Kings in Generalized Tournaments

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Thesis Defense

Background

tournaments

Definition (Maurer 1980)

An **tournament** consists of a set of **vertices** V, and a set of **edges** E which consists of ordered pairs of vertices, such that

- for all vertex a, $(a, a) \notin E$.
- for all vertices a, b, $(a, b) \in E$ or $(b, a) \in E$, but not both.

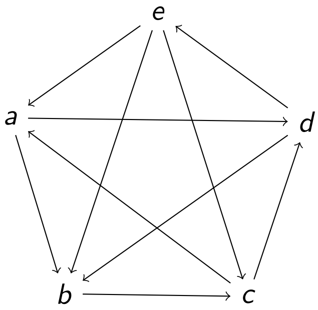


Figure: example of a tournament.

Background

tournaments

If (a, b) in edge set E, we write it as $a \to b$ or a beats b.

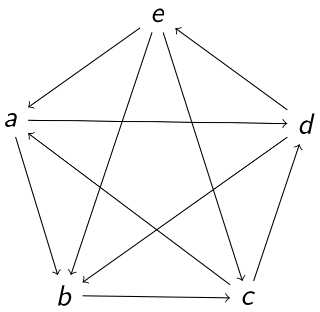


Figure: example of a tournament.

Background

kings

Definition (Maurer 1980)

A **king** (or **2-king**) in a oriented graph is a vertex that can beat every vertex by at most 2 steps.

$$\begin{tikzcd}
 & b \arrow[r] & c \\
k \arrow[ur] \arrow[dr] & & \\
 & a &
\end{tikzcd}$$

Background

kings

The kings in this graph are a, d, e.

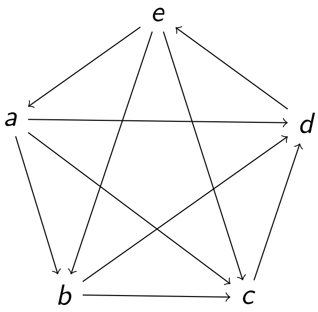
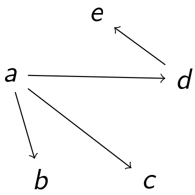


Figure: example of a tournament.

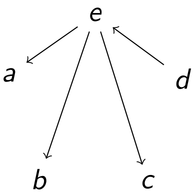
Background

kings

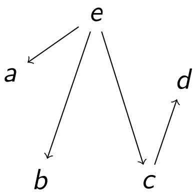
The kings in this graph are a, d, e:



(a) a is a king.



(b) d is a king.



(c) e is a king.

Generalization of Tournaments

relation between three generalizations

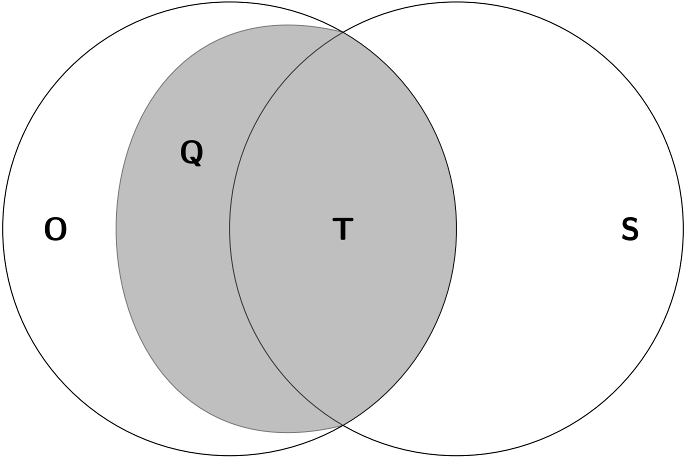


Figure: the relationship between tournaments, semi-complete digraphs, oriented graphs, and quasi-transitive oriented graphs.

Quasi-transitive Oriented Graph

oriented graph

Definition

An **oriented graph** consists of a set of **vertices** V, and a set of **edges** E which consists of ordered pairs of vertices, such that

- for all vertex a, $(a, a) \notin E$.
- for all vertices $\{a, b\}$, if a beats b, then b does not beat a.

$$
\begin{tikzcd}
\mathrm{d} \arrow[r] & \mathrm{b} \arrow[dr] & & \mathrm{a} \arrow[ll] \\
& & \mathrm{c} \arrow[ur] &
\end{tikzcd}
$$

Figure: example of an oriented graph.

Quasi-transitive Oriented Graph

oriented graph

Definition

A vertex a is **adjacent** to vertex b if $a \to b$ or $b \to a$.

Definition

There exists a **tie** between a and c, if a is not adjacent to c.

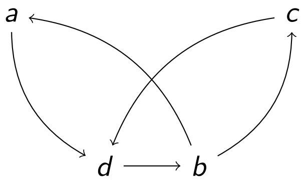


Figure: another example of an oriented graph.

Quasi-transitive Oriented Graph

oriented graph

King in this graph is d, e:

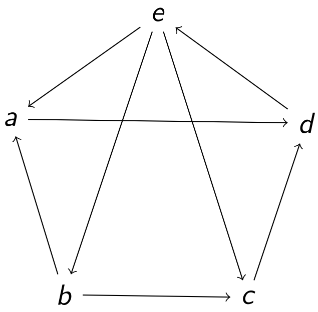


Figure: an example of kings.

Quasi-transitive Oriented Graph

definition

Definition (Bang-Jensen and Huang 1995)

Quasi-transitive oriented graph is an oriented graph such that if $a \to b$ and $b \to c$ then a and c are adjacent.

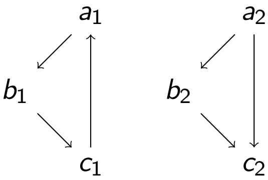


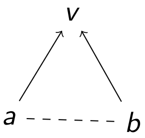
Figure: examples of quasi-transitive oriented graphs.

Quasi-transitive Oriented Graph

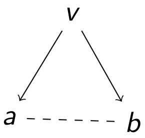
ties

Lemma

In Quasi-transitive oriented graphs, If vertex a ties with vertex b, then for any other vertex v that does not tie with a or b, $v \to \{a, b\}$ or $\{a, b\} \to v$



(a) $v \to \{a, b\}$.

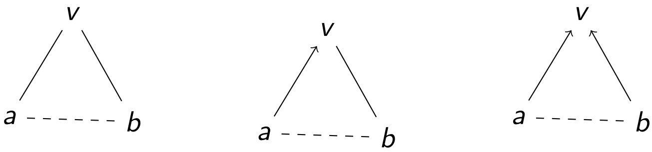


(b) $\{a, b\} \to v$.

Figure: an example where a ties with b.

Quasi-transitive Oriented Graph

tie transmits arrows



(a) both a and b are adjacent to v.

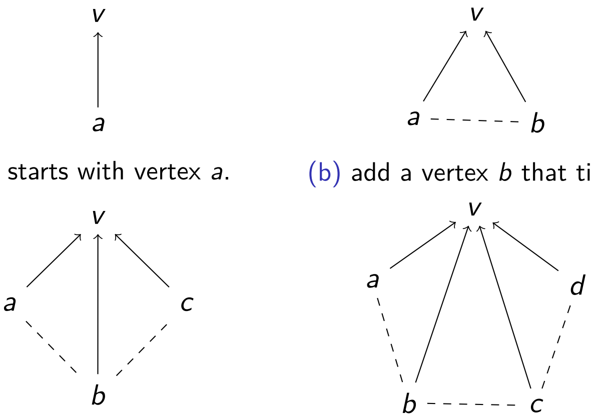
(b) let $a \to v$.

(c) then b has to beat v.

Figure: arrow direction transmitted from a to b.

Quasi-transitive Oriented Graph

tie transmits arrows



(a) starts with vertex a. (b) add a vertex b that ties a. (c) add another vertex c ties b. (d) add another vertex d ties c.

Figure: tie is transmitting the direction of arrow.

Quasi-transitive Oriented Graph

tie paths

Definition

In a digraph, a **tie path** from vertex a_0 to vertex a_n, or a tie path between vertices a_0 and a_n, is a sequence of vertices $[a_0, a_1, a_2, \ldots, a_{n-1}, a_n]$, such that for all $0 \leq k < n$, a_k ties a_{k+1}. Note that all the a_i's do not have to be distinct.

$$a \text{ - - - } b \text{ - - - } c \text{ - - - } d$$

Figure: a tie path from a to b.

Quasi-transitive Oriented Graph

tie paths

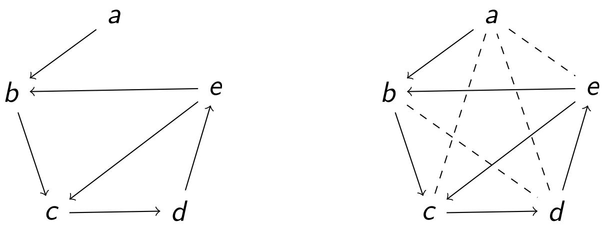


Figure: tie path example.

Quasi-transitive Oriented Graph

tie paths

Lemma

In Quasi-transitive oriented graphs, If vertex a ties with vertex b, then for any other vertex v that does not tie with a or b,
$v \to \{a, b\}$ *or* $\{a, b\} \to v$

Lemma

For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.

Quasi-transitive Oriented Graph

tie paths

Lemma

For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.

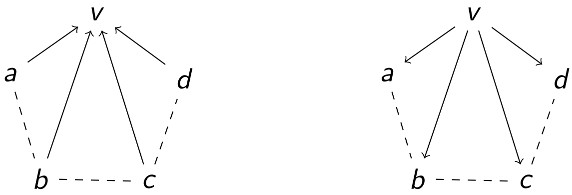
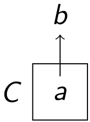


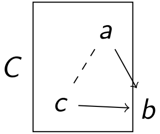
Figure: a tie path from a to d.

Quasi-transitive Oriented Graph

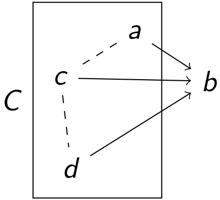
tie components



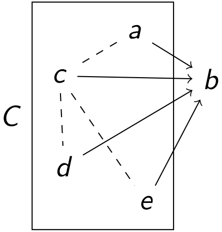
(a) *a* beats *b*.



(b) direction transmitted from *a* to *c*.



(c) transmitted from *a* to *c* to *d*.



(d) transmitted from *a* to *c* to *e*.

Figure: the arrow direction to *b* was transmitted in tie component *C*.

Quasi-transitive Oriented Graph

tie components

Definition

In a digraph G, a **tie component of vertex** a: $C(a)$, all vertices v such that there exists a tie path between a and v.

Definition

In a digraph G, a tie component $C(a)$ is a **trivial tie component** if $C(a)$ only contains a itself and no other vertex.

Corollary

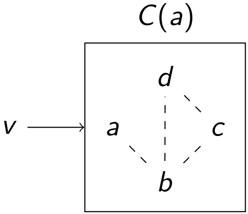
In a digraph, if vertex v is not in tie component $C(a)$, then v is adjacent to $C(a)$.

Quasi-transitive Oriented Graph

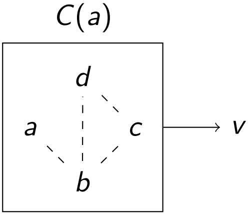
tie components

Lemma

In a quasi-transitive oriented graph, for any tie component $C(a)$ *and any vertex* v *such that* $v \notin C(a)$*, then either* $C(a) \rightarrow v$ *or* $v \rightarrow C(a)$.



(a) $v \rightarrow C(a)$.

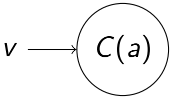


(b) $C(a) \rightarrow v$.

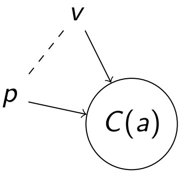
Quasi-transitive Oriented Graph

tie components

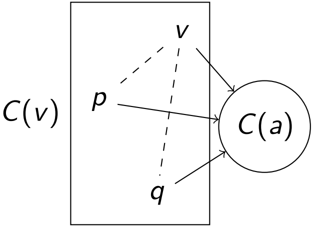
Consider tie component $C(v)$. Without loss of generosity, assume $v \to C(a)$



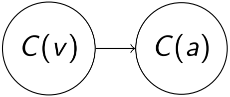
(a) direction is $v \to C(a)$.



(b) transmitted from v to p.



(c) transmitted from v to q.



(d) $C(v) \to C(a)$.

Quasi-transitive Oriented Graph

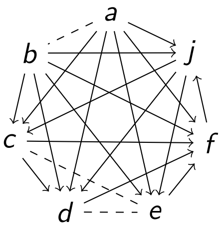
condensation to tournaments

Theorem

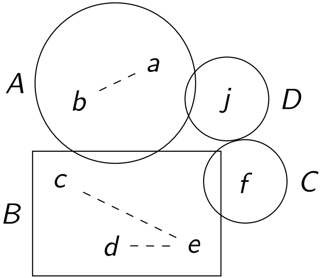
For any two distinct tie components $C(v)$ and $C(a)$ in a quasi-transitive oriented graph, $C(v) \to C(a)$ or $C(a) \to C(v)$.

Quasi-transitive Oriented Graph

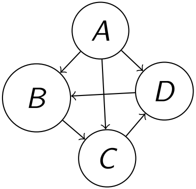
condensation to tournaments



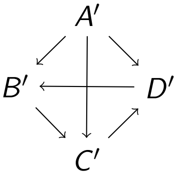
(a) a quasi-transitive oriented graph.



(b) find its tie components.



(c) components beat each other.



(d) components are just like vertices.

Quasi-transitive Oriented Graph

condensation to tournaments

Definition

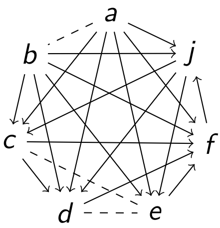
A **tie component condensation** is a function $f : Q \to T$, where Q is a quasi-transitive oriented graph, T is a directed graph. And f maps $V(Q)$ to $V(T)$ surjectively, such that:

- if $a, b \in V(Q)$ are in the same tie component, then $f(a) = f(b)$.
- if $a, b \in V(Q)$ are in 2 distinct tie components A, B respectively, then
  - $f(a) \to f(b)$ if $A \to B$.
  - $f(b) \to f(a)$ if $B \to A$.
  - otherwise, tie condensation do not exists.

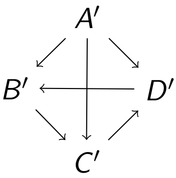
Quasi-transitive Oriented Graph

condensation to tournaments

- if $a, b \in V(Q)$ are in the same tie component, then $f(a) = f(b)$.
- if $a, b \in V(Q)$ are in 2 distinct tie components A, B respectively, then
  - $f(a) \to f(b)$ if $A \to B$.
  - $f(b) \to f(a)$ if $B \to A$.
  - otherwise, tie condensation do not exists.



(a) before condensation.



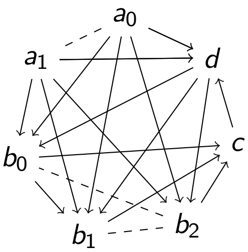
(b) after condensation.

Quasi-transitive Oriented Graph

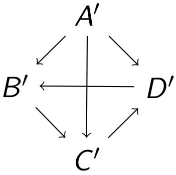
condensation to tournaments

Theorem

*Tie component condensation of given quasi-transitive oriented graph Q will always result in a unique tournament T, where T is called the **underlying tournament of** Q.*



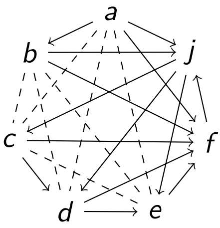
(a) quasi-transitive oriented graph Q.



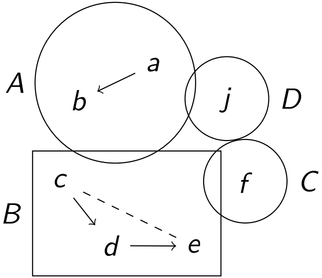
(b) underlying tournament of Q.

Quasi-transitive Oriented Graph

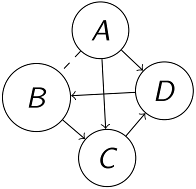
graph condensation



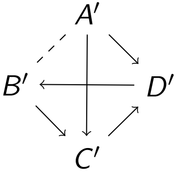
(a) an oriented graph G.



(b) components of G.



(c) relationship between components.



(d) condensed graph H.

Quasi-transitive Oriented Graph

graph condensation

Theorem

Given a quasi-transitive graph Q and its tie component condensation f, consider the set of all the condensations $f_k : G_k \to T_k$, where G_k has the same tie structure as Q and T_k is a tournament: $F = \{f_0, f_1, \ldots, f_{n-1}, f_n\}$. f is an efficient condensation in F.

Quasi-transitive Oriented Graph

property of kings

Theorem

If vertex k is a king in quasi-transitive oriented graph G, then k is adjacent to every other vertex in G.

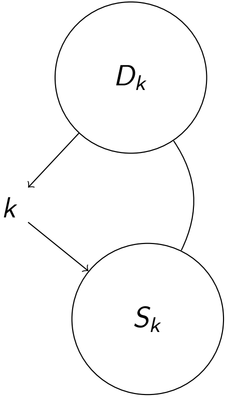


Figure: the rich structure of a king in a quasi-transitive oriented graph.

Quasi-transitive Oriented Graph

property of kings

Theorem

A vertex k is a king in a quasi-transitive oriented graph if and only if

- *k is in a trivial tie component.*
- *the result of k after tie component condensation is a king in the underlying tournament.*

Thank You

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References

Bang-Jensen, Jørgen and Jing Huang (Oct. 1, 1995). "Quasi-transitive digraphs". In: *Journal of Graph Theory* 20.2, pp. 141–161. ISSN: 1097-0118. DOI: 10.1002/jgt.3190200205. URL: http://onlinelibrary.wiley.com/doi/10.1002/jgt.3190200205/abstract (visited on 02/08/2018).

Maurer, Stephen B. (Mar. 1, 1980). "The King Chicken Theorems". In: *Mathematics Magazine* 53.2, p. 67. ISSN: 0025570X. DOI: 10.2307/2689952. URL: http://www.jstor.org/stable/10.2307/2689952?origin=crossref (visited on 01/27/2018).