must be in $\mathfrak{p}$. If it is x^{n-1}, then we perform the same argument with x and x^{n-2}, etc. until we reach the conclusion that $x \in \mathfrak{p}$. Thus, $r(\mathfrak{a}) \subset \mathfrak{p}$, so $\mathfrak{p} \in V(r(\mathfrak{a}))$. For the other direction, $V(r(\mathfrak{a})) \subset V(\mathfrak{a})$ is obvious because $\mathfrak{a} \subset r(\mathfrak{a})$. Thus, $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$, as desired.

ii) $V(0) = X$, because every prime ideal contains 0. $V(1) = \emptyset$, because if an ideal contains 1, then it contains all of A and it cannot be prime.

iii) If $\mathfrak{p} \in V(\cup_{i\in I} E_i)$, then $\mathfrak{p} \in V(E_i)$, $i \in I$, so $p \in \cap_{i\in I} V(E_i)$. If $\mathfrak{p} \in \cap_{i\in I} V(E_i)$, then $\cap_{i\in I} E_i \in \mathfrak{p}$, so $E_i \in \mathfrak{p}$, $i \in I$, so $\cup_{i\in I} E_i \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\cup_{i\in I} E_i)$. Thus, $V(\cup_{i\in I} E_i) = \cap_{i\in I} V(E_i)$, as desired.

iv) Since $\mathfrak{ab} \subset \mathfrak{a} \cap \mathfrak{b}$, it is clear that $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{ab})$. Now suppose $\mathfrak{p} \in V(\mathfrak{ab})$. Now suppose $x \in \mathfrak{a} \cap \mathfrak{b}$. Then clearly $x^2 \in \mathfrak{ab}$. So since $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$. Thus, $V(\mathfrak{ab}) \in V(\mathfrak{a} \cap \mathfrak{b})$. Thus, $V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b})$. This could also have been proven by noting that $r(\mathfrak{ab}) = r(\mathfrak{a} \cap \mathfrak{b})$ and using part i. Next, let $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Suppose $\mathfrak{p}$ was a prime ideal such that $\mathfrak{a} - \mathfrak{p} \neq \emptyset, \mathfrak{b} - \mathfrak{p} \neq \emptyset$. Let $x \in \mathfrak{a} - \mathfrak{p}$ and $y \in \mathfrak{b} - \mathfrak{p}$. Then $xy \in \mathfrak{ab} \subset \mathfrak{p}$, but $x, y \notin \mathfrak{p}$, a contradiction. Thus, $\mathfrak{p}$ must contain either $\mathfrak{a}$ or $\mathfrak{b}$, so $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. Next, suppose $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. WLOG, suppose $\mathfrak{p} \in V(\mathfrak{a})$. Then clearly $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Thus, $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, as desired.

1.16

Problem

Draw pictures of Spec $\mathbb{Z}$, Spec $\mathbb{R}$, Spec $\mathbb{C}[x]$, Spec $\mathbb{R}[x]$, Spec $\mathbb{Z}[x]$.

Solution

TODO

1.17

Problem

TODO

Solution

TODO

1.18

Problem

TODO

Solution

TODO

1.19

Problem

A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec A is irreducible if and only if the nilradical of A is a prime ideal.

Solution

TODO

1.20

Problem

TODO

Solution

TODO

Chapter 2

2.1

Problem

Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution

The tensor product is generated by all possible tensor products of pairs of generators from the two rings we are tensoring. Both rings are cyclic and are generated by 1. So the tensor product is generated by $1 \otimes 1$. Now, note that $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$, and $n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0$. Thus, the order of $1 \otimes 1$ must divide both m and n. However, since m and n are coprime, the only such positive number is 1. Thus, $1 \otimes 1$ has order 1, so $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is a ring that contains only one element, which is therefore 0, as desired.

2.2

Problem

Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution

Consider the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$. We now tensor this with M to get $0 \to \mathfrak{a} \otimes_A M \to A \otimes_A M \to A/\mathfrak{a} \otimes_A M \to 0$. By the first isomorphism theorem, we therefore have that $A/\mathfrak{a} \otimes_A M \cong A \otimes_A M/\mathfrak{a} \otimes_A M$. Clearly, $A \otimes_A M \cong M$ and $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$, so we have $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$, as desired.

2.3

Problem

Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution

Let $\mathfrak{m}$ be the maximal ideal of A and let $k = A/\mathfrak{m}$ be the residue field. Then $M_k = k \otimes_A M \cong M/\mathfrak{m}M$ by 3.2. Since A is a local ring, $\mathfrak{m}$ is the only maximal ideal of A, so by Nakayama's lemma, if $M_k = 0$, then $M = 0$.

Since $M \otimes_A N = 0$, $(M \otimes_A N)_k = M_k \otimes_A N_k = 0$. Since M_k and N_k are both vector spaces over a field, this implies that either $M_k = 0$ or $N_k = 0$, which implies that either $M = 0$ or $N = 0$, as desired.

2.4

Problem

Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution

$\Rightarrow$: Suppose that M is flat. The maximal ideals of M are $\{\times_{i \neq j} M_i \times \mathfrak{m}_j | j \in I, \mathfrak{m}_\mathrm{j} \in M_j\}$, where $\mathfrak{m}_j$ are the maximal ideals of M_j. The quotients of all these maximal ideals must also be flat. These are all $M_j/\mathfrak{m}_j$, over all $j \in I$ and over all $\mathfrak{m}_j$ which are maximal in M_j. For a specific j, this implies that all the maximal ideals are flat. This implies that M_j is flat. So this direction is done.

$\Leftarrow$: Just go the other direction.

2.5

Problem

Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution

$A[x]$ is just the infinite direct sum of A, considered as an A-module, so $A[x]$ is flat if A is flat as an A-algebra, by problem 2.4. But for any A-module M, $M \otimes_A A \cong M$, so clearly A is flat. Thus, $A[x]$ is a flat A-algebra.

2.6

Problem

For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r (m_t \in M)$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x] \cong A[x] \otimes_A M$.

Solution

From Proposition 2.14, we know that $(M + N) \otimes P \cong (M \otimes P) + (N \otimes P)$. We can consider $M[x]$ to be an infinite direct sum of M, and $A[x]$ to be an infinite direct sum of A. From Proposition 2.14, we know that when M is a module of A, $A \otimes M \cong M$. These two facts immediately imply the desired conclusion, $M[x] \cong A[x] \otimes_A M$.

2.7

Problem

Let $\mathfrak{p}$ be a prime ideal of A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

Solution

Let $f, g \in A[x] - \mathfrak{p}[x]$. Let

$$f = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

and let

$$g = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$$

Suppose a_i is the coefficient with smallest index that is not in $\mathfrak{p}$, and let b_j be defined likewise. These must both exist because $f, g \notin \mathfrak{p}[x]$. Then consider the coefficient of x^{i+j} in fg. WLOG, let $i < j$. Then the coefficient of x^{i+j} in fg is

$$\sum_{r=0}^{i-1} a_r b_{i+j-r} + a_i b_j + \sum_{s=0}^{j-1} a_{i+j-s} b_s$$

Both the sums are in $\mathfrak{p}$ because $a_r \in \mathfrak{p}$ for $r < i$ and $b_s \in \mathfrak{p}$ for $s < j$, by the definitions of i and j. But a_i and b_j are not in $\mathfrak{p}$, and $\mathfrak{p}$ is prime, so $a_i b_j$ is not in $\mathfrak{p}$. Thus, fg is not in $\mathfrak{p}[x]$. Thus, $\mathfrak{p}[x]$ is prime in $A[x]$, as desired.

2.8

Problem

i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution

i) Let $0 \to B \to C \to D \to 0$ be an exact sequence of A-modules. Since M is flat,

$$0 \to B \otimes_A M \to C \otimes_A M \to D \otimes_A M \to 0$$

is exact. Since N is flat,

$$0 \to (B \otimes_A M) \otimes_A N \to (C \otimes_A M) \otimes_A N \to (D \otimes_A M) \otimes_A N \to 0$$

is also exact. By Proposition 2.14, $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$, so therefore we have

$$0 \to B \otimes_A (M \otimes_A N) \to C \otimes_A (M \otimes_A N) \to D \otimes_A (M \otimes_A N) \to 0$$

is exact. Thus, $M \otimes_A N$ is flat, as desired.

ii) Let $0 \to C \to D \to E \to 0$ be an exact sequence of A-modules. Since B is a flat A-algebra,

$$0 \to C \otimes_A B \to D \otimes_A B \to E \otimes_A B \to 0$$

is exact. Since N is a flat B-module,

$$0 \to C \otimes_A B \otimes_B N \to D \otimes_A B \otimes_B N \to E \otimes_A B \otimes_B N \to 0$$

But $B \otimes_B N \cong N$, so we get

$$0 \to C \otimes_A N \to D \otimes_A N \to E \otimes_A N \to 0$$

which implies that N is flat as an A-module, as desired.

2.9

Problem

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

Solution

Let $u_1, u_2, \cdots, u_m$ be the generators of M' and let $v_1, v_2, \cdots v_n$ be the generators of M''. Let $w_i = f(u_i), 1 \leq i \leq m$. Since M surjects onto M'', for each of the v_i, there is an x_i such that $f(x_i) = v_i$.

Let f be the map from M to M'' in the exact sequence. For each $m \in M$, either m goes to 0 or $f(m)$ is a finite sum $s_1 v_1 + s_2 v_2 + \cdots + s_n v_n$. In the first case, it is in the submodule generated by $w_1, w_2, \cdots, w_m$. In the second case, it is in the submodule generated by $x_1, x_2, \cdots, x_n$. In all cases, m is in the submodule generated by $w_1, w_2, \cdots, w_m, x_1, x_2, \cdots, x_n$. Thus M is finitely generated as desired.

2.10

Problem

Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution

TODO

2.11

Problem

Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.

Solution

Let $\mathfrak{m}$ be a maximal ideal of A, let $k = A/\mathfrak{m}$, and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes_k \phi : k \otimes_k A^m \to k \otimes_k A^n$ is an isomorphism of vector spaces of dimensions m and n. Thus, $m = n$.

2.12

Problem

Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that Ker (ϕ) is finitely generated.

Solution

Ker $(\phi) \subset M$ and M is finitely generated, so Ker (ϕ) must be finitely generated.

2.13

Problem

Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution

TODO

Chapter 3

3.1

Problem

Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution

$\Leftarrow$: Let $m/t \in S^{-1}M$. Now note that $m/t = 0/1 \Leftrightarrow mu = 0$ for some $u \in S$. But since $sM = 0$, $ms = 0$ for all $m \in M$, so setting $u = s$ establishes that $m/t = 0$, so $S^{-1}M = 0$.

$\Rightarrow$: Suppose that $S^{-1}M = 0$. Then for all $m/1 \in S^{-1}M$, we have $m/1 = 0/1$, so there exists a $u \in S$ such that $mu = 0$. Let $e_1, e_2, \cdots, e_n$ be the generators of M. Let their corresponding annihilators in S be $u_1, u_2, \cdots, u_n$. Then $u_1 u_2 \cdots u_n$ annihilates every element of M. Thus, we are done.

3.2

Problem

Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1+\mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Solution

I will show that $\mathfrak{a}$ is contained in every maximal ideal of A. Suppose not. Then there would exist $a \in \mathfrak{a}$, $m \in M$ such that $a+m = 1$. This implies that a is a unit. Thus, $\mathfrak{a}$ is contained in every maximal ideal of A. It is also clear that $\mathfrak{a}$ and S are disjoint. By the one-to-one correspondence between prime ideals of A and prime ideals of $S^{-1}A$, this means that $S^{-1}\mathfrak{a}$ is contained in all the maximal ideals of $S^{-1}A$. The one-to-one correspondence applies to maximal ideals because they are disjoint from $S = 1 + \mathfrak{a}$. If $m = 1 + a$, this implies that a is a unit. Therefore, $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, as desired.

3.3

Problem

Let A be a ring, let S and T be two multiplicatively closed subsets of A, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Solution

Let $f : (ST)^{-1}A \to U^{-1}(S^{-1}A)$ be defined by $f(a/st) = (a/s)/(t/1)$. This map is clearly surjective. Now suppose that $f(a_1/s_1t_1) = f(a_2/s_2t_2)$. Then $(a_1/s_1)/(t_1/1) = (a_2/s_2)/(t_2/1)$. Then there is some $a_3/s_3 \in S^{-1}A$ such that $((a_1/s_1)(t_2/1) - (a_2/s_2)(t_1/1))a_3/s_3 = 0 \Rightarrow (a_1t_2/s_1 - a_2t_1/s_2)a_3/s_3 = 0 \Rightarrow (a_1t_2s_2 - a_2t_1s_1)/s_1s_2 \cdot (a_3/s_3) = 0 \Rightarrow a_1t_2s_2a_3 - a_2t_1s_1a_3 = 0$ This means that $a_1/(s_1t_1) = a_2/(s_2t_2)$, so f is injective as well. Thus, f is an isomorphism, as desired.

3.4

Problem

Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Solution

TODO

3.5

Problem

Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Solution

Suppose $x \in A$ was nilpotent and $\neq 0$. It must be contained in every prime ideal, so let $\mathfrak{p}$ be some prime ideal of A. Let $y \in A - \mathfrak{p}$, which is clearly not nilpotent, and consider the element $x/y \in A_\mathfrak{p}$. Suppose that $x^n = 0$. Then $(x/y)^n = x^n/y^n = 0/y^n$, which is equal to $0/1$, the zero element of $A_\mathfrak{p}$. This violates the assumption that $A_\mathfrak{p}$ has no nonzero nilpotent element. Thus we are done.

3.6

Problem

Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

Solution

Σ has maximal elements by Zorn's lemma, because $A - \{0\}$ is a multiplicatively closed subset that contains all the others, and inclusion is a partial order. Suppose $S \in \Sigma$ is maximal. Consider $A - S$. If this was not prime, then S would not be multiplicatively closed, because there would exist $x, y \in S$ such that $xy \in A - S$. Thus, $A - S$ is prime. Suppose that $A - S$ was not minimal, and suppose $T \subset S$ was a proper subset of

S forming a prime ideal. Then $A - T$ would be multiplicatively closed and contain S, which contradicts the fact that S is maximal. Thus, $A - S$ is a minimal prime ideal, as desired.

For the other direction, suppose $A - S$ is a minimal prime ideal. Then clearly S is multiplicatively closed by the definition of prime ideals. If S was not maximal, then suppose $S \subset T$ for some T that is multiplicatively closed. Then $A - T$ would be prime and be contained in $A - S$ which contradicts the fact that $A - S$ is minimal. Thus, we are done.

3.7

Problem

A multiplicatively closed subset S of a ring A is said to be *saturated* if

$$xy \in S \Leftrightarrow x, y \in S.$$

Prove that

i) S is saturated $\Leftrightarrow A - S$ is a union of prime ideals.

ii) If S is any multiplicatively closed subset of A, there is a unique smallest saturated multiplicatively closed subset $\bar{S}$ containing S, and that $\bar{S}$ is the complement in A of the union of prime ideals which do not meet S. ($\bar{S}$ is called the *saturation* of S.)

If $S = 1 + \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of A, find $\bar{S}$.

Solution

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i) Suppose $A - S$ is a union of prime ideals. Let $x, y \in S$. Then xy cannot be in any of the prime ideals comprising $A - S$, so $xy \in S$. Next, suppose $xy \in S$. WLOG, if $x \in A - S$, then x is in some prime ideal in $A - S$, so xy is also in the same prime ideal, implying $xy \in A - S$, a contradiction. Thus, $x, y \in S$. Thus, S is saturated.

For the other direction, suppose S is saturated. Let T be the union of all prime ideals which do not meet S. Let $x \in (A - S) - T$, and let $y \in S$. If $xy \in S$, then $x \in S$, a contradiction. If $xy \in T$, then xy is in some prime ideal, but both x and y are not in that prime ideal, a contradiction. Thus, $xy \in (A - S) - T$. One can clearly see that $(A - S) - T$ must be a prime ideal, and it does not meet S, a contradiction because we defined T be the union of all prime ideals that do not meet S. Thus, $(A-S)-T$ must be the empty set, so $A-S=T$, so $A-S$ is the union of prime ideals, as desired.

ii) Let T be the union of all prime ideals that do not meet S, and consider $A-T$. This contains S, and is saturated by part i. Suppose there was a smaller set U containing S that is saturated. Then let $x \in (A-T)-U$. Then applying the same argument from part i, we see that $(A-T)-U$ is a prime ideal which does not meet S, and hence must be in T, a contradiction. Thus, T is in fact the unique smallest saturated subset containing S, as desired.

$1 \bar{+} \mathfrak{a}$ is the complement of the union of all prime ideals that do not meet $1+\mathfrak{a}$.

3.8

Problem

Let S, T be multiplicatively closed subsets of A, such that $S \subset T$. Let $\phi : S^{-1}A \to T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

i) ϕ is bijective.

ii) For each $t \in T, t/1$ is a unit in $S^{-1}A$.

iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.

iv) T is contained in the saturation of S (Exercise 7)

v) Every prime ideal which meets T also meets S.

Solution

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i $\Rightarrow$ ii: Since ϕ is surjective, for each $a/t \in T^{-1}A$, there exists $a_1/s \in S^{-1}A$ such that a_1/s, considered as an element of $T^{-1}A$, is equal to a/t. So $\exists u \in T$ such that $u(ta_1 - sa) = 0$. Since ϕ is injective, if $a_1/s_1 = a_2/s_2$ considered as elements of T, then they are equal considered as elements of S. In other words, if there exists $t_3 \in T$ such that $t_3(s_2a_1 - s_1a_2) = 0$, then there exists a $s_3 \in S$ such that $s_3(s_2a_1 - s_1a_2) = 0$. ii) TODO

iii) TODO
iv) TODO
v) TODO

3.9

Problem

The set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals (see Chapter 1, Exercise 14). Show that every minimal prime ideal of A is contained in D.

The ring $S_0^{-1}A$ is called the total ring of fractions of A. Prove that

i) S_0 is the largest multiplicatively closed subset of A for which the homomorphism $A \to S_0^{-1}A$ is injective.

ii) Every element in $S_0^{-1}A$ is either a zero-divisor or a unit.

iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e., $A \to S_0^{-1}A$ is bijective).

Solution

Let $\mathfrak{p}$ be a minimal prime ideal. Then $A - \mathfrak{p}$ is a maximal multiplicatively closed subset of A. This means that it contains S_0, which means that $\mathfrak{p} \subset D$.

i) First, let us establish that $A \to S_0^{-1}A$ is injective. Suppose $a_1/s_1 - a_2/s_2 = 0$. Then there exists $u \in S_0$ such that $u(a_1s_2 - a_2s_1) = 0$. But this contradicts the fact that u is a non-zero-divisor. Thus, this is indeed injective. To see that it is the largest such set, suppose we had a set S_1 with $S_0 \subset S_1$ which contained a zero-divisor z. Then I claim that $0/z = 1/z$. To see this, note that $0/z - 1/z = 0 \Leftrightarrow \exists u \in S_1, u(-z) = 0$. But such a u clearly exists because z is a zero-divisor. So the map is no longer injective.

ii) If a/s is a zero-divisor, then there exists a_1/s_1 such that $aa_1/ss_1 - 0/1 = 0$, so in other words, $\exists u \in S_0$ such that $uaa_1 = 0$. But this is impossible because this would imply u is a zero-divisor, unless $aa_1 = 0$. This happens if and only if a is a zero-divisor. So if a is not a zero-divisor, then a/s is not a zero-divisor, for all $s \in S_0$. And these are precisely all the non-zero-divisors in $S_0^{-1}A$. Now clearly $a \in S_0$, so we can just take s/a as the inverse of a/s and see that it is a unit. Thus we are done.

iii)

Chapter 4

4.1

Problem

Solution

Problem

If $\mathfrak{a} = r(\mathfrak{a})$, then $\mathfrak{a}$ has no embedded prime ideals.

Solution

Let $\mathfrak{a} = \cap \mathfrak{q}_i$ be the primary decomposition of $\mathfrak{a}$. Then $r(\mathfrak{a}) = r(\cap \mathfrak{q}_i) = \cap r(\mathfrak{q}_\mathfrak{i})$. Let $r(\mathfrak{q}_\mathfrak{i}) = \mathfrak{p}_i$. Then $r(\mathfrak{a}) = \mathfrak{a} = \cap \mathfrak{p}_i$. Since all prime ideals are primary, this is the primary decomposition of $\mathfrak{a}$. Clearly, there are no embedded ideals, because if there were, we could just remove them from this representation and it would remain the same. Thus, $\mathfrak{a}$ has no embedded ideals, as desired.