Individual Investigation

Research question:

"How mathematics is used in Discrete Fourier Transform and

Discrete Z-transform"

Word count: 3958

Abstract

Digital signal processing has a significant role in the scientific world mostly in every aspect of individuals' lives. DSP deals with the transmission of signals such as sound and digital frequencies, and this involves the use of mathematical tools. In general, one must convert real life signals to processed signals so that computers or other devices could understand them. Despite a variety of ways to transform real-life signals, two major ways of converting signals are: Fourier transform and Z-transform (Smith, 1999). Even in this category, analysing discrete signals is called Discrete Fourier Transform and Discrete Z-transform.

The research question is: "How mathematics is used in Discrete Fourier Transform and Discrete Z-transform". I first explained Discrete Fourier Transform, and the extended version with the use of complex number, Complex Discrete Fourier Transform. Then, I dived into derivation of the equation of Z-transform with pure mathematics. Even though the transforms I have addressed is not the complete transform that is used in real life, due to many different kinds of ways of expressing various transforms. However, this paper will examine how mathematics is used behind those methods, and the relationship and difference between two methods.

(188 words)

The graphs and figures used in this essay are created on Desmos and GoodNotes unless stated otherwise.

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1. Introduction to Digital Signal Processing

A signal can be continuous or discrete and it can be periodic or aperiodic. This creates four different categories of signals: Continuous-Aperiodic, Continuous-Periodic, Discrete-Aperiodic and Discrete-Periodic (Smith, 1999). However, the type of signal that will be examined in this paper is:

> Periodic-Discrete: This category includes "discrete signals that repeat themselves in a periodic fashion from negative to positive infinity" (Smith, 1999). This particular signal is used due to a wide use of periodic-discrete signal in the study of Fourier transform (Smith, 1999).

Fourier transform that uses this signal is called the **Discrete Fourier Transform**, which will be extended to **Complex Fourier Transform** in the later section. Z-transform also converts discrete time signals, similar to Discrete Fourier Transform, but has a major difference in how mathematics is used behind it. The mathematical term: **transform**, is defined as a procedure that allows both the input and output to have a range of values (Smith, 1999). This will be a fundamental definition of all transforms including Fourier transform and Z-transform that will be examined in this essay.

2. Fourier Transform

2.1 Important notations in Fourier Transform

As mentioned earlier, periodic-discrete signals can undergo transformation by using Fourier transform, specifically called Discrete Fourier Transform (DFT). This transform only uses ordinary numbers and algebra for the synthesis and decomposition of signals (Smith, 1999). The input signal contains the signal being decomposed, while the outputs obtain the amplitudes of the component sine and cosine waves. In this context, the input signal exists in a specific domain called **time domain.** On the other hand, the output signal exists in **frequency domain.** The conversion of input to output is called forward DFT and going back from output to input is called inverse DFT. A summary is shown below:

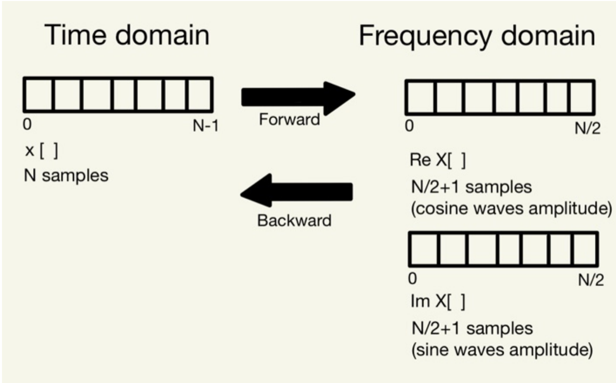

Fig 1. The diagram of a brief summary of Fourier transform

In the time domain, $x[\ ]$, is defined as consisting of N points running from 0 to $N-1$. In the frequency domain, the DFT generates two signals, the real and imaginary part, written as $ReX[\ ]$ and $ImX[\ ]$. The number of samples in the time domain is generally represented by the variable N, where N can be any positive integer (Smith 1999).

The following is an example of time domain:

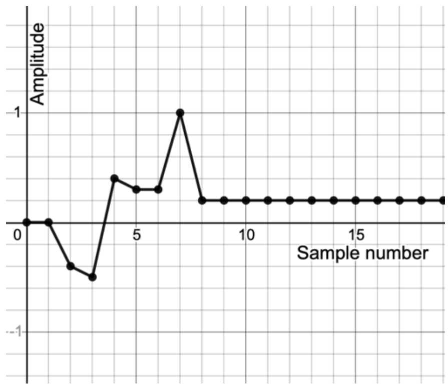

Fig 2. An example of time domain signal

These are important notations in DFT (Julius O, 2007):

1. The frequency domain: f, where f takes on $\frac{N}{2}+1$ equally spaced values, $0 \leq f \leq 0.5$
2. Real part of the frequency domain: $ReX[f]$

   Imaginary part of the frequency domain: $ImX[f]$
3. The index, k, represents k^{th} frequency, which is determined by the sampling frequency and number of samples
4. f can be also represented as $\frac{k}{N}$, where N is the total number of samples

5. Another way of expressing frequency is, ω, natural frequency, which is radian frequency interval $\frac{2\pi}{NT}$, $T =$ sampling interval (s)
6. $n = n^{th}$ sample e.g. $n = 1$ means 2nd sample where $0 \leq n \leq a, a \in \mathbb{Z}^+$

2.2 DFT Basis Functions

The sine and cosine waves are used in the DFT as the basis functions (Smith, 1999):

$$c_k[i] = \cos\left(\frac{2\pi ki}{N}\right) \quad s_k[i] = \sin\left(\frac{2\pi ki}{N}\right)$$

$c_k[i]$ is the cosine wave for the amplitude held in $ReX[k]$, and $s_k[i]$ is the sine wave for the amplitude held in $ImX[k]$. The time input, which is equal to the length of time, i, means that when $i = 32$, there are 32 samples taken from $i = 0$ to 31 as time starts from 0 second. For instance, $c_1[\ ]$ is the cosine wave that makes one complete cycle in N sampling points, $c_5[\ ]$ is the cosine wave that undergoes five complete cycles in N sampling points.

However, there are some exceptions when it comes to the transform. When $k = 0$, $c_0[\ ]$, it gives a cosine wave of zero frequency, which is a constant value of 1. For $s_0[\ ]$, it gives zero. This means that $ReX[0]$ holds the average value of all the points in the time domain signals, and $s_0[\ ]$ becomes irrelevant to transform and always set to zero (Smith, 1999). In this case, the amplitude that $ReX[0]$ is called **DC offset** (Smith, 1999).

Now, using the previous information, the synthesis equation of real and imaginary parts can be written:

$$x[i] = \sum_{k=0}^{N/2} Re\bar{X}\,[k] \cos(\frac{2\pi ki}{N}) + \sum_{k=0}^{N/2} Im\bar{X}\,[k] \sin(\frac{2\pi ki}{N})$$

In this synthesis equation, $x[i]$ is the signal being synthesized, with the index, $i,$ running from 0 to $N-1$ samples. $Re\bar{X}[k]$ and $Im\bar{X}[k]$ hold the amplitudes of the cosine and sine waves with k running from 0 to $\frac{N}{2}$.

The difference between $Re\bar{X}[k]$ and $ReX[k]$ is that the former is the amplitude needed for the synthesis while the latter is the frequency domain of a signal. Therefore, a slight adjustment can clarify the difference between two by the equation below (Smith, 1999):

$$Re\bar{X}[k] = \frac{ReX[k]}{\frac{N}{2}} \quad Im\bar{X}[k] = -\frac{ImX[k]}{\frac{N}{2}}$$

Except for two special cases (Smith, 1999):

$$Re\bar{X}[0] = \frac{ReX[0]}{N}$$

$$Re\bar{X}[N/2] = \frac{ReX[\frac{N}{2}]}{N}$$

These are the conversion between the time domain and the frequency domain. They are collectively called **Discrete Fourier Transform** (Smith, 1999). However, the only problem is that the value of k is only an integer, $k \in R$. This brings a problem when a complex number is involved, which creates a complex frequency domain where $ReX[k]$ becomes the real part of the complex frequency spectrum, and $ImX[k]$ becomes that of imaginary part.

2.3 Complex Discrete Forward Fourier Transform

The following is the equation of real discrete Forward Fourier Transform (Smith, 1999):

$$ReX[k] = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\cos(\frac{2\pi kn}{N})$$

$$ImX[k] = -\frac{2}{N}\sum_{n=0}^{N-1} x[n]\sin(\frac{2\pi kn}{N})$$

This is similar to the equation of forward transform of DFT in the previous section, however, the term $\frac{2}{N}$ is included in this transform due to the conversion of $ReX[k]$ to $Re\bar{X}[k]$, which was discussed in the previous section. However, this will be not considered as it is mostly used in computing to minimize programming errors (Smith, 1999).

In order to deal with complex numbers, it is necessary to use substitution method instead of putting real numbers into the forward transform.

According to Euler's formula (Woit, 2019):

$$e^{ix} = \cos(x) + i\sin(x)$$

Expressing each cosine and sine in terms of e^{jx}:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Recalling that sinusoids have outputs that are in frequency domain having different periods, the expressions are now modified by having natural frequency, ω :

$$\cos(\omega x) = \frac{1}{2}\ e^{i(-\omega)x} + \frac{1}{2}e^{i\omega x} \qquad \sin(\omega x) = \frac{1}{2}ie^{i(-\omega)x} + \frac{1}{2}ie^{i\omega x}$$

The forward complex Fourier transform is given by (University of Oxford, 2021):

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-\frac{i2\pi kn}{N}}\ or\ X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n](\cos\frac{2\pi kn}{N} - i\sin\frac{2\pi kn}{N})$$

2.4 Phase change in complex Discrete Fourier Transform

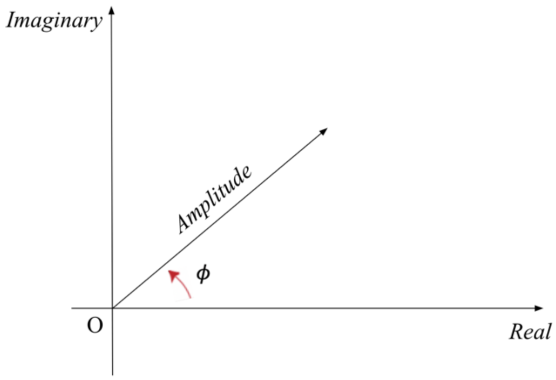

Fig 3. Real and Imaginary plane

The previous transforms did not mention about phase shift of the waves. As shown in figure 8, phase and amplitude can be described by a complex number (Rochester Institute of Technology, n.d.). Which means, when the $\phi = 0$, amplitude $= 1$. And this gives a real number as it lies on the real number axis. $X[k]$ can be represented as:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{i2\pi kn}{N}}$$

$$\text{Let } \frac{i2\pi kn}{N} = b_n,$$

$$X[k] = x[0] \cdot [\cos\ (-b_0) + i\sin\,(-b_0)] + \cdots$$

$$X[k] = R_k + I_k i$$

$$R_k = \sum_{n=0}^{N-1} x[n]\cos(-b_n), I_k = \sum_{n=0}^{N-1} x[n]\sin(-b_n)$$

Amplitude equals to (Rochester Institute of Technology, n.d.):

$$\text{Amplitude} = \sqrt{{R_k}^2 + {I_k}^2}$$

For $\phi = 45°\ (\pi/4)$, by using $\tan\phi = \frac{\text{Im}}{\text{Real}}$, it gives $\frac{\sqrt{2}}{2}$ for real part and $\frac{\sqrt{2}}{2}i$ for the imaginary part. For instance, the amplitude when $\phi = 45°$ can be expressed:

$$x[n]e^{\frac{2\pi ikn}{N}} = \frac{\sqrt{2}}{2}\cos(\frac{2\pi nk}{N}) - \frac{\sqrt{2}}{2}\sin\left(\frac{2\pi nk}{N}\right) + \frac{\sqrt{2}}{2}\cos\left(\frac{2\pi nk}{N}\right)i - \frac{\sqrt{2}}{2}\sin(\frac{2\pi nk}{N})i$$

In this case, the part highlighted with blue is the real part of time, while the part highlighted with black is imaginary time. This equation can be used for different k and n values. This is essentially used when the waves have phase shift. An example of DFT when a sine wave ($\sin x$) that has a frequency of 1 Hz (one cycle per second), amplitude $= 1$ and a sampling frequency of 8 Hz for 8 samples is used. By undergoing a discrete Fourier Transform:

$X[k]$	Term expression	Result
$X[0]$	$x[0]\cdot e^{-\frac{i2\pi(0)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(0)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(0)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(0)(7)}{8}}$	0
$X[1]$	$x[0]\cdot e^{-\frac{i2\pi(1)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(1)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(1)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(1)(7)}{8}}$	$0 - 4i$
$X[2]$	$x[0]\cdot e^{-\frac{i2\pi(2)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(2)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(2)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(2)(7)}{8}}$	0
$X[3]$	$x[0]\cdot e^{-\frac{i2\pi(3)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(3)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(3)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(3)(7)}{8}}$	0
$X[4]$	$x[0]\cdot e^{-\frac{i2\pi(4)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(4)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(4)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(4)(7)}{8}}$	0
$X[5]$	$x[0]\cdot e^{-\frac{i2\pi(5)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(5)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(5)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(5)(7)}{8}}$	0
$X[6]$	$x[0]\cdot e^{-\frac{i2\pi(6)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(6)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(6)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(6)(7)}{8}}$	0
$X[7]$	$x[0]\cdot e^{-\frac{i2\pi(7)(0)}{8}} + x[1]\cdot e^{-\frac{i2\pi(7)(1)}{8}} + x[2]\cdot e^{-\frac{i2\pi(7)(2)}{8}} + \cdots + x[7]\cdot e^{-\frac{i2\pi(7)(7)}{8}}$	$0 + 4i$

Table 1. The table showing the values of $X[k]$

Table 1 shows how DFT transforms time domain to frequency domain. However, any frequency that is larger than Nyquist limit experience aliasing effect in which a signal is not sampled fast enough to construct an accurate waveform record (Harvey, 2021). Nyquist limit equals to one half of the sampling rate, which, in this case is 4, and then double the values before the Nyquist limit (Harvey, 2021). Thus, $k > 4,$ where $k \in \mathbb{Z}$ will now be ignored giving new values expressed in magnitude:

$\boldsymbol{X[k]}$	**Result**	**Amplitude**	**Average**
$X[0]$	0	0	0
$X[1]$	$0 - 8\text{i}$	8	1
$X[2]$	0	0	0
$X[3]$	0	0	0
$X[4]$	0	0	0

Table 2. The table showing the amplitude and average

Table 2 shows the amplitude in the frequency domain. The amplitude has to be divided by the number of samples as eight samples is used in DFT (Smith, 1999), which gives 1 as the final amplitude. If the result from table 1 is plotted on the complex plane:

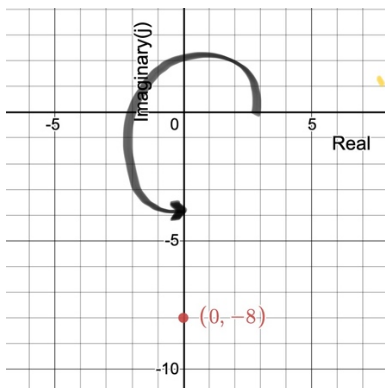

Fig 4. The graph plotting $\mathbf{0 - 8}\boldsymbol{j}$

In Figure 8, the black arrow shows that the phase angle is $\frac{3\pi}{2}$. To verify that this transform is true, we can check by going back. Since the angle is based on when cosine is zero, $\cos 0 = 1$, if there is a phase angle of $\frac{3\pi}{2}$ we can go back to get the original input:

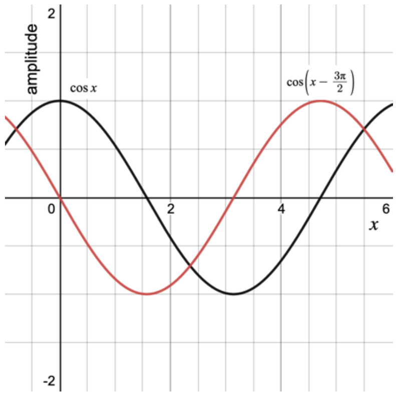

Fig 5. Shifting $\cos(x)$ by $\frac{3\pi}{2}$ radians

As can be seen from figure 5, cos (x) looks like sin (x) after being shifted by $\frac{3\pi}{2}$ radians. This $\sin(x)$ is the input, and from this it is verified that our output is valid.

2.5 Graphs of $Re\ X[\ ]$ and $Im\ X[\ ]$

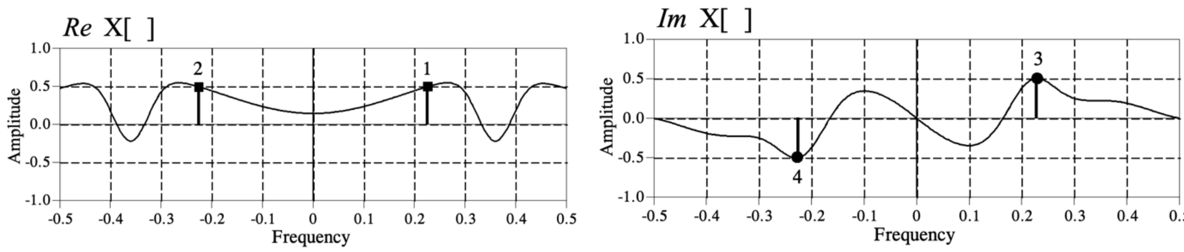

Fig 5. A graph of $Re\ X[\ ]$ and a graph of $Im\ X[\ ]$ (Smith, 1999)

The two graphs above shows examples of $Re\ X[\ ]$ and $Im\ X[\ ]$. $Re\ X[\ ]$ has an even function while $Im\ X[\ ]$ has an odd function. This gives an analysis that the graph of $Re\ X[\ ]$ shows two identical values in both negative and positive frequencies. On the other hand, $Im\ X[\ ]$ shows the same value but having an opposite sign. This gives an important evidence that the imaginary part after transformation will not be created as the sum of positive and negative frequency will cancel out the values making zero. This concept will be discussed further in the next section.

2.6 Complex Discrete Fourier Inverse Transform

Even though the forward Fourier transform enables the transform of time domain to frequency domain, it is not a complete transform when one is not able to go back to its original state from frequency to time domain. The inverse complex DFT is given by:

$$x[n] = \sum_{n=0}^{N-1} X[k]e^{-\frac{j2\pi kn}{N}}$$

Using Euler's relation and what was written in the previous section:

$$x[n] = \sum_{n=0}^{N-1} ReX[k](\cos\frac{2\pi kn}{N} + i\sin\frac{2\pi kn}{N}) - \sum_{n=0}^{N-1} ImX[k](\sin\frac{2\pi kn}{N} - i\cos\frac{2\pi kn}{N})$$

When going back to time domain from frequency domain to reconstruct the cosine wave at a frequency f, this requires a positive frequency and a negative frequency from the real part of the frequency spectrum. If we let $f, \frac{k}{N} = 0.3$, the positive frequency at 0.3 contributes a cosine wave and an imaginary sine wave to the time domain:

$$0.5\cos(2\pi 0.3n) + 0.5\, i\sin(2\pi 0.3n)$$

Similar to the positive frequency, the negative frequency at -0.3 also contributes a cosine and an imaginary sine wave:

$$0.5\cos(2\pi(-0.3)n) + 0.5i\sin(2\pi(-0.3)n)$$

Using the identity $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, expression above can be rewritten:

$$0.5\cos(2\pi \cdot 0.3n) - 0.5i\sin(2\pi \cdot 0.3n)$$

Adding all the real parts and imaginary parts together gives:

$$0.5\cos(2\pi \cdot 0.3n) + 0.5\cos(2\pi \cdot 0.3n) = \cos(2\pi \cdot 0.3n)$$

$$0.5i\sin(2\pi \cdot 0.3n) - 0.5i\sin(2\pi \cdot 0.3n) = 0$$

Giving the resultant time domain signal: $\cos(2\pi \cdot 0.3n)$

Applying the same method for a sine wave in the time domain:

From positive frequency: $-0.5\sin(2\pi \cdot 0.3n) - 0.5i\cos(2\pi \cdot 0.3n)$

From negative frequency: $-0.5\sin(2\pi \cdot 0.3n) + 0.5i\cos(2\pi \cdot 0.3n)$

Which gives the resultant time domain signal: $-\sin(2\pi \cdot 0.3n)$

This demonstrates that, in Complex Discrete Fourier Transform, the negative and positive frequencies have the same sign in the real part, but opposite signs in the imaginary part. This gives a result of imaginary parts being eliminated. In real life, the physical meaning of the imaginary part does not exist, but it is only allowed to exist by complex mathematics, as shown above (Smith, 1999).

In a summary, the equations of Real Discrete Fourier Transform are:

Analysis (Forward transform)

$$ReX[k] = \frac{2}{N}\sum_{n=0}^{N-1} x[n]\cos(\frac{2\pi kn}{N}), ImX[k] = -\frac{2}{N}\sum_{n=0}^{N-1} x[n]\sin(\frac{2\pi kn}{N})$$

Synthesis (Backward transform)

$$x[n] = \sum_{k=0}^{\frac{N}{2}} Re\bar{X}[k]\cos(\frac{2\pi kn}{N}) + \sum_{k=0}^{\frac{N}{2}} Im\bar{X}[k]\sin(\frac{2\pi kn}{N})$$

The equations of Complex Discrete Fourier Transform are:

Analysis (Forward transform)

$$x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{-\frac{i2\pi kn}{N}}$$

Synthesis (Backward transform)

$$x[n] = \sum_{n=0}^{N-1} X[k] e^{-\frac{i2\pi kn}{N}}$$

3. Z-Transform

3.1 Products of distances from zero

Similar to Discrete Fourier Transform discussed in previous section, there is another method in DSP, which is called Z-transform. In order to understand Z-transform, it is essential to understand how Z-transform is established mathematically. First, let there is graph of $f(x) = (x-1)(x+2)$.

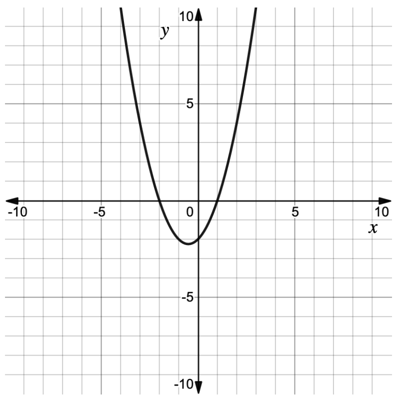

Fig 6. The graph of $f(x)$

Generally, one calculates y value of $f(x)$ by substituting a x value. However, there is another way to approach to calculation of y, which is using the distances from zeros of $f(x)$. Let there is a variable d which is the distance from zeros of $f(x)$. For example, when looking at one point $(3, 0)$, d_1 from $(1, 0)$ is 2 units and d_2 from $(-2, 0)$ is 5 units.

Then, $d_1 \times d_2 = 10$, we can see that it is equivalent to the y-value of $f(x)$ when $x = 3$. This means that the product of the distances from two zeros at certain point is the same as the y of $f(x)$ of that point. This rule applies in the same way to the other kinds of functions as well. For example, the rational function denoted as $y = \frac{f(x)}{x}$. The graph of $\frac{f(x)}{x}$ is shown below:

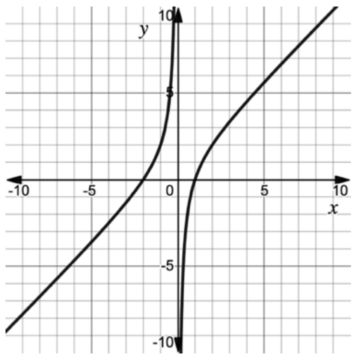

Fig 7. The graph of $\frac{f(x)}{x}$

The rule applies the same for the rational function in figure 2, but only if when adding the distance from $x = 0$. This means that when we are looking at the distances between a point $(3, 0)$ and other zeros, we have to consider three distance variables:

$d_1 = 5$, which is from $(-2, 0)$

$d_2 = 2$, which is from $(1, 0)$

$d_3 = 3$, which is the distance from $(0, 0)$ (the pole)

The multiple of d-variables of the graph will be:

$$\frac{d_1 \times d_2}{d_3} = \frac{5 \times 2}{3} = \frac{10}{3}$$

In this case, $d_1 \times d_2$ must be divided by d_3 as it is the denominator of $\frac{f(x)}{x}$.

When we substitute $x = 3$ to $\frac{f(x)}{x}$:

$$\frac{f(x)}{x} = \frac{(x-1)(x+2)}{x}$$

$$\frac{f(3)}{3} = \frac{(3-1)(3+2)}{3}$$

$$= \frac{10}{3}$$

Real-world signals often involve various peaks such as a range of peaks in stock market prices and sounds. It is important to find a general trend to identify the long-term changes, and one of the generalizations that can be done is by taking a moving average of signals.

3.2 Moving average

Let us define $g(x) = \cos(x)$, where the points are sampled every $\frac{\pi}{3}$ radians. Four points sampled are:

$$\mathrm{A}(0, 1),\ \mathrm{B}\left(\frac{\pi}{3}, 0.5\right),\ \mathrm{C}(\frac{2\pi}{3}, -0.5),\ \mathrm{D}(\pi, -1)$$

This gives a graph:

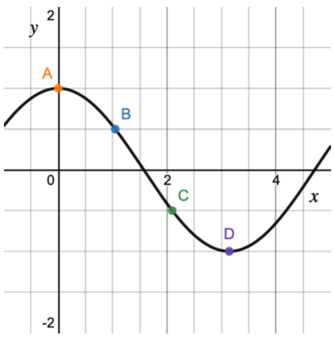

Fig 8. The graph of $g(x)$ with points A, B, C, D

Given that a sequence $\{a_i\}_{i=1}^{N}$, an n-moving average a new sequence $\{s_i\}_{i=1}^{N+n-1}$ defined from a_i by taking the arithmetic mean of subsequence of n terms (Weisstein, 2021).

$$s_i = \frac{1}{n}\sum_{j=i}^{i+n-1} a_j$$

For instance, for moving average of $n = 2$ of $g(x)$ is:

$$\text{Average value of } x = \frac{A_x+B_x}{2}$$
$$\text{Average value of } y = \frac{f(A_x)+f(B_x)}{2}$$

Which gives a moving average for A and B:

$$(x, y) = \left(\frac{1 + 0.5}{2}\right), \left(\frac{1.0472 + 0}{2}\right) = (0.52, 0.75)$$

Applying the same calculation for points B, C and C, D; the new coordinates are respectively:

$$\mathrm{A}'(0.524, 0.750), \mathrm{B}'(1.571, 0.00), \mathrm{C}'(2.618, -0.750)$$

By putting these coordinates on the plane, we can obtain another function $g'(x) = \frac{\sqrt{3}}{2}\cos(x)$ which is illustrated below:

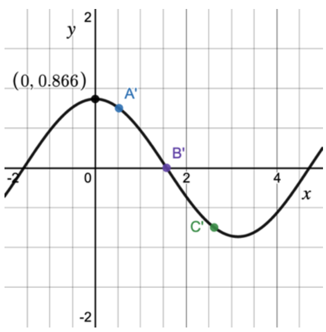

Fig 9. The graph of $g'(x)$

After plotting the moving averages, the graph of $g'(x)$ looks more less sharp and generalized. Now, define $h(x) = \cos(x)$, where a sample is taken every $\frac{\pi}{4}$ radians. The four points are:

$$\mathrm{A}(0,1),\ \mathrm{B}\left(\frac{\pi}{2},0\right),\ \mathrm{C}(\pi,-1),\ \mathrm{D}\left(\frac{3\pi}{2},0\right)$$

Using the same method used for $g(x)$, it gives: $\mathrm{A}'(0.7854, 0.5), B'(2.3562, -0.5), \mathrm{C}'(3.9270, -0.5)$

Which gives a graph:

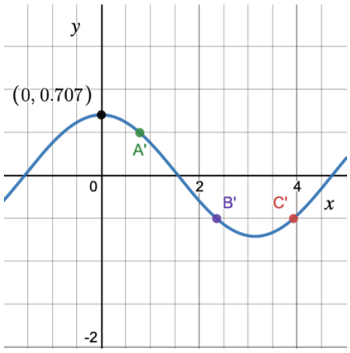

Fig 10. The graph of $h'(x)$

With reference to figure 5, $h(x)$ becomes less sharp in $h'(x)$. But one thing to take note is that the magnitude of function after plotting the moving averages is dependent on the magnitude that divides the *x* value. Although the graphs of $g(x)$ and $h(x)$ are the same cosine graphs, after calculating moving averages the graphs drawn have different magnitudes respectively $\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{2}}{2}$.

3.3 Basics of Z-transform

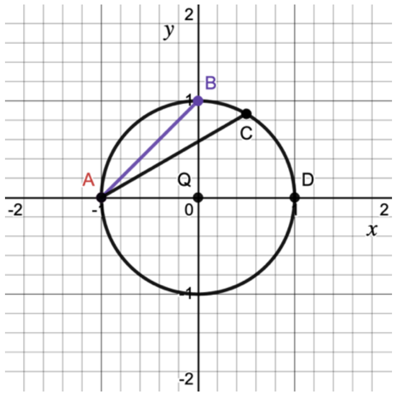

Fig 11. Unit circle with A, B, C

With reference to figure 6, three points are shown: A, B and C. Q is the origin of the plane and D is another imaginary point for the demonstration of the angle.

A, B and C have different θ with respect to line DQ. θ represents the sampling interval from the function $\cos(x)$:

$$\angle DQC = 60° \ (\pi/3 \text{ rad})$$

$$\angle DQB = 90° \ (\pi/2 \text{ rad})$$

$$\angle DQC = 180° \ (\pi \text{ rad})$$

Given that the amplitude of cosine graph after taking moving average when $\theta = \pi$ is zero, the distances between points A and B, and A and C in the unit circle are:

$$|\mathrm{A} - \mathrm{A}| = 0$$

$$|\mathrm{A} - \mathrm{B}| = \sqrt{2}$$

$$|\mathrm{A} - \mathrm{C}| = \sqrt{3}$$

This means that the distance between A and a moving point on a unit circle divided by two will give the magnitude of a graph after taking moving averages. The moving point will move dependent on the sampling interval $x \leq \pi$. Then the amplitude of $\cos(x)$ when x is any point on the unit circle:

$$y(z) = \frac{|\mathrm{A} - x|}{2}, y(z) \text{ represents amplitude}$$

Following is a unit circle with points $A, O,$ and Z:

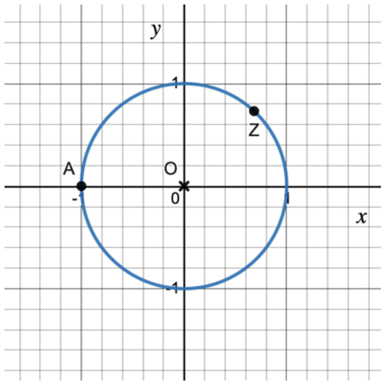

Fig 12. A unit circle with a point Z

With the reference to figure 7, by using of vector addition of Z:

$$\overrightarrow{AZ} = \overrightarrow{AO} + \overrightarrow{OZ}$$

$$\overrightarrow{AZ} = \mathrm{z} + 1$$

$$y(z) = \frac{1}{2}(z + 1)$$

Add the distance from pole as z, where $y(z)$ is the distance from zero:

$$\frac{y(z)}{x(z)} = \frac{\frac{1}{2}(z + 1)}{z}$$

$$\frac{y(z)}{x(z)} = \frac{1}{2}(1 + z^{-1})$$

$$y(z) = \frac{1}{2} x(z)(1 + z^{-1})$$

$$y(z) = \frac{1}{2} [x(z) + z^{-1} x(z)] \dots \dots \dots \dots (1)$$

$$y(n) = \frac{x(n) + x(n-1)}{2} \dots \dots \dots \dots (2)$$

In this case, $z + 1$ is the distance from zero while z is the distance from the pole. (1) shows that the output $y(z)$ can be calculated by using distance of $z + 1$ and distance from the pole. (2) shows that output $y(n)$ can be calculated by using the separation between two x values. Two methods are significantly related to Z-transform, and both gives the same output. Which means:

$Z^{-1}(x) = \mathrm{X(n-1)}$, which gives a relationship: $Z^{-n} = X(N - n)$

For instance, the equation showing a moving average with four points:

$$\frac{x(n) + x(n-1) + x(n-2) + x(n-3)}{4} = y(n)$$

If this is converted in terms of $Z^{-1}(x)$:

$$\frac{x(z) + z^{-1}x(z) + z^{-2}x(z) + z^{-3}x(z)}{4} = y(z) \dots \dots \dots (a)$$

By simplifying the equation:

$$\frac{Z^3}{Z^3} \times \frac{1 + z^{-1} + z^{-2} + z^{-3}}{4} = \frac{y(z)}{x(z)}$$

$$\frac{z^3 + z^2 + z^1 + 1}{4z^3} = \frac{y(z)}{x(z)} \dots \dots \dots (3)$$

(3) gives zeros: $z = i, z = -i, z = -1$ and the pole $= 0$. This is important because the distances that are used to calculate amplitude is based on the zeros from this equation. In addition, the equation (a) provides the general formula of Z-transform (Yogananda, 2021) given that there are infinite number of points on the complex plane. This is expressed as:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Instead of $y(z)$, $X(z)$ is used. An example of this transform using a function of $\cos(x)$:

By taking the sample every $\frac{\pi}{4}$ rad from $\cos{(x)}$, the new output can be calculated by using the distances from input z to zeros in which is the value on a unit circle when the angle from the x axis is $\frac{\pi}{4}$ rad (45°):

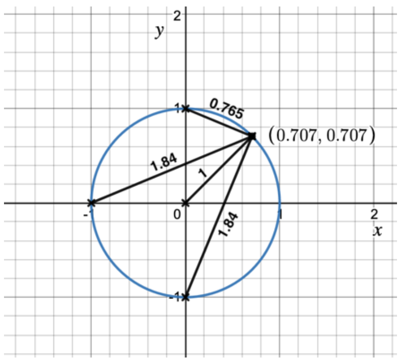

Fig 13. The output when the sampling interval is $\frac{\pi}{4}$ of $\cos{(x)}$

As shown in the graph, the output of the reconstructed signal now can be calculated by using the distances from the Z and the origin:

$$\frac{1.84 \times 0.765 \times 1.84 \times 1}{4} = 0.65$$

0.65 is the amplitude of new reconstructed signal. However, the only limitation of the method is that it is only applied to simple sinusoids that have a period of 2π. So minor change can fix this problem.

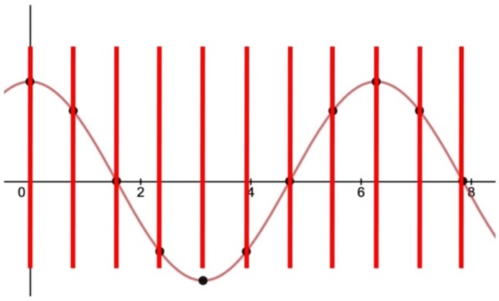

Fig 14. A graph with fixed lines every $\frac{\pi}{4}$ radians

Consider there are fixed lines every $\frac{\pi}{4}$ radians. Now, instead of changing the sampling interval, the frequency of the graph will change. Which means, even though the graph does not have the same period, the sample is taken from the fixed lines above and this gives the same output.

3.4 Application of Z-transform

Z-transform is used when filtering out the noise to get the wanted part of the sound without interference. By using audacity, my voice is recorded can be plotted as discrete signal:

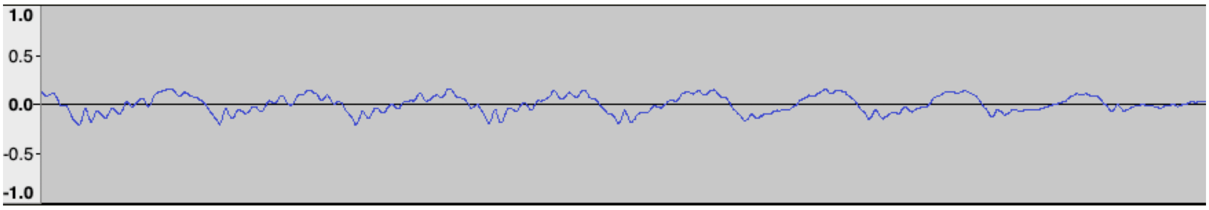

Fig 15. A graph of sound recorded on Audacity

However, when the sound wave in the form of sin (x) as following is added:

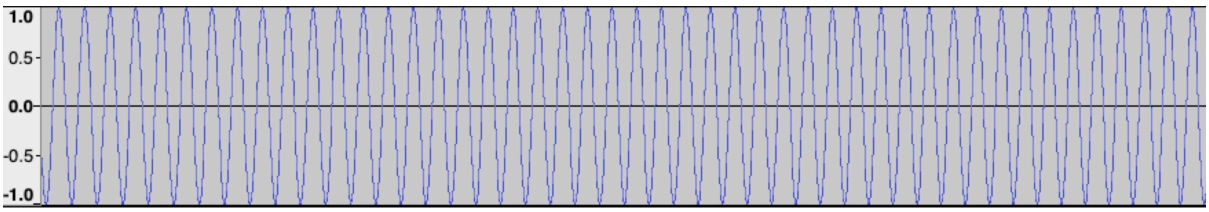

Fig 16. A graph of $\sin(x)$

It creates a new sound that is in different shape:

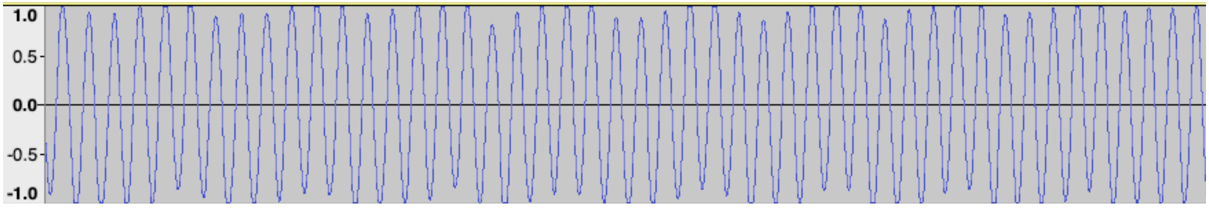

Fig 17. $\sin(x)$ added to the voice recorded

By using a frequency spectrum analysis which is a tool in Audacity, we can see that there is a pitch at 1000 Hz that dominates the sound sample:

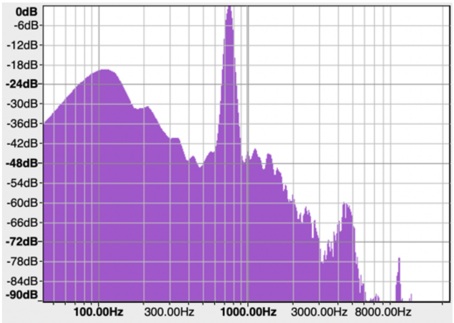

Fig 18. Frequency analysis of previous sound graph

In order to go back to obtain original sound, it is important to remove the pitch from the analysis, so a tool in audacity called Equalization is used at 1000 Hz:

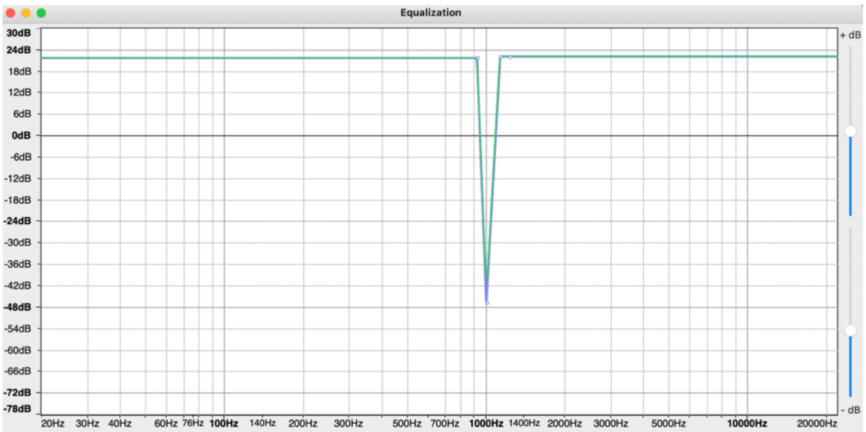

Fig 19. A screenshot of Equalization tool

When equalization tool is applied, it gives a result as following:

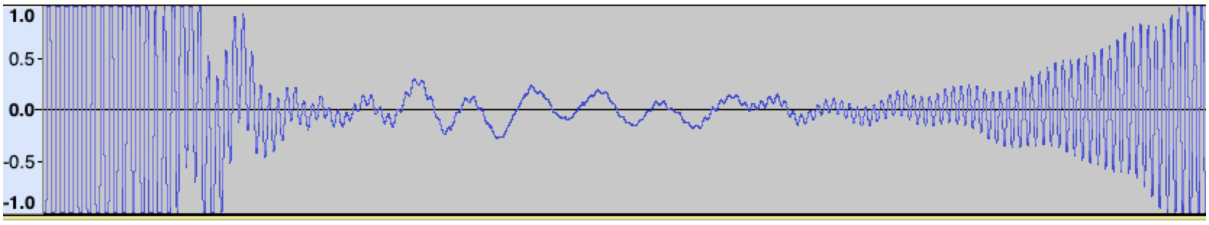

Fig 20. Sound after equalization was applied

As figure 11 shows, we can now see the part of original sound. Even though it does not reveal whole part, it still recovers some parts of the original sound. An important thing that can be found from this is that finding original sound is significantly related to Z-transform. For example, if there is a specific frequency to remove, by using the unit circle, plot a point near the unit circle as following:

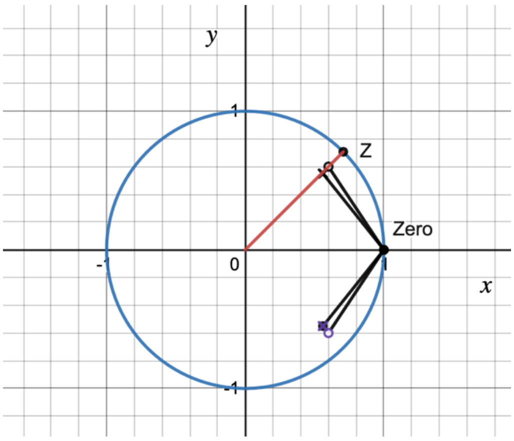

Fig 21. A unit circle having points of a specific frequency and a pole

The circle represents a specific frequency and the cross is the pole that is placed slightly under the circle. The two points (which are illustrated as hollow circles) are connected to zero and the zero will now move to $1 \rightarrow -1$ by recording the values where frequency refers to the x-axis and output amplitude to the y-axis. By moving the zero, it creates a graph:

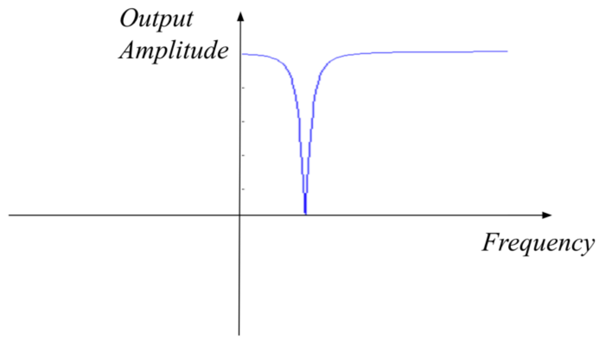

Fig 22. A graph of notch filter when zero in fig 21 moves to -1

This is very similar to the equalization tool in audacity. This is how Z-transform is used in real life, working as a tool to remove unwanted frequency or other signals that need modifications to obtain other signals. One aspect is that Z-transform provides filter to obtain original signal or to remove interfered frequency of a sample.

4. Conclusion

The investigation lead in this paper shows how mathematic is used in Discrete Fourier Transform and Z-transform. The complex plane, which I was not sure how complex number like $\sqrt{-1}$ is used in real life. However, it is found that the complex number is a body of DFT, as there are two parts, the real part and the imaginary part in signals. What stood out in this essay was the application of pure mathematics, how Euler's relation and basic unit circle was extended to the equations of Complex DFT and Z-transform.

It is distinct that DFT and ZT has some similarities but different math is used. DFT mostly uses cosine and sine function with the use of imaginary number i. However, discrete ZT uses unit circle to find out the amplitude of reconstructed signal, which is the value in the frequency domain. Even though I have explained the fundamental equations of DFT and one aspect of ZT, it could be improved by using more examples in relation to computing, which could help to visualise more complicated sinusoids and real life signals. The fact that there are a range of properties in both DFT and ZT such as circular shift, modulation (Selesnick, 2021), there were so many aspects of both methods that could have been discussed in this paper. In addition, when explaining DFT, the frequency domain was set to $-0.5 < f < 0.5$, for the better explanation. It does not have to be in that range, other values can be used as well.

Furthermore, the resultant time signal that shows imaginary parts being cancelled out, could be analysed further to explain the reasons why imaginary part of time domain produces a frequency domain with and odd real part and an even real part. Also, other Fourier transforms, such as Fourier series could be explored as well since it has a strong relationship with DFT as the sampled signal of DFT is proportional to the Fourier series of coefficient (Julius O, 2007).

In summary, even though the extent of this paper is somewhat limited in terms of providing multiple examples and extensions, it does give a valid answer to the research question, the use of mathematics behind DFT and Discrete ZT. The essay enhances the basic understanding of Discrete Fourier Transform, provides insight to new concepts such as Time domain and Frequency domain, and fundamentally for ZT, showing the derivation by using pure mathematics. This paper shows applications of DSP nowadays, exhibiting strong connections to mathematics, and without all the mathematics behind, it would be impossible to transform signals one way or another.

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