Numerical Analysis

Solutions of Equations in One Variable

OL Say

ol.say@itc.edu.kh

Institute of Technology of Cambodia

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Outline

1. The Bisection Method

Bisection or Binary-search Method

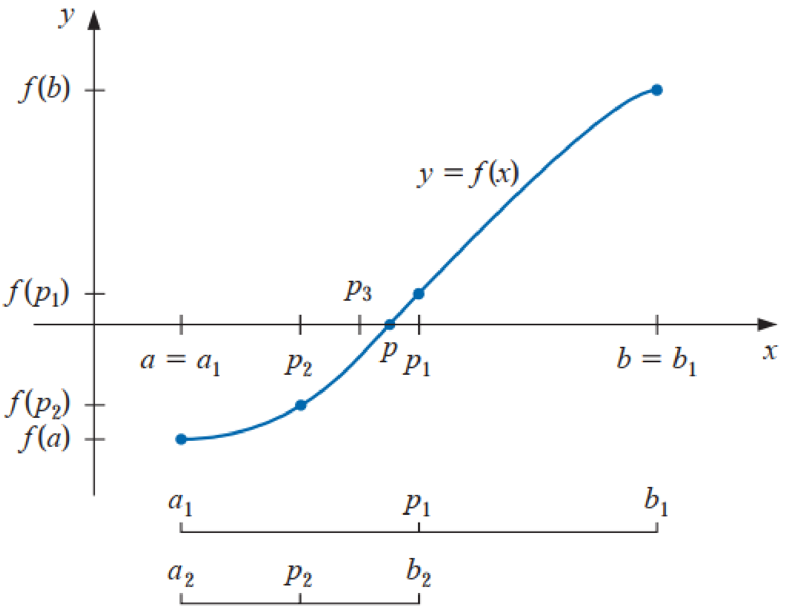
Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

1. If $f(p_1) = 0$, then $p = p_1$, and we are done.
2. If $f(p_1) = 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
   a. If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
   b. If $f(p_1)$ and $f(a_1)$ have opposite sign, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$.

1. The Bisection Method



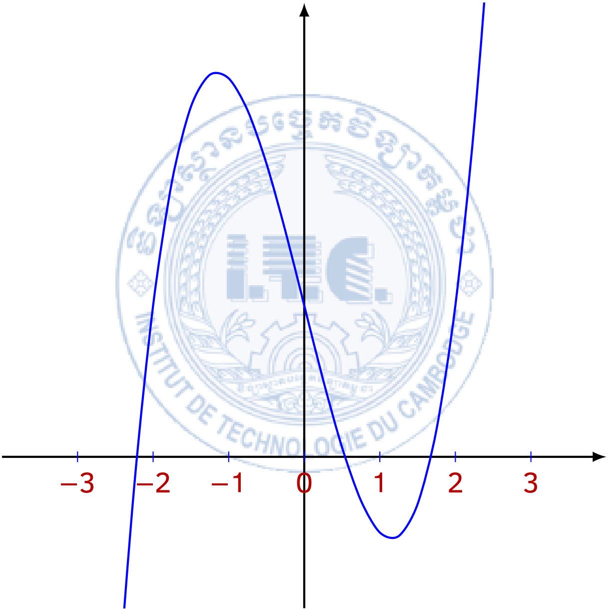
Theorem 1

Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1.$$

Example 2

Determine the number of iterations necessary to solve $x^3 - 4x + 2 = 0$ with accuracy 10^{-2} using $a_1 = 0$ and $b_1 = 1$.


−3
−2
−1
0
1
2
3

Proof.

Let n be the number of iterations necessary to solve the equation with accuracy 10^{-2}. We want $|p_n - p| \leq 10^{-2}$, but $|p_n - p| \leq \frac{b-a}{2^n}$.

So, we just choose such that $\frac{b-a}{2^n} \leq 10^{-2}$.

$$n \geq \log_2 10^2 = \frac{2 \ln 10}{\ln 2} \approx 6.64$$

Thus, $n = 7$ and after the 7-th iterations, we get $x = 0.5391888725571334$. $\square$

Definition 3 (Fixed Point)

The number p is a fixed point for a given function g if $g(p) = p$.

Example 4

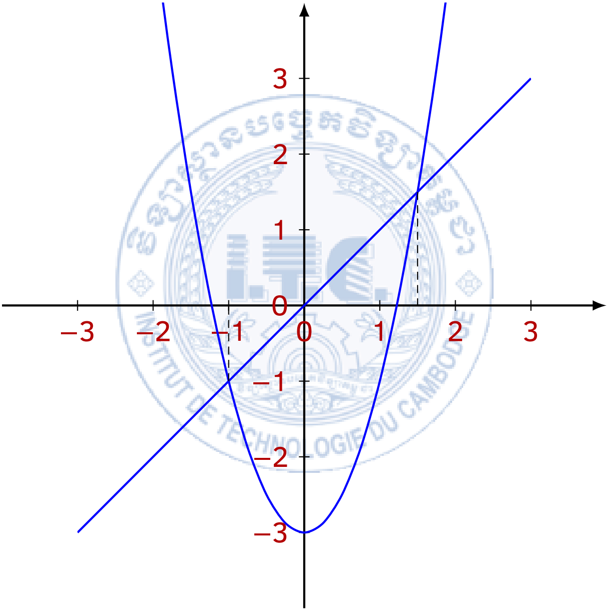
Determine any fixed points of the function $f(x) = 2x^2 - 3$.

Proof.

Let p be a fixed point of f. Then,

$$f(p) = p \Leftrightarrow 2p^2 - 3 = p \Leftrightarrow p = -1, p = \frac{3}{2}.$$

□

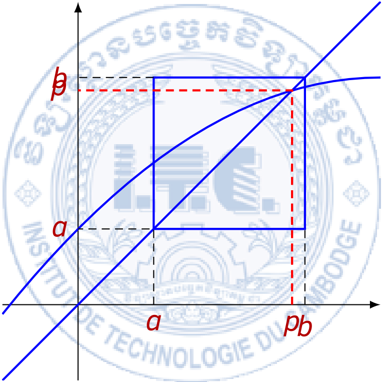

3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

Theorem 5

1. *If* $g \in C[a,b]$ *and* $g(x) \in [a,b]$ *for all* $x \in [a,b]$, *then* g *has at least one fixed point in* $[a,b]$.
2. *If, in addition,* $g(x)$ *exists on* (a,b) *and a positive constant* $k < 1$ *exists with*

$$|g'(x)| \leq k, \ \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$.


b
p
a
a
p
b

Theorem 6 (Fixed-Point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

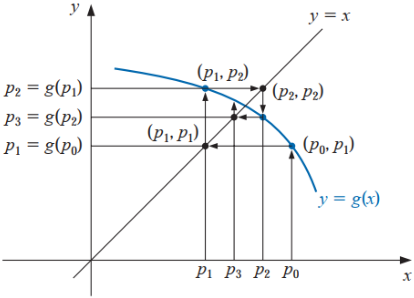
$$|g'(x)| \leq k, \ \textit{for all } x \in (a, b).$$

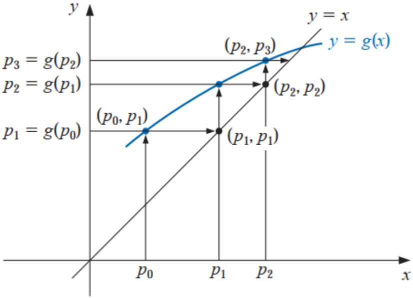
Then for any number $p_0$in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \ n \geq 1,$$

converges to the unique fixed point p in $[a, b]$.

2. Fixed-Point Iteration





Corollary 7

If g satisfies the hypotheses of Fixed-Point Theorem, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1.$$

Example 8

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. There are many ways to changethe equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. It is not important for you to derive thefunctions shown here, but you should verify that the fixed point of each is actually a solution to the original equation, $x^3 + 4x^2 - 10 = 0$. For instance,

1. $g_1(x) = x - x^3 - 4x^2 + 10$
2. $g_2(x) = \left(\dfrac{10}{x} - 4x\right)^{1/2}$
3. $g_3(x) = \dfrac{1}{2}(10 - x^3)^{1/2}$
4. $g_4(x) = \left(\dfrac{10}{4+x}\right)^{1/2}$
5. $g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

1. Our theorem cannot guarantee the convergence of choice g_1.
2. Our theorem cannot guarantee the convergence of choice g_2.
3. $g_3'(x) = -\frac{3}{4}x^2(10-x^3)^{-1/2} < 0$ on $[1,2]$. However, $|g_3'(2)| \approx 2.12$, so the criterion $|g_3'(x)| \le k < 1$ fails on $[1,2]$. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ with $p_0 = 1.5$ on the interval $[1, 1.5]$.

$$1 < 1.28 \approx g_3(1.5) \le g(x) \le g(1) = 1.5$$

for all $x \in [1, 1.5]$. This means that g_3 maps the interval $[1, 1.5]$ into itself and moreover $|g_3'(x)| \le |g_3'(1.5)| \approx 0.66$ on $[1, 1.5]$. In this cases, the convergence is guaranteed by the theorem.
4. $|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| \le \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15$ for all $x \in [1,2]$.
5. $g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x} = x - \frac{f(x)}{f'(x)}$ will be discussed in the following section.

Fixed-Point Iteration

To find a solution to $p = g(p)$ given an initial approximation p_0 :
INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

1. Set $i = 1$.
2. While $i \leq N_0$ do steps 3–6.
3. If $|p - p_0| < TOL$ then
   a. OUTPUT p
   b. STOP.
4. Set $i = i + 1$.
5. Set $p_0 = p$.
6. OUTPUT 'The method failed after N_0 iterations.'

Table: Fixed-Point Iteration: $g_4(x) = \sqrt{\frac{10}{4+x}}, x_0 = 1.5$

step	x	f(x)
0	1.500000000000000	2.375000000000000
1	1.348399724926484	-0.275636863770030
2	1.367376371991283	0.035480981304289
3	1.364957015402487	-0.004507521778089
4	1.365264748113442	0.000573597719557
5	1.365225594160525	-0.000072976739745
6	1.365230575673434	0.000009284815373
7	1.365229941878183	-0.000001181301048
8	1.365230022515568	0.000000150296234
9	1.365230012256122	-0.000000019122099
10	1.365230013561425	0.000000002432893

Newton's Method

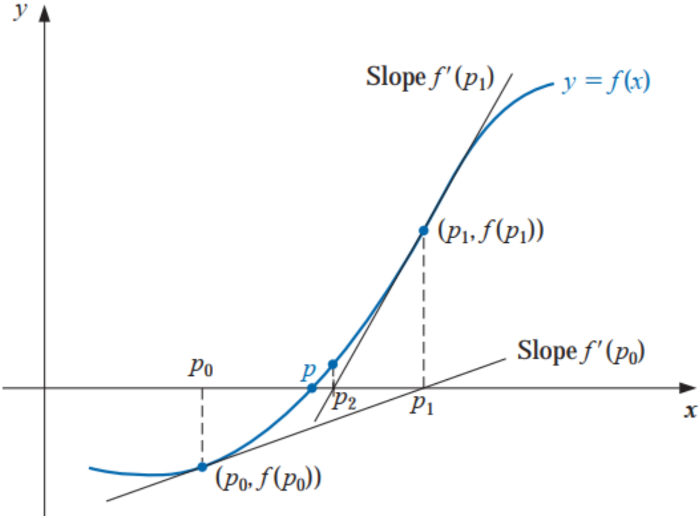
Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small." Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$ and Newton's Method is derived by assuming that $|p - p_0|$ is small, $p \approx p_0 - \dfrac{f(p_0)}{f'(p_0)} := p_1$. This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1.$$

3. Newton's Method



Theorem 9 (Convergence of the Newton's Method)

Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.

The stopping-technique inequalities given with the Bisection method are applicable to Newton's method. That is, select a tolerance $\varepsilon > 0$, and construct $p_1, \ldots, p_N$ until

$$|p_N - p_{N-1}| < \varepsilon, \tag{1}$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0 \tag{2}$$

$$\text{or, } |f(p_N)| < \varepsilon. \tag{3}$$

Note that none of the above inequalities give precise information about the actual error $|p_N - p|$.

Newton's

To find a solution to $f(x) = 0$ given an initial approximation p_0 :
INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

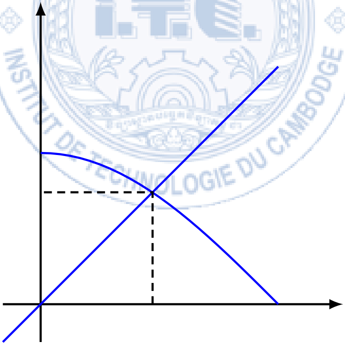
1. Set $i = 1$.
2. While $i \leq N_0$ do steps 3–6.
3. Set $p = p_0 - f(p_0)/f'(p_0)$.
4. If $|p - p_0| < TOL$ then
   a. OUTPUT p
   b. STOP
5. Set $i = i + 1$
6. Set $p_0 = p$
7. OUTPUT 'The method failed after N_0 iterations'

Example 10

Consider the function $f(x) = \cos x - x$. Approximate a zero of f using

1. a fixed-point method, and
2. Newton's method.

Figure: $y = \cos x, y = x$



Proof.

1. A solution to this root-finding problem is also a solution to the fixed-point problem $x = \cos x$, and the graph in above figure implies that a single fixed-point p lies in $[0, \pi/2]$. In this case, we choose $p_0 = \pi/4 \in [0, \pi/2]$.

$$p_n = g(p_{n-1}) = \cos(p_{n-1}), \quad p_0 = \frac{\pi}{4}$$

   At $n = 7, p_7 \approx 0.7361282565008520$.

2. We have $f'(x) = -\sin x - 1$.

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{\cos(p_{n-1}) - p_{n-1}}{-\sin(p_{n-1}) - 1}, \ p_0 = \frac{\pi}{4}$$

   At $n = 3, p_3 = 0.7390851332151610$.

The best we could conclude from these results is that $p \approx 0.74$. □

Table: Fixed-Point's Iteration: $g(x) = \cos x, x_0 = \pi/4$

step	x	$f(x)$
0	0.785398163397448	-0.078291382210901
1	0.707106781186548	0.053137815889083
2	0.760244597075630	-0.035577116186504
3	0.724667480889126	0.024052404900358
4	0.748719885789484	-0.016159041197242
5	0.732560844592242	0.010903366723052
6	0.743464211315294	-0.007335954814442
7	0.736128256500852	0.004945430582858
8	0.741073687083710	-0.003329528091135
9	0.737744158992575	0.002243605803296
10	0.739987764795871	-0.001510956071317

Table: Newton's Method: $f(x) = \cos x - x, x_0 = \pi/4$

step	x	$f(x)$
0	0.785398163397448	-0.078291382210900
1	0.739536133515238	-0.000754874682502
2	0.739085178106010	-0.000000075129866
3	0.739085133215161	-0.000000000000000
4	0.739085133215160	0.000000000000000
5	0.739085133215160	0.000000000000000

The Secant Method

By definition of derivative,

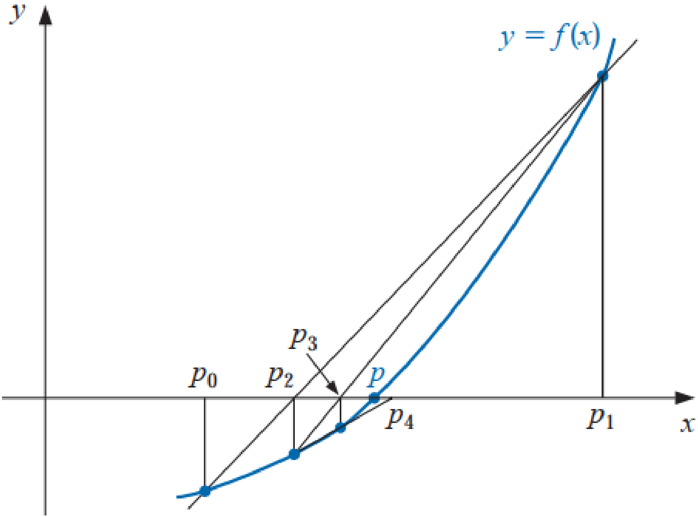
$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If p_{n-2} is close to p_{n-1}, then

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's formula gives the Secant method

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$



y
$y = f(x)$
p_3
p_0
p_2
p
p_4
p_1
x

Secant Method

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1 :
INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

1. Set $i = 2; q_0 = f(p_0); q_1 = f(p_1)$
2. While $i \leq N_0$ do steps 3–6.
3. Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$.
4. If $|p - p_1| < TOL$, then
    a. OUTPUT p;
    b. STOP.
5. Set $i = i + 1$.
6. Set $p_0 = p_1; q_0 = q_1; p_1 = p; q_1 = f(p)$.
7. OUTPUT ‘The method failed after N_0 iterations’

Example 11

Find a solution to $\cos x - x = 0$ on $[0, \pi/2]$ for which $|\cos x - x| < 10^{-16}$ by

1. using the Bisection Method with $p_0 = \pi/4$;
2. using the Fixed-Point Iterations with $p_0 = \pi/4$;
3. using the Newton's Method with $p_0 = \pi/4$;
4. using the Secant Method with $p_0 = 0.5, p_1 = \pi/4$.

Table: Bisection Method $\cos x - x = 0, x_0 = 0, x_1 = \pi/4$

step	x	f(x)
0	0.0000000000000000	1.0000000000000000
1	0.3926990816987241	0.5311804508125626
2	0.5890486225480862	0.2424209897544590
3	0.6872233929727672	0.0857870603899697
⋮	⋮	⋮
49	0.7390851332151605	0.0000000000000003
50	0.7390851332151605	0.0000000000000003
51	0.7390851332151605	0.0000000000000003
52	0.7390851332151607	0.0000000000000000

Table: Fixed-Point Iteration $\cos x - x = 0, x_0 = \pi/4$

step	x	f(x)
0	1.5000000000000000	-1.4292627983322972
1	0.0707372016677029	0.9267619655388831
2	0.9974991672065860	-0.4550941748673661
3	0.5424049923392199	0.3140647166081081
⋮	⋮	⋮
83	0.7390851332151603	0.0000000000000004
84	0.7390851332151608	-0.0000000000000002
85	0.7390851332151606	0.0000000000000001
86	0.7390851332151607	0.0000000000000000

Table: Newton-Raphson's Method $\cos x - x = 0, x_0 = \pi/4$

step	x	f(x)
0	0.7853981633974483	-0.0782913822109007
1	0.7395361335152383	-0.0007548746825027
2	0.7390851781060102	-0.0000000751298666
3	0.7390851332151610	-0.0000000000000007
4	0.7390851332151606	0.0000000000000001
5	0.7390851332151607	0.0000000000000000

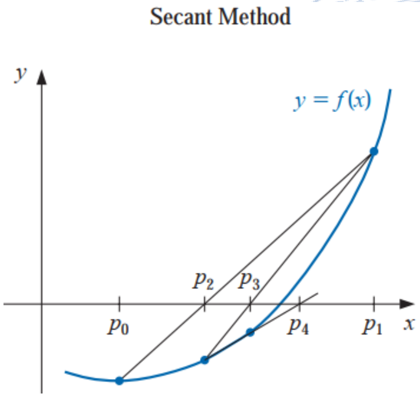
Table: Secant Method: $f(x) = \cos x - x, x_0 = 0.5, x_1 = \pi/4$

i	x0	x1	f(x1)
0	0.5000000000000000	0.7853981633974483	-0.0782913822109007
1	0.7853981633974483	0.7363841388365822	0.0045177185221702
2	0.7363841388365822	0.7390581392138897	0.0000451772159638
3	0.7390581392138897	0.7390851493372764	-0.0000000269821671
4	0.7390851493372764	0.7390851332150645	0.0000000000001609
5	0.7390851332150645	0.7390851332151607	0.0000000000000000

- The method of False Position (also called Regula Falsi) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations.
- Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.
- First choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$.
- The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of theline joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$.
- To decide which secant line to use to compute p_3, consider $f(p_2) \cdot f(p_1)$, or more correctly $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1)$.
  a. If $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$.

- (b) If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.

Figure: Secant Method and The Method of False Position



To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:
INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

1. Set $i = 2; q_0 = f(p_0); q_1 = f(p_1)$.
2. While $i \leq N_0$ do step 3–7.
3. Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$.
4. If $|p - p_1| \leq TOL$ then
   - OUTPUT p;
   - STOP.
5. Set $i = i + 1; q = f(p)$.
6. If $q \cdot q_1 < 0$ then set $p_0 = p_1; q_0 = q_1$.
7. Set $p_1 = p; q_1 = q$.
8. OUTPUT ‘Method failed after N_0 iterations.’

Example 12

Find a solution to $\cos x - x = 0$ on $[0, \pi/2]$ for which $|\cos x - x| < 10^{-10}$ by

1. using the Method of False Position with $p_0 = 0.5, p_1 = \pi/4$.
2. using the Secant Method with $p_0 = 0.5, p_1 = \pi/4$.
3. using the Newton's Method with $p_0 = \pi/4$;

Table: The Method of False Position

	x0	x1	f(x1)
0	0.5000000000	0.7853981634	-0.0782913822
1	0.7853981634	0.7363841388	0.0045177185
2	0.7853981634	0.7390581392	0.0000451772
3	0.7853981634	0.7390848638	0.0000004509
4	0.7853981634	0.7390851305	0.0000000045
5	0.7853981634	0.7390851332	0.0000000000

Table: Second Method

	x0	x1	f(x1)
0	0.5000000000	0.7853981634	-0.0782913822
1	0.7853981634	0.7363841388	0.0045177185
2	0.7363841388	0.7390581392	0.0000451772
3	0.7390581392	0.7390851493	-0.0000000270
4	0.7390851493	0.7390851332	0.0000000000

Table: Newton's Method

	x	f(x)
0	0.7853981634	-0.0782913822
1	0.7395361335	-0.0007548747
2	0.7390851781	-0.0000000751
3	0.7390851332	-0.0000000000

Theorem 13 (Fundamental Theorem of Algebra)

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root.

Corollary 14

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, \ldots, m_k$, such that $\sum_{i=1}^{k} m_i = n$ and $P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}$.

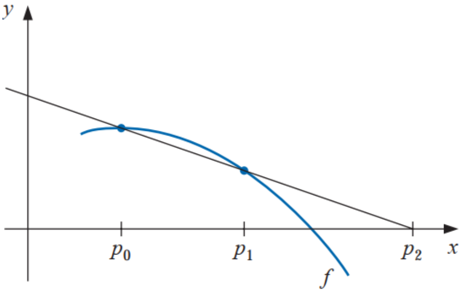
Corollary 15

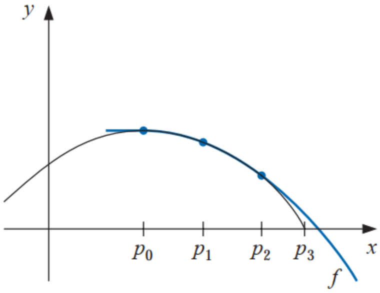
Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x.

Theorem 16

If $z = a + bi$ is a complex zero of multiplicity m of the polynomial $P(x)$ with real coefficients, then $z = a - bi$ is also a zero of multiplicity m of the polynomial $P(x)$, and $(x^2 - 2ax + a^2 + b^2)^m$ is a factor of $P(x)$.

The Secant method begins with two initial approximations p_0 and p_1 and determines the next approximation p_2 as the intersection of the x-axis with the line through $(p_0, f(p_0))$ and $(p_1, f(p_1))$. Müller's method uses three initial approximations, p_0, p_1, and p_2, and determines the next approximation p_3 by considering the intersection of the x-axis with the parabola through $(p_0, f(p_0)), (p_1, f(p_1))$, and $(p_2, f(p_2))$.





The derivation of Müller's method begins by considering the quadratic polynomial $P(x) = a(x - p_2)^2 + b(x - p_2) + c$ that passes through the three points. The constant a, b and c can be derived as

$$c = f(p_2)$$
$$b = \frac{(p_0 - p_2)^2[f(p_1) - f(p_2)] - (p_1 - p_2)^2[f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}$$
$$a = \frac{(p_1 - p_2)[f(p_0) - f(p_2)] - (p_0 - p_2)[f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}$$

To determine p_3, a zero of P, we apply the quadratic formula to $P(x) = 0$.

$$p_3 - p_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

The Müller's Method

In Müller's method , the sign is chosen to agree with the sign of b. Chosen in this manner, the denominator will be the largest in magnitude and will result in p_3 being selected as the closest zero of P to p_2. Thus

$$p_3 = p_2 - \frac{2c}{b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}}$$

where a, b and c are given above.

To find a solution to $f(x) = 0$ given three approximations, p_0, p_1, and p_2 :
INPUT p_0, p_1, p_2; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

1. Set $h_1 = p_1 - p_0;$
   $h_2 = p_2 - p_1;$
   $\delta_1 = (f(p_1) - f(p_0))/h_1;$
   $\delta_2 = (f(p_2) - f(p_1))/h_2;$
   $d = (\delta_2 - \delta_1)/(h_2 + h_1);$
   $i = 3.$
2. While $i \leq N_0$ do steps 3–7.
3. $b = \delta_2 + h_2 d;$
   $D = (b^2 - 4f(p_2)d)^{1/2}.$
4. If $|b - D| < |b + D|$ then set $E = b + D$ else set $E = b - D.$

5. Set $h = -2f(p_2)/E;$

   $p = p_2 + h.$

6. If $|h| < TOL$ then OUTPUT p and STOP.

7. Set $p_0 = p_1;$

   $p_1 = p_2;$

   $p_2 = p;$

   $h_1 = p_1 - p_0$

   $h_2 = p_2 - p_1$

   $\delta_1 = (f(p_1) - f(p_0))/h_1;$

   $\delta_2 = (f(p_2) - f(p_1))/h_2;$

   $d = (\delta_2 - \delta_1)/(h_2 + h_1);$

   $i = i + 1.$

8. OUTPUT 'Method failed after N_0 iterations.'

Example 17

Approximate roots of $x^4 - 3x^3 + x^2 + x + 1 = 0$ using Muller's method with $|p_n - p_{n-1}| < 10^{-10}$ and

1. $p_0 = -0.5, p_1 = 0, p_2 = 0.5$
2. $p_0 = 0.5, p_1 = -0.5, p_2 = 0$
3. $p_0 = 0.5, p_1 = 1, p_2 = 1.5$
4. $p_0 = 1.5, p_1 = 2, p_2 = 2.5$

Table: Muller's Method: $p_0 = -0.5, p_1 = 0, p_2 = 0.5$

	p	f(p)
0	-0.5000000000+0.0000000000j	1.1875000000+0.0000000000j
1	0.0000000000+0.0000000000j	1.0000000000+0.0000000000j
2	0.5000000000+0.0000000000j	1.4375000000+0.0000000000j
3	-0.1000000000+0.8888194417j	-0.0112000000+3.0148755464j
4	-0.2880151881+0.2382530457j	0.6445573559-0.0434768946j
5	-0.3744124231+0.3742351304j	0.2327137783-0.2209969944j
6	-0.3470404269+0.4521998200j	-0.0358246384-0.0216552418j
7	-0.3392167459+0.4464985276j	0.0002952307-0.0007092150j
8	-0.3390929916+0.4466301312j	-0.0000003885-0.0000005423j
9	-0.3390928378+0.4466301000j	-0.0000000000+0.0000000000j
10	-0.3390928378+0.4466301000j	0.0000000000-0.0000000000j

Table: Muller's Method: $p_0 = 0.5, p_1 = -0.5, p_2 = 0$

	p	f(p)
0	0.5000000000+0.0000000000j	1.4375000000+0.0000000000j
1	-0.5000000000+0.0000000000j	1.1875000000+0.0000000000j
2	0.0000000000+0.0000000000j	1.0000000000+0.0000000000j
3	-0.1000000000-0.8888194417j	-0.0112000000-3.0148755464j
4	-0.4921457099-0.4470307000j	-0.1691207751+0.7367331512j
5	-0.3522257126-0.4841324442j	-0.1786006615-0.0181872218j
6	-0.3402285705-0.4430356274j	0.0119760808+0.0105562188j
7	-0.3390946788-0.4466564890j	-0.0001055719-0.0000387260j
8	-0.3390928334-0.4466301006j	0.0000000054-0.0000000180j
9	-0.3390928378-0.4466301000j	0.0000000000+0.0000000000j
10	-0.3390928378-0.4466301000j	0.0000000000-0.0000000000j

Table: Muller's Method: $p_0 = 0.5, p_1 = 1, p_2 = 1.5$

	p	f(p)
0	0.5000000000	1.4375000000
1	1.0000000000	1.0000000000
2	1.5000000000	-0.3125000000
3	1.4063269672	-0.0485133690
4	1.3887833343	0.0017410073
5	1.3893896196	0.0000030492
6	1.3893906833	-0.0000000000
7	1.3893906833	-0.0000000000

Table: Muller's Method: $p_0 = 1.5, p_1 = 2, p_2 = 2.5$

	p	f(p)
0	1.5000000000	-0.3125000000
1	2.0000000000	-1.0000000000
2	2.5000000000	1.9375000000
3	2.2473316390	-0.2450656380
4	2.2865220950	-0.0144639245
5	2.2887754750	-0.0001247201
6	2.2887949939	0.0000000112
7	2.2887949922	0.0000000000
8	2.2887949922	0.0000000000

Consider a nested brackets method for evaluating polynomial.

$$\begin{aligned} P_4(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \\ &= a_0 + x\{a_1 + x[a_2 + x(a_3 + xa_4)]\} \end{aligned}$$

The computational sequence becomes

$$\begin{aligned} &P_0(x) = a_4; P_0'(x) = 0 \\ &P_1(x) = a_3 + xP_0(x); P_1'(x) = P_0(x) + xP_0'(x) \\ &P_2(x) = a_2 + xP_1(x); P_2'(x) = P_1(x) + xP_1'(x) \\ &P_3(x) = a_1 + xP_2(x); P_3'(x) = P_2(x) + xP_2'(x) \\ &P_4(x) = a_0 + xP_3(x); P_4'(x) = P_3(x) + xP_3'(x) \end{aligned}$$

For a polynomial of degree n, the procedure can be summarized as

$$\begin{aligned} &P_0(x) = a_n; P_0'(x) = 0 \\ &P_k(x) = a_{n-k} + xP_{k-1}(x); P_k'(x) = P_{k-1}(x) + xP_{k-1}'(x),\ k = 1, 2, \dots, n \end{aligned}$$

Compute the value of $P(x) = \sum_{k=1}^{n} a_k x^k$ and its derivative at x_0.

INPUT Coefficients $a_0, \ldots, a_n$ and x_0.
OUTPUT The value $P(x_0), P'(x_0)$ and $P''(x_0)$.

1. Set $p = a_n; dp = 0; ddp = 0$.
2. For k from 1 to n,
    a. set $ddp = 2dp + ddp * x_0$
    b. set $dp = p + dp * x_0$
    c. set $p = a_{n-k} + p * x_0$
3. OUTPUT p, dp, ddp.

Laguerre's Method is a root-finding algorithm which converges to a complex root from any starting position. To motivate the formula, consider an n-th order polynomial and its derivatives,

$$P_n(x) = (x - x_1) \cdots (x - x_n)$$

$$P_n'(x) = P(x) \left(\frac{1}{x - x_1} + \cdots + \frac{1}{x - x_n} \right)$$

$$\Rightarrow \frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \cdots + \frac{1}{x - x_n} \equiv G(x)$$

$$\Rightarrow \frac{P''(x)}{P(x)} - \left[\frac{P_n'(x)}{P_n(x)} \right]^2 = -\frac{1}{(x - x_1)^2} - \cdots - \frac{1}{(x - x_n)^2} \equiv -H(x)$$

Now make "a rather drastic set of assumptions" that the root x_1 being sought is a distance a from the current best guess, so

$$a \equiv x - x_1$$

while all other roots are at the "same distance" b, so

$$b \equiv x - x_i, \quad \forall i = 2, \dots, n.$$

This allows G and H to be expressed in terms of a and b as

$$G \equiv \frac{1}{a} + \frac{n-1}{b}$$
$$H \equiv \frac{1}{a^2} + \frac{n-1}{b^2}.$$

Solving these equations for a, we get

$$a = \frac{n}{G \pm \sqrt{(n-1)(nH - G^2)}}$$

where the sign is taken to give the largest magnitude for the denominator. To apply the method, calculate a for a trial value x, then use $x - a$ as the next trial value, and iterate until a becomes sufficiently small.

The algorithm of the Laguerre method to find one root of a polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree n is:
INPUT Choose an initial guess $a_0, \ldots, a_n; x_0$.
OUTPUT Approximated zero of the polynomial $P(x)$.

1. For k from 1 to $maxiter$:
   a. Set $p = P(x_0), dp = P'(x_0), ddp = P''(x_0)$
   b. If $|p| < TOL$, OUTPUT x_0; STOP.
   c. Set $G = \dfrac{dp}{p}$;
   d. Set $H = G^2 - \dfrac{ddp}{p}$;
   e. Set $F = \sqrt{(n-1)(nH - G^2)}$;
   f. If $|G + F| > |G - F|$: $a = \dfrac{n}{G + F}$.
   g. Else $a = \dfrac{n}{G - F}$.

- h. Set $x_0 = x_0 - a$.
- i. If $|a| < TOL$: OUTPUT x_0; STOP.

2. OUTPUT "Too many iterations."

Example 18

Use Laguerre's method to find a root of $5 - 4x^2 + x^4 = 0$ with initialized value $x_0 = 0$.

Table: Laguerre's Method: $5 - 4x^2 + x^4 = 0,\ x_0 = 0$

	x	P(x)
0	0.0000000000+0.0000000000j	5.0000000000+0.0000000000j
1	0.0000000000-0.9128709292j	2.3611111111-0.0000000000j
2	0.0000000000-1.5602819207j	1.1887725854+0.0000000000j
3	0.2999131406-1.5073929784j	0.2156975903+0.3296359528j
4	0.3437219030-1.4555255341j	-0.0011842083+0.0008201527j
5	0.3435607497-1.4553466902j	0.0000000001+0.0000000000j

After a root r of $P_n(x) = 0$ has been computed, it is desirable to factor the polynomialas follows:

$$P_n(x) = (x - r)P_{n-1}(x)$$

This procedure, known as "deflation" or "synthetic division," involves nothing more thancomputing the coefficients of $P_{n-1}(x)$. If we let

$$P_{n-1}(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$$

then

$$a_0 + a_1 x + \cdots a_n x^n = (x - r)(b_0 + b_1 x + \cdots + b_{n-1} x^{n-1})$$

Equating the coefficients of like powers of x, we obtain

$$b_{n-1} = a_n, b_{n-2} = a_{n-1} + r b_{n-1}, \ldots, b_0 = a_1 + r b_1$$

Horner's Deflation Algorithm

Find $P_{n-1}(x)$ where $P_n(x) = (x - r)P_{n-1}(x)$ such that r $P_n(x)$ are given.
INPUT Coefficients $a_0, a_1, \ldots, a_n$ and a zero r of
$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.
OUTPUT Coefficients $b_0, b_1, \ldots, b_{n-1}$.

1. Set $b_{n-1} = a_n$.
2. For k from $n - 2$ to -1 with step -1 : $b_k = a_{k+1} + rb_{k+1}$
3. OUTPUT $b_0, b_1, \ldots, b_{n-1}$.

Example 19

We are given $P(x) = 26 - 47x + x^2 - 12x^3 - 36x^4 + 37x^5 - 11x^6 + 2x^7$. Deflate the polynomial $P(x)$ into $(x-2)Q(x)$. That is determining the polynomial $Q(x)$.

Example 20

Write a python program to find all (real and complex) roots of the equation

a. $26 - 47x + x^2 - 12x^3 - 36x^4 + 37x^5 - 11x^6 + 2x^7 = 0$
b. $8x^6 + 52x^5 + 110x^4 + 55x^3 - 70x^2 - 44x + 24 = 0$

with the help of Laguerre's method and synthetic division algorithm.