Kings in Generalized Tournaments

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Background

4.1 Directed Graph

Definition 4.1. *a* ***directed graph*** *,* ***digraph*** *or* ***graph*** *consists of a vertex set $V(G)$, and an edge set (ordered pair of vertices) $E(G)$.*

For example, in Fig. 4.1, if we call this digraph G, then the vertex set or $V(G)$ is $\{a, b, c, d, e\}$, and the edge set or $E(G)$ is $\{(a, a), (a, b), (b, c), (c, e), (e, a), (e, c)\}$. We sometimes refer to "edges" as "arrows".

4.2 Beating Relations

To simplify the notation, we think of an edge as a *beating relation*:

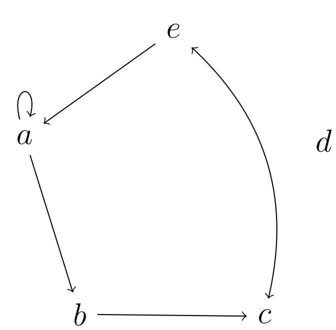


Figure 4.1: example of a directed graph.

Definition 4.2. *In a directed graph G, if $(a,b) \in E(G)$ then we say a* ***beats*** *(* ***dominates*** *) b or $a \to b$.*

Definition 4.3. *A* ***path*** *or a* ***walk*** *from vertex a_0 to vertex a_n is a sequence of vertices $[a_0, a_1, \ldots, a_{n-1}, a_n]$ such that $\forall 0 \leq k < n$, $a_k \to a_{k+1}$. We will sometimes write this path as $a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$.*

Definition 4.4. *For a given path P, if the sequence has $n+1$ vertices, then we say the* ***length of path*** *P is n.*

The length of a path measures the number of edges on this path. For example, in Fig. 4.1, there is a path $P = e \to a \to b \to c$. And there are edges $(e,a), (a,b), (b,c)$ on this path, therefore the length of this path is 3. There is another path $P' = e \to c$ goes from e to c. This path P' only has length 1. Therefore, path P' is shorter than path P, and if we go through every possible path from e to c, we can find out that P' is the *shortest path* from e to c.

Definition 4.5. *In a directed graph G,* ***a beats (dominates) b by n steps*** *, if the shortest path from a to b has length n.*

In Fig. 4.1, $e \to a \to b$, and e does not beat b by 1 step, therefore e beats b by 2 steps.

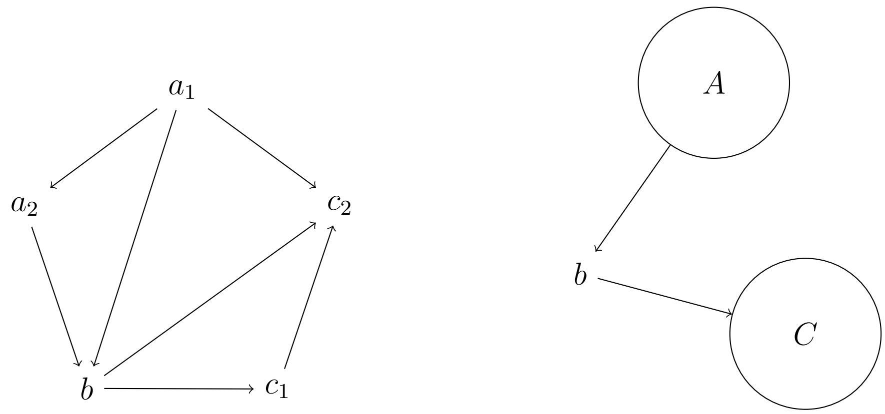
However, although $e \to a \to b \to c$, e *does not* beat c by 3 steps, because the shortest path from e to c is $e \to c$. Therefore, e beats c by 1 step.

Definition 4.6. *In a directed graph, vertex a is* ***adjacent*** *to vertex b if $a \to b$ or $b \to a$ or both.*

Definition 4.7. *In a directed graph, for two distinct vertices a, b, if a is not adjacent to b, then a* ***ties*** *b.*

In Fig. 4.1, c is adjacent to e; a is adjacent to b; b is adjacent to c. Whereas, c ties a, because there is no edge (c,a) and no edge (a,c) in this digraph.

Since we don't always know the exact structure of the graph, it is sometimes useful to look at the beating relationships between sets of vertices. We then define beating, adjacency, and tie between vertex sets.



(a) (b)

Figure 4.2: we can draw graph (a) as graph (b).

Definition 4.8. *In digraph G, we write $A \rightarrow B$ where A and B are disjoint subsets of $V(G)$, when every vertex in A beats every vertex in B. We also write $A \rightarrow b$ as a shorthand for $A \rightarrow \{b\}$, and $a \rightarrow B$ as a shorthand for $\{a\} \rightarrow B$.*

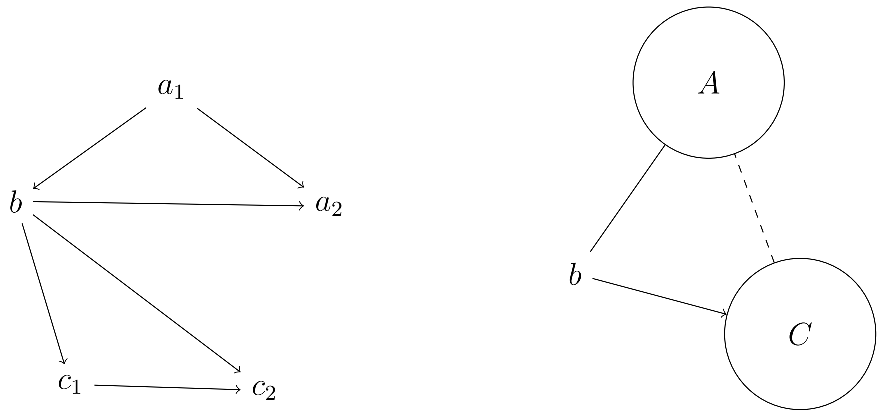
Definition 4.9. *In digraph G, A is adjacent to B where A and B are disjoint subsets of $V(G)$, when every vertex in A is adjacent to every vertex in B. We also write "A is adjacent to b" as a shorthand for "A is adjacent to $\{b\}$", and "a is adjacent to B" as a shorthand for "$\{a\}$ is adjacent to B".*

Definition 4.10. *In digraph G, A ties B where A and B are disjoint subsets of $V(G)$, when every vertex in A ties every vertex in B. We also write "A ties b" as a shorthand for "A ties $\{b\}$" and "a ties B" as a shorthand for "$\{a\}$ ties B"*

In Fig. 4.2, we show how we draw the beating relationship with subsets $A = \{a_1, a_2\}$ and $C = \{c_1, c_2\}$. Notice, in Fig. 4.2b, set A and set C are not adjacent. This *does not* means that set A ties set C; it only means that we have not drawn out the relationship between these 2 sets in Fig. 4.2b.

We draw adjacency between sets (for example, set A is adjacent to set $\{b\}$) using a solid edge without arrow. We draw tie between sets (for example, set A ties set C) using a dashed edge without arrow. (see Fig. 4.3)

Definition 4.11. *The **submissive set** (**dominant set**) of a vertices v in graph G is the set of vertices in G that are beaten by v (beat v), formally it is $\{e \in V(G) \mid v \rightarrow e\}$*



(a) (b)

Figure 4.3: we can draw graph (a) as graph (b).

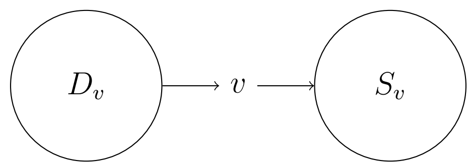


Figure 4.4: D_v is the dominant set of v; S_v is the submissive set of v

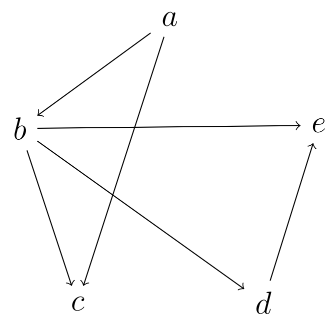
(or $\{e \in V(G) \mid e \to v\}|$*). We will denote the submissive set of* v *as* S_v*, and the dominant set of* v *as* D_v*. (See Fig. 4.4)*

Definition 4.12. ***Out-degree*** *(* ***in-degree*** *) of a vertex* v *in graph* G *is the size of the submissive set (dominant set) of* v*.*

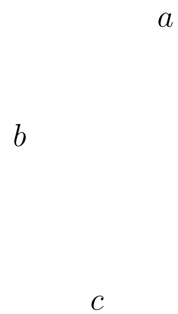
For example, in Fig. 4.1, $S_a = \{a, b\}$, therefore vertex a has out-degree 2; $D_a = \{e, a\}$, therefore vertex a has in-degree 2. Vertex d has in-degree 0, and out-degree 0 because d is not adjacent to any other vertex in the graph. Therefore, D_d and S_d are both empty sets.

Notice it is very natural to relate out-degree and in-degree with the "power" or "strength" of a vertex. However, we will discuss in later sections and chapters that a vertex with larger out-degree does not necessarily have more power, by our definition of "king" (defined in Section 4.4), However, the property that "vertices with largest out-degree are kings" is true in some nice families of graphs (see Section 4.4 and Section 5.2).

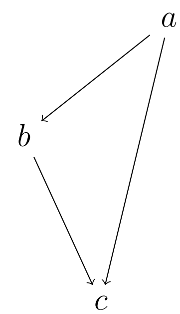
Definition 4.13. *A* ***induced subgraph*** *or* ***vertex-induced subgraph*** H *of digraph*



(a) start with graph G.



(b) let $V(H) = \{a, b, c\}$.



(c) induced subgraph H.

Figure 4.5: the process to create an induced subgraph.

G is a subgraph of G such that: for all pairs of vertices $a, b \in V(H)$, if $(a, b) \in E(G)$, then $(a, b) \in E(H)$.

In Fig. 4.5, we show the process of creating an induced subgraph. We redraw the vertices in $V(H)$, that is a, b, c, and then simply copy the edges between those vertices, that is, $(a, b), (b, c), (a, c)$ from G to get H.

This definition can be viewed in 2 ways:

- Take the vertices a, b, c together with the edges between them to form subgraph H.
- Take away vertices e, d together with all the edges having e or d as an end point, the rest of the graph is the subgraph H.

For all the families of graphs discussed in this paper (oriented graph, semi-complete digraph, tournament,quasi-transitive oriented graph), induced subgraphs preserve the property of the original graphs.

4.3 Oriented Graph and Tournament

Definition 4.14. *An* ***oriented graph*** *is a digraph, such that:*

1. *for all vertices a, a does not beat itself.*
2. *for all pairs of adjacent vertices a, b, if a beats b, then b does not beat a.*

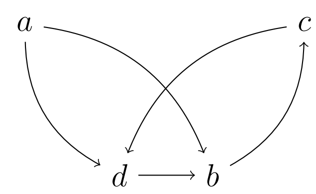


Figure 4.6: example of an oriented graph.

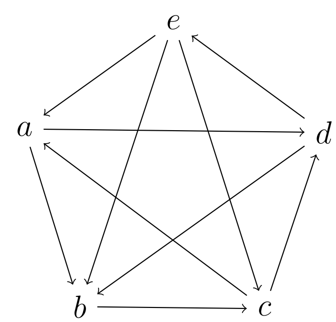


Figure 4.7: example of a tournament.

We show an oriented graph in Fig. 4.6. This definition of oriented graph can be viewed as a mathematical model of a round-robin tournament with ties allowed. A team cannot compete with itself, therefore property 1 holds; each pair of teams a and b competes exactly once, either a beats b, b beats a, or there is a tie between a and b. therefore property 2 holds.

Therefore, it is very natural to view an edge going from a to b as a "beats" b, and two non-adjacent vertices a, b as a tie between a and b.

Definition 4.15. *A* ***tournament*** *(sometimes called* ***semi-complete oriented graph*** *) is an oriented graph without ties.*

We give a simple example of a tournament in Fig. 4.7. A tournament can be interpreted as, what we call in the real world, a round-robin tournament, where every pair of vertices compete and the competition results in exactly one winner and one loser.

In [2], the author uses tournament to model a flock of chicken, between every two chickens there is a "pecking relation" (we call it the "beating relation"). Given two chickens, one chicken has to peck the other chicken, or be pecked by the other chicken, but not both.

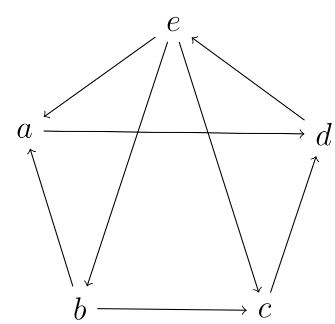


Figure 4.8: an example of kings.

4.4 Kings

Definition 4.16. *a* ***king*** *in a digraph is a vertex that beats every other vertex by 1 or 2 steps.*

In Fig. 4.8, the kings in this graph are vertices e and d. Vertex e beats a, b, c by one step, and beats d by 2 steps, so e is a king. Vertex d beats e by one step, and e beats a, b, c, so d beats a, b, c by 2 steps, which makes d a king.

The other vertices (a, b, c) are not kings, because a cannot beat c by one or two steps ($a \to d \to e \to c$ is a shortest path); b cannot beat e by one or two steps ($b \to c \to d \to e$ is a shortest path); c cannot beat b by one or two steps ($c \to d \to e \to b$ is a shortest path).

There are two points to note in this example:

- Fig. 4.8 shows that larger out-degree does not associate with more "power" or "strength". Vertex d only has out-degree 1, and d is a king. Whereas vertex b has out-degree 2, but b is not a king. Vertex d is more "powerful" than b, even though d has smaller out-degree. (trying to understand this phenomenon will be an interesting practice)
  - A vertex with the *highest* out-degree in a digraph may not be a king. In Fig. 4.9, vertex a has out-degree 3, which is the highest out-degree in this graph. However, it is not a king, since it cannot beat k by 1 or 2 steps
  - Also, a vertex with *lowest* out-degree in a digraph may be a king. In Fig. 4.9,

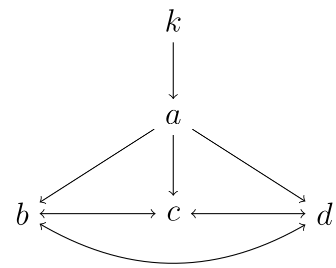


Figure 4.9: the vertex with smallest out-degree is the only king.

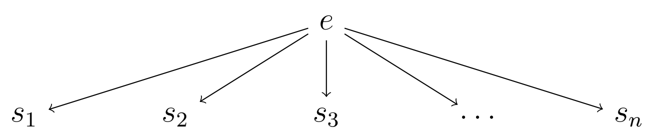


Figure 4.10: the vertex e is the emperor of this graph.

the vertex k has the smallest out-degree in the graph, however it is the only king in this graph.

- Fig. 4.8 shows that a digraph may have more than one king. In fact, [2] proved the probability that every vertex in a random tournament is a king approaches 1 as the number of vertices in the graph approaches ∞.

Definition 4.17. *An* ***emperor*** *in a digraph is a vertex that beats every other vertex. (see Fig. 4.10)*

Definition 4.18. *an* ***(n, k) digraph*** *(* ***(n, k) oriented graph*** *,* ***(n, k) tournament*** *, etc.) is a digraph (oriented graph, tournament, etc.) with n vertices and k kings.*

The following work has been done on kings in tournaments. These theorems will be useful when we investigate properties of kings in generalized tournaments in later chapters.

Theorem 4.1. *A tournament has only one king if and only if that king is an emperor. [2]*

Theorem 4.2. *A tournament cannot have exactly 2 kings. [2]*

Theorem 4.3. *In a tournament, a vertex with largest out-degree is a king. [2]*

Corollary 4.4. *In a tournament, there always exists at least 1 king. [2]*

Theorem 4.5. *for all integers* $n \geq k \geq 1$, (n, k)*-tournaments exist with the following exceptions:* $(n, 2)$ *with any* $n \geq 2$ *and* $(4, 4)$. *[2]*

Chapter 5

Semi-complete Digraph

5.1 Definitions

In Chapter 4, we introduced one way to model "tie" as the non-adjacency between 2 vertices. However, if we think of "beating" as a weak order (like "subset" relation), then we can define "double tie" to capture this idea:

Definition 5.1. *There exists a* ***double tie*** *between vertices a and b if $a \to b$ and $b \to a$.*

With the definition of double tie, we can then formalize another model of round-robin tournament with double ties. This kind of graph is called "semi-complete digraph".

Definition 5.2. *A* ***semi-complete digraph*** *is a digraph where every vertex is adjacent to every other vertex, but does not beat itself. see Fig. 5.1*

This definition says, that a semi-complete digraph is a digraph such that between each pair of vertices, there exists at least one edge between them.

Definition 5.3. *In a semi-complete digraph, if $a \to b$ and b does not beat a, then vertex a* ***strictly beats*** *vertex b.*

Tournaments are a special case of semi-complete digraphs. In tournaments, for each pair of vertices a, b a always strictly beats b or b strictly beaten by b. We can view a

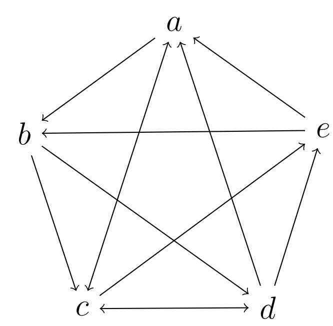


Figure 5.1: an example of semi-complete digraph.

tournament as a semi-complete digraph that only has strict beatings; no double tie allowed. In other words, tournaments is an oriented graph that is also a semi-complete digraph, hence the name "semi-complete oriented graph" (see Definition 4.15)

Definition 5.4. *In a semi-complete digraph, the* ***double tie set of vertex v*** *is the set of vertices that double ties v. We denote the double tie set of v as DT_v.*

For example, in Fig. 5.1, there is a double tie between a and c, and another double tie between c and d. Every other beating relation between any other pair of vertices are "strict beating" relations. For example, a strictly beats b, d strictly beats e, and b strictly beats d.

For vertex c, T_c (double tie set of c) is $\{a, d\}$. The set of vertices that strictly beat c, which can be expressed as $D_c - T_c$, is $\{b\}$. The set of vertices that are strictly beaten by c, which can be expressed as $S_c - T_c$, is $\{e\}$.

5.2 Properties

Although tournaments are a special case of semi-complete digraphs, many useful properties of tournaments are also true for semi-complete digraphs.

Lemma 5.1. *for every vertex v in any semi-complete digraph G, $\{S_v - T_v, T_v, D_v - T_v, \{v\}\}$ forms a partition of G.*

Proof. $D_v - T_v$ is the set of vertices that strictly beat v, $S_v - T_v$ is the set of vertices that are strictly beaten by v, T_v is all the vertices that double tie with v. They are

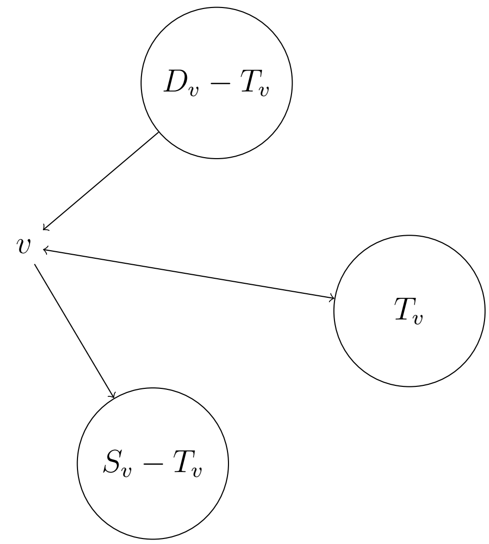


Figure 5.2: illustration of Lemma 5.1.

clearly disjoint by definition, and all of them are disjoint to $\{v\}$ by definition.

Every vertex in $V(G)$ needs to be in one of the sets $S_v - T_v, T_v, D_v - T_v, \{v\}$, because every other vertex needs to be adjacent to v. □

See Fig. 5.2 for a visualization for the proof of Lemma 5.1.

Lemma 5.1 and Fig. 5.2 are useful in proofs of many properties of semi-complete digraphs.

When we move on to general oriented graphs, the lack of this property in oriented graphs will make the structure of oriented graphs harder to work with.

Theorem 5.2. *All vertices with the maximum out-degree in a semi-complete digraph are kings.*

Proof. Let v be a vertex with the maximum out-degree in a semi-complete digraph G.

Suppose v is not a king in G. Since v is not a king and v beats every vertex in S_v by exactly 1 step, then by Lemma 5.1, there exists a vertex $d \in D_v - T_v = V(G) - S_v - \{v\}$ that is not beaten by v by 1 or 2 steps. Therefore there cannot be any vertex in S_v that beats d. Because d is adjacent to every vertex in S_v d strictly dominates v and S_v. Hence $|S_d| \geq 1 + |S_v|$. Therefore $|S_d| > |S_v|$, that is, d has larger out-degree than v. Contradiction. □

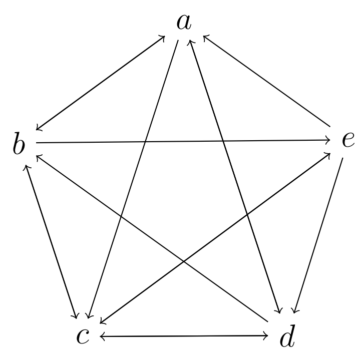


Figure 5.3: every vertex has the same out-degree.

Fig. 5.3 shows that there can be more than one vertex with maximum out-degree. In fact, in this graph, every vertex has the same out-degree 3, therefore all of the vertices have the maximum out-degree. By Theorem 5.2, all vertices in this graph are kings.

Corollary 5.3. *For any non-empty semi-complete digraph G, there exists at least one king.*

Proof. This corollary is a result of Theorem 5.2. because out-degrees are non-negative integers, and $V(G)$ is not empty. Thus there exists at least one vertex with maximum out degree. □

Theorem 5.4. *In a semi-complete digraph G, every vertex with a non-empty dominant set is beaten by a king.*

Proof. Let v be a vertex in G, such that D_v is not empty. Consider the subgraph induced by the vertices in D_v; this subgraph is also a semi-complete digraph. By Corollary 5.3, there is a king k in the induced subgraph of D_v. We will show that k is also a king of G.

- k dominates D_v by 1 or 2 steps, because k is a king in D_v.
- k dominates v by exactly 1 step, because k is in D_v.
- k dominates T_v and S_v by 1 or 2 steps, because k dominates v, which beats all vertices in T_v and S_v.

Then, by Lemma 5.1, we know that k dominates every vertex in the graph by 1 or 2 steps. Therefore k is a king in G and k beats v. □

$$\mathrm{a} \longleftrightarrow \mathrm{b}$$

Figure 5.4: both vertices a and b are emperors and kings.

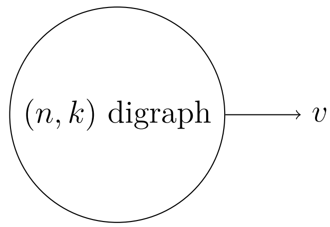


Figure 5.5: an $(n+1, k)$ semi-complete digraph by adding a new vertex v

Corollary 5.5. *If a semi-complete digraph has only one king, then that king is an emperor.*

Proof. Suppose G is a semi-complete digraph with only one king k, and k is not an emperor.

Then there exists v in the graph, such that k does not beats v. Therefore, $v \to k$, since there is no tie in the graph. Therefore, k has an non-empty dominate set. Then by Theorem 5.4, k is beaten by another king. Therefore there exists more than one king. Contradiction. □

Notice, unlike in tournaments, the converse of Corollary 5.5 is not true In Fig. 5.4, we show we can have more than one kings that are emperors.

Theorem 5.6. (n, k) *semi-complete digraphs exist for all* $n \geq k \geq 1$*, where* n, k *are integers.*

Proof. We prove this theorem by induction. Construct an (n, k) semi-complete digraph:

- When there is only one vertex in the graph then the graph is a $(1, 1)$ semi-complete digraph.
- See Fig. 5.5. We can add one vertex that is not a king by adding a vertex that is strictly beaten by all the vertices in the original graph. In other words, we can construct a $(n+1, k)$ semi-complete digraph from any (n, k) semi-complete digraph.

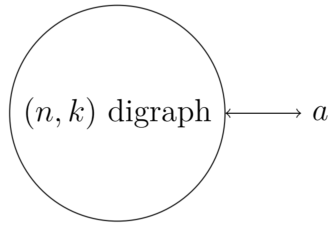
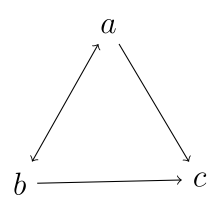


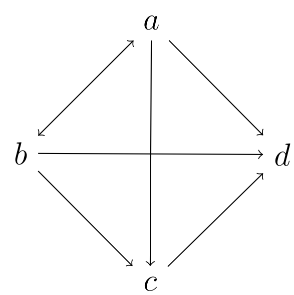
Figure 5.6: an $(n+1, k+1)$ semi-complete digraph by adding a new king a.

(a) start with $(1, 1)$ digraph.

(b) add king b.



(c) add non-king c.



(d) add non-king d.

Figure 5.7: to construct a $(4, 2)$ semi-complete digraph.

- See Fig. 5.6. We can add one king by adding one vertex that double ties all the vertices in the original graph. In other words, we can construct an $(n+1, k+1)$ semi-complete digraph from any (n, k) semi-complete digraph.

Therefore, we can obtain any (n, k) flock by: start with a $(1, 1)$ semi-complete digraph first add $k-1$ kings to get a (k, k) semi-complete digraph; then add $n-k$ non-king vertices to get an (n, k) semi-complete digraph. $\square$

In Fig. 5.7, we give an example of how to construct a $(4, 2)$ semi-complete digraph using the inductive algorithm we introduced in Theorem 5.6:

1. Start with a single vertex, which is a $(1, 1)$ semi-complete digraph.
2. Add a king b, by letting it double tie with a which gives us a $(2, 2)$ semi-complete digraph.

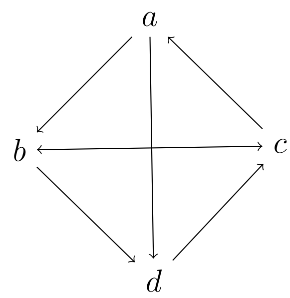


Figure 5.8: a $(4, 4)$ semi-complete digraph with only one double tie

3. Add a non-king c, by letting every vertex in the $(2, 2)$ semi-complete digraph beat c. which gives us a $(3, 2)$ semi-complete digraph.
4. Add a non-king d, by letting every vertex in the $(3, 2)$ semi-complete digraph beat d, which gives us a $(4, 2)$ semi-complete digraph.

Using the method in Theorem 5.6, when we add the i_{th} king, we are creating $(i-1)$ ties, so adding the 2_{nd} king to the k kings would require a total of $1 + 2 + \cdots + (k-1)$ double ties. The summation can be simplified to $\frac{k(k-1)}{2}$. Thus, the number of double ties blows up with the increase of k (number of kings).

Therefore we want to investigate how to use double ties more "efficiently": what is the minimum number of double ties needed to get an (n, k) semi-complete digraph.

Lemma 5.7. *All $(n, 2)$ semi-complete digraphs with $n \geq 2$ can be constructed with only one double tie.*

Proof. use the method described in Theorem 5.6.

First add 1 king to a $(1, 1)$ semi-complete digraph to get a $(2, 2)$ semi-complete digraph (one double tie added). Then add $(n-2)$ non-king vertices to get an $(n, 2)$ semi-complete digraph (no double tie added).

Therefore we add only one double tie to construct any $(n, 2)$ semi-complete digraph.

□

Lemma 5.8. *there exists a $(4, 4)$ semi-complete digraph with only one double tie.*

Proof. Fig. 5.8 gives an example of a $(4, 4)$ semi-complete digraph with only one double tie. The only double tie is between b and c. □

Theorem 5.9. *for all $n \geq k \geq 1$ exists an (n, k) semi-complete digraph with at most one double tie.*

Proof. Recall Theorem 4.5, there exists an (n, k) tournament with the exception of $(n, 2)$ and $(4, 4)$ Because tournaments are also semi-complete digraphs, therefore, we can use Theorem 4.5 along with Lemma 5.7 and Lemma 5.8, to get the result.

The proofs of Theorem 4.5, Lemma 5.7, and Lemma 5.8 provide constructions for the these semi-complete digraphs. Therefore we can construct an (n, k) semi-complete digraph with at most one double tie, for all $n \geq k \geq 1$. $\square$

Chapter 6

General Oriented Graph

The properties of semi-complete digraphs basically inherit the properties of tournaments, and most of the proofs are almost the same.

We then move on to another family of graphs that are less similar to tournaments: oriented graphs. We will investigate general properties of oriented graph in this chapter. In later Chapter 7, we will focus on specific families of oriented graph

Lemma 6.1. *There does not exist a* $(4, 4)$ *general oriented graph.*

Lemma 6.2. *There does not exist a* $(2, 2)$ *oriented graph and a* $(3, 2)$ *oriented graph*

The previous lemmas can be proved by listing all the oriented graph with 2, 3, and 4 vertices. We will not show the proofs in this thesis, because the proofs are too long and uninteresting.

Lemma 6.3. *There exists* $(n, 2)$ *oriented graph for* $n \geq 4$

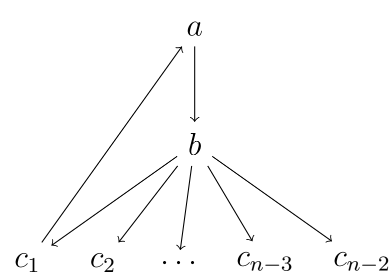


Figure 6.1: only a and b are kings for $n \geq 4$.

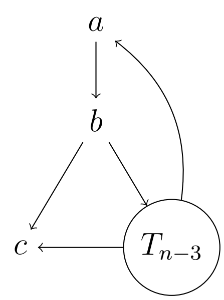


Figure 6.2: the constructive proof for Lemma 6.5

Proof. We can see in Fig. 6.1 that c_1 cannot dominate c_2 in 2 steps. and other c_i cannot dominate any other vertex, because there is no edge going out of them.

a is a king because $a \to b \to c_k$, where k is a integer and $k \le n-2$, therefore a beats b by one step, and beats all the c_k by 2 steps. b beats every c_k by 2 steps, where k is a positive integer and $k \le n-2$, and b beats a by 2 steps: $b \to c_1 \to a$. □

Theorem 6.4. *There exists an (n, k) oriented graph for all the $n \ge k \ge 1$, with the exception of $(2, 2)$, $(2, 3)$ and $(4, 4)$ oriented graph.*

Proof. Theorem 4.5 shows that there exists (n, k) tournament for all $n \ge k \ge 1$ with the exception of $(n, 2)$, and $(4, 4)$.

Because tournaments are also oriented graphs and by Lemma 6.3, Lemma 6.2, Lemma 6.1, we can conclude that the theorem is correct. □

We generalize the result from [2] on tournaments to oriented graphs and show that there are only finite number of (n, k) oriented graphs that cannot be constructed.

Following the idea from Chapter 5, one of the question to ask is that how can we use ties more "efficiently". The construction method in the proof of Lemma 6.3 is pretty inefficient.

Here we present a better way to construct the graph.

Lemma 6.5. *There exists $(n, 2)$ oriented graph with only one tie, for $n \ge 4$*

Proof. See Fig. 6.2, the T_{n-3} is a tournament of $n-3$ vertices. In this graph, the only tie is between a and c and the only kings are a and b.

a is a king because, $a \to b \to c$ and $a \to b \to T_{n-3}$, therefore a beats b in 1 step and a beats c and T_{n-3} in 2 steps. b is a king because, $b \to c$ and $b \to T_{n-3} \to a$ (because T_{n-3} is not empty), therefore b beats c and T_{n-3}, and b beats a in 2 steps.

c is not a king, because it has out-degree 0. Any vertex v in T_{n-3} cannot be a king, because the closest path between v and c is $v \to a \to b \to c$ which has length 3. □

With Lemma 6.5, we can prove the following theorem.

Theorem 6.6. *There exists an* (n, k) *oriented graph with one tie for all the* $n \geq k \geq 1$*, with the exception of* $(2, 2)$*,* $(2, 3)$ *and* $(4, 4)$ *oriented graph.*

Proof. Almost the same proof as Theorem 6.4, just substitute Lemma 6.3 with Lemma 6.5. □

in other words, Theorem 6.6, states that for all (n, k) oriented graphs that can be constructed, can be constructed with only one tie.

Chapter 7

Quasi-transitive Oriented Graph

7.1 Definitions

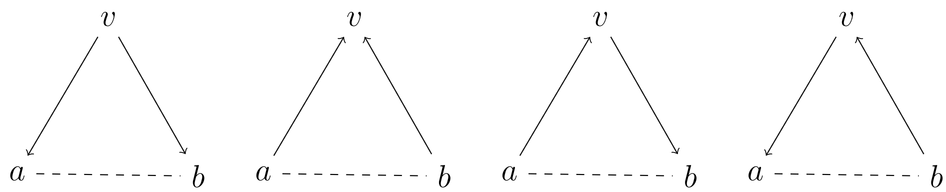
Chapter 6 is a rather short chapter, because the behavior of oriented graph is so unpredictable that it is hard to produce theorems about them. Therefore we decided to focus on a special case of oriented graph, which is called "quasi-transitive oriented graph":

Definition 7.1. *a* ***quasi-transitive oriented graph*** *G is an oriented graph such that, for all vertices $a, b, c \in V(G)$, if $a \to b \to c$, then a is adjacent to c. See Fig. 7.1*

Because every vertex is adjacent to each other in tournaments, therefore tournaments is a special case of quasi-transitive oriented graph.

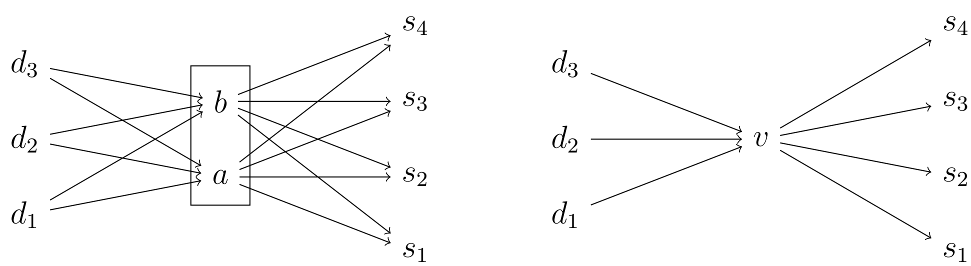
Although quasi-transitive oriented graph can be quite strange (for example, every graph with no edge is a quasi-transitive oriented graph), there are many connections between quasi-transitive oriented graphs and tournaments [1].

Figure 7.1: if $a \to b \to c$ then a is adjacent to c.



(a) $v \to \{a, b\}$. (b) $\{a, b\} \to v$. (c) $a \to v \to b$. (d) $b \to v \to a$.

Figure 7.2: all the possible beating relation between vertex v and vertices a, b



(a) all the vertices are adjacent to a, b. (b) the set $\{a, b\}$ behaves like v.

Figure 7.3: only look at all the vertices adjacent to both a and b, then $\{a, b\}$ behaves like a vertex.

7.2 Ties and Tie Paths

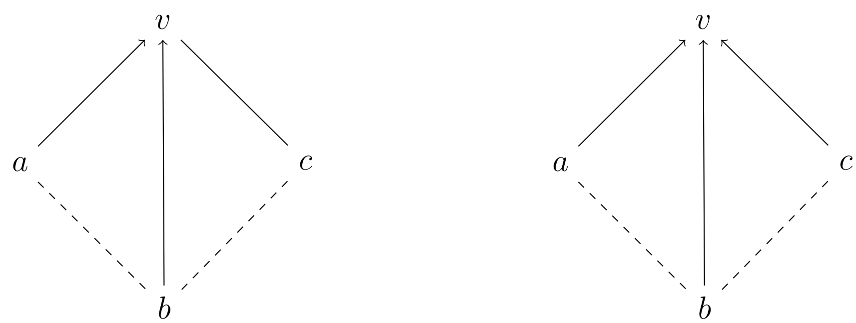
Properties of ties in quasi-transitive digraphs are vastly important in this paper.

Lemma 7.1. *In a quasi-transitive oriented graph G, if vertex a ties vertex b, then for all vertex v that is adjacent to both a and b, either v beats both a and b, or v is beaten by both a and b ($\{a, b\} \to v$ or $v \to \{a, b\}$).*

Proof. Assume there exists one vertex v such that this lemma does not hold.

Because v is adjacent to both a and b, we show all the possible beating relationships between v and a, b in Fig. 7.2. Because G is a quasi-transitive oriented graph, if $a \to v \to b$ or $b \to v \to a$ then a, b have to be adjacent. Therefore, only possible beating relationships are $\{a, b\} \to v$ or $v \to \{a, b\}$. □

Fig. 7.3a shows a nice visualization of Lemma 7.1. If we only draw out all the vertices that are adjacent to both a and b, represented by $d_1, d_2, d_3, s_1, s_2, s_3, s_4$ in this graph, then a and b are either beaten by them or beat them. The behavior of $\{a, b\}$ is somewhat like a vertex, see Fig. 7.3b.



(a) add a vertex c ties b, but not v. (b) by Lemma 7.1, $c \to v$.

Figure 7.4: ties are "transmitting" arrow directions.

However, this nice visualization in Fig. 7.3 has a very strong prerequisite, that is, all the vertices need to be adjacent to both a and b. What will happen when we add another vertex that ties either a or b but is adjacent to v? We will then create a visualization for this problem.

In Fig. 7.4a we add a vertex c that ties b, but needs to be adjacent to v (we use an edge without arrow to denote adjacency). We then try to figure out the edge directed between v and c. Because b, c are both adjacent to v, and c ties b, therefore either $\{b, c\} \to v$ or $v \to \{b, c\}$. However, in the graph, $b \to v$, therefore $c \to v$. Thus, we get Fig. 7.4b.

We find out that because of the ties between a, b, c the arrows direction between v, a and v, b, got "transmitted" to v, c via the Lemma 7.1. We formalize this "arrow transmission" idea:

Definition 7.2. *In a digraph, a* ***tie path*** *from vertex a_0 to a_n, or a tie path between vertex a_0 and a_n, is a sequence of vertices $[a_0, a_1, a_2, \ldots, a_{n-1}, a_n]$, such that for all $0 \leq k < n$, a_k ties a_{k+1}. If there is $n+1$ vertices in that sequence, we will say* ***the length of the tie path*** *is n.*

A tie path is pretty similar to a path in undirected graph; just substitute all the edges in a path with ties. See Fig. 7.5.

We first prove some general properties of digraph, these lemmas will be useful later:

Lemma 7.2. *In a digraph, if there exists a tie path from vertex a to vertex b and there exists a tie path from vertex b to vertex c, then there exists a tie path from a to c.*

(a) a path from a to d. (b) a tie path from a to d.

Figure 7.5: tie paths are just like paths.

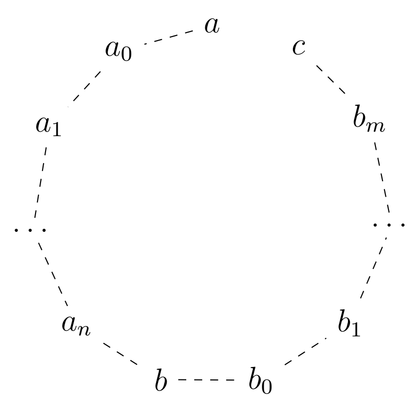


Figure 7.6: tie path from a to b, and from b to c.

Proof. Write the tie path from a to b as: $[a, a_0, a_1, \ldots, a_n, b]$, and the tie path from b to c as: $[b, b_0, b_1, \ldots, b_m, c]$. Then there exists a tie path $[a, a_0, a_1, \ldots, a_n, b, b_0, b_1, \ldots, b_m, c]$ from a to c. See Fig. 7.6. $\square$

Lemma 7.3. *In a digraph, if there exists a tie path $[a_0, a_1, \ldots, a_{n-1}, a_n]$, then for all $0 \leq p \leq n$ and $0 \leq q \leq n$, and p does not equal q, then there exists a tie path between p and q.*

Proof. Without loss of generality, assume $p < q$.

Then we can find a tie path $[a_p, a_{p+1}, \ldots, a_{q-1}, a_q]$, see Fig. 7.7. $\square$

After looking at Fig. 7.4, we can hypothesize that tie paths can transmit arrow directions across the whole path in quasi-transitive oriented graph. See Fig. 7.8.

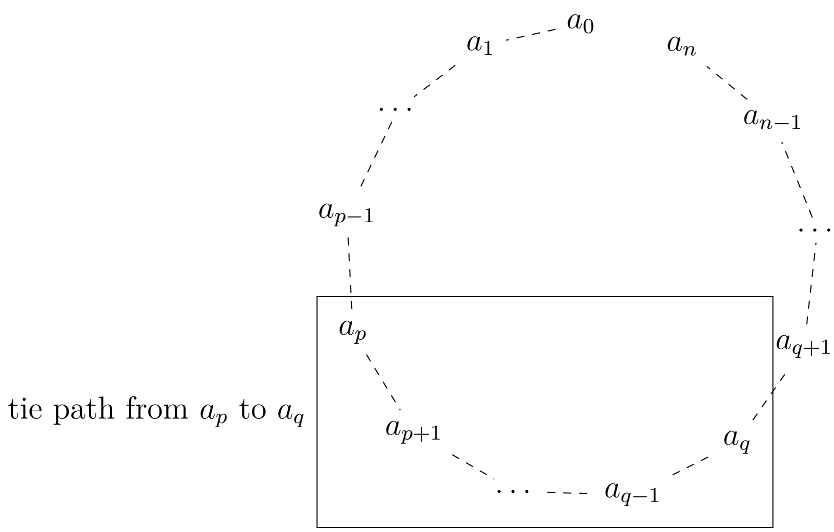
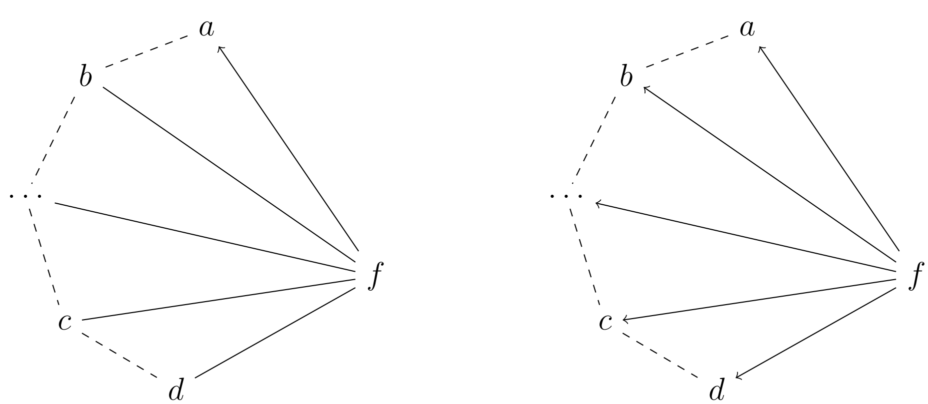


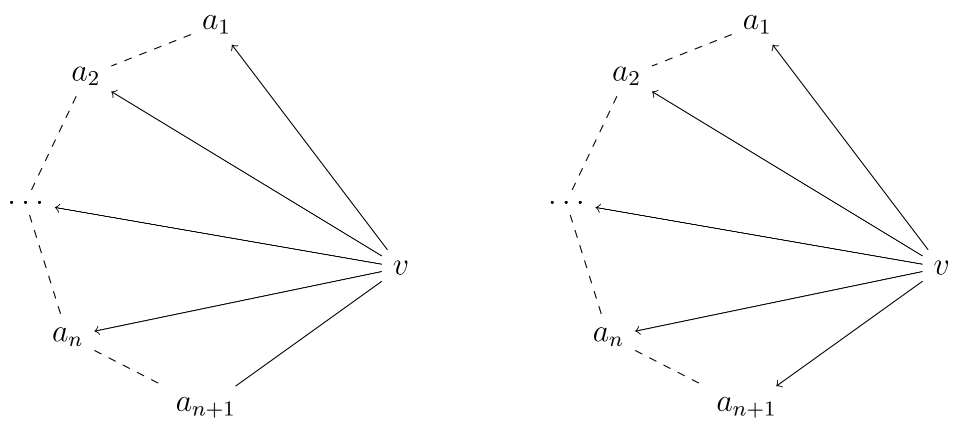
Figure 7.7: the tie path from a_0 to a_n.



(a) a is beaten by f, and other vertices adjacent to f

(b) the arrow direction is "transmitted" through the tie path

Figure 7.8: tie path transmits the arrow direction.



(a) a_{n+1} is adjacent to v. (b) then a_{n+1} has to be beaten by v.

Figure 7.9: the induction step of the proof of Lemma 7.4

Lemma 7.4. *For every tie path in a quasi-transitive oriented graph, take a vertex v that is adjacent to all the vertex on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.*

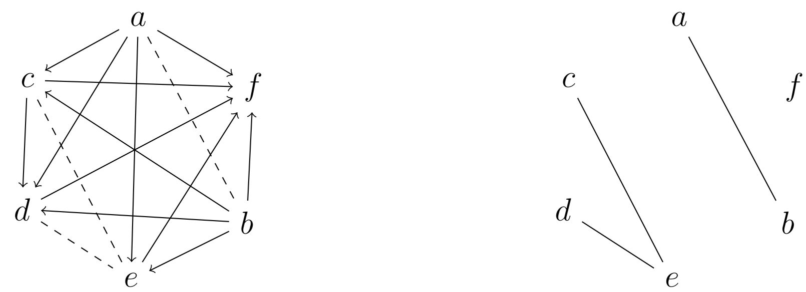
Proof. We can prove this lemma via induction, just like we did in Fig. 7.4.

Start with tie path of length 1, then we only have a_0 ties a_1 in this tie path, and v that is adjacent to both of them. The lemma holds because Lemma 7.1.

Assume this lemma holds for any tie path of length n, then we need to prove this theorem holds for tie path of length $n+1$. See Fig. 7.9, take a tie path $[a_0, a_1, \ldots, a_n, a_{n+1}]$. because this lemma holds for tie path of length n, then it is true for tie path $[a_0, a_1, \ldots, a_n]$. Because a_n ties a_{n+1}, and v is adjacent to both of them. Therefore, if $a_n \to v$ then $a_{n+1} \to v$ and if $v \to a_n$ then $v \to a_{n+1}$. Therefore this lemma still holds for tie path of length $n + 1$.

Therefore this lemma is always true. □

Lemma 7.4 generalizes Lemma 7.1, however, it is still far from elegant, because we still requires the vertex v to be adjacent to the whole tie path.



(a) distinct tie components are $\{a, b\}, \{c, d, e\}, \{f\}$

(b) connected components are $\{a, b\}, \{c, d, e\}, \{f\}$

Figure 7.10: the tie component in (a) is the connected component in (b).

7.3 Tie Components

We keep exploring the possibility of tie paths and the idea of "arrow direction transmitted by ties". We will be able to define the following structure with the intuition of "path" and "connected component" in undirected graph.

Definition 7.3. *In a digraph G, a* ***tie component*** *of vertex a: $C(a)$ contains the vertex a and all vertices v such that there exists a tie path between a and v. a is called the* ***representation*** *of $C(a)$*

In Fig. 7.10, we first change every tie in Fig. 7.10a to an undirected edge and then remove all the directed edges to obtain Fig. 7.10b, and tie component in Fig. 7.10a exactly corresponded to connected components in Fig. 7.10b. Therefore, just like tie path is similar to path in undirected graph, tie component is very similar to connected component in undirected graphs, and we can bring all the properties and intuitions of connected component to tie component.

We then prove some of these useful properties of tie components in digraphs. Notice, theses theorems work in all digraphs, not just in quasi-transitive oriented graphs.

Lemma 7.5. *For any two vertices a and b in a digraph G if there exists a tie path between a and b, then $C(a) = C(b)$.*

Proof. Take any vertex v in $C(a)$, then there exists a tie path between a and v. Because there is a tie path between a and b, thus there exists a tie path between b and v by Lemma 7.2. Then $v \in C(b)$. Therefore $C(a) \subseteq C(b)$

By symmetry, $C(b) \subseteq C(a)$, then $C(a) = C(b)$ □

Lemma 7.6. *In a digraph, there does not exists a tie path between two vertex a, b, then $C(a)$ and $C(b)$ disjoint.*

Proof. Assume $C(a)$ and $C(b)$ are not disjoint. Then there exists vertex $v \in C(a)$ and $v \in C(b)$. Therefore, there exists a tie path between v and a and there exists a tie path between v and b. Then there exists a tie path between a and b by Lemma 7.2. Contradiction. □

Lemma 7.7. *In a digraph, if vertex v not in tie component $C(a)$, then v is adjacent to $C(a)$.*

Proof. recall v is adjacent to $C(a)$ means v is adjacent to every vertex in $C(a)$.

Assume there exists a vertex $b \in C(a)$ that is not adjacent to v, that is, v ties b. Then there exists a tie path $[v, b]$ between v and b. Because $b \in C(a)$, then there exists a tie path between a and b. Therefore, by Lemma 7.2, there exists a tie path between v and a, and $v \in C(a)$. Contradiction. □

Theorem 7.8. *Distinct tie components of a digraph G forms a partition of the $V(G)$.*

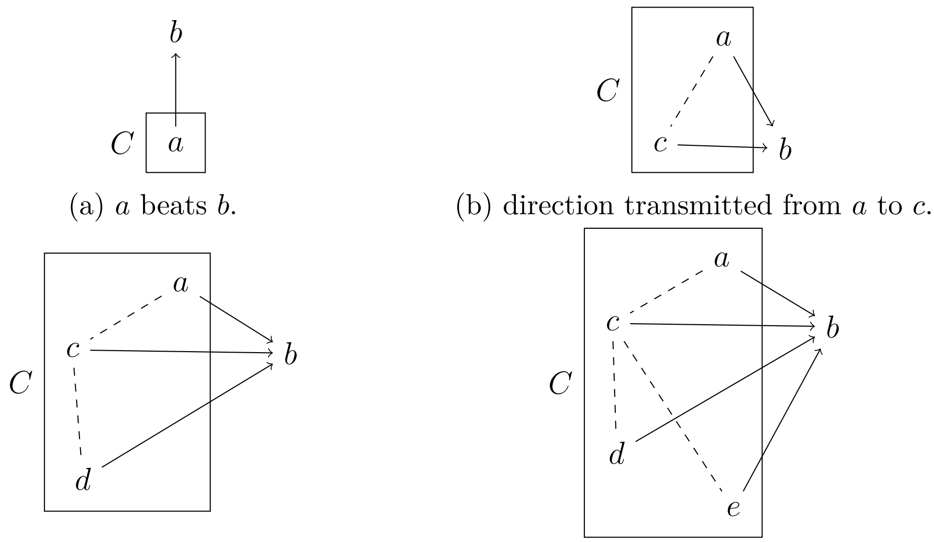
Proof. Prove that distinct tie components are disjoint: by Lemma 7.5 and Lemma 7.6, any two tie component $C(a), C(b)$ either equal or disjoint. Because distinct tie components cannot equal each other, therefore all distinct tie components are disjoint.

Prove that every vertex v in a tie component: $v \in C(v)$ by definition of tie component. □

Theorem 7.9. *Given a quasi-transitive oriented graph Q, the partition formed by distinct tie components is unique.*

Proof. Suppose there exists two different partition P and P', then there exists a tie component $C(a) \in P$ that is not in P'.

take a because P' partitions the graph, then there exists $C(a')$ such that $a \in C(a')$. Therefore, by Lemma 7.5, and Lemma 7.6, then $C(a)$ and $C(a')$ either disjoint or equal. Because a in both $C(a)$ and $C(a')$, therefore $C(a) = C(a')$. Because the tie component $C(a) \in P$ that is not in P', contradiction. □



(a) a beats b.

(b) direction transmitted from a to c.

(c) transmitted from a to c to d.

(d) transmitted from a to c to e.

Figure 7.11: the arrow direction to b was transmitted in tie component C.

Follow our intuition that tie “transmits” arrow directions in quasi-transitive oriented graph, we can see that for a tie component C and a vertex $v \notin C$, $C \to v$ or $v \to C$. See Fig. 7.11, for a visualization.

We then formalize and generalize this idea into two theorems:

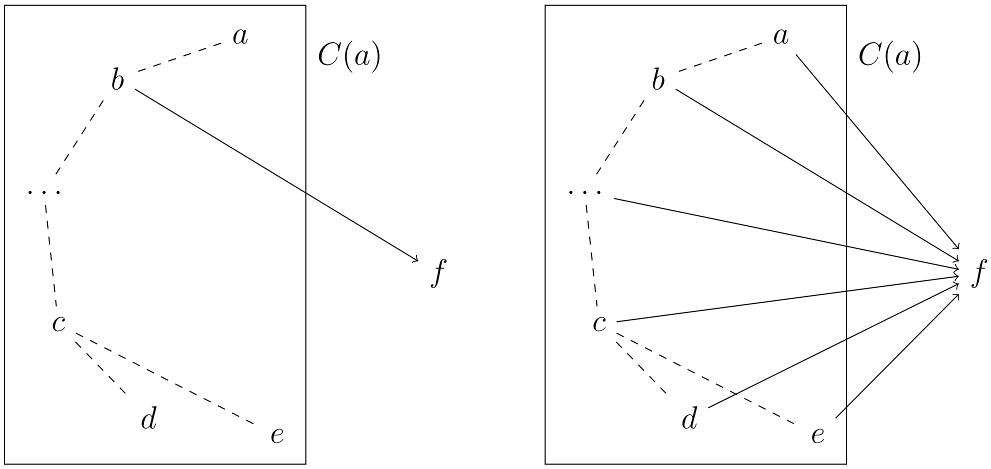
Lemma 7.10. *In a quasi-transitive oriented graph, for any tie component $C(a)$ and any vertex v such that $v \notin C(a)$, then either $C(a) \to v$ or $v \to C(a)$.*

Proof. Because v is not in $C(a)$, then v is adjacent to every vertex in $C(a)$ by Lemma 7.7. Take any vertex $b \in C$, there exists a tie path between a and b, and by Lemma 7.3, for every vertex c on this tie path, there exists a tie path between c and a, therefore $c \in C(a)$, thus v is adjacent to every vertex c on the tie path. Therefore $v \to \{a, b\}$ or $\{a, b\} \to v$, by Lemma 7.4.

Case 1, $a \to v$: for any vertex $b \in C(a)$, $b \to v$, by above argument. Therefore $C(a) \to v$.

Case 2, $v \to a$: then for any vertex $b \in C(a)$, $v \to b$, by above argument. Therefore $v \to C(a)$.

Case 3, v ties a: v cannot tie a because $v \notin C(a)$. $\square$



(a) a single vertex in $C(a)$ beats f. (b) component $C(a)$ has to beat f.

Figure 7.12: a vertex b beats f will force the component $C(a)$ to beats f.

Lemma 7.11. *In a quasi-transitive oriented graph, for any tie component $C(a)$, If there exists one vertex $v \in C(a)$ that beats (be beaten) a vertex $v' \notin C(a)$. Then $C(a) \to v'$ ($v' \to C(a)$). See Fig. 7.12*

Proof. Because $v' \notin C(a)$, then either $v' \to C(a)$ or $C(a) \to v'$ by Lemma 7.10.

Case 1, $v \to v'$: because $v \in C(a)$, then v' cannot beat all of $C(a)$. Therefore $C(a) \to v'$.

Case 2, $v' \to v$: because $v \in C(a)$, then v' cannot be beaten by all of $C(a)$. Therefore $v' \to C(a)$. □

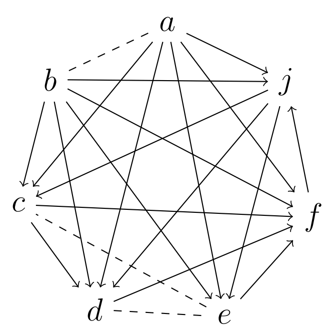
To further generalize Lemma 7.10, we prove the following theorem:

Theorem 7.12. *For any two distinct tie components $C(a)$ and $C(b)$ in a quasi-transitive oriented graph, $C(a) \to C(b)$ or $C(b) \to C(a)$.*

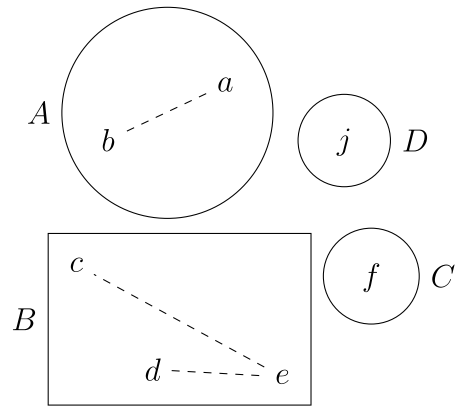
Proof. Take any vertex a' in $C(a)$. By Theorem 7.8, a' is not in $C(b)$. Then because of Lemma 7.10, $a' \to C(b)$ or $C(b) \to a'$.

Case 1, $a' \to C(b)$: take any element $b' \in C(b)$. By Lemma 7.11, $C(a) \to b'$. Therefore $C(a)$ beats every vertex in $C(b)$, then $C(a) \to C(b)$.

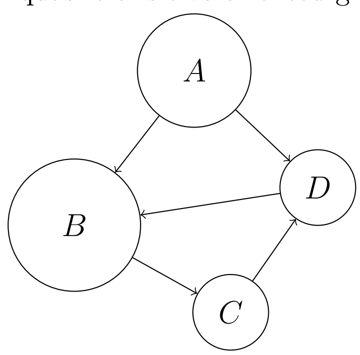
Case 2, $C(b) \to a'$: by almost the same reasoning, $C(b) \to C(a)$. □



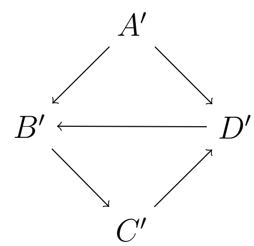
(a) a quasi-transitive oriented graph.



(b) find its tie components.



(c) components beats each other.



(d) components are just like vertex.

Figure 7.13: condense the tie components and get a tournament.

After couple pages of theorems, we can finally take a pause and understand what we are saying here. We combine Theorem 7.8 and Theorem 7.12, then try to understand it visually.

See Fig. 7.13. For every quasi-transitive digraph, we can split it into tie components, and every tie component either beats another tie component or be beaten by another tie component. Then, tie components are kind of like vertices. What even more exciting is that there cannot be ties between tie components. The graph formed by tie components is a tournament.

7.4 Graph Condensations

Definition 7.4. *A* ***condensation*** *is a function $f : G \to H$, where G and H are oriented graphs, and f maps $V(G)$ to $V(H)$ surjectively, such that*

- *for any two vertices* $a, b \in V(G)$*, if* $f(a) \to f(b)$ *in* H *then* $a \to b$ *in* G*.*
- *for any two vertices* $a, b \in V(G)$*, if* $f(a)$ *ties* $f(b)$ *in* H *then* a *ties* b *in* G*.*

We call G *the* ***uncondensed graph*** *, and* H *the* ***condensed graph*** *.*

Two examples of condensations are identity condensation and trivial condensation.

Identity condensation i maps an oriented graph G to itself, such that i maps every vertex to itself, and $f(a) \to f(b)$ if $a \to b$ and $f(a)$ ties $f(b)$ if a ties b. The identity condensation preserves all the edges and ties, and did not add any ties. We can see that this map is a condensation by definition.

Another example is trivial condensation t, which maps any oriented graph G into a single point. Because for any two vertices $a, b \in G$, $t(a) = t(b)$, therefore by definition, $t(a)$ does not tie or beat $t(b)$. Therefore, this mapping also satisfy the definition of condensation.

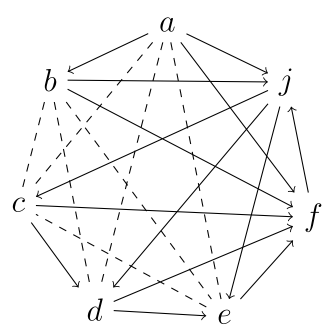
Definition 7.5. *Given oriented graph* G *as the uncondensed graph of condensation* f*, A* ***component*** *of vertex* $a \in V(G)$ *is the pre-image of* $f(a)$*, in other words, the component of* a *is* $\{x \in V(G) \mid f(x) = f(a)\}$*.*

A component of vertex a is a set of vertices that are condensed into the same vertex as a. For example, the component of vertex a in oriented graph G identity condensation is a set that only contains a itself, and the component of a vertex a in oriented graph G is set of all the vertices, because every vertex in G are all mapped onto a single vertex, therefore all the vertices in G are condensed into the same vertex as a.

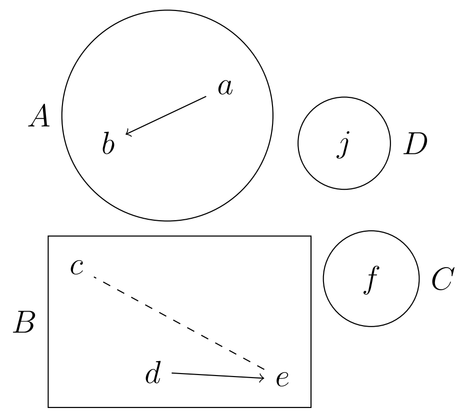
In Fig. 7.14, we show a general example of condensation. In this example, the distinct components in G are A, B, C and D, and they are condensed into vertices A', B', C' and D' respectively. The component of a is A, the component of d is B, and the component of c is also B, because both c and d are mapped into vertex B' in oriented graph G.

We proof couple theorem to help the understanding of condensation and components:

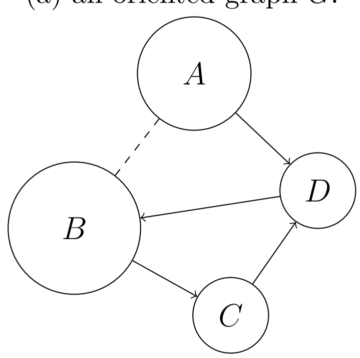
Theorem 7.13. *Given any oriented graph* G *and condensation* f*, distinct components in* G *partitions* $V(G)$*.*



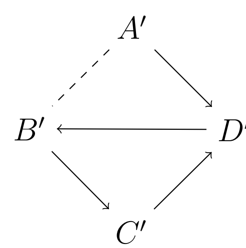
(a) an oriented graph G.



(b) components of G in this condensation.



(c) find the relationship between components.



(d) condensed graph H.

Figure 7.14: an example of graph condensation.

Proof. Prove distinct components disjoint: take any two vertices $a, b \in V(G)$, find component of a: A, and component of b: B. Assume there exists $v \in A$ and $v \in B$. Therefore $f(v) = f(a)$, and $f(v) = f(b)$, therefore $f(b) = f(a)$, and $B = \{x \in V(G) \mid f(x) = f(b)\} = \{x \in V(G) \mid f(x) = f(a)\} = A$. Therefore, A and B are not distinct. Contradiction.

Prove every vertex are in a component: for every vertex $v \in V(G)$, v is in the component of v by definition. □

Lemma 7.14. *Given any oriented graph G and condensation f, for any two distinct components A and B,*

- *for all $a \in A, b \in B$, if a ties b, then $f(a)$ ties $f(b)$.*
- *for all $a \in A, b \in B$, if $a \to b$, then $f(a) \to f(b)$*

Proof. Because A and B are distinct components, then by Theorem 7.13, A and B are disjoint, therefore for any vertex $a \in A$, and any vertex $b \in B$, $f(a) \neq f(b)$.

Given vertex $a \in A$ ties vertex $b \in B$. Then $f(a)$ cannot beat $f(b)$, otherwise a should beat b, by definition of condensation. Also, $f(b)$ cannot beat $f(a)$, otherwise b should beat a, therefore $f(a)$ ties $f(b)$.

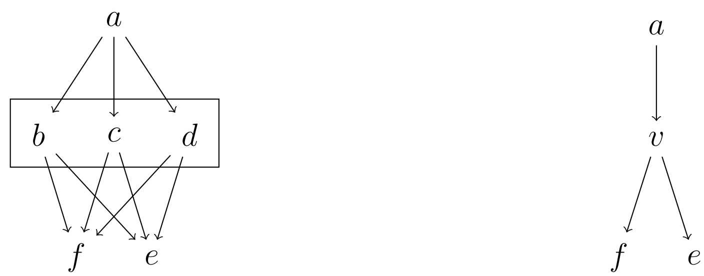
Given vertex $a \in A$ beats vertex $b \in B$. Then $f(a)$ cannot tie $f(b)$, otherwise a should tie b Also $f(b)$ cannot beat $f(a)$, otherwise b should beat a. Therefore $f(a) \to f(b)$. □

Theorem 7.15. *Given any oriented graph G and condensation f, for any two distinct components A and B,*

- *if any vertex $a \in A$ ties any vertex $b \in B$, then A ties B.*
- *if any vertex $a \in A$ beats any vertex $b \in B$, then $A \to B$.*

Proof. Because A and B are distinct components, then by Theorem 7.13, A and B are disjoint, therefore for any vertex $a \in A$, and any vertex $b \in B$, $f(a) \neq f(b)$.

Given vertex $a \in A$ ties vertex $b \in B$, by Lemma 7.14, $f(a)$ ties $f(b)$. Thus, for every vertex $a' \in A$, and every vertex $b' \in B$, because $f(a') = f(a)$ and $f(b') = f(b)$, and $f(a)$ ties $f(b)$, therefore $f(a')$ ties $f(b')$, then a' ties b'. Therefore, A ties B.



(a) every vertex either beats $\{b, c, d\}$ or is beaten by them.

(b) vertices $\{b, c, d\}$ in Fig. 7.15a behaves exactly like v in this graph

Figure 7.15: $\{b, c, d\}$ behaves like a vertex.

Given vertex $a \in A$ beats vertex $b \in B$. by Lemma 7.14, $f(a) \to f(b)$. Thus for every vertex $a' \in A$, and every vertex $b' \in B$, because $f(a') = f(a)$ and $f(b') = f(b)$, and $f(a) \to f(b)$, therefore $f(a') \to f(b')$, then $a' \to b'$. Therefore, A ties B. $\square$

Corollary 7.16. *Given any oriented graph G and condensation f, for any two distinct components A and B, either $A \to B$, or $B \to A$, or A ties B.*

Proof. Because components are not empty (component of v always contains v itself), we can take $a \in A$, and $b \in B$. Because G is an oriented graph, then either $a \to b$, $b \to a$, or a ties b.

According to Theorem 7.15, if $a \to b$, then $A \to B$; if $b \to a$, then $B \to A$; if a ties b, then A ties B. $\square$

We first see Fig. 7.15, we show an example of a set of vertices $\{b, c, d\}$ behaves like the vertex v. We then try to visualize Corollary 7.16, with the help of Fig. 7.14, we find out that graph condensation just takes the idea in Fig. 7.15 to the next level: all the components behaves likes a vertex. A condensation is just to find out those sets of vertices that behave like a single vertex (components), and then condense them into a single vertex.

Notice although we can define identity condensation and trivial condensation on all the oriented graphs, not all the oriented graphs have other condensations defined on them.

For example, graph in Fig. 7.16 cannot have any condensation defined on them except for identity condensation and trivial condensation, because we cannot find any

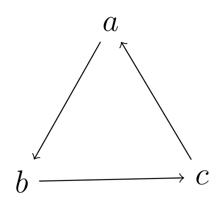
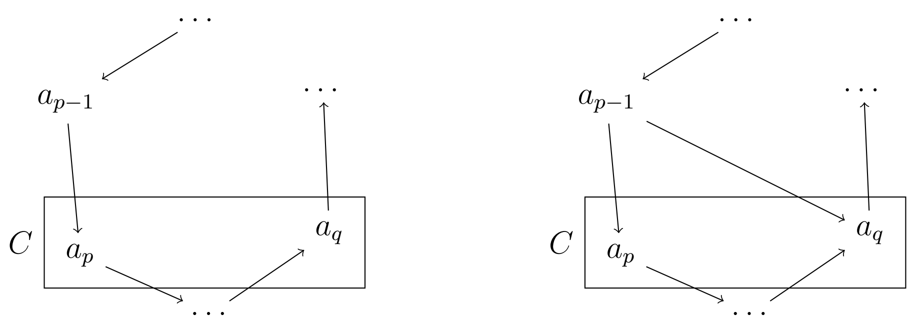


Figure 7.16: an oriented graph with only identity condensation and trivial condensation



(a) a_p and a_q are in the same component. (b) by Theorem 7.15, $a_{p-1} \to a_q$.

Figure 7.17: the path took a "detour" at a_p.

components that is not the whole graph or a single vertex. For example, if we find a set $\{a, c\}$, vertex b both beats a vertex in the set (b beats a), and are beaten by a vertex in the set (c is beaten by$b$). Therefore the set $\{a, c\}$ is not a component.

Condensation is a relatively strong transformation, because it preserves many properties of the graph. We will only prove some theorems that will be useful in later sections.

Lemma 7.17. *Given a condensation $f : G \to H$, for all $a_0, a_n \in V(G)$, such that a_0 and a_n are in two distinct components, and the shortest path from a_0 to a_n is $P : a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$. Then vertices $a_0, a_1, \ldots, a_{n-1}, a_n$ are all in different components.*

Proof. Take the first a_p such that there exists a_q are in the same component C as a_p where $p < q \leq n$.

We then show that $a_{p-1} \in C$: assume $a_{p-1} \notin C$, then because $a_{p-1} \to a_p$ and Theorem 7.15, $a_{p-1} \to C$. Thus $a_{p-1} \to a_q$, and path $a_0 \to a_1 \to \cdots \to a_{p-2} \to a_{p-1} \to a_q \to a_{q+1} \to \cdots \to a_{n-1} \to a_n$ exists and shorter than P. See Fig. 7.17. Contradiction. □

Theorem 7.18. *Given a condensation $f : G \to H$, for any $a_0, a_n \in V(G)$, such that a_0 and a_n are in two distinct components. The shortest path from a_0 to a_n is $a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$ in G if and only if the shortest path from $f(a_0)$ to $f(a_n)$ is $f(a_0) \to f(a_1) \to \cdots \to f(a_{n-1}) \to f(a_n)$ in H.*

Proof. Prove $\Rightarrow$: assume $P : f(a_0) \to f(a_1) \to \cdots \to f(a_{n-1}) \to f(a_n)$ is not the shortest path, then there exists another path P' from $f(a_0)$ to $f(a_n)$: $f(a_0) \to f(b_1) \to f(b_2) \to \cdots \to f(b_{m-1}) \to f(b_m) \to f(a_n)$. Because a_0 and a_n are in distinct components, then the path P' does not have length 0. If P' is shorter than P, the path $a_0 \to b_1 \to b_2 \to \cdots \to b_{m-1} \to b_m \to a_n$ exists (by definition of condensation) and will be shorter than $a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$. Since $a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$ is the shortest path from a_0 to a_n, contradiction.

Prove $\Leftarrow$: assume the shortest path is $a_0 \to b_1 \to b_2 \to \cdots \to b_m \to a_n$. Because of Lemma 7.17, $\{a_0, b_1, b_2, \ldots, b_m, a_n\}$ are all in different components. Then by Lemma 7.14, $f(a_0) \to f(b_1) \to f(b_2) \to \cdots \to f(b_m) \to f(a_n)$ exists and will be shorter than $f(a_0) \to f(a_1) \to \cdots \to f(a_{n-1}) \to f(a_n)$. Because $f(a_0) \to f(a_1) \to \cdots \to f(a_{n-1}) \to f(a_n)$ is the shortest path, contradiction. □

Corollary 7.19. *Given a condensation $f : G \to H$, for all $a_0, a_n \in G$, such that a_0 and a_n are in two distinct components, a_0 beats a_n by n steps in G if and only if $f(a_0)$ beats $f(a_n)$ by n steps in H*

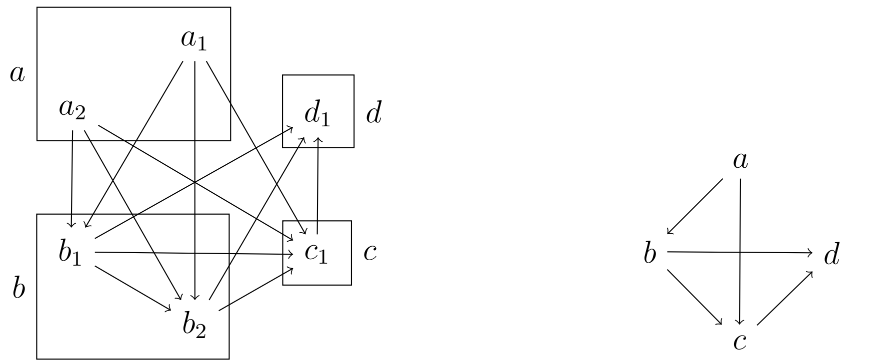
Proof. Result of Theorem 7.18. □

Corollary 7.20. *Given a condensation $f : G \to H$, vertex k is a king in G if and only if*

- *k is a king in the induced subgraph of the component of k.*
- *$f(k)$ is a king in H.*

Proof. Denote component of k as C. Denote the induced subgraph of C as G_c.

Prove $\Rightarrow$: If k is a king, then it beats every vertex in $V(G)$ by 1 or 2 steps. First, if k beats every vertex outside of C by 1 or 2 steps, because of Corollary 7.19, then $f(k)$ beats every vertex in H in 1 or 2 steps, therefore $f(k)$ is a king in H. Then if k beats every vertex $v \in C$ in 1 or 2 steps in G, we split into 2 cases:



(a) the uncondensed graph. (b) the condensed graph.

Figure 7.18: information lost during condensation.

- if k beats v by 1 step, then k beats v by 1 step in G_c.
- if k beats v by 2 steps, then there exists a such that $k \to a \to v$. $a \in C$, because of Theorem 7.15. Therefore, k beats v by 2 steps in G_c.

Then k is a king in G_c, and $f(k)$ is a king in H.

Prove $\Leftarrow$: If $f(k)$ is a king in H, then $f(k)$ beats every other vertex $f(v)$ in H, then k beats every other vertex that is not in C by 1 or 2 steps, by Corollary 7.19. If k is a king in G_c, then k beats every vertex in C by 1 or 2 steps. Therefore, k beats every vertex in $V(G)$ by 1 or 2 steps, then k is a king in G. $\square$

Although condensation is a very strong transformation, some information in the graph does get lost via this transformation. For example, the beating relationships between all the vertices in the same components are lost.

In Fig. 7.18, we show an example of condensation, where the vertices a_1, a_2 are condensed into a, b_1, b_2 are condensed into b, c_1 is condensed into c, and d_1 is condensed into d. If we just look at the condensed graph, we noticed that we cannot recreate the beating relationship of a_1 and a_2, therefore the information about the beating relationship is lost. This fact is true for all the vertices in the same component. Another example in this graph is that we cannot know the beating relationship between b_1 and b_2 just by the uncondensed graph.

Notice the beating relationship between components are not lost. For example, if we want to know the beating relationship between b_1 and a_2, we first observed that

$a \to b$ in the condensed graph, then by definition of a condensation, a_2 have to beat b_1 in the uncondensed graph. Another example is that we can know that a_1 ties d_1 because a ties d in the condensed graph.

Therefore, a "efficient condensation" should keep as many vertices as possible. One of the most "efficient" condensation is the identity condensation, because it does not lose any information about this graph. However, identity condensation also do not mutate the graph at all, therefore it is not very practical.

Definition 7.6. *Given a set of condensation $F = \{f_0, f_1, \ldots, f_n\}$ where $f_k : G_k \to H_k$ and all the $V(G_k)$ are of the same size for $0 \leq k \leq n$, an **efficient condensation** $f : G \to H$ in F is a condensation such that H has largest vertex set in $\{H_0, H_1, H_2, \ldots H_n\}$*

Notice in this definition, all the G_k's where $0 \leq k \leq n$ are not necessarily distinct. One example of efficient condensation is given a graph G and all the condensation defined on G, that is $\{f_0, f_1, \ldots, f_n\}$ where $f_k : G \to H_k$. Then the identity condensation, is an efficient condensation in this set, since it keeps all the vertices.

7.5 Tie Component Condensations

In the previous section we talked about the general property of graph condensations. In this section, we will focus on a specific type of condensation mentioned in previous sections and Fig. 7.13.

Definition 7.7. *A **tie component condensation** is a family of function $f : Q \to T$, where Q is a quasi-transitive oriented graph, T is a directed graph. And f maps $V(Q)$ to $V(T)$ surjectively, maps $E(Q)$ to $E(T)$ surjectively, such that:*

- *if $a, b \in V(Q)$ in the same tie component, then $f(a) = f(b)$.*
- *if $a, b \in V(Q)$ in 2 distinct tie component A, B respectively, then*
  - *$f(a) \to f(b)$ if $A \to B$.*
  - *$f(b) \to f(a)$ if $B \to A$.*

Corollary 7.21. *All tie component condensations are graph condensation, where the components of the uncondensed graphs are the tie component of the uncondensed graphs.*

Corollary 7.22. *for any tie component condensation $f : Q \to T$, T will always be a tournament.*

Proof. Because of Theorem 7.12, for any two distinct vertices $f(a), f(b) \in T$, either $f(a) \to f(b)$ or $f(b) \to f(a)$. Therefore, T is a tournament. □

Fig. 7.13 is a great example of tie component condensation. In this condensation, the function g maps a, b to A', maps c, d, e to B', and maps f to C', and j to D'.

It is also helpful to note that the tie component C and D are trivial tie components, since they only contain vertex f and j respectively.

Corollary 7.23. *Given a quasi-transitive digraph Q, there only exists a unique tie component condensation $f : Q \to T$*

Proof. Tie component condensation exists by definition, tie component condensation unique by Theorem 7.9. □

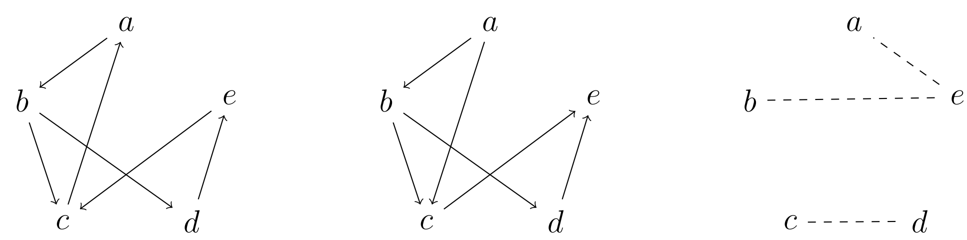
Definition 7.8. *Given a quasi-transitive digraph Q, we call the tie component condensation $f : Q \to T$ the tie component condensation on Q.*

Definition 7.9. *For a given quasi-transitive oriented graph Q, we call the result of tie component condensation of Q the* ***underlying tournament*** *of Q.*

It is very surprising that every quasi-transitive oriented graph can always be condensed into a tournament, which is one of the most understood family of digraphs. But just to say that a condensation exists is not surprising enough, since for every graph, there exists a trivial condensation, which also always result in a tournament (a single vertex is, by definition, a tournament). Therefore, we need to investigate the efficiency of the tie component condensation.

Definition 7.10. *For two orientated graphs G, H, such that*

- $V(G) = V(H)$,



(a) graph G. (b) graph H. (c) “tie structures” of G, H.

Figure 7.19: G and H has the same tie.

- *a ties b in H if and only if a ties b in G,*

then we say G, H has the same ***tie structure*** *.*

Graph with same tie structure means that if we draw out all the ties from two graph, the graph formed by the ties are the same. In other words, these two graphs can only differ by the orientation of some of their edges.

See Fig. 7.19, in Fig. 7.19a and Fig. 7.19b, we show two graph G and H with the same tie structure, In Fig. 7.19c, we draw out all the ties from the previous two graphs, and discover that they are the same. the only differences between these two graphs are that $c \to a$, $e \to c$ in G, however $a \to c$, $c \to e$ in H, that is, the only differences between G and H are the orientations of these two edges.

Corollary 7.24. *For G and H with the same tie structure, if there exists a tie path between a, b in G, then there exists a tie path between a, b in H.*

Proof. since given any two vertices $p, q \in G$, if p ties q in G then p ties q in H. Therefore if there exists a tie path $[a, a_0, a_1, \ldots, a_n, b]$ in G, then the same tie path exists in H $\square$

Theorem 7.25. *Given a quasi-transitive graph Q and its tie component condensation f, consider the set of all the condensation $f_k : G_k \to T_k$, where G_k has the same tie structure as Q and T_k is a tournament: $F = \{f_0, f_1, \ldots, f_{n-1}, f_n\}$. f is an efficient condensation in F.*

Proof. Because G_k are has the same tie structure as Q, therefore all the G_k has the same number of vertices.

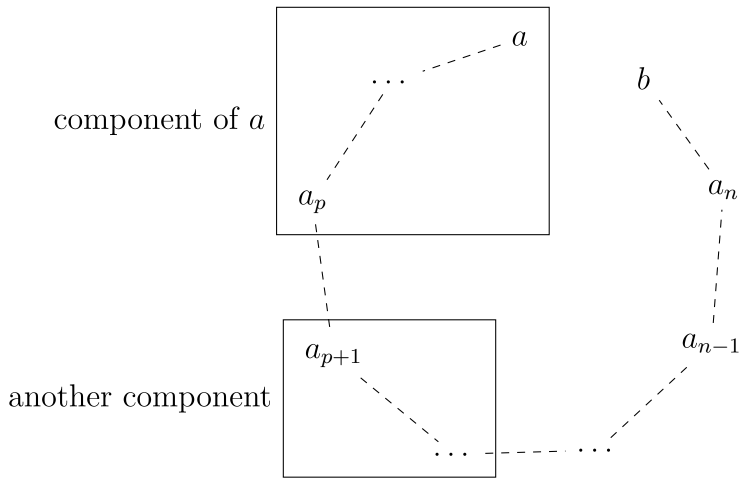


Figure 7.20: vertices a_p and a_{p+1} crosses components.

if there exists another condensation $f' : G' \to T'$ such that T' has more vertices than the underlying tournament T of Q. Then there exists two vertices a, b in the same component in Q, and in different components in G', because the number of component in the uncondensed graph is the same as the number of vertices in the condensed graph.

Because a, b in the same component in Q, there exists a tie path between a and b in Q. By Corollary 7.24, there exists a tie path between a and b in G'. Because a and b are in different components in G', there exists a point on the tie path between a and b that "crosses components" in G'.

Formally, denote the tie path between a, b as: $[a, a_0, a_1, \ldots, a_n, b]$, then there exists a_p such that a_p and a_{p+1} are not in the same component. Because a_p ties a_{p+1} and they are in distinct components, then by Lemma 7.14, $f'(a)$ ties $f'(a')$. Therefore T' is not a tournament. Contradiction. $\square$

Theorem 7.25 states that tie component condensation is not only the most efficient condensation to tournaments on any quasi-transitive orientated graph, tie component condensation is the most efficient in all the condensations defined on all orientated graphs with the same tie structure.

To put it in other way, for all the graph with the same tie structure, the quasi-transitive orientated graphs are the ones that can be condensed into tournaments most efficiently.

7.6 Kings

Kings in quasi-transitive oriented graph have very nice property, because these two definitions are deeply connected. The definition of king states that it can beats every vertex by one or two steps, and the definition of quasi-transitive oriented graph states that if a vertex beats another vertex by two steps, then they are adjacent.

After a brief looking that we can came up with the following lemma:

Lemma 7.26. *In a quasi-transitive oriented graph, king is adjacent to every vertex.*

Proof. Given a king k, and another vertex v, then there are two possibilities:

Case 1, k beats v by one step: therefore $k \to v$, k and v are adjacent.

Case 2, k beats v by two steps: therefore exists vertex a, such that $k \to a \to v$. Then by definition of quasi-transitive oriented graph, k needs to be adjacent to v $\square$

Lemma 7.27. *In a quasi-transitive oriented graph, for any king k, $\{D_k, S_k, \{k\}\}$ partitions the graph.*

Proof. Proof disjoint: pretty obvious. D_k, S_k disjoint because we are working in oriented graph. D_k, S_k disjoint with $\{k\}$ because a vertex cannot beat itself.

Proof the union is the whole vertex set: by Lemma 7.26 $\square$

Lemma 7.28. *In a quasi-transitive oriented graph, for any king k, D_k, S_k are adjacent.*

Proof. By definition of quasi-transitive oriented graph, because every vertex in D_k beats k and then beats every vertex in S_k, every vertex in D_k is adjacent to every vertex in S_k. $\square$

Theorem 7.29. *In a quasi-transitive oriented graph, if we have a king k, then*

- *$D_k, S_k, \{k\}$ partitions the vertex set.*
- *D_k and S_k is adjacent.*

See Fig. 7.21, this figure shows the rich structure of a king in quasi-transitive oriented graph. We were able to almost see the partition structure mentioned in Chapter 5, and Lemma 5.1.

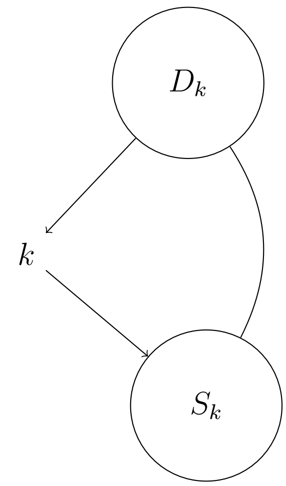


Figure 7.21: the rich structure of a king in quasi-transitive oriented graph.

The vertex in D_k cannot ties everything out side of D_k, and so is S_k. Intuitively, if there is a king in quasi-transitive oriented graph, then this graph becomes "very connected", the only places ties can appear are inside the induced subgraph of D_k and S_k.

Since we have deduced nice structure about ties in quasi-transitive oriented graph, it is only logical to combine the property of king with the property of tie (tie component). Hence come the final theorem of this paper:

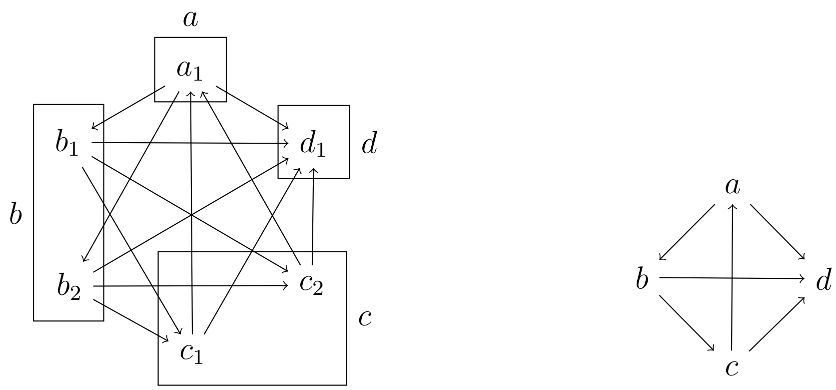
Theorem 7.30. *A vertex k is a quasi-transitive oriented graph if and only if*

- *k is in a trivial tie component.*
- *k is a king in the underlying tournament.*

Proof. By Corollary 7.20, k is a king if and only if

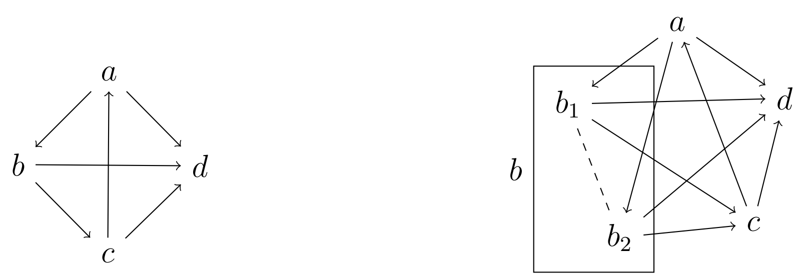
- k is a king in the induced graph of its tie component.
- k is a king in the underlying tournament.

Show if k is a king in the induced graph of its tie component, then it is in a trivial tie component: because of Lemma 7.26, k is a king in its tie component only if k does not tie with any other vertex in its component. If k is not in a trivial tie component, it has to tie at least another vertex to have a tie path going to other vertices. Therefore k has to be in a trivial tie component.



(a) quasi-transitive oriented graph G. (b) the underlying tournament of G.

Figure 7.22: a quasi-transitive digraph and its underlying tournament.



(a) start with a tournament with 3 kings. (b) then change king b to $\{b_1, b_2\}$.

Figure 7.23: construct a quasi-transitive oriented graph with 2 kings.

Show if k is is in a trivial tie component, then k in a king in the induced graph of its tie component: true by definition of king; for every graph with one vertex, that vertex is a king in the graph. $\square$

In Fig. 7.22, we show a quasi-transitive oriented graph G and its underlying tournament. In the underlying tournament the kings are a, b, c, but only a_1 can be a king, since vertices b and c do not correspond to trivial tie components. (tie component $\{b_1, b_2\}$ are condensed into b, and tie component $\{c_1, c_2\}$ are condensed into c). Notice, although vertex d_1 is in a trivial tie component, d_1 is not a king in G, because in the underlying tournament, d (the image of d_1 under tie condensation) is not a king.

This theorem not only gives us an way to identify king in quasi-transitive oriented graph, but it helps us to construct a quasi-transitive oriented graph with certain number of kings.

For example, if we want to construct a quasi-transitive oriented graph with k kings, we first start with a quasi-transitive graph with n kings, where $n > k$, and then we change $n-2$ vertices into "non-trivial tie components". See Fig. 7.23, we start with a tournament with 3 kings $\{a, b, c\}$, then we change the king b into a non-trivial tie component $\{b_1, b_2\}$. Then the only kings left are a and c.

This way of constructing quasi-transitive oriented graphs is very useful, since we can control the number of ties in the graph. For a tie components with n vertices, the number of ties in the component can be any number between $n-1$ (ties forms a spanning tree of the component) and $\frac{n(n+1)}{2}$ (there exists a tie between every two vertices). And there is no tie between different tie components.

Another significant usage of Theorem 7.30 is to use the properties of kings in tournaments to prove properties of kings in quasi-transitive oriented graphs.

Definition 7.11. *a* ***great king*** *k in a digraph G, is the only king in graph G, and k is not an emperor.*

Corollary 7.31. *If there exists a great king in a quasi-transitive oriented graph G, then there exists at least two ties in G.*

Proof. Assume k is the great king in G, then there exists a vertex $v \to k$. denote the underlying tournament of G as T, and the tie component condensation of G as f.

By Lemma 7.14, $f(v) \to f(k)$. Therefore, by Theorem 7.30 $f(k)$ is a king, but not an emperor in T. Because of Theorem 4.1 and Theorem 4.2, there exists at least 3 kings in T, because k is the only king in G, there has to exist at least 2 non-trivial tie components. Therefore, G have at least 2 ties. $\square$

Chapter 8

Further Problems

The existence of a (n, k, t) digraph.

Definition 8.1. *a (n, k, t) digraph is a digraph with n chickens, k kings and t ties.*

What is the property of a king with very low out-degree?

Can we extend these result to infinite graph?

How hard is a condensation? How many graph only have identity condensation and trivial condensation on them?

List of Figures

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