MATH1052

Multivariate Calculus and Ordinary Differential Equations

WORKBOOK

How to use this workbook

This workbook is designed for use during lectures. It contains copies of the slides that will be shown in lectures, giving you time to listen and think in class, rather than spending the entire time writing.

There are exercises, definitions and examples in the workbook for you to fill in. These will be covered in lectures, and you should write down all the information given. The lecturer will make it clear when they are covering something that you need to complete. Take particular note of any highlighted formulae or results such as

$$1 + 1 = 2.$$

These are formulae or results we expect you to remember for your exams.

The completed workbook will act as a study guide to assist you in working through assignments and preparing for the mid-semester and final exams. For this reason, it is very important to attend lectures.

For further information about the course, including copies of assignments and worksheets, please go to the course website through `https://learn.uq.edu.au`

Overview

MATH1052 is a second course in calculus. It will enable you to use and visualise functions of two or more variables. Most importantly it extends your knowledge of calculus so you can differentiate multivariable functions and find their maxima and minima. These ideas are basic to the complex optimisation problems that occur when mathematics is used in economics, finance, and engineering.

MATH1052 also introduces ordinary differential equations (ODEs); one of the basic tools in mathematical modelling. ODEs are used to describe how things change with time. In engineering and the physical sciences ODEs are used to work out the motion of particles and satellites, the rates of chemical reactions, or to model electric fields. In biology they are used to model populations (to determine safe harvesting levels, for example) and to model epidemics.

The textbook for the course is Calculus (8th Edition) (the references in the parentheses correspond to the 7th Edition) by James Stewart (CENGAGE Learning). This is the only text you need for the course.

Contents

1 Functions of Several Variables

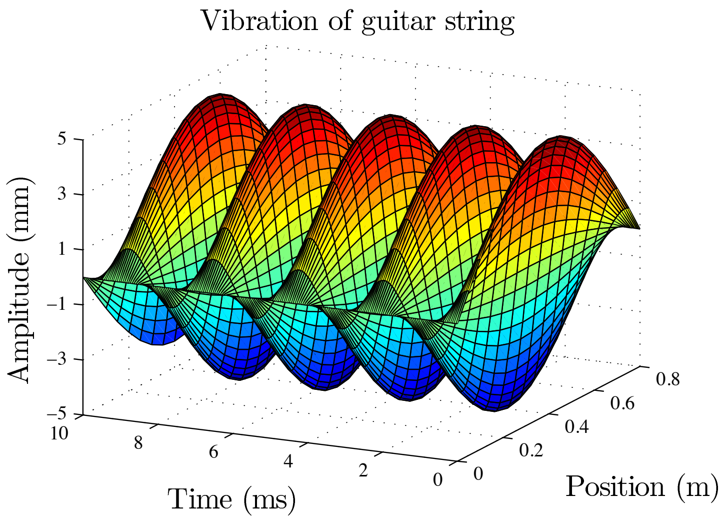
1.1 Introduction

From MATH1051 or earlier mathematics courses, you should be familiar with functions f of one variable. The tools of calculus were useful because you could:

- Sketch the graph $y = f(x)$ of f.
- Find the minima and maxima of f.
- Analyse the slope of f by calculating f'.
- Find approximations of f using Taylor series.
- Find solutions to $f(x) = 0$.

In this module of MATH1052, we shall undertake these same basic ideas, but in the framework of functions of several variables (multivariate functions). Most of the theory for this section is covered in Stewart, Chapter 14 (Chapter 14).

Many familiar formulae are essentially just functions of more than one variable. For example, the volume V of a box is a function of its width, height and depth: $V(w,h,d) = whd$, and the profile of a guitar string is a function of time and position along the string: $f(x,t) = A\sin(\alpha x)\cos(\beta t)$.



1.1.1 Visualising functions

Consider the volume of a cylinder as a function of two variables:

$$V(r,\ell) = \pi r^2 \ell.$$

A first method of visualising this function is to plot V for fixed ℓ and varying r.

Using Matlab, the **ezplot** function is the simplest graphing aid:

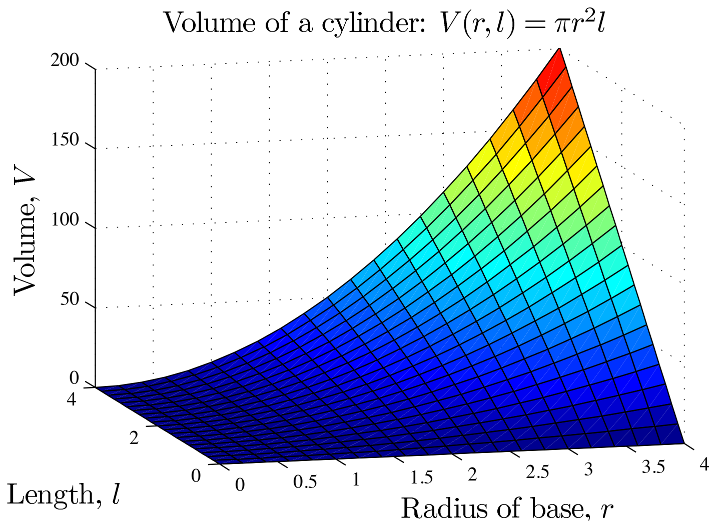
```
ezplot('pi*r^2*1',[0,5])
hold on; ezplot('pi*r^2*2',[0,5])
hold on; ezplot('pi*r^2*0.5',[0,5])
```

Alternatively, we could keep r fixed and plot V as a function of ℓ:

```
ezplot('pi*1^2*l',[0,5])
hold on;
ezplot('pi*0.5^2*l',[0,5])
ezplot('pi*2^2*l',[0,5])
```

Better still is visualising the surface by sketching the graph of $z = f(x, y)$. In Matlab, the **ezsurf** function will prove to be our quickest tool:

```
ezsurf('pi*r^2*l',[0,4,0,4])
```



Note how you can change the point of view of the plot with the 'rotate' button.

In Matlab we can plot contour lines for the volume function using the **ezcontour** function:

```
ezcontour('pi*r^2*l',[0,2,0,2])
```

Using these tools, you should be able to replicate the plots in Stewart, Section 14.1 (Section 14.1).

We shall start the course by looking at quadratic functions of two variables. These have contours which are circles, ellipses, parabolas or hyperbolas.

1.1.2 Main points

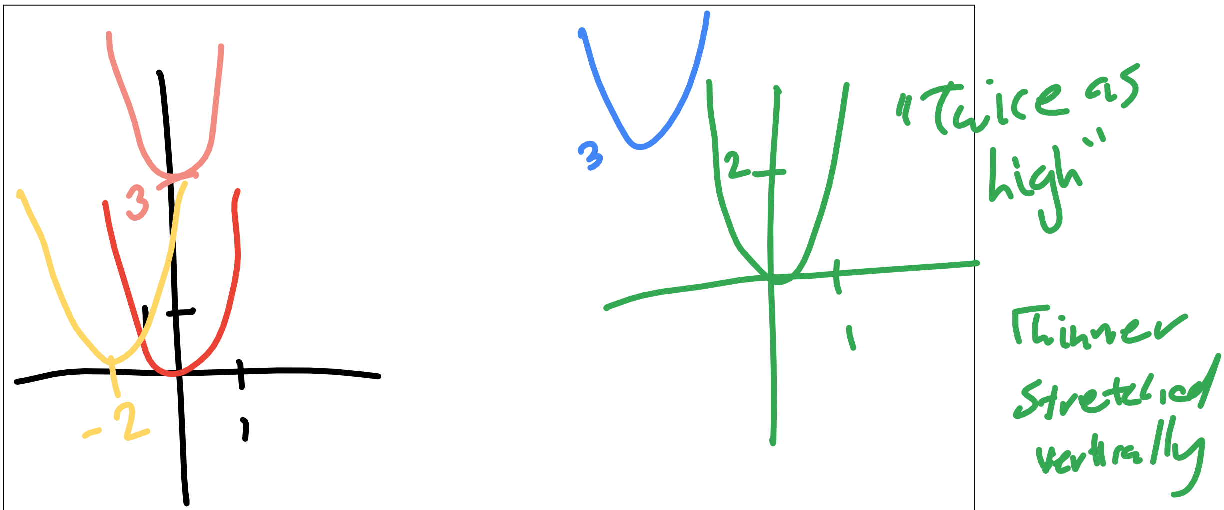
- What is a function of several variables and why are they useful?
- Methods to visualise a function of two variables.
- Matlab commands **ezplot**, **ezsurf** and **ezcontour**.

1.2 Conic Sections

The goal of this section is for you to become an expert in graphing parabolas, circles, ellipses and hyperbolas (without use of a graphics calculator, of course). The key-trick to this is "completing the square". See also Stewart, Section 10.5 (Section 10.5).

1.2.1 Parabolas

Example: Sketch the following parabolas:
$y = x^2$, $y = x^2 + 3$, $y = (x+2)^2$, $y = 2x^2$, $y = 2(x+2)^2 + 3$.



Putting a quadratic $y = ax^2 + bx + c$ in the form

$$y = a(x - h)^2 + k,$$

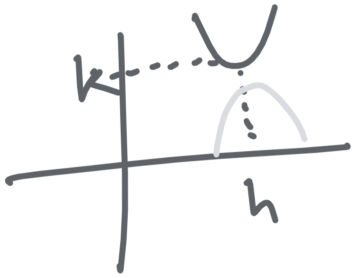
is called completing the square.

Question: What is the interpretation of the point (h, k)?

$V = (h, k)$ is the vertex of the parabola

$\cup$ $a > 0$

$\cap$ $a < 0$



Completing the square makes use of the identity

$$x^2 + 2xh + h^2 = (x + h)^2.$$

Example: Sketch the parabola $y = x^2 + 6x + 8$.

$ac = 8$ $\quad 4 \times 2$

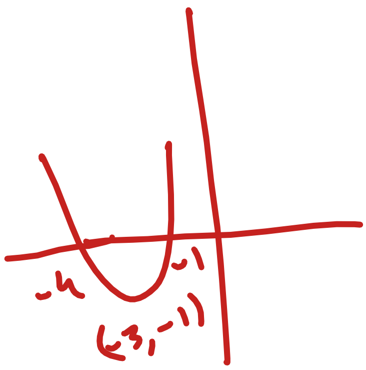
$b = 6$ $\quad 4 + 2$

$(x+4)(x+2)$

$y = x^2 + 2 \cdot 3x + 3^2 - 3^2 + 8$

$y = (x+3)^2 - 1$

$V = (-3, -1)$



$x^2+2xh+h^2=(x+h)^2$

Example: Sketch the parabola $y=-3x^2-6x+1$.

$y=-3(x^2+2x)+1$

$y=-3(x^2+2x+1^2-1)+1$

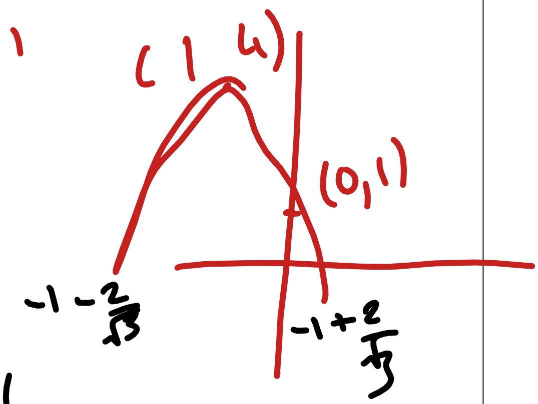
$=-3(x+1)^2+3+1$

$=-3(x+1)^2+4$

$3(x+1)^2=4$

$(x+1)^2=\frac{4}{3}$

$x+1=\pm\sqrt{\frac{4}{3}}$ $\quad x=\frac{\pm 2}{3}-1$



Example: Prove the quadratic formula. In other words, given $ax^2+bx+c=0$, by completing the square, show that

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

$y=ax^2+bx+c=a(x^2+\frac{b}{a}x)+c$

$=a(x^2+2x\frac{b}{2a})+c$

$=a(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2})+c$

$=a(x+\frac{b}{2a})^2-\frac{b^2}{\quad}$

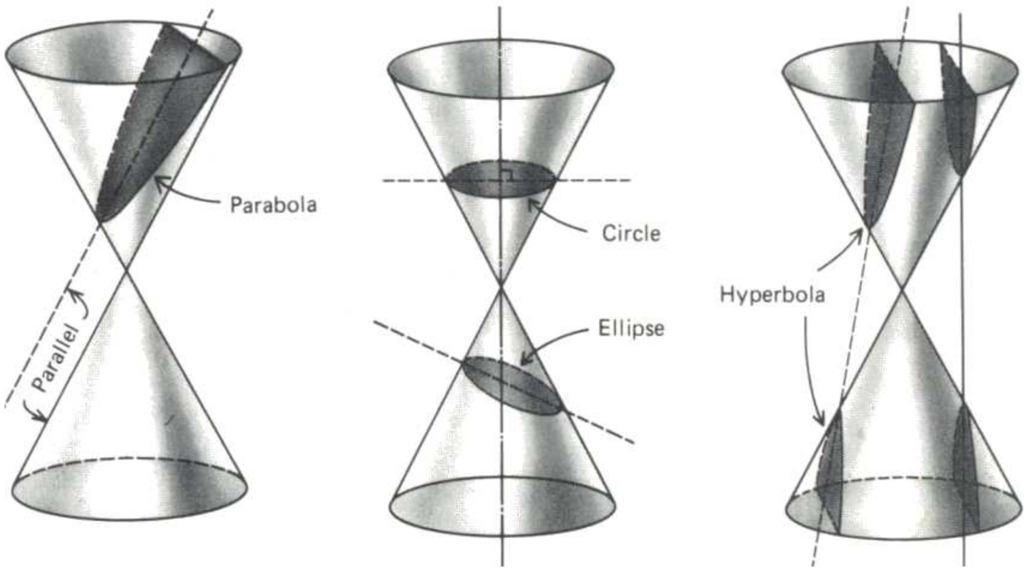
Parabolas, circles, ellipses and hyperbolas are all examples of conic sections; they all arise by intersecting a double cone such as $z^2 = x^2+y^2$, with a plane $z = -ex-dy+l$.

See the diagrams in Stewart p. 714 (p. 694). Slice an upright cone horizontally and you get a circle. Tip your knife a bit (with a gradient of magnitude less than 1) and you will get an ellipse. Keep tipping ($|\text{gradient}| = 1$), and you will get a parabola. Continue tipping ($|\text{gradient}| > 1$), and you will cut out an hyperbola.

Formally then, the general equation for a conic is given by the formula,

$$(l - ex - dy)^2 = x^2 + y^2,$$

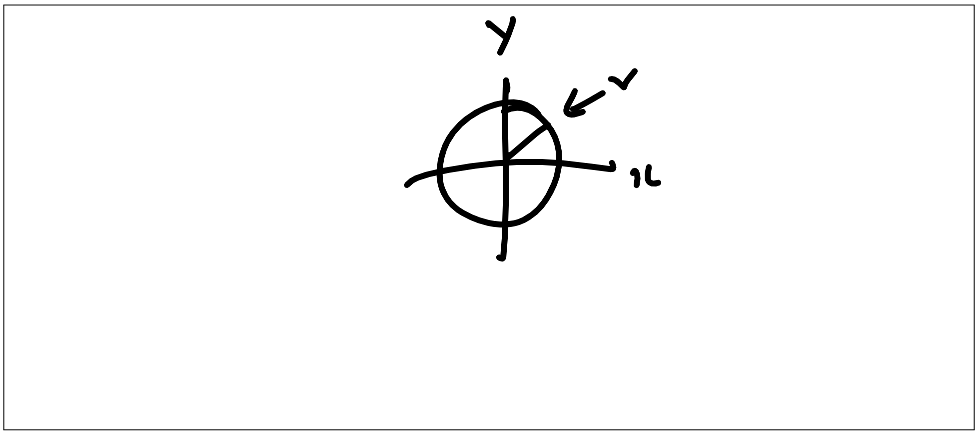
with a focus at the origin. We will explore this in more detail in a tutorial sheet.



1.2.2 Circles

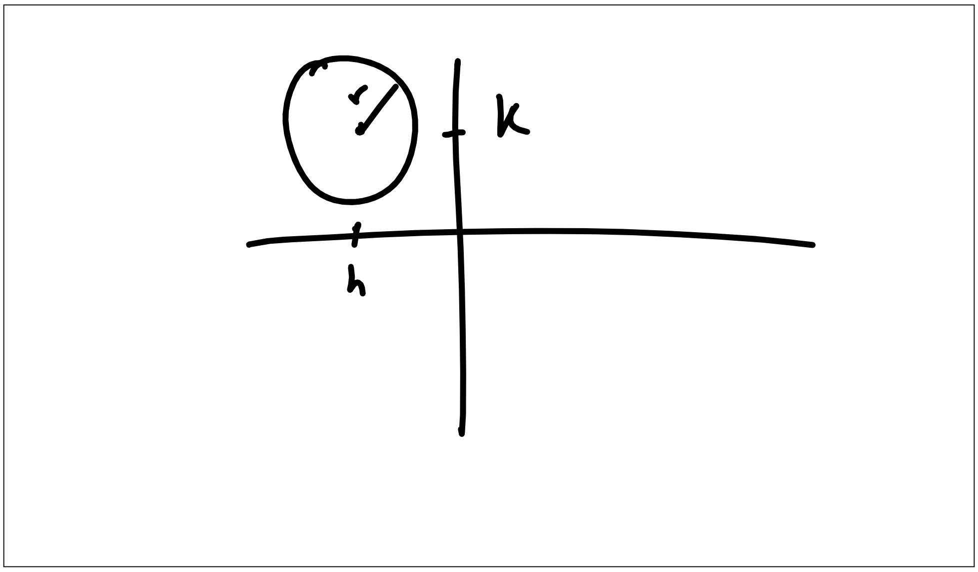
A circle centered at the origin $(0, 0)$ with radius r is given by the equation

$$x^2 + y^2 = r^2.$$



A circle with radius r and center (h, k) corresponds to

$$(x - h)^2 + (y - k)^2 = r^2.$$



$(x-h)^2+(y-k)^2=r^2$

Example: Sketch the circle $x^2 + 10x + y^2 - 4y + 20 = 0$.

$x^2 + 2\cdot 5x + 5^2 - 5^2$

$+ \; y^2 - 2\cdot 2x + 2^2 - 2^2 \quad + 20$

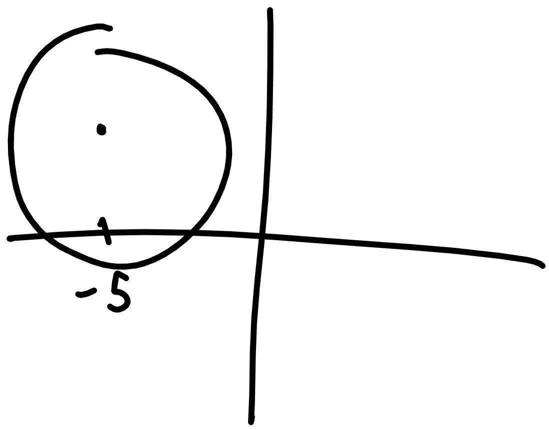
$= (x+5)^2 + (y-2)^2 - 25 - 4 + 20 = 0$

$= (x+5)^2 + (y-2)^2 = 3^2$

$r = 3$

$C = (-5, 2)$

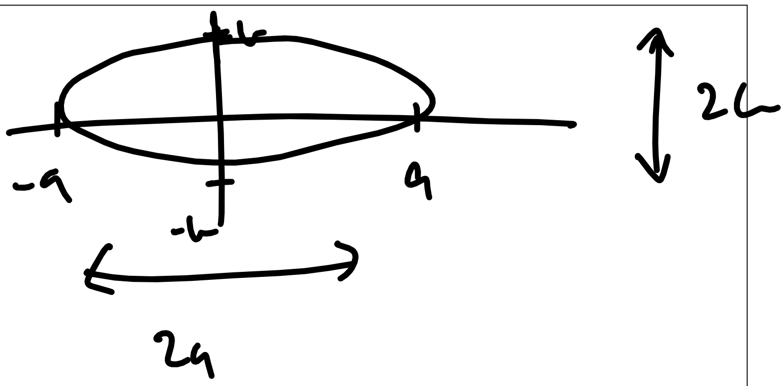
centre



1.2.3 Ellipses

In standard form, an ellipse centered at the origin is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



An ellipse may also be defined as the set of points in a plane the sum of whose distances from two fixed focal points F_1 and F_2 is a constant, and you should definitely ask your lecturer to use two pins and a piece of string to demonstrate this. If $a \geq b$, then the foci are at $(\pm c, 0)$ where $c^2 = a^2 - b^2$. These lie on the major axis (x-axis here). The ellipse intersects the x axis at the vertices $(\pm a, 0)$, and the y-axis at the vertices $(0, \pm b)$.

Question: What is the equation of an ellipse centered at the point (h, k)?

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Example: Sketch the ellipse $x^2 - x + 9y^2 = 0$.

$$x^2 - x + 9y^2 = 0$$

$$\left(x - 2\cdot\tfrac{1}{2}x + \left(-\tfrac{1}{2}\right)^2\right) - \left(\tfrac{1}{2}\right)^2 + 9y^2 = 0$$

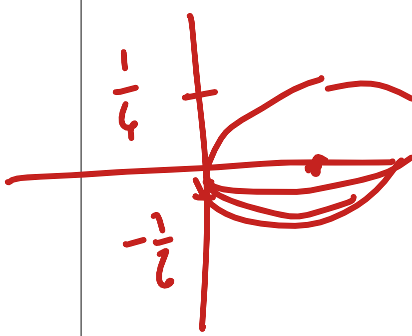
$$\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + 9y^2 = 0$$

$$\left(x - \tfrac{1}{2}\right)^2 + 9y^2 = \tfrac{1}{4}$$

$$4\left(x - \tfrac{1}{2}\right)^2 + 36y^2 = 1$$

$$\frac{\left(x - \frac{1}{2}\right)^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{36}} = 1$$

$$\frac{\left(x - \frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{\left(\frac{1}{6}\right)^2} = 1$$



$c = \left(\tfrac{1}{2}, 0\right)$

$a = \tfrac{1}{2}$ $b = \tfrac{1}{6}$

We could check this in Matlab with the command

```
ezplot('x^2-x+9*y^2=0',[-1,2],[-1,1])
```

1.2.4 Hyperbolas

A hyperbola centered at the origin, with asymptotes $y = \pm bx/a$, is given by the equation

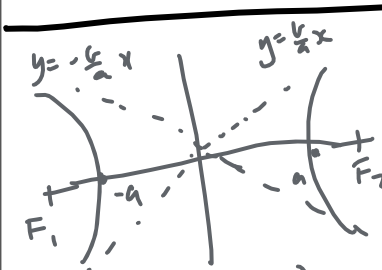
$$① \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or by} \quad ② \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

eq. for asymptotes

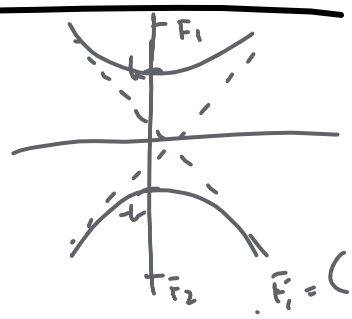
x-int: ① $y=0,\ x = \pm a$ ② $y=0 \quad -\frac{x^2}{a^2} = 1 \Rightarrow$ No sol

y-int: ① $x=0 \quad -\frac{y^2}{b^2} = 1 \Rightarrow$ No sol. ② $x=0 \quad y = \pm b$

ASYMPTOTES: $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \quad y \to \infty$; $y^2/b^2 = 1 + x^2/a^2 \quad x \to \infty$ $\Big\}$ $\frac{x^2}{a^2} \approx \frac{y^2}{b^2} \Rightarrow \sqrt{y^2} \approx \sqrt{\frac{b^2}{a^2} x^2} \Leftrightarrow y \approx \pm \frac{b}{a} x$



$F_i = (\pm c, 0)$



$F_i = (0, \pm c)$

A hyperbola may also be defined as the set of points in a plane, the difference of whose distances from two fixed focal points F_1 and F_2 is a constant. For such a hyperbola, the foci are at $(\pm c, 0)$ and $(0, \pm c)$ respectively, where $c^2 = a^2 + b^2$.

Question: What is the equation of a hyperbola centered at the point (h, k)?

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

$F_i = (\pm c + h, k) \qquad F_i = (h, \pm c + k)$

$|V_1 F_2| - |V_1 F_1| = 2a$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Example: Sketch the hyperbola $x^2 - 4y^2 = 9$.

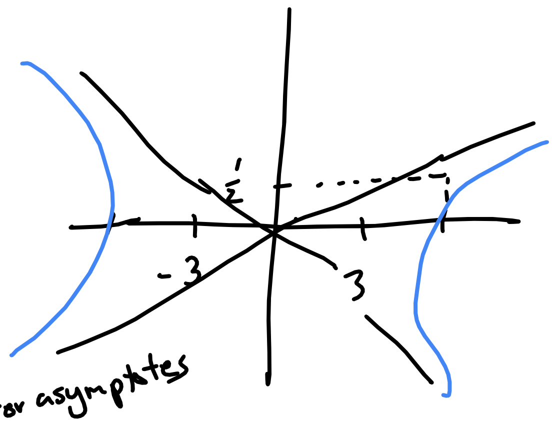
$$\frac{x^2}{9} - \frac{4y^2}{9} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{9/4} = 1$$

$a^2 = 9 \quad a = 3 \ (\pm)$

$b^2 = 9/4 \quad b = 3/2$

use to solve for asymptotes



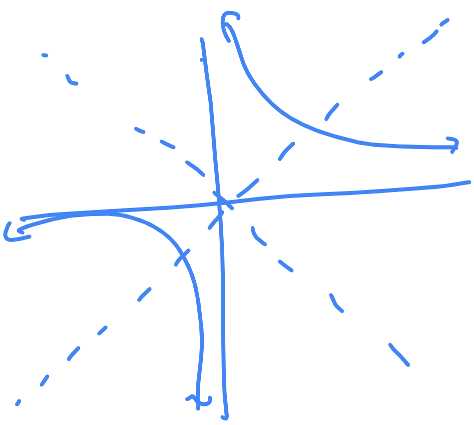
$$\frac{x^2}{\cancel{9}} = \frac{4y^2}{\cancel{9}}$$

$x = \pm 2y$

$y = \pm \frac{1}{2} x$

line going thru origin

Another kind of hyperbola is the very familiar $y = 1/x$.



$xy = 1$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

1.2.5 Quiz

Can you match each equation with one of the graphs?

$y = x^2+2x+1$
$y=(x+1)^2$

(a) $x^2 - y^2 - 1 = 0$ $x^2-y^2=1$

(b) $x + y^2 - 1 = 0$ $y=\pm\sqrt{1-x}$

(c) $x^2 - 2x + y^2 = 0$

(d) $x^2 + 2x - y + 1 = 0$.

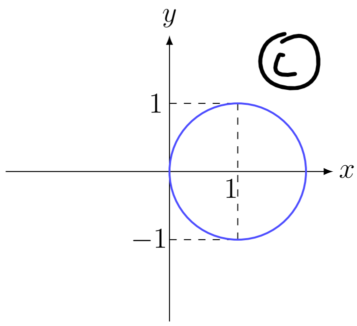
(e) $x^2 - y^2 + 4 = 0$. $x^2-y^2=-4$

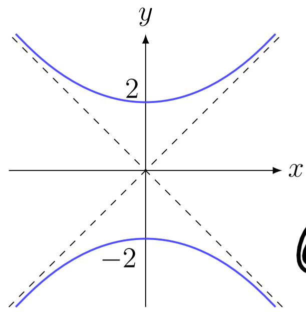
(f) $4x^2 - 8x + y^2 = 0$.

$x^2-2x+y^2=0$
$(x^2-2\cdot 1x+1^2 -1^2+y^2$
$(x-1)^2-1+y^2=0$
$(x-1)^2+y^2=1$

$-x^2+y^2=4$
$-\frac{x^2}{4}+\frac{y^2}{4}=1$
$x=0$
$y^2=4$
$y=\pm 2$

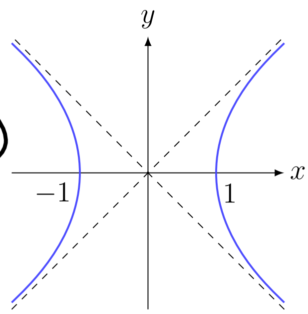
c

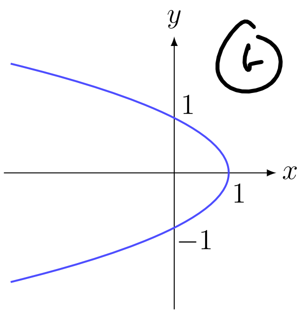




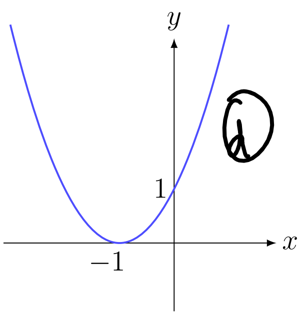
e

a

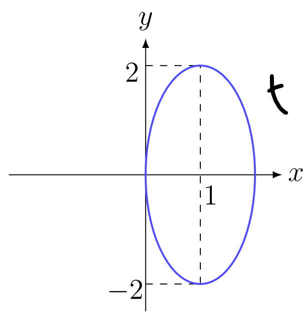




b $y^2=1-x$



d



f

$4x^2-8x+y^2=0$
$4(x^2-2x+1^2-1^2)+y^2$
$4(x-1)^2-4+y^2=0$
$4(x-1)^2+y^2=4$

Double-check each answer by putting each equation in standard form.

1.2.6 Main points

- By completing the square, you should be able to

  (i) sketch any curve of the form $ax^2 + by^2 + cx + dy + e = 0$,

  (ii) identify this curve as a straight line (or lines), circle, parabola, ellipse or hyperbola, and

  (iii) point out its main features such as turning points, asymptotes and intercepts with the x- and/or y-axes.

- You should also be able to obtain the equation of a curve of the above form from a sketch with features given.

CIRCLE $(x-h)^2 + (y-k)^2 = r^2$ $A=B=1$ $A=B$ $C=(h, k)$ radius r

ELLIPSE $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $A \neq B$

HYP. $\pm \frac{(x-h)^2}{a^2} \mp \frac{(y-k)^2}{k} = 1$ A & B have diff signs.

PARABOLA $a(x-h)^2 = (y-k)$ $\rightarrow$ $V = (h, k)$ $a>0$ $\ddot{\cup}$ $a<0$ $\ddot{\cap}$

$a(y-k)^2 = (x-h)$

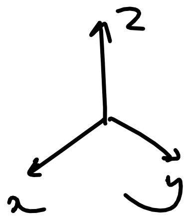
$V = (h, k)$ $a>0$ $\subset$ $a<0$ $\supset$

1.3 Contour Diagrams

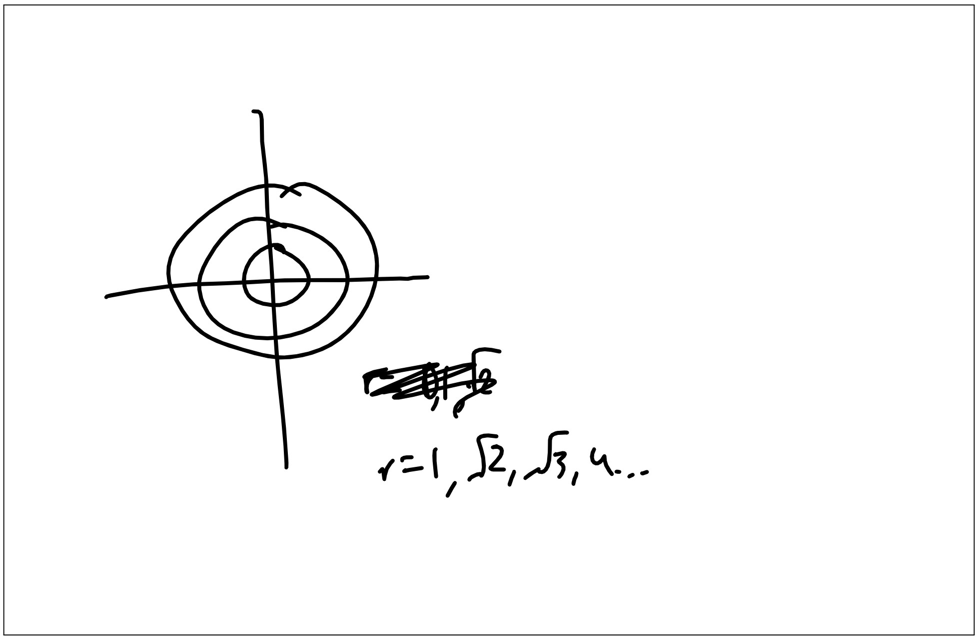
Geographical maps have curves of constant height above sea level, or curves of constant air pressure (isobars), or curves of constant temperature (isothermals). Drawing contours is an effective method of representing a 3-dimensional surface in two dimensions. We now look at functions f of two variables. A contour is a curve corresponding to the equation $z = f(x, y) = C$, see also Stewart, Section 14.1 (Section 14.1).

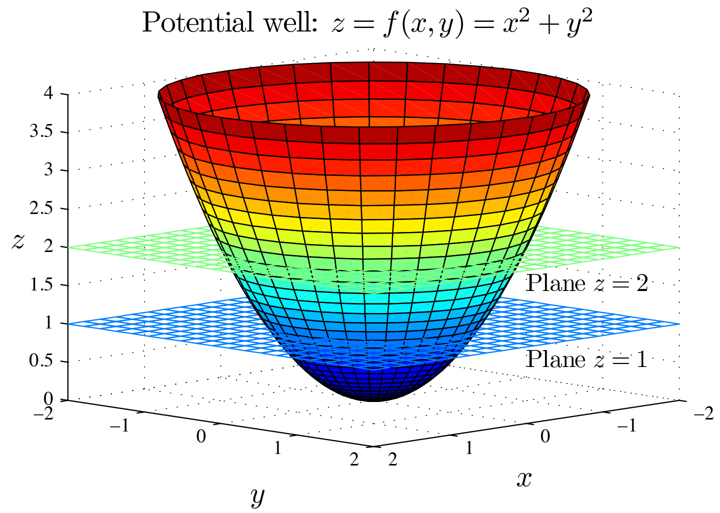
Consider the surface $z = f(x, y) = x^2 + y^2$ sliced by horizontal planes $z = 0$, $z = 1$, $z = 2, \ldots$

Plane	Contour	Description
$z = 0$	$f(x,y) = x^2+y^2=0$	Point $(0,0)$
$z = 1$	$f(x,y) = x^2+y^2=1$	Centre $C = (0,0)$ $r=1$
$z = 2$	$x^2+y^2=2$	$r=\sqrt{2}$
$z = 3$	$x^2+y^2=3$	$r=\sqrt{3}$
$z = 4$	$x^2+y^2=4$	$r=2$



Note that as the radius increases, the contours are more closely spaced.





Example: Draw a contour diagram of f given by

$$f(x, y) = \sqrt{x^2 + y^2}.$$

In Matlab, the command is

```
ezcontour('sqrt(x^2+y^2)',[-10,10,-10,10]);
```

If horizontal planes are equally spaced, say $z = 0, c, 2c, 3c, \ldots$, it is not hard to visualise the surface from its contour diagram. Spread-out contours mean the surface is quite flat and closely spaced ones imply a steep climb.

$z=0$ $\sqrt{x^2+y^2}=0$ $\Rightarrow P(0,0)$

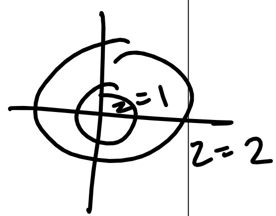
$z=1$ $x^2+y^2=1^2$ $\Rightarrow$ Circle $(0,0)$ $r=1$

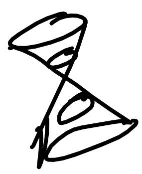
$z=2$ $\sqrt{x^2+y^2}=2$ $= x^2+y^2=2^2$ $r=2$

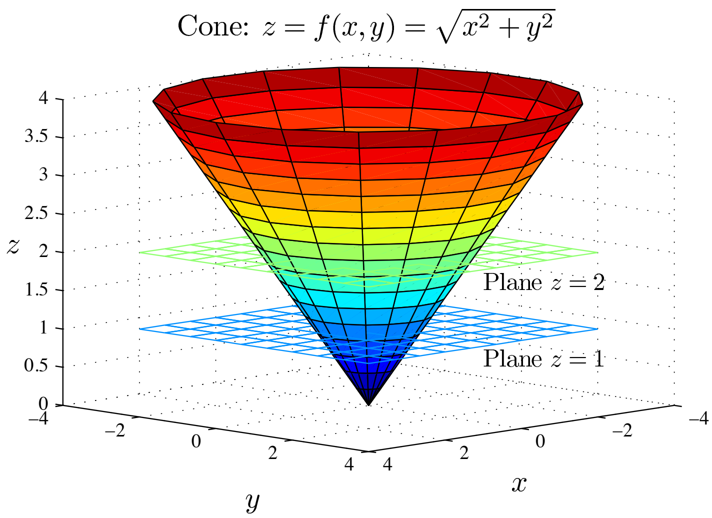
$z=3$ $x^2+y^3=3^2$ $(0,0)$ $r=3$

$z=n$ $x^2+y^2=n^2$ $(0,0)$

$r=n$

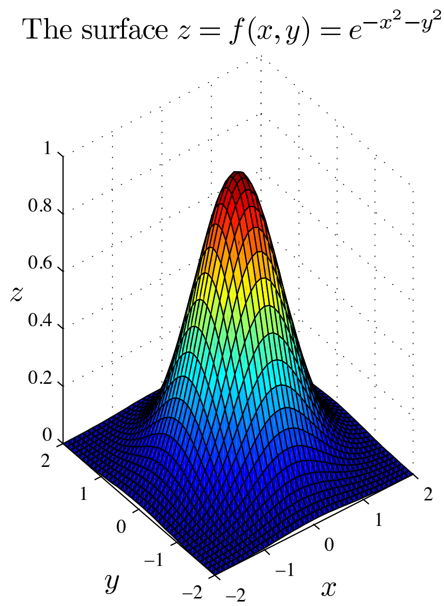


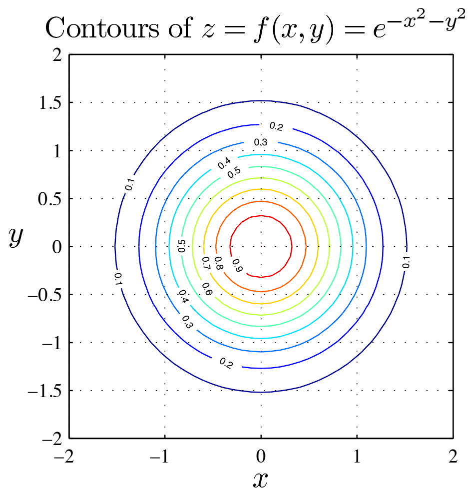




Note that the contours of the last two functions were all circles. Such surfaces have circular symmetry. When x and y only appear as x^2+y^2 in the definition of f, then the graph of f has circular symmetry about the z axis. The height z depends only on the radial distance $r = \sqrt{x^2+y^2}$.

Example: $z = f(x,y) = \mathrm{e}^{-x^2-y^2}$





Example: Draw a contour diagram for $z = f(x, y) = x^2 + 4y^2 - 2x + 1$. $(x-1)^2+4y^2$

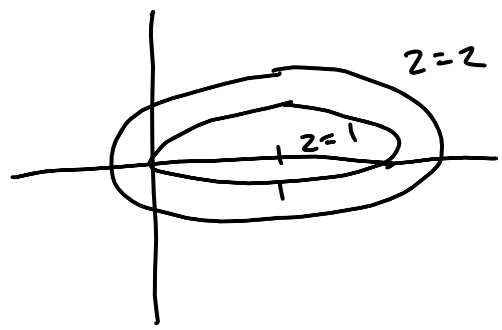
$z = (x-1)^2 + 4y^2$ $\qquad z_0 = (x-1)^2 + 4y^2$

$z = z_0$, $z_0 < 0$ no sol.

$z_0 = 0 \Rightarrow x = 1$ $y = 0$ $(1, 0)$

$z_0 > 0$

Plane	Contour	Description
$z = 1$	$(x-1)^2 + 4y^2 = 1$ $(x-1)^2 + \frac{y^2}{(1/2)^2} = 1$	$a = 1$ $b = \frac{1}{2}$
$z = 2$	$(x-1)^2 + 4y^2 = 2$ $\frac{(x-1)^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{\frac{1}{2}})^2} = 1$	$a = \sqrt{2}$ $b = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$
$z = 3$	$\frac{(x-1)^2}{(\sqrt{3})^2} + \frac{y^2}{(\sqrt{3}/2)^2} = 1$	$a = \sqrt{3}$ $b = \frac{\sqrt{3}}{2}$



Example: A saddle $z = x^2 - y^2$ has hyperbolic contours.

$z = 0:$ $x^2-y^2=0$ $x^2=y^2$ $y=\pm x$

$z = 1:$ $x^2-y^2=1$ $a=b=1$ $y=\pm\frac{b}{a}x=\pm x$ $c=(0,0)$

1st st form

$(-1,0)$ $(1,0)$ x-int

$z = 2:$ $x^2-y^2=2$ $\frac{x^2}{2}-\frac{y^2}{2}=1$ $a=b=\sqrt{2}$

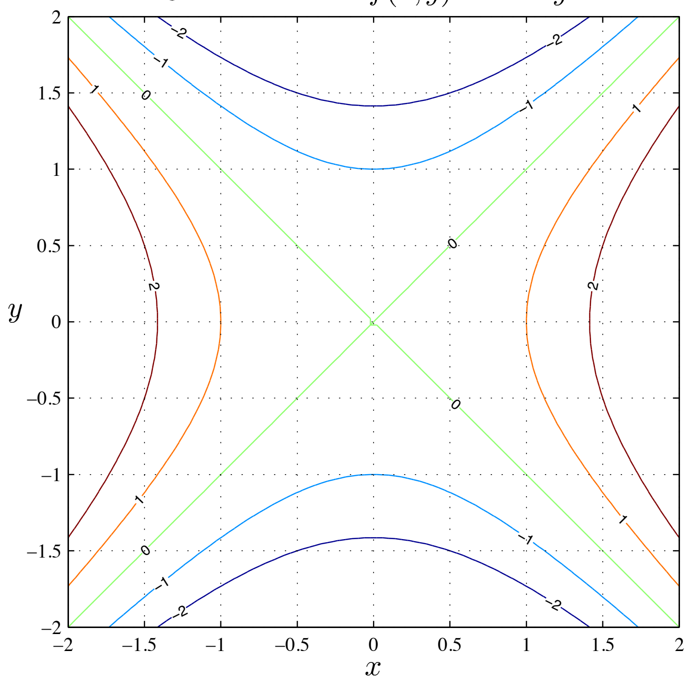
Asym. $y=\pm x$

$z = -1:$ $x^2-y^2=-1 \equiv y^2-x^2=1 \to$ 2nd st f

$a=b=1$ Asym. $y=\pm x$ $(0,1)(0,-1)$

$z = -2:$ $\frac{y^2}{2}-\frac{x^2}{2}=1$ $a=b=\sqrt{2}$

Contours of $z = f(x,y) = x^2 - y^2$



Example: Sketch the contour diagram for $z = f(x, y) = 9x^2 - 4y^2 + 2$ with contours at $z = -2$, 2, 6 and 10.

$z = 9x^2 - 4y^2 + 2$

$z = -2 \quad -2 = 9x^2 - 4y^2 + 2$

$-4 = 9x^2 - 4y^2$

$1 = -\frac{9}{4}x^2 + y^2$

$x = 0, \ y = \pm 1$

$z = 2 \quad 2 = 9x^2 - 4y^2 + 2$

$0 = 9x^2 - 4y^2$

$4y^2 = 9x^2 \qquad y = \pm\frac{3}{2}x$

$y^2 = \frac{9}{4}x^2$

$z = 6$

$6 = 9x^2 - 4y^2 + 2$

$4 = 9x^2 - 4y^2$

$1 = \frac{9}{4}x^2 - y^2$

$y = 0, \quad \frac{9}{4}x^2 = 1, \ x^2 = \frac{4}{9}$

$x = \pm\frac{2}{3}$

$z = 10 \quad 10 = 9x^2 - 4y^2 + 2$

$8 = 9x^2 - 4y^2$

$1 = \frac{9x^2}{8} - \frac{1}{2}y^2$

$y = 0, \ \frac{9}{8}x^2 = 1 \quad x^2 = \frac{8}{9}$

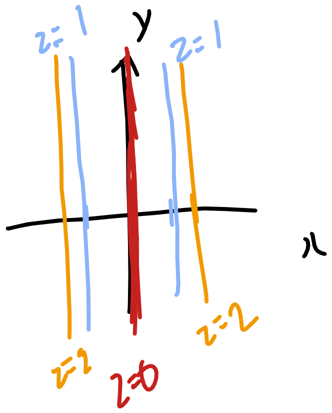
$x = \pm\frac{2\sqrt{2}}{3}$

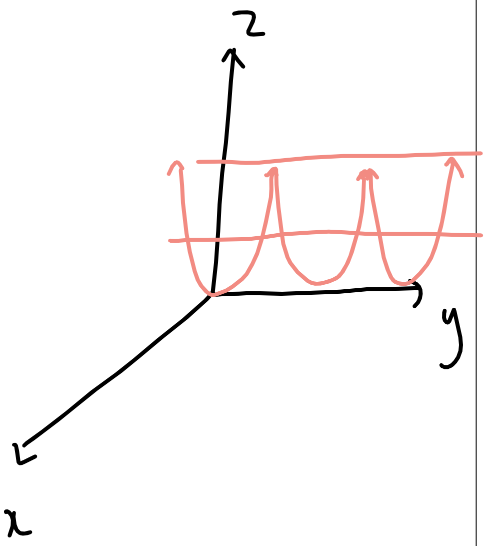
Example: Sketch a contour diagram for $z = f(x, y) = x^2$ and use this to sketch the graph of f.

case $z=0$. $f(x,y)=x^2=0 \Rightarrow x=0$

case $z=1$ $f(x,y)=x^2=1 \Rightarrow x=\pm 1$

case $z=2$ $f(x,y)=x^2=2 \Rightarrow x=\pm\sqrt{2}$





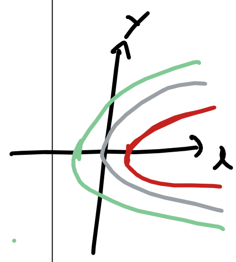
Example: Sketch a contour diagram for $z = f(x, y) = x - y^2$.

case $z=0$ $z=f(x,y)=x-y^2=0$ $y^2=x$ $y=\pm\sqrt{x}$ $(0,0)$

case $z=1$ $z=f(x,y)=x-y^2=1$ $x=y^2+1$ $(1,0)$

case $z=-1$ $-1=x-y^2$

$x=y^2-1$ $(-1,0)$

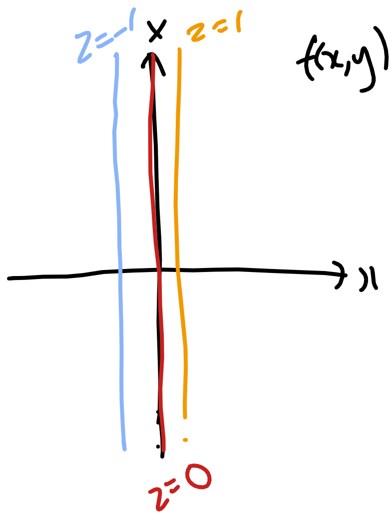


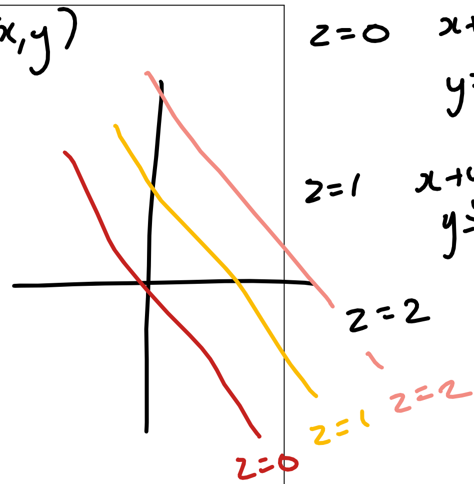
Example: Sketch contour diagrams of $z = f(x, y) = x$ and $z = g(x, y) = x + y$.

$z=0$, $x=0$

$z=1$, $x=1$

$z=-1$, $x=-1$





$z=0$ $x+y=0$

$y=-x$

$z=1$ $x+y=1$

$y=-x+1$

$z=2$ $y=-x+2$

Note: Contour diagrams of functions whose graphs are planes consist of equidistant parallel lines. (Equidistant lines in $\mathbb{R}^2$ are in fact always parallel, unlike $\mathbb{R}^3$. Why?)

3D - can have skew lines - don't intersect

keep two planes of the same distance

and are not parallel.

1.3.1 Main points

- ~~You should be able to plot contour diagrams in Matlab using the **ezcontour** function (and its variants).~~
- You should be able to recognise circular symmetry in an equation.
- You should be able to match contour diagrams with functions.
- You should be able to sketch simple contour diagrams.

1.4 Cross-sections of a Surface

look at XZ plane

vertical slices

A cross-section is the intersection of a surface with a vertical plane such as $y = C$, see also Stewart Section 12.6 (Section 12.6).

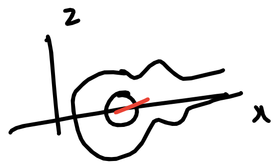
Example:

The height z of a vibrating guitar string can be expressed as a function of horizontal distance x, and time t

magnitude

$$z = f(x, t) = A \sin(\pi x) \cos(2\pi t) \qquad \text{where} \qquad 0 < x < 1.$$

space time

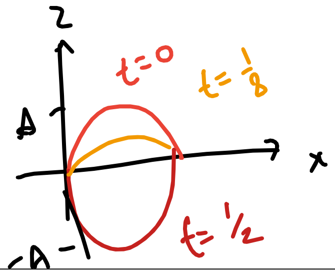


The snapshots where t is constant are cross-sections of the 'surface'.

$t=0$

$z = f(x,0) = A \sin(\pi x) \cos 0$

$= A \sin(\pi x)$

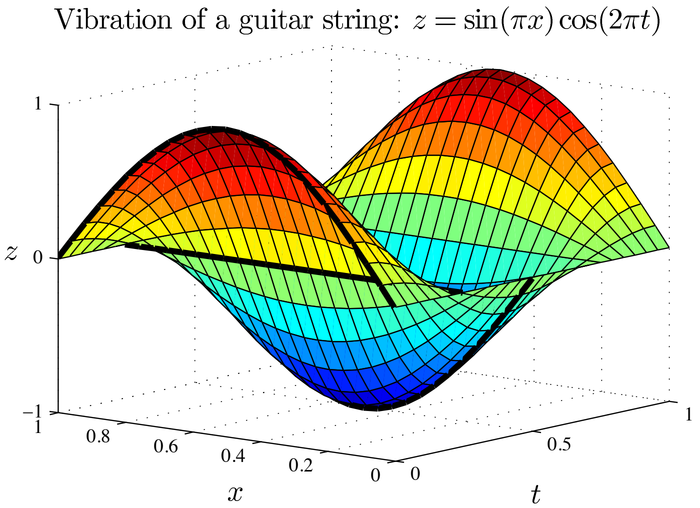


Varying time we get

$$\begin{aligned}
t &= 0: & z &= A \sin(\pi x) \\
t &= \frac{1}{8}: & z &= \frac{1}{2}\sqrt{2} A \sin(\pi x) \\
t &= \frac{1}{4}: & z &= 0 \\
t &= \frac{3}{8}: & z &= -\frac{1}{2}\sqrt{2} A \sin(\pi x) \\
t &= \frac{1}{2}: & z &= -A \sin(\pi x).
\end{aligned}$$

These represent sine curves, with amplitudes between 0 and A.

We can also consider the cross-sections in x. For instance $x = \frac{1}{2}$ (at the top of the sine wave), then $z = A \cos(2\pi t)$ which equals the amplitude of the sine wave.



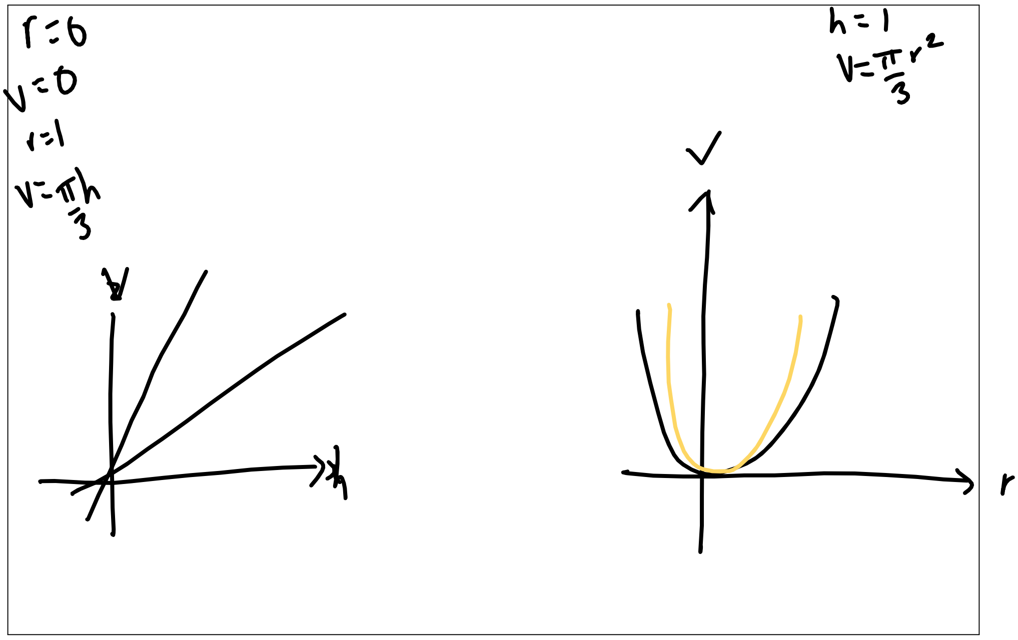
A=1

Matlab can be used to make a movie of the 2-dimensional surface by plotting cross-sections at different t values in sequence. The sequence of plots can be stored in a vector and played as a movie using the following code:

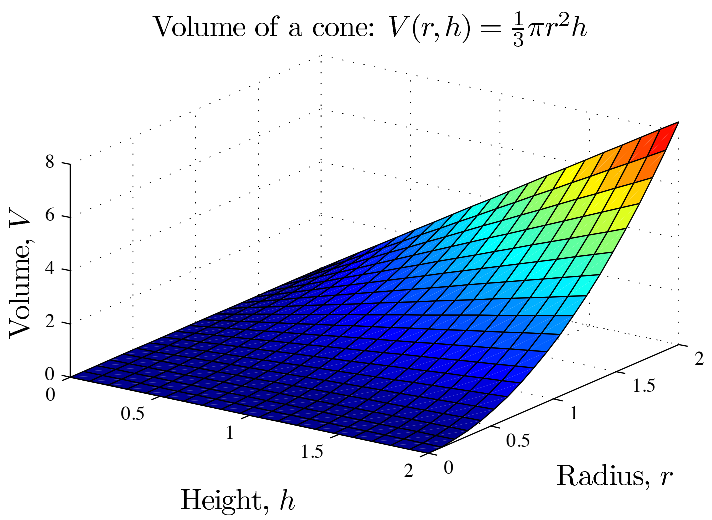
```
x=(0:0.25:1);
for j=1:100
   t=j/25;
   z=sin(pi*x)*cos(2*pi*t);
   plot(x,z);axis([0,1,-1,1]);
   M(j)=getframe;
end
```

Note: **ezplot** cannot be used to do this because Matlab gets confused about which of t, x is a variable and which is a number.

Example: Sketch the cross-sections of the surface $V = \frac{1}{3}\pi r^2 h$ (volume of a cone).

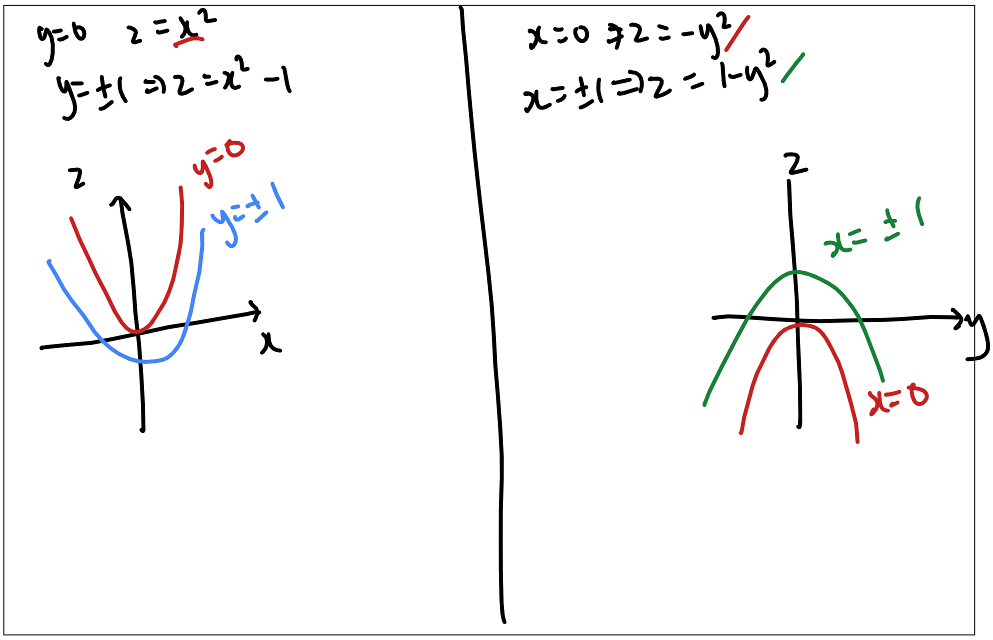


Here is the 3-dimensional picture from Matlab.



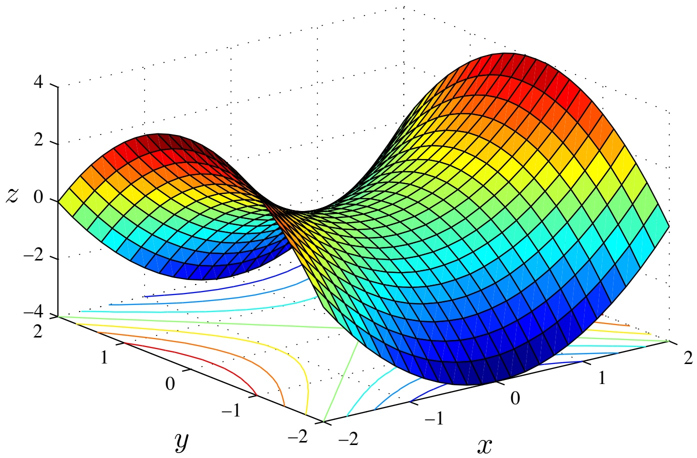
Example: The cross-sections of a saddle $z = x^2 - y^2$ are parabolas. For $y = y_0$ they point up: $z = x^2 - (y_0)^2$; and for $x = x_0$ they point down: $z = -y^2 + x_0^2$.

The surface is tricky to draw, unless you are an equestrian.



Here is the Matlab plot of the saddle $z = x^2 - y^2$ and its contours.

A saddle: $z = x^2 - y^2$



Example: Use cross-sections to sketch the graph of $z = f(x, y) = x^2$.

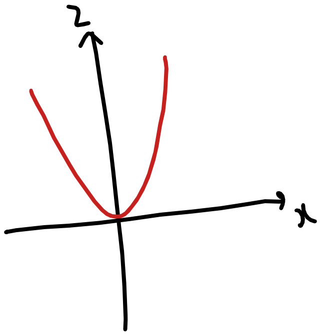
$z = f(x,y) = x^2$

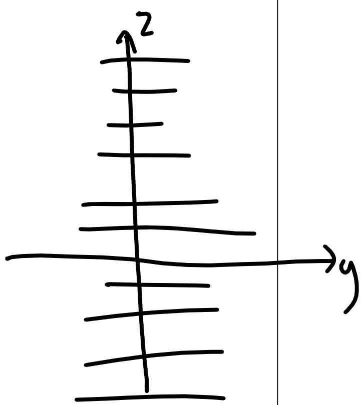
$y = 0, \; z = x^2$

$y = 1, \; z = x^2$

$x = 0, \; z = 0$

$x = 1, \; z = 1$





1.4.1 Main points

- You should be able to construct cross-sections of multivariate functions.
- Cross sections are 2-dimensional graphs.
- Animation of cross-sections is another way to visualise multivariate functions.

2 Partial Derivatives and Tangent Planes

We will need to consider derivatives of functions of more than one variable. To do this, we first check how the familiar concepts of limits and continuity extend to functions of more than one variable. This material is covered in Section 14.2 (Section 14.2) of Stewart.

2.1 Limits and Continuity

2.1.1 Review of the 1-variable case

Let $f : D \to \mathbb{R}$ be a function with domain D an open subset of $\mathbb{R}$. For $a \in D$ we say that the limit $\lim_{x\to a} f(x)$ exists if and only if, (i) the limit from the left exists, (ii) the limit from the right exists, and (iii) these two limits coincide, i.e.,

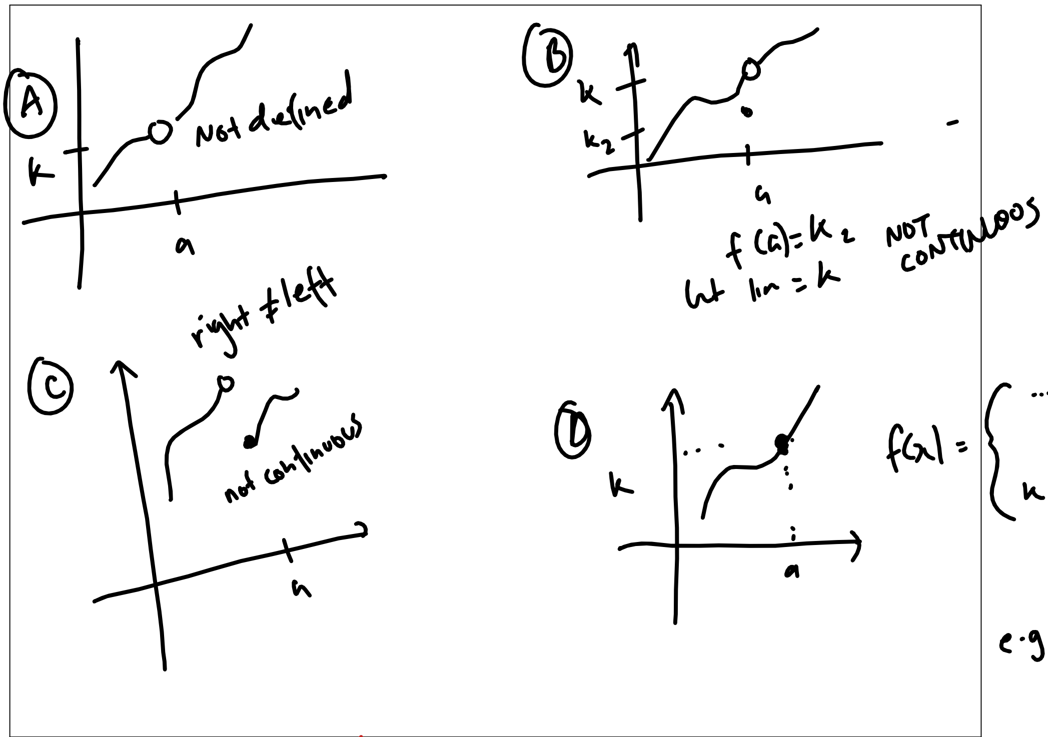
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

Furthermore, if the limit exists and is equal the actual value of f at a, i.e., if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a),$$

we say that f is continuous at $x = a$.

If f is continuous on all of D we say that f is a continuous function on D.

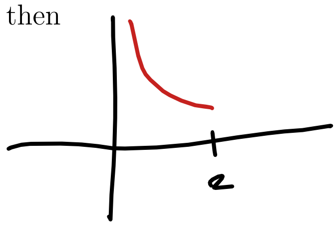


$$f(x) = \begin{cases} \ldots & x \neq a \\ k & x = a \end{cases}$$

e.g. $\dfrac{x-2}{x-2}$

	A	B	C	D
lim at x=a	✓	✓	✗	✓
cont. at x=a	✗	✗	✗	✓

We can also consider the limit for points on the boundary of the domain D of a function. For example, if $f : (0, 2) \to \mathbb{R}$ is defined by $f(x) = 1/x$, then



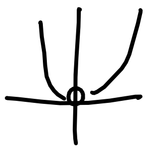
$$\lim_{x\to2^-} f(x) = \frac{1}{2}$$

but

$$\lim_{x\to0^+} f(x) \to \infty$$

does not exist.

Another instructive example is $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ given by $f(x) = x^2$. The domain is now the punctured real line, i.e., $D = (-\infty, 0) \cup (0, \infty)$, but



$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 0.$$

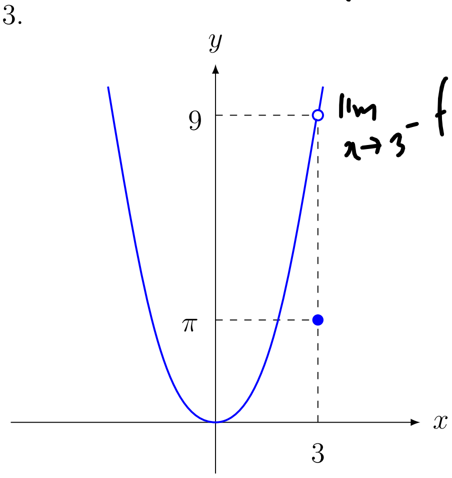
In this situation we also say that $\lim_{x\to0} f(x)$ exists and in fact one can *fix the hole* by defining $f(0) = 0$, to extend f to a continuous function on all of $\mathbb{R}$.

Important remark: Never, ever compute $\lim_{x\to a} f(x)$ by blindly substituting $x = a$ in f. For example, if

$$f(x) = \begin{cases} x^2 & \text{for } x \neq 3 \\ \pi & \text{for } x = 3, \end{cases}$$

$a=3$

then $\lim_{x\to3} f(x)$ exists and is given by 9 which is not equal to π: the function f is not continuous at $x = 3$.



$\lim_{x\to3^-} f(x) = 9 \neq \lim_{x\to3^+} f(x)$

As a second example, if $f : D \to \mathbb{R}$ with $D = \mathbb{R}\backslash\{1\}$ is given by

$$f(x) = \frac{x^2 - 1}{x - 1},$$

$\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$ $\lim_{x\to1}$ $x+1 = 2$

then $\lim_{x\to1} f(x) = 2$. Those (and there will be some) who write "This limit gives 0/0 which does not exist" should hang their heads in shame.

2.1.2 Multivariable limits

When f is a function of more than one variable, the situation is more interesting. There are more than two ways to approach a given point of interest. Consider the function

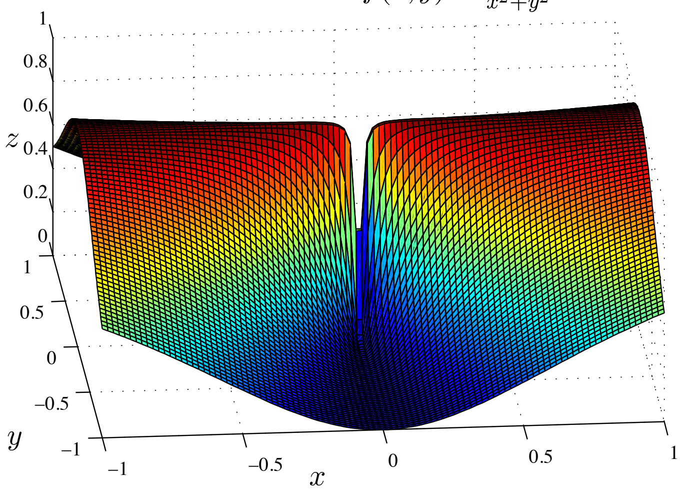
$$f(x,y) = \frac{x^2}{x^2+y^2}$$

with domain given by $\mathbb{R}^2 \backslash \{(0,0)\}$.

To see the graph of f in Matlab, type

```
ezsurf('(x^2/(x^2+y^2)')
```

The surface $z = f(x,y) = \frac{x^2}{x^2+y^2}$



Next we consider the limit as $(x,y) \to (0,0)$. $f(0,0)$

(i) Approaching the origin along $y = 0$:

$f(x,0) = \frac{x}{x^2+0} = 1 \quad \forall x \in \mathbb{R}$

$\lim_{(x,0)\to(0,0)} f(x,0) = \lim_{x\to 0} 1 = 1 = L_1$

(ii) Approaching the origin along $x = 0$:

$f(0,y) = \frac{0}{0^2+y} = 0 \quad \forall y \in \mathbb{R}$

$\lim_{(0,y)\to(0,0)} f(0,y) = \lim_{y\to 0} 0 = 0 = L_2$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Since $L_1 \neq L_2$ then $\lim_{(x,y)\to(0,0)} f(x,y) = DNE$

In general, for the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, it is necessary that every path in D approaching (a,b) (the point (a,b) itself may or may not be in D) gives the same limiting value. This gives us the following method for finding if a limit does not exist.

Test for showing no limit exists

$$\text{If } \begin{cases} f(x,y) \to L_1 & \text{as } (x,y) \to (a,b) \text{ along the path } C_1 \in D \\ f(x,y) \to L_2 & \text{as } (x,y) \to (a,b) \text{ along the path } C_2 \in D \end{cases}$$

such that $L_1 \neq L_2$, then the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Important remark: The above notation is somewhat deficient and perhaps one should write

$$\lim_{(x,y)\to_D(a,b)} f(x,y)$$

to indicate that only paths in D terminating in (a,b) (which itself may or may not be in D) are considered. For example, if $f(x,y) = x^2+y^2$ with $D = \{(x,y): x^2+y^2 < 1\}$ then $\lim_{(x,y)\to(1,0)} f(x,y)$ exists and is 1. However, if

$$f(x,y) = \begin{cases} x^2+y^2 & \text{for } D = \{(x,y): x^2+y^2 < 1\} \\ 0 & \text{for } D = \{(x,y): x^2+y^2 > 1\} \end{cases}$$

then $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist.

Example: Let $D = \mathbb{R}\backslash\{(0,0)\}$ and $f : D \to \mathbb{R}$ be given by $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$$\lim_{(x,y)\to(0,0)} x^2 - y^2 = 0 \qquad \lim_{(x,y)\to(0,0)} = 0$$

$x=0$

$$f(0,y) = \frac{-y^2}{y^2} = -1 = L_1 \quad \forall y$$

$y=0$ $\qquad L_1 \neq L_2$

$$f(x,0) = \frac{x^2}{x^2} = 1 = L_2$$

DNE

Example: With the same D as above but now $f(x,y) = \dfrac{xy}{x^2 + y^2}$, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$$x=0, \; f(0,y) = \frac{0}{y^2} = 0 = L_1$$

$$y=0 \quad f(x,0) = \frac{0}{x^2} = 0 = L_2$$

$$y=x, \; f(x,x) = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2} = L_3$$

Since $L_1 \neq L_3 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ DNE

Important remark: There are infinitely many paths terminating in a given point, say (a, b), in $\mathbb{R}^2$, raising the question if one can ever prove that $\lim_{(x,y)\to(a,b)} f(x, y)$ does exist. The good news is that there are methods that can deal with infinitely many paths simultaneously. The bad news is that these methods (typically ϵ-δ proofs) are not part of this course. See Stewart Sec 14.2 (Sec 14.2), Example 4 for a rigorous ϵ-δ proof that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0,$$

where

$$f(x, y) = \frac{3x^2y}{(x^2 + y^2)}$$

and $D = \mathbb{R}^2\backslash\{(0, 0)\}$.

Example: Give a non-rigorous proof that the above limit is indeed correct by writing $x = r\cos\theta$ and $y = r\sin\theta$.

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \frac{3(\cancel{r}\cos\theta)^2\, r\sin\theta}{\cancel{r^2}(\cos^2\theta + \sin^2\theta)}$$

$$= 3r\sin\theta\cos^2\theta, \ \forall\theta$$

$$-1 \le \sin\theta\cos^2\theta \le 1$$

$$-3r \le 3r\sin\theta\cos^2\theta \le 3r$$

$$0 = \lim_{r\to 0} -3r \le \lim_{r\to 0} f(r\cos\theta, r\sin\theta) \le \lim_{r\to 0} 3r = 0$$

By Squeeze theorem

$$\lim_{r\to 0} f(r\cos\theta, \sin\theta) = \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

2.1.3 Multivariable continuity

Definition Let $f : D \to \mathbb{R}$ be a function with domain D, an open subset of $\mathbb{R}^2$. Let $(a, b) \in D$. Then $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),$$

i.e., the limit $(x, y) \to (a, b)$ of $f(x, y)$ exists and is equal to $f(a, b)$.

If a function is continuous on all of D we say simply that it is continuous on D. Most of the functions we will consider are continuous. For example, polynomials in x and y are continuous on $\mathbb{R}^2$. As a rule of thumb, if a function with domain D is defined by a single expression it will be continuous on D.

Example: Returning to the first example on page 44, where $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ and $D = \mathbb{R}^2 \backslash \{(0, 0)\}$, is $f(x, y)$ a continuous function?

$\forall (x_0, y_0) \in D \quad (x_0, y_0) \neq (0,0)$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{x^2-y^2}{x^2+y^2} = \frac{\lim x^2-y^2}{\lim x^2+y^2} = \frac{x_0^2 - y_0^2}{x_0^2 + y_0^2}$$

f is cts at (x_0, y_0)

Example: If we edit the above example by instead defining f on all of $\mathbb{R}^2$ by taking $f(0, 0) = 0$, then is f a continuous function?

$$f(x,y) = \begin{cases} x^2-y^2/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$f(x,y)$ not cont. at $(x,y)=(0,0)$

See page 46

2.1.4 Main points

- You should be able to show when a limit does not exist.
- You should understand continuity of multivariate functions.

2.2 Partial Derivatives

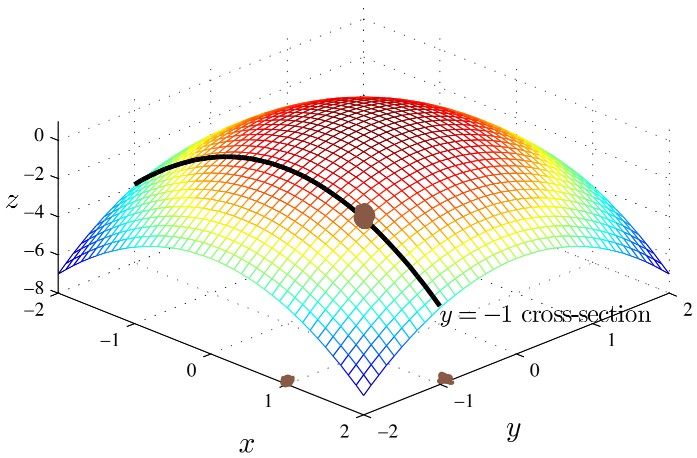
This material is covered in Stewart, Section 14.3 (Section 14.3).

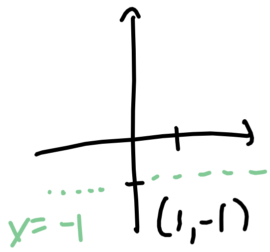
2.2.1 Slope in the x-direction

(x, y, z)

Consider the surface $z = f(x, y) = 1 - x^2 - y^2$ and the point $P = (1, -1, -1)$ on the surface. Use the "y-is-constant" cross-section through P to find the slope in the x-direction at P.

The surface $z = f(x,y) = 1 - x^2 - y^2$





$f(x, -1) = 1 - x^2 - (-1)^2 = -x^2 = f(x)$

$h'(x) = -2x \qquad h'(1) = 2 \qquad x = 1$

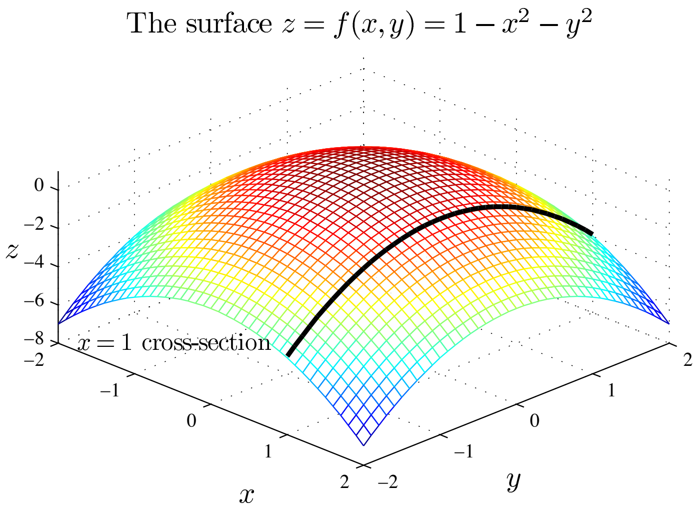
$-2 = \lim\limits_{g \to 0} \dfrac{h(1+g) - h(1)}{g} = \lim\limits_{g \to 0} \dfrac{f(1+h, -1) - f(1, -1)}{g} = \dfrac{\partial f}{\partial x}(1, -1) = f_x(1, -1)$

The slope in the x-direction, with y held fixed, is called the partial derivative of f with respect to x at the point (a, b)

$$\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

2.2.2 Slope in the y-direction

Use the "x-is-constant" cross-section to find the slope at $P = (1, -1, -1)$ in the y direction, i.e., where $x = 1$.



$f(1,y) = 1 - 1 - y^2 = -y^2 = h(y)$

$h'(y) = -2y, \; h'(-1) = -2(-1) = 2$

$$2 = h'(-1) = \lim_{g \to 0} \frac{h(-1+g) - h(-1)}{g}$$

$$= \lim_{g \to 0} \frac{f(1, -1+g) - f(1,-1)}{g}$$

Similarly, the slope in the y-direction, with x held fixed, is called the partial derivative of f with respect to y at the point (a, b)

$$\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

Important remark: Normal rules of differentiation apply, we simply think of the variables being held fixed as constants when doing the differentiation.

Example: Find the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ of $f(x, y) = x \sin y + y \cos x$.

$$\frac{\partial f}{\partial x} = \sin y - y \sin(x)$$

$$\frac{\partial f}{\partial y} = x \cos y + \cos x$$

Example: Given $f(x, y) = xy^3 + x^2$, find $f_x(1, 2)$ and $f_y(1, 2)$.

$$f_x(x,y) = y^3 + 2x$$

$$f_y(x,y) = 3xy^2 + 0$$

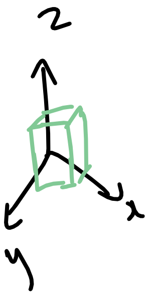
$$f_x(1,2) = 2^3 + 2 \cdot 1 = 16$$

$$f_y(1,2) = 3(1)(2)^2 = 12$$

2.2.3 Partial derivatives for $f(x, y, z)$

Example: The volume of a box $V(x, y, z) = xyz$.

If x changes by a small amount, say Δx, denote the corresponding change in V by ΔV. We can easily visualise that $\Delta V = yz\Delta x$.



$$\Delta V = V(x+\Delta x, y, z) - V(x, y, z)$$
$$= (\cancel{x}+\Delta x)yz + -\cancel{x}yz$$
$$= \Delta x(yz)$$

$$\frac{\Delta V}{\Delta x} = \frac{\Delta x \cdot yz}{\Delta x} = yz$$

$$\frac{\partial V}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \lim_{\Delta x \to 0} yz = yz$$
$$= \lim_{\Delta x \to 0} \frac{V(x+\Delta x, y, z) - V(x, y, z)}{\Delta x}$$

Therefore,

$$\frac{\Delta V}{\Delta x} = yz.$$

Letting $\Delta x \to 0$ we have $\dfrac{\partial V}{\partial x} = yz$.

For partial derivatives only one independent variable changes and all other independent variables remain fixed.

Example: Parallel resistance

In an electrical circuit, the combined resistance R, from three resistors R_1, R_2 and R_3 connected in parallel, is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \quad R(R_1, R_2, R_3)$$

What is the rate of change of the total resistance R with respect to R_1?

Set $C = \frac{1}{R_2} + \frac{1}{R_3}$

$$\frac{1}{R} = \frac{1}{R_1} + C$$

$$R = (R_1^{-1} + C)^{-1}$$

$$\frac{\partial R}{\partial R_1} = -1(R_1^{-1} + C)^{-2} \times (-1)R_1^{-2}$$

$$= \left[(R_1^{-1} + C) \cdot R_1\right]^{-2} = (1 + CR_1)^{-2}$$

$$= \left(1 + R_1\left(\frac{1}{R_2} + \frac{1}{R_3}\right)\right)^{-2}$$

$$= \left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right)^{-2}$$

$$\frac{\partial R}{\partial R_2} = \left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_3}\right)^{-2}$$

2.2.4 Higher order derivatives

The second order partial derivatives of f, if they exist, are written as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

CLAIRAUT'S THEOREM

If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Example: Returning to the example on page 49 for which $f(x, y) = x \sin y + y \cos x$, calculate all of the second order partial derivatives of f and show that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

$$f_x = \frac{\partial f}{\partial x} = \sin y - y \sin x$$

$$f_y = \frac{\partial f}{\partial y} = x \cos y + \cos x$$

$$f_{xx} = 0 - y \cos x$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(\sin y - y \sin x) = \cos y - \sin x$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x \cos y + \cos x) = \cos y - \sin x$$

2.2.5 Main points

- You should know the definition and meaning of partial derivatives.
- You should be able to evaluate partial derivatives of functions.

2.3 The Tangent Plane

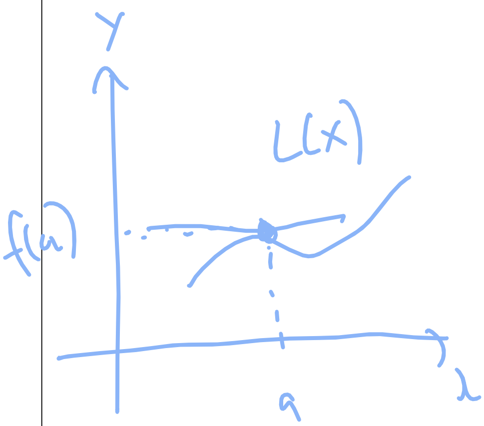
This section is covered in Stewart, Section 14.4 (Section 14.4).

2.3.1 Review for $f(x)$

Recall that if $y = f(x)$ then the tangent line at the point $(a, f(a))$ is given by

$$\underline{y - f(a) = f'(a)(x - a)} \quad \Leftrightarrow \quad y = f(a) + f'(a)(x - a). \Rightarrow L(x)$$

$y - y_0 = k(x - x_0)$



$(a, f(a))$

$y = mx + b$

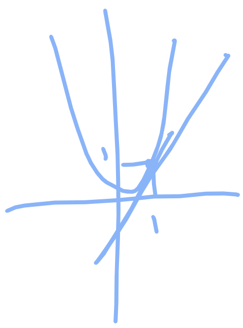
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ f(a) \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ f'(a) \end{pmatrix}$

$\forall x \in (a - \varepsilon, a + \varepsilon)$

$f(x) \approx L(x)$

Example: Find the tangent line to $y = f(x) = x^2$ at $x = 1$.

$y = f(a) + f'(a)(x - a)$



$y = 1 + 2\cancel{a}(x - 1)$

$y = 1 + 2x - 2$

$y = 2x - 1$

2.3.2 Equation for a tangent plane

In general, the equation of the tangent plane to a given surface $z = f(x, y)$ at $(a, b, f(a, b))$, is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

or, equivalently,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ f_x(a,b) \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ f_y(a,b) \end{pmatrix}$$

$$(x, y, z) = \big(a, b, f(a, b)\big) + \lambda\big(1, 0, f_x(a, b)\big) + \mu\big(0, 1, f_y(a, b)\big), \quad \lambda, \mu \in \mathbb{R}.$$

Indeed, the first two components of this vector equation for the tangent plane imply $\lambda = x - a$ and $\mu = y - b$. Substituting this into the third component gives

$$z = f(a, b) + \lambda f_x(a, b) + \mu f_y(a, b) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

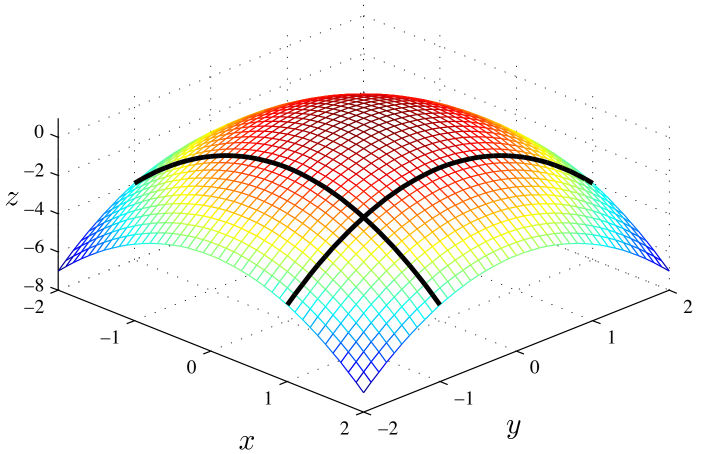
We also note that if we write $\Delta x = x - a$, $\Delta y = y - b$ and $\Delta z = z - f(a, b)$ then the vector equation for the tangent plane may be rewritten as

$$(\Delta x, \Delta y, \Delta z) = \lambda\big(1, 0, f_x(a, b)\big) + \mu\big(0, 1, f_y(a, b)\big), \quad \lambda, \mu \in \mathbb{R}.$$

This shows that if $\Delta x = 1$ and $\Delta y = 0$ (i.e., $\lambda = 1$ and $\mu = 0$) then $\Delta z = f_x(a, b)$ and if $\Delta x = 0$ and $\Delta y = 1$ (i.e., $\lambda = 0$ and $\mu = 1$) then $\Delta z = f_y(a, b)$, matching our interpretation of f_x and f_y as the respective slopes of f in the x or y direction.

Example: Find the equation for the tangent plane to the surface $z = 1 - x^2 - y^2$ at the point $P = (1, -1, -1)$.

The surface $z = 1 - x^2 - y^2$



$x \; y \; z$

$P(1, -1, -1)$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-a)$

$f(1,-1) = -1$

$f_x(x,y) = -2x, \; f_x(1,-1) = -2$

$f_y(x,y) = -2y, \; f_y(1,-1) = 2$

$z = -1 + (-2)(x-1) + 2(y-(-1))$

$= -1 - 2x + 2 + 2y + 2 = -2x + 2y + 3$

Example: What is the plane tangent to the surface $z = f(x, y) = 4 - x^2 + 4x - y^2$ at $(1, 1)$?

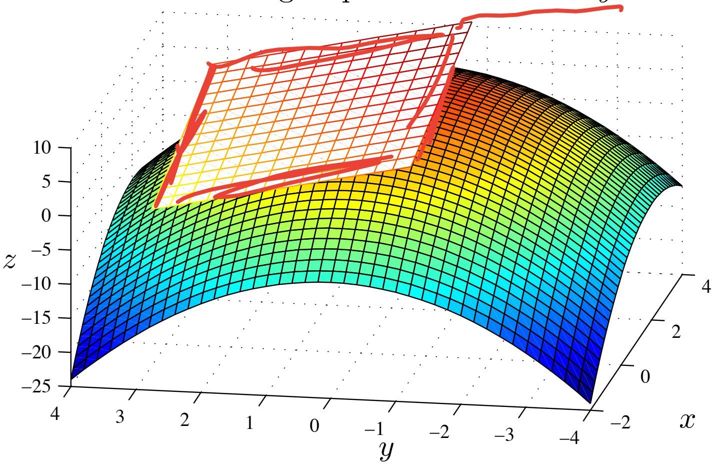
$f(1,1) = 6$ $\quad f_x(1,1)$ $\quad f_y(1,1)$

$f_x(x,y) = -2x+4$ $\quad f_x(1,1) = 2$

$f_y(x,y) = -2y$ $\quad f_y(1,1) = -2$

$z = 6+2x-2y$

The surface $z = 4 - x^2 + 4x - y^2$
and the tangent plane $z = 6 + 2x - 2y$



$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Example: Find the tangent plane of $z = f(x, y) = \mathrm{e}^{-x^2-y^2}$ at $(x, y) = \overset{a\ \ b}{(1, 3)}$.

$f(1,3) = e^{-1-9} = e^{-10}$

$f_x(x,y) = e^{-x^2-y^2} \cdot (-2x) \qquad f_x(1,3) = -2e^{-16}$

$f_y(x,y) = e^{-x^2-y^2} \cdot (2y) \qquad f_y(1,3) = 6e^{-16}$

$z = e^{-16} - 2e^{-10}(x-1) - 6e^{-10}(y-3)$

$z = e^{-16}(-2x - 6y + 21)$

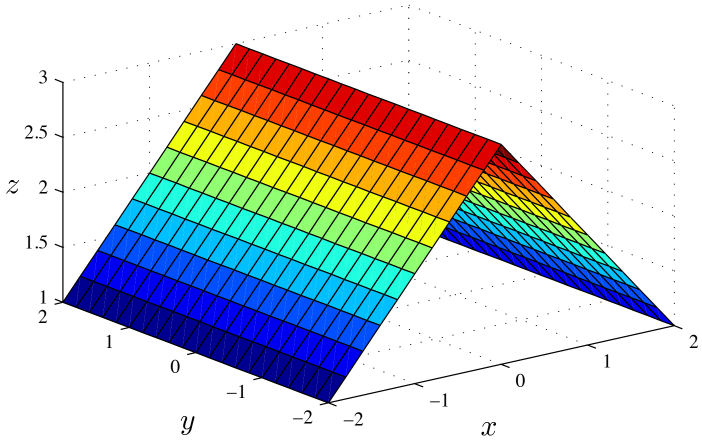
2.3.3 Smoothness

|x| is continuous
not differentiable at x=0

Can we always find partial derivatives and tangent planes?

Example: Simple cusp-like functions are not smooth:

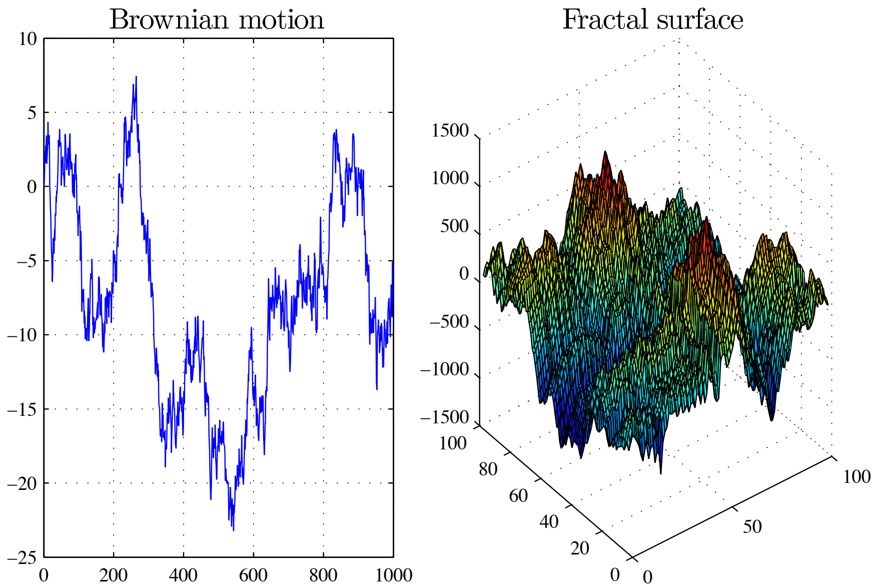
Cusp-like surface: $z = 3 - |x|$
$f_x(x = 0)$ is undefined



A surface $z = f(x, y)$ is smooth at (a, b) if f, f_x and f_y are all continuous at (a, b). When you zoom in close enough to a smooth surface it looks like a plane. walking on Earth

One way to see this is to look at the contours. The contours of a plane are straight parallel lines, the same perpendicular distance apart. As you zoom into a smooth surface the contours straighten out. This means that close to (a, b) the surface is approximated by a plane; in fact it can be approximated by the tangent plane. Do straight contours imply smoothness?

Example: Brownian motion is not smooth. Look at the figure below. No matter how much you zoom in, it always looks rough—in fact, Brownian motion is a fractal. There are surface-analogues to Brownian motion, demonstrated with the fractal surface below.



2.3.4 Main points

- You should know how to find a tangent plane to a smooth surface, and recognise when a tangent plane or partial derivatives do not exist.

2.4 Linear Approximations

See Stewart, Section 14.4 (Section 14.4).

Approximate f(x) using a straight line

2.4.1 Review for $f(x)$

In science and engineering practice, a function $f(x)$, of one variable, is sometimes approximated by a straight line.

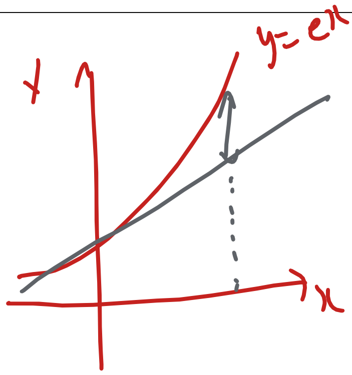
In particular, if we know an exact value $(a, f(a))$, then the tangent line through this point is usually an accurate approximation close to the point. It is called a linear or first-order approximation

$(a, f(a)) \to (x, f(x))$

$$f(x) \simeq f(a) + f'(a)(x - a).$$

RHS: Tangent line at $x = a$

Example: Estimate $e^{0.1}$.



$x_0 = 0$

$x = 0.1$

$L(x) = f(0) + f'(0)(x - 0)$

$f'(x) = e^x \quad f'(0) = 1 \quad f(0) = 1$

$L(x) = 1 + x \quad \forall x \in (0 - \varepsilon, 0 + \varepsilon) \quad f_x \approx L(x)$

$f(0.1) = e^{0.1} \approx L(0.1) = 1 + 0.1$

$= 1.1$

Actual value of $e^{0.1}$ is $1.10517\ldots$

TAYLOR SERIES:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

2.4.2 Linear approximations for $f(x, y)$

The corresponding linear or first-order approximation for a function f of two variables, near a known point (a, b) is the tangent plane. The linear approximation (i.e., tangent plane) is accurate close to the known point, provided f is smooth.

The linear approximation to f at (a, b) is

$$f(x, y) \simeq f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

RHS: Tangent Plane at (a,b)

Example: The temperature in a region at is given by $T(x, y) = 100 - x^2 - y^2$. Find the linear approximation to $T(x, y)$ near $(0, 5)$.

$T(0,5) = 100 - 0 - 25 = 75$

$T_x(x,y) = -2x \qquad T_x(0,5) = 0$

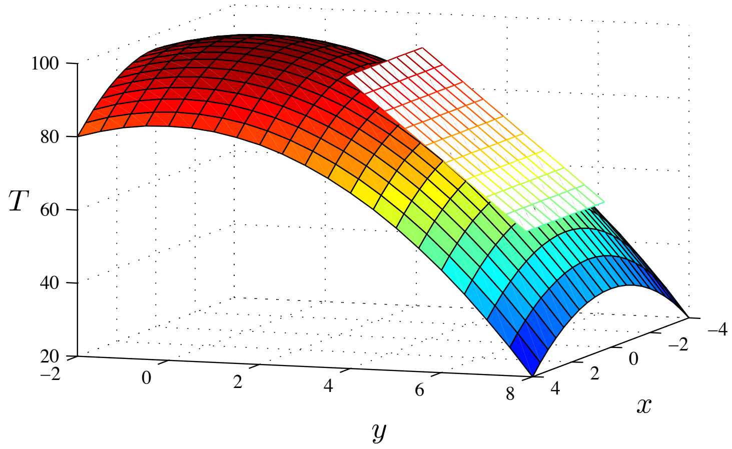
$T_y(x,y) = -2y \qquad T_y(0,5) = -10$

$L(x,y) = T(0,5) + T_x(0,5)(x-0) + T_y(0,5)(y-5)$

$= 75 - 10(y-5) \qquad 125 - 10y$

$= 75 - 10y + 50 \qquad T(x,y) \approx 125 - 10y \quad \forall (x,y) \in B_\delta(0,5)$

The surface $T(x, y) = 100 - x^2 - y^2$
and the tangent plane $T = 125 - 10y$



Example: Find the tangent plane to $z = \mathrm{e}^{-x^2}\sin y$ at $(1, \frac{\pi}{2})$ and use it to find an approximate value for $\mathrm{e}^{-(0.9)^2}\sin(1.5)$.

$f(x,y) = e^{-x^2}\sin y$ $(0.9, 1.5) \leftarrow (1, \frac{\pi}{2})$

$f(1, \frac{\pi}{2}) = e^{-1}$

$f_x(x,y) = e^{-x^2}\sin y\,(-2x)$ $f_x(1, \frac{\pi}{2}) = -2e^{-1}$

$f_y(x,y) = e^{-x^2}\cos y$ $f_y(1, \frac{\pi}{2}) = 0$

$L(x,y) = f(1, \frac{\pi}{2}) + f_x(1, \frac{\pi}{2})(x-1) + f_y(1, \frac{\pi}{2})(y - \frac{\pi}{2})$

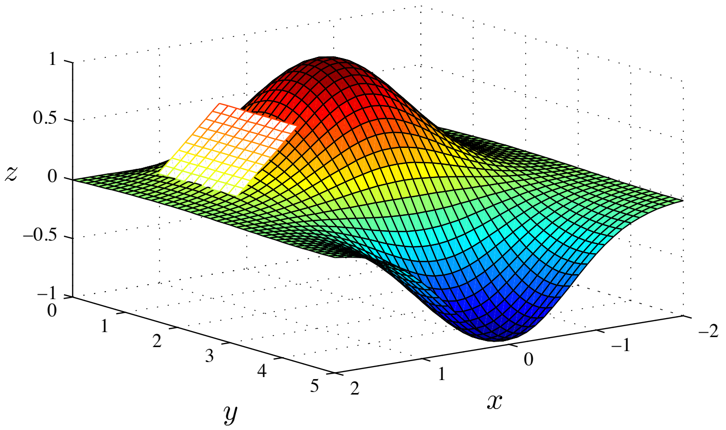
$= e^{-1} - 2e^{-1}(x-1) = e^{-1} - 2xe^{-1} + 2e^{-1}$

$= \frac{1}{e}(1 - 2x + 2)$ $f(0.9, 1.5)$

$\approx \frac{1}{e}(-2x+3)$ $\approx e^{-1}(-1.8+3)$

≈ 0.4

The surface $z = \mathrm{e}^{-x^2}\sin(y)$
and the tangent plane $z = \frac{3-2x}{\mathrm{e}}$



2.4.3 Estimating small changes

We may use the equation for the tangent plane to infer that

$$\underbrace{\Delta z}_{f(x,y)} \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y,$$

where $\Delta x = x - a$, $\Delta y = y - b$ and $\Delta z = z - f(a,b)$ represent a small change in x, y and z respectively. This is useful for estimating small changes in z arising from small changes in x and y.

Example: Electric power is given by $P(E, R) = E^2/R$ where E is the voltage and R is the resistance. Find a linear approximation for $P(E,R)$ if $E \simeq 200$ (in Volts) and $R \simeq 400$ (in Ohms). Use this to find the effect that a change in E and R has on P.

$$P(E,R) \approx L(E,R) = P(200,400) + P_E(200,400)(E-200) + P_R(200,400)(R-400)$$

$$\Delta P = P(E,R) - P(200,400) \approx P_E(200,400)(E-200) + P_R(20\quad 00)(R-400)$$

$$P_E = \frac{2E}{R} \qquad P_R = -\frac{E^2}{R^2}$$

$$P_E(200,400) = \frac{2 \times 200}{400} = 1$$

$$P_r = -(200)^2/400^2 = -1/4$$

$$\Delta P \approx (E-200) - \frac{1}{4}(R-400)$$

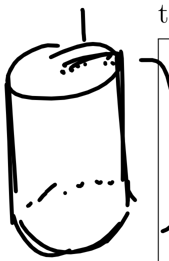
2.4.4 Estimating error

Δx $x-a$ Δy $y-b$

If the error in x is at most E_1 and in y is at most E_2, then a reasonable estimate of the worst-case error in the linear approximation of f at (a, b) is

$$|E| \approx |f_x(a,\, b)E_1| + |f_y(a,\, b)E_2|.$$

Example: Suppose when making up a metal barrel of base radius 1 (in metres) and height 2 (in metres), you allow for an error of 5% in radius and height. Estimate the worst-case error in volume.



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$V(r,h) = \pi r^2 h$ Δx $\quad V(1,2) = 2\pi$

$$|E| \approx |V_r(1,2)E_1| + |V_h(1,2)E_2|$$

$V_r(r,h) = 2r\pi h \quad V_r(1,2) = 4\pi$

$V_h(r,h) = \pi r^2 \quad V_h(1,2) = \pi$

$E_1 = 0.05 \cdot 1 = 0.05$ Δr $\quad E_2 = 0.05 \times 2 = 0.1$ Δh

$$|E| \approx |4\pi (0.05)| + |\pi (0.1)|$$

$$\approx 0.3\pi$$

2.4.5 Extra reading: Differentials

If we reduce our small changes Δx and Δy down to infinitesimal changes $\mathrm{d}x$ and $\mathrm{d}y$, we may rewrite the linear approximation (tangent plane) as

$$\mathrm{d}z = \mathrm{d}f = f_x(a,b)\,\mathrm{d}x + f_y(a,b)\,\mathrm{d}y.$$

This infinitesimal change $\mathrm{d}z$ is called the total differential. Note that this equation is not just an approximation.

Example: Work done by a gas

Let the work done by an expanding gas be given by $W = PV$, where P denotes pressure and V volume.

An expression for the total differential $\mathrm{d}W$ is then

$$\begin{aligned}\mathrm{d}W &= \frac{\partial W}{\partial P}\,\mathrm{d}P + \frac{\partial W}{\partial V}\,\mathrm{d}V \\ &= V\,\mathrm{d}P + P\mathrm{d}V.\end{aligned}$$

2.4.6 Main points

- You should be able to calculate finite small deviations of a function given a small change in one of the variables. Note that this is a computationally more efficient method of approximation than simply evaluating the function at x and $x + \Delta x$, and then taking the difference.
- You should be able to estimate error using the linear approximation.

2.5 Gradients and Directional Derivatives

This material is covered in Stewart, Section 14.6 (Section 14.6).

2.5.1 Directional derivatives

The partial derivative f_x (or f_y) corresponds to the slope of $f(x, y)$ in the x-direction (or y-direction). We now turn our attention to the question of slopes in arbitrary directions, such as $\boldsymbol{i} + 2\boldsymbol{j}$ or $-\boldsymbol{j}$.

Let $\boldsymbol{u} = (u_1, u_2)$ be an arbitrary unit vector in $\mathbb{R}^2$, i.e., $\|\boldsymbol{u}\| = 1$. We have already seen that the equation of the tangent plane at (a, b) of a function f may be written as

$$(\Delta x, \Delta y, \Delta z) = \lambda\big(1, 0, f_x(a, b)\big) + \mu\big(0, 1, f_y(a, b)\big), \quad \lambda, \mu \in \mathbb{R},$$

where $\Delta x = x - a$, $\Delta y = y - b$ and $\Delta z = z - f(a, b)$. This allows us to easily describe the slope of f at the point (a, b) in the direction of $\boldsymbol{u}$: we must "measure" Δz when $\Delta x = u_1$ and $\Delta y = u_2$. Hence we take $\lambda = u_1$ and $\mu = u_2$ in order to find that

$$\begin{aligned} \Delta z &= u_1 f_x(a, b) + u_2 f_y(a, b) \\ &= \big(f_x(a, b), f_y(a, b)\big) \cdot \boldsymbol{u}. \end{aligned}$$

This is usually denoted as $f_{\boldsymbol{u}}(a, b)$ and known as the slope of f at the point (a, b) in the direction of $\boldsymbol{u}$, or the directional derivative of f at (a, b) in the direction of $\boldsymbol{u}$.

Because it is not always convenient to work with unit vectors, we more generally have that the directional derivative of f at (a, b) in the direction of an arbitrary nonzero vector $\boldsymbol{u}$ is given by

$$f_{\boldsymbol{u}}(a, b) = \big(f_x(a, b), f_y(a, b)\big) \cdot \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}.$$

Example: Find the directional derivative of $f(x, y) = 4 - x^2 - 4y^2$ at (a, b) in the $(1, 1)$ direction.

$f_x(x,y) = -2x \quad f_x(a,b) = -2a \quad \vec{v} = (1,1)$

$|\vec{v}| = \sqrt{2}$

$f_y(x,y) = -8y \quad f_y(a,b) = -8b$

$\begin{pmatrix} -2a \\ -8b \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{-2a - 8b}{\sqrt{2}}$

$\vec{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Example: If $f(x, y) = x^2 - 3y^2 + 6y$, find the slope at $(1, 0)$ in the direction $\boldsymbol{i} - 4\boldsymbol{j}$.

$f_x(x,y) = 2x \quad f_x(1,0) = 2$

$f_x(x,y) = -6y + 6 \quad f_y(1,0) = 6$

$||\vec{v}|| = \sqrt{17}$

$(2,6) \cdot (1,-4) = 2 - 24 = \frac{-22}{\sqrt{17}}$

2.5.2 The gradient vector ∇f

The gradient vector or simply gradient of f is a vector with the partial derivatives as components

$$\operatorname{grad} f = \nabla f = (f_x,\, f_y) = f_x \boldsymbol{i} + f_y \boldsymbol{j}.$$

Example: Find the gradient of $f(x, y) = x^2 - 3(y-1)^2 + 3$.

$f_x(x,y) = 2x$

$f_y(x,y) = -6(y-1)$

$\nabla f(x,y) = (-2x, -6y+6)$

Similarly, for a function $f(x, y, z, w)$ of four variables,

$$\operatorname{grad} f = \nabla f = (f_x, f_y, f_z, f_w).$$

Note that directional derivative can be conveniently expressed in terms of the gradient as

$$f_{\boldsymbol{u}} = \nabla f \cdot \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}.$$

Example: Find $f_{(1,-1)}(0, 1)$ for $f(x, y) = x - x^2y^2 + y$.

$f_x(x,y) = (1 - 2xy^2)\big|_{(x,y)=(0,1)} = 1$

$f_y(x,y) = (2yx^2+1)\big|_{(x,y)=(0,1)} = 1$

$\|\vec{u}\| = \sqrt{2}$

$f_{(1,-1)}(0,1) = \nabla f(0,1) = (1,1) \cdot \frac{(1,-1)}{\sqrt{2}}$

$= \frac{1-1}{\sqrt{2}} = 0.$

Example: Find the directional derivative of $g(x, y) = \mathrm{e}^{x^2} \cos y$ at $(1, \pi)$ in the direction $-3\boldsymbol{i} + 4\boldsymbol{j}$.

$g_x(x,y) = 2x e^{x^2} \cos y \qquad g_x(1,\pi) = -2e$

$g_y(x,y) = -e^{x^2} \sin y \qquad g_y(1,\pi) = 0$

$g_{(-3,4)}(1,\pi) = (-2e, 0) \cdot \dfrac{(-3,4)}{5}$

$= \dfrac{6e}{5}$

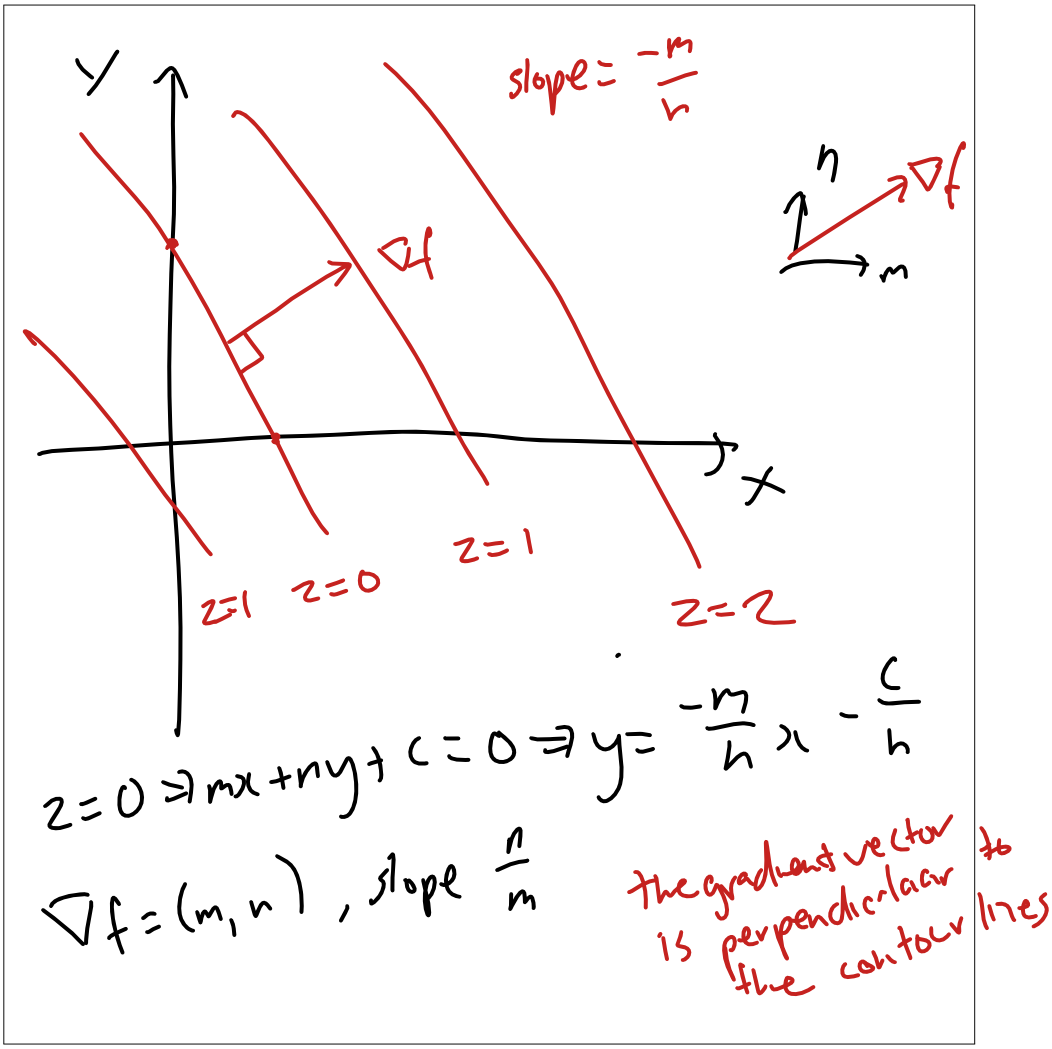
$\vec{v} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\|\vec{v}\| = \sqrt{9+16}$

$= \sqrt{25} = 5$

2.5.3 Properties of the gradient vector

Example: Consider the contour diagram of a plane $z = f(x, y) = mx + ny + c$. For $n \neq 0$ the contours have slope $-m/n$ in the xy-plane.

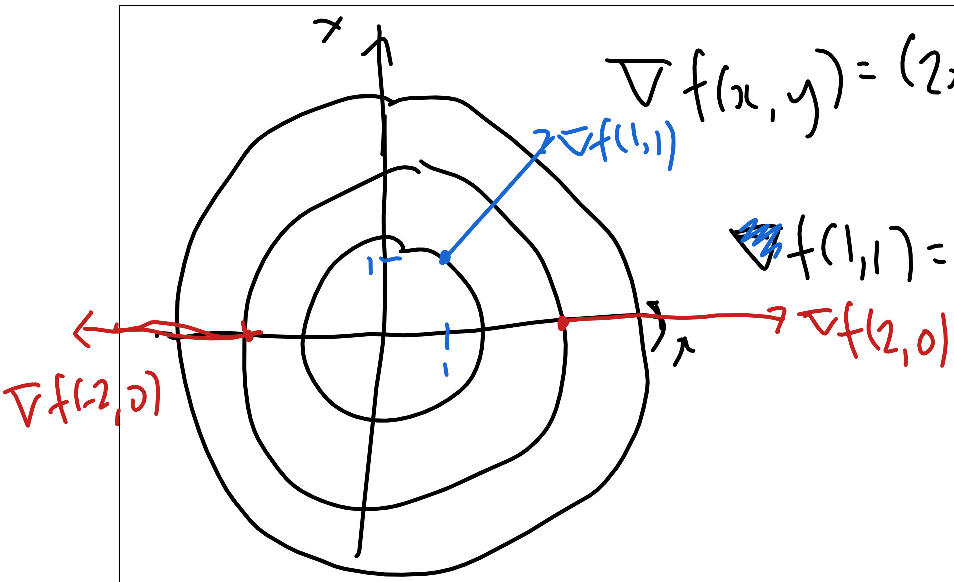


The gradient vector, $m\boldsymbol{i} + n\boldsymbol{j}$ ($m \neq 0$), has slope n/m and so is perpendicular to the contours. It points in the direction of increasing f. In fact, the direction in which it points is the direction of greatest slope.

Example: Consider $f(x, y) = x^2 + y^2$. The contours are circles centered at the origin, and $\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j}$ points radially out. Once again, ∇f points in the direction of greatest slope, perpendicular to the contour lines.

$\nabla f(x,y) = (2x, 2y)$

$\nabla f(1,1) = (2,2)$



$\nabla f(2,0) = (4,0)$

$\nabla f(-2,0) = (-4,0)$

In general, two important properties of the gradient of a function are:

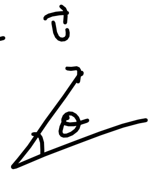
> The gradient $\nabla f(a, b)$ is **perpendicular to the contour line through** (a, b) and points in the direction of increasing f. In fact, the direction and magnitude of **steepest slope** at (a, b) are given by $\nabla f(a, b)$ and $\|\nabla f(a, b)\|$.

We can understand these two facts by considering the value of $\cos\theta$ in

$$f_{\boldsymbol{u}} = \nabla f \cdot \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \|\nabla f\| \cos\theta.$$

What's direction of steepest slope? i.e.
what's $\vec{v}$ s.t $|f_{\vec{v}}|$ is largest?

$$f_{\vec{v}} = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|} = \|\nabla f\| \frac{\|\vec{v}\|}{\|\vec{v}\|} \cos\theta$$



$$|f_{\vec{v}}| = \|\nabla f\| |\cos\theta|$$

greatest when $\cos\theta = 1$ θ :(

ie direction of $\vec{v}$ same as ∇f

In this case $|f_{\vec{v}}| = \|\nabla f\| \to f_{\vec{v}} = \|\nabla f\|$

↑
largest steepest slope.

Example: $T(x, y) = 20 - 4x^2 - y^2$ describes the temperature on the surface of a metal plate.

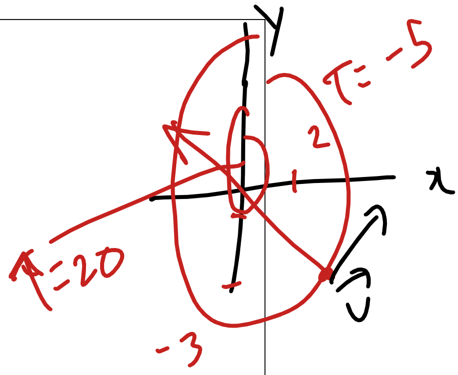
In which direction away from the point $(2, -3)$ does the temperature increase most rapidly?

In which directions away from the point $(2, -3)$ is the temperature not changing?

• Most rapid increase in the dir ∇T

$\nabla T(x,y) = (-8x, -2y)$

$\nabla T(2,-3) = (-16, 6)$



• Temperature in perpendicular direction to ∇T (contours of T)

e.g. $\vec{v}$ $(6,16)$

Indeed $\vec{v} \cdot \nabla T(2,3) = (6,16)\cdot(-16,6)$
$= 6(-16) + 16(6) = 0$

Example: A team from a large British oil company is mapping the ocean floor to assist in the plugging of a leaking oil well in the Gulf of Mexico. Using sonar, they develop the model

$$D(x, y) = 1700 - 30x^2 - 50\sin\left(\frac{\pi y}{2}\right),$$

where x and y are distance in kilometres, D is depth in metres, and $-2 \le x \le 2$ and $-2 \le y \le 2$.

(a) The well is located at $(1, 0.5)$. What is its depth?

$D(1,0.5) = 1700 - 30 - 50\sin\left(\frac{\pi(0.5)}{2}\right)$ $\frac{\pi}{4}$

$= 1670 - 50\frac{\sqrt{2}}{2} = 1676 - 25\sqrt{2} \approx 1635$

(b) Determine the slope of the ocean floor in the positive x-direction and in the positive y-direction in the area considered by the clean-up team.

$D_x(x,y) = -60x$

$D_y(x,y) = -50\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}y\right) = -25\pi\cos\left(\frac{\pi}{2}y\right)$

(c) Determine the magnitude and direction of greatest rate of change of depth at the position of the well.

$\nabla D(x,y) = \left(-60x, -25\pi\cos\left(\frac{\pi}{2}y\right)\right)$

DIRECTION greatest ROC of depth $\hookrightarrow \nabla D(1,0.5) = \left(-60, -25\pi\frac{\sqrt{2}}{2}\right)$

MAGNITUDE $\|\nabla D(1,0.5)\| = \sqrt{60^2 + \left(\frac{25\pi\sqrt{2}}{2}\right)^2}$

≈ 71.7

2.5.4 Main points

- You should be able to find the slope of a function in a given direction.
- You should be able to find the direction and rate of maximal increase of a function.
- You should understand directional derivative and gradient.

2.6 The Chain Rule and Applications

The chain rule is covered in Stewart, Section 14.5 (Section 14.5).

2.6.1 Review for $f(x)$

For related functions of one variable, such as $y = f(u)$ and $u = g(x)$, we may use the chain rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{or} \qquad f'(x) = f'(u)\,g'(x).$$

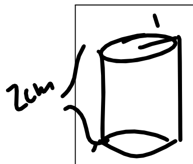
Example: Find $y'(x)$ where $y = f(x) = (x^2 + 1)^5$.

$u(x) = x^2 + 1$ $\qquad f(u) = u^5$

$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx} = 5u^4(2x)$$
$$= 10x(x^2+1)^4$$

Example: Suppose the radius of a cylinder decreases at a rate of $r'(t) = -2$ cm/s. How fast is the volume decreasing when $r = 1$ cm and $h = 2$ cm?

Fixed height



$V = 2\pi r^2$

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt}$$
$$= (4\pi r)(-2)$$
$$\left.\frac{dV}{dt}\right|_{\substack{r=1 \\ r'=-2}} = (4\pi \cdot 1)(-2)$$
$$= -8\pi$$

2.6.2 The chain rule for $f(x, y)$

Given $f(x, y)$ with x and y functions of t. Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t),\, y(t+\Delta t)) - f(x(t),\, y(t))}{\Delta t}.$$

Now if f is smooth and Δf is small, we can relate it to Δx and Δy through the linear approximation

$$\Delta f \simeq f_x \Delta x + f_y \Delta y.$$ ← from tangent plane approximation

Hence

$$\frac{\Delta f}{\Delta t} \simeq f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t}.$$

Now we let $\Delta t \to 0$, and provided $x(t)$ and $y(t)$ are smooth,

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = f_x \frac{\mathrm{d}x}{\mathrm{d}t} + f_y \frac{\mathrm{d}y}{\mathrm{d}t}}$$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

which is the chain rule for $f(x(t), y(t))$.

Example: Continuing the previous example, suppose that not just the radius but also the height h is decreasing: $\frac{\mathrm{d}h}{\mathrm{d}t} = -1\,\mathrm{cm/s}$. What is the rate of change in volume?

$V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} + \frac{dV}{dh}\frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{\substack{h=2\\ r=1}} = 4\pi(-2) + \pi(-1) = -9\pi \ \mathrm{cm}^3/\mathrm{s}$$

The chain rule can be extended to any number of dimensions.

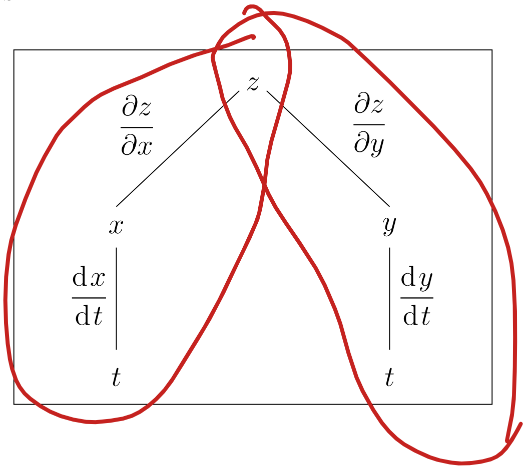
Example: If $V(a(t)),\, b(t),\, c(t)) = abc$ is the volume of a box then find $\frac{\mathrm{d}V}{\mathrm{d}t}$.

$$\frac{dV}{dt} = \frac{dV}{da}\frac{da}{dt} + \frac{dV}{db}\frac{db}{dt} + \frac{dV}{dc}\frac{dc}{dt}$$

$$= bc\,a'(t) + ac\,b'(t) + ab\,c'(t)$$

2.6.3 Larger Chain Rules

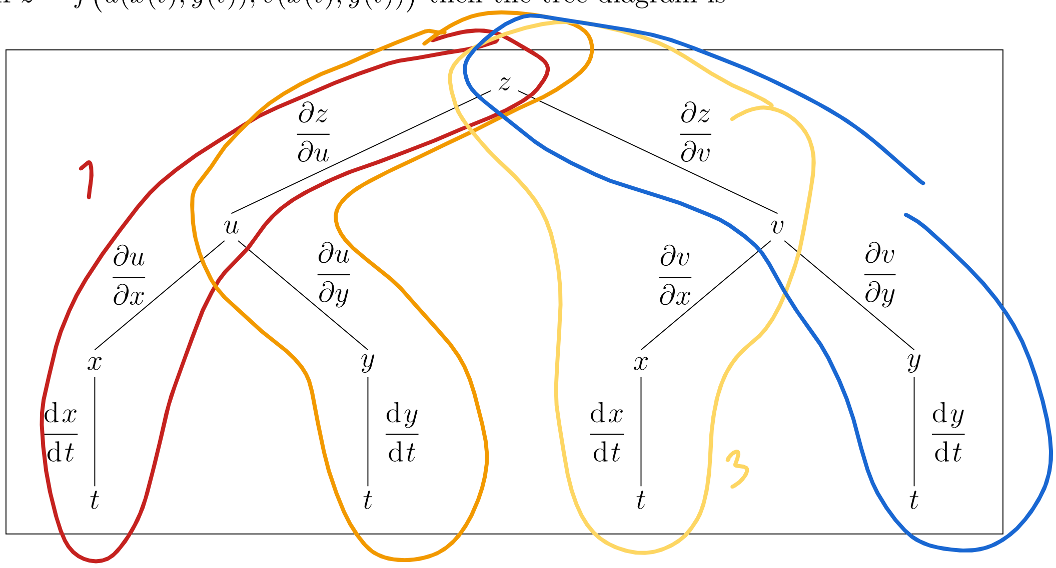
To keep check of what depends on what, you can draw a tree diagram for $z = f(x(t), y(t))$ as follows:



Hence

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

If $z = f\big(u(x(t), y(t)), v(x(t), y(t))\big)$ then the tree diagram is



Hence

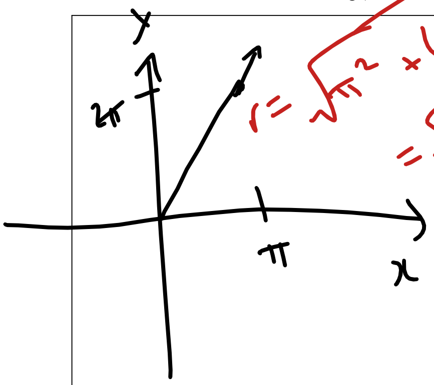
$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right)\\ &= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\frac{dy}{dt}.\end{aligned}$$

1 3 2 4

Example: Find the slope of $z = f(x, y) = \sin x \cos y$ in the radial direction at $(\pi, 2\pi)$. That is, find $\dfrac{\partial z}{\partial r}$.

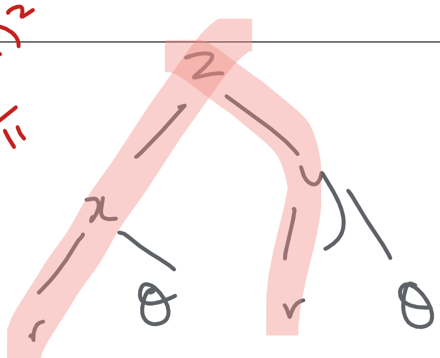
$x = r\cos\theta$

$y = r\sin\theta$



$r = \sqrt{\pi^2 + (2\pi)^2}$

$= \sqrt{5}\,\pi$



$\cos\theta = \dfrac{\pi}{r}$

$= \dfrac{1}{\sqrt{5}}$

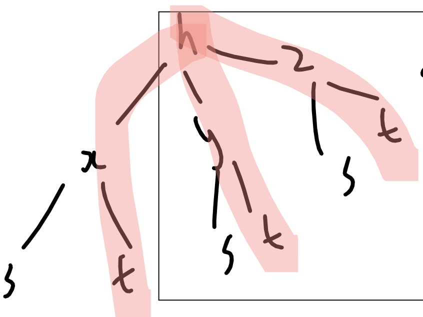
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= (\cos x \cos y)\cos\theta + (-\sin x \sin y)\sin\theta$$

$$\left.\frac{\partial z}{\partial r}\right|_{\substack{x=\pi\\ y=2\pi}} = (\cos\pi)(\cos 2\pi)\frac{1}{\sqrt{5}} + (-\sin\pi \sin 2\pi)\sin\theta$$

$$= \frac{(-1)}{\sqrt{5}} + 0 \times \sin\theta$$

Example: Given $h = f(x, y, z)$ and each of x, y and z depend on both s and t, find an expression for $\dfrac{\partial h}{\partial t}$.



$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial t}$$

2.6.4 Implicit Differentiation

Both $y + \sin(x + y) = 0$ and $x^{11} - y^7 + xy^2 + 1 = 0$ correspond to curves in $\mathbb{R}^2$.

In neither of these examples can we explicitly solve for y as a function of x (or vice versa). This raises the following question: Can we compute $y'(x)$?

The answer to this question is affirmative and the method to do this goes by the name of implicit differentiation.

First we will try to find $y'(x)$ by "brute force". It is only advisable to use this method if you have a PhD in applications of the chain rule.

Example: Find $y'(x)$ on the curve $y + \sin(x + y) = 0$.

$$y + \sin(x+y) = 0$$

$$\sin(x+y) = -y$$

$$x = \arcsin(-y) - y$$

$$\frac{dx}{dy} = \frac{d}{dy}\left(\sin^{-1}(y) - y\right) = \frac{-1}{\sqrt{1-y^2}} - 1$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{-1}{\sqrt{1-y^2}} - 1}$$

Important remark: Although we have found $y'(x)$ it is almost never possible to express the answer in terms of x only (as this requires y to be solved for x). Extreme care has thus to be taken in using the above formula for $y'(x)$. It only has meaning for points (x, y) that lie on the curve $y + \sin(x + y) = 0$, such as $(0, 0)$ or $(-1 - \pi/2, 1)$.

So what if you do not have PhD in applications of the chain rule?

The idea is to view a given curve C in $\mathbb{R}^2$ (such as $y + \sin(x + y) = 0$) as the contour $z = 0$ of a function $F(x, y)$. That is, we define $F(x, y) = y + \sin(x + y)$. We already know that, for any given point (x, y) on C, $\nabla F(x, y)$ corresponds to a vector perpendicular to C at that point.

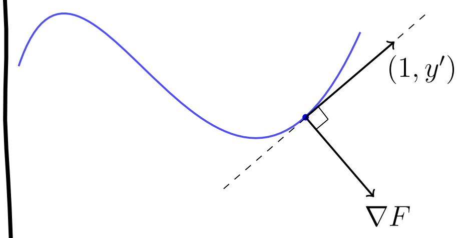
Alternatively,

$$\frac{dF(x,y(x))}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$F(x,y(x)) = 0$ constant on curve

$$\frac{dF}{dx} = 0$$

$$F_x + F_y y' = 0$$



In other words,

$$\nabla F(x, y) \cdot (1, y'(x)) = 0.$$

$$F_x + F_y y' = 0$$

Solving this equation for $y'(x)$ gives

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}.}$$

Example: Find $y'(x)$ on the curve $y + \sin(x + y) = 0$.

$F(x,y) = y + \sin(x+y)$

$F_x = \cos(x+y)$ $\quad F_y = 1 + \cos(x+y)$

$$y'(x) = \frac{dy}{dx} = \frac{-\cos(x+y)}{1+\cos(x+y)}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

Example: The Folium of Descartes is given by $x^3 + y^3 = 6xy$. Find $y'(x)$.

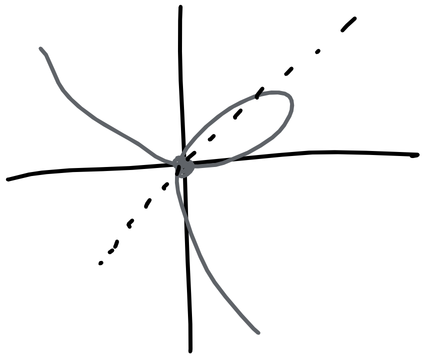
$F(x,y) = x^3 + y^2 - 6xy$ $\quad F(x,y) = 0$

$F_x = 3x^2 - 6y$

$F_y = 2y - 6x$

$$y'(x) = \frac{-3x^2 - 6y}{3y^2 - 6x}$$

when $3y^2 - 6x \neq 0$



symmetry about $x = y$

Question: What happens at $x = y = 0$?

2.6.5 Main points

- You should be able to differentiate a multivariable function whose variables depend on multivariable functions.
- You should be able to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ from implicit forms.

3 Max and Min Problems on Surfaces

The content of this chapter is found in Stewart, Sections 14.7 and 14.8 (Sections 14.7 and 14.8).

3.1 Quadratic Approximation

In the previous chapter, we looked at the first-order, or linear approximation to a function. In this section we introduce the second-order, or quadratic approximation to a function.

3.1.1 Review for $f(x)$

We saw in Section 2.4.1 that the value of $f(x)$ near $x = a$ can be approximated using a straight line:

$$f(x) \simeq f(a) + f'(a)(x - a).$$

Adding a second-order term will improve this approximation

@$x=a$ value

$$f(x) \simeq f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

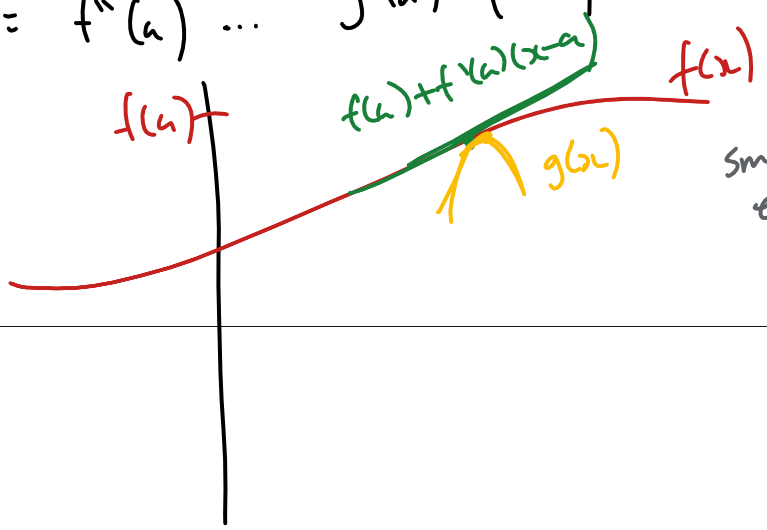
slope

This quadratic (or second-order) approximation corresponds to the approximation of f by the parabola through $(a, f(a))$ with slope and second derivative at a matching those of f. Unless f is very wild around $x = a$ this is generally quite a good approximation for $x - a$ not too large.

$$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$g'(x) = f'(a) + \frac{f''(a)}{2}\,2(x-a) = f'(a) + f''(a)(x-a)$$

$$g''(x) = f''(a) \quad \dots \quad g''(a) = f''(a)$$



smaller error w Quadratic eq!

Example: Compute $e^{0.1}$ using a linear and a quadratic approximation.

$f(x) = e^x \qquad a = 0$

$f'(x) = e^x \qquad$ linear approx: $f(a) + f'(a)(x-a)$

$1 + x$

$f''(x) = e^x$

for $x = 0.1$, $e^{0.1} = f(0.1) \approx 1 + 0.1 = 1.1$

QUADRATIC APPROX.

$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = 1 + x + \frac{1}{2}(x)^2$

$= 1 + x + \frac{x^2}{2}$

for $x = 0.1$ $\quad e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2} = 1.105$

Example: In the same graph, sketch $f(x) = \cos x$ as well as its linear and quadratic approximations around 0.

$f'(x) = -\sin x \qquad f'(0) = 0$

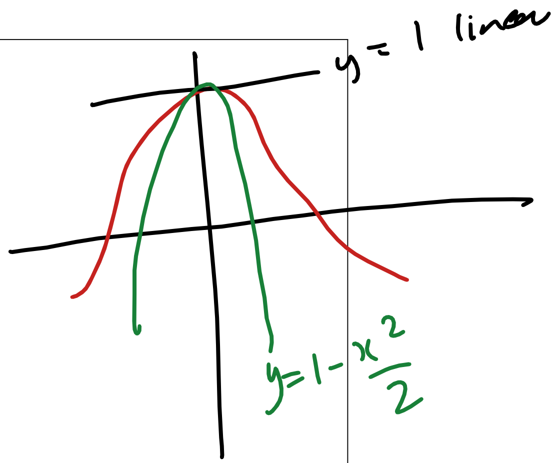
$f''(x) = -\cos x \qquad f''(0) = -1$

linear approx

$\cos x \approx 1 + 0 = 1$

Quadratic approx

$1 + 0 - \frac{1}{2}x^2 = 1 - \frac{x^2}{2}$



3.1.2 Quadratic approximations of $f(x,y)$

This material is covered in Stewart p. 1010 (p. 980).

The quadratic or second-order approximation to $f(x,y)$ around (a,b) is a function of the form

$$Q(x,y) = h + mx + ny + px^2 + qxy + ry^2$$

such that $Q(a,b) = f(a,b)$ and such that all first order and second order partial derivatives of f and Q agree at (a,b).

It is not hard to verify that this gives

$$Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2.$$

Example: Check that $Q_{xx}(a,b) = f_{xx}(a,b)$ and $Q_{xy}(a,b) = f_{xy}(a,b)$.

$Q_x(x,y) = 0 + f_x(a,b) + 0 + \frac{1}{2} f_{xx}(a,b)\, 2(x-a) + f_{xy}(a,b)(y-b) + 0$

$Q_{xx}(x,y) = 0 + f_{xx}(a,b) + 0$

$Q_{xx}(a,b) = f_{xx}(a,b)$

$Q_{xy}(a,b) = f_{xy}(a,b)$

Exercise check $Q_{yy}(a,b) = f_{yy}(a,b)$

Example: Find the quadratic approximation around $(0, 0)$ of

$$f(x, y) = 1 - x^2 + y^2 + xy + x^3 + x^2y^2.$$

$f(0,0) = 1$

$f_x(0,0) = -2x + y + 3x^2 + 2y^2x \Big|_{(0,0)} = 0$

$f_y(0,0) = 2y + x + 2yx^2 \Big|_{(0,0)} = 0$

$f_{xx}(0,0) = -2 + 6x + 2y^2 \Big|_{(0,0)} = -2$

$f_{xy}(0,0) = 1 + 4xy \Big|_{(0,0)} = 1$

$f_{yy}(0,0) = 2 + 2x^2 \Big|_{(0,0)} = 2$

QUADRATIC APPROX

$1 + 0(x-0) + 0(y-0) - \frac{2}{2}(x-0)^2 + 1(x-0)(y-0) + \frac{2}{2}(y-0)^2$

$= 1 - x^2 + xy + y^2$

Quadratic and any higher order terms.

If $|x|, |y| < 0.1$ then we are ignoring terms on the order of $(0.1)^3 = 0.001$.

In this example the function is a polynomial in x and y and we can immediately read off the quadratic approximation. In the next example we really need to use the formula (or do we?).

Example: Find the linear and quadratic approximations around $(0,0)$ of

$$f(x,y) = \mathrm{e}^{-x^2-y^2}.$$ Not a polynomial

$f(0,0) = 1$

$f_x(0,0) = -2x e^{-x^2-y^2}\Big|_{(0,0)} = 0$

$f_y(0,0) = -2y e^{-x^2-y^2}\Big|_{(0,0)} = 0$

$f_{xx}(0,0) = -2e^{-x^2-y^2} + 4x^2 e^{-x^2-y^2}\Big|_{(0,0)} = -2$

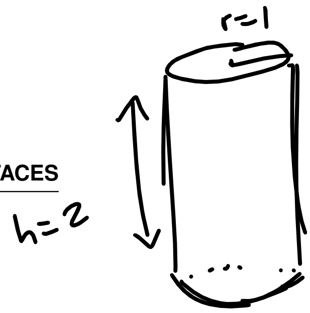
$f_{xy}(0,0) = 4xy e^{-x^2-y^2}\Big|_{(0,0)} = 0$

$f_{yy} = -2e^{-x^2-y^2} + 4y^2 e^{-x^2-y^2}\Big|_{(0,0)} = -2$

$L(x,y) \approx 1 + 0x + 0y \approx 1$

$Q(x,y) \approx 1 + \frac{(-2)}{2}x^2 + 0(x)(y) + \frac{(-2)}{2}(y)^2$

$= 1 - x^2 - y^2$



Example: Consider, again, a barrel with base of radius 1 and height 2.

$$
\begin{aligned}
V(r,h) &= \pi r^2 h &\quad\Rightarrow\quad V(1,2) &= 2\pi\\
\frac{\partial V}{\partial r} &= 2\pi r h &\quad\Rightarrow\quad \frac{\partial V}{\partial r}(1,2) &= 4\pi\\
\frac{\partial V}{\partial h} &= \pi r^2 &\quad\Rightarrow\quad \frac{\partial V}{\partial h}(1,2) &= \pi\\
\frac{\partial^2 V}{\partial r^2} &= 2\pi h &\quad\Rightarrow\quad \frac{\partial^2 V}{\partial r^2}(1,2) &= 4\pi\\
\frac{\partial^2 V}{\partial r\partial h} &= 2\pi r &\quad\Rightarrow\quad \frac{\partial^2 V}{\partial r\partial h}(1,2) &= 2\pi\\
\frac{\partial^2 V}{\partial h^2} &= 0.
\end{aligned}
$$

Hence the quadratic approximation around $(1,2)$ is

$$
\begin{aligned}
V(r,h) &\simeq Q(r,h)\\
&= 2\pi + 4\pi\overbrace{(r-1)}^{\Delta r} + \pi\overbrace{(h-2)}^{\Delta h} + 2\pi\overbrace{(r-1)^2}^{(\Delta r)^2} + 2\pi\overbrace{(r-1)}^{\Delta r}\overbrace{(h-2)}^{\Delta h}.
\end{aligned}
$$

Let $\Delta r = r-1$ and $\Delta h = h-2$. Use $Q(r,h)$ to approximate the change ΔV in the volume of the barrel when $\Delta r = 0.05$ and $\Delta h = 0.1$.

To approx ΔV, subtract $V(1,2)$ from $Q(r,h)$

$$\Delta V \approx \cancel{2\pi} + 4\pi(0.05) + \pi(0.1) + 2\pi(0.05)^2 + 2\pi(0.05)(0.1)$$

$$\cancel{-2\pi} = \pi(0.2 + 0.1 + 0.005 + 0.1) = \pi(0.315)$$

3.1.3 Main points

- You should be able to find the quadratic approximation given a function of two variables.
- In Matlab, you should be able to plot both the linear and quadratic approximations of a function, and make a comparison of the errors.
- The quadratic approximation is superior to the linear approximation.

3.2 Critical Points and Optimisation

3.2.1 Review for $f(x)$

Smooth functions f of one variable have local maxima and minima where $f(x)$ has zero slope, i.e., where the first derivative of f is zero:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 0.$$

The second derivative test tells us whether $f(x)$ is concave up (a minimum) or concave down (a maximum):

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} > 0 \Rightarrow \text{local minimum}, \quad \cup \quad x^2$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} < 0 \Rightarrow \text{local maximum}. \quad \cap \quad -x^2$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0 \Rightarrow ? \quad \text{either, smth else}$$

Example: Find the local maxima and minima of $f(x) = x^2 - x^4$.

$f'(x) = 2x - 4x^3 \rightarrow$ local max and min will occur when $f'(x) = 0$

$2x(1 - 2x^2) = 0$

$x = 0$ or $2x^2 = 1$

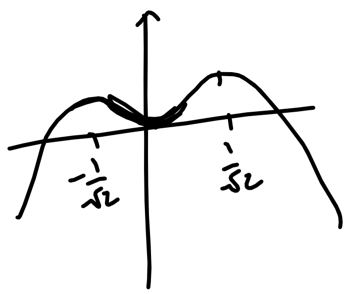
$x = \pm \frac{1}{\sqrt{2}}$

$f''(x) = 2 - 12x^2$

$x = 0 \Rightarrow f''(x) = 2 > 0$

$\Rightarrow$ local min

$x = \pm \frac{1}{\sqrt{2}} \Rightarrow f''(x) = -4 \Rightarrow$ local max



Note, however, that the local minimum in the above example is not a global minimum. At the local minimum, $x = 0$ and $f(0) = 0$. But for $|x| > 1$, we have $f(x) < 0$.

If we define the domain as $[-2, 2]$, the global minimum occurs on the boundary at $f(\pm 2) = 4 - 16 = -12$.

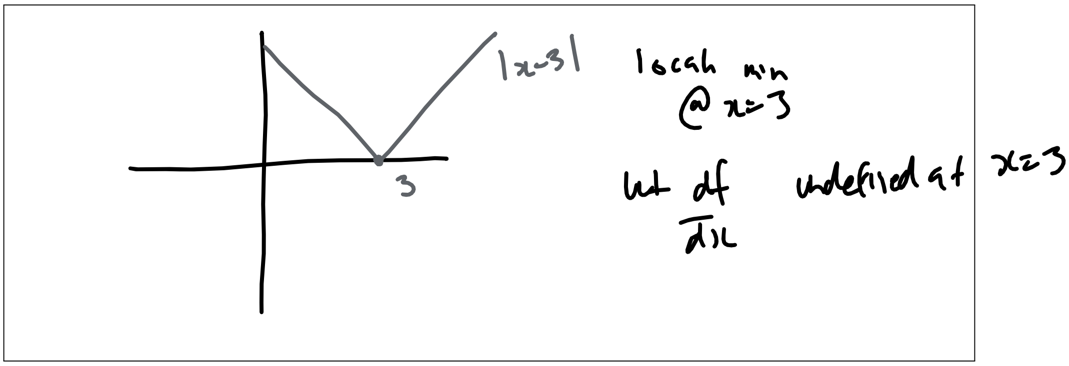
The global maximum, however, is at the local maximum, i.e. $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

The global maximum or minimum of a continuous function $f(x)$ occurs either at a local minimum, a local maximum, or on the boundary of its domain.

In some circumstances there may be local maxima or minima at points where the derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not defined (at cusps, for instance).

If function is not differentiable, cannot find $f'(x)=0$

Example: $f(x) = |x - 3|$.



A more general definition of a local minimum or maximum is:

- A local minimum occurs at a point a where $f(x) \geq f(a)$ for all x sufficiently close to a.
- A local maximum occurs at a point a where $f(x) \leq f(a)$ for all x sufficiently close to a.

If f is continuous, a local maximum or minimum occurs either when $f'(x) = 0$ or when $f'(x)$ is undefined.

3.2.2 Critical points for $f(x, y)$

In this section we will find and classify critical points of a function of two variables, see Stewart, Section 14.7 (Section 14.7). (There are also some useful Matlab functions which perform minimisation, such as **fmin** and **fminsearch**).

First we define local maxima and local minima:

- $f(x, y)$ has a *local maximum* at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) sufficiently close to (a, b)
- $f(x, y)$ has a *local minimum* at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) sufficiently close to (a, b).

If f is differentiable, a local maximum or minimum can only occur in points where the corresponding tangent plane is horizontal, i.e., where

$$\nabla f = \mathbf{0}.$$

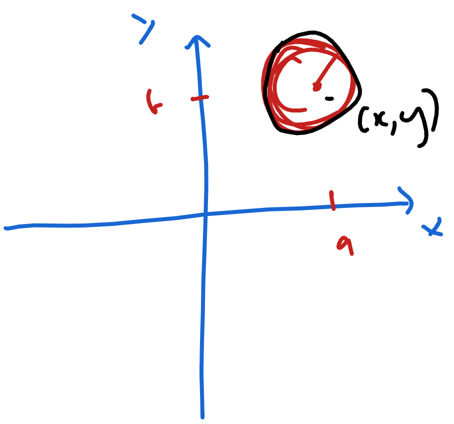
Note that if either one of $\dfrac{\partial f}{\partial x}$ or $\dfrac{\partial f}{\partial y}$ is undefined then ∇f is undefined.

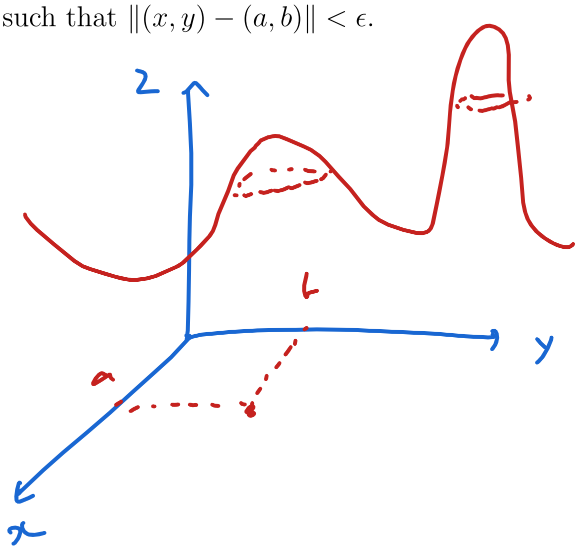
Definition: A critical point of $f(x, y)$ is a point where either $\nabla f = (0, 0)$ or where ∇f is undefined.

3D crit point of $f(x,y,z)$ is a point where either $\nabla f = \vec{0}$ or ∇f undefined

Local maxima and minima occur at critical points, but not all critical points are local maxima or minima. → SADDLE POINTS

Remark: (For the mathematically inclined.) The rather vague description "sufficiently close" can be made much more precise. In the simplest case, when $f : \mathbb{R}^2 \to \mathbb{R}$, it goes as follows. The function f has a *local maximum* at (a, b) if there exists an $\epsilon > 0$ such that $f(a, b) \geq f(x, y)$ for all (x, y) such that $\|(x, y) - (a, b)\| < \epsilon$.





Example: Find the critical points of $f(x, y) = x^2 + 3x + y^2$.

$\nabla f(x,y) = (2x+3, 2y)$ always defined

crit point when $\nabla f = \vec{0}$

$\iff$

$2x+3=0 \iff x = -\frac{3}{2}$

$2y=0 \qquad\quad y=0$

$(-\frac{3}{2}, 0)$ only crit point.

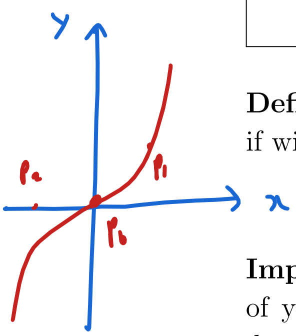
Example: Find the critical points of $f(x, y) = x^2 + 3x - y^2$.

$\nabla f(x,y) = (2x+3, -2y)$ always defined

$2x+3=0 \iff x = -3/2$

$-2y=0 \qquad\quad y=0$

$(-\frac{3}{2}, 0)$ only crit point



Definition. A function f has a saddle point at P_0 if P_0 is a critical point of f and if within any distance of P_0 (no matter how small) there are points P_1 and P_2 with

$$f(P_1) > f(P_0) \quad \text{and} \quad f(P_2) < f(P_0).$$

Important remark: Not all saddle points look like a "horse saddle", and in one of your Matlab sessions you will be asked to look at a so-called monkey saddle, described by an equation of the form $z = x^3 - 3xy^2$.

Example: Find all the critical points of

$$f(x,y) = x^2 + y^2(1-x)^3$$

and use a quadratic approximation to investigate the nature of the critical points.

$\nabla f(x,y) = \left(2x - 3y^2(1-x)^2,\ 2y(1-x)^3\right)$

crit points $\nabla f = \vec{0} = (0,0)$

$2x - 3y^2(1-x^2) = 0 \quad \ldots ①$

$2y(1-x)^3 = 0 \quad \ldots ②$

② $2y(1-x)^3 \implies y=0$ or $x=1$

If $y=0$

① $2x = 0$

$x=0$

$(0,0)$

only crit point

If $x=1$

$2 - 3y^2(0) = 0$

impossible

$f_{xx} = 2 - 3y^2(1-x)(-1) \quad f_{xx}(0,0) = 2$

$f_{xy} = -6y(1-x)^2 \quad f_{xy}(0,0) = 0$

$f_{yy} = 2(1-x)^3 \quad f_{yy}(0,0) = 2$

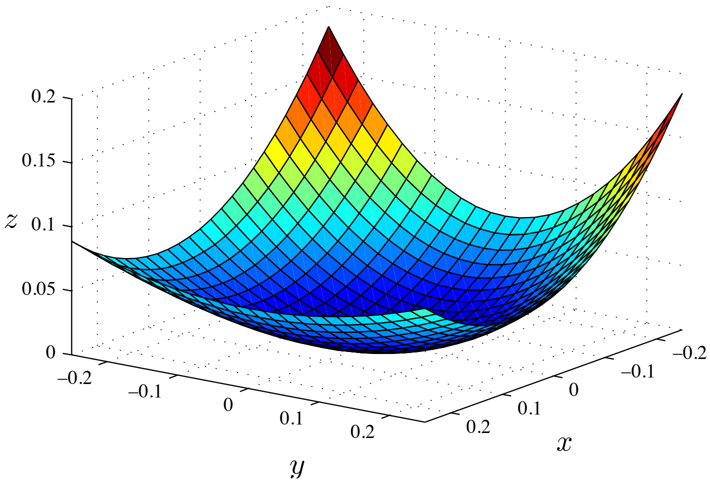
QUADRATIC APPROX

$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2} f_{xx}(0,0)x^2$
$\quad + f_{xy}(0,0)xy + \frac{1}{2} f_{yy}(0,0)y^2$
$\quad = 0 + x^2 + 0 + y^2$

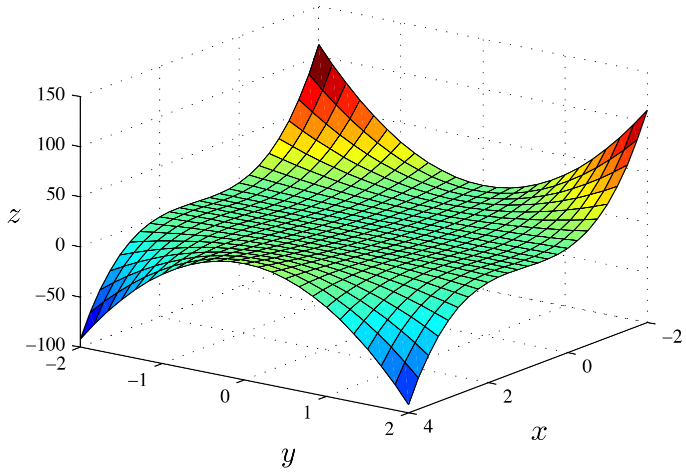
at $(0,0)$ x^2+y^2 attains a minimum

Also $(0,0)$ is a local minimum for $f(x,y)$

Local view of the surface $z = f(x, y) = x^2 + y^2(1 - x)^3$



The surface $z = f(x, y) = x^2 + y^2(1 - x)^3$



The plus or minus signs in front of x^2 and y^2 in the quadratic approximation can indicate whether f has a maximum, minimum or saddle at a given critical point.

$$f(x,y) \approx f(a,b) + \boxed{f_x(a,b)(x-a) + f_y(a,b)(y-b)} = 0 \text{ @crit point}$$
$$+ \tfrac{1}{2} f_{xx}(a,b)(x-a)^2 + \boxed{f_{xy}(a,b)(x-a)(y-b)}$$
$$+ f_{yy}(a,b)(y-b)^2$$

cross term

If cross term $= 0$

$$f(x,y) \approx f(a,b) + \frac{f_{xx}(a,b)}{2} \underbrace{(x-a)^2}_{\geq 0} + \frac{1}{2} f_{yy}(a,b) \underbrace{(y-b)}_{\geq 0}$$

$+x^2 \quad +y^2 \Rightarrow$ local min

$-x^2 \quad -y^2 \Rightarrow$ local max

$\pm x^2 \quad \mp y^2 \Rightarrow$ saddle

However, when $Q(x, y)$ has cross-terms such as xy one needs to be extremely careful.

Example: Examine the critical point $(0, 0)$ of $f(x, y) = x^2 + xy + y^2$.

$f(x,y) = x^2 + 2x(\tfrac{1}{2}y) + y^2 = (x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2 \geq 0$

$f(0,0) = 0$ so $(0,0)$ is a minimum

Next consider

$$f(x, y) = x^2 + xy + \frac{y^2}{8} = \left(x + \frac{1}{2}y\right)^2 - \frac{1}{8}y^2.$$

This function has a saddle point at $(0, 0)$.

Question: Can we find a general rule to determine the nature of critical points?

$\frac{q}{p} = 2a \Rightarrow a = \frac{q}{2p}$

Let Q be the quadratic function

$$
\begin{aligned}
Q(x,y) &= px^2 + qxy + ry^2 \\
&= p\left(\left(x + \frac{qy}{2p}\right)^2 - \frac{q^2y^2}{4p^2} + \frac{ry^2}{p}\right) \\
&= p\left(\left(x + \frac{qy}{2p}\right)^2 + \frac{(4pr - q^2)}{4p^2}\, y^2\right).
\end{aligned}
$$

$= p(x^2 + \frac{q}{p}xy + \frac{r}{p}y^2)$

$p((x+ay)^2 + \dots)$

≥ 0 ≥ 0

Then

$\pm$ $+$

- $p > 0$ and $4pr - q^2 > 0$ implies that f has a local minimum;
- $p < 0$ and $4pr - q^2 > 0$ implies that f has a local maximum;
- $4pr - q^2 < 0$ implies that f has a saddle.

critical point

Applying this argument to our quadratic approximation formula for f around (a,b):

$$
Q(x,y) = f(a,b) + \underbrace{\frac{1}{2}f_{xx}(a,b)}_{p}(x-a)^2 + \underbrace{f_{xy}(a,b)}_{q}(x-a)(y-b) + \underbrace{\frac{1}{2}f_{yy}(a,b)}_{r}(y-b)^2
$$

gives what is known as the second-derivative test.

$4pr - q^2 = 4\cdot\frac{1}{2}f_{xx}(a,b)\frac{1}{2}f_{yy}(a,b) - f_{xy}^2(a,b)$

Theorem: Assume that f and its first- and second-order partial derivatives are all continuous at (a,b) and $\nabla f(a,b) = (0,0)$. Define the 2×2 determinant

$$
D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}.
$$

Then:

- $f_{xx}(a,b) > 0$ and $D > 0$ implies that (a,b) is a local minimum;
- $f_{xx}(a,b) < 0$ and $D > 0$ implies that (a,b) is a local maximum;
- $D < 0$ implies that (a,b) is a saddle point;
- if $D = 0$ the second-derivative test is inconclusive.

Remark: (For the connoisseurs.) The matrix

$$
\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}
$$

is known as the Hessian and its determinant as the discriminant.

The beauty of this test is that we do not even have to write down the quadratic approximation.

Example: Find and classify the critical points of $f(x, y) = e^{-x^2+y^2}$.

$\nabla f(x,y) = (-2x e^{-x^2+y^2}, 2y e^{-x^2+y^2})$

$\nabla f = \vec{0}$ if $(x,y) = (0,0)$

$f_{xx} = -2e^{-x^2+y^2} + 4x^2 e^{-x^2+y^2} \Rightarrow f_{xx}(0,0) = -2$

$f_{xy} = -2x(2y)e^{-x^2+y^2} \Rightarrow f_{xy}(0,0) = 0$

$f_{yy} = 2e^{-x^2+y^2} + 4y^2 e^{-x^2+y^2} \Rightarrow f_{yy}(0,0) = 2$

$$D = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4$$

$D < 0 \Rightarrow (0,0)$ SADDLE POINT

Example: Find and classify the critical points of $f(x, y) = x^2 + y^2 - 2x^2y^2$.

$\nabla f(x,y) = (2x - 4xy^2, 2y - 4yx^2)$

$\nabla f = \vec{0}$ if

$2x - 4xy^2 = 0$ $\iff$ $2x(1-2y^2) = 0$ ... (1)

$2y - 4x^2y = 0$ $\quad$ $2y(1-2x^2) = 0$ ... (2)

(1) $\Rightarrow x = 0$ or $1 - 2y^2 = 0$

$x = 0$ or $y = \pm\frac{1}{\sqrt{2}}$

• if $x = 0$ (2) $\Rightarrow 2y(1-0) = 0 \Rightarrow y = 0$

$(0,0)$ crit pt

• If $y = \pm\frac{1}{\sqrt{2}}$ (2) $\Rightarrow 2 \cdot \frac{1}{\sqrt{2}}(1 - 2x^2) = 0$

$\Rightarrow 1 - 2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

4 more crit points: $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

5 critical points

$f_{xx} = 2 - 4y^2$ $\quad f_{xy} = -8xy$

$f_{yy} = 2 - 4x^2$

At $(0,0)$

$f_{xx} = 2$ $\quad f_{xy} = 0$ $\quad f_{yy} = 2$ $\quad D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$f_{xx} > 0 \Rightarrow (0,0)$ LOCAL MIN

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$f_{xx} = 0$ $\quad f_{xy} = -4$ $\quad f_{yy} = 0$

$D = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0$ $\quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ SADDLE

other points are also saddles

3.2.3 Global maxima and minima

To find the global maximum or minimum values of f on a closed and bounded domain D, we need to compare both

- the values of f at the critical points (where $\nabla f = (0,0)$ or undefined), and
- the extreme values of f on the boundary of D.

Example: What are the global maximum and minimum values of the function

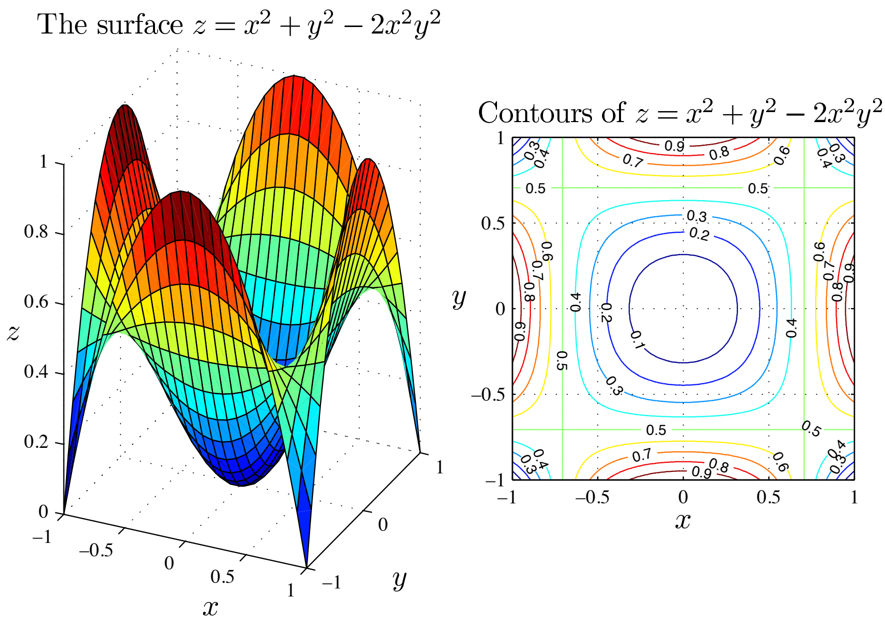
$$f(x,y) = x^2 + y^2 - 2x^2y^2$$

on the domain $[-1,1] \times [-1,1]$?

crit points (from previous example)
$(0,0)$ local min, other crit pts are saddles...
$f(0,0) = 0$
• boundary: $x = \pm 1$ $y = \pm 1$
i) $x = 1$ $f(1,y) = 1 + y^2 - 2y^2 = 1 - y^2$ greatest value @ $y = 0$ $f(1,0) = 1$
ii) $x = -1$ $f(-1,y) = 1 + y^2 - 2y^2$ greatest value @ $y = 0$ $f(-1,0) = 1$
iii) $y = 1$ $f(x,1) = x^2 + 1 - 2x^2 = 1 - x^2$ greatest value @ $x = 0$ $f(0,1) = 1$
lowest @ $x = \pm 1$ $f(\pm 1, 1) = 0$
iv) $y = -1$ $f(x,-1) = x^2 + 1 - 2x^2 = 1 - x^2$
greatest @ $x = 0$ $f(0,-1) = 1$
lowest @ $x = \pm 1$
$f(\pm 1, 1) = 0$

4 global max: occur at $(1,0), (-1,0)$, $(0,1)$ $(0,-1)$ max value is 1

5 global min: occur @ $(0,0)$ $(\pm 1, \pm 1)$ min value is 0


The surface $z = x^2 + y^2 - 2x^2y^2$
z
x
y
Contours of $z = x^2 + y^2 - 2x^2y^2$
x
y

3.2.4 The Matlab command fminsearch

The *Nelder–Mead* algorithm is a search method built into the Matlab function **fminsearch**. One advantage of this method is that it works with non-smooth functions as well, so it extends beyond traditional calculus methods.

The syntax is that **fminsearch** will try to minimise a function f that you must store as a function in an M-file. Then you give **fminsearch** a starting point and let it run.

For example, to minimise $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$, use the following code:

```
function output = f(x)
output = (x(1)-1)^2 + (x(2)-2)^2 + (x(3)-3)^2;
```

Note that `x(1)` corresponds to x, `x(2)` to y and `x(3)` to z. This is because **fminsearch** requires that f takes a single vector as input (in this example the three-dimensional vector `x`).

Now, to run the minimisation, call

```
fminsearch('f',[0,5,10])
```

where $x = 0, y = 5, z = 10$ is the initial guess.

For functions with several local minima, **fminsearch** can get stuck within a local minimum. Starting it with different initial guesses or restarting it when it thinks it has converged may help it to look for "better" minima. How would you use **fminsearch** to find a maximum value of f?

3.2.5 Main points

can only happen

- You should know that the local minimum or maximum is obtained when either all the partial derivatives are zero or some of the partial derivatives are not defined.
- You should know how to find critical points and classify them using either the quadratic approximation, or the second derivative test.
- You should be able to find global maxima and minima of a function on a bounded domain.
- You should be competent finding extrema with **fminsearch**.

3.3 Constrained Optimisation and Lagrange Multipliers

3.3.1 Lagrange multipliers

In practical problems one often needs to maximise or minimise a function subject to certain constraints, see also Stewart, Section 14.8 (Section 14.8).

Example: Find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 1$.

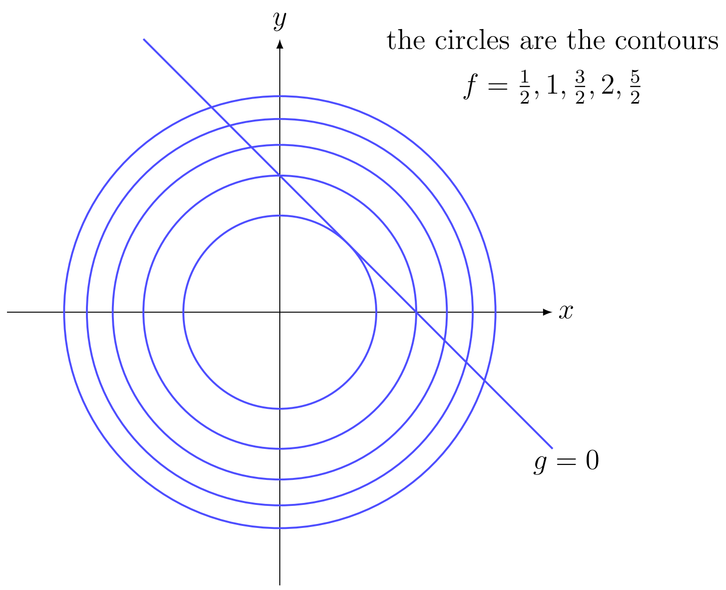
We want to minimise the function f given by $f(x, y) = x^2 + y^2$, subject to the constraint $x + y = 1$.

In this baby-example we can explicitly solve the constraint-equation: $y = 1 - x$, so

$$F(x) := f(x, 1 - x) = x^2 + (1 - x)^2 = 2x^2 - 2x + 1,$$

which is a function of x alone. The critical points of F occur when $F'(x) = -2+4x = 0$, yielding $x = 1/2$. Since $F''(x) = 4 > 0$ it follows that F has a minimum at $x = 1/2$. Therefore, f subject to the constraint $x + y = 1$ achieves its minimum value at $x = y = 1/2$, with actual value also $1/2$.

There is a much better way to solve the above problem not requiring the explicit solution of the constraint-equation. The trick is to define a second function, say g, such that the constraint-equation corresponds to $g(x, y) = 0$. In our example, $g(x, y) = x + y - 1$. Now graph the contour plot of $f(x, y) = x^2 + y^2$ and add to this graph the single contour $g(x, y) = 0$:



The minimum occurs where the contour $g(x, y) = 0$ touches one of the contours of f, which means that ∇f and ∇g are parallel. Hence

$$\boxed{\nabla f = \lambda \nabla g}$$

for some λ, known as the Lagrange multiplier.

Example: Find the minimum value of $x^2 + y^2$ subject to the constraint $x + y = 1$.

$f(x,y) = x^2 + y^2$ $\qquad$ $g(x,y) = x + y - 1$

At min $\nabla f = \lambda \nabla g$

$\nabla f(x,y) = (2x, 2y)$ $\qquad$ $\nabla g(x,y) = (1,1)$

So $\begin{cases} 2x = \lambda \cdot 1 \\ 2y = \lambda \cdot 1 \end{cases}$ $\qquad$ So $2x = \lambda = 2y \Rightarrow x = y$ — 1

Plug into constraint

$x + x = 1$, $x = \frac{1}{2}$

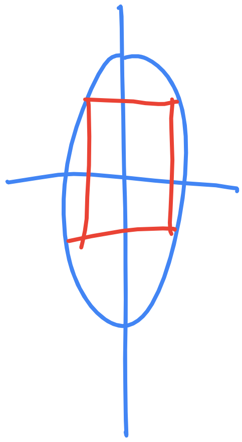
$\lambda = 1$

Thus minimum is $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Important remark: When solving a constrained optimisation problem with Lagrange multipliers, it is typically not necessary to find the value for λ. So, when solving your equations obtained from $\nabla f = \lambda \nabla g$ and the constraint-equation, your aim is really to eliminate λ in order to solve for the variables in your problem.

Important remark: If upon using the method of Lagrange multipliers you obtain simple, linear equations to solve then consider yourself lucky. In some cases you will in fact obtain nonlinear equations that you need to solve. If this happens we suggest the first thing you should try is taking the ratio of your equations obtained from $\nabla f = \lambda \nabla g$. This allows you to immediately eliminate λ from your equations and proceed from there. The next example nicely illustrates this point.

Example: Let $A = 4xy$ describe the area of a rectangle centred on the origin with width $2x$ and height $2y$. Maximise the area of this rectangle subject to the constraint that it is inscribed within the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



$f(x,y) = 4xy \qquad g(x,y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$

$\nabla f(x,y) = (4y, 4x) \qquad \nabla g(x,y) = \left(\dfrac{2x}{a^2}, \dfrac{2y}{b^2}\right)$

At maximum

$\nabla f = \lambda \nabla g$

$4y = \lambda\left(\dfrac{2x}{a^2}\right)$ (1)

$4x = \lambda\left(\dfrac{2y}{b^2}\right)$ (2)

• If $x=0$, then $y=0$
so $f(x,y)=0$ ⇝ min area

• So at max, $x \neq 0$ ⇒

Divide $\dfrac{(1)}{(2)} \Rightarrow \dfrac{\cancel{4}y}{\cancel{4}x} = \dfrac{\cancel{2}x}{a^2}\,\dfrac{b^2}{\cancel{2}y} \Rightarrow \dfrac{y}{x} = \dfrac{x\,b^2}{y\,a^2} \Rightarrow y^2 = \dfrac{b^2}{a^2}x^2$ — (3)

constraint $\Rightarrow \dfrac{x^2}{a^2} + \dfrac{\frac{b^2}{a^2}x^2}{b^2} = 1 \Rightarrow \dfrac{2x^2}{a^2} = 1 \Rightarrow x = \pm\dfrac{a}{\sqrt{2}}$ → take positive root $x = \dfrac{a}{\sqrt{2}}$

(3) $y^2 = \dfrac{b^2}{a^2}\,\dfrac{a^2}{2} = \dfrac{b^2}{2} \Rightarrow y = \dfrac{b}{\sqrt{2}}$

Maximum occurs at $(x,y) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

Max area is $4xy = 4\left(\frac{a}{\sqrt{2}}\right)\left(\frac{b}{\sqrt{2}}\right) = 2ab$

The method of Lagrange multipliers can be extended to any number of dimensions.

Example: Let $V(x, y, z) = xyz$ describe the volume of a box with vertices at $(0, 0, 0)$, $(x, 0, 0)$, $(0, y, 0)$, $\ldots$, (x, y, z), with $x, y, z > 0$. Maximise the volume of the box subject to the constraint that the vertex (x, y, z) lies in the plane $z+2x+3y = 6$.

$f(x,y,z) = xyz \qquad g(x,y,z) = z+2x+3y-6 = 0$

$\nabla f (yz, xz, xy) \qquad \nabla g (2, 3, 1)$

at max $\nabla f = \lambda \nabla g \quad \lambda \in \mathbb{R}$

$yz = 2\lambda$ ①

$xz = 3\lambda$ ② (at maximum, $x, y, z \neq 0$ otherwise volume $= 0$ so $\lambda \neq 0$)

$xy = \lambda$ ③

$\frac{①}{②} \Rightarrow \frac{yz}{xz} = \frac{2\lambda}{3\lambda} \Rightarrow \frac{y}{x} = \frac{2}{3} \Rightarrow \left[y = \frac{2}{3}x\right]$

$\frac{①}{③} \Rightarrow \frac{xyz}{xy} = \frac{2\lambda}{\lambda} \Rightarrow \frac{z}{x} = 2 \Rightarrow \left[z = 2x\right]$

Plug into constraint:

$2x + 2x + 3\left(\frac{2}{3}x\right) = 6 \Rightarrow 6x = 6 \Rightarrow x = 1$

So $y = \frac{2}{3}$, $z = 2$

$\therefore$ Max occurs at $(x, y, z) = (1, \frac{2}{3}, 2)$

max volume is $1\left(\frac{2}{3}\right) \times 2 = \frac{4}{3}$

Example: The distance between a point and a plane.

We now return to a problem considered in Chapter 2 — that of finding the distance between a point and a plane. Recall that when we say distance, we always mean minimum distance.

Let $P = (x_1, y_1, z_1)$ be a point and $\Pi : \ ax + by + cz = d$ a plane. We want to compute the distance between P and Π.

If we take an arbitrary point $Q = (x, y, z)$ in $\mathbb{R}^3$ then the square of the distance between P and Q is given by

$\nabla f(x,y,z) = (2(x-x_1), 2(y-y_1), 2(z-z_1))$

$$D^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 =: f(x, y, z).$$

We want to minimise this distance subject to the constraint that Q lies in the plane Π, i.e., subject to the constraint $ax+by+cz = d$. Following the method of Lagrange multipliers we thus define $g(x, y, z) = ax + by + cz - d$ and compute $\nabla f = \lambda \nabla g$ to get

$\nabla g = (a, b, c)$

$$\begin{cases} 2(x - x_1) = \lambda a \rightarrow (x-x_1)^2 = \left(\frac{\lambda a}{2}\right)^2 \ (1) \\ 2(y - y_1) = \lambda b \ (2) \\ 2(z - z_1) = \lambda c. \ (3) \end{cases} \qquad (*)$$

First we use $(*)$ to find

$$D^2 = f(x, y, z) = (\lambda a/2)^2 + (\lambda b/2)^2 + (\lambda c/2)^2 = \left(\frac{\lambda}{2}\right)^2 (a^2 + b^2 + c^2)$$

from which we still need to eliminate λ. Second we use $(*)$ to find

$$x = \lambda a/2 + x_1, \quad y = \lambda b/2 + y_1, \quad z = \lambda c/2 + z_1.$$

Substituting this into the constraint equation $ax + by + cz - d = 0$ we find that

$\frac{\lambda a^2}{2} + ax_1 + \frac{\lambda b^2}{2} + by_1 + \frac{\lambda c^2}{2} + cz_1 - d$

$$\hookrightarrow \frac{\lambda}{2} = \frac{d - ax_1 - by_1 - cz_1}{a^2 + b^2 + c^2}.$$

$D = \frac{d - ax_1 - by_1 - cz_1}{\sqrt{a^2+b^2+c^2}}$ MIN DISTANCE

Therefore,

$$D^2 = \left(\frac{\lambda}{2}\right)^2 (a^2 + b^2 + c^2) = \frac{(d - ax_1 - by_1 - cz_1)^2}{a^2 + b^2 + c^2}$$

as before.

If needed, we can also obtain the point on the plane that is closest to (x_1, y_1, z_1).

Example: Use the method of Lagrange multipliers to find the distance between the point $(1, 2, 3)$ and the plane $x + y + z = 1$.

$x_1 = 1 \quad y_1 = 2 \quad z_1 = 3 \qquad a = b = c = d = 1$

$$D^2 = \frac{(d - ax_1 - by_1 - cz_1)^2}{a^2 + b^2 + c^2} = \frac{(1 - 1\cdot 1 - 1\cdot 2 - 1\cdot 3)^2}{1^2 + 1^2 + 1^2}$$

$$D^2 = \frac{(-5)^2}{3} = \frac{25}{3} \quad \therefore \quad D = \frac{5}{\sqrt{3}}$$

~~p. 137~~ sol for previous example somewhere..

3.3.2 Main points

- You should be able to apply the method of Lagrange multipliers to solve problems in constrained optimisation.
- You should understand the method of Lagrange multipliers geometrically in terms of gradient vectors.

4 Ordinary Differential Equations

4.1 Introduction

This material is covered in Stewart, Section 9.1 (Section 9.1).

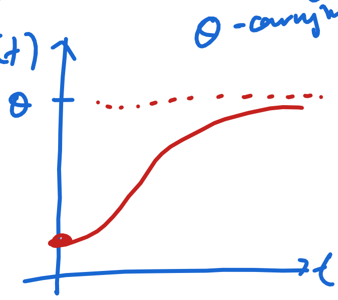
In almost all fields of scientific endeavour, quantifying time-dependent occurrences almost inevitably comes down to a differential equation (DE). A DE is an equation which contains one or more derivatives of an unknown function. There are two types of DEs:

- ordinary differential equations (ODEs), where the unknown function is a function of only one variable, and
- partial differential equations (PDEs) where the unknown function is a function of more than one variable. We will not consider PDEs in this course.

P(t) population at time t

θ - carrying capacity

4.1.1 Examples of ODEs



- Unbounded population growth: $P'(t) = kP$.
  exponential growth $(k>0)$
- Bounded population growth: $P'(t) = kP(1 - P/\theta)$.
- Motion due to gravity: $my''(t) = -mg$.
  y(t) position at time t
- Spring system: $mx''(t) = -kx$.

In ODEs, one often takes t instead of x for the independent variable, where t denotes time. Also, derivatives, such as $x'(t)$ and $x''(t)$ with respect to time are often written as $\dot{x}$ and $\ddot{x}$ respectively.

$x'(t) = \dot{x}(t) = \frac{dx}{dt}(t)$ $\qquad$ $x' = \dot{x} = \frac{dx}{dt}$

t dependence is implicit.

4.1.2 How do ODEs arise?

Population modelling. Assume a population grows at a constant rate proportional to the size of the population. If $P = P(t)$ stands for the population at time t, then the model states that

proportional to

$$\frac{\mathrm{d}P}{\mathrm{d}t} \propto P,$$

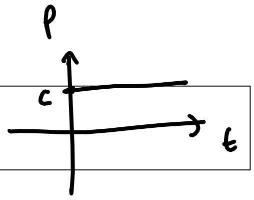
i.e.,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,$$

where k is the growth constant. If $k > 0$ the population is growing (think humans) and if $k < 0$ the population is decreasing (think tigers).

Question: What happens for $k = 0$?

$dP/dt = 0 \times P = 0$ so $P(t) = C$



Newton's eureka moment. Newton's second law of motion states that

$$\text{mass} \times \text{acceleration} = \text{force}.$$

Let $y = y(t)$ be the vertical displacement from ground at time t of an apple of mass m, soon to land on Newton's head. Then the acceleration is $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2}$ so that

$$m \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -mg,$$

where g is the constant of acceleration at the surface of the earth due to gravity.[1]

[1] By convention, we've take the direction "up" to be positive, so the downward force of gravity has a negative sign.

4.1.3 Solution to an ODE

Suppose that we are given an ODE for y which is an unknown function of x. Then $y = f(x)$ is said to be a solution to the ODE if the ODE is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Example: Show that $y = y(x) = A\exp(x^2/2)$ is a solution to the ODE $y' = xy$.

Check

$y'(x) = Ae^{(x^2/2)} \times \frac{\cancel{2}x}{\cancel{2}} = y(x)x$

$\downarrow$ $y(x)$

so $y' = yx$ ✓

When asked to solve an ODE, you are expected to find *all* possible solutions. This means that you need to find the general solution to the ODE. For an ODE that involves only the unknown function y and its first derivative, the general solution will involve *one* arbitrary constant.

You should already know how to solve ODEs of the form

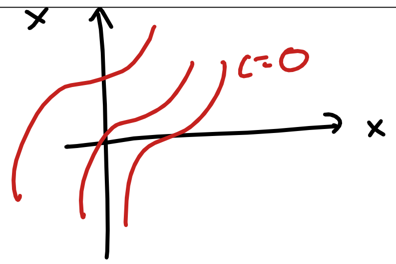
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \quad \text{or} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = g(x).$$

Example: Find the general solution to the differential equation $y' = x^2$.

$y' = x^2$

$y = \int x^2 \, dx = \frac{x^3}{3} + C$

$C \in \mathbb{R}$



4.1.4 Initial value problems

An initial value problem (IVP) is the problem of solving an ODE subject to some initial conditions of the form $y(t_0) = a$, $y'(t_0) = b$, etc.

The solution to an initial value problem no longer contains arbitrary constants from the general solution to the ODE — these are determined by the initial conditions of the problem at hand.

Example: A flow-meter in a pipeline measures flow-through as $2+\sin t$ litres/second. How much fluid passes through the pipeline from time zero up to time T?

flow thru is volume per unit of time (in litres)

IVP $\left[\ \frac{dV}{dt} = 2 + \sin(t) \qquad V(0) = 0 \ \text{IC}\right.$

$V(t) = \int 2 + \sin(t)\, dt = 2t - \cos t + c$

IC yields $0 = V(0) = 2(0) - \cos(0) + c = -1 + c, \ c = 1$ //

so $V(T) = 2T - \cos T + 1$

4.1.5 The order of an ODE

The order of the highest-order derivative in an ODE defines the order of the ODE.

Exercise: What is the order of each of these ODEs?

Order

- $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ — 1
- $m\ddot{x} = -kx$ — 2
- $x(y'')^2 + y'y''' + 4y^5 = yy'$ — 3
- $y' = xy$ — 1

first order

Observation: Order of an ODE determines the number of constants in general sol'n

4.1.6 Riding your bike at constant speed

Find the position of your bike (at time t) if you are travelling at a constant speed of 60 km/h along a perfectly straight road.

If $x = x(t)$ is the distance you have travelled at time t then the corresponding ODE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 60. \qquad (*)$$

We can directly integrate to find $x(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 60 \qquad \Rightarrow \qquad \int \mathrm{d}x = \int 60\,\mathrm{d}t.$$

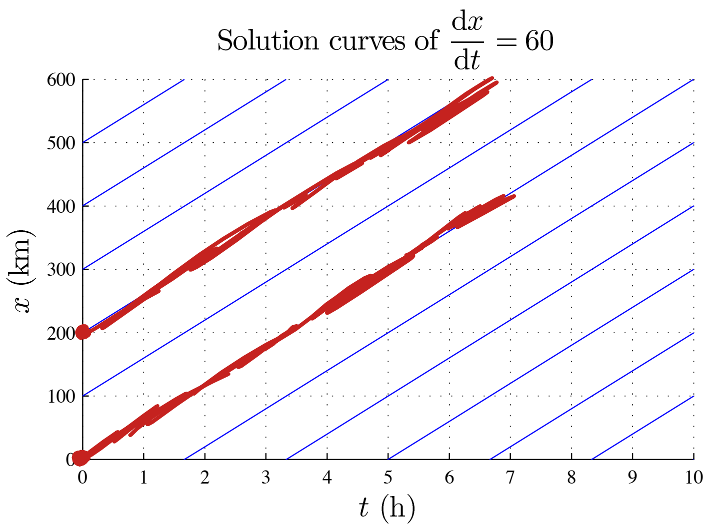
This yields the general solution to the ODE

$$x(t) = 60t + C.$$

$x(0) = C$

To determine the constant C we need more information, such as your initial position at time 0, in which case we would be solving an IVP.

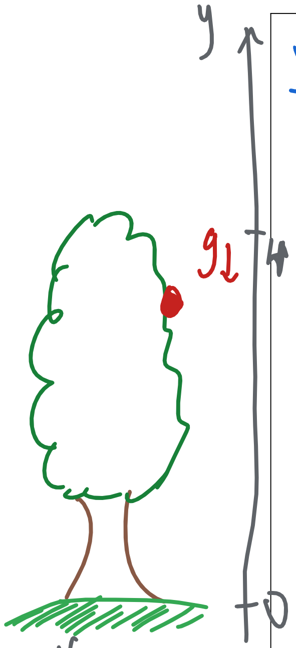
For different values of C we get different solutions, and below we graph some of these. If $C = 0$ then $y = 60t$. All the other solutions are parallel to this line.



The curves $y = 60t + C$ are called solution curves to the ODE $(*)$. Note that in this particular case all curves are straight lines with slope 60.

4.1.7 Motion of Projectiles

Example: Consider an apple falling under gravity. Find an expression for the height y of the apple at time t.



$y(t)$ vertical position of apple at time t

acceleration $y''(t)$

Newton's law $\cancel{m} y''(t) = -\cancel{m} g$

$y''(t) = -g$ SECOND ORDER DE

integrate $y'(t) = -gt + C$, $C \in \mathbb{R}$

$y(t) = -g\frac{t^2}{2} + Ct + D$, $D \in \mathbb{R}$

$y'(0) = C$ $\quad y'$

$y(0) = 0 + 0 + D$

IC

$y(0) = H$

$y'(0) = 0$ $\rightarrow$ $C = 0,\ D = H$

$y(t) = -\frac{g t^2}{2} + H$

Example: Now suppose you are throwing apples over your fence to your neighbour. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = 0, y(0) = 0$ and the initial velocity is $\dot{x}(0) = u$, $\dot{y}(0) = v$.

Horizontally no acceleration (g acts vertically)

$\ddot{x}(t) = 0$

$\dot{x}(t) = a$, IC: $\dot{x}(0) = u \Rightarrow a = u$

$x(t) = at + b$, IC: $x(0) = 0 + b \Rightarrow b = 0$

$x(t) = ut$

Vertically (as previous ex)

$\ddot{y} = -g$ ↳ $y(t) = -\frac{gt^2}{2} + ct + d$

$\dot{y}(0) = c \Rightarrow c = v$

$y(0) = d$ $\quad d = 0$

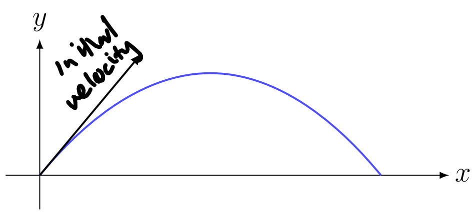
So $(x(t), y(t)) =$

$(ut, -gt^2/2 + vt)$

NOTE: $t = \frac{x}{u}$, $y = \frac{-g}{2}\left(\frac{x}{u}\right)^2 + v\left(\frac{x}{u}\right)$

$= -g\frac{x^2}{2u^2} + \frac{v}{u}x$

PARABOLA ∩



4.1.8 Extra reading: Realistic models

More realistic models of projectile motion deal with air resistance.

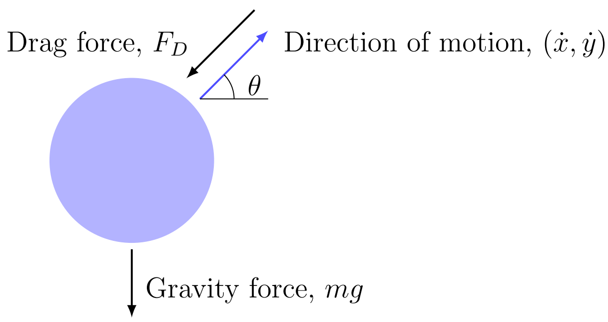
- For fast moving objects a good model includes air resistance proportional to square of the velocity
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -g - f\frac{\mathrm{d}y}{\mathrm{d}t}\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|.$$
- For slower objects, a good enough model has air resistance proportional to velocity.

Realistic models may also include the fact that gravity diminishes as you move away from the earth's surface (Newton's inverse square law).

Example: A model with air resistance

Consider the motion of a ball subject to air resistance proportional to the *square* of the speed. Note that the air resistance vector is always directed *against* the direction of motion.

Decompose the coordinates into x and y directions, and again apply Newton's law $F = ma$.



Note that

$$\sin\theta = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \qquad \text{and} \qquad \cos\theta = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}.$$

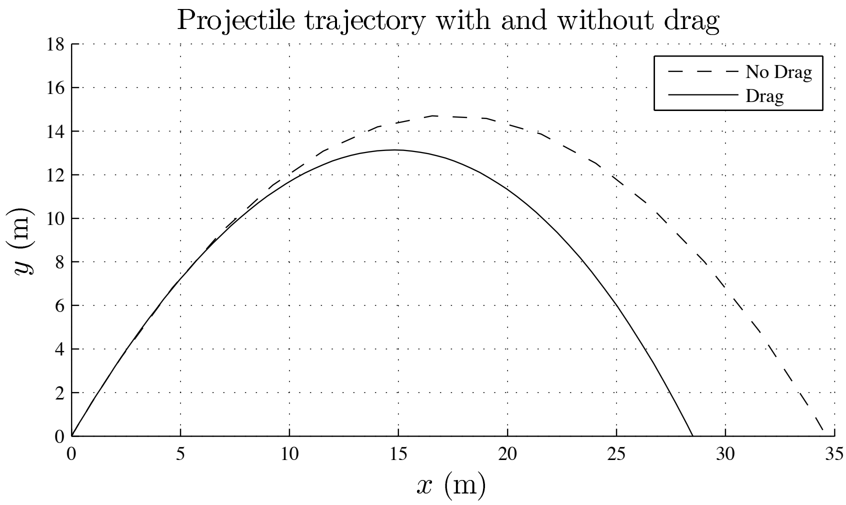
Hence $F = ma$ in the y direction gives

$$\begin{aligned}
& m\ddot{y} = -mg - F_D \sin\theta \\
\Rightarrow \quad & m\ddot{y} = -mg - kv^2 \sin\theta \\
\Rightarrow \quad & m\ddot{y} = -mg - k(\dot{x}^2+\dot{y}^2) \times \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\
\Rightarrow \quad & \ddot{y} = -g - \frac{k}{m}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}.
\end{aligned}$$

In much the same way

$$\ddot{x} = -\frac{k}{m}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}.$$

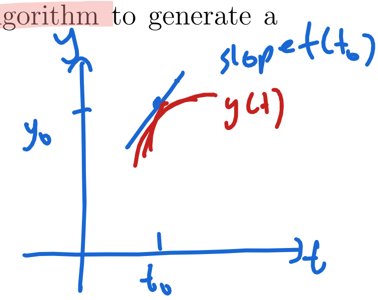
This is a coupled system of ODEs which is extremely difficult to solve analytically. However, numerical solutions are easy to obtain, and you can undertake this a little later in the semester. A sample trajectory in the xy-plane is shown here.



4.1.9 Analytical and numerical solutions

To solve an ODE (or IVP) analytically means to give a solution curve in terms of continuous functions defined over a specified interval, where the solution is obtained exactly by analytic means (e.g., by integration). The solution satisfies the ODE (and initial conditions) on direct substitution.

To solve an ODE (or IVP) numerically means to use an algorithm to generate a sequence of points which approximates a solution curve.



Important remark: As we have already seen, the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

can be solved analytically. The solution simply is

$$y = \int f(t)\,\mathrm{d}t.$$

Now this may seem like a cop-out because all we are saying is that the solution is given by the anti-derivative of $f(t)$, and the above solution is as informative as the actual ODE. In practice one hopes to be able perform the above integral to get a more explicit form of the solution. Depending on the form of f this may either be done exactly or by numerical means.

4.1.10 Main points

- You should understand what a differential equation (DE) is and how they can arise.
- You should understand what it means for a function to be a solution to an ODE.
- You should know what a general solution is.
- You should know what an initial value problem (IVP) is.
- You should be able to solve simple differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = g(x)$.
- You should understand what is meant by the order of a differential equation.
- You should be able to derive, using Newton's second law, equations of motion in a force field (such as gravity).
- You should understand the difference between an analytical and numerical solution to an ODE.

4.2 Slope Fields and Equilibrium Solutions

This material is covered in Stewart, Section 9.2 (Section 9.2).

We have seen that in order to solve

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

we only need to integrate. However, for the more general first-order ODE

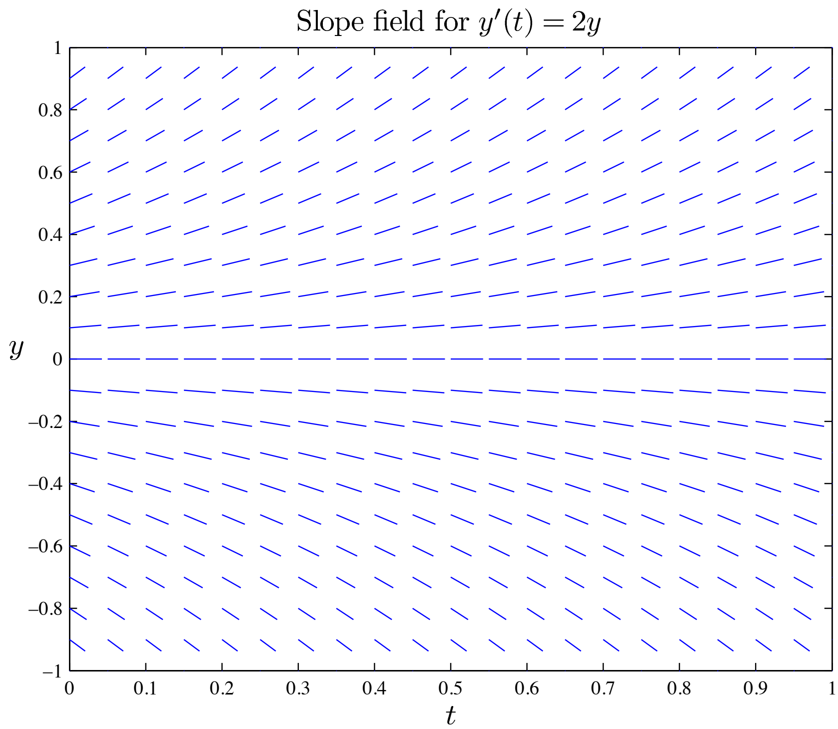
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$$

this no longer works. Nonetheless, the ODE gives a qualitative picture of the solution by noting that at $(t, y) = (a, b)$ the slope of $y(t)$ is $f(a, b)$. So what one can do is as follows:

- In the ty-plane, at $(t, y) = (a, b)$ draw a small straight line with slope $f(a, b)$.
- Repeat the process for many different values of (a, b).
- The resulting diagram is called the slope field.

Note the slope field can be generated without having to solve the ODE.

4.2.1 Example: the slope field of $y' = 2y$



From the slope field of $y' = 2y$ we can see that $y = 0$ is one of the solution curves. It is a constant or equilibrium solution.

4.2.2 Equilibrium solutions

An equilibrium solution is a constant solution $y(t) = c$ to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y).$$

The graph of an equilibrium solution is a horizontal line. Such a line has a slope of zero, i.e., $y' = 0$. This can only happen if $\underbrace{f(t, y) = 0}_{\text{for every } t}$ has a solution $y = c$ for some real constant c.

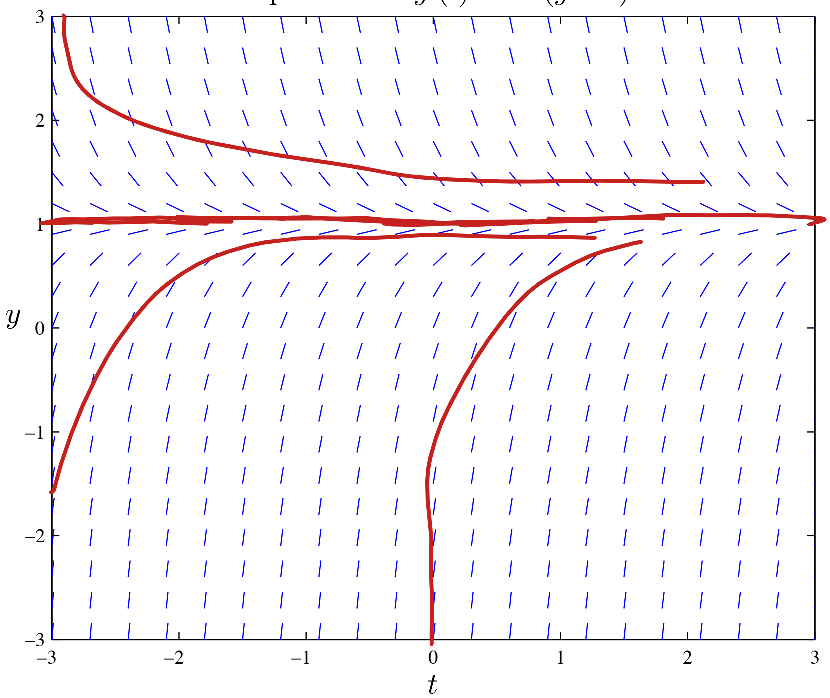
Example: Find the equilibrium solutions of $y'(t) = -3(y-1)$.

For equil. sol., need $y'(t)=0$ for all $t \in \mathbb{R}$

$-3(y-1)=0 \quad y=1$

$y(t)=1$ is the equil solution

Slope field for $y'(t) = -3(y-1)$



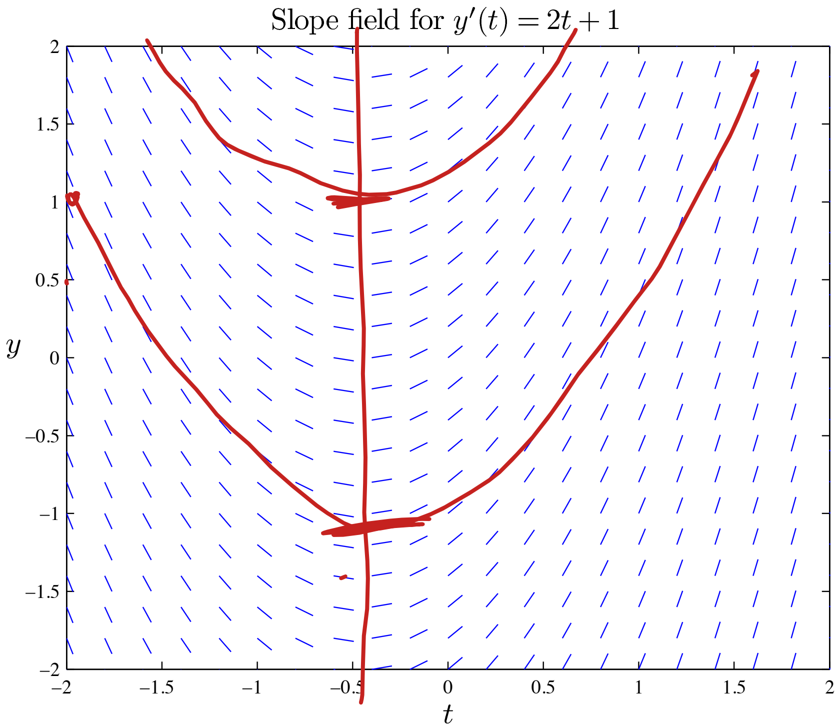
Example: Find the equilibrium solutions of $y'(t) = 2t + 1$.

Equil sol: $y'(t) = 0 \quad \forall\, t \in \mathbb{R}$

$2t+1=0$ can only happen for $t = -\frac{1}{2}$

So no equilibrium sol'n $y' = 2t+1$

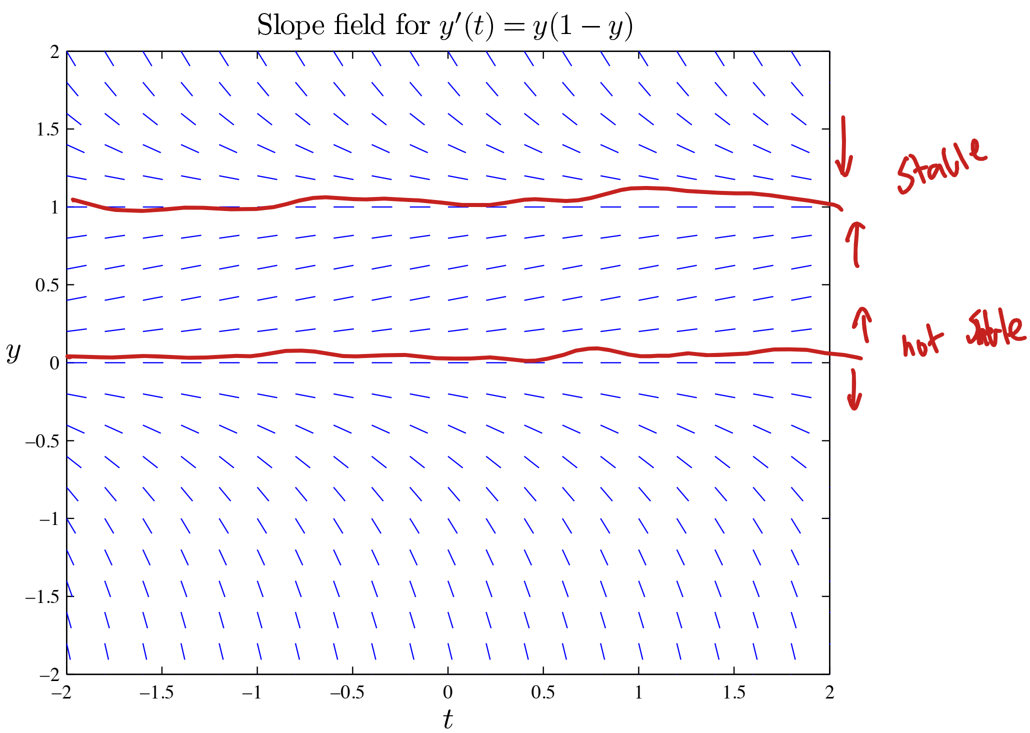
(Slope of a solution cannot be 0 away from $t = -\frac{1}{2}$)



Example: Find the equilibrium solutions of $y' = y(1-y)$.

$y' = 0$, $y(1-y) = 0$

$y = 0$, $y = 1$ two equil solutions



4.2.3 Stability of equilibrium solutions

A pencil sitting balanced vertically is in an equilibrium state. But make one small perturbation and it will topple over. This is an unstable equilibrium. On the other hand, a pendulum hanging vertically is also in an equilibrium state. But if you perturb it slightly, it will eventually (with friction) return to its equilibrium. This is a stable equilibrium.

From the slope field, you can decide if an equilibrium solution is stable or not by looking at whether solution curves will tend toward the equilibrium solution or away from it as time increases.

Formally, an equilibrium solution $y(t) = y_0$ to the differential equation $y' = f(t, y)$ is stable if the initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \qquad \underline{y(0)} = y_0 \pm \epsilon$$

IC near y_0

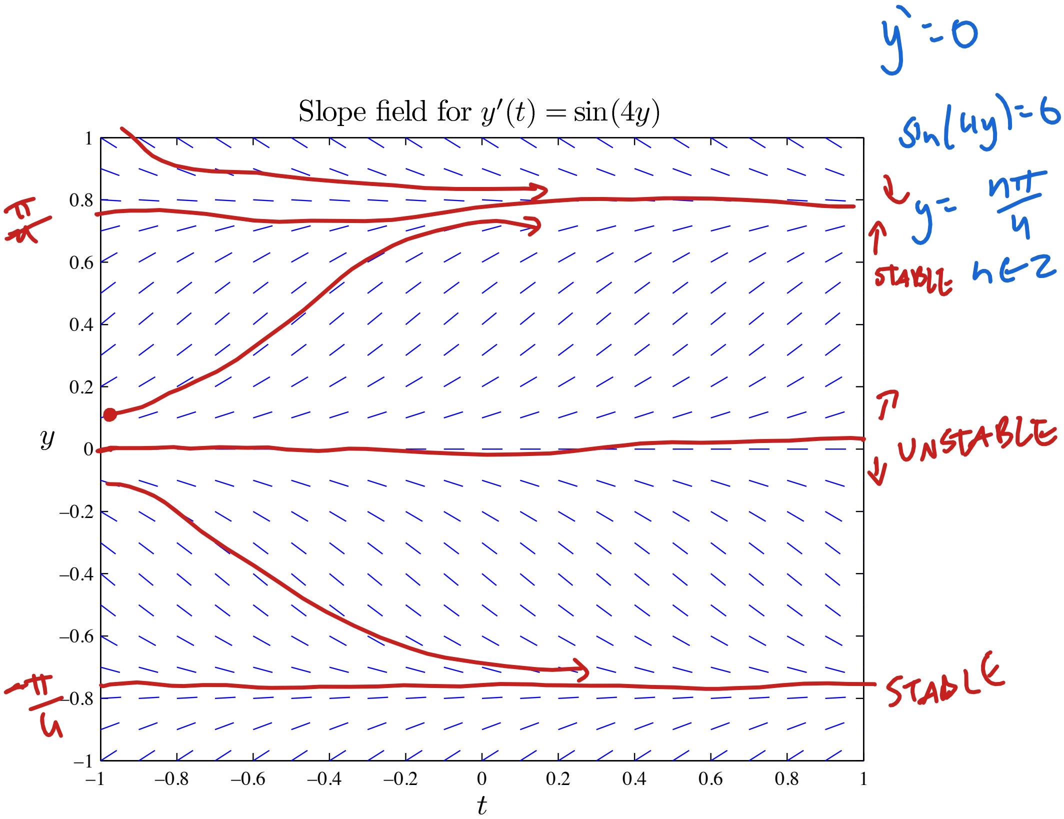
has a solution $y(t)$ which satisfies

$$\underline{\lim_{t \to \infty} y(t)} = y_0.$$

In other words, if you start sufficiently close to a stable equilibrium solution, then you will approach that equilibrium solution.

asymptically

Example: Here is the slope field for $y' = \sin(4y)$.



For this ODE, which equilibrium solutions are stable and which are unstable?

$\frac{\pi n}{2}$ UNSTABE $\frac{\pi n}{2} \pm \frac{\pi}{4}$ STABLE $n \in \mathbb{Z}$

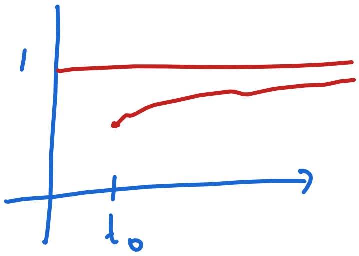
Important remark: It is an important point that **equilibrium solutions cannot be crossed** by other solution curves. In fact, no solutions curves can cross each other because this would mean that in some point(s) $y'(t)$ has more than one value. Equilibrium solutions therefore partition the solution space.

4.2.4 Long-term behaviour

Exercise: True or False? Assume the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3(y-1)$$

has initial condition $y(t_0) = c$ with $c < 1$. Then $y(t) < 1$ for all $t > t_0$.

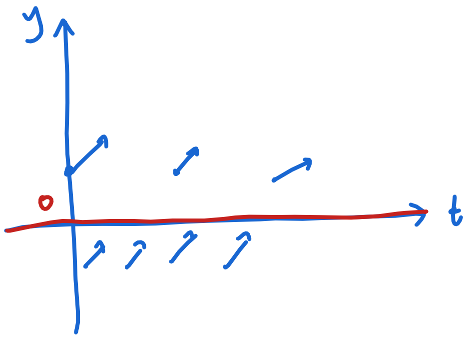


True bc graph of $y(t)$ cannot cross eq. sol $y=1$

Example: Consider the ODE $y' = y^2$. It has one equilibrium solution $y = 0$. stable?

For $y > 0$, $y' > 0$ so that $y(t)$ is increasing and hence any solution with $y > 0$ tends away from $y = 0$. not stable

For $y < 0$, $y' > 0$ so that $y(t)$ is increasing and hence any solution with $y < 0$ approaches $y = 0$.



"semi-stable"

4.2.5 Main points

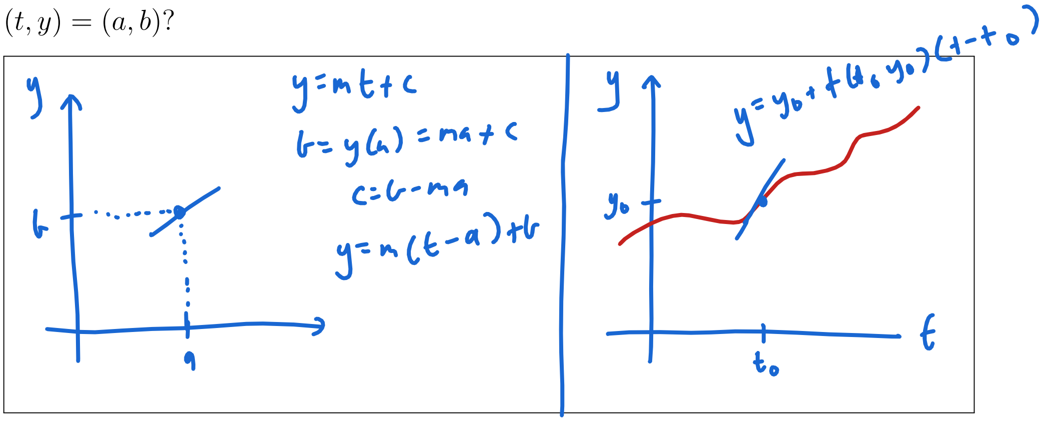
- You should understand how to generate and interpret slope fields.
- You should understand what is meant by an equilibrium solution and how to find one.
- You should be able to determine the stability of an equilibrium solution.
- You should understand that uniqueness of solutions to an IVP implies that solutions cannot cross.

4.3 Euler's Method for Solving ODEs Numerically

Stewart, Section 9.2 (Section 9.2) covers Euler's method. The method gives a simple approximate solution to an ODE and is closely related to the notion of a slope field.

4.3.1 Euler's method using tangent lines

What is the equation of the straight line with slope m that passes through the point $(t, y) = (a, b)$?



Euler's method uses tangent lines as approximations to solution curves. The tangent line to a solution curve of $y' = f(t, y)$ at (t_0, y_0) is

slope $y'(t_0) = f(t_0, y_0)$

$$\boxed{y = y_0 + \underbrace{f(t_0, y_0)}_{m}(t - t_0).}$$

This approximates the curve when t is close to t_0.

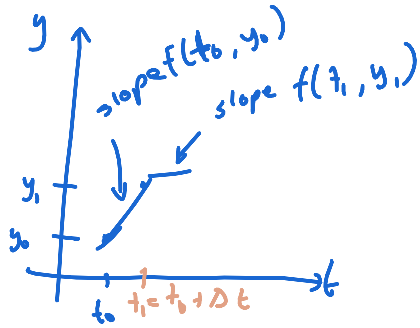
Now imagine a family of solution curves to the differential equation. We can calculate an approximate value for y at some later time by taking lots of small steps in time. At each step we will use the tangent line to the solution through our current point. This is Euler's method.

Using Δt as the step size for our algorithm, let

$$t_1 = t_0 + \Delta t, \quad t_2 = t_1 + \Delta t, \quad t_3 = t_2 + \Delta t, \; \ldots \;, t_n = t_{n-1} + \Delta t.$$

To find approximate y values at these times, use the following equations in sequence:

$$\begin{aligned} y_1 &= y_0 + f(t_0, y_0)\Delta t \\ y_2 &= y_1 + f(t_1, y_1)\Delta t \\ y_3 &= y_2 + f(t_2, y_2)\Delta t, \end{aligned}$$



and so on.

Example: Use Euler's method with $\Delta t = 0.2$ to find an approximate solution to $y(0.6)$ for the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \underbrace{2t}_{}^{f(t,y)}, \qquad y(0) = 0.$$

Compare your answer with the actual value.

$t_0 = 0$

$t_1 = t_0 + \Delta t = 0.2$

$t_2 = t_1 + \Delta t = 0.2 + 0.2 = 0.4$

$t_3 = t_2 + \Delta t = 0.6$

$y_0 = y(0) = 0$, $f(t_0, y_0) = f(0,0) = 2\cdot 0 = 0$

$y_1 = y_0 + f(t_0, y_0)\Delta t = 0 + 0(0.2) = 0$

$\hookrightarrow f(t_1, y_1) = f(0$

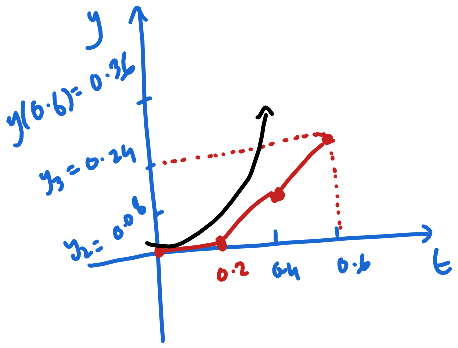
$y_2 = y_1 + f(t_1, y_1)\Delta t = 0 + 0.4(0.2) = 0.08$

$y_3 = y_2 + f(t_2, y_2)\Delta t = 0.08 + 0.8(0.2) = 0.24$

approx value of $y(0.6)$

of $y(0.6)$

$f(t_2, y_2) = 2t_2 = 2(0.4) = 0.8$



Analytic solution:

$y'(t) = 2t$

$y(t) = t^2 + C$

$y(0) = 0$, $C = 0$

$y(t) = t^2$ PARABOLA

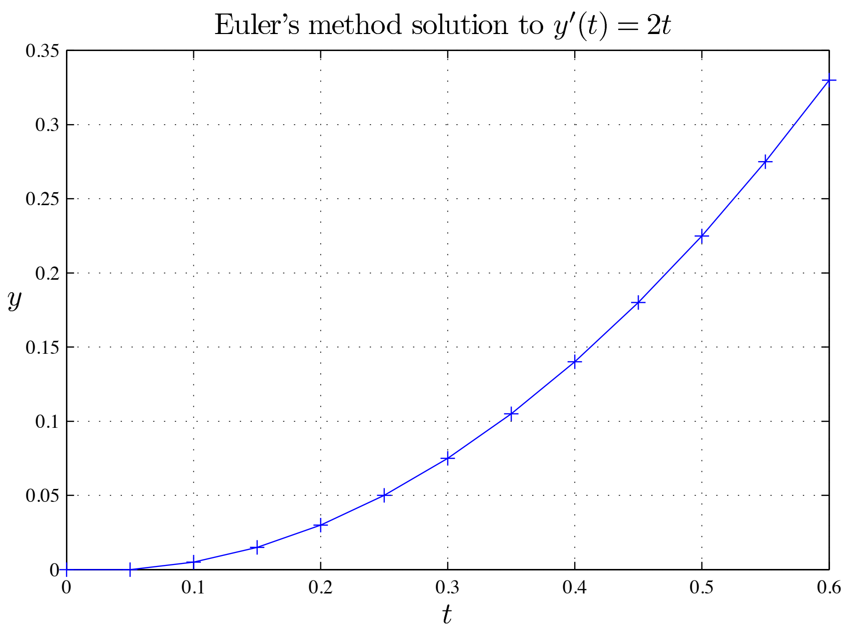
The method is only accurate if you make Δt small, which means a large number of steps is usually required. For this you might want to use Matlab. Define a t vector and an initial y value, then use the **for** command to iterate. In the example $\Delta t = 0.05$ and 12 steps have been chosen to take t to 0.6.

```
t=(0:0.05:0.6);
y(1)=0;
for i=1:12
    y(i+1)=y(i)+2*t(i)*0.05;
end
y(13)
plot (t,y,'-')
```

Using Matlab we find $y(0.6) = 0.33$ which is much much better than the previous estimate value for $y(0.6)$.

Note that we start with $y(1) = 0$, whereas in the theoretical work we write $y(0) = 0$. Matlab does not allow you to put an index of zero. Keep this in mind in case you ever get the Matlab error message:

```
???  Index into matrix is negative or zero.
```



4.3.2 Euler's method using Matlab; an example

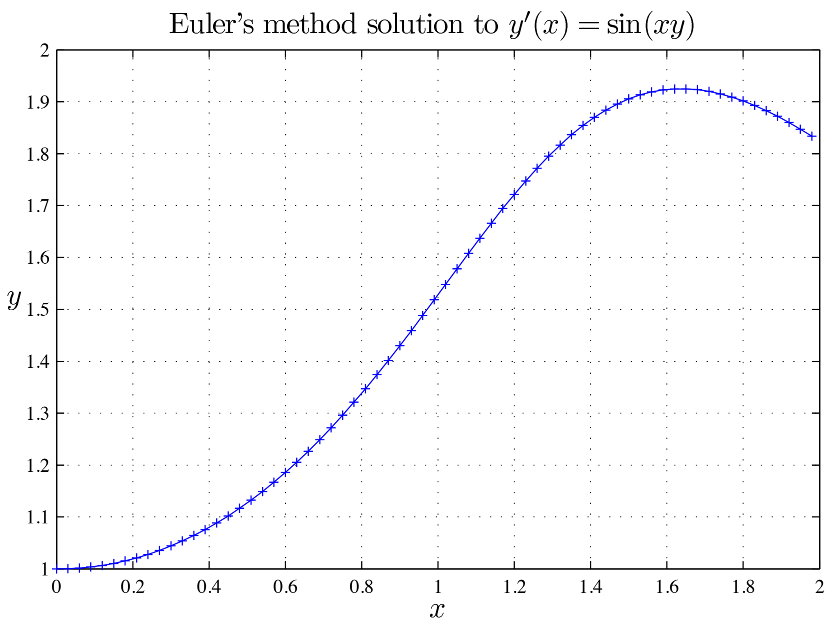
Consider the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy), \qquad y(0) = 1.$$

Matlab code to estimate $y(2)$, using Euler's method with step size $\Delta x = 0.01$ is:

```
x=(0:0.01:2);
y(1)=1;
for i=1:200
    y(i+1)=y(i)+sin(x(i)*y(i))*0.01;
end
y(201)
plot (x,y,'-')
```

Using Matlab we find $y(2) = 1.8243$.



4.3.3 Demonstrating errors in Euler's method

Use Euler's method to approximate the solution curve of the initial value problem

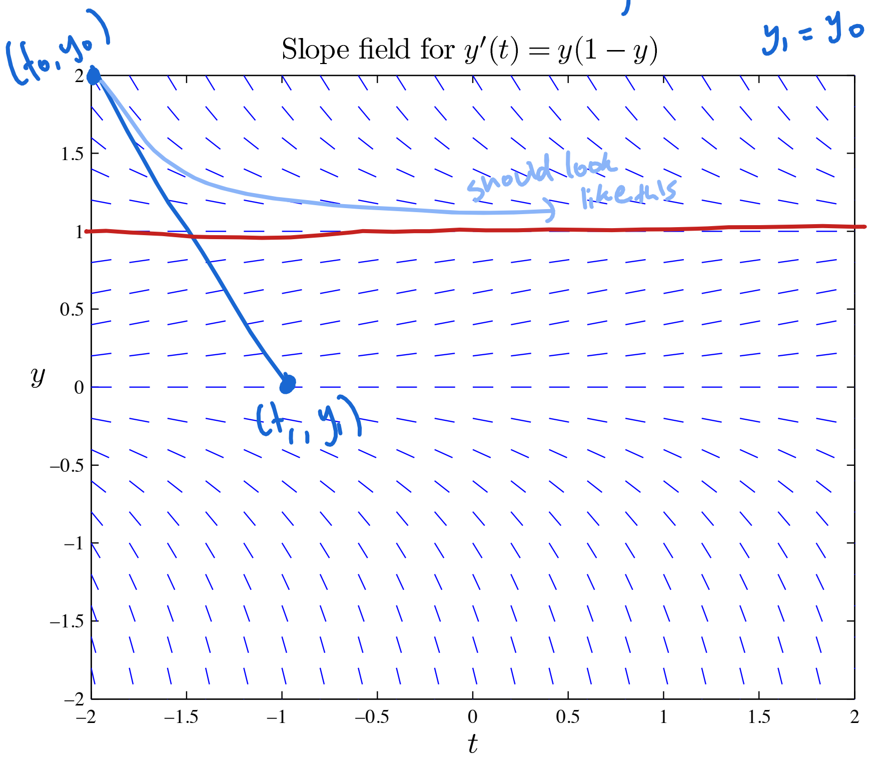
take $\Delta t = 1$

$$y' = y(1-y), \qquad y(\underset{t_0}{-2}) = \underset{y_0}{2}.$$

$f(t_0, y_0) = f(-2,2) = (2)(-1)$
$= -2$

Use the slope field below as a guide to draw in the approximate solution curve. What goes wrong in this case? $y_{n+1} = y_n + f(t_n, y_n)\Delta t$

$y_1 = y_0 + f(t_0, y_0)\Delta t$
$\Rightarrow y_1 = 2 + (-2) = 0$



approx solution crosses equil. sol! (bad ☹)

TAKE SMALLER Δt!

4.3.4 Main points

- You should understand how slope fields lead to Euler's method.
- You should know how to approximate a solution to an IVP using Euler's method by hand.
- You should know how to implement Euler's method in Matlab.
- You should understand that errors can be a problem in the approximate solution, especially if a large step size is used.

4.4 Separable ODEs

Separable first-order ODEs are one of several classes of ODE we will study in MATH1052. It is very important that you become skilled at identifying and solving this type of ODE. See also Stewart, Section 9.3 (Section 9.3).

4.4.1 Definition

A first-order ODE is called separable if it can be written in the form

$\frac{dy}{dt} = f(t,y)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

↳ product form

Exercise: Which of the following ODEs are separable?

SEPARABLE?

- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)$, ✓ $f(x)=1 \quad g(y)=y(1-y)$
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x+y}$, ✓ $f(x)=e^x \quad g(y)=e^y$
- $y' = \mathrm{e}^{(x+y)^2}$, ✗
- $\dot{y} = \dfrac{ty+y}{t^2}$, $= \dfrac{y(t+1)}{t^2}$ ✓ $f(t)=\dfrac{t+1}{t^2} \quad g(y)=y$
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = ty + y^2$. $= y(t+y)$ ✗

4.4.2 Solving separable ODEs

$\frac{dy}{dx} = f(x)g(y)$, $y = y(x)$

$= \int \frac{1}{g(y(x))} y'(x)\, dx = \int \frac{1}{g(y)}\, du$

↑ substitution

1. Rewrite the equation as $\dfrac{1}{g(y)}\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.
2. Integrate both sides with respect to x: $\displaystyle\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x$.
3. Note the integral on the left is a "substitution", so that we can replace the last equation by $\displaystyle\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x$. ← Integration problem
4. If one is lucky (or very smart, or both) one or both of the integrals can actually be evaluated.
5. If one is even more lucky (or even smarter) one can then finally explicitly express y as a function of x.

Important remark: The final two steps will not always be possible. Whenever you are asked to solve a separable ODE or IVP in this course you are supposed to always "go as far as possible". For good reasons we assume you are infinitely smart; not being able to compute simple integrals is never a valid excuse.

For the IVP $y'(x) = f(x)g(y)$, $y(x_0) = y_0$, we again might have the problem that we can't evaluate one of both of the integrals needed to solve the separable ODE. In this case, we can still write down an actual solution, be it a rather implicit one. Indeed, the solution to the IVP is given by

$$\boxed{\int_{y_0}^{y} \frac{\mathrm{d}t}{g(t)} = \int_{x_0}^{x} f(t)\,\mathrm{d}t.}$$

definite integrals

Example: Solve the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$. $= x\left(\frac{1}{y}\right)$ $\quad f(x)=x \quad g(y)=y$

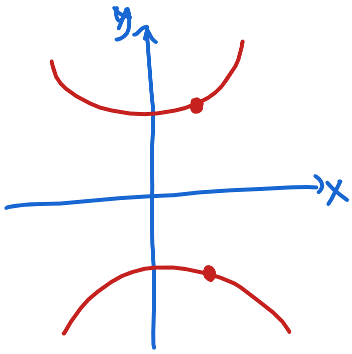
$y \cdot \frac{dy}{dx} = x$

$\int y \, dy = \int x \, dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C, \ C \in \mathbb{R}$

$y = \pm\sqrt{x^2+2C}$

sign determined by IC



The solution we found for the above example contains an arbitrary constant so is the general solution. In the case of an IVP this constant will be fixed to find one, particular solution.

Example: Solve the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}, \quad y(0) = 3.$

$y = \pm\sqrt{x^2+2c}$ $\qquad$ bc $y(0)>0$ $y=\sqrt{x^2+2c}$

$y(0) = \sqrt{2c} = 3 \Rightarrow 2c = 9$

~~y=~~ sol $y(x) = \sqrt{x^2+9}$

$f(x) = \sin x \quad g(y) = \frac{1}{y}$

ODE is separable

Example: Solve the IVP $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sin x}{y}, \quad y(0) = 1.$

$y \dfrac{dy}{dx} = \sin x$

$\int y \, dy = \int \sin x \, dx$

$\dfrac{y^2}{2} = -\cos x + c, \; c \in \mathbb{R}$

$y = \pm \sqrt{-2\cos x + 2c}$

$y(0) > 0 \quad \Rightarrow \quad y = \sqrt{-2\cos x + 2c}$

$1 = y(0) = \sqrt{-2 + 2c}$

$1 = -2 + 2c$

$2c = 3$

Sol. $y(x) = \sqrt{-2\cos x + 3}$

4.4.3 Singular solutions

$\frac{dy}{dx} = f(x)g(y)$

In step 1 of our recipe for solving separable ODEs we rewrote $y'(x) = f(x)g(y)$ as

$$\frac{1}{g(y)} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

Since we can't divide by zero, this means that our method is only valid provided $g(y) \neq 0$.

If there is an a such that $g(a) = 0$ then the ODE will have the equilibrium solution $y(x) = a$. This is easy to check:

If $y(x)=a$ then $\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = f(x)g(y)$

and $f(x)g(y) = f(x)g(a) = 0$

so $y(x)=a$ is an equilibrium solution

Such a solution is *also* known as a singular solution because it won't generally arise from our earlier recipe. Whenever you are asked to solve a separable ODE you will have to check for singular/equilibrium solutions.

Example: Solve the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)$, $y(0) = y_0$. This corresponds to the logistic equation.

$f(t)=1$ $g(y)=y(1-y)$ so ODE is separable

- Since $g(0)=g(1)=0$ then $y(t)=0$ and $y(t)=1$ are equilibrium solutions.
- For $y \neq 0$ and $y \neq 1$ $\frac{1}{y(1-y)} \frac{dy}{dt} = 1$

$\int \frac{dy}{y(1-y)} = \int dt$ $\left[\frac{1}{y} + \frac{1}{1-y} = \frac{1-y+y}{y(1-y)} = \frac{1}{y(1-y)}\right]$ PARTIAL FRACTIONS

$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int 1 \, dt$

$\ln|y| - \ln|1-y| = t + C$, $C \in \mathbb{R}$ $\Rightarrow \ln\left|\frac{y}{1-y}\right| = t + C$

$\left|\frac{y}{1-y}\right| = e^{t+C} = e^t e^C$

(*) $\frac{y}{1-y} = Ae^t$ $(A = \pm e^C)$

$$y = (1-y)Ae^t = Ae^t - yAe^t$$

$$(1+Ae^t)\,y = Ae^t \qquad \text{(general sol)}$$

$$y = \frac{Ae^t}{1+Ae^t}$$

If $y(0) = y_0$ (*) $\dfrac{y_0}{1-y_0} = Ae^0 = A$

Solution to IVP is

$$y(t) = \frac{\frac{y_0}{1-y_0}e^t}{1+\frac{y_0}{1-y_0}e^t} = \frac{y_0 e^t}{1-y_0+y_0e^t}$$

The logistic equation has many uses in population dynamics, physical science and economics.

seperable

Example: Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$. $\quad f(x)=1 \quad g(y)=y^2$

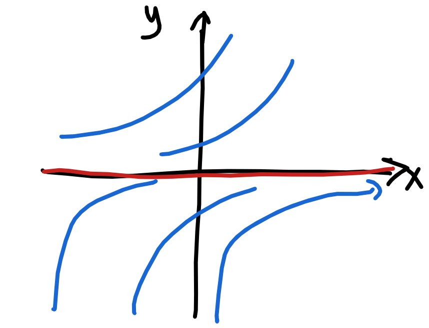
Eq. sols $y=0$

for $y \neq 0$, $\frac{1}{y^2}\frac{dy}{dx} = 1$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C \quad C \in \mathbb{R}$$

$$y = \frac{-1}{x+C}$$



$y=0$
Singular solution

4.4.4 Main points

- You should be able to identify a first-order separable ODE.
- You should know how to solve a separable ODE.
- You should understand that equilibrium and singular solutions are equivalent and must be checked for when solving a separable ODE.

4.5 Applications: Law of Cooling, Population Growth

Here we look at using some simple ODEs to model cooling and population dynamics. See Stewart Section 9.4 (Section 9.4).

4.5.1 Newton's law of cooling

Newton's law of cooling states that the rate at which a "body" cools is proportional to the temperature difference between the body and its surrounding medium.

If T is the temperature of the body and T_m the temperature of the surrounding medium then, according to Newton,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_m), \qquad k > 0.$$

Here the constant k is chosen such that if $T > T_m$, $T'(t)$ is negative, describing cooling.

Example: Solve this ODE with the intial condition $T(0) = T_0$.

ODE is seperable $f(t) = -k$ $g(T) = T - T_m$

equil sol: $\frac{dT}{dt} = 0 \Rightarrow T = T_m$

For $T \neq T_m$

$$\int \frac{dT}{T - T_m} = \int -k \, dt$$

$$\ln|T - T_m| = -kt + C \quad C \in \mathbb{R}$$

$$|T - T_m| = e^{-kt} e^{C} \Rightarrow T - T_m = Ae^{-kt}, \; A \in \mathbb{R}$$

$T = T_m + Ae^{-kt}$ general solution

IC: if $T(0) = T_0$ then $T_0 = T(0) = T_m + A \Rightarrow A = T_0 - T_m$

and $T(t) = T_m + (T_0 - T_m)e^{-kt}$

In real-life problems the cooling constant k is usually not known and you will have to infer it from measurements.

Example: CSI Victoria Carl W.'s body was discovered in his cell at Barwon Prison, Victoria on the 19th of April 2010. The coroner's report stated that:

- C.W.'s body discovered at 11am on 19 April 2010.
- Body temperature at 11am: 34.8°C.
- Body temperature at 12.30pm: 34.1°C.
- Temperature in C.W.'s cell: 21.1°C.

Use the above information to determine the time of Carl's death, assuming his body temperature at the time of death was 37°C.

Newton's law of cooling:

$T(t) = T_m + (T_0 - T_m)e^{-kt}$

Let t be the time since 11am (in hours)

Then $T_0 = 34.8$ so

$T(t) = 21.1 + (34.8 - 21.1)e^{-kt}$

$= 21.1 + 13.7e^{-kt}$ what's k?

Know $T(1.5) = 34.1$ so

$34.1 = 21.1 + 13.7e^{-k(1.5)}$

$\frac{13}{13.7} = e^{-k(1.5)} \Rightarrow k = \frac{-\ln\left(\frac{13}{13.7}\right)}{1.5}$ $k \sim 0.035$

So $T(t) = 21.1 + 13.7e^{-0.035t}$

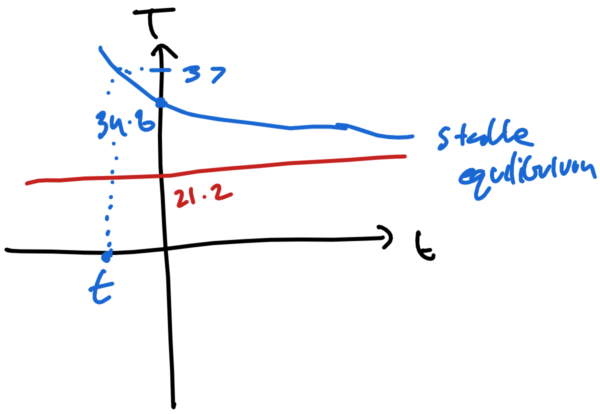
Time of death is given by 11am + t

$$37 = T(t) = 21.1 + 13.7e^{-0.035t}$$

$$t = \ln\left(\frac{37 - 21.1}{13.7}\right)\left(\frac{-1}{0.035}\right)$$

$$t \sim -4.255$$

(about 4h 15min before 11am ie 6:45am)



4.5.2 Malthus: "...doomed to misery and vice"

t in years

Without inhibiting factors the rate at which the human population grows is proportional to its existing population; if P is the population at time t then

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP,$$

separable IC : $P(2000)=6$

where r is the growth rate (per human per year, say).

The world population is currently growing at a rate of approximately $r = 0.0125$. If we take as initial condition the year 2000 with roughly 6 billion people on the planet, and ignore global warming so that we may take the total area of land on Earth to be constant at 150 million square kilometres, find in which year each human will have, on average, one square metre of land left to stand on.

$\frac{dP}{dt} = 0.0125P$ separable (equilibrium sol: $P = 0$)

$\int \frac{dP}{dt} = \int 0.0125\,dt \Rightarrow \ln|P| = 0.0125t + C$

$\Rightarrow P = Ae^{0.0125t}$ $\quad A = e^{C}$

IC: $6 = P(2000) = Ae^{(0.0125)(2000)} = Ae^{25}$

$1\,km^2 = 10^6\,m^2$

$A = 6e^{-25} \Rightarrow p(t) = 6e^{0.0125t - 25}$

want to find year when there will be $(150{,}000{,}000$ square km$)(10^6)$

$= 150\,000$ billion people

solve for t

$150\,000 = 6e^{0.0125t - 25}$

$\ln(25\,000) = 0.0125t - 25$

$t = \frac{\ln(25\,000) + 25}{0.0125}$ $\sim \boxed{\text{YEAR } 2810}$

4.5.3 Verhulst to the rescue; down with the Malthusian law

The problem with the Malthusian population model is that it does not contain a "damping" factor reflecting issues such as over-population, limited food or water supply, etc. The logistic model of Verhulst is the simplest model that has such built-in damping:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1 - \frac{P}{\theta}\Big),$$

where θ is some very large constant, known as the carrying capacity.

Note that when P is small(ish) then P/θ is small and we approximately have

$$\frac{\mathrm{d}P}{\mathrm{d}t} \approx rP$$

as before. When the population becomes close to the carrying capacity (i.e., P/θ very close to 1 so that not just $1 - P/\theta \approx 0$ but also $P(1 - P/\theta) \approx 0$) then growth comes almost to a standstill.

If we introduce new variables $y = P/\theta$ and $\tau = rt$ then the ODE becomes

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y(1 - y).$$

We have already seen that if $y(0) = y_0$ then this has solution

$$y = \frac{y_0\,\mathrm{e}^{\tau}}{1 - y_0 + y_0\,\mathrm{e}^{\tau}}.$$

If $P(0) = P_0$ in our original variables then we have

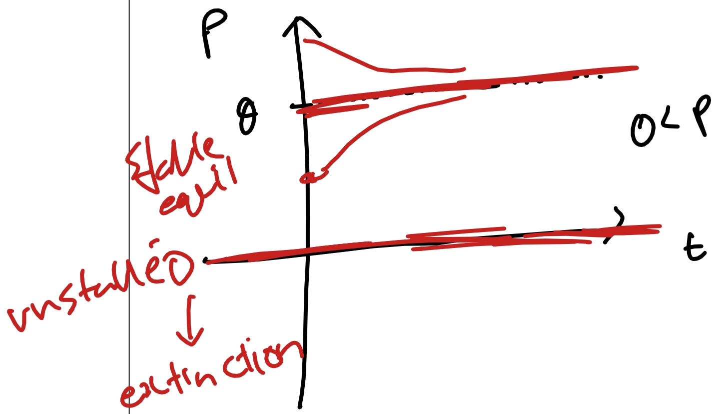
$$P = \frac{\theta P_0\,\mathrm{e}^{rt}}{\theta - P_0 + P_0\,\mathrm{e}^{rt}}.$$

Example: Identify the equilibrium (or singular) solutions to the logistic population model, and interpret these solutions in terms of population growth. Include a diagram.

Equilibrium sol'si $\frac{dP}{dt}=0$

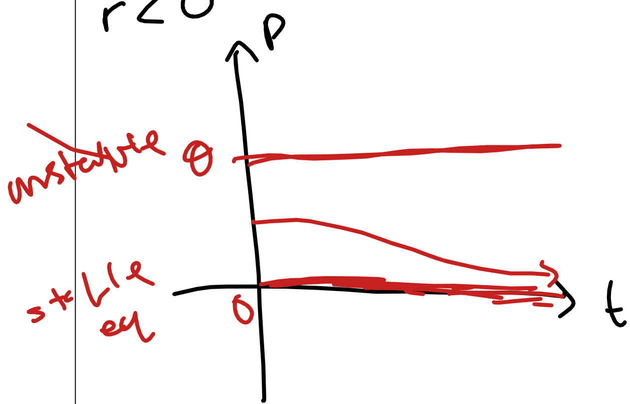
$rP\left(1-\frac{P}{\theta}\right)=0 \Rightarrow P=0 \qquad 1-\frac{P}{\theta}=0 \Rightarrow P=\theta$

$r>0$



$0<P<\theta \qquad P>0 \qquad 1-\frac{P}{\theta}>0 \qquad \frac{dP}{dt}>0$

$r<0$



Example: The Pacific Halibut Fishery uses a logistic model plus an extra term to take into account their harvesting, which is at a rate proportional to the existing population.

If E is the constant of proportionality for harvesting, their model is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\Big(1 - \frac{P}{\theta}\Big) -.EP.$$

What restriction must be put on E so that the population will not die out?

$$\frac{dP}{dt} = rP\left(1-\frac{P}{\theta}\right) - EP$$

$$= (r-E)P - \frac{rP^2}{\theta} = (r-E)P\left(1 - \frac{r}{(r-E)\theta}P\right)$$

$$= r'P\left(1 - \frac{P}{\theta'}\right) \qquad r' = r - E$$

$$\theta' = \frac{(r-E)\theta}{r} = \left(1 - \frac{E}{r}\right)\theta$$

logistic model

θ' carrying capacity

stable equilibrium provided $r' > 0$ (ie $r > E$)

In this case $\theta' = \left(1 - \frac{E}{r}\right)\theta > 0$

For population to persist, the harvesting rate E should be smaller than natural growth (otherwise the population will eventually become extinct).

4.5.4 Extra reading: numerically solving ODEs

Matlab has some more sophisticated numerical methods for solving ODEs than Euler's method. Matlab's **ode45** uses a Runge–Kutta fourth-order integration technique. To use it you need to write two little programs. The first one, say logistic.m, defines the differential equation

```
function dy=logistic(t,y)
dy=y*(1-y);
```

The second program will numerically solve the ODE and plot your solution.

```
y0=0.1;
tfinal =6;
[t,y]= ode45 ('logistic', tfinal, y0);
plot (t,y)
```

4.5.5 Main points

- You should be able to apply Newton's law of cooling and solve the resulting IVP.
- You should know how to model population growth and solve the resulting IVP.
- You should be able to interpret the obtained solutions.

4.6 Solving Linear First Order Equations

If you cannot solve a differential equation by separation, your next plan of attack should be to test if it is linear and use the integration factor method described below. Further exposition and examples are given in Stewart Section 9.5 (Section 9.5).

4.6.1 Definition

$f(y) = my + b$

A first-order ODE is linear if it can be put in the form

$y' + p(x)y = q(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)$$

for some functions $p(t)$ and $q(t)$. do not depend on y

Exercise: Which of the following are linear?

- ✓ $y' = x$ $\quad \frac{dy}{dx} = x \quad p(x)=0 \quad q(x)=x$ ✓
- ✓ $y' = \log x$ $\quad p(x)=0 \quad q(x)=\log x$
- ✗ $y' = y^2$ quadratic term
- ✓ $y' = y$ $\quad y' - y = 0 \quad p(x)=-1 \quad q(x)=0$
- ✗ $y'y = 1$
- ✓ $y' = y + 3\sin x$
- ✓ $y' = t^5 - t^2y$ $\quad \frac{dy}{dt} + t^2 y = t^5 \quad p(t)=t^2, \ q(t)=t^5$
- ✗ $y' = \sin y + \mathrm{e}^x$
- ✗ $(y')^2 = y$

Recall the product rule for differentiation (written here in reverse order):

$$f\,\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}\,g = \frac{\mathrm{d}}{\mathrm{d}t}(f\,g).$$

This formula is key to solving first-order linear ODEs.

Example: Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \Big(\dfrac{2t}{t^2+1}\Big)y = t^2 + 1.$

- Multiply ODE by (t^2+1)

$$(t^2+1)\frac{dy}{dt} + 2ty = (t^2+1)^2$$

$$\frac{d}{dt}\left((t^2+1)y\right) = t^4 - 2t^2 + 1$$

integrate wrt t

$$(t^2+1)y = \int t^4 + 2t^2 + 1\,dt$$

$$= \frac{t^5}{5} + \frac{2}{3}t^3 + t + C \quad C \in \mathbb{R}$$

general sol: $y = \dfrac{1}{t^2+1}\left(\dfrac{t^5}{5} + \dfrac{2}{3}t^3 + t + C\right)$

The lucky thing (it wasn't luck really) about the previous example was that after multiplication by t^2+1, the equation exactly matched the product rule. This raises the following natural question.

Question: Can we always multiply a first-order linear ODE by some function $I(t)$, so that its left-hand side matches the left-hand side of the product rule? Yes!

This question can be answered in the affirmative and any function $I(t)$ that does the trick is known as an integrating factor.

Given the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)$$

we multiply by the yet-to-be-found integrating factor $I(t)$:

$I(t)\frac{dy}{dt} + I'(t)y = \frac{d}{dt}(Iy) \overset{?}{=}$

$$I(t)\frac{\mathrm{d}y}{\mathrm{d}t} + I(t)p(t)y = I(t)q(t). \qquad (*)$$

We think of I as g and y as f. Then the first term on the left is $f'g$ and consequently we want the second term on the left to be $fg' = yI'$. Given that this second term is in fact Ipy, we infer that I must satisfy the ODE

$$I'(t) = I(t)p(t).$$

This is good news since this is a separable ODE which we already know how to solve:

$\frac{I'}{I} = p$

$\ln|I| = \int p(t)\,dt$

$$I(t) = \exp\Big(\int p(t)\,\mathrm{d}t\Big).$$

Important remark: Any integrating factor will do. Hence when you try to solve $I' = Ip$, i.e., when you try to compute the above integral (assuming you can explicitly do so), you do not need to worry about keeping track of constants. For example, if $p(t) = 2t$, you would get

↳ eg $C=0$

$$I(t) = \exp\Big(2\int t\,\mathrm{d}t\Big) = \mathrm{e}^{t^2+C} = \mathrm{e}^C\mathrm{e}^{t^2} = B\mathrm{e}^{t^2},$$

where $B = \exp(C) \neq 0$. When you substitute this in $(*)$ the constant B would simply be an overall factor which may thus be divided out. It is therefore perfectly okay to write on your assignment: `Because constants of integration do not matter,`

$$I(t) = \exp\Big(2\int t\,\mathrm{d}t\Big) = \mathrm{e}^{t^2}.$$

To confuse your tutor, you might even write: `Because constants of integration do not matter,`

$$I(t) = \exp\Big(2\int t\,\mathrm{d}t\Big) = \mathrm{e}^{t^2+65\pi^{123}}.$$

SOLVING FIRST ORDER LINEAR ODE

Summary

To solve a first-order linear ODE:

standard

1. Write the equation in the form $\dfrac{\mathrm{d}y}{\mathrm{d}t} + py = q$.
2. Find an integrating factor $I = \exp\left(\int p\,\mathrm{d}t\right)$. $I = e^{\int p(x)\,dx}$
3. Multiply the ODE by I and apply the product rule to get $\dfrac{\mathrm{d}}{\mathrm{d}t}(Iy) = Iq$.
4. Integrate both sides with respect to t. (This time, don't forget about constants of integration!).

Example: Solve the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y + t$.

① $\dfrac{dy}{dt} - y = t$

② $I(t) = e^{\int p(t)dt} = e^{\int(-1)dt} = e^{-t+C}$

③ Multiply ODE by $I(t)$:

$$e^{t}\left(e^{-t}\frac{dy}{dt} - e^{-t}y\right) = \left(e^{-t}t\right)e^{t}$$

$$\frac{d}{dt}\left(e^{-t}y\right) = e^{-t}t$$

④ Integrate (PARTS)

$$e^{-t}y = \int \underbrace{e^{-t}}_{v'}\,\underbrace{t}_{u}\,dt = uv - \int vu'$$

$$= te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t} + C \quad C \in \mathbb{R}$$

General sol.

$$y = e^{t}\left(-te^{-t} - e^{-t} + C\right)$$

$$y = -t - 1 + Ce^{t}, \quad C \in \mathbb{R}$$

Example: Solve the IVP $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = 1, \quad y(1) = 2.$

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^2} \quad (x \neq 0)$ First order linear ODE

2. $I(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x| + C}$ ← Choose C = constant = 0 for I.F

$= x \quad$ (bc IC has $x > 0$)

3. $x \dfrac{dy}{dx} + y = \dfrac{1}{x}$

so $\dfrac{d}{dx}(xy) = \dfrac{1}{x}$

4. Integrate

$xy = \int \frac{1}{x}\, dx = \ln|x| + C, C \in R$

$y = \dfrac{\ln|x| + C}{x}$

$\dfrac{IC}{2} = y(1) = \dfrac{\ln|1| + C}{1} = C \Rightarrow C = 2$

solution:

$y(x) = \dfrac{\ln|x| + 2}{x}$

Example: Solve the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} - x^2 y = x^2$.

- Integrating factor

$$I(x) = e^{\int -x^2\,dx} = e^{-\frac{x^3}{3} + c} \leftarrow \text{choose } c = 0$$

- multiply both sides by $e^{-x^3/3}$ & use prod rule

$$\frac{d}{dx}\left(e^{-x^3/3} y\right) = e^{-x^3/3} x^2$$

- Integrate

SUBSTITUTION $u = -\frac{1}{3}x^3 \quad du = -x^2\,dx$

$$e^{-x^3/3} y = \int e^{-x^3/3} x^2\,dx = -\int e^u\,du = -e^u + C$$

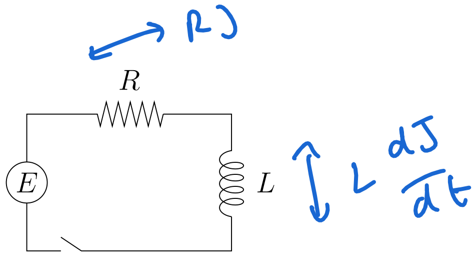
$$= -e^{-\frac{1}{3}x^3} + C, \quad C \in \mathbb{R}$$

- general solution

$$y(x) = -1 + Ce^{x^3/3}$$

4.6.2 Worked-out example: RL circuit

The ODE describing the current in a circuit with a resistor and an inductor (known as an RL circuit) turns out to be linear. To obtain this equation consider the change in voltage around the circuit. The voltage drop due to a resistor is RJ where R is the resistance (in Ohms) and J is the current (in Amperes).



The voltage drop due to the inductor is $L\dfrac{\mathrm{d}J}{\mathrm{d}t}$ where L is the inductance (in Henries).

According to Kirchhoff's Law the sum of the voltage drops is equal to the supplied voltage $E(t)$ (in Volts). Hence

$$L\frac{\mathrm{d}J}{\mathrm{d}t} + RJ = E(t). \qquad (*)$$

Example: Suppose in an RL circuit a battery supplies a constant voltage of 80V, the inductance is 2H and the resistance is 10Ω. (i) Find an expression for $J(t)$, and (ii) determine the the current after 1 second if $J(0) = 0$.

Solution: We first put $(*)$ into standard form:

$$\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{RJ}{L} = \frac{E(t)}{L}.$$

An integrating factor for this ODE is

$$I(t) = \exp(R/L\int \mathrm{d}t) = \exp(Rt/L).$$

Multiplying the ODE by this factor and using the product rule gives

$$\mathrm{e}^{Rt/L}\frac{\mathrm{d}J}{\mathrm{d}t} + \mathrm{e}^{Rt/L}\frac{RJ}{L} = \frac{E(t)}{L}\mathrm{e}^{Rt/L} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{Rt/L}J\right) = \frac{E(t)}{L}\mathrm{e}^{Rt/L}$$

Integrating the previous equation leads to

$$\mathrm{e}^{Rt/L}J = \frac{1}{L}\int E(t)\mathrm{e}^{Rt/L}\mathrm{d}t \quad \Rightarrow \quad J(t) = \frac{1}{L}\mathrm{e}^{-Rt/L}\int E(t)\mathrm{e}^{Rt/L}\mathrm{d}t.$$

So far we have treated everything completely generally. Now we use that the battery supplies constant voltage, i.e., E does not depend on t. This yields

$$J(t) = \frac{E}{L}\mathrm{e}^{-Rt/L}\int \mathrm{e}^{Rt/L}\mathrm{d}t \quad \Rightarrow \quad J(t) = \frac{E}{L}\mathrm{e}^{-Rt/L}\Big(\frac{L}{R}\mathrm{e}^{Rt/L} + C\Big),$$

so that we finally obtain

$$J(t) = \frac{E}{R} + D\mathrm{e}^{-Rt/L}.$$

$D = \frac{E}{L}c$

Since $J(0) = 0$, the constant $D = -E/R$:

$$J(t) = \frac{E}{R}\Big(1 - \mathrm{e}^{-Rt/L}\Big).$$

$0 = J(0) = \frac{E}{R} + D$

Since $E = 80$, $R = 10$ and $L = 2$:

$$J(t) = 8\Big(1 - \mathrm{e}^{-5t}\Big).$$

We can finally determine $J(1)$ as $J(1) = 8(1 - \exp(-5))$ Amperes.

Remember that we derived a completely general solution to the ODE for any applied voltage $E(t)$:

$$J(t) = \frac{1}{L}\mathrm{e}^{-Rt/L}\int E(t)\mathrm{e}^{Rt/L}\mathrm{d}t.$$

So if we had a non-constant voltage such as $E(t) = E_0\sin(\omega t)$, we would be able to find the solution for this case by evaluating the integral.

4.6.3 Main points

- You need to be able to identify a first-order linear ODE.
- You need to be able to solve a first-order linear ODE using an integrating factor.

4.7 Linear Second-Order Differential Equations with Constant Coefficients

Second-order ODEs are covered in Stewart in Sections 17.1 and 17.2 (Sections 17.1 and 17.2). They often correspond with unforced (homogeneous) oscillators in our applications, but they arise in many other problems.

4.7.1 Definition

If the unknown function and its derivatives appear linearly in an ODE, the ODE is said to be linear. Otherwise it is nonlinear.

Exercise: Which of the following ODEs are linear?

since it not y

- ✓ $y'' + y' = y \log x$
- ✗ $y' = y^2 + y''$
- ✗ $y''y' = y$ unknown
- ✗ $y'y = y''$
- ✓ $y' + x^3y'' = xy + 3\sin x$
- ✓ $y' = y''t^5 - t^2y$
- ✗ $y'' = \sin y + e^x$
- ✗ $y'' + (y')^2 = y$

A second-order linear ODE is an ODE that can be written in the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = r(t).$$

The functions $p(t)$ and $q(t)$ are called the coefficients of the ODE and if $r(t) = 0$ the ODE is called homogeneous.

4.7.2 The superposition principle

If $y_1(t)$ and $y_2(t)$ are solutions of

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = 0,$$

then so is

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

Proof

$$\frac{d^2}{dt^2}(c_1y_1 + c_2y_2) + p(t)\frac{d}{dt}(c_1y_1 + c_2y_2) + q(t)(c_1y_1 + c_2y_2) =$$

$$c_1y_1'' + c_2y_2'' + p(t)c_1y_1' + p(t)c_2y_2' + q(t)c_1y_1 + q(t)c_2y_2 =$$

$$c_1\underbrace{(y_1'' + p(t)y' + q(t)y_1)}_{\text{soln.}\therefore = 0} + c_2\underbrace{(y'' + p(t)y_2' + q(t)y_2)}_{0} =$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

Theorem: If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions (i.e., $y_2(t) \neq Cy_1(t)$) of the homogeneous second-order linear ODE

not multiples of each other

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = 0$$

then the **general solution** is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Important remark: For first-order ODEs we have seen that the general solution contains one arbitrary constant. The above result suggests that for second-order ODEs the general solution should contain *two* arbitrary constants, and this is in fact the case.

Solving for IVP – two pieces of info given for IC

4.7.3 Reduction of order

The previous theorem states that all we need to do in order to solve a homogeneous linear second-order ODE is to find two linearly independent solutions. Unfortunately, that is where the good news ends: no general method (such as the integrating factor method for linear first-order ODEs) is known to find two such solutions.

There is however a method to find a second solution once a first solution has been found. So, if you are clever enough to guess one solution (and a guess it will have to be, given that we have no general method) you can obtain a second, linearly independent solution. This goes by the name of reduction of order.

Assume your smart little sister has guessed that $y_1(x)$ is a solution to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0.$$

To find a second solution $y_2(x)$ you set

$$y_2(x) = u(x)y_1(x).$$

with $u(x)$ a yet-to-be-determined function. Substituting this in the ODE gives

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0.$$

This may also be written as

$$u''y_1 + u'(y_1 p + 2y_1') + u(y_1'' + py_1' + qy_1) = 0.$$

But y_1 is a solution to the ODE so the third term actually vanishes! Thus

$$u'' + u'\left(p + 2\frac{y_1'}{y_1}\right) = 0.$$

This is a first-order linear ODE in $v = u'$ which we known how to solve using an integrating factor.

Example: One solution to the ODE $x^2y'' + 2xy' - 6y = 0$ is $y(x) = x^2$. Use reduction of order to find the general solution.

Standard form: $y'' + \frac{2}{x}y' - \frac{6y}{x^2} = 0 \qquad x \neq 0$

Look for solution of the form $y_2(x) = u(x)\,x^2$

Plug into (*)

$$(ux^2)'' + \frac{2}{x}(ux^2)' - 6\frac{ux^2}{x^2} = 0$$

$$(2xu + u'x^2)' + \frac{4xu + 2u'x^2}{x} - 6u = 0$$

$$\cancel{2u} + 2xu' + u''x^2 + 2xu' + \cancel{4u} + 2u'x - \cancel{6u} = 0$$

$$u''x^2 + 6u'x = 0$$

Let $v = u' \Rightarrow v'x^2 + 6vx = 0$

$$\frac{v'}{v} = \frac{-6}{x} \Rightarrow \ln|v| = -6\ln|x| + C$$

$$e^{\ln|v|} = e^{-6\ln|x| + C} = e^C\left(e^{\ln|x|}\right)^{-6} = e^C|x|^{-6}$$

$$\Rightarrow |v| = e^C|x|^{-6} \qquad v = Dx^{-6} \quad D \in \mathbb{R}$$

$$u' = v = Dx^{-6}$$

$$u = \frac{Dx^{-5}}{-5} + C \overset{\text{or } E}{=} E + Fx^{-5} \qquad E, F \in \mathbb{R}$$

$$y_2 = u(x)x^2 = Ex^2 + Fx^{-3} \qquad E, F \in \mathbb{R}$$

Since $y_1(x) = x^2$ and $y_2(x)$ are solns, take $E = 0$, $F = 1$: $y_2(x) = x^{-3}$

so $y(x) = c_1x^2 + c_2x^{-3}$ is a general solution

4.7.4 Homogeneous linear second-order ODEs with constant coefficients

We next discuss how to solve a homogeneous linear second-order ODE with constant coefficients:

$$ay'' + by' + cy = 0, \qquad a \neq 0.$$

The trick is to try to find a solution of the form $y = \mathrm{e}^{\lambda t}$. Substituting this into the ODE gives

$$a\lambda^2 \mathrm{e}^{\lambda t} + b\lambda \mathrm{e}^{\lambda t} + c\mathrm{e}^{\lambda t} = 0.$$

Since $\mathrm{e}^{\lambda t} \neq 0$ we may divide both sides of the above equation by this term to be left with the quadratic equation

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{a\lambda^2 + b\lambda + c = 0.}$$

This equation is called the characteristic equation of the ODE.

Important remark: When you are solving ODEs of this type on your assignments and exams, you don't have to *derive* the characteristic equation. It is fine to simply start from the characteristic equation and solve the ODE from there.

Various scenarios can now arise, which we discuss in more detail below.

Case 1: Two distinct real roots

If $b^2 - 4ac > 0$ the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has two distinct real roots, say λ_1 and λ_2.

$y_1 = e^{\lambda_1 t}$

In this case the general solution to the ODE is

$y_2 = e^{\lambda_2 t}$

$$\boxed{y = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}.}$$

$c_1, c_2 \in \mathbb{R}$

Question: What will this solution look like for very large t?

• If $\lambda_1, \lambda_2 < 0$, $y(t) \xrightarrow[t \to \infty]{} 0$

• If $\lambda_1 > 0$ if $c_1 > 0$, $y(t) \to \infty$, $t \to \infty$

$c_1 < 0$ $y(t) \xrightarrow[t \to \infty]{} -\infty$

$c_1 = 0$ .... depends on

whether $\begin{cases} \lambda_2 > 0, < 0 \\ c_2 \end{cases}$

Example: Solve the ODE $y'' - 3y' + 2y = 0$.

Characteristic equation

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, \ \lambda = 1$

So $y(x) = C_1 e^{2x} + C_2 e^{x}$ $\quad C_1, C_2 \in \mathbb{R}$

Case 2: A single root of multiplicity 2

If $b^2 - 4ac = 0$ the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has one root, say μ, of multiplicity 2. (Of course $\mu = -b/(2a)$ but it is more convenient to work with μ).

This is somewhat of a problem because now we only have single solution to the ODE: $y = \mathrm{e}^{\mu t}$.

Fortunately, we know that reduction of order will give us a a second linearly independent solution, so that we try $y = u(t)\mathrm{e}^{\mu t}$. By the product rule

$$y' = u'\mathrm{e}^{\mu t} + \mu u\mathrm{e}^{\mu t}$$
$$y'' = u''\mathrm{e}^{\mu t} + 2\mu u'\mathrm{e}^{\mu t} + \mu^2 u\mathrm{e}^{\mu t}.$$

Substituting this into $ay'' + by' + cy = 0$ gives

$$\Big[\big(a\mu^2 + b\mu + c\big)u + \big(2a\mu + b\big)u' + au''\Big]\mathrm{e}^{\mu t} = 0.$$

Since μ is a solution to the characteristic equation, $a\mu^2 + b\mu + c = 0$. Moreover, since $\mu = -b/2a$, $2a\mu + b = 0$. Hence we are left with just

$$u'' = 0.$$

This is easily solved as

$$u(t) = (c_1 + c_2 t).$$

Using this with $c_1 = 0$ and $c_2 = 1$ we now have two linearly independent solutions:

$$y_1(t) = \mathrm{e}^{\mu t} \quad \text{and} \quad y_2(t) = t\,\mathrm{e}^{\mu t},$$

where $\mu = -b/2a$.

Important remark: In actual problems you do not have to repeat the above derivation. It is enough for you to claim that if the characteristic equation has a single root of multiplicity 2 then the above are a pair of linearly independent solutions and

$$\boxed{y(t) = (c_1 + c_2 t)\mathrm{e}^{\mu t}}$$

is the general solution.

depends on signs of μ, c_1, c_2

Question: What will this solution look like for very large t?

For nonzero sol's

If $\mu < 0 \Rightarrow y(t) \to 0$ as $t \to \infty$

If $\mu > 0 \Rightarrow |y(t)| \to \infty$ as $t \to \infty$

If $c_2 > 0$, $y(t) \to \infty$ as $t \to \infty$

$c_2 < 0$, $y(t) \to -\infty$ as $t \to \infty$

If $c_2 = 0$ depends on sign of c_1

If $\mu = 0$ depends on sign of c_2 as above.

Example: Solve the IVP $y'' + 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 0.$

Char eq: $\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0 \quad \lambda = -3$

So $y(t) = (c_1 + c_2 t) e^{-3t} \quad c_1, c_2 \in \mathbb{R}$

is the general sol to ODE

IC:

$2 = y(0) = (c_1 + c_2 \cdot 0) e^0 = c_1 \Rightarrow c_1 = 2$

$y'(t) = c_2 e^{-3t} + (c_1 + c_2 t)(-3) e^{-3t}$

$0 = y'(0) = c_2 e^0 + (c_1 + c_2 \cdot 0)(-3) e^0$

$= c_2 - 3c_1 \Rightarrow c_2 = 3c_1 = 6$

$y(t) = (2 + 6t) e^{-3t}$

Case 3: Complex roots

$i = \sqrt{-1}$ $\quad i^2 = -1$

If $b^2 - 4ac < 0$ the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has a pair of complex conjugate roots, say $\alpha \pm \mathrm{i}\beta$.

$\alpha, \beta \in \mathbb{R}$

This leads to the general solution

$$y(t) = \left(c_1 \mathrm{e}^{\mathrm{i}\beta t} + c_2 \mathrm{e}^{-\mathrm{i}\beta t}\right) \mathrm{e}^{\alpha t}.$$

$y = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$

For those who want the solution to be manifestly real we note that

$e^{i\pi} + 1 = 0$

$$\mathrm{e}^{\pm \mathrm{i}\beta t} = \cos(\beta t) \pm \mathrm{i} \sin(\beta t),$$

Eular formula

so that

REAL $\quad$ IMAGINARY

$$y(t) = \Big((c_1 + c_2)\cos(\beta t) + (c_1 - c_2)\mathrm{i}\sin(\beta t)\Big) \mathrm{e}^{\alpha t}.$$

Introducing new constants b_1 and b_2 by $b_1 = c_1 + c_2$ and $b_2 = (c_1 - c_2)\mathrm{i}$ we thus get

$$\boxed{y(t) = \Big(b_1 \cos(\beta t) + b_2 \sin(\beta t)\Big) \mathrm{e}^{\alpha t}.}$$

Example: Solve $y'' + 9y = 0$.

$\begin{pmatrix} \alpha = 0 \\ \beta = 3 \end{pmatrix}$

char eq: $\lambda^2 + 9 = 0$ $\quad \lambda^2 + 9 = 0$ $\quad \lambda = \pm 3i$

general sol:

$y(t) = (b_1 \cdot b_2(3t) + b_2 \sin(3t)) e^{0 \cdot t}$

$= b_1 \cos(3t) + b_2 \sin(3t)$ $\quad b_1, b_2 \in \mathbb{R}$

Example: Solve the IVP $y'' - 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

char eq:

$\lambda^2 - 4\lambda + 5 = 0 \qquad \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\alpha = 2$ - real
$\beta = 1$ - imaginary

general soln is:
$y(t) = (b_1 \cos t + b_2 \sin t)e^{2t} \qquad b_1, b_2 \in \mathbb{R}$

IC $y(0) = 1 = (b_1 \cdot 1 + b_2 \cdot 0)e^0 = b_1 \Rightarrow b_1 = 1$

$y'(t) = (-b_1 \sin t + b_2 \cos t)e^{2t} + (b_1 \cos t + b_2 \sin t) \times 2e^{2t}$

$0 = y'(0) = (0 + b_2)e^0 + (b_1 + 0) \cdot 2e^0$

$0 = b_2 + 2b_1 = b_2 + 2 \Rightarrow b_2 = -2$

soln to IVP is
$y(t) = (\cos t - 2\sin t)e^{2t}$

Challenge problem: Solve the ODE $y'''' + y''' - y'' + y' - 2y = 0$.

$y(t) = e^{2t}$

$\lambda^4 \cancel{e^{2t}} + \lambda^3 \cancel{e^{2t}} - \lambda^2 \cancel{e^{2t}} + \lambda \cancel{e^{2t}} - 2 = 0$

hint
try $\lambda = 1$

$\lambda^4 + \lambda^3 - \lambda^2 + \lambda - 2 = 0$

4.7.5 Extra reading: numerical solutions

Numerical solutions of second-order ODEs usually require one second-order equation to be split into two first-order equations. Then the two coupled ODEs are solved together. For example, the equation

$$z'' + z' - z = \cos(x), \quad z(0) = 1, \quad z'(0) = 2$$

can be converted into:

$$\begin{aligned} z' &= y \\ y' &= -y + z + \cos(x) \end{aligned}$$

with initial conditions $z(0) = 1$, $y(0) = 2$. Then an Euler method can be used on these equations. For instance, the code to implement Euler's method would look like:

```
deltax=0.1;
x(1)=0;
z(1)=1;
y(1)=2;
for i = 1:100
   x(i+1) = x(i) + deltax;
   z(i+1) = z(i) + deltax*y(i);
   y(i+1) = y(i) + deltax*(-y(i) + z(i) + cos(x(i)));
end
plot(x,z)
```

4.7.6 Main points

- You should be able to identify a linear second-order ODE.
- You should understand the superposition principle and the general solution theorem for homogeneous linear second-order ODEs.
- You should know how to solve homogeneous linear second-order ODEs with constant coefficients.

4.8 Simple Harmonic Motion and Damped Oscillations

This section deals with the modelling of oscillators by second-order ODEs. This material is covered in Stewart Section 17.3 (Section 17.3).

4.8.1 The undamped spring

A spring of (natural) length L is stretched a distance s by a weight with mass m.

In equilibrium we take the position of the weight (viewed as a point-particle) to be $x = 0$, with the positive x-axis pointing down.

If we pull down the weight and then release it, it will start to oscillate.

If we assume the weight-spring-system moves free of any resistance (no air-resistance and no internal friction in the spring) we say the spring is undamped.

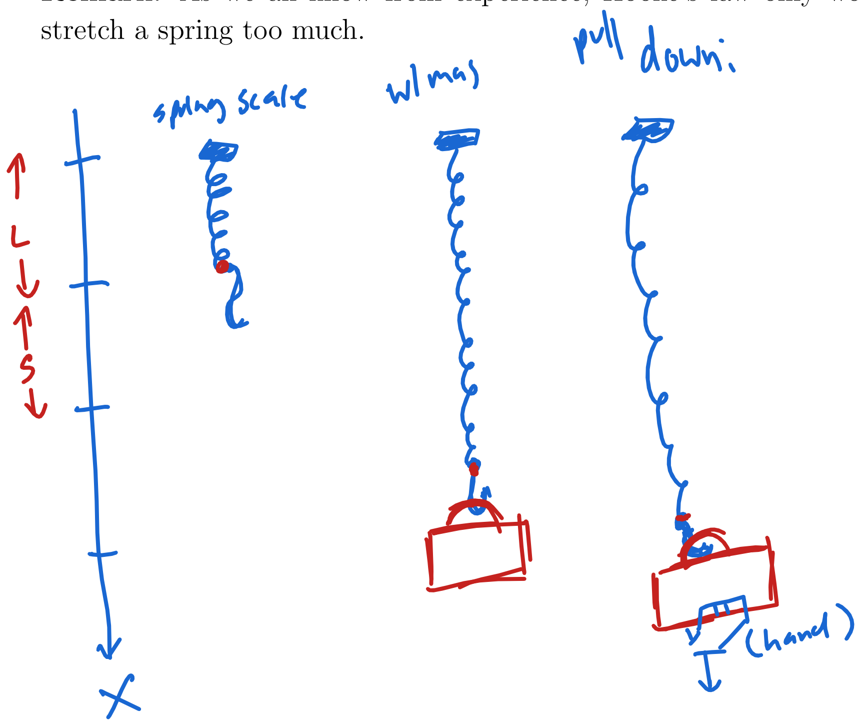
To derive an ODE for the undamped spring we must combine Newton's second law and Hooke's law.

According to Hooke's linear spring law the restoring force T exerted by the spring ("trying to unstretch") is proportional to the distance d it is stretched:

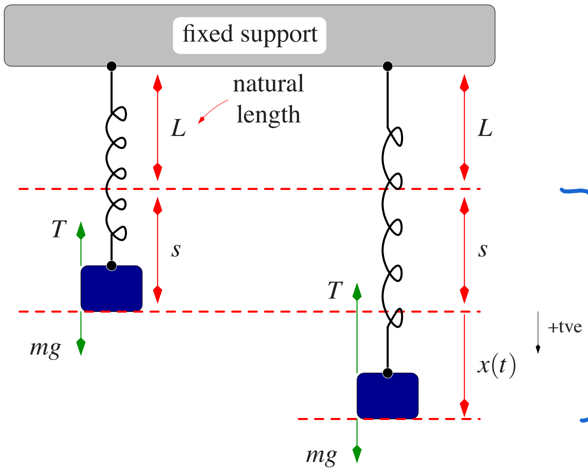
$$T = -kd.$$

Here $k > 0$ is measured in N m^{-1} and is called the spring constant. Since $k > 0$, the tension in the spring acts to restore the spring to its natural length. The larger the spring constant, the more tightly-coiled the spring.

Remark: As we all know from experience, Hooke's law only works if we do not stretch a spring too much.



@ equilibrium s



$s + x$

According to Hooke's law the total force F is given by

$$F = \underbrace{-k(s+x)}_{\text{restoring force}} + \underbrace{mg}_{\text{gravitational force}},$$

with $x = x(t)$ the displacement out of equilibrium and g ($\approx$ 9.8 m/s^2) the acceleration on the Earth's surface.

Before we pulled down the weight the system was in equilibrium, so that $(x=0)$

$$ks = mg.$$

We are thus left with

$$F = -kx. \qquad (*)$$

According to Newton's second law of motion the total force equals mass times acceleration:

$$F = mx''. \qquad (**)$$

Equating $(*)$ and $(**)$ leads to the equation of motion

$$\boxed{mx'' + kx = 0}$$

$x'' + \frac{k}{m} x = 0$

(which holds throughout the Universe, not just on Earth).

The equation of motion only depends on the ratio of k and m, and it is customary to introduce

$$\omega^2 = \frac{k}{m},$$

$\omega = \sqrt{\frac{k}{m}}$

where $\omega > 0$ is known as the angular frequency.

Then the equation of motion for the undamped spring is thus given by

$$\boxed{x'' + \omega^2 x = 0.}$$

This is a homogeneous second-order linear ODE with constant coefficients.

The corresponding characteristic equation is given by

$$\lambda^2 + \omega^2 = 0$$

$\left(\alpha + i\beta \quad \alpha = 0, \; \beta = \omega\right)$

with purely imaginary roots

$$\lambda_1 = \mathrm{i}\omega \quad \text{and} \quad \lambda_2 = -\mathrm{i}\omega.$$

Two linearly independent solutions are thus

$$\mathrm{e}^{\mathrm{i}\omega t} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}\omega t}$$

or

$$\cos(\omega t) \quad \text{and} \quad \sin(\omega t).$$

and the general solution is

$$\boxed{x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).}$$

Although we could stop here it is standard to rewrite the above solution in the alternative form

$$x(t) = A\cos(\omega t - \phi),$$

where the two constants c_1 and c_2 have been traded for two new constants $A > 0$ and $\phi \in (-\pi, \pi]$, known as the amplitude and phase shift.

Recall $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

So $x(t) = A\cos(\omega t - \phi)$

$= A\cos\omega t\cos(-\phi) - \sin(\omega t)\sin(-\phi)$

$= (\underbrace{A\cos\phi}_{c_1})\cos\omega t + (\underbrace{A\sin\phi}_{c_2})\sin(\omega t)$

$c_1 = A\cos\phi \qquad c_2 = A\sin\phi$

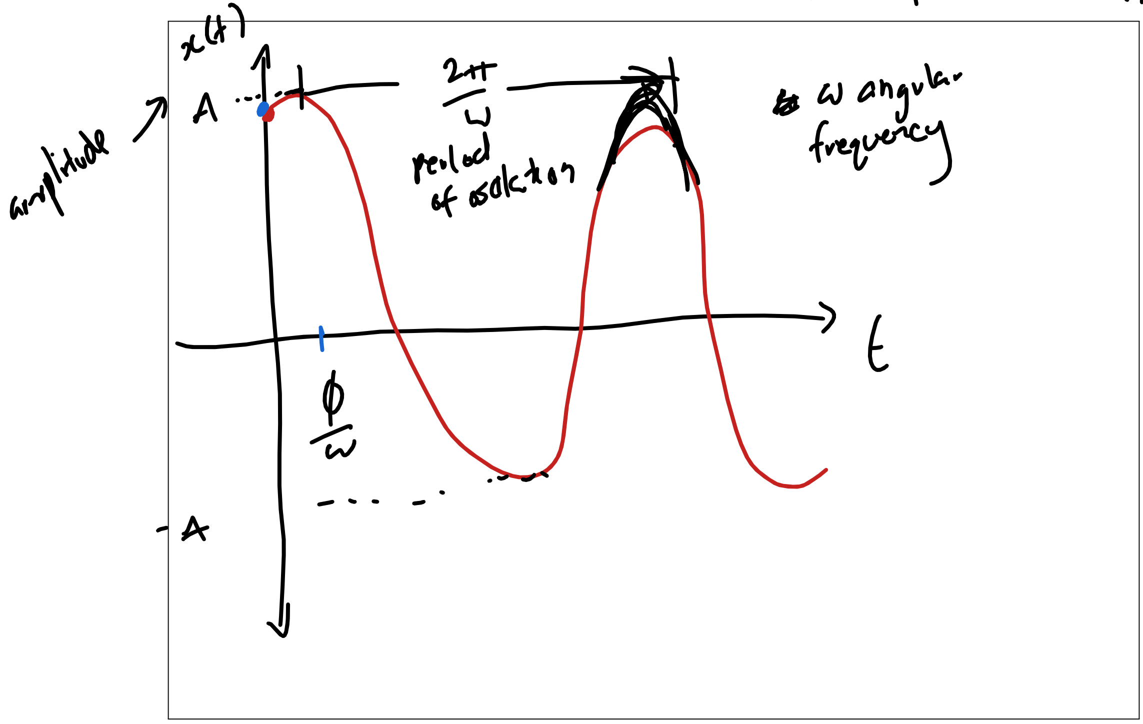
$c_1^2 + c_2^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2$

$A = \sqrt{c_1^2 + c_1^2} \qquad \tan\phi = \dfrac{c_2}{c_1}$

amplitude $\qquad$ phase shift

Example: Find the relation between c_1, c_2 and A, ϕ.

$x(t) = A\cos(\omega t - \phi)$



Example: A mass of 9kg is attached to a spring with spring constant 4N/m. The spring is pulled down 1m and given an initial upward kick of −0.5 m/s. Solve for position as a function of time and explicitly determine the amplitude of the oscillations.

$m = 9 \quad k = 4 \quad$ IC: $x(0) = 1$

$x'(0) = -0.5$

Eq of motion

$x'' + \frac{k}{m} x = 0$

$x(t) = ?$
$A = ?$

$x'' + \frac{4}{9} x = 0$

Char eq: $\lambda^2 + \frac{4}{9} = 0 \qquad \lambda = \pm \frac{2}{3} i$

- general solution
  $x(t) = c_1 \cos\left(\frac{2}{3} t\right) + c_2 \sin\left(\frac{2}{3} t\right) \qquad c_1, c_2 \in \mathbb{R}$
- IC $\quad 1 = x(0) = c_1$
  $x'(t) = -\frac{2}{3} c_1 \sin\left(\frac{2}{3} t\right) + \frac{2}{3} c_2 \cos\left(\frac{2}{3} t\right)$
  $-0.5 = x'(0) = \frac{2}{3} c_2 \Rightarrow c_2 = -\frac{3}{4}$
- soln to IVP is
  $x(t) = \cos\left(\frac{2}{3} t\right) - \frac{3}{4} \sin\left(\frac{2}{3} t\right)$
- Amplitude
  $A = \sqrt{c_1^2 + c_2^2} = \sqrt{1^2 + \left(-\frac{3}{4}\right)^2} = \frac{5}{4}$

$\left(\tan\phi = \frac{-3}{4}\right)$

4.8.2 The damped spring

If you try the "spring-weight" system at home it is unlikely to display true simple harmonic motion. If that were possible you could build a perpetuum mobile.

To make the model more realistic we need to take into account the effect of damping. This is caused by air-resistance and mechanical friction of the spring.

The most common model of damping assumes that it corresponds to a force F_d, proportional to the velocity of the moving object, and opposite to the direction of movement. Hence

$$F_d = -\beta x',$$

where $\beta > 0$ is the damping constant measured in N s m^{-1}.

The total force F for the damped spring is thus given by

$$F = \underbrace{-\beta x'}_{\text{damping force}} \quad \underbrace{-k(s+x)}_{\text{restoring force}} + \underbrace{mg}_{\text{gravitational force}} .$$

As before,

$$ks = mg,$$

so that

$$F = -\beta x' - kx.$$

Thanks to Newton's second law this yields the equation of motion

$$\boxed{mx'' + \beta x' + kx = 0.}$$

$x'' + \frac{\beta}{m} x' + \frac{k}{m} x = 0$

This equation only depends on the ratios of k/m and β/m, and it is customary to not only set

$$\omega^2 = \frac{k}{m} > 0$$

but also

$$2p = \frac{\beta}{m} > 0.$$

Then the equation of motion for the damped spring is given by

$$\boxed{x'' + 2p\,x' + \omega^2 x = 0.}$$

This is a homogeneous second-order linear ODE with constant coefficients.

The corresponding characteristic equation is given by

$$\lambda = \frac{-2p \pm \sqrt{4p^2 - 4\omega^2}}{2}$$

$$\lambda^2 + 2p\lambda + \omega^2 = 0,$$

which has discriminant

$$D = 4(p^2 - \omega^2).$$

There are three different cases to consider.

- $D < 0$ (i.e., $p < \omega$). complex conj roots
  This is the underdamped or weakly-damped spring.
- $D = 0$ (i.e., $p = \omega$).
  This is the critically-damped spring.
- $D > 0$ (i.e., $p > \omega$).
  This is the overdamped or strongly-damped spring.

4.8.3 Underdamping

$p < \omega$

For $p < \omega$ the characteristic equation

$$\lambda^2 + 2p\lambda + \omega^2 = 0$$

$\alpha \pm i\gamma$ $\alpha = -p$, $\gamma = \sqrt{\omega^2 - p^2} = \omega_p$

has two complex conjugate roots

$$\lambda_1 = -p + \mathrm{i}\sqrt{\omega^2 - p^2} \quad \text{and} \quad \lambda_2 = -p - \mathrm{i}\sqrt{\omega^2 - p^2}.$$

Two linearly independent solutions are thus

$$\mathrm{e}^{-pt}\cos(\omega_p t) \quad \text{and} \quad \mathrm{e}^{-pt}\sin(\omega_p t),$$

where

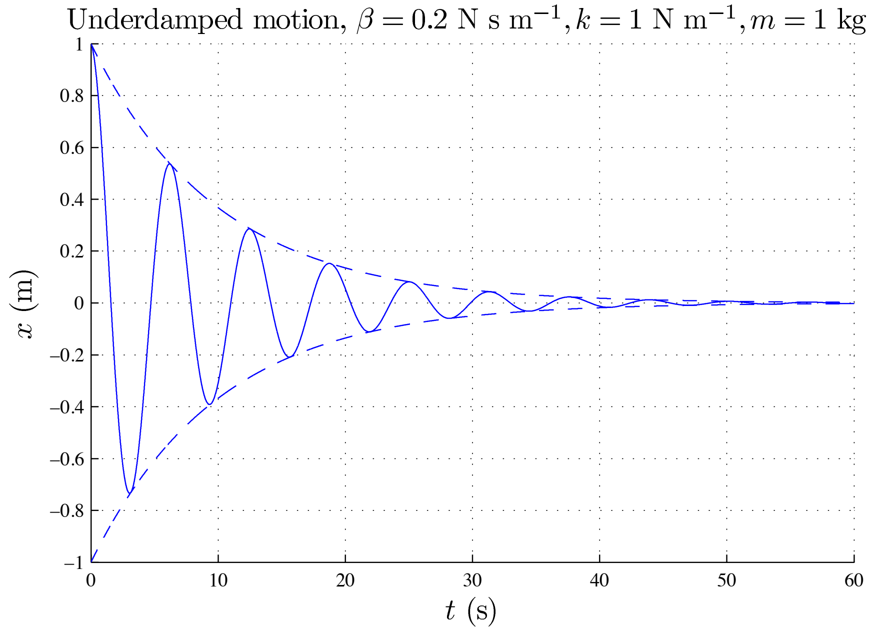
not ang'l vel.
$$\omega_p = \sqrt{\omega^2 - p^2} = \omega\sqrt{1 - (p/\omega)^2}.$$

The general solution is

exponential decay

$$x(t) = \mathrm{e}^{-pt}\Big(c_1\cos(\omega_p t) + c_2\sin(\omega_p t)\Big).$$

This figure shows an example of underdamped motion:



It is best to write the general solution in terms of a single trigonometric function:

$$x(t) = \underbrace{A\mathrm{e}^{-pt}}_{A(t)} \cos(\omega_p t - \phi) = A(t)\cos(\omega_p t - \phi).$$

Here $A(t)$ is a time-dependent amplitude (as shown in the above figure with the dashed lines):

$$A(t) = A\mathrm{e}^{-pt}$$

and ω_p is the frequency of the underdamped spring.

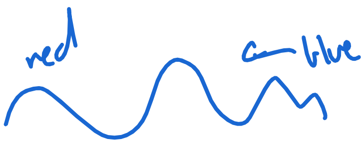
Since

$$\frac{\omega_p}{\omega} = \sqrt{1 - (p/\omega)^2} < 1$$

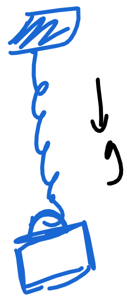
$\omega_p < \omega$

the damping of the spring leads to a frequency red-shift and an increase in the period of the spring.

When p approaches ω the period tends to infinity so that we may expect that the critically-damped spring does not display oscillatory motion. This is the case considered in the next section.



Example: A mass of 1kg is attached to a spring hanging under gravity with damping constant 0.2 and spring constant 4. Find the position of the mass after time t if it is pulled down 1m from the equilibrium position and released without kick.



$m=1$ $\beta=0.2$ $k=4$; IC: $x(0)=1$, $x'(0)=0$

Equation of motion || Characteristic eq:

$x''+0.2x'+4x=0$ || $\lambda^2+0.2\lambda+4=0$

$$\lambda=\frac{-0.2\pm\sqrt{0.04-16}}{2}=-0.1\pm\sqrt{0.01-4}=-0.1\pm i\sqrt{3.99}$$

general sol:

$$x(t)=e^{-0.1t}\left(c_1\cos(\sqrt{3.99}\,t)+c_2\sin(\sqrt{3.99}\,t)\right)$$

IC:

$1=x(0)=e^{0}(c_1\cdot 1+c_2\cdot 0)=c_1$

$$x'(t)=-0.1e^{-0.1t}\left(c_1\cos\sqrt{3.99}\,t+c_2\sin(\sqrt{3.99}\,t)\right)+e^{-0.1t}\left(-c_1\sin(\sqrt{3.99}\,t)\sqrt{3.99}+c_2\cos(\sqrt{3.99}\,t)\sqrt{3.99}\right)$$

IC $\Rightarrow$ $0=x'(0)=-0.1c_1+c_2\sqrt{3.99}$

$$c_2=\frac{0.1}{\sqrt{3.99}}$$

$$x(t)=e^{-0.1t}\left(\cos(\sqrt{3.99}\,t)+\frac{0.1}{\sqrt{3.99}}\sin(\sqrt{3.99}\,t)\right)$$

4.8.4 Critical damping

$\lambda = \frac{-2p \pm \cancel{\sqrt{4p^2-4\omega^2}}}{2}$

For $p = \omega$ the characteristic equation

$$\lambda^2 + 2p\lambda + \omega^2 = 0$$

has one root of multiplicity two: $\lambda = -p$. Two linearly independent solutions are thus

$$\mathrm{e}^{-pt} \quad \text{and} \quad t\,\mathrm{e}^{-pt}$$

and the general solution is

$$\boxed{x(t) = \mathrm{e}^{-pt}(c_1 + c_2 t).}$$

$x'(t) = e^{-pt}(-pc_1 - pc_2 t + c_2)$

Unlike the underdamped spring it is straightforward to express c_1 and c_2 in terms of the initial conditions $x(0)$ and $x'(0)$:

$x'(0) = -px(0) + c_2$

$$c_1 = x(0) \quad \text{and} \quad c_2 = x(0)p + x'(0).$$

Therefore

$$x(t) = \mathrm{e}^{-pt}(x(0) + x(0)pt + x'(0)t).$$

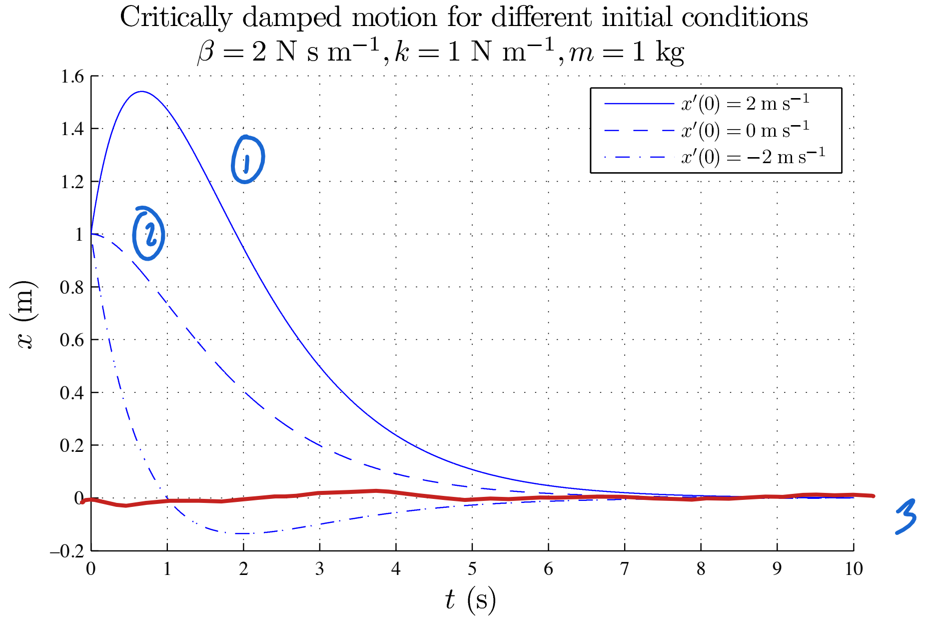
The linear polynomial $x(0) + x(0)pt + x'(0)t$ vanishes for

$$t = -\frac{x(0)}{x(0)p + x'(0)}.$$

overshooting.

This corresponds to a positive time t if $x'(0) < -x(0)p$.

Depending on the initial conditions, the critically-damped spring behaves as shown in this figure:



1 • The top-most curve corresponds to $x(0)$ and $x'(0)$ both positive, so that the weight is not just pulled down a distance $x(0)$ but is also given a downward kick on release.

2 • The middle curve corresponds to $x'(0) = 0$ so that we only pull down the weight and then release it.

3 • The third curve corresponds to $x'(0) < 0$ so that upon release the weight is kicked upward. Since the spring *overshoots the equilibrium* we must also have that $x'(0) < -x(0)p$.

Example: A mass of 1kg is attached to a spring hanging under gravity, with spring constant 4. What value of the damping constant implies critical damping?

$m=1 \quad k=4 \quad \beta=?$ $\qquad x''+\beta x'+4x=0$

$mx''+\beta x'+kx=0$

$\lambda^2+\beta\lambda+4=0$

$D=\beta^2-16$

critical damped when $D=0 \Rightarrow \beta^2=16$, $\beta=4$

4.8.5 Overdamping

$p > \omega$

don't use β again (it's damping constant)

For $p > \omega$ the characteristic equation

$$\lambda^2 + 2p\lambda + \omega^2 = 0$$

has two distinct real roots

$$\lambda_1 = -p - \sqrt{p^2 - \omega^2} =: -\alpha$$
$$\lambda_2 = -p + \sqrt{p^2 - \omega^2} =: -\cancel{\beta}\gamma$$

with $\alpha > \cancel{\beta}\gamma > 0$.

Two linearly independent solutions are thus

$$\mathrm{e}^{-\alpha t} \quad \text{and} \quad \mathrm{e}^{-\cancel{\beta}\gamma t}$$

and the general solution is

$$x(t) = c_1 \mathrm{e}^{-\alpha t} + c_2 \mathrm{e}^{-\cancel{\beta}\gamma t}.$$

For generic initial conditions $x(0)$ and $x'(0)$, the constants c_1 and c_2 are both nonzero. Assuming this, we have that for sufficiently large t

(the other term is much smaller as $t \to \infty$)

$$x(t)_{\text{critial damping}} \sim c_2 t \mathrm{e}^{-pt}$$

and

$$x(t)_{\text{overdamped}} \sim c_2 \mathrm{e}^{-\cancel{\beta}\gamma t}.$$

Since $\beta = p - \sqrt{\ldots} < p$ we see that the overdamped spring takes longer to return to equilibrium than the critically damped spring.

The overdamped spring describes a door-closer that is too tight. It may take a long time for the door to actually close because the strong damping restricts (nearly) all movement.

A tradeswoman will try to adjust a door-closer such that the damping will be as close as possible to critical damping. As always, Maths Rules!

Example: A mass of 1kg is attached to a spring hanging under gravity with damping constant 5 and spring constant 4. Find the position of the mass after time t if it is pulled down 1m from the equilibrium position and released without kick.

$m=1\,kg$ $\beta=5$ $k=4$ IC $= x(0)=1$ $x'(0)=0$

eq of motion: $x''+5x'+4x=0$

char eq: $\lambda^2+5\lambda+4=0$

$(\lambda+4)(\lambda+1)=0$ $\lambda_1=-4, \lambda_2=-1$

general sol:

$x(t)=c_1e^{-4t}+c_2e^{-t}$ $c_1, c_2 \in \mathbb{R}$

$x'(t)=-4c_1e^{-4t}-c_2e^{-t}$

IC $= x(0)=c_1+c_2$ $(*)$

$0=x'(0)=-4c_1-c_2$ $(**)$

adding $(*)+(**)$

$1=-3c_1 \Rightarrow c_1=-1/3$

$(*) \Rightarrow c_2=1-c_1=4/3$

so $x(t)=-\frac{1}{3}e^{-4t}+\frac{4}{3}e^{-t}$

4.8.6 Summary of motion of a spring:

The equation of motion of a mass m attached to a damped spring is described by the ODE

$$x'' + 2p\,x' + \omega^2 x = 0,$$

where

$$\omega^2 = \frac{k}{m} > 0, \qquad 2p = \frac{\beta}{m} > 0$$

and k is the spring constant and β the damping constant.

The roots of the characteristic equation $\lambda^2 + 2p\lambda + \omega^2 = 0$ determine the type of motion:

- $p = 0$; No damping — pure oscillatory or harmonic motion.
  The roots are $\pm \mathrm{i}\omega$ and the general solution is given by
  $$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \cos(\omega t - \phi).$$
- $p < \omega$; Underdamped motion.
  The roots are $-p \pm \mathrm{i}\omega_p$, $\omega_p = \omega\sqrt{1 - (p/\omega)^2}$ and the general solution is given by
  $$x(t) = \mathrm{e}^{-pt}\Big(c_1 \cos(\omega_p t) + c_2 \sin(\omega_p t)\Big) = A\mathrm{e}^{-pt} \cos(\omega_p t - \phi).$$
- $p = \omega$; Critically damped motion.
  The single root (of multiplicity 2) is given by $-p$ and the general solution is given by
  $$x(t) = \mathrm{e}^{-pt}(c_1 + c_2 t).$$
- $p > \omega$; Overdamped motion.
  The two roots (denoted $-\alpha$ and $-\beta$) are $-p \pm \sqrt{p^2 - \omega^2}$ and the general solution is given by
  $$x(t) = c_1 \mathrm{e}^{-\alpha t} + c_2 \mathrm{e}^{-\beta t}.$$

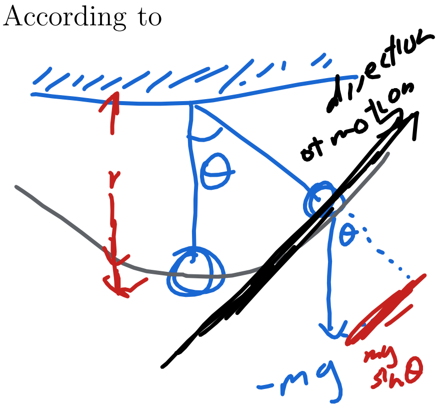
4.8.7 The pendulum

Some of you may have seen those beautiful old pendulum clocks. In such a clock a pendulum of mass m is suspended from a pivot by a long, thin metal rod.

Now suppose that the pendulum moves in an arc of a circle with radius r. If the pendulum makes an angle (say in the anti-clockwise direction) θ with its equilibrium position, then it has travelled a distance $r\theta$ from this position. According to Newton's second law of motion

acceleration

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\theta) = F,$$



where F is the force in the direction of motion. Hence

$$r\cancel{m}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\cancel{m}g\sin\theta.$$

$r\theta'' + g\sin\theta = 0$

This is a nonlinear ODE which displays very interesting nonlinear behaviour. If we however assume that θ is very small (the pendulum does not swing very far out, unlike your little sister on the swing perhaps), then we have $\sin\theta \simeq \theta$. Making this linear → approximation we get

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{r}\theta = 0.$$

Setting $g/r = \omega^2$ as before we obtain the same ODE describing simple harmonic motion.

linear

Question: What is the period of the undamped pendulum? → general sol $\theta(t) = c_1 e^{-i\omega t} + c_2 e^{-i\omega t}$
↳ can be written as sin, cos (complex)

$\theta'' + \omega^2\theta = 0$

char eq $\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$ $\begin{pmatrix} \alpha = 0 \\ \beta = \omega \end{pmatrix}$

$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$ $A, B \in \mathbb{R}$

period T: $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g}}$

Remark: It is interesting to note that the ODE describing the undamped pendulum does not depend on the mass of the pendulum.

As before one can make the model more realistic by assuming damping proportional to velocity.

$\gamma(r\theta)'$

Example: If the damping constant is γ show that this leads to the ODE

linear

$$\theta'' + 2p\,\theta' + \omega^2\theta = 0,$$

where

$$\omega^2 = \frac{g}{r} > 0, \qquad 2p = \frac{\gamma}{m} > 0.$$

approx $\omega\theta$

$$rm\theta'' = -mg\underbrace{\sin\theta} - \gamma_r\theta'$$

$$\theta'' + \underbrace{\frac{g}{r}}_{\omega^2}\theta + \underbrace{\frac{\gamma}{m}}_{2p}\theta' = 0$$

$$\theta'' + 2p\theta' + \omega^2\theta = 0$$

4.8.8 Main points

- You should know how to derive and solve the equations of motion for a damped oscillator given just the mass, damping and spring constants.
- You should also be able to classify the motion into under, over or critically damped, and know what these notions mean.

4.9 Inhomogeneous Linear Second-Order ODEs

4.9.1 Inhomogeneous linear second-order ODEs with constant coefficients

Inhomogeneous second-order ODEs, also known as ODEs with forcing, are covered in Stewart in Section 17.2 (Section 17.2). They are very common in applications and often they require a bit of ingenuity to solve.

We will only consider inhomogeneous linear second-order ODEs with constant coefficients, i.e., equations of the form

$$ay'' + by' + cy = r(x),$$

where r is a continuous function.

First we outline the general strategy of solving such equations.

1. First set $r(x) = 0$ and solve the corresponding homogeneous ODE. Its solution, $y_H(x) = c_1 y_1(x) + c_2 y_2(x)$, is known as the complementary function.

2. Find a solution, $y_P(x)$ to the inhomogeneous equation. Any solution will do and goes by the name of a particular solution.

3. The general solution to the inhomogeneous equation is then given by

$$y(x) = y_H(x) + y_P(x).$$

Clearly the remaining question is: How does one find a particular solution?

4.9.2 Method of Undetermined Coefficients

When the functional form of y_p is known, the ODE can be solved using undetermined coefficients.

We will show how to do this when the inhomogeneous term $r(x)$:

- is a polynomial.
- is a polynomial times an exponential.
- is a sum of terms.
- is a simple trigonometric function.

4.9.3 $r(x)$ is a polynomial

Example: Solve the ODE

$$y'' + y' - 2y = x^2 - 2x + 3.$$

The polynomial on the right hand side is of degree 2. Let's first "guess" that the functional form of y_p is also a polynomial of degree 2, with unknown coefficients. We write $y_p = ax^2 + bx + c$.

Differentiate this function 2 times to get

$$y_p' = 2ax + b$$
$$y_p'' = 2a.$$

Substituting back in to the ODE, we get

$$2a + (2ax + b) - 2(ax^2 + bx + c) = x^2 - 2x + 3.$$

Expanding, and grouping like powers of x together yields

$$-2ax^2 + (2a - 2b)x + (2a + b - 2c) = x^2 - 2x + 3.$$

Two polynomials are equal if and only if their coefficients are also equal. This gives us three equations and 3 unknowns, which we can solve.

$$-2a = 1 \implies a = -1/2.$$
$$2a - 2b = -2 \implies b = 1/2$$
$$2a + b - 2c = 3 \implies c = -7/4.$$

Thus,

$$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}.$$

So our initial guess was correct. The ODE's right hand side was a quadratic, and so was the particular solution. But where did our guess come from?

Theorem: Consider an ODE of the form $ay'' + by' + cy = r(x)$, with $c \neq 0$.
If $r(x)$ is a polynomial of degree n, y_p is also.

general solution

$y = y_H + y_P \leftarrow$ coefficient stuff

$\uparrow$

λ stuff

To see this, suppose we have an ODE of the form

$$ay'' + by' + cy = \sum_{j=0}^{n} \alpha_j x^j,$$

with $a, b, c \neq 0$.

We can differentiate both sides n times with respect to x;

$$\frac{d^n}{dx^n}(ay'' + by' + cy) = \frac{d^n}{dx^n} \sum_{j=0}^{n} \alpha_j x^j$$

$$ay^{(n+2)} + by^{(n+1)} + cy^{(n)} = n!\,\alpha_n$$

Since we don't require the most general solution, we may take $y_p^{(n)}$ to be a constant, which implies $y_p^{(n+1)} = y_p^{(n+2)} = 0$.

This yields;

$$y_p^{(n)} = \frac{n!\,\alpha_n}{c} = d_1$$

$$y_p^{(n-1)} = d_1 x + d_2$$

$$y_p^{(n-2)} = \frac{d_1 x^2}{2} + d_2 x + d_3$$

$$\vdots$$

$$y_p = \frac{d_1 x^n}{n!} + \frac{d_2}{(n-1)!} x^{n-1} + \dots + d_{n+1}$$

and so the theorem is proved.

RE-DO

Example: Find a particular solution of $3y'' - 2y' + 4y = 2x^3 + 1$.

guess $y_p = ax^3+bx^2+cx+d$

$y'_p = 3ax^2+2bx+c$

$y''_p = 6ax+2b$

Sub into ODE

$3(6ax+2b)-2(3ax^2+2bx+c)+4(ax^3+bx^2+cx+d)$

$18ax+6b-6ax^2-4bx-2c+4ax^3+4bx^2+4cx+4d=2x^3+1$

$4ax^3+(-6a+4b)x^2+(18a-4b+4c)x+6b-2c+4d$

$4a=2$, $a=\frac{1}{2}$

$-3+4b=0$, $4b=3$, $b=\frac{3}{4}$

$9-3+4c=0$, $6+4c=0$, $4c=-6$, $c=-\frac{3}{2}$

$6(\frac{3}{4})+3+4d=1$

$\frac{9}{2}+\frac{6}{2}+\frac{8}{2}d=1$

$\frac{15}{2}+\frac{8d}{2}=\frac{2}{2}$

$d=-\frac{13}{8}$

$\frac{8d}{2}=-\frac{13}{2}\times\left(\frac{2}{8}\right)$

$y_p=\frac{1}{2}x^3+\frac{3}{4}x^2-\frac{3}{2}x+\frac{17}{8}$

4.9.4 $r(x)$ is a polynomial times an exponential

Example: Find a particular solution to $y'' + y' - 2y = 4x\mathrm{e}^{2x}$.

We have a degree one polynomial, multiplied by e^{2x}. Let's guess that y_p is of the form

$$y_p = (ax + b)e^{2x}.$$

This means

$$y'_p = (2ax + 2b + a)e^{2x}$$
$$y''_p = (4ax + 4b + 4a)e^{2x}.$$

Substituting these into the ODE, we have

$$\begin{aligned} y_p'' + y_p' - 2y_p &= 4xe^{2x} \\ (4ax + 4b + 4a)\cancel{e^{2x}} + (2ax + 2b + a)\cancel{e^{2x}} - 2(ax + b)\cancel{e^{2x}} &= 4x\cancel{e^{2x}} \\ \implies 4ax + 4b + 5a &= 4x + 0. \end{aligned}$$

This gives us two equations, and two unknowns

$$\begin{aligned} 4a = 4 &\implies a = 1 \\ 4b + 5a = 0 &\implies b = -\frac{5}{4}. \end{aligned}$$

Hence, $y_p = \left(x - \frac{5}{4}\right) e^{2x}$.

Where did this guess come from?

Theorem: Consider an ODE of the form $ay'' + by' + cy = r(x)$.

If $r(x) = e^{\alpha x} \sum_{j=0}^{n} \beta_j x^j$,

$y_p = e^{\alpha x} \sum_{j=0}^{n} \gamma_j x^j$ for undetermined coefficients γ_j.

(subject to some caveats, discussed below)

To prove this, suppose we have an ODE of the form

$$ay'' + by' + cy = e^{\alpha x} \sum_{j=0}^{n} \beta_j x^j.$$

Let

$$\begin{aligned} y_p &= u(x)e^{\alpha x} \\ \implies y_p' &= (u' + \alpha u)e^{\alpha x} \\ \implies y_p'' &= (u'' + 2\alpha u' + \alpha^2 u)e^{\alpha x}. \end{aligned}$$

Substituting these into the left hand side of the ODE, we have

$$e^{\alpha x}\left[(au'' + 2a\alpha u' + a\alpha^2 u) + (bu' + b\alpha u) + (cu)\right] = e^{\alpha x}\sum_{j=0}^{n} \beta_j x^j$$

Dividing both sides by the exponential and grouping like terms of u, we have:

$$au'' + (2a\alpha + b)u' + (a\alpha^2 + b\alpha + c)u = \sum_{j=0}^{n} \beta_j x^j.$$

$$= \beta_0 x^0 + \beta_1 x^1 + \beta_2 x^2 + \ldots + \beta_n x^n$$

This is a second order ODE, where the RHS is a polynomial. But we've previously established that this means $u(x)$ is a polynomial of degree n.[2]

Thus, the theorem is proved.

[2]In fact, we've only proved this is true when $a\alpha^2 + b\alpha + c \neq 0$. How might things change if this were not the case?

4.9.5 $r(x)$ is the sum of the previous two cases

Example: Find the general solution of $y'' + y' - 2y = (x^2 - 2x + 3) + (4x\mathrm{e}^{2x}.)$

$$y_p = \left(-\frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}\right) + \left(\left(x - \frac{5}{4}\right)e^{2x}\right)$$

Check for H/W

Theorem: If y_{P_1} is a particular solution of $ay'' + by' + cy = r_1(x)$ and y_{P_2} is a particular solution of $ay'' + by' + cy = r_2(x)$, then $y_P = y_{P_1} + y_{P_2}$ is a particular solution of $ay'' + by' + cy = r_1(x) + r_2(x)$.

To prove this, suppose we know y_{P_1} and y_{P_1} such that

$$\begin{aligned} ay''_{P_1} + by'_{P_1} + cy_{P_1} &= r_1(x) \\ ay''_{P_2} + by'_{P_2} + cy_{P_2} &= r_2(x). \end{aligned}$$

Let $z(x) = y_{P_1}(x) + y_{P_2}(x)$. Then

$$\begin{aligned} z' &= y'_{P_1} + y'_{P_2} \\ z'' &= y''_{P_1} + y''_{P_2}, \end{aligned}$$

and

$$az'' + bz' + cz = a(y_{p_1}'' + y_{p_2}'') + b(y_{p_1}' + y_{p_2}') + c(y_{p_1} + y_{p_2})$$
$$= (a y_{p_1}'' + b y_{p_1}' + c y_{p_1}) + (a y_{p_2}'' + b y_{p_2}' + c y_{p_2})$$
$$= r_1(x) + r_2(x)$$

Thus, $z(x) = y_p$, and the theorem is proved.

4.9.6 $r(x)$ is a simple trigonometric function

Example: Find a particular solution to $y'' + 4y' + 5y = 12\cos x + 4\sin x$.

$$y_p = a\cos x + b\sin x$$
$$y_p' = -a\sin x + b\cos x$$
$$y_p'' = -a\cos x - b\sin x$$
$$(-a\cos x - b\sin x) + 4(-a\sin x + b\cos x) + 5(a\cos x + b\sin x)$$
$$= 12\cos x + 4\sin x$$
$$4a + 4b = 12 \quad (1)$$
$$-4a + 4b = \quad (2)$$
$$b = 2$$
$$a = 1$$
$$y_p = \cos x + 2\sin x$$

Once again, where did our guess come from?

Theorem: Consider an ODE of the form $ay'' + by' + cy = r(x)$, with $a, b, c \neq 0$.
If $r(x)$ is a linear combination of cos and sin terms, then so is y_p.
(subject to some caveats, discussed below)

Consider the ODE

$$ay'' + by' + cy = \alpha\cos(nx) + \beta\sin(nx)\diamond$$

First, let's observe that

$$\begin{aligned}\cos(nx) &= \frac{e^{inx} + e^{-inx}}{2}\\ \sin(nx) &= \frac{e^{inx} - e^{-inx}}{2i}.\end{aligned}$$

(To see why, recall Euler's formula, $e^{ix} = \cos x + i\sin x$.)

Thus,

$$\begin{aligned}r(x) &= \alpha\cos(nx) + \beta\sin(nx)\\ &= \alpha\left(\frac{e^{inx} + e^{-inx}}{2}\right) + \beta\left(\frac{e^{inx} - e^{-inx}}{2i}\right)\\ &= \left(\frac{\alpha}{2} + \frac{\beta}{2i}\right)e^{inx} + \left(\frac{\alpha}{2} - \frac{\beta}{2i}\right)e^{-inx}.\end{aligned}$$

From section 4.9.4, we know that the simplified ODE

$$ay'' + by' + cy = \left(\frac{\alpha}{2} + \frac{\beta}{2i}\right)e^{inx},$$

has solution $y_{p_1} = \gamma e^{inx}$, for some γ.[3]

Furthermore, $y_{p_2} = \delta e^{-inx}$ solves

$$ay'' + by' + cy = \left(\frac{\alpha}{2} - \frac{\beta}{2i}\right)e^{-inx}.$$

[3] again, when would this solution not work?

Thus, using section 4.9.5 the solution to $\diamond$ is

$$
\begin{aligned}
y_{P1} + y_{P2} &= \gamma e^{inx} + \delta e^{-inx} \\
&= \gamma(\cos(nx) + i\sin(nx)) \\
&\quad + \delta(\cos(nx) - i\sin(nx)) \\
&= (\gamma + \delta)(\cos(nx)) + i(\gamma - \delta)\sin(nx)
\end{aligned}
$$

and the theorem is proved.

4.9.7 Main points

- You should be able to find a particular solution for the cases we have talked about.
- You should know how to find the general solution as the sum of a particular solution and the complementary function.

4.10 The Energy principle

4.10.1 Conservation of energy

In many cases, the ODE describing a physical system can be simplified.

Consider an object with mass m subject to a force, $F(t)$. From Newton's Second Law of Motion, the object's trajectory is described as

$$\boxed{m\frac{d^2y}{dt^2} = F(y,t).}$$

Often, the force F depends **only** on position $y(t)$. We write

$$\boxed{m\frac{d^2y}{dt^2} = F(y(t)). \quad (\star)}$$

By the Fundamental Theorem of Calculus, if F is a continuous function, it has an anti-derivative $\phi(y)$. Hence, $\dfrac{d\phi}{dy} = F(y)$.

We multiply both sides by $\dfrac{dy}{dt}$

$$m\frac{d^2y}{dt^2}\frac{dy}{dt} = \frac{d\phi}{\cancel{dy}}\frac{\cancel{dy}}{dt}.$$

Applying the chain rule, the LHS becomes

chain rule

$$\boxed{m\frac{dy}{dt}\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2\right)}$$

while the RHS is

$$\frac{d\phi}{dy}\frac{dy}{dt} = \frac{d\phi}{dt}$$

Equation ($\star$) may then be written as

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2\right) = \frac{d}{dt}\phi$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - \phi\right) = 0$$

$$\Rightarrow \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - \phi(y) = E$$

E is the total energy, or simply the energy of the system. It is constant in time, or in physics parlance, **conserved**.

To simplify the equation, let's introduce a function $V(y) = -\phi(y)$. Energy can now be expressed as the **sum** of two terms, rather than the difference

$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + V(y(t)) = E.$$

The first term, $\frac{1}{2}m\left(\frac{dy}{dt}\right)^2$ is a function of velocity (mass is a constant). It is the contribution to total energy due to the object's motion, and is called kinetic energy (K.E). Note that $\left(\frac{dy}{dt}\right)^2 \geq 0$, so kinetic energy is non-negative.

The second term, $V(y(t))$ is a function of position. It's called potential energy (P.E.), because it has the potential to be converted into kinetic energy.

There are several advantages to adopting this formalism:

1. Newton's second law of motion is a second order ODE. The above technique allows any system with a continuous, position-dependent force to be described as a separable, first order ODE. Any such system is solveable in principle.

2. Energy E is seen as an integration constant, calculated from initial position and velocity.

3. Given knowledge of the energy, position can be calculated from velocity, and vice-versa.

We can re-state the result in another way.

If we write

$$y_f = y(t_f) \text{ (Final position)} \qquad v_f = \frac{dy}{dt}(t_f) \text{ (Final velocity)}$$
$$y_0 = y(t_0) \text{ (Initial position)} \qquad v_0 = \frac{dy}{dt}(t_0) \text{ (Initial velocity)},$$

Then we have

$$\frac{1}{2}mv_f^2 + V(y_f) = E = \frac{1}{2}mv_0^2 + V(y_0)$$

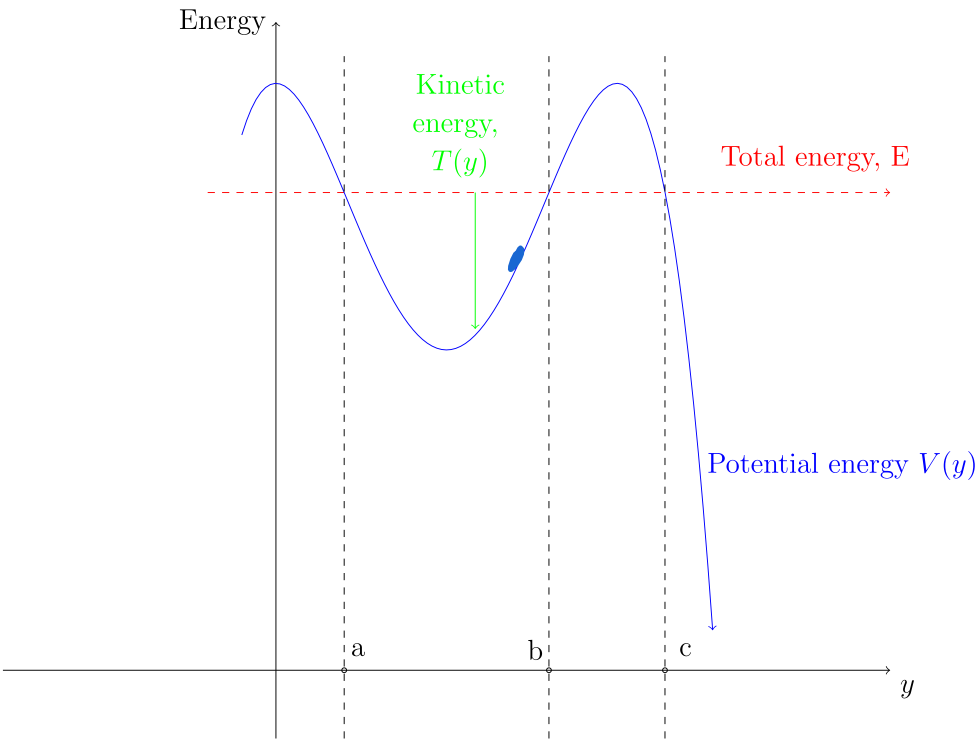
$$\Rightarrow \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2\right) + \left(V(y_f) - V(y_0)\right) = 0$$

Δ kinetic energy $+\ \Delta$ potential energy $= 0$

This calculation shows energy cannot be created or destroyed, but can be changed from one form to another.

4.10.2 Potential Energy diagrams

Since kinetic energy is non-negative, it follows that total energy can never be less than potential energy. We can visualise this for a physical system by graphing V against y using a potential energy diagram:



$F = \frac{d\phi}{dy} = \frac{-dv}{dy}$

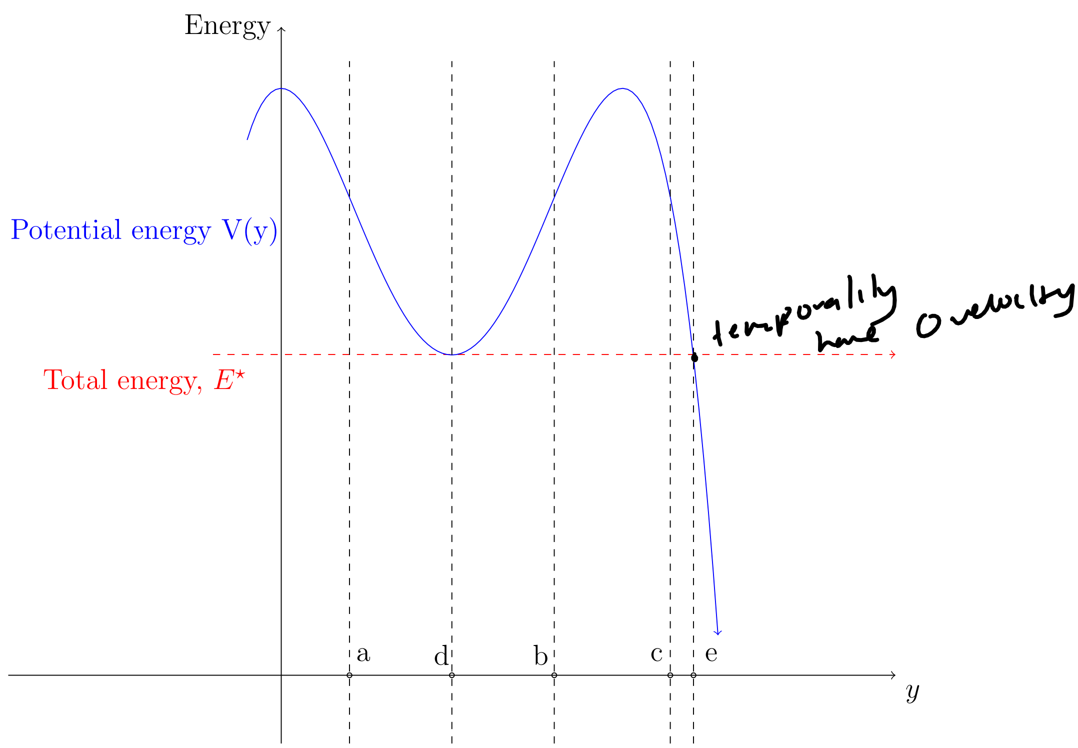
Potential energy diagrams provide a visual description of a system's dynamics. Picture a ball rolling on the hilly terrain imposed by the shape of $V(y)$. If the ball is placed where V has an upward slope ($V'(y) > 0$), your intuition tells you it should be pushed to the left. Indeed, since the force is $F(y) = -V'(y) < 0$, your intuition is correct. For the same reason, if V has a downward slope, it is pushed to the right.

$\Rightarrow \frac{1}{2}mv^2$

For $a < y < b$, $V(y) < E$. This means kinetic energy is positive, and hence motion can take place here. The same is true when $y > c$. On the other hand, the system does not have enough energy to occupy positions $y < a$, or $b < y < c$.

Positions a, b, and c are called turning points, because an object in this system reverses its direction here. Since $V(y) = E$ at these points, kinetic energy must be zero, and so an object must be temporarily at rest.

A system with the same potential energy function $V(y)$ but a different total energy $E^\star$ will behave differently. In the graph below, motion is only possible for $y > e$.



Example: Show that equilibrium solutions to equation ($\star$) can only occur where kinetic energy is 0, and only at critical points of V.

Let $y = \alpha$ be an equilibrium solution

Then $\dot{y} = 0$, so $KE = \frac{1}{2} m 0^2 = 0$

Also $\ddot{y} = 0$, and so $F = ma = m \times 0 = 0 = -V'(\alpha)$

Thus, $V'(\alpha) = 0$ so α must be a crit pt of V

An object at point d in the graph above, with total energy $E^\star$, represents an equilibrium solution.

4.10.3 Conservation of energy under constant gravity.

Consider the acceleration of a particle under a constant gravitational force. (For motion near the Earth's surface, gravitational force is approximately constant).

By Newton's second law ($ma = F$), the ODE governing this sytem is

$$m\frac{d^2y}{dt^2} = -mg. \quad (\#)$$

$g = 9.8\ m/s^2$

There is a negative sign on the right of this equation because gravity pulls down towards the earth, but y is measured as distance up from the the earth. Now also recall that V, the potential energy has been defined as the negative of the integral of the force with respect to distance.

$F = -dV/dt$

$$V(y) = -\int F(y)\, dy$$

$$= -\int -mg\, dy$$

$$= mgy + C$$

Setting c equal to 0, the ODE (#) may be written;

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy + c\right) = 0$$

$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy = E$$

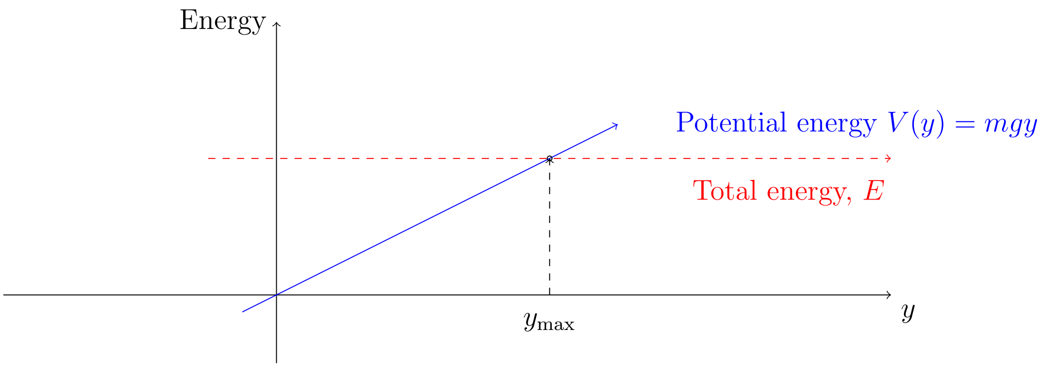
$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy + C = E'$$

As y decreases, the P.E. of the particle decreases and the K.E. increases so that the total energy is conserved.

$$\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + mgy = E' - C$$

$$= E$$

The energy diagram looks like this:



The potential energy function $V(y)$ is a straight line, with slope mg. Given a fixed total energy E, the system cannot occupy positions $y > y_{\max}$. The turning point (and thus the maximum height an object can reach) is $y_{\max}$.

Example: A projectile is launched vertically upward from ground level, with initial speed $y = 98\text{m/s}$. What is the maximum height the projectile will reach?

$\frac{dy}{dt}(t=0) = 98 \qquad y(0) = 0$

Hence, using energy equation with $g = 9.8$

the IC give $\frac{1}{2} m (98)^2 + m \cdot g \cdot 0 = E$

At max height $\frac{dy}{dt} = 0$

$$\frac{1}{2} m \times 0^2 + mgy_{max} = E = \frac{m}{2}(98)^2$$

$$y_{max} = \frac{\frac{m}{2}(98)^2}{mg} = \frac{(98)^2}{2 \times 9.8}$$

490 meters

4.10.4 Elastic potential energy.

$ma = F$

Recall the undamped mass-spring system with equation;

$$m\frac{d^2x}{dt^2} = -kx.$$

$V(x) = -\int F(x)\,dx$

From page (175), the solution of this ODE is

$$x(t) = A\cos(\omega t - \phi).$$

where $\omega = \sqrt{k/m}$.

The potential energy corresponding to the spring force $F = -kx$ can be calculated as

$$V(x) = -\int (-kx)\,dx$$
$$= \frac{1}{2}kx^2 + C$$
$$= \frac{1}{2}m\omega^2x^2 + C$$

Setting $c = 0$, the system is characterised by the energy equation

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega^2x^2 \; + C = E'. \clubsuit$$

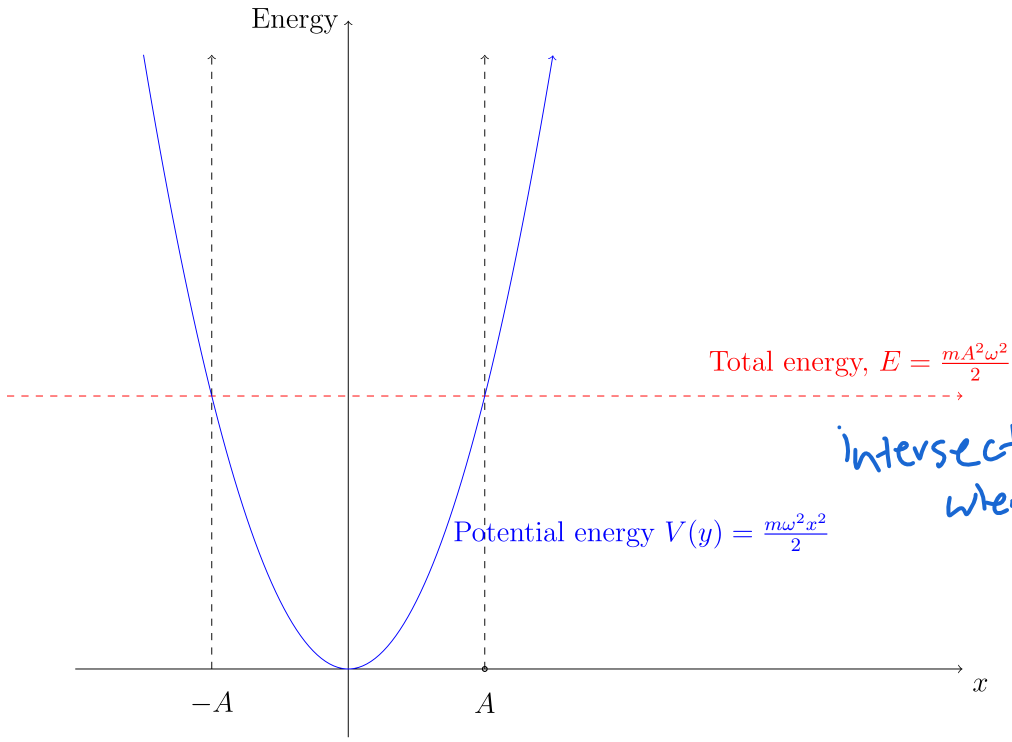
$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega^2x^2 = E' - C$$

Since $x(t) = A\cos(\omega t - \phi)$, $\frac{dx}{dt} = -A\omega\sin(\omega t - \phi)$. Equation $\clubsuit$ becomes

$$E = \frac{1}{2}m\left(A^2\omega^2\sin^2(\omega t + -\phi)\right) + \frac{1}{2}m\omega^2\left(A^2\cos^2(\omega t + -\phi)\right)$$
$$= \frac{m}{2}A^2\omega^2(1) = \frac{m}{2}A^2\omega^2$$

The energy of this system is proportional both to square of the angular frequency, and to the square of the amplitude.

A potential energy diagram provides another way to see this result:



intersect when $x = \pm A$

The amplitude A is the maximum extent of the object's oscillation.

Hence, it represents a turning point, where $V(x) = E$, and kinetic energy is 0.

Hence we can calculate:

$$\boxed{V(A) = \frac{1}{2} m\omega^2 A^2 = E}$$

Example: Recall the undamped, oscillating spring system from page 177. It had a mass of 9kg, and a spring constant of 4N/m. It was pulled down 1m and given an initial upward kick of -0.5m/s.

How fast is the mass moving when it passes through its equilibrium position?

$\omega = \frac{2}{3}$ and $A = 5/4$

Recall that $E = \frac{m}{2}\omega^2 A^2$

At equilibrium point

$$E = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{m}{2}\omega^2 \times 0^2 = \frac{m}{2}\left(\frac{dx}{dt}\right)^2$$

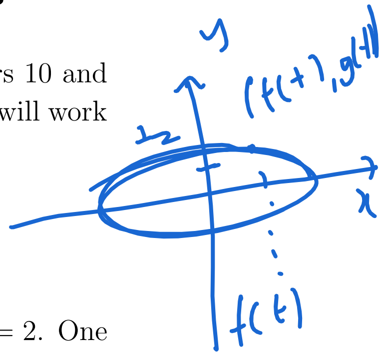
Hence, $\frac{m}{2}\omega^2 A^2 = \frac{m}{2}\left(\frac{dx}{dt}\right)^2$

$$\left|\frac{dx}{dt}\right| = \sqrt{\omega^2 A^2} = \omega A = \left(\frac{2}{3}\right)\left(\frac{5}{4}\right) = \frac{5}{6}$$

4.10.5 Main points

- You should understand how to find the energy equation for any position-dependent force.
- You should be able to determine position and velocity from such an equation.
- You should also be able to draw a potential energy diagram, for a position-dependent force.
- You should be able to interpret system behaviour and determine equilibrium solutions from an energy diagram.

5 Parametrisation of Curves and Line Integrals



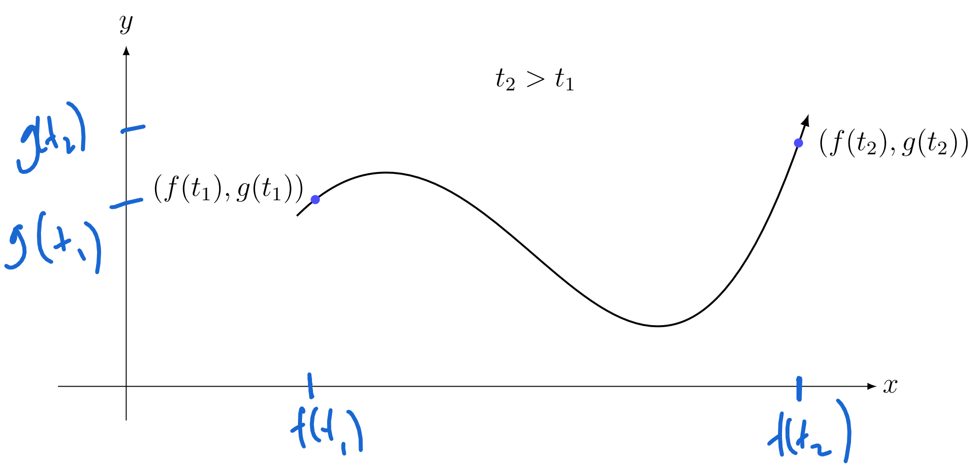
Stewart covers parametrisation in Chapters 10 and (some of) 12 (Chapters 10 and (some of) 12). There are some features of Matlab, especially plotting, that will work with parametric forms.

5.1 Parametrisation of Curves

Imagine some curve in the xy-plane, for example, $x^2 + 2y^2 = 1$ or $3x + y = 2$. One way to think of the curve is as the trajectory of a moving particle, such that at time t the particle is at the point $(x(t), y(t))$ on the curve. In other words, we describe the curve by

$$\begin{cases} x = f(t) \\ y = g(t). \end{cases}$$

Such equations for the curve are called parametric equations, and the auxiliary variable t, is called a parameter. The functions f and g are said to provide a parametrisation of the curve.

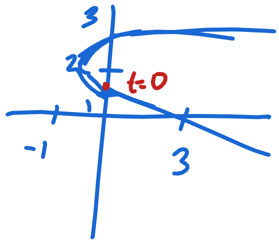


Before we look at some examples, it should be clear from our mental picture of a moving particle that a parametrisation of a curve is not unique. For example,

$$\begin{cases} x = F(t) \\ y = G(t), \end{cases}$$

where, say $F(t) = f(3 - t)$ and $G(t) = g(3 - t)$, is another parametrisation of the same curve; we simply have changed the velocity and initial (i.e., $t = 0$) position of the imaginary particle.

5.1.1 Sketching parametric curves



Example: Sketch the curve parametrised as

$$\begin{cases} x = t^2 - 2t & (1) \\ y = t + 1. & (2) \end{cases}$$

Give an equation for the curve in terms of just the Cartesian coordinates x and y and give two alternative parametrisations of the curve.

(2) $t = y - 1$ plug into (1) $\Rightarrow x = (y-1)^2 - 2(y-1)$

$= y^2 - 2y + 1 - 2y + 2 \qquad x = y^2 - 4y + 3 = (y-2)^2 - 1$

Another parametrisation

Set $s = t+1 \Rightarrow t = s-1$

$x = (s-1)^2 - 2(s-1) = s^2 - 2s + 1 - 2s + 2 = s^2 - 4s + 3$

$y = s - 1 + 1 = s$

$s = 0 \Rightarrow y = 0$

3. Another parametrisation (reversing time)

Set $s = -t \Rightarrow t = -s$

then $x = (-s)^2 + 2s = s^2 + 2s$

$y = -s + 1$

$s = 0,\ y = 1$

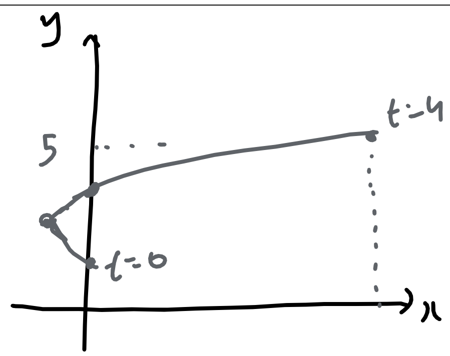
Sometimes one has reasons to restrict t to a finite interval. In general, the curve with parametric equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad a \leq t \leq b$$

starts at the point $(f(a), g(a))$ and finishes at the point $(f(b), g(b))$. One has to be careful though, since even a finite time-interval can still give rise to an infinitely-long curve.

$x=(y-2)^2-1$

Example: Sketch the curve $\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases} \qquad 0 \leq t \leq 4.$



$x(0)=0$ $\quad x(1)=1-2=-1$

$y(0)=1$ $\quad y(1)=2$

$x(4)=8$

$y(4)=5$

Example: Sketch the Archimedean spiral given by $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$ $\quad 0 \leq t \leq \dfrac{\pi}{2}$

and verify that it may also be described by $x^2 + y^2 = \arctan^2(y/x)$.

$x\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$

$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

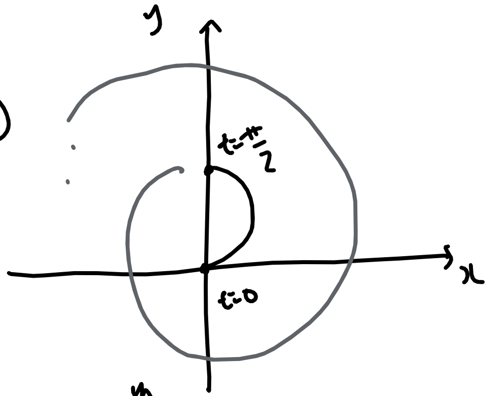
Note that

$x^2+y^2 = t^2\cos^2 t + t^2\sin^2 t$

$= t^2(\cos^2 t + \sin^2 t) = t^2$ (1)

$\frac{y}{x} = \frac{t\sin t}{t\cos t} = \tan t$

$t = \arctan\left(\frac{y}{x}\right)$ (2)



(1) & (2) $\Rightarrow x^2+y^2 = \arctan^2\left(\frac{y}{x}\right)$

If you like spirals you could also try to sketch

$$\begin{cases} x = \dfrac{\cos t}{t} \\ y = \dfrac{\sin t}{t} \end{cases} \quad t > 0; \qquad \begin{cases} x = \dfrac{\cos t}{\sqrt{t}} \\ y = \dfrac{\sin t}{\sqrt{t}} \end{cases} \quad t > 0; \qquad \begin{cases} x = \mathrm{e}^{-t}\cos t \\ y = \mathrm{e}^{-t}\sin t \end{cases} \quad t \in \mathbb{R}.$$

For the true connoisseurs there even is the amazing Cornu spiral; who said maths couldn't be fun?

$$\begin{cases} x = \displaystyle\int_0^t \cos(s^2)\,\mathrm{d}s \\ y = -\displaystyle\int_0^t \sin(s^2)\,\mathrm{d}s, \end{cases} \qquad t > 0.$$

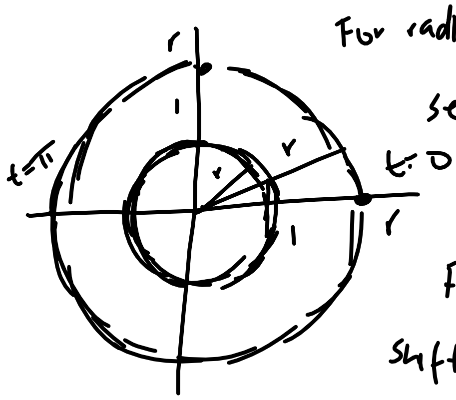
5.1.2 Parametrising circles and ellipses

Example: Parametrise the circle of radius r centered at (a, b).

For $(a,b) = (0,0)$ $r = 1$

want $x^2 + y^2 = 1$

set $x(t) = \cos t$ $y(t) = \sin t$ $0 \le t < 2\pi$



For radius r $x^2 + y^2 = r^2$

set $x(t) = r\cos t$

$y(t) = r\sin t$ $0 \le t < 2\pi$

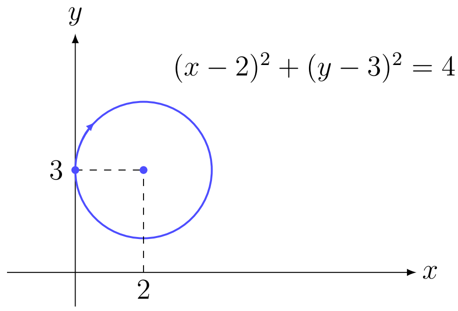
For any center (a,b)

shift horizontally by a

vertically by b

$x(t) = a + r\cos t$ $y(t) = b + r\sin t$ $0 \le t < 2\pi$

Example: Parametrise the circle with radius 2 centered at $(2,3)$, such that the imaginary particle tracing this circle is travelling in the clockwise direction and at time $t=0$ is at $(0,3)$.

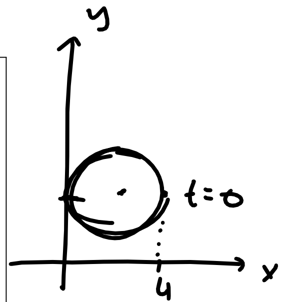


A parametrisation:

$x(t)=2+2\cos t \qquad 0 \le t < 2\pi$
$y(t)=3+2\sin t$

$t=0$
$x(0)=4$
$y(0)=3$

$t=\pi$
$x(\pi)=0$
$y(\pi)=3$

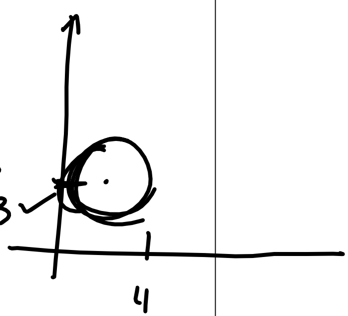


• shift t by π

$x(t)=2+2\cos(t+\pi) \qquad 0 \le t < 2\pi$
$y(t)=3+2\sin(t+\pi)$

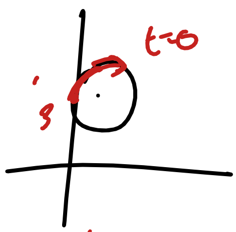
counterclockwise!

$t=0$
$x(0)=6$
$y(0)=3$



· measure angle backwards

$x(t)=2+2\cos(-t+\pi)$
$y(t)=3+2\sin(-t+\pi)$



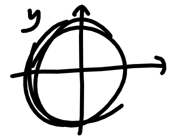
alternative: flip y coordinate about $y=3$ (center)

$x(t)=2+2\cos(t+\pi)$
$y(t)=3-2\sin(t+\pi)$

Example: Parametrise the ellipse

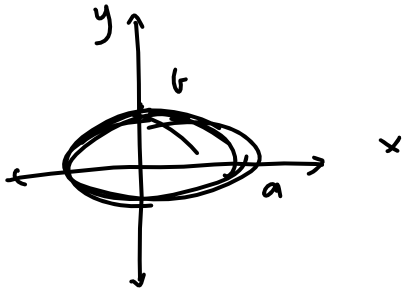
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

unit circle



$x(t) = \cos t$
$y(t) = \sin t$

Scale x coord by a
y coord by b



$x(t) = a\cos t$
$y(t) = b\sin t$

$0 \le t < 2\pi$

Check:

$$\frac{x(t)^2}{a^2} + \frac{y(t)^2}{b^2} = \frac{\cancel{a^2}\cos^2 t}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2 t}{\cancel{b^2}} = 1$$

A much more fancy parametrisation may be given using Jacobi's elliptic functions sn, cn and dn. You will hopefully learn about these functions in a more advanced mathematics course.

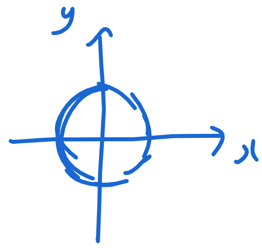
5.1.3 Parametric curves in $\mathbb{R}^3$

In general, a curve in three dimensions can be parametrised in the form

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t). \end{cases}$$

A helix is a three-dimensional curve that has shape of a wound spring, or a piece of string wrapped around a cylinder.

One possible parametrisation of the helix is



$$\begin{cases} x = r\cos(\pi t) \\ y = r\sin(\pi t) \\ z = t \end{cases}$$

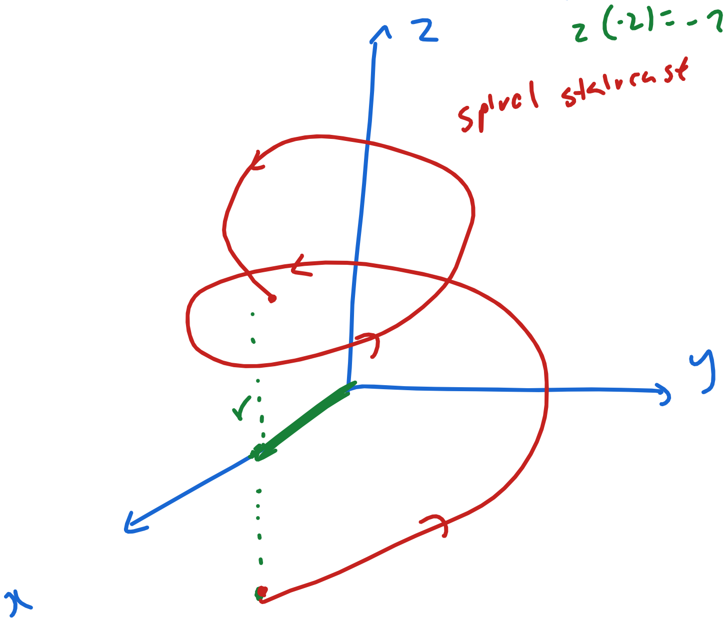
for $t \in \mathbb{R}$.

$x(2) = r\cos(2\pi) = r$
$y(-2) = 0$
$z(-2) = -2$

Example: Sketch the helix for $t \in [-2, 2]$.

spiral staircase

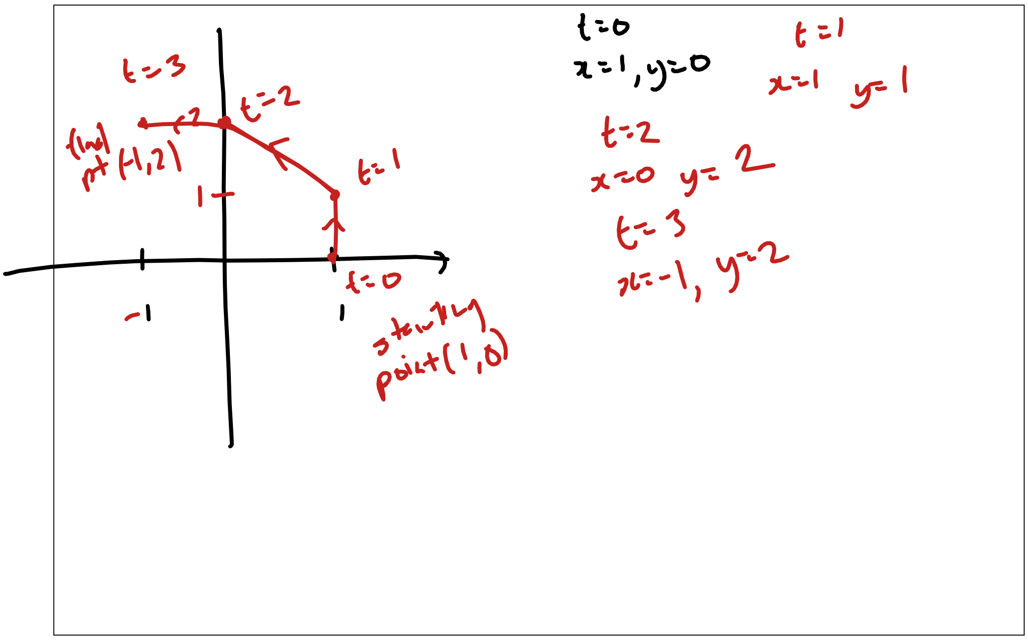
$x(2\pi) = r$ (cos)
$y(2) = 0$
$z(2) = 2$



5.1.4 Piecewise linear paths

Example: Sketch the curve parametrised by

$$\begin{aligned}
x(t) &= 1, & y(t) &= t, & 0 &\le t \le 1\\
x(t) &= 2-t, & y(t) &= t, & 1 &< t \le 2\\
x(t) &= 2-t, & y(t) &= 2, & 2 &< t \le 3.
\end{aligned}$$



5.1.5 Plotting parametrised equations in Matlab

The ezplot function handles parametric plots. To plot an ellipse, the command is simply

```
ezplot('2*cos(t)','1/3*sin(t)')
```

To plot a spiral, simply use

```
ezplot('exp(-t/100)*cos(t)','exp(-t/100)*sin(t)')
```

Executing this command Matlab uses a default range for the t values. To get a longer spiral, $0 \le t \le 100$, run

```
ezplot('exp(-t/100)*cos(t)','exp(-t/100)*sin(t)',[0,100])
```

5.1.6 Main points

- You should understand what a parametric representation of a curve is.
- You should be able to parametrise circles and ellipses.
- You should be able to sketch curves from their parametric representation. This includes being able to sketch parametric representations of parabolas, circles, ellipses, spirals, helices and piecewise linear paths.
- You should be able to plot parametric forms in Matlab.

5.2 Position Vectors, Velocity and Acceleration

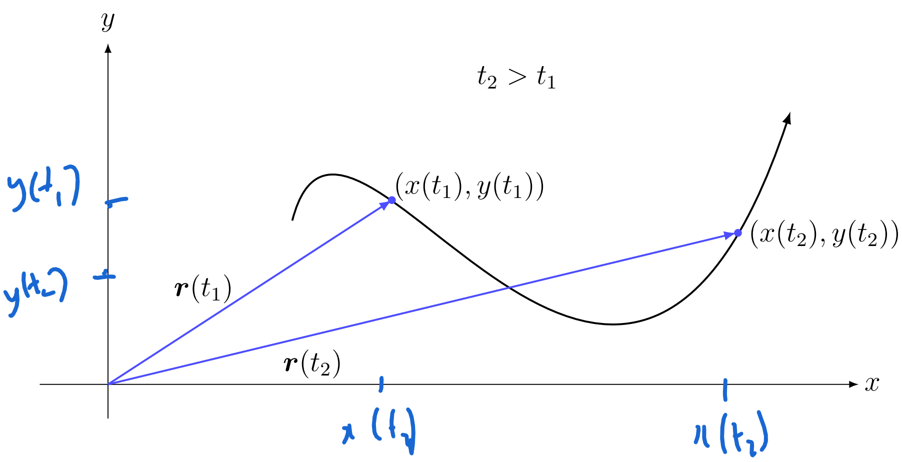
Stewart covers the following material in Section 13.4 (Section 13.4). Parametric equations are used to describe position, velocity and acceleration.

5.2.1 Position vector

The position vector

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

traces out the path given by the parametrisation $(x(t), y(t))$.



Similarly in three dimensions, the position vector

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

traces out the path given by the parametrisation $(x(t), y(t), z(t))$.

Radial distance is the magnitude of position:

$$\text{radial distance} = r(t) = ||\mathbf{r}(t)|| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}.$$

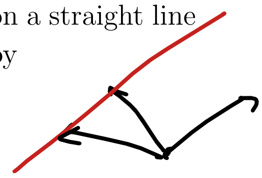
Example: Find the position vector of a car moving along the helical path

$$x = \cos 5t, \quad y = \sin 5t, \quad z = t.$$

$\vec{r}(t) = \cos(5t)\,\vec{i} + \sin 5t\,\vec{j} + t\vec{k}$

5.2.2 Parametrising straight lines

We know from previous work that the position vector of any point on a straight line parallel to the vector $\boldsymbol{v}$ and passing through the point $\boldsymbol{r}_0$ is given by



$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v}.$$

Example: Find the position vector of any point on the line passing through the point $(0, 1, 3)$ and parallel to the vector $2\boldsymbol{i} - 3\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$.

$\vec{r}_0 = (0,1,3)$ $\vec{r} = 2\vec{i} - 3\vec{j} + \frac{1}{2}\vec{k}$

$\vec{r}(t) = (0\vec{i} + \vec{j} + 3\vec{k}) + t(2\vec{i} - 3\vec{j} + \frac{1}{2}\vec{k})$

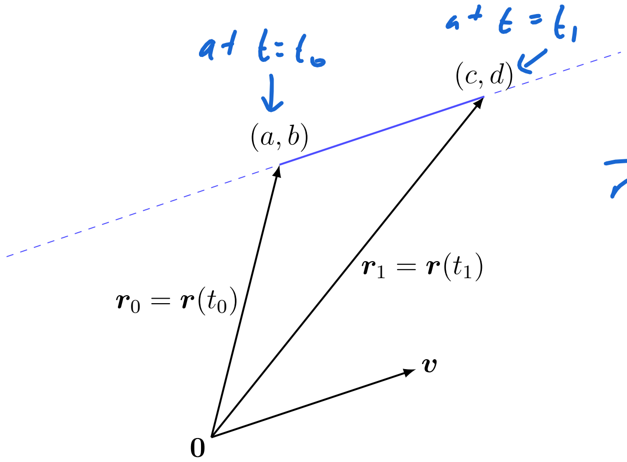
$= 2t\,\vec{i} + (1-3t)\,\vec{j} + (3 + \frac{1}{2}t)\,k$

Say we want to parametrise the straight line between the points (a, b) and (c, d). We also require that (a, b) corresponds to parameter value $t = t_0$, and (c, d) corresponds to $t = t_1$, with $t_1 = t_0 + \Delta t$ and $\Delta t > 0$.

Let $\boldsymbol{r}_0$ be the position vector for (a, b) and $\boldsymbol{r}_1$ the position vector for (c, d). The vector

$$\boldsymbol{v} = \boldsymbol{r}_1 - \boldsymbol{r}_0$$

is a direction vector for the line that goes through (a, b) and (c, d).



$\vec{r}_0 = (a, b)$
$\vec{v}_1 = (c, d)$

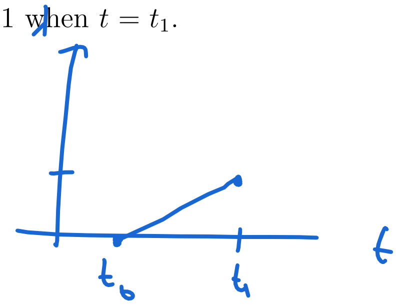
We know that the position vector for points on the line through $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ is

$\vec{r}_0 + \lambda(\vec{r}_1 - \vec{r}_0)$

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v} = (1 - \lambda)\boldsymbol{r}_0 + \lambda \boldsymbol{r}_1, \qquad \lambda \in \mathbb{R}.$$

So all we need to do to parametrise the portion of the line from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$ is choose a relationship between λ and t that gives $\lambda = 0$ when $t = t_0$ and $\lambda = 1$ when $t = t_1$. The linear relation between λ and t that does this is

$$\lambda = \frac{t - t_0}{\Delta t}. \quad = \frac{t - t_0}{t_1 - t_0}$$

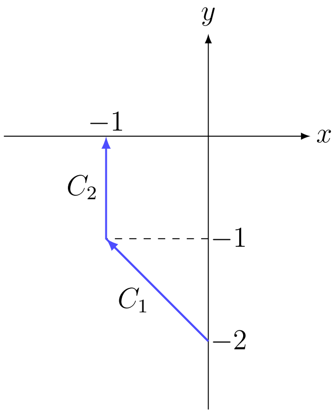


A parametrisation for the line segment is therefore

$$\boxed{\boldsymbol{r}(t) = \boldsymbol{r}_0 + \frac{t - t_0}{\Delta t}\boldsymbol{v}, \qquad t_0 \le t \le t_1.}$$

$$= \vec{r}_0 + \frac{t - t_0}{t_1 - t_0}(\vec{r}_1 - \vec{r}_0)$$

Example: Parametrise the path given in the following diagram, so that path C_1 is given for $0 \le t \le 1$ and the path C_2 is given for $1 \le t \le 3$.



a) $t \in [0,1]$ $\quad \vec{r}_0 = (0,-2)$ $\quad \vec{r}_1 = (-1,-1)$

$t_0 = 0 \quad t_1 = 1$

$$\vec{r}(t) = \vec{r}_0 + \frac{t-t_0}{t_1-t_0}(\vec{r}_1 - \vec{r}_0)$$

$$= (0,-2) + \frac{t-0}{1-0}\big((-1,-1)-(0,-2)\big)$$

$$= (0,-2) + t(-1,1) = \begin{pmatrix}-t \\ -2+t\end{pmatrix}$$

b) $t \in [1,3]$ $\quad \vec{r}_0 = (-1,-1)$ $\quad \vec{r}_1 = (-1,0)$

$t_0 = 1 \quad t_1 = 3$

$$\vec{r}(t) = \begin{pmatrix}-1 \\ -1\end{pmatrix} + \frac{t-1}{3-1}\big((-1,0)-(-1,-1)\big)$$

$$= \begin{pmatrix}-1 \\ -1\end{pmatrix} + \frac{t-1}{2}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}-1 \\ \frac{t}{2} - \frac{3}{2}\end{pmatrix}$$

parametrisation :)

$$\vec{r}(t) = \begin{cases} (-t, -2+t) & 0 \le t \le 1 \\ (-1, \frac{t}{2} - \frac{3}{2}) & 1 \le t \le 3 \end{cases}$$

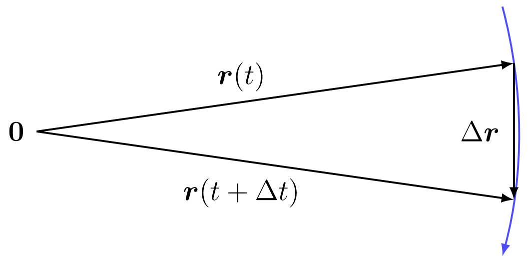
5.2.3 Velocity vector

The vector tangent to the path of motion, with magnitude equal to the speed is the velocity vector. If $\boldsymbol{r}(t)$ is the position vector, then

$$\Delta \boldsymbol{r} = \boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)$$

is approximately tangent to the curve traced out by $\boldsymbol{r}(t)$. The approximation gets better as $\Delta t \to 0$.

time derivative of position



The velocity vector $\boldsymbol{v}$ is given by

$$\begin{aligned}\boldsymbol{v}(t) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} \\ &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}.\end{aligned}$$

tangent to curve

In component-form, if $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, then the velocity is

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\boldsymbol{k}.$$

Speed is the magnitude of velocity:

$$v(t) = \|\boldsymbol{v}(t)\| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

can be in diff directions

Note that if an object moves with constant velocity, this means that it has constant speed and direction, i.e., it travels in a straight line.

5.2.4 Acceleration

The acceleration vector is defined as

$$\boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,\boldsymbol{i} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\,\boldsymbol{j} + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\,\boldsymbol{k},$$

$\frac{d^2v}{dt^2}$

with magnitude

$$a(t) = \|\mathbf{a}\| = \sqrt{\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right)^2 + \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)^2 + \left(\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\right)^2}.$$

Example: The velocity of an object is $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and at $t = 0$ the object passes through $(-2, 1, 0)$. Find the position vector.

velocity $\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}j + \frac{dz}{dt}k = 2i + 3j + 4k$

position $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$x(t) = \int \frac{dx}{dt}\, dt = \int 2\, dt = 2t + c_1$

$x(0) = -2 \Rightarrow c_1 = -2$

$y(t) = \int \frac{dy}{dt}\, dt = \int 3\, dt = 3t + c_2$ $\qquad c_1, c_2, c_3 \in \mathbb{R}$

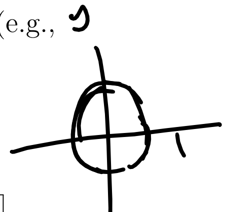
$y(0) = 1 \Rightarrow c_2 = 1$

$z(0) = \int \frac{dz}{dt}\, dt = \int 4\, dt = 4t + c_3$

$z(0) = 0 \quad c_3 = 0$

position vector $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} = (2t - 2)\vec{i} + (3t + 1)\vec{j} + (4t)\vec{k}$

Example: Find the velocity vector of a car moving along the helical path (e.g., the Indooroopilly Shopping Centre carpark)



$\frac{dx}{dt} = -5\sin 5t \quad \frac{dy}{dt} = 5\cos 5t \quad \frac{dz}{dt} = 1$

$$x = \cos 5t, \quad y = \sin 5t, \quad z = t.$$

5x faster

$\vec{v}(t) = -5\sin(5t)\vec{i} + 5\cos(5t)\vec{j} + \vec{k}$

constant vertical velocity

$\Downarrow$

no acceleration in vertical direction

acceleration

$\vec{a}(t) = \frac{d\vec{v}}{dt} = -25\cos(5t)\vec{i} - 25\sin(5t)\vec{j} + 0\vec{k}$

speed $= \sqrt{(-5\sin(5t))^2 + (5\cos(5t))^2 + 1^2}$

$= \sqrt{25\sin^2 5t + 25\cos^2 5t + 1}$

$= \sqrt{25 + 1} = \sqrt{26}$

Example: The position vector of an object undergoing uniform circular motion with radius A and angular frequency ω is

$$\mathbf{r}(t) = A\cos(\omega t)\boldsymbol{i} + A\sin(\omega t)\boldsymbol{j}.$$

Show that its acceleration, speed, and distance from the origin are related by

$$a = \frac{v^2}{r}.$$

The force which changes the direction of an object's motion is called the Centripetal force.

5.2.5 Extra reading: stone on a string

Imagine you spin a stone on a string above your head so that it moves in a circle of radius 1m at a height of 2m above the ground at a constant speed with period π seconds. Suppose the string breaks when $t = 0$.

Find the position, velocity and acceleration vectors of the stone both before and after the string breaks.

Before the string breaks

$$\boldsymbol{r}(t) = \cos 2t\boldsymbol{i} + \sin 2t\boldsymbol{j} + 2\boldsymbol{k}$$

$$\boldsymbol{v}(t) = -2\sin 2t\boldsymbol{i} + 2\cos 2t\boldsymbol{j}$$

$$\mathbf{a}(t) = -4\cos 2t\boldsymbol{i} - 4\sin 2t\boldsymbol{j},$$

so that $\boldsymbol{r}(0) = \boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{v}(0) = 2\boldsymbol{j}$.

After the string breaks the stone moves under gravity

$$m\boldsymbol{a}(t) = -mg\boldsymbol{k} \quad \Rightarrow \quad \boldsymbol{a}(t) = -g\boldsymbol{k} \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -g\boldsymbol{k}.$$

Hence

$$\boldsymbol{v} = -gt\boldsymbol{k} + \boldsymbol{v}(0) = 2\boldsymbol{j} - gt\boldsymbol{k}.$$

Since $\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}$ this in turn implies that

$$\boldsymbol{r}(t) = 2t\boldsymbol{j} - \frac{1}{2}gt^2\boldsymbol{k} + \boldsymbol{r}(0)$$

so that

$$\boldsymbol{r}(t) = \boldsymbol{i} + 2t\boldsymbol{j} + \Big(2 - \frac{1}{2}gt^2\Big)\boldsymbol{k}.$$

5.2.6 Quiz

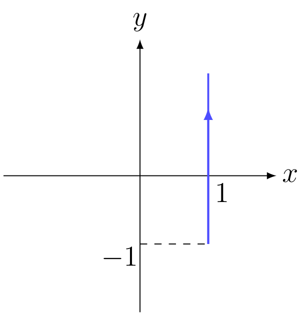
Can you match each position vector with one of the curves? Then locate the points corresponding to $t = 0$.

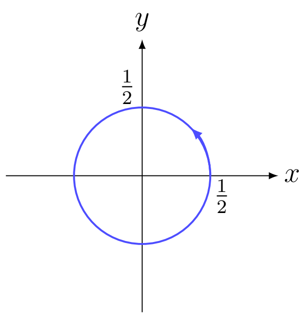
(a) $\boldsymbol{r}(t) = (-t, 3/2 - t)$

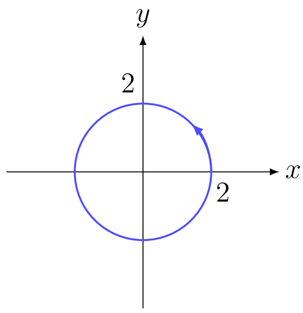
(b) $\boldsymbol{r}(t) = (1, t - 1)$

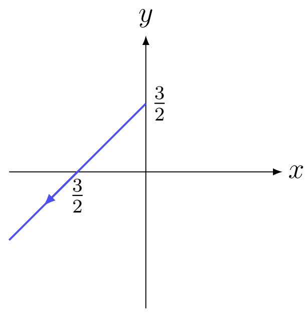
(c) $\boldsymbol{r}(t) = \dfrac{1}{2}(-\sin t, \cos t)$

(d) $\boldsymbol{r}(t) = 2(\cos 3t, \sin 3t)$.






y
2
x
2


y
3/2
x
3/2

5.2.7 Main points

- You should understand the relationship between a position vector and the parametric equations of a curve.
- You should understand the relationship between position, velocity and acceleration vectors.
- You should be able to parametrise lines and piecewise linear paths.

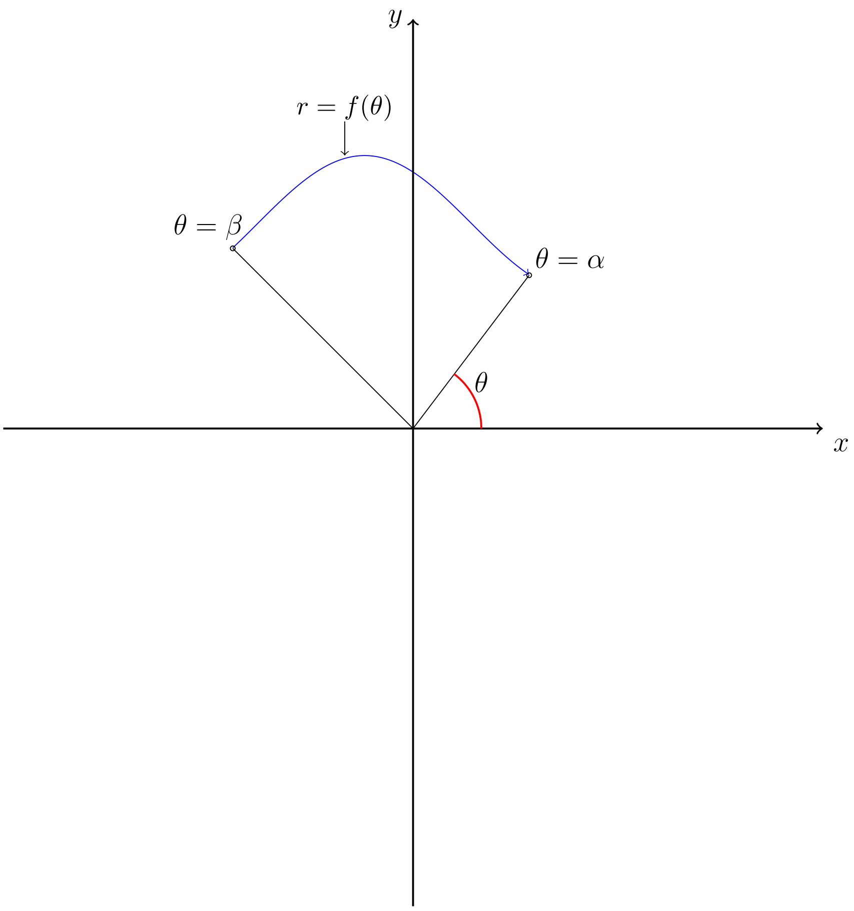
5.3 Polar curves

This material is covered in Stewart, Section 10.4-10.6.

So far, we have mostly discussed curves of the form $y = f(x)$. A polar curve on the other hand, is the set of points whose polar coordinates satisfy an equation of the form

$$r = f(\theta).$$

Here r is the distance from the origin, and θ is the angular displacement from the positive x-axis.



Any polar curve can be parameterised by;

Example: Express the polar curve $r = \dfrac{l}{1 + e\cos\theta}$ in cartesian form.

On page 17, we saw that a conic section could be formed by the intersection of a plane $z = l - ex - dy$, and the double cone, $z^2 = x^2 + y^2$, resulting in the formula

$$(l - ex - dy)^2 = x^2 + y^2.$$

Question: What kind of a curve is $r = \dfrac{l}{1 + e\cos\theta}$?

5.3.1 Main points

- You should understand polar curves, and their uses.
- You should be able to identify conic sections expressed in their polar form.

5.4 Line Integrals and Work Done

The content of this section is covered in Stewart over a few sections: arc length in Sections 8.1 and 10.4 (Sections 8.1 and 10.4) and general line integrals in 16.2 (16.2).

5.4.1 Length of a curve

Say we have a curve in $\mathbb{R}^3$ parametrised by

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b.$$

We aim to calculate the length of the curve. This is commonly referred to as the arc length.

We can work out how to do this by first approximating the curve as a series of straight lines. We can draw a picture of what we are doing for a curve in $\mathbb{R}^2$, but we can do the same thing for a curve in $\mathbb{R}^3$.

In this case, with Δs_i representing the length of a small segment of the curve corresponding to a small change Δt in the parameter from t_i to $t_i + \Delta t$, and given $\boldsymbol{r}(t) = (x(t), y(t))$ we have the approximation

$$\Delta s_i \approx \left\| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right\|_{t=t_i} \Delta t.$$

The same formula applies in $\mathbb{R}^3$ and then the length of the curve is approximately the sum of the lengths of each line segment:

This approximation improves for smaller (and hence more) line segments. In the limit when the number of line segments tends to infinity, we obtain an exact result, giving the arc length formula

$$\text{arc length } = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

This formula is often written more succinctly using the notion of a line integral. If the function $s(t)$ gives the length of the curve C between $\boldsymbol{r}(a)$ and $\boldsymbol{r}(t)$ — so that the arc length we are after is nothing but $s(b)$ — then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

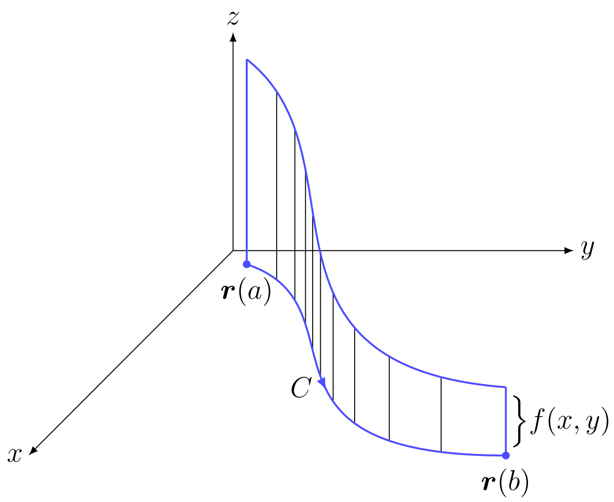
Then the length of the arc between $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$ is given by the line integral

$$\text{arc length } = \int_C \mathrm{d}s.$$

In general, if $f(x, y)$ is defined on a smooth curve C in $\mathbb{R}^2$ then the line integral of f along C is

$$\int_C f(x, y)\,\mathrm{d}s.$$

If C is parametrised by $\boldsymbol{r}(t)$, $a \leq t \leq b$ and $f(x, y) > 0$, then this line integral gives the area of a ribbon whose base is the curve C and height above the point (x, y) is $f(x, y)$.



Line integrals are a generalisation of one-dimensional definite integrals.

Example: Find the length of the helix $\begin{cases} x = \cos 10t \\ y = \sin 10t \\ z = t \end{cases} \qquad 0 \leq t \leq \pi.$

Example: Use the arc length formula to show that the length of a semicircular arc of radius 2 is 2π.

Example: Find the length of the spiral

$$\begin{cases} x(t) = \mathrm{e}^{-t/10} \cos t \\ y(t) = \mathrm{e}^{-t/10} \sin t \end{cases} \qquad \text{for } t \geq 0.$$

Solution:

We compute

$$\begin{aligned} \boldsymbol{r}(t) &= \mathrm{e}^{-t/10}(\cos t\, \boldsymbol{i} + \sin t\, \boldsymbol{j}) \\ \boldsymbol{v}(t) &= \mathrm{e}^{-t/10}\left(-\frac{1}{10}\cos t - \sin t\right)\boldsymbol{i} + \mathrm{e}^{-t/10}\left(-\frac{1}{10}\sin t + \cos t\right)\boldsymbol{j} \\ \|\boldsymbol{v}(t)\| &= \sqrt{\left(\frac{1}{100} + 1\right)\mathrm{e}^{-t/5}} = \frac{1}{10}\sqrt{101}\,\mathrm{e}^{-t/10}. \end{aligned}$$

Hence

$$\text{arc length } = \int_0^\infty \|\boldsymbol{v}(t)\|\, \mathrm{d}t = \frac{1}{10}\sqrt{101}\int_0^\infty \mathrm{e}^{-t/10}\mathrm{d}t.$$

The improper integral can be computed as

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-t/10}\mathrm{d}t &= \lim_{A\to\infty}\int_0^A \mathrm{e}^{-t/10}\mathrm{d}t \\ &= -10\lim_{A\to\infty}\left[\mathrm{e}^{-t/10}\right]_0^A \\ &= -10\lim_{A\to\infty}\left(\mathrm{e}^{-A/10} - 1\right) \\ &= 10. \end{aligned}$$

Therefore the arc length of the spiral is $\sqrt{101}$.

Example: Find the length of the parabola $y = x^2$ for x between 0 and 2.

Solution: First we parametrise the parabola as $x = t$, $y = t^2$ for $0 \leq t \leq 2$. Hence

$$\boldsymbol{r}(t) = (t, t^2) \quad \Rightarrow \quad \boldsymbol{v}(t) = (1, 2t)$$

$$\Rightarrow \|\boldsymbol{v}(t)\| = \sqrt{1 + (2t)^2} \quad \Rightarrow \quad \text{arc length } = \int_0^2 \sqrt{1+(2t)^2}\,\mathrm{d}t.$$

(If you are really clever you can perhaps show that this gives $\sqrt{17} + \operatorname{arcsinh}(4)/4$.)

5.4.2 Work done by a constant force

In one dimension, the work done by a constant force F in moving an object along a straight line of length d is $W = Fd$.

In two or three dimensions, the work done by a constant force in moving a particle along a straight line from P to Q is $W = F \cdot \overrightarrow{PQ}$.

5.4.3 Work done over a curve

We now consider the more general case of the work done by a force field

$$\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k}$$

which moves an object along a curve C.

First we give an approximation: divide C into n arcs, so that the ith arc has length Δs_i.

We approximate Δs_i as a straight line (for Δs_i small enough), and $\boldsymbol{F}$ as constant over Δs_i by evaluating $\boldsymbol{F}$ at a specified point P_i on the arc. Let $\boldsymbol{T}(P_i)$ be a unit tangent vector to the curve C at the point P_i. We then approximate the direction over the arc as the vector $\boldsymbol{T}(P_i)\Delta s_i$.

Thus the work done over the ith arc is approximately

$$W_i \approx \boldsymbol{F}(P_i) \cdot \boldsymbol{T}(P_i)\Delta s_i.$$

Summing up over all arcs gives an approximation to the total work done:

$$W \approx \sum W_i \approx \sum \boldsymbol{F}(P_i) \cdot \boldsymbol{T}(P_i)\Delta s_i.$$

Taking the limit $\Delta s_i \to 0$ gives

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s.$$

This last integral, which is an integral over the curve C, is again known as a line integral.

To evaluate the line integral, we use a parametrisation of the curve C. Let C be parametrised by

$$\boldsymbol{r}(t) = (x(t), y(t), z(t)), \qquad a \leq t \leq b.$$

We let Δt represent a small change in t corresponding to Δs_i. Let P_i correspond to parameter value t_i.

As we saw previously, as $\Delta s_i \to 0$,

$$\Delta s_i \approx \|\boldsymbol{r}'(t_i)\|\Delta t,$$

and we can write the unit tangent vector to the curve at the point P_i in terms of the parametrisation as

$$\boldsymbol{T}(P_i) = \frac{\boldsymbol{r}'(t_i)}{\|\boldsymbol{r}'(t_i)\|}.$$

Hence the work done over the ith arc is approximately

$$W_i \approx \boldsymbol{F}(\boldsymbol{r}(t_i)) \cdot \boldsymbol{r}'(t_i)\Delta t.$$

Summing up over all arcs gives an approximation to the total work done:

$$W \approx \sum W_i \approx \sum \boldsymbol{F}(\boldsymbol{r}(t_i)) \cdot \boldsymbol{r}'(t_i)\Delta t.$$

Finally, taking the limit as $\Delta t \to 0$, we have

$$W = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t.$$

It is common to express these integrals as

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}.$$

This is merely a notational convenience. This notation means

$$\text{work done} = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t,$$

where C is parametrised by $\boldsymbol{r}(t)$, $a \le t \le b$.

Example: Evaluate $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F} = (xy, yz, zx)$ and C is parametrised as

$$\begin{cases} x = t \\ y = t^2 \qquad 0 \le t \le 1. \\ z = t^3 \end{cases}$$

Example: Find the work done by gravity on a 2 kg mass along an arc of the circle $x^2 + z^2 = h^2$ from $(-h, 0, 0)$ to $(0, 0, h)$:
a) anticlockwise from $(0, 0, h)$ to $(-h, 0, 0)$.

b) clockwise from $(0, 0, h)$ to $(-h, 0, 0)$.

Does the direction we choose matter?

5.4.4 Main points

- You should know how to use a line integral to calculate the arc length of a curve.
- You should know how to use a line integral to calculate the work done by a force field in moving a particle along a parametrised path.

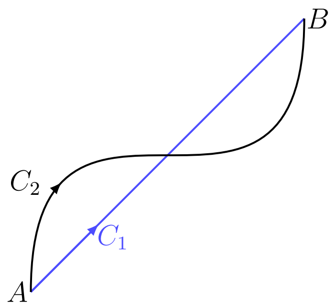
5.5 Gradient Fields

Gradient fields are a special type of force field which are conservative, and they give a relatively easy way to determine if the field is conservative. Stewart covers this in Sections 16.1 and 16.3 (Sections 16.1 and 16.3).

5.5.1 Conservative fields

If $\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is independent of the path taken, then $\boldsymbol{F}$ is called a conservative field.

That is, if $\boldsymbol{F}$ is conservative then the line integral $\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ will give the same result for any path C that you choose between A and B.

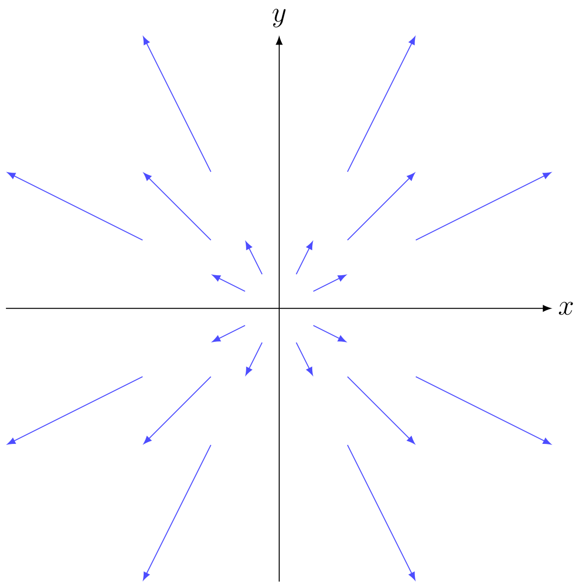


Example: Gravity is a conservative field because

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -mg \int_A^B \mathrm{d}z = -mg(z(B) - z(A)),$$

and it does not matter how you get from A to B. Any constant field must be conservative by the same reasoning.

Example: Take the field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$, which you can picture as a set of vectors $\boldsymbol{F}(x, y)$ at each point (x, y).



Since

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = (x\boldsymbol{i} + y\boldsymbol{j}) \cdot (\mathrm{d}x\boldsymbol{i} + \mathrm{d}y\boldsymbol{j}) = x\,\mathrm{d}x + y\,\mathrm{d}y,$$

we get

$$\begin{aligned} \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_A^B x\,\mathrm{d}x + y\,\mathrm{d}y \\ &= \frac{1}{2}\int_A^B \mathrm{d}(x^2 + y^2) = \frac{1}{2}\Big[x^2 + y^2\Big]_A^B. \end{aligned}$$

Once again $\boldsymbol{F}$ is a conservative field. But it was chosen in rather a special way. $\boldsymbol{F}$ is a gradient field, i.e., $\boldsymbol{F} = \nabla f$ for some f. In this case $f(x, y) = (x^2 + y^2)/2$ since

$$\boldsymbol{F} = \nabla\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = (x, y).$$

Theorem: Any gradient field $\boldsymbol{F}$ is conservative and hence $\displaystyle\int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$ is path-independent.

To show this, let $\boldsymbol{F} = \nabla f = \dfrac{\partial f}{\partial x}\boldsymbol{i} + \dfrac{\partial f}{\partial y}\boldsymbol{j}$. Then using the chain rule,

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= \frac{\mathrm{d}f}{\mathrm{d}t}.\end{aligned}$$

The work done is given by

$$\begin{aligned}\int_A^B \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int_a^b \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t\\ &= \int_A^B \mathrm{d}f\\ &= f(B) - f(A).\end{aligned}$$

Important remark: The above means that finding the work done by a conservative field really comes down to finding a function f such that $\boldsymbol{F} = \nabla f$. This function f is called a potential function for the force field $\boldsymbol{F}$. If you can find a potential function f such that $\nabla f = \boldsymbol{F}$ then

$$\int_A^B \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = f(B) - f(A).$$

Important remark: This equation is by far the simplest way to evaluate the work done by a conservative field. When asked to evaluate $\int_A^B \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ for a conservative field you are expected to use the above formula and not a parametrisation of the path.

Important remark: It turns out that the converse is also true; conservative fields (subject to certain conditions) all have scalar potentials.

Example: Find the work done by the electric field

$$\boldsymbol{E} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{x^2 + y^2 + z^2}$$

with potential

$$f = \frac{1}{2}\log(x^2 + y^2 + z^2)$$

along any path from $(1, 0, 0)$ to $(1, 1, 1)$ that does not pass through $(0, 0, 0)$.

Example: Evaluate the work done by the force $\boldsymbol{F}$ from $(0,0)$ to $(1,2)$, where

$$\boldsymbol{F} = \nabla f \quad \text{and} \quad f(x,y) = xy.$$

Example: Show that $\boldsymbol{F}(x,y) = (x+y, x+2y)$ is a gradient field.

Example: Show that $\boldsymbol{F}(x,y) = \dfrac{(x+y)}{2}\,\boldsymbol{i} + \dfrac{y}{2}\,\boldsymbol{j}$ is not a gradient field.

It turns out that you don't have to look for a potential function to check whether a force field $\boldsymbol{F} = (F_1, F_2)$ is a gradient field. If we assume that $\boldsymbol{F}$ is a gradient field with potential function f then

$$\boldsymbol{F} = (F_1, F_2) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

We know that $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$ when the second order partial derivatives are continuous, which means that the components of our force field $\boldsymbol{F}$ must satisfy

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

In fact, the components of a vector field in $\mathbb{R}^2$ satisfy this condition *if and only if* $\boldsymbol{F}$ is a gradient field. This means that we can simply check this condition to determine whether a force field is a gradient field.

You will look at this result in more detail in 2001.

Example: Determine whether the force field $\boldsymbol{F} = (y^2 - 2, 2xy)$ is conservative. Evaluate the work done by $\boldsymbol{F}$ in moving a particle along the path $x = 2t$, $y = t^2$, $0 \leq t \leq 2$.

5.5.2 Main points

- You should understand the relation between a gradient field and its potential function.
- You should understand that conservative fields and gradient fields are equivalent and that the work done by these fields in moving a particle between two points is independent of the path taken.
- You should know that the potential function will give you an easy method to calculate line integrals for gradient fields.
- You should know how to check whether a force field is a gradient field.
- You should know how to find a potential for a conservative force field.

5.6 Work Done or Path Integrals for Non-conservative Fields

This material is covered in Stewart, Section 16.2 (Section 16.2).

Line integrals for forces that are not conservative depend on the chosen path, which makes them harder to evaluate. To work them out we need to parametrise the path, by say t, and convert the integral to one over t. We have already seen how to do this in section 5.3.

Let $\boldsymbol{F}(x, y) = F_1(x, y)\boldsymbol{i} + F_2(x, y)\boldsymbol{j}$ be a non-conservative field and let $x(t)$ and $y(t)$ provide a parametrisation of the path along which we wish to integrate. Then

$$\boxed{\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,\mathrm{d}t.}$$

Or, in component form,

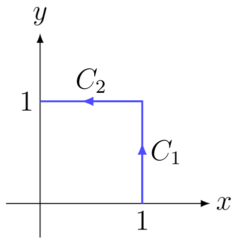
$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y$$

and

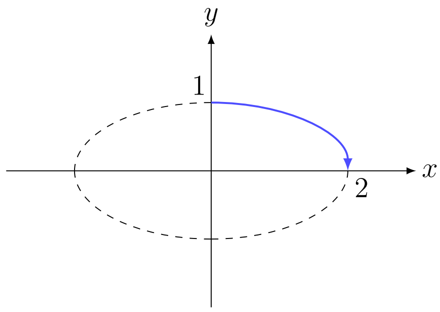
$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \left(F_1(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + F_2(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t.$$

There are some important properties of line integrals that can be useful when evaluating the work done by a non-conservative field:

Example: Evaluate $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x, y) = \left(\frac{x+y}{2}\right) \boldsymbol{i} + \frac{y}{2} \boldsymbol{j}$ and the path is as shown in the figure below.



Example: Evaluate $\displaystyle\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, were $F = (x+y)\boldsymbol{i} + y\boldsymbol{j}$ and C is the arc of the ellipse $\dfrac{x^2}{4} + y^2 = 1$ from $(0,1)$ to $(2,0)$ as shown in the figure below.



5.6.1 Main points

- You should know how to recognise a non-conservative field.
- You should know how to evaluate the work done by a non-conservative field by performing a parametrisation of the path.

5.7 Conservation of Energy in three-dimensional Space

We combine the work of sections (4.10) and (5.5) to discuss conservation of energy in three-dimensional space.

We've seen that if $\mathbf{F}(\mathbf{r})$ is conservative, there exists a scalar potential function $f(\mathbf{r})$ such that

$$\mathbf{F} = \nabla f.$$

We'll take $V(\mathbf{r}) = -f(\mathbf{r})$ here. If an object of mass m is subject to an external force $\mathbf{F}$, Newton's second law of motion $\mathbf{F} = m\mathbf{a}$ states that;

Using the same process as section (4.10.1), we apply to both sides the dot product of $\dot{\mathbf{r}}$;

$$-\nabla V \cdot \dot{\mathbf{r}} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}$$

By the multidimensional chain rule, the LHS is while the RHS is

Hence, the second law of motion under conservative forces becomes

$$
\begin{aligned}
-\frac{dV}{dt} &= \frac{d}{dt}\frac{m}{2}||\dot{\mathbf{r}}||^2 \\
\implies 0 &= \frac{d}{dt}\left(\frac{m}{2}||\dot{\mathbf{r}}||^2 + V(\mathbf{r})\right) \\
\implies E &= \frac{m}{2}||\dot{\mathbf{r}}||^2 + V(\mathbf{r}),
\end{aligned}
$$

where E is a constant.

This is the Energy Equation. It's the three-dimensional form of what we presented on page 200. The term $\frac{m}{2}||\dot{\mathbf{r}}||^2$ is kinetic energy, while $V(\mathbf{r})$ is potential energy. The integration constant, E is the total energy.

Important remark: The total energy of a particle moving under the influence of a conservative force is constant in time, or equivalently, conserved. Energy can be converted from one form into another, but it can be neither created, nor destroyed.

Example: A particle subject to a conservative force has an initial speed of 10 m/s. After a long, winding journey, it arrives back at its starting point. What is its final speed?

5.7.1 Central forces

A force $\mathbf{F}(x, y, z)$ is called central if it has the form:

$$\mathbf{F} = F(r)\hat{\mathbf{r}} = \frac{F(r)}{r}\mathbf{r}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Important remark: The magnitude of a central force is dependent only on the distance of the object from the origin. Such a force is called attractive if it acts towards the origin (i.e. $F(r) < 0$). A repulsive force acts away from the origin ($F(r) > 0$).

Theorem: If $F(r)$ is continuous over some domain D,
then the central field $\boldsymbol{F} = F(r)\hat{\mathbf{r}}$ is conservative throughout D.

To prove this, first consider the gradient of the radial distance,

Let $f(r)$ be the anti-derivative of $F(r)$, i.e. let $F(r) = f'(r)$. Note that f must exist, since F is continuous.

We now show that the central force $\mathbf{F}(r) = F(r)\hat{\mathbf{r}}$ is a gradient field (and hence conservative), with potential function $f(r)$.

We set

$V(r) = -f(r)$ to be the potential energy associated with $\mathbf{F}$, so that

$$\mathbf{F}(\mathbf{r}) = -\nabla V(r).$$

Important remark: Central forces arise in a number of physical situations, such as planetary motion due to gravity, interactions involving charged particles, and rotating spring systems.

Example: Newton's theory of Gravity

Suppose a large object of mass M is held fixed at the origin in some coordinate system, with a smaller object of mass m free to move. Then the gravitational force between two objects is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Find an expression for the potential energy associated with this system.

5.7.2 Angular momentum

Consider a particle with mass m, velocity $\mathbf{v}$ and position $\mathbf{r}$. We say its angular momentum is:

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}).$$

This is a vector quantity, with associated scalar $L = ||\mathbf{L}||$.

If a force $\mathbf{F}$ is exerted on the particle, we have from Newton's second law

where $\mathbf{r} \times \mathbf{F}$ is called the torque.

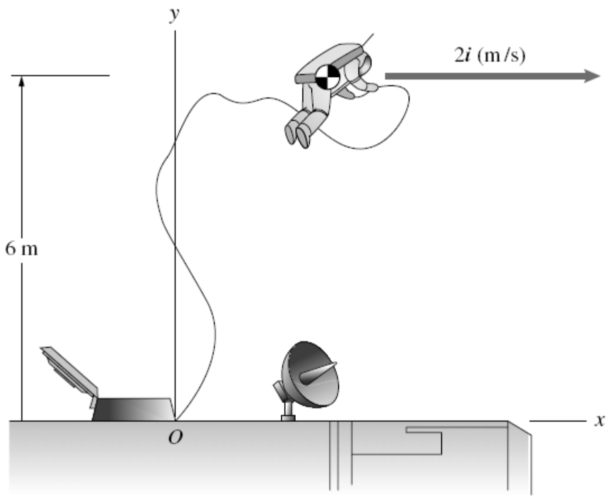
Theorem: Under all central forces,
the angular momentum vector $\mathbf{L}$ is conserved.

Proof; Recall that for a central force,

$$\mathbf{F} = \frac{F(r)}{r}\mathbf{r}.$$

Since $\mathbf{L} = (L_1, L_2, L_3)$ is constant in time, it is a conserved quantity. Thus, central forces conserve both Energy E, and Angular Momentum $\mathbf{L}$.

Example: An astronaut of mass M is moving in the xy plane at the end of a 10m tether attached to a large space station at the origin (pictured).

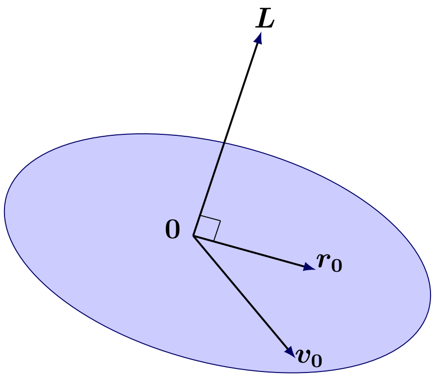


a) What is the astronaut's angular momentum before the tether becomes taut?

b) What is the astronaut's speed immediately after the tether becomes taut?

Theorem: If a central force acts on an object in three dimensions, its motion is restricted to a plane.

To see this, first consider the plane P containing the initial position vector $\mathbf{r}_0 = \mathbf{r}(t_0)$ and initial velocity $\mathbf{v}_0 = \mathbf{v}(t_0)$. Since the angular momentum vector is given by $\mathbf{L} = m(\mathbf{r}_0 \times \mathbf{v}_0)$, it is perpendicular to P.

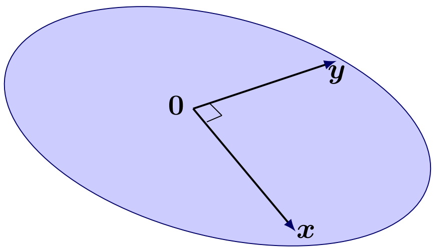


Since $\mathbf{L}$ is a conserved vector quantity, if $\mathbf{r}_1 = \mathbf{r}(t_1)$, and $\mathbf{v}_1 = \mathbf{v}(t_1)$,

$$\mathbf{L} = m(\mathbf{r}_0 \times \mathbf{v}_0) = m(\mathbf{r}_1 \times \mathbf{v}_1).$$

Hence, $\mathbf{r}_1$ must be orthogonal to $\mathbf{L}$, and so must also lie in P.

We choose P to be the xy - plane.

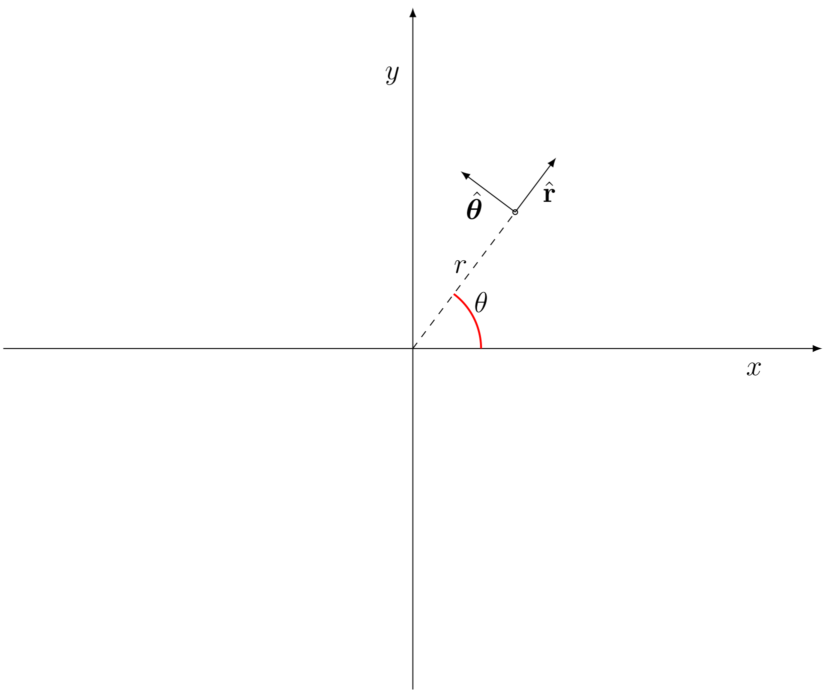


The precise orientation of the xy axes is arbitrary. In practice, we try to choose an orientation which simplifies our problem. For instance, we could define an object's starting point to be on the x axis.

Position-dependent unit vectors for central force fields

The acceleration of an object in a central force field depends only on the radial distance, r. For central force problems we use polar coordinates in $\mathbb{R}^2$, and set

$$\begin{aligned} x &= r\cos(\theta), \\ y &= r\sin(\theta). \end{aligned}$$



We can express the position vector in a different way;

where $\hat{\mathbf{r}} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$.

We can see that $\hat{\mathbf{r}}$ is a unit vector, which is not fixed in space. It changes according to the position of the object being studied. Polar coordinates are an example of curvilinear coordinates.

We can check that another unit vector is given by

$$\hat{\boldsymbol{\theta}} = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j},$$

and that

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \end{vmatrix} = \mathbf{k}.$$

The vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ thus form a basis for $\mathbb{R}^2$. These vectors are orthogonal to each other, and normal (of length 1), so we term this basis orthonormal. (The familiar cartesian system defined by $\mathbf{i}$ and $\mathbf{j}$ is also orthonormal).

Unlike a cartesian co-ordinate system however, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary according to the angle an object makes with the x axis:

We then obtain the following expressions for position, velocity, and angular momentum:

5.7.3 Centrifugal energy and the effective potential.

Since all central forces are conservative, the energy equation on page 260 becomes;

$$E = \frac{1}{2}m||\mathbf{v}||^2 + V(r),$$

where the potential V is now dependent only on r, the scalar distance from the origin, rather than the position vector $\mathbf{r}$.

Using polar coordinates to express velocity, we have

$$\begin{aligned} E &= \frac{1}{2}m||\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}||^2 + V(r) \\ E &= \frac{1}{2}m\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right)\cdot\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) + V(r) \\ &= \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right) + V(r) \text{ (since } \hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = 0). \end{aligned}$$

This is a useful expression, but still rather complicated to solve for position. As it stands, the energy equation depends on the rate of change of **two** unknown functions of time, $r(t)$ and $\theta(t)$.

To make further progress, we use the constancy of angular momentum in polar coordinates, $L = mr^2\dot{\theta}$ and eliminate $\dot{\theta}$;

This is a first order, separable ODE involving only the variable r. It looks very much like the equation for a particle moving in one dimension (section 4.10.1).

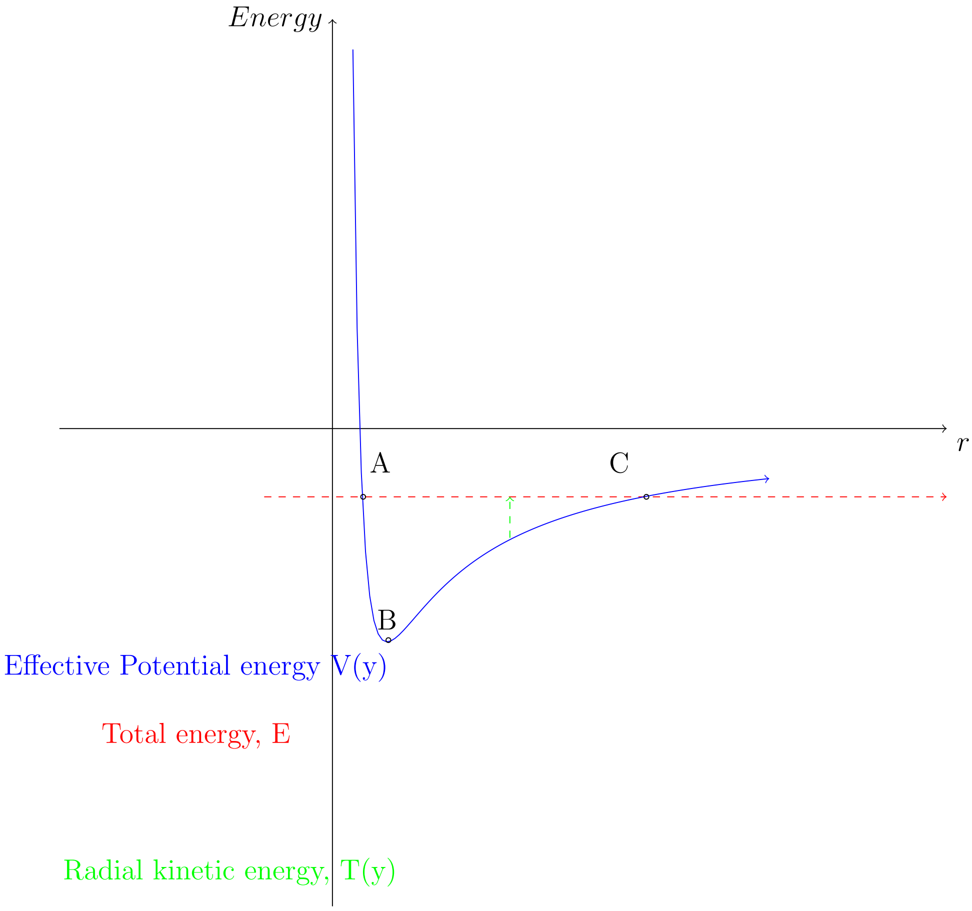
In principle, we can solve this for $r(t)$ (given initial conditions). Once this is known, the equation $L = mr^2\dot{\theta}$ is itself a first order separable ODE, which we can use to solve for $\theta(t)$. Once again, constants of motion allow us to "integrate" the equation of motion.

We call $V_{\text{eff}} = \dfrac{L^2}{2mr^2} + V(r)$ the effective potential of this 1d problem.

We can also think of the total energy being shared across three terms:

- Radial kinetic energy $\dfrac{m}{2}\dot{r}^2$, the energy stored in motion towards or away from the origin.
- centrifugal energy $\dfrac{L^2}{2mr^2}$, the energy stored as rotational motion around the origin.
- Potential energy $V(r)$, discussed earlier.

We can draw the potential energy diagram of this three-dimensional system on a single plane.



If in the above diagram, the energy line passed through the point B, this would indicate an equilibrium solution for our ODE. This would not mean the object was stationary, however. It would simply indicate that r was constant, and that the object was undergoing **circular motion**.

So to recap, the conserved quantities we've identified allow us to significantly reduce the complexity of our problems.

1. First, the conservation of energy E under a conservative force allowed us to turn Newton's law into a first order ODE.

2. Second, the conservation of angular momentum $\mathbf{L}$ under a central force in three dimensions allowed two further simplifications:

   a) Conservation of the direction of $\mathbf{L}$ implied that all motion must take place in a two-dimensional plane, not all of $\mathbb{R}^3$.

b) Conservation of the magnitude of $\mathbf{L}$ gave us two ODEs for $r(t)$ and $\theta(t)$, which can be solved one after another.

Important remark: If $L = mr^2\dot{\theta} = 0$, then $\dot{\theta} = 0$ for all time. In such a case, centrifugal energy is zero, and all motion takes place along a radial from the origin. Such a system is then truly one-dimensional. For the rest of these lectures, it is assumed that $L \neq 0$.

5.7.4 Trajectory

From the previous section, we can solve for $r(t)$ and $\theta(t)$ for objects in a central force field. In many circumstances we'd also like to visualise the trajectory of the object. Since $\mathbf{L} \neq \mathbf{0}$, it follows that $\dot{\theta} \neq 0$. Hence, the object's motion is always clockwise, or always anticlockwise. We can therefore think of r as being dependent on $\theta(t)$, rather than t directly, and turn our attention to finding $r(\theta)$.

[4]

Using the chain rule, we find,

Now that we have an expression for $\dfrac{dr}{d\theta}$, we can re-write the energy equation:

$$\frac{L^2}{2m}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{L^2}{2mr^2} + V(r) = E.$$

[4] Formally, $\dot{\theta} > 0$ implies $\theta(t)$ is an increasing function of t. All increasing functions are one-to-one, and hence invertible. This means $r(t)$ can be written $r(t) = r(\theta(\theta^{-1}(t)))$.

We now have a first order, separable ODE which we can use to solve for $r(\theta)$. We'll call this the Trajectory Equation.

For many central force problems, we can simplify things by introducing a variable substitution, $u = r^{-1}$. Using the chain rule,

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -r^{-2}\frac{dr}{d\theta}.$$

The trajectory equation becomes

5.7.5 Motion in a gravitational field.

In this section, we'll find the trajectory of an object of mass m in the gravitational system described on page 263.

Previously, we derived the gravitational potential energy, $V(r) = -\dfrac{GMm}{r}$.

Thus, the gravitational trajectory equation becomes

$$\left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2GMm^2}{L^2}u = \frac{2mE}{L^2}.$$

Setting $l = \dfrac{L^2}{GMm^2}$, we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2}{l}u = \frac{2mE}{L^2}.$$

Completing the square:

Now let $e^2 = \dfrac{2mEl^2}{L^2} + 1$, and $v = u - 1/l$. Then ODE becomes

$$\left(\frac{dv}{d\theta}\right)^2 + v^2 = \frac{e^2}{l^2}.$$

This is a separable ODE. You can verify that the solution is

This yields

$$r = \frac{l}{1 + e\cos\theta}.$$

This is an equation for a conic section, symmetric about the $x-$axis, (page 231). Thus, we conclude the following:

> **Conclusion:** The orbits of all objects acting under such a gravitational field must be conic sections
> (either a circle, ellipse, parabola, or one branch of a hyperbola),
> with focus at the origin.

This was an empirical law, first derived by Johannes Kepler in 1609. It is called Kepler's First Law of Planetary Motion.[5]

Given a trajectory of the form $r = \dfrac{l}{1 + e\cos\theta}$, we can find the distance from the origin when $\theta = \pm\pi/2$, .

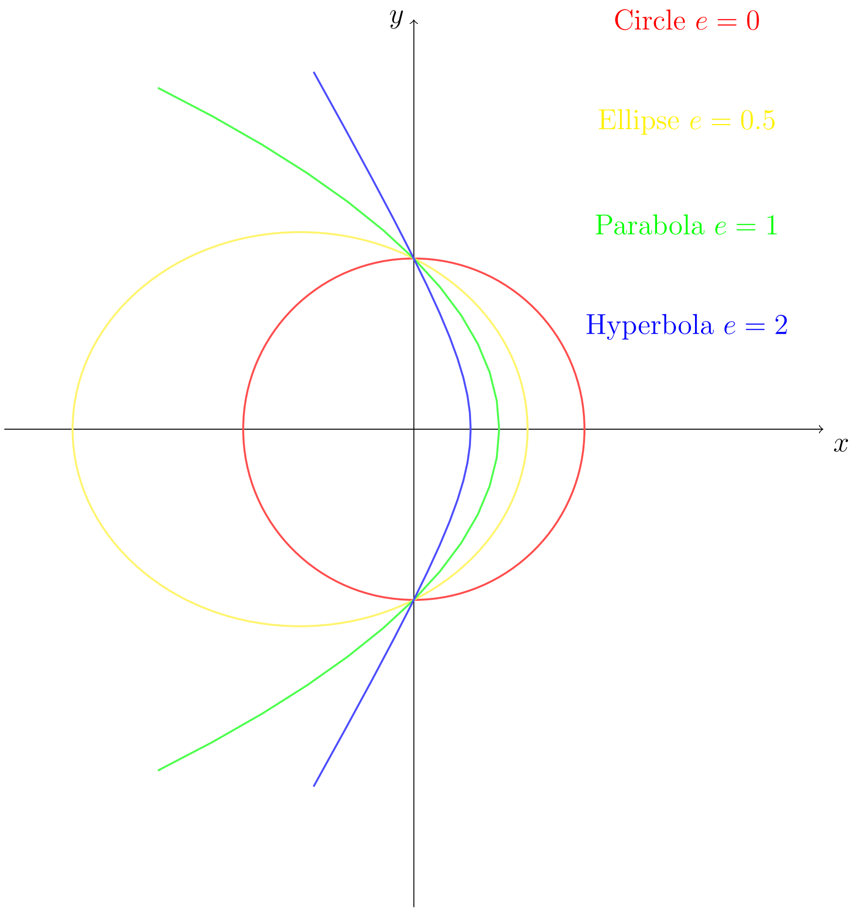
$$r(\pm\pi/2) = \frac{l}{1 + e * 0} = l.$$

This occurs for points of the conic section on the y-axis.

Question: What is the distance from the origin when $\theta = 0$ (points on the x-axis)?

A variety of orbits of this type are pictured below, symmetric about the x-axis. Note that each of these curves intersect on the y-axis, so l is the same for all of these.

[5] Of course, Kepler's law predates Newton's Inverse Square Law of Gravitation. Subsequently, Newton used Kepler's data to **derive** the inverse square law. Thus, historically speaking, by deriving Kepler's law from Newton, we're actually working backward!



The dimensionless orbital parameter e is called the eccentricity. It is a measure of how far the conic section varies from a simple circle. The parameter l is called the semi-latus rectum, with units of distance. It acts as a scaling factor, controlling the overall size of the conic.

The Sun's gravitational pull forces every nearby object into a gravitational orbit described above.[6] Therefore, the orbits are conic sections. The following table shows orbital parameters for the 8 major planets of our solar system. The unit of distance used for l is scaled for Earth's orbit, and is called the astronomical unit (AU). 1 AU$= 149,597,870.7$km

Important remark: Objects in a circular or elliptical orbit (i.e. with $0 \leq e < 1$) are said to be bound. There is some distance from the origin r_{max} that they will

[6]In fact, this is only approximately true. Since the planets also act gravitationally on the sun, as well as on each other, there are additional small sources of gravitational force which are not at the origin. Hence, the actual orbits are a little more complicated than we're describing here. However, since the the Sun makes up 99.8% of the solar system's mass, its effects gravitationally dominate the solar system.

Planet	e	l
Mercury	0.2056	0.371
Venus	0.0067	0.723
Earth	0.0172	1.000
Mars	0.0935	1.511
Jupiter	0.0480	5.190
Saturn	0.0558	9.518
Uranus	0.0540	18.918
Neptune	0.0110	30.008

Table 1: Orbital parameters of planets.

never exceed. We call this distance Apoapsis. By contrast, parabolic and hyperbolic orbits ($e \geq 1$) are unbound, with no apoapsis. All planets in our solar system have bound orbits. Otherwise, they would pass through the solar system only once.

Question: Find an expression for the angular momentum L and energy E from the observed observed orbital parmeters l and e.

5.7.6 Main points

- You should understand the energy equation in three-dimensional space.
- You should know what a central force field is, and why it conserves angular momentum.
- You should be able to show that conserved angular momentum implies motion takes place in a plane.
- You should be able to construct the polar form of the energy equation, and use that to derive an ODE for an object's trajectory.
- You should be able to solve this ODE in the case of a massive gravitational object acting from the origin.

6 Review: Lines and Planes

The following material on Lines and Planes is covered in Queensland Specialist Mathematics/MATH1050 (Mathematical Foundations II). We will assume knowledge of this material in MATH1052.

6.1 Scalar equation for a plane

The equation of a plane in $\mathbb{R}^3$ can be expressed using scalars or vectors. In this first section we will discuss scalar equations.

Note that in $\mathbb{R}^3$, we adopt the convention to take the z-axis pointing upwards, and the (x, y)-plane to be horizontal.

Horizontal planes

The x- and y-axes lie in the horizontal plane $z = 0$. All other horizontal planes are parallel to $z = 0$, and are given by the equation $z = c$.

Vertical planes

A vertical plane has the form $ax + by = d$; it depends on x and y only and z does not appear. If you are not told that this is an equation of a plane, or, equivalently, an equation in $\mathbb{R}^3$, then you cannot distinguish it from the equation of a line in $\mathbb{R}^2$.

Consider the plane $x + y = 1$. First imagine the line $x + y = 1$ in the xy-plane. Then the plane $x + y = 1$ in $\mathbb{R}^3$ contains this line, and is parallel to the z-axis.

Arbitrary planes

The general equation of a plane in $\mathbb{R}^3$ is given by

$$ax + by + cz = d$$

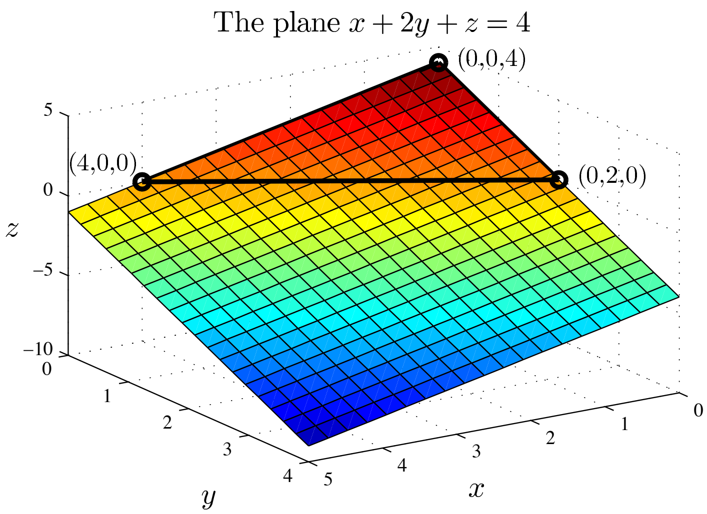
with a, b, c, d fixed real numbers. If the plane is not vertical, i.e., $c \neq 0$, this equation can be rearranged so that z is expressed as a function of x and y:

$$z = F(x,\, y) = -(a/c)x - (b/c)y + (d/c) = mx + ny + z_0.$$

To plot the plane in MATLAB, simply plot $F(x, y)$ using **ezsurf**.

The easiest way to sketch the plane by hand is to use the triangle method: If all of $a, b, c \neq 0$ the plane $ax + by + cz = d$ intercepts each axis at precisely one point. These three points make up a triangle which fixes the plane.

Example: The plane $x + 2y + z = 4$ intersects the x-axis at $x = 4$, the y-axis at $y = 2$ and the z-axis at $z = 4$.

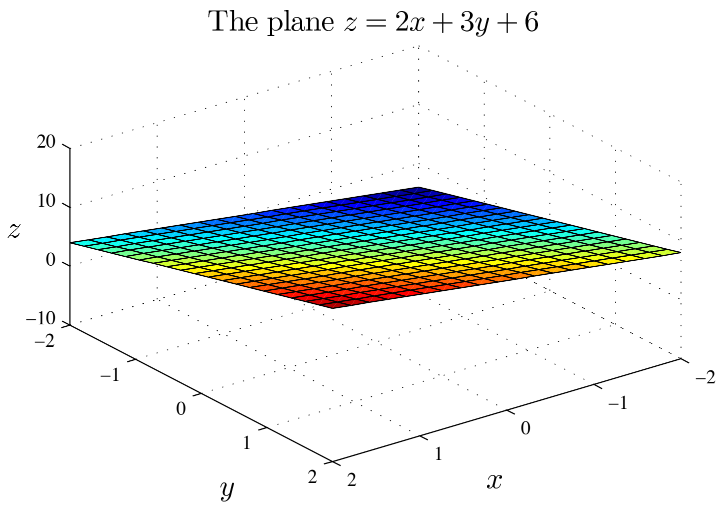


The triangle method is based on the simple fact that *any* three points that lie in a plane uniquely determine this plane *provided these three points do not lie on a single straight line.*

Example: Consider the plane $2x + 3y - z + 6 = 0$.

In MATLAB, the graphing command is simply:

```
ezsurf('2*x+3*y+6',[-2,2,-2,2])
```



Important remark: It is customary to say *the* equation of a plane, even though it is not unique. Multiplying the equation of a plane by a nonzero constant gives another equation for the same plane. For example, $x - 2y + 3z = 4$ and $-2x + 4y - 6z = -8$ are equations of the same plane.

Example: Find the equation of the plane through $(0, 0, 5)$, $(1, 3, 2)$ and $(0, 1, 1)$.

Let $z = ax + by + d$.

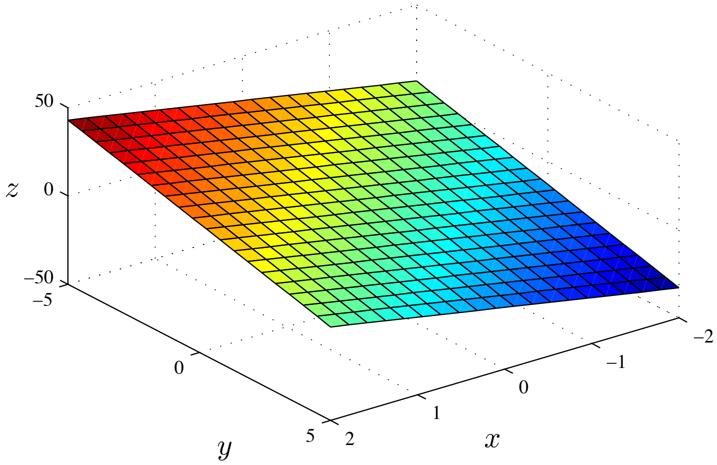
The first point gives $d = 5$.

The 2nd point gives $2 = a + 3b + 5$ and the 3rd point gives $1 = b + 5$.

Solving simultaneously gives $a = 9$ and $b = -4$.

So the plane is $z = 9x - 4y + 5$.

The plane $z = 9x - 4y + 5$

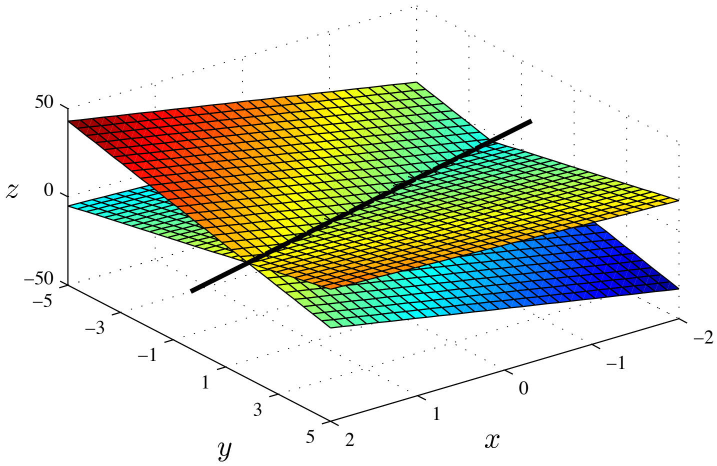


The MATLAB command to produce a plot of this plane would be:

```
ezsurf('9*x-4*y+5',[-2,2,-5,5])
```

The following picture shows the intersection of the planes $2x + 3y - z + 6 = 0$ and $z = 9x - 4y + 5$. Note that the intersection is a line.

The intersection of $z = 9x - 4y + 5$ and $z = 2x + 3y + 6$



Remember that to produce two or more plots on the same figure use the MATLAB command:

```
hold on;
```

6.2 Vector equation of a plane I

Revision of vectors

Pythagoras gives the distance d between the points $(x_0,\, y_0)$ and $(x_1,\, y_1)$ as

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

In three dimensions, the distance d between $(x_0,\, y_0,\, z_0)$ and $(x_1,\, y_1,\, z_1)$ is

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

The norm of a vector is its length. In MATLAB, you can directly call the norm of a vector:

```
v=[1 2 3]
v =
     1     2     3
norm(v)
ans =
    3.7417
```

Recall that the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a}\cdot\boldsymbol{b} = (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3.$$

It is also true that

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta,$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\|\boldsymbol{a}\|$ is the norm of $\boldsymbol{a}$:

$$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

If $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.

The projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is "the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$":

$$\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = \|\boldsymbol{b}\|\cos\theta \times \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\,\boldsymbol{a}.$$

Recall that the cross product $\boldsymbol{a}\times\boldsymbol{b}$ of two vectors in $\mathbb{R}^3$ is a vector orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$ given by

$$\boldsymbol{a}\times\boldsymbol{b} = (a_1, a_2, a_3)\times(b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

This is best memorised using a 3 by 3 determinant.

The cross product points in the direction given by the right-hand-rule.

It is also true that

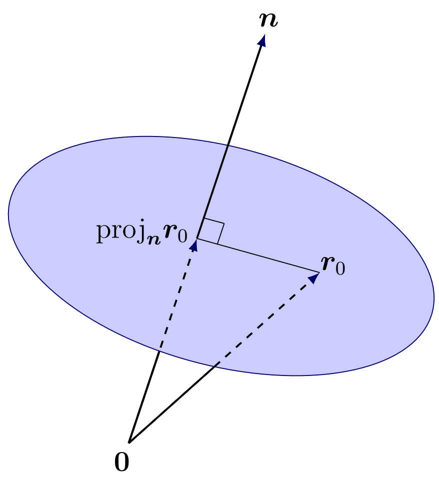
$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin\theta.$$

This means that $\|\boldsymbol{a} \times \boldsymbol{b}\|$ gives the area of the parallelogram spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector equation of a plane I

We have seen in Section 6.1 that any three points in $\mathbb{R}^3$ which do not lie on a straight line determine a plane. There is another geometric way to view a plane, which gives rise to the vector equation of a plane.

Let P be a plane and $\boldsymbol{n}$ be a vector perpendicular to P. Such a vector is called a normal to P. Let $\boldsymbol{r}_0$ be a vector from the origin to a point $\boldsymbol{r}_0$ in P and $\mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r}_0$ its projection onto $\boldsymbol{n}$. Then the length of this projection vector, i.e., $\|\mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r}_0\|$ is the distance of the plane to the origin.



If we had taken another vector, say $\boldsymbol{r}$, from the origin to a point $\boldsymbol{r}$ in P then

$$\mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r} = \mathrm{proj}_{\boldsymbol{n}}\boldsymbol{r}_0.$$

In other words, the plane P is given by the collection of *all* points r whose corresponding vectors $\boldsymbol{r}$ have the same orthogonal projection onto $\boldsymbol{n}$ as $\boldsymbol{r}_0$. Hence

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0.$$

This is the vector equation for a plane.

The vector and scalar equations are in fact the same, as shown next.

We start with the vector equation, where $\boldsymbol{n} = (a, b, c)$ is a normal, $\boldsymbol{r}_0 = (x_0, y_0, z_0)$ and $\boldsymbol{r} = (x, y, z)$. Then

$$\begin{aligned} \boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0 \\ (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\ ax + by + cz = d \end{aligned}$$

where $d = \boldsymbol{n} \cdot \boldsymbol{r}_0 = ax_0 + by_0 + cz_0$.

Important remark: Given a plane $ax + by + cz = d$ we now have a geometric interpretation of the vector (a, b, c): it is a normal to the plane. Obviously the normal is not unique since any vector of the form $k(a, b, c)$, where k is a nonzero scalar, is also a normal.

Our understanding of normal vectors allows us to compute angles between planes, defined as the angle between their respective normal vectors.

Important remark: By convention, the angle between two planes cannot exceed $\pi/2$.

6.3 Equations for a line

There are three common ways to represent a line:

(a) the vector representation;

(b) the parametric representation;

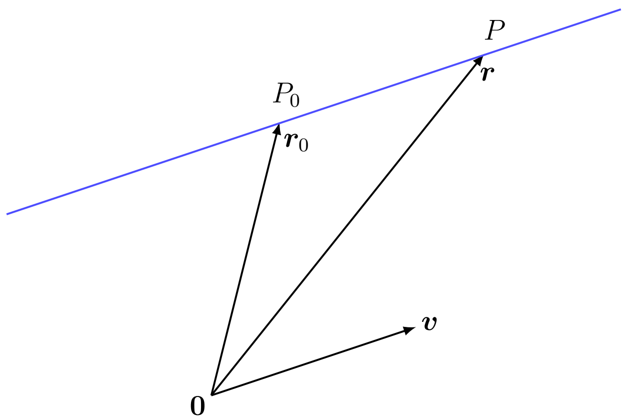
(c) the symmetric equations, obtained by eliminating parameters in (b).

The representation that is best depends on the particular problem at hand.

(a) Vector representation

Let $P_0 = (x_0, y_0, z_0)$ be a point on the line L with corresponding position vector $\boldsymbol{r}_0$ and let $\boldsymbol{v}$ be a vector parallel to L, known as a direction vector. For an arbitrary point $P = (x, y, z)$ lying on L we have

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0).$$



Since $\overrightarrow{P_0P}$, is parallel to $\boldsymbol{v}$, we have $\overrightarrow{P_0P} = \lambda\boldsymbol{v}$ for some scalar λ. Hence

$$\boldsymbol{r} = \boldsymbol{r}_0 + \overrightarrow{P_0P} = \boldsymbol{r}_0 + \lambda\boldsymbol{v}.$$

The equation

$$\boxed{\boldsymbol{r} = \boldsymbol{r}_0 + \lambda\boldsymbol{v}, \qquad \lambda \in \mathbb{R}}$$

is know as the vector equation of the line L.

Important remark: As with *the* equation of a plane, *the* vector equation of a line is not unique. One can choose *any* point P_0 on the line as “starting point”and one can multiply the vector $\boldsymbol{v}$ by any nonzero constant.

(b) Parametric representation

The parametric representation of a line is a scalar representation of the vector equation $\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v}$. Writing each vector in component form,

$$\begin{aligned} \boldsymbol{r} &= (x, y, z) \\ \boldsymbol{r}_0 &= (x_0, y_0, z_0) \\ \boldsymbol{v} &= (a, b, c) \end{aligned}$$

gives

$$(x, y, z) = (x_0, y_0, z_0) + \lambda(a, b, c).$$

Matching components results in three scalar equations

$$\begin{cases} x = x_0 + a\lambda \\ y = y_0 + b\lambda \\ z = z_0 + c\lambda \end{cases} \qquad (*)$$

known as the parametric equations of a line.

(c) Symmetric equations

The parameter λ can be eliminated from the parametric equations of a line. For example, by eliminating λ from each of the three equations $x = 5 + \lambda$, $y = 1 + 4\lambda$, $z = 3 - 2\lambda$ for the line L, we obtain

$$\lambda = x - 5 = \frac{y - 1}{4} = \frac{z - 3}{-2}.$$

The equations

$$x - 5 = \frac{y - 1}{4} = \frac{z - 3}{-2}$$

are known as the symmetric equations of L.

What these equation really are is a set of two non-identical, non-parallel planes

$$x - 5 = \frac{y - 1}{4} \quad \text{and} \quad x - 5 = \frac{z - 3}{-2},$$

which, as we know, must intersect to give a line.

The general form of the symmetric equations of the line $(*)$ from the previous page is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

but some care is needed when one (or two) of a, b or c is equal to zero. For example, if $a = 0$ the above needs to be replaced by

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

and so on.

To plot a line in three-dimensional space, use the parametric form. The MATLAB command is **ezplot3**. As input it takes three functions of a parameter, and the range of the parameter.

Example: Plot the line

$$\begin{cases} x = 100 + 200t \\ y = 200 + 300t \\ z = 300 - 10t \end{cases}$$

for t between -100 and 100. The MATLAB command would be

```
ezplot3('100+200*t','200+300*t','300-10*t',[-100,100])
```

Remember, you can grab and rotate the resulting axes with these three-dimensional plots.

6.4 Parallel, skew and orthogonal lines.

Two lines are parallel if, when written as $\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v}$, and $\boldsymbol{s} = \boldsymbol{s}_0 + \mu \boldsymbol{u}$, the direction vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linear multiples of each other, i.e., $\boldsymbol{u} = k\boldsymbol{v}$ for some nonzero scalar k.

Two straight lines in three dimensional space rarely intersect. Non parallel, non-intersecting lines are called skew lines.

Example: Check that the lines L_1 and L_2 given by

$$L_1 : \begin{cases} x = 1 + \lambda \\ y = -2 + 3\lambda \\ z = 4 - \lambda \end{cases} \qquad L_2 : \begin{cases} x = 2\mu \\ y = 3 + \mu \\ z = -3 + 4\mu \end{cases}$$

for $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, are skew lines.

Two lines are orthogonal if their direction vectors are orthogonal. An easy test for orthogonality is to take the dot product of the direction vectors; if and only if this gives zero are the lines orthogonal.

Example: Check that the lines L_1 and L_2 given by

$$\begin{aligned} L_1: &\quad \frac{x-1}{2} = y-3 = \frac{z+5}{4}, \\ L_2: &\quad 2-x = \frac{5-y}{2} = z-5. \end{aligned}$$

are orthogonal.

Further theory

Unlike lines, planes can not be skew in 3 dimensions. Given two planes, the only possibilities are that they intersect, or that they are parallel. If two planes are parallel, then their normal vectors must be parallel. If two (distinct) planes intersect, then they intersect in a line.

6.5 Vector equation of a plane II

Previously we discussed the scalar equation of a plane, which is an equation of the form $ax + by + cz = d$, and the vector equation of a plane, which is an equation of the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$.

There is a second type of vector equation of a plane, which is similar in form to the vector equation of a line. One fixes a point P_0 on the plane with corresponding position vector $\boldsymbol{r}_0$, but now not one, but two (non-parallel) vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ parallel to the plane are required to fully determine the plane:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{u} + \mu \boldsymbol{v}, \qquad \lambda, \mu \in \mathbb{R}.$$

Important remark: In some sense the above vector equation of a plane is even less unique than the vector equation of a line. Again one can take $\boldsymbol{r}_0$ to be any point on the plane and one can multiply both $\boldsymbol{u}$ and $\boldsymbol{v}$ by nonzero constants. But unlike a line, we can also replace $\boldsymbol{u}$ and $\boldsymbol{v}$ by any other pair of (independent) vectors parallel to the plane. For example, $(x, y, z) = (1, 0, 1) + \lambda(0, 1, 2) + \mu(3, 1, -1)$ and $(x, y, z) = (4, 1, 0) + \lambda(-1, 0, 1) + \mu(-3, 1, 5)$ are different vector representations of the same plane! So do not panic if in your exam the person next to you writes down a very different-looking equation; they still might have the right answer...

6.6 Distance from a point to a plane

A common problem that arises in applications is to find the distance from a point to a plane, where, by distance, we always mean minimum distance.

If $P = (x_1, y_1, z_1)$ is a point, with corresponding position vector $\boldsymbol{p}$, and Π is a plane (with normal $\boldsymbol{n} = (a, b, c)$) given by $ax + by + cz = d$, then the formula for the distance between P and Π is

$$D = \frac{|\boldsymbol{n} \cdot \boldsymbol{p} - d|}{\|\boldsymbol{n}\|} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Important remark: It is best not to try to memorise such a formula because, oddly enough, under exam pressure small details often subtly change for the worse. Also, it depends on one's choice of representation of a plane. For example, sometimes a plane is expressed as $ax + by + cz + d = 0$ in which case the numerator contains $+d$ instead of $-d$. Best is to *understand* the actual derivation given below.

One derivation of the above distance formula uses orthogonal projections. Below we present an alternative method, which is not quite as slick, but which is very easy to carry out for explicit examples, even under exam conditions.

First we construct the line L through the point P and orthogonal to the plane Π:

$$L: \quad \boldsymbol{r} = \boldsymbol{p} + \lambda \boldsymbol{n}, \qquad \lambda \in \mathbb{R}.$$

Next we determine the point, say Q, where L intersects Π. That is, we substitute the equation for L into the equation $\boldsymbol{r} \cdot \boldsymbol{n} = d$ for Π:

$$(\boldsymbol{p} + \lambda \boldsymbol{n}) \cdot \boldsymbol{n} = d.$$

Solving for λ yields

$$\lambda = \frac{d - \boldsymbol{p} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|^2},$$

so that the point Q, with position vector $\boldsymbol{q}$, is given by

$$\boldsymbol{q} = \boldsymbol{p} + \frac{d - \boldsymbol{n} \cdot \boldsymbol{p}}{\|\boldsymbol{n}\|^2} \boldsymbol{n}.$$

The distance between P and Π is now the distance between P and Q. But

$$\boldsymbol{p} - \boldsymbol{q} = \frac{\boldsymbol{n} \cdot \boldsymbol{p} - d}{\|\boldsymbol{n}\|^2} \boldsymbol{n}$$

so that

$$D^2 = \|\boldsymbol{p} - \boldsymbol{q}\|^2 = \frac{(\boldsymbol{n} \cdot \boldsymbol{p} - d)^2}{\|\boldsymbol{n}\|^4} \boldsymbol{n} \cdot \boldsymbol{n} = \frac{(\boldsymbol{n} \cdot \boldsymbol{p} - d)^2}{\|\boldsymbol{n}\|^2}.$$

The formula now follows by taking the square root on both sides *and remembering that a distance can never be negative*, whereas $\boldsymbol{n} \cdot \boldsymbol{p} - d$ can.