Kings in Generalized Tournaments

C. Zhang.

Department of Mathematics
Wheaton College Student

Thesis Defense

Background

Definition (Maurer 1980)

An **tournament** consists of a set of **vertices** V, and a set of **edges** E which consists of ordered pairs of vertices, such that

- for all vertex a, $(a, a) \notin E$.
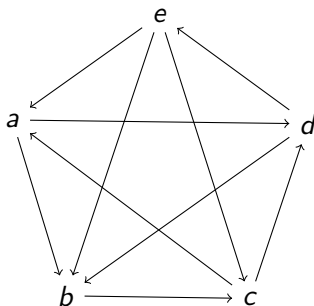- for all vertices a, b, $(a, b) \in E$ or $(b, a) \in E$, but not both.

Figure: example of a tournament.

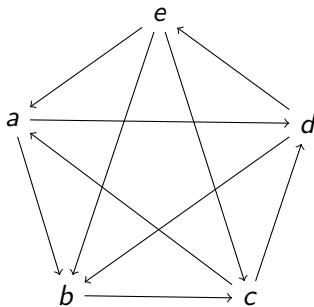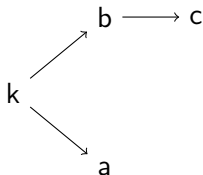If (a, b) in edge set E, we write it as $a \to b$ or a beats b.

Figure: example of a tournament.

Definition (Maurer 1980)

A **king** (or **2-king**) in a oriented graph is a vertex that can beat every vertex by at most 2 steps.
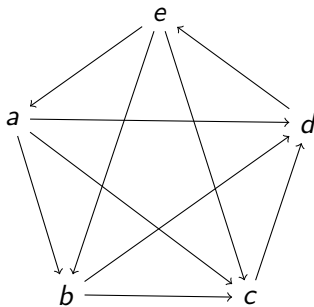
The kings in this graph are a, d, e.

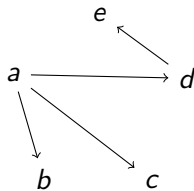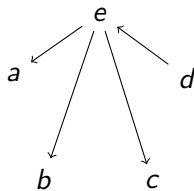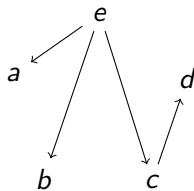Figure: example of a tournament.

The kings in this graph are a, d, e:

(a) a is a king.

(b) d is a king.

(c) e is a king.

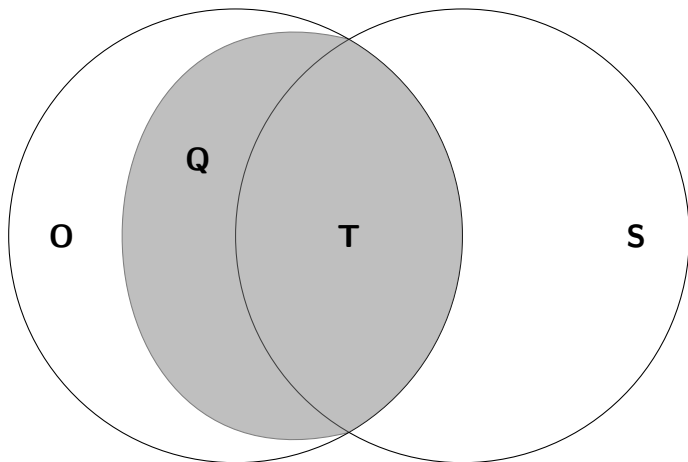# Generalization of Tournaments

relation between three generalizations

Figure: the relationship between tournaments, semi-complete digraphs, oriented graphs, and quasi-transitive oriented graphs.

Definition

An **oriented graph** consists of a set of **vertices** V, and a set of **edges** E which consists of ordered pairs of vertices, such that

- for all vertex a, $(a, a) \notin E$.
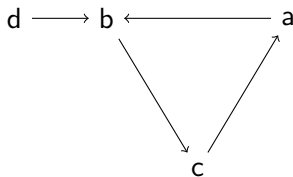- for all vertices $\{a, b\}$, if a beats b, then b does not beat a.

Figure: example of an oriented graph.

Quasi-transitive Oriented Graph

oriented graph

Definition
A vertex a is **adjacent** to vertex b if $a \rightarrow b$ or $b \rightarrow a$.

Definition
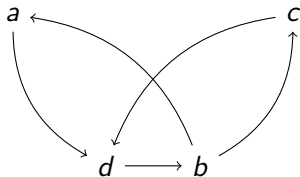There exists a **tie** between a and c, if a is not adjacent to c.

Figure: another example of an oriented graph.

Quasi-transitive Oriented Graph
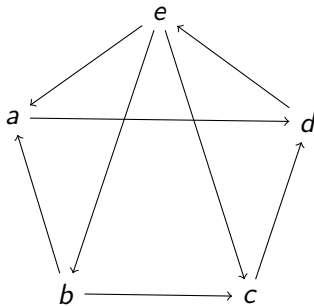
oriented graph

King in this graph is d, e:

Figure: an example of kings.

Quasi-transitive Oriented Graph

definition

Definition (Bang-Jensen and Huang 1995)

Quasi-transitive oriented graph is an oriented graph such that if $a \to b$ and $b \to c$ then a and c are adjacent.
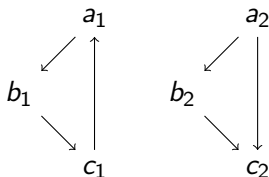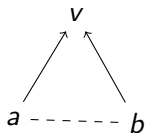

Figure: examples of quasi-transitive oriented graphs.
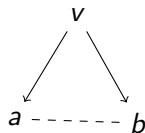
Quasi-transitive Oriented Graph

ties

Lemma
In Quasi-transitive oriented graphs, If vertex a ties with vertex b, then for any other vertex v that does not tie with a or b,
$v \to \{a, b\}$ *or* $\{a, b\} \to v$
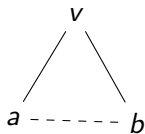

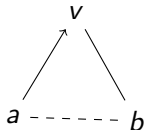(a) $v \to \{a, b\}$. (b) $\{a, b\} \to v$.

Figure: an example where a ties with b.
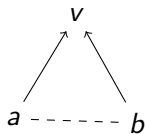
Quasi-transitive Oriented Graph

tie transmits arrows

(a) both *a* and *b* are adjacent to *v*.

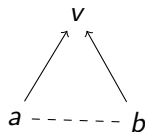(b) let $a \to v$.

(c) then *b* has to beat *v*.

Figure: arrow direction transmitted from *a* to *b*.

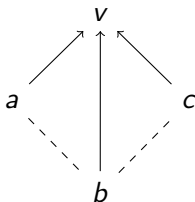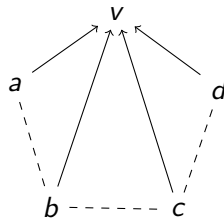# Quasi-transitive Oriented Graph

tie transmits arrows

(a) starts with vertex a.

(b) add a vertex b that ties a.

(c) add another vertex c ties b.

(d) add another vertex d ties c.

Figure: tie is transmitting the direction of arrow.

Definition

In a digraph, a **tie path** from vertex a_0 to vertex a_n, or a tie path between vertices a_0 and a_n, is a sequence of vertices $[a_0, a_1, a_2, \ldots, a_{n-1}, a_n]$, such that for all $0 \leq k < n$, a_k ties a_{k+1}. Note that all the a_i's do not have to be distinct.

$$a \text{ --- } b \text{ --- } c \text{ --- } d$$

Figure: a tie path from a to b.

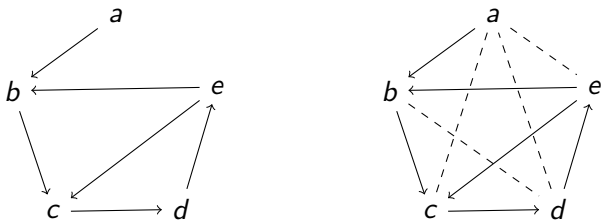# Quasi-transitive Oriented Graph

tie paths

Figure: tie path example.

Quasi-transitive Oriented Graph

Lemma
In Quasi-transitive oriented graphs, If vertex a ties with vertex b, then for any other vertex v that does not tie with a or b, $v \rightarrow \{a, b\}$ or $\{a, b\} \rightarrow v$

Lemma
For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.

Lemma

For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.
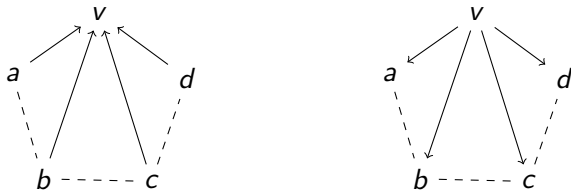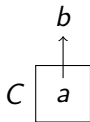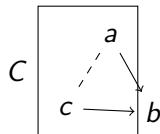

Figure: a tie path from a to d.
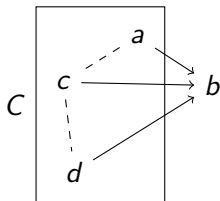
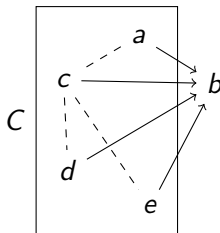# Quasi-transitive Oriented Graph

tie components

(a) a beats b.

(b) direction transmitted from a to c.

(c) transmitted from a to c to d. (d) transmitted from a to c to e.

Figure: the arrow direction to b was transmitted in tie component C.

Definition
In a digraph G, a **tie component of vertex** a: $C(a)$, all vertices v such that there exists a tie path between a and v.

Definition
In a digraph G, a tie component $C(a)$ is a **trivial tie component** if $C(a)$ only contains a itself and no other vertex.
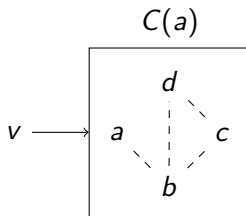
Corollary
In a digraph, if vertex v is not in tie component $C(a)$, then v is adjacent to $C(a)$.
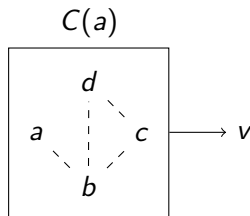
Quasi-transitive Oriented Graph

tie components

Lemma
In a quasi-transitive oriented graph, for any tie component $C(a)$ and any vertex v such that $v \notin C(a)$, then either $C(a) \to v$ or $v \to C(a)$.

(a) $v \to C(a)$.   (b) $C(a) \to v$.
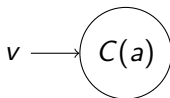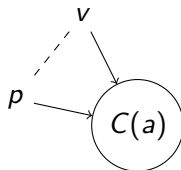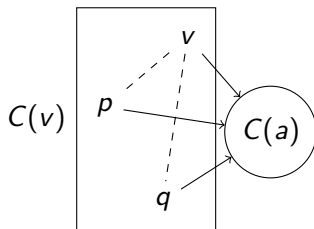
Quasi-transitive Oriented Graph

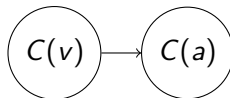Consider tie component $C(v)$. Without loss of generosity, assume $v \to C(a)$

(a) direction is $v \to C(a)$.

(b) transmitted from v to p.

(c) transmitted from v to q.
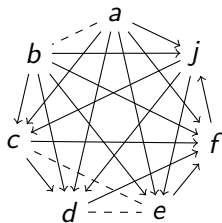
(d) $C(v) \to C(a)$.

Quasi-transitive Oriented Graph

condensation to tournaments

Theorem

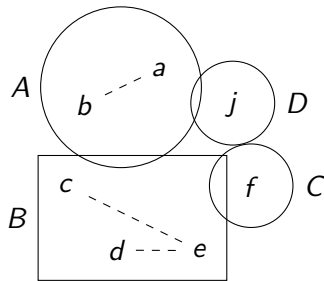*For any two distinct tie components $C(v)$ and $C(a)$ in a quasi-transitive oriented graph, $C(v) \to C(a)$ or $C(a) \to C(v)$.*
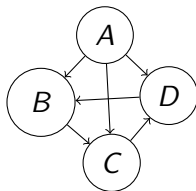
Quasi-transitive Oriented Graph
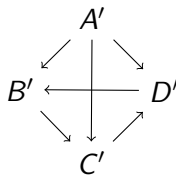
condensation to tournaments

(a) a quasi-transitive oriented graph.

(b) find its tie components.
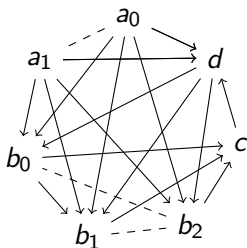
(c) components beat each other.

(d) components are just like vertices.

Quasi-transitive Oriented Graph

condensation to tournaments

Theorem

*Tie component condensation of given quasi-transitive oriented graph Q will always result in a unique tournament T, where T is called the **underlying tournament of** Q.*

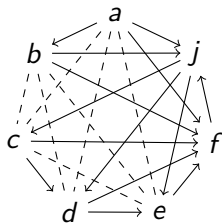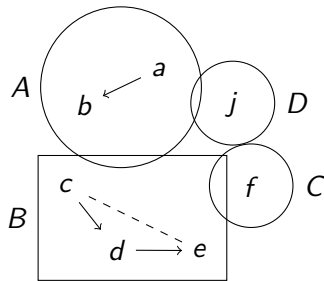

(a) quasi-transitive oriented graph Q.
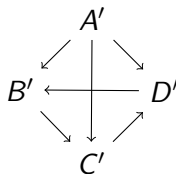
(b) underlying tournament of Q.

Quasi-transitive Oriented Graph

graph condensation

(a) an oriented graph G.

(b) components of G.
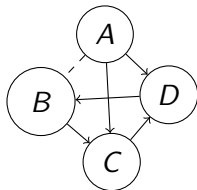
(c) relationship between components.
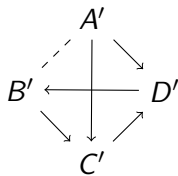
(d) condensed graph H.

Quasi-transitive Oriented Graph

graph condensation

Theorem

Given a quasi-transitive graph Q and its tie component condensation f, consider the set of all the condensations $f_k : G_k \to T_k$, where G_k has the same tie structure as Q and T_k is a tournament: $F = \{f_0, f_1, \ldots, f_{n-1}, f_n\}$. f is an efficient condensation in F.

Quasi-transitive Oriented Graph

property of kings

Theorem

If vertex k is a king in quasi-transitive oriented graph G, then k is adjacent to every other vertex in G.
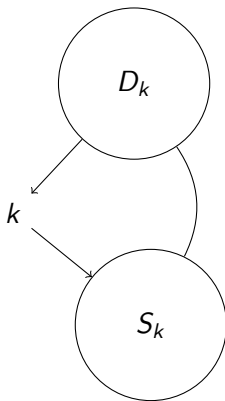

Figure: the rich structure of a king in a quasi-transitive oriented graph.

Quasi-transitive Oriented Graph
property of kings

Theorem
A vertex k is a king in a quasi-transitive oriented graph if and only if

- *k is in a trivial tie component.*
- *the result of k after tie component condensation is a king in the underlying tournament.*

Thank You

- Personal Email: zcsx.ok@outlook.com

- Institution Email: zhang_cheng@wheatoncollege.edu

References

📄 Bang-Jensen, Jørgen and Jing Huang (Oct. 1, 1995). "Quasi-transitive digraphs". In: *Journal of Graph Theory* 20.2, pp. 141–161. ISSN: 1097-0118. DOI: 10.1002/jgt.3190200205. URL: http://onlinelibrary.wiley.com/doi/10.1002/jgt.3190200205/abstract (visited on 02/08/2018).

📄 Maurer, Stephen B. (Mar. 1, 1980). "The King Chicken Theorems". In: *Mathematics Magazine* 53.2, p. 67. ISSN: 0025570X. DOI: 10.2307/2689952. URL: http://www.jstor.org/stable/10.2307/2689952?origin=crossref (visited on 01/27/2018).