Mathematics HL Investigation: Deriving the equation of the motion of double pendulum

1. Introduction

In the field of dynamic systems, a double pendulum is a pendulum with another pendulum attached to the end of the mass, as shown in figure 1. When I first encountered the nature of the motion of double pendulum, I was amazed that the pendulum moves in a chaotic movement which was distinctly different to the movement of conventional pendulums that moves in a simple harmonic motion. Even though it is difficult to predict the movement of double pendulum through observation, it is possible to derive the motion of the pendulum using mathematics behind the physics theory of pendulum. As I have been interested in double pendulum from the study of simple harmonic motion in IB Physics, I wanted to challenge myself mathematically in my math IA to derive the equation of motion of double pendulum by starting from relevant equations that I could use.

Before finding out the motion of double pendulum, there are some assumptions that must be considered for a system to use relevant equations [1]:

1. The masses are point mass
2. Strings are massless
3. Gravity is present

In this investigation, only two degree of freedom will be examined, which means the equation motion of the system is described only through two coordinates, not in the form of (x, y, z) which is three-dimensional space. Furthermore, the equations found from the investigation will help in providing answer to the chaotic movement of double pendulum.

2. Derivation of the motion equation

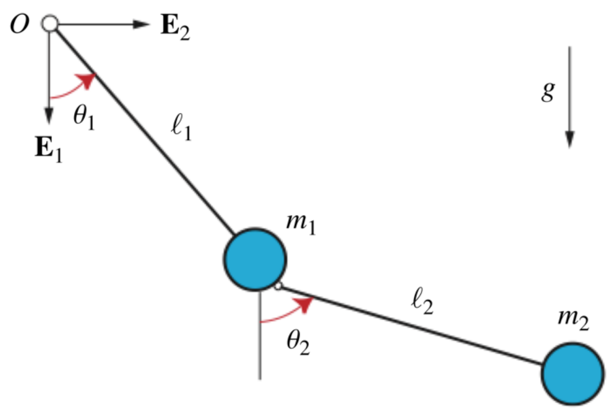

Fig 1. The diagram of double pendulum

Assume that assumptions above are true, the equation of individual motions of masses are determined. From figure 1, the movement of m_1 and m_2 can be broken down into (x, y), where O is the origin. By using the law of sines in a right-angle triangle, (x, y) of m_1 and m_2 can be expressed:

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

Which gives the coordinate of m_1 and m_2:

$$m_1: (l_1 \sin\theta_1\,, -l_2 \cos\theta_1)$$
$$m_2: (l_1 \sin\theta_1 + l_2 \sin\theta_2, l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Finding the location of double pendulum is complex as there is dramatic increase of kinetic energy due to the masses over time [3]. Lagrange system is described in terms of the generalized coordinates and generalized velocities [2], which will be presented soon. Lagrange (*L*) is defined [3]:

$$L = T - V$$

Where *T* is the total kinetic energy (KE) and *V* is the potential energy (PE).

An accent is used when expressing a time derivative:

$$\frac{d}{dt}x = \dot{x}, \frac{d}{dt}(\dot{x}) = \ddot{x}$$

Then the generalized velocities of m_1 and m_2 can be derived by using differentiation:

$$\dot{x}_1 = \dot{\theta}_1 l_1 \cos\theta_1$$
$$\dot{y}_1 = \dot{\theta}_1 l_1 \sin\theta_1$$
$$\dot{x}_2 = \dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_2 l_2 \cos\theta_2$$
$$\dot{y}_2 = \dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2$$

Now the expression of total potential energy of the system is also required. The potential object of any mass from the ground is:

$$\mathrm{PE} = mgh$$

Which gives the total potential energy of the system, represented by V:

$$V = m_1 g y_1 + m_2 g y_2 \ \mathbf{(1)}$$

By substituting y_1 and y_2 that were derived in the previous section to equation **(1)**

$$V = m_1 g[-l_2 \cos\theta_1] - m_2 g[l_1 \cos\theta_1 + l_2 \cos\theta_2]$$
$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

The expression of total kinetic energy is now derived. The kinetic energy of a mass with a velocity v is:

$$\mathrm{KE} = \frac{1}{2} m v^2$$

Which gives the expression of the total KE of the system, represented by T:

$$T = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$

$$T = \frac{1}{2}m_1\left(\sqrt{{\dot{x}_1}^2 + {\dot{y}_1}^2}\right)^2 + \frac{1}{2}m_2\left(\sqrt{{\dot{x}_2}^2 + {\dot{y}_2}^2}\right)^2$$

$$T = \frac{1}{2}m_1({\dot{x}_1}^2 + {\dot{y}_1}^2) + \frac{1}{2}m_2({\dot{x}_2}^2 + {\dot{y}_2}^2)$$

$\dot{x}_1, \dot{y}_1, \dot{x}_2$ and $\dot{y}_2$ can be rewritten as:

$$T = \frac{1}{2}m_1[(l_1\dot{\theta}_1)^2(\cos^2\theta_1 + \sin^2\theta_1)] + \frac{1}{2}m_2[\left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2$$

$$\text{Since } \cos^2\theta_1 + \sin^2\theta_1 = 1$$

$$T = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2[\left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2$$

Expanding the squares in the equation:

$$T = \tfrac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \tfrac{1}{2}m_2[{l_1}^2{\dot{\theta}_1}^2(\cos^2\theta_1 + \sin^2\theta_1) + {l_2}^2{\dot{\theta}_2}^2(\cos^2\theta_2 + \sin^2\theta_2) + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

Trigonometric identity states:

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos\theta_1 - \theta_2$$

Showing that *T* could be simplified further:

$$T = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2[{l_1}^2{\dot{\theta}_1}^2 + {l_2}^2{\dot{\theta}_2}^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)]$$

Substituting *T* and *V* that were found earlier to Lagrange:

$$L = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2[{l_1}^2{\dot{\theta}_1}^2 + {l_2}^2{\dot{\theta}_2}^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)] + (m_1 + m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

Because the system has two generalized coordinates θ_1 and θ_2 associated with two degree of freedom of system, we must Lagrange must be expressed into two separate respect to θ_1 and θ_2. Lagrange equation is going to be used to find separate equations for θ_1 and θ_2. Euler-Lagrange's equation states that [3]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \text{for } q = \theta_1 \text{ and } \theta_2 \ldots\ldots\ldots \mathbf{(2)}$$

Using **(2)** when $q = \theta_1$:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 {l_1}^2 \dot{\theta}_1 + m_2 {l_2}^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2){l_1}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \cdot (\dot{\theta}_1 - \dot{\theta}_2) \mathbf{(3)}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 \ \mathbf{(4)}$$

Using **(3)** and **(4)** for substitution to **(2)**:

$$(m_1 + m_2){l_1}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

Simplified by dividing both sides by l_1:

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0 \ \mathbf{(5)}$$

This is the first equation we can obtain. Now, we will take the derivative for θ_2 to **(2)**:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 {l_2}^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 {l_2}^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \cdot (\dot{\theta}_1 - \dot{\theta}_2) \mathbf{(6)}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 \ \mathbf{(7)}$$

Using **(6)** and **(7)** for substitution to Lagrange's equation:

$$m_2 {l_2}^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

Term $m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)$ cancels out:

$$m_2 {l_2}^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

Divide both sides by l_2:

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \ \mathbf{(8)}$$

This is the second equation we can obtain. In order to find the motion, we must find solutions of $\theta_1(t)$ which is the angular displacement for vertical axis. In addition for $\theta_2(t)$, which is the angular displacement for horizontal axis, is required to find the solution. In addition, it is necessary to find $\dot{\theta}_1(t)$ and $\dot{\theta}_2(t)$ as well, which are the second order equations that represents angular velocity; solving them analytically is a challenge. Analytically means solving the two equations mathematically, so with the help of the computer program SciLab with substitution method, we are able to visualize the movement of the system.

3. Converting second order differential equation to first order

Since there are two second order differential equations that need to be solved, solving them mathematically will be a challenge. In this investigation, computational software SciLab will be used for calculations and for the creation of the graphs that show the motion of double pendulum system based on the equations that are derived. The second order differential equations will be converted to first order differential equations, hence, computer program will be used to calculate the value of θ_1 and θ_2. First the variables of angle will be represented as y_n:

$$\begin{aligned} y_1 &= \theta_1 \\ y_2 &= \dot{\theta}_1 \\ y_3 &= \theta_2 \\ y_4 &= \dot{\theta}_2 \end{aligned}$$

Let $\dot{y_1} = y_2 \dots\dots\dots$ **(9)**. First, rearrange **(5)**:

$$(m_1 + m_2)l_1\ddot{\theta}_1 = -m_2l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2l_2\,\dot{\theta}_2^{\;2}\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\sin\theta_1$$

Rearrange **(8)** as well, but multiply both sides by $\cos(\theta_1 - \theta_2)$:

$$\begin{aligned} m_2l_1\ddot{\theta}_1\cos^2(\theta_1 - \theta_2) = &- m_2l_2\theta_2\cos(\ddot{\theta}_1 - \theta_2) + m_2l_1\dot{\theta}_1^{\;2}\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) \\ &- m_2g\sin\theta_2\cos(\theta_1 - \theta_2) \end{aligned}$$

Combine two equations **(5)** and **(8)** by using subtraction

$$\begin{aligned} LHS &= (m_1 + m_2)l_1\ddot{\theta}_1 - m_2l_1\ddot{\theta}_1\cos^2(\theta_1 - \theta_2) \\ RHS &= -m_2l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2l_2\,\dot{\theta}_2^{\;2}\sin(\theta_1 - \theta_2) \\ &\quad - (m_1 + m_2)g\sin\theta_1 - m_2l_2\theta_2\cos(\ddot{\theta}_1 - \theta_2) + m_2l_1\dot{\theta}_1^{\;2}\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) \\ &\quad - m_2g\sin\theta_2\cos(\theta_1 - \theta_2) \end{aligned}$$

Which gives:

$$\begin{aligned} \ddot{\theta}_1[(m_1 + m_2)l_1 - m_2l_1\cos^2(\theta_1 - \theta_2)] = &- m_2l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2l_2\,\dot{\theta}_2^{\;2}\sin(\theta_1 - \theta_2) \\ &- (m_1 + m_2)g\sin\theta_1 - m_2l_2\theta_2\cos(\ddot{\theta}_1 - \theta_2) + m_2l_1\dot{\theta}_1^{\;2}\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) \\ &- m_2g\sin\theta_2\cos(\theta_1 - \theta_2) \end{aligned}$$

Simply:

$$\begin{aligned} \ddot{\theta}_1 = [&-m_2l_1\dot{\theta}_1^{\;2}\sin(\theta_1 - \theta_3)\cos(\theta_1 - \theta_3) + m_2g\sin\theta_2\cos(\theta_1 - \theta_3) - -m_2l_2\dot{\theta}_2^{\;2}\sin(\theta_1 - \theta_3) \\ &- (m_1 + m_2)g\sin\theta_1][(m_1 + m_2)\,l_1 - m_2l_1\cos^2(\theta_1 - \theta_3)]^{-1} \end{aligned}$$

Replacing all the variables ($\theta_1, \theta_2, \dot{\theta}_1$ and $\dot{\theta}_2$) by y_1 to y_4:

$$\dot{y_2} = [-m_2 l_1 {y_2}^2 \sin(y_1 - y_3)\cos(y_1 - y_3) + m_2 g \sin y_3 \cos(y_1 - y_3) - -m_2 l_2 {y_4}^2 \sin(y_1 - y_3) - (m_1 + m_2) g \sin y_1][(m_1 + m_2)\, l_1 - m_2 l_1 \cos^2(y_1 - y_3)]^{-1} \ldots\ldots\ldots (\mathbf{10})$$

Let $\dot{y_3} = y_4 \ldots\ldots\ldots (\mathbf{11})$, rearrange **(8)**:

$$l_2 \ddot{\theta_2} = -l_1 \ddot{\theta_1} \cos(\theta_1 - \theta_2) + l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2$$

Rearranging **(5)**:

$$m_2 l_2 \ddot{\theta_2} \cos(\theta_1 - \theta_2) = -(m_1 + m_2) l_1 \ddot{\theta_1} - m_2 l_2 \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin\theta_1$$

Multiply $\frac{1}{m_1+m_2}\cos(\theta_1 - \theta_2)$ both sides:

$$\frac{m_2 l_2}{m_1 + m_2} \ddot{\theta_2} \cos^2(\theta_1 - \theta_2) = [-(m_1 + m_2) l_1 \ddot{\theta_1} - m_2 l_2 \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin\theta_1][\frac{1}{m_1 + m_2}\cos(\theta_1 - \theta_2)]$$

Combine two rearranged equations by using subtraction:

$$LHS = l_2 \ddot{\theta_2} \cos(\theta_1 - \theta_2) - \frac{m_2 l_2}{m_1 + m_2} \ddot{\theta_2} \cos^2(\theta_1 - \theta_2) = \ddot{\theta_2}[l_2 - \frac{m_2 l_2}{m_1 + m_2}\cos^2(\theta_1 - \theta_2)]$$

$$RHS = -l_1 \ddot{\theta_1} \cos(\theta_1 - \theta_2) + l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 - \left[-l_1 \ddot{\theta_1} \cos(\theta_1 - \theta_2) - \frac{m_2 l_2}{m_1+m_2} \dot{\theta_2}^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) - g \sin\theta_1 \cos(\theta_1 - \theta_2)\right]$$

Simplify further:

$$RHS = l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 + \left(\frac{m_2 l_2}{m_1 + m_2}\right) \dot{\theta_2}^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g \sin\theta_1 \cos(\theta_1 - \theta_2)$$

Which gives a final equation:

$$\ddot{\theta_2} = \left[l_2 - \frac{m_2 l_2}{m_1 + m_2}\cos^2(\theta_1 - \theta_2)\right]^{-1} [l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 + \left(\frac{m_2 l_2}{m_1 + m_2}\right) \dot{\theta_2}^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + g \sin\theta_1 \cos(\theta_1 - \theta_2)]$$

Replacing all the variables ($\theta_1, \theta_2, \dot{\theta_1}$ and $\dot{\theta_2}$) by y_1 to y_4:

$$\dot{y_4} = \left[l_2 - \frac{m_2 l_2}{m_1 + m_2}\cos^2(y_1 - y_3)\right]^{-1} [l_1 {y_2}^2 \sin(y_1 - y_3) - g \sin y_3 + \left(\frac{m_2 l_2}{m_1 + m_2}\right) {y_4}^2 \sin(y_1 - y_3)\cos(y_1 - y_3) + g \sin y_1 \cos(y_1 - y_3)] \ldots\ldots\ldots (\mathbf{12})$$

4. Extension: Computer Simulation

Now, all the equations are expressed in first order ordinary differential equations, in terms of y_1, y_2, y_3 and y_4. Two equations will be used in SciLab for plotting the graph of the motion of double pendulum starting from a specific condition. The screenshots below show the coding of computer program that gives the graph of double pendulum based on the equations we have derived:

Constants

```
clear;clc;clf();
g=9.81;             // Acceleration due to gravity
l1=1;               // Length of body 1
l2=1;               // Length of body 2
m1=0.5;             // Mass of body 1
m2=0.5;             // Mass of body 2
t=[0:0.005:6*%pi]   // Time Period
th1=120;            // Initial displacement of b1
th2=0;           // Initial displacement of b2
v1=0;               // Initial velocity of m1
v2=0;               // Initial velocity of m2
yi=(%pi/180)*[th1; 0; th2; 0];
```

Equations from the previous section (in terms of y1 to y4)

```
function [ydash]=f(t,y)
    ydash(1)= y(2)
    ydash(2)= (1/((m1+m2)*l1-m2*l1*(cos(y(1)-y(3)))^2))*((-m2*l1*y(2)*y(2)*sin(y(1)-y(3))*cos(y(1)-y(3)))+(m2*g*sin(y(3))*cos(y(1)-y(3)))-(m2*l2*y(4)*y(4)*sin(y(1)-y(3)))-(m1+m2)*g*sin(y(1)) )
    ydash(3)= y(4)
    ydash(4)= (1/(l2-(m2*l2*cos(y(1)-y(3))*cos(y(1)-y(3))/(m1+m2))))*((l1*y(2)*y(2)*sin(y(1)-y(3)))-(g*sin(y(3)))+((m2*l2/(m1+m2))*y(4)*y(4)*sin(y(1)-y(3))*cos(y(1)-y(3)))+(g*sin(y(1))*cos(y(1)-y(3))))
endfunction
t0=0
y=ode(yi,t0,t,f)
//Trajectory
x1=l1*sin(y(1,:));
y1=-l1*cos(y(1,:));
```

Graphing command

```
x2=l1*sin(y(1,:))+l2*sin(y(3,:))
y2=-l1*cos(y(1,:))-l2*cos(y(3,:));
subplot(2,2,1)
xgrid
plot(x1,y1,'-ob')
plot(x2,y2,'-or')
xtitle(['Trajectory'])
```

Figure 2. The screenshot of the script of computer program

The first few lines set the constants that are used to the equation derived in the previous section. The values: $g = 9.81, l_1$ and $l_2 = 1$, $m_1 = 0.5, m_2 = 0.5$, and $\theta_2 = 0°$ are going to be constant, only θ_1 will vary in order to examine how the motion changes. Refer to Figure 1 for confirming which variables refer to what specific parts of the double pendulum system. Using command: function, we can use (9), (10), (11), (12) as functions so that we can substitute whatever initial angle we want to put in. The graphs that will be drawn consist of x-axis that represents time period and y-axis the amplitude of the pendulum at an incident of time. In other words, the movement of mass is going to be tracked on the graph every time period.

By using the program written above, the graphs of $\theta_1 = 30°, 40°, 50°, 60°$ and $70°$ will be shown:

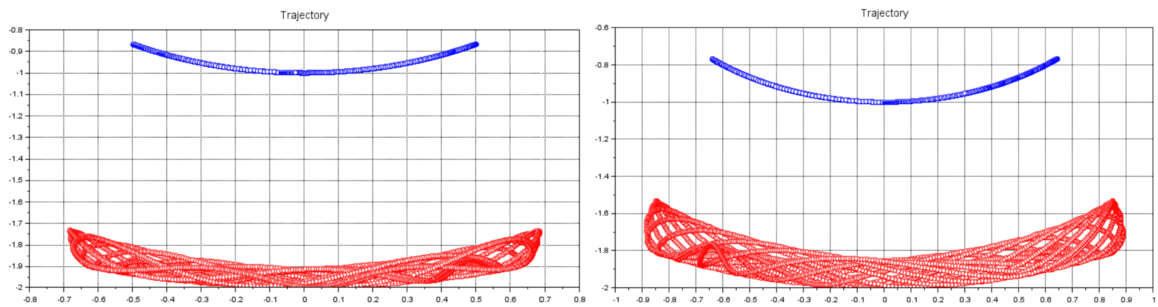

Figure 3 and 4. The left figure is when $\theta_1 = 30°$ and the right figure is when $\theta_1 = 40°$

As can be seen from figure 3 and 4, the blue dots show the movement of mass 1 while red dots show the movement of mass 2. It can be seen that the movement of mass 1 is somehow restricted, as it exhibits a parabolic trajectory that has line of symmetry in the middle. However, mass 2 moves in a little bit of uneven movement, as the movement is not identical to the left and to the right. This shows that mass 1 still generally have a pattern like parabola, however, the movement is differentiated for every one cycle of the movement of pendulum.

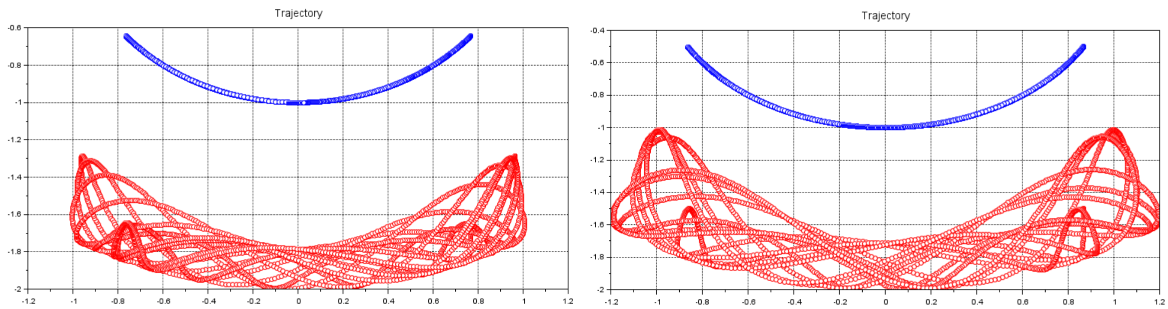

Figure 5 and 6. The left figure is when $\boldsymbol{\theta_1} = 50°$ and the right figure is when $\boldsymbol{\theta_1} = 60°$

Figure 5 and figure 6 start to show different behaviours compared to figure 3 and 4. As the angle of mass 1 increases, the movement of mass 2 starts to get a huge variation in its trajectory pattern. For example, when $\theta_1 = 60°$, the red dots are not in a parabola anymore, there are several high amplitudes and spikes which do not occur periodically. However, the movement of mass 1 is still parabolic, only the amplitude of its motion is different as the angle that the mass is launched is different respectively.

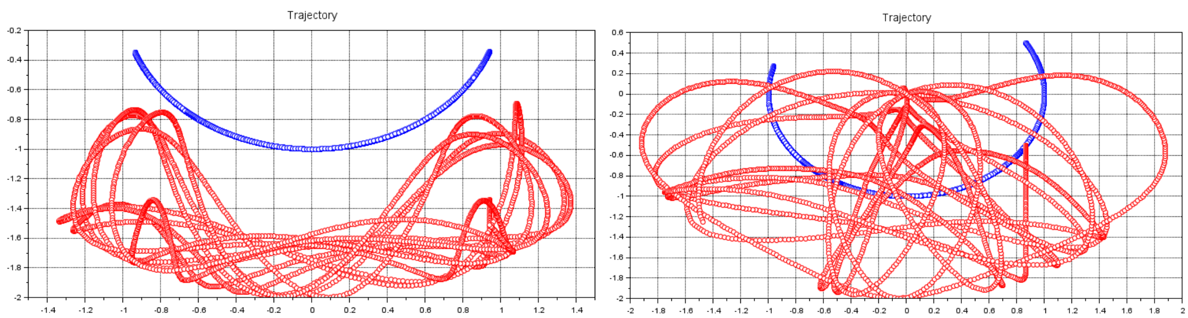

Figure 7 and 8. The left figure is when $\boldsymbol{\theta_1} = 70°$ and the right figure is when $\boldsymbol{\theta_1} = 120°$

When $\theta_1 = 70°$, it can be seen that the movement of mass 2 is not symmetrical anymore, since there is a spike for the right side which the left side does not have. The movement of mass 1 is still in a parabolic shape, and given that figure 3 to 8 showed the same pattern it can be concluded that even though θ_1 changes, mass 1 will have a movement in a parabola. However, mass 2 moves in a completely different way compared to previous angles, and when θ_1 becomes huge like 120°, the system shows a chaotic movement that likely not have a pattern of the movement. Going back to the original question, the reason why the double pendulum exhibits chaotic movement can be answered through this investigation, at certain angles of θ_1 the mass 2 will move in a chaotic movement as shown in figure 7.

5. Conclusion

This investigation examined the derivation of the equation of motion of double pendulum by using calculus and Lagrange's system and equation. A range of variables are covered in order to construct the motion equation such as θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$. Because the pendulum does not move at a constant speed, the derivative of x and y were found to obtain generalized velocities, which were eventually used to find the potential and kinetic energy of the system. The difference between kinetic energy and potential energy provides the equation respect to θ_1 and θ_2, and using Lagrange equation we can find the relative change of two variables. Hence, by using the substitution method we can find the solution for θ_1 and θ_2, which can be substituted into the first equation derived: m_1 ($l_1 \sin\theta_1$, $-l_2 \cos\theta_1$) and m_2 ($l_1 \sin\theta_1 + l_2 \sin\theta_2$, $l_1 \cos\theta_1 + l_2 \cos\theta_2$).

The key equations in this investigation are (9), (10), (11) and (12) because it helps to calculate the value of angles which eventually gives the coordinate of mass 1 and mass 2 over time. Even though a high level of calculus and some further research on Lagrange equation were required during this investigation, it was valuable experience and it was intriguing to prove the chaotic movement of double pendulum system mathematically, using the equation I have derived.

6. Possible investigations

This investigation only focuses on the trajectory of mass 1 and mass 2 over time, as can be seen in figure 3 to 8. However, there are much more factors to investigate such as finding how the angle affects the velocity of masses and angular displacement of masses by drawing relevant graphs. In addition, there are other ways of solving second order simultaneous equations for θ_1 and θ_2, such as using numerical methods such as Runge-Kutta method or Euler method.

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