Kings in Generalized Tournaments

BY

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Contents

Chapter 1

Abstract

This thesis explores how to find and construct kings in three generalizations of tournament: semi-compelete digraphs, oriented graphs and quasi-transitive oriented graphs.

In Chapter 5 and Chapter 6, We present a way to interpret semi-compelete digraphs and oriented graphs as tournaments with "ties" (we call the "ties" in semi-compelete digraphs "double ties", and the "ties" in oriented graphs "ties"). In Chapter 5, we prove there exists an (n, k) semi-complete digraphs if and only if $n \geq k \geq 1$, and all the (n, k) semi-compelete digraphs that exists can be constructed with at most 1 double tie. In Chapter 6, we prove there exists an (n, k) oriented for all $n \geq k \geq 0$ except $(1, 0)$, $(2, 2)$, $(3, 2)$, and $(4, 4)$ oriented graphs and all the (n, k) oriented graphs that exists can be constructed with at most 1 tie.

The main focus of this thesis is quasi-transitive oriented graph, which is discussed in Chapter 7. We showed a interesting fact that all the quasi-transitive oriented graphs can be condensed into tournaments by "tie component condensations". Then, we showed that the tie component condensation on a quasi-transitive oriented graph is a most efficient condensation to tournament in all the condensations to tournaments defined on all the oriented graph with the same tie structure. Finally we prove that the kings in quasi-transitive oriented graph Q is related to the kings in the "underlying tournament of Q" (result of Q after tie component condensation). This result gives us a way to understand the properties of kings in quasi-transitive oriented graphs using the properties of king in tournaments.

Chapter 2

Acknowledgment

Chapter 3

Introduction

3.1 Summary of Previous Works

In [6], the author introduced the idea of kings (or 2-kings) in tournaments, and proved that kings in tournaments have many nice properties. In the end of [6], the author proposed several ways to generalize kings, one of them is s-kings where s is a positive integer. When we say “king” in a graph theory context, we usually mean 2-king. In [7], the author further investigated the properties of kings in tournaments, and proved many interesting theorem about them.

With the more and more understanding for the properties of kings in tournaments, people begin to curious about generalized tournaments.

In [3], the author proposed a generalization of tournaments called quasi-transitive digraphs (a super set of quasi-transitive oriented graphs studied in Chapter 7). Then the same author investigated the properties of 3-kings in quasi-transitive digraph in [2]. And [5] generalized the results in [2] to $(k+1)$-kings and k-quasi-transitive digraphs.

In [1], the author surveys various types of generalized tournaments, including semi-complete digraphs, oriented graphs and quasi-transitive digraphs. However, this article did not focus on the properties of kings in these generalizations of tournaments.

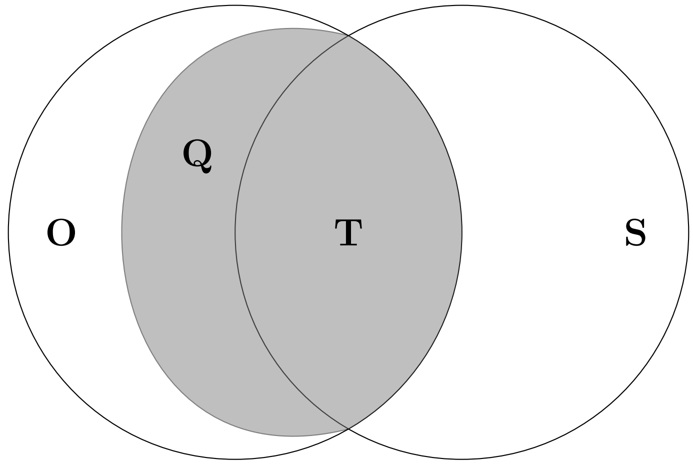


Figure 3.1: the relationship between tournaments, semi-complete digraphs, oriented graphs, and quasi-transitive oriented graphs.

3.2 Structure of This Thesis

The core content of this paper is in Chapter 4, Chapter 5, Chapter 6, and Chapter 7. The latter three chapters focused on kings in three generalization of tournaments: semi-complete digraph (see Definition 5.2), oriented graph (see Definition 4.14), and quasi-transitive oriented graph (see Definition 7.1), respectively.

It is helpful to know the relation of these three generalizations of tournaments. In Fig. 3.1, we show the relationship between these graphs. The "O" represents the set of all oriented graphs; the "Q" represents the set of all quasi-transitive oriented graphs; the "T" represents the set of all tournaments; the "S" represents the set of all semi-compelete digraphs. tournaments are a subset of all three generalized tournaments. Quasi-transitive oriented graphs are a subset of oriented graphs, and the intersection between oriented graphs and semi-compelete digraphs is exactly all the tournaments (hence the name "semi-compelete oriented graph", see Definition 4.15).

In Chapter 4, we discussed some of the terminology and previously-proven theorems that are used in other chapters. Chapter 5, Chapter 6, and Chapter 7 are all independent from each other. We recommend the readers to start with Chapter 4, and then choose the chapter that interest them the most to continue reading.

Chapter 5 and Chapter 6 are two relatively easy chapter, comparing to Chapter 7. These two chapters mainly focused on constructing (n, k) semi-complete digraphs and

(n, k) oriented graph (see Definition 4.18).

Chapter 7 is the most interesting chapter. Unfortunately, it is harder than Chapter 5 and Chapter 6. This chapter is proof heavy and moves at a faster pace than the previous two chapters. In this chapter we proved the four most important results in this thesis; namely Corollary 7.21, Corollary 7.22 (these 2 results can be merged into one), Theorem 7.24, and Theorem 7.29.

Chapter 4

Background

4.1 Directed Graph

Definition 4.1. *a **directed graph**, **digraph** or **graph** consists of a vertex set* $\boldsymbol{V(G)}$*, and an edge set (ordered pair of vertices)* $\boldsymbol{E(G)}$*.*

For example, in Fig. 4.1, if we call this digraph G, then the vertex set or $V(G)$ is $\{a, b, c, d, e\}$, and the edge set or $E(G)$ is $\{(a, a), (a, b), (b, c), (c, e), (e, a), (e, c)\}$. We sometimes refer to "edges" as "arrows".

4.2 Beating Relations

To simplify the notation, we think of an edge as a *beating relation*:

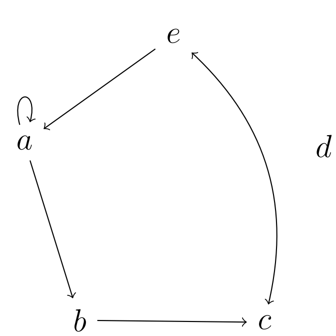


Figure 4.1: example of a directed graph.

Definition 4.2. *In a directed graph G, if $(a, b) \in E(G)$ then we say a* ***beats (dominates) b*** *or $\boldsymbol{a \to b}$.*

Definition 4.3. *A* ***path*** *or a* ***walk*** *from vertex $\boldsymbol{a_0}$ to vertex $\boldsymbol{a_n}$ is a sequence of vertices $[\boldsymbol{a_0, a_1, \ldots, a_{n-1}, a_n}]$ such that for all $\boldsymbol{0 \leq k < n,\ a_k \to a_{k+1}}$. We will sometimes write this path as $\boldsymbol{a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n}$.*

Definition 4.4. *For a given path P, if the sequence has $n+1$ vertices, then we say the* ***length of path $\boldsymbol{P}$*** *is $\boldsymbol{n}$.*

The length of a path measures the number of edges on this path. For example, in Fig. 4.1, there is a path $P = e \to a \to b \to c$. And there are edges $(e, a), (a, b), (b, c)$ on this path, therefore the length of this path is 3. There is another path $P' = e \to c$ goes from e to c. This path P' only has length 1. Therefore, path P' is shorter than path P, and if we go through every possible path from e to c, we can find out that P' is the *shortest path* from e to c.

Definition 4.5. *In a directed graph G,* ***a beats (dominates) b by n steps****, if the shortest path from $\boldsymbol{a}$ to $\boldsymbol{b}$ has length $\boldsymbol{n}$.*

In Fig. 4.1, $e \to a \to b$, and e does not beat b by 1 step, therefore e beats b by 2 steps.

However, although $e \to a \to b \to c$, e *does not* beat c by 3 steps, because the shortest path from e to c is $e \to c$. Therefore, e beats c by 1 step.

Definition 4.6. *In a directed graph, vertex a is* ***adjacent*** *to vertex $\boldsymbol{b}$ if $\boldsymbol{a \to b}$ or $\boldsymbol{b \to a}$ or both.*

Definition 4.7. *In a directed graph, for two distinct vertices a, b, if a is not adjacent to b, then a* ***ties b****.*

In Fig. 4.1, c is adjacent to e; a is adjacent to b; b is adjacent to c. Whereas, c ties a, because there is no edge (c, a) and no edge (a, c) in this digraph.

Since we don't always know the exact structure of the graph, it is sometimes useful to look at the beating relationships between sets of vertices. We then define beating, adjacency, and tie between vertex sets.

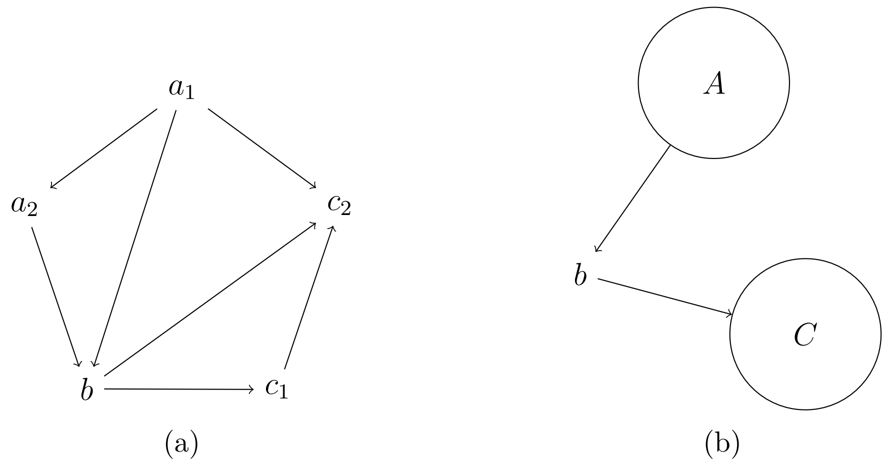


Figure 4.2: we can draw graph (a) as graph (b).

Definition 4.8. *In digraph G, we write $A \to B$ where A and B are disjoint subsets of $V(G)$, when every vertex in A beats every vertex in B. We also write $A \to b$ as a shorthand for $A \to \{b\}$, and $a \to B$ as a shorthand for $\{a\} \to B$.*

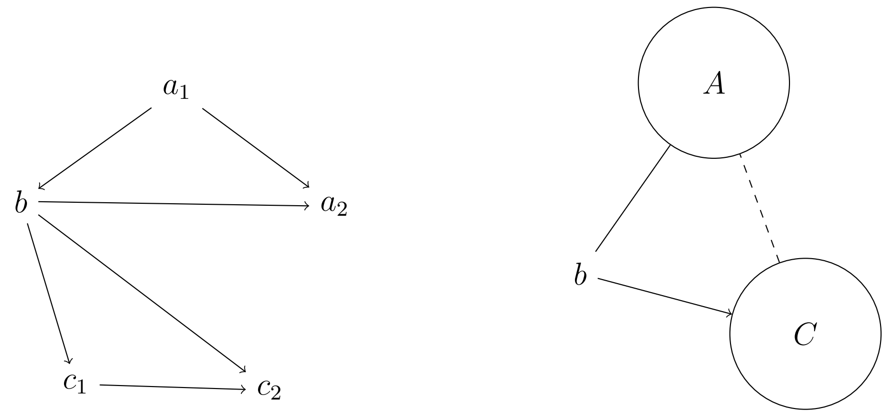
Definition 4.9. *In digraph G, A is adjacent to B where A and B are disjoint subsets of $V(G)$, when every vertex in A is adjacent to every vertex in B. We also write "A is adjacent to b" as a shorthand for "A is adjacent to $\{b\}$", and "a is adjacent to B" as a shorthand for "$\{a\}$ is adjacent to B".*

Definition 4.10. *In digraph G, A ties B where A and B are disjoint subsets of $V(G)$, when every vertex in A ties every vertex in B. We also write "A ties b" as a shorthand for "A ties $\{b\}$" and "a ties B" as a shorthand for "$\{a\}$ ties B"*

In Fig. 4.2, we show how we draw the beating relationship with subsets $A = \{a_1, a_2\}$ and $C = \{c_1, c_2\}$. Notice, in Fig. 4.2b, set A and set C are not adjacent. This *does not* means that set A ties set C; it only means that we have not drawn out the relationship between these 2 sets in Fig. 4.2b.

We draw adjacency between sets (for example, set A is adjacent to set $\{b\}$) using a solid edge without arrow. We draw tie between sets (for example, set A ties set C) using a dashed edge without arrow. (see Fig. 4.3)

Definition 4.11. *The **submissive set (dominant set)** of a vertices $\boldsymbol{v}$ in graph $\boldsymbol{G}$ is the set of vertices in $\boldsymbol{G}$ that are beaten by $\boldsymbol{v}$ (beat $\boldsymbol{v}$), formally it is $\boldsymbol{\{e \in V(G) \mid}$*



(a) (b)

Figure 4.3: we can draw graph (a) as graph (b).

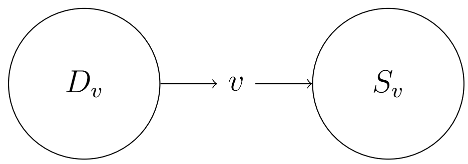


Figure 4.4: D_v is the dominant set of v; S_v is the submissive set of v

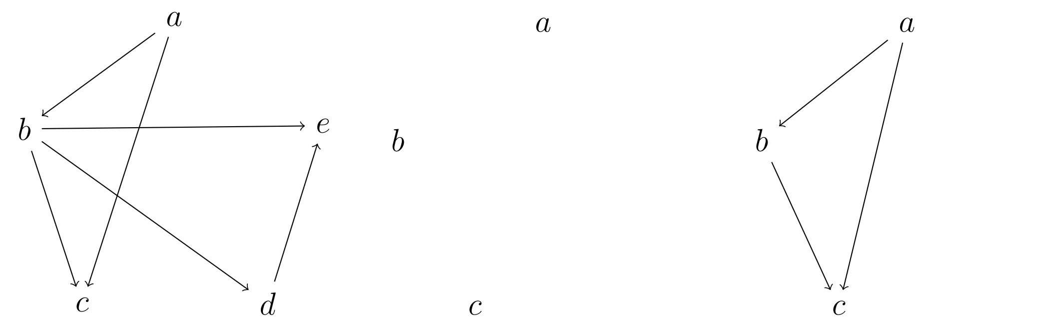
$\boldsymbol{v \to e\}}$ *(or* $\boldsymbol{\{e \in V(G) \mid e \to v\}}|$*). We will denote the submissive set of* $\boldsymbol{v}$ *as* $\boldsymbol{S_v}$*, and the dominant set of* $\boldsymbol{v}$ *as* $\boldsymbol{D_v}$*. (See Fig. 4.4)*

Definition 4.12. ***Out-degree*** *(**in-degree**) of a vertex* $\boldsymbol{v}$ *in graph* $\boldsymbol{G}$ *is the size of the submissive set (dominant set) of* $\boldsymbol{v}$*.*

For example, in Fig. 4.1, $S_a = \{a, b\}$, therefore vertex a has out-degree 2; $D_a = \{e, a\}$, therefore vertex a has in-degree 2. Vertex d has in-degree 0, and out-degree 0 because d is not adjacent to any other vertex in the graph. Therefore, D_d and S_d are both empty sets.

Notice it is very natural to relate out-degree and in-degree with the “power” or “strength” of a vertex. However, we will discuss in later sections and chapters that a vertex with larger out-degree does not necessarily have more power, by our definition of “king” (defined in Section 4.4), However, the property that “vertices with largest out-degree are kings” is true in some nice families of graphs (see Section 4.4 and Section 5.2).

Definition 4.13. *A* ***induced subgraph*** *or* ***vertex-induced subgraph H*** *of digraph*



(a) start with graph G. (b) let $V(H) = \{a, b, c\}$. (c) induced subgraph H.

Figure 4.5: the process to create an induced subgraph.

$\boldsymbol{G}$ is a subgraph of $\boldsymbol{G}$ such that: for all pairs of vertices $\boldsymbol{a, b} \in \boldsymbol{V(H)}$, if $\boldsymbol{(a, b)} \in \boldsymbol{E(G)}$, then $\boldsymbol{(a, b)} \in \boldsymbol{E(H)}$.

In Fig. 4.5, we show the process of creating an induced subgraph. We redraw the vertices in $V(H)$, that is a, b, c, and then simply copy the edges between those vertices, that is, $(a, b), (b, c), (a, c)$ from G to get H.

This definition can be viewed in 2 ways:

- Take the vertices a, b, c together with the edges between them to form subgraph H.
- Take away vertices e, d together with all the edges having e or d as an end point, the rest of the graph is the subgraph H.

For all the families of graphs discussed in this paper (oriented graph, semi-complete digraph, tournament,quasi-transitive oriented graph), induced subgraphs preserve the property of the original graphs.

4.3 Oriented Graph and Tournament

Definition 4.14. *An* ***oriented graph*** *is a digraph, such that:*

1. *for all vertices $\boldsymbol{a}$, $\boldsymbol{a}$ does not beat itself.*
2. *for all pairs of adjacent vertices $\boldsymbol{a, b}$, if $\boldsymbol{a}$ beats $\boldsymbol{b}$, then $\boldsymbol{b}$ does not beat $\boldsymbol{a}$.*

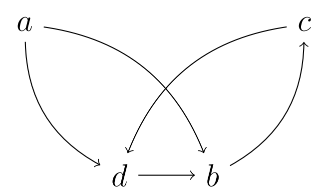


Figure 4.6: example of an oriented graph.

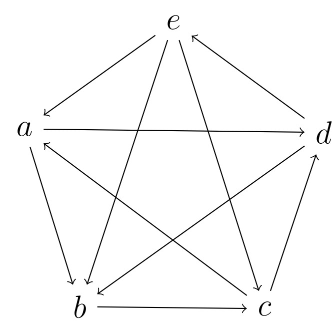


Figure 4.7: example of a tournament.

We show an oriented graph in Fig. 4.6. This definition of oriented graph can be viewed as a mathematical model of a round-robin tournament with ties allowed. A team cannot compete with itself, therefore property 1 holds; each pair of teams a and b competes exactly once, either a beats b, b beats a, or there is a tie between a and b. therefore property 2 holds.

Therefore, it is very natural to view an edge going from a to b as a "beats" b, and two non-adjacent vertices a, b as a tie between a and b.

Definition 4.15. *A* ***tournament*** *(sometimes called* ***semi-complete oriented graph****) is an oriented graph without ties.*

We give a simple example of a tournament in Fig. 4.7. A tournament can be interpreted as, what we call in the real world, a round-robin tournament, where every pair of vertices compete and the competition results in exactly one winner and one loser.

In [6], the author uses tournament to model a flock of chicken, between every two chickens there is a "pecking relation" (we call it the "beating relation"). Given two chickens, one chicken has to peck the other chicken, or be pecked by the other chicken, but not both.

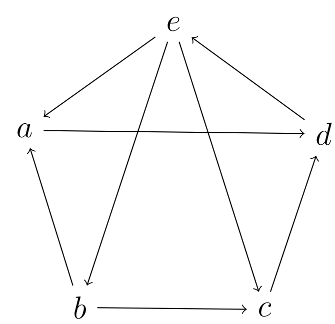


Figure 4.8: an example of kings.

4.4 Kings

Definition 4.16. *a **king** in a digraph is a vertex that beats every other vertex by 1 or 2 steps.*

In Fig. 4.8, the kings in this graph are vertices e and d. Vertex e beats a, b, c by one step, and beats d by 2 steps, so e is a king. Vertex d beats e by one step, and e beats a, b, c, so d beats a, b, c by 2 steps, which makes d a king.

The other vertices (a, b, c) are not kings, because a cannot beat c by one or two steps $(a \to d \to e \to c$ is a shortest path); b cannot beat e by one or two steps $(b \to c \to d \to e$ is a shortest path); c cannot beat b by one or two steps $(c \to d \to e \to b$ is a shortest path).

There are two points to note in this example:

- Fig. 4.8 shows that larger out-degree does not associate with more "power" or "strength". Vertex d only has out-degree 1, and d is a king. Whereas vertex b has out-degree 2, but b is not a king. Vertex d is more "powerful" than b, even though d has smaller out-degree. (trying to understand this phenomenon will be an interesting practice)

  - A vertex with the *highest* out-degree in a digraph may not be a king. In Fig. 4.9, vertex a has out-degree 3, which is the highest out-degree in this graph. However, it is not a king, since it cannot beat k by 1 or 2 steps

  - Also, a vertex with *lowest* out-degree in a digraph may be a king. In Fig. 4.9,

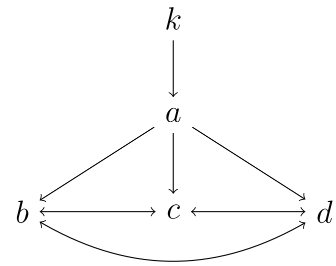


Figure 4.9: the vertex with smallest out-degree is the only king.

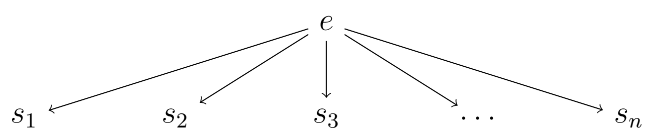


Figure 4.10: the vertex e is the emperor of this graph.

the vertex k has the smallest out-degree in the graph, however it is the only king in this graph.

- Fig. 4.8 shows that a digraph may have more than one king. In fact, [6] proved the probability that every vertex in a random tournament is a king approaches 1 as the number of vertices in the graph approaches ∞.

Definition 4.17. *An* ***emperor*** *in a digraph is a vertex that beats every other vertex. (see Fig. 4.10)*

Definition 4.18. *an* $\boldsymbol{(n,k)}$ ***digraph*** *(*$\boldsymbol{(n,k)}$ ***oriented graph****,* $\boldsymbol{(n,k)}$ ***tournament****, etc.) is a digraph (oriented graph, tournament, etc.) with* $\boldsymbol{n}$ *vertices and* $\boldsymbol{k}$ *kings.*

The following work has been done on kings in tournaments. These theorems will be useful when we investigate properties of kings in generalized tournaments in later chapters.

Theorem 4.1. *A tournament has only one king if and only if that king is an emperor. [6]*

Theorem 4.2. *A tournament cannot have exactly 2 kings. [6]*

Theorem 4.3. *In a tournament, a vertex with largest out-degree is a king. [6]*

Corollary 4.4. *In a tournament, there always exists at least 1 king. [6]*

Theorem 4.5. *for all integers* $n \geq k \geq 1$*,* (n, k)*-tournaments exist with the following exceptions:* $(n, 2)$ *with any* $n \geq 2$ *and* $(4, 4)$*. [6]*

Chapter 5

Semi-complete Digraph

5.1 Definitions

In Chapter 4, we introduced one way to model "tie" as the non-adjacency between 2 vertices. However, if we think of "beating" as a weak order (like "subset" relation), then we can define "double tie" to capture this idea:

Definition 5.1. *There exists a* ***double tie*** *between vertices* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *if* $\boldsymbol{a} \rightarrow \boldsymbol{b}$ *and* $\boldsymbol{b} \rightarrow \boldsymbol{a}$.

With the definition of double tie, we can then formalize another model of round-robin tournament with double ties. This kind of graph is called "semi-complete digraph".

Definition 5.2. *A* ***semi-complete digraph*** *is a digraph where every vertex is adjacent to every other vertex, but does not beat itself. see Fig. 5.1*

This definition says, that a semi-complete digraph is a digraph such that between each pair of vertices, there exists at least one edge between them.

Definition 5.3. *In a semi-complete digraph, if* $a \rightarrow b$ *and* b *does not beat* a*, then vertex* a ***strictly beats*** *vertex* $\boldsymbol{b}$.

Tournaments are a special case of semi-complete digraphs. In tournaments, for each pair of vertices a, b a always strictly beats b or b strictly beaten by b. We can view a

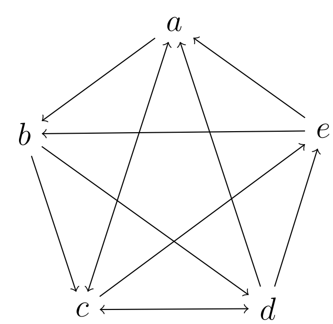


Figure 5.1: an example of semi-complete digraph.

tournament as a semi-complete digraph that only has strict beatings; no double tie allowed. In other words, tournaments is an oriented graph that is also a semi-complete digraph, hence the name "semi-complete oriented graph" (see Definition 4.15)

Definition 5.4. *In a semi-complete digraph, the **double tie set of vertex v** is the set of vertices that double ties **v**. We denote the double tie set of **v** as* $\boldsymbol{DT_v}$.

For example, in Fig. 5.1, there is a double tie between a and c, and another double tie between c and d. Every other beating relation between any other pair of vertices are "strict beating" relations. For example, a strictly beats b, d strictly beats e, and b strictly beats d.

For vertex c, DT_c (double tie set of c) is $\{a, d\}$. The set of vertices that strictly beat c, which can be expressed as $D_c - DT_c$, is $\{b\}$. The set of vertices that are strictly beaten by c, which can be expressed as $S_c - DT_c$, is $\{e\}$.

5.2 Properties

Although tournaments are a special case of semi-complete digraphs, many useful properties of tournaments are also true for semi-complete digraphs.

Lemma 5.1. *for every vertex v in any semi-complete digraph G, $\{S_v - DT_v, DT_v, D_v - DT_v, \{v\}\}$ forms a partition of G.*

Proof. $D_v - DT_v$ is the set of vertices that strictly beat v, $S_v - DT_v$ is the set of vertices that are strictly beaten by v, DT_v is all the vertices that double tie with v. They are

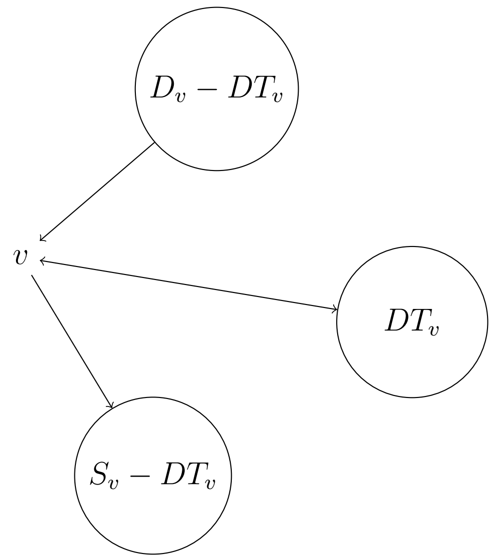


Figure 5.2: illustration of Lemma 5.1.

clearly disjoint by definition, and all of them are disjoint to $\{v\}$ by definition.

Every vertex in $V(G)$ needs to be in one of the sets $S_v - DT_v, DT_v, D_v - DT_v, \{v\}$, because every other vertex needs to be adjacent to v. □

See Fig. 5.2 for a visualization for the proof of Lemma 5.1.

Lemma 5.1 and Fig. 5.2 are useful in proofs of many properties of semi-complete digraphs.

When we move on to general oriented graphs, the lack of this property in oriented graphs will make the structure of oriented graphs harder to work with.

Theorem 5.2. *All vertices with the maximum out-degree in a semi-complete digraph are kings.*

Proof. Let v be a vertex with the maximum out-degree in a semi-complete digraph G.

Suppose v is not a king in G. Since v is not a king and v beats every vertex in S_v by exactly 1 step, then by Lemma 5.1, there exists a vertex $d \in D_v - DT_v = V(G) - S_v - \{v\}$ that is not beaten by v by 1 or 2 steps. Therefore there cannot be any vertex in S_v that beats d. Because d is adjacent to every vertex in S_v d strictly dominates v and S_v. Hence $|S_d| \geq 1 + |S_v|$. Therefore $|S_d| > |S_v|$, that is, d has larger out-degree than v. Contradiction. □

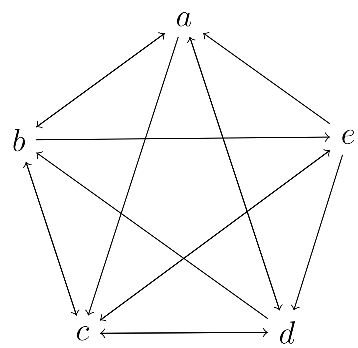


Figure 5.3: every vertex has the same out-degree.

Fig. 5.3 shows that there can be more than one vertex with maximum out-degree. In fact, in this graph, every vertex has the same out-degree 3, therefore all of the vertices have the maximum out-degree. By Theorem 5.2, all vertices in this graph are kings.

Corollary 5.3. *For any non-empty semi-complete digraph G, there exists at least one king.*

Proof. This corollary is a result of Theorem 5.2. because out-degrees are non-negative integers, and $V(G)$ is not empty. Thus there exists at least one vertex with maximum out degree. □

Theorem 5.4. *In a semi-complete digraph G, every vertex with a non-empty dominant set is beaten by a king.*

Proof. Let v be a vertex in G, such that D_v is not empty. Consider the subgraph induced by the vertices in D_v; this subgraph is also a semi-complete digraph. By Corollary 5.3, there is a king k in the induced subgraph of D_v. We will show that k is also a king of G.

- k dominates D_v by 1 or 2 steps, because k is a king in D_v.
- k dominates v by exactly 1 step, because k is in D_v.
- k dominates DT_v and S_v by 1 or 2 steps, because k dominates v, which beats all vertices in DT_v and S_v.

Then, by Lemma 5.1, we know that k dominates every vertex in the graph by 1 or 2 steps. Therefore k is a king in G and k beats v. □

a $\longleftrightarrow$ b

Figure 5.4: both vertices a and b are emperors and kings.

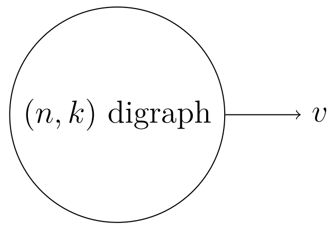


Figure 5.5: an $(n+1, k)$ semi-complete digraph by adding a new vertex v

Corollary 5.5. *If a semi-complete digraph has only one king, then that king is an emperor.*

Proof. Suppose G is a semi-complete digraph with only one king k, and k is not an emperor.

Then there exists v in the graph, such that k does not beats v. Therefore, $v \to k$, since there is no tie in the graph. Therefore, k has an non-empty dominate set. Then by Theorem 5.4, k is beaten by another king. Therefore there exists more than one king. Contradiction. □

Notice, unlike in tournaments, the converse of Corollary 5.5 is not true In Fig. 5.4, we show we can have more than one kings that are emperors.

Theorem 5.6. (n, k) *semi-complete digraphs exist for all* $n \geq k \geq 1$*, where* n, k *are integers.*

Proof. We prove this theorem by induction. Construct an (n, k) semi-complete digraph:

- When there is only one vertex in the graph then the graph is a $(1, 1)$ semi-complete digraph.
- See Fig. 5.5. We can add one vertex that is not a king by adding a vertex that is strictly beaten by all the vertices in the original graph. In other words, we can construct a $(n+1, k)$ semi-complete digraph from any (n, k) semi-complete digraph.

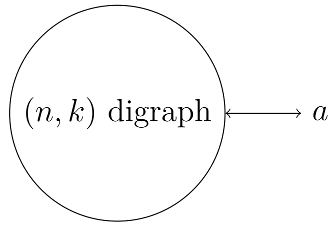
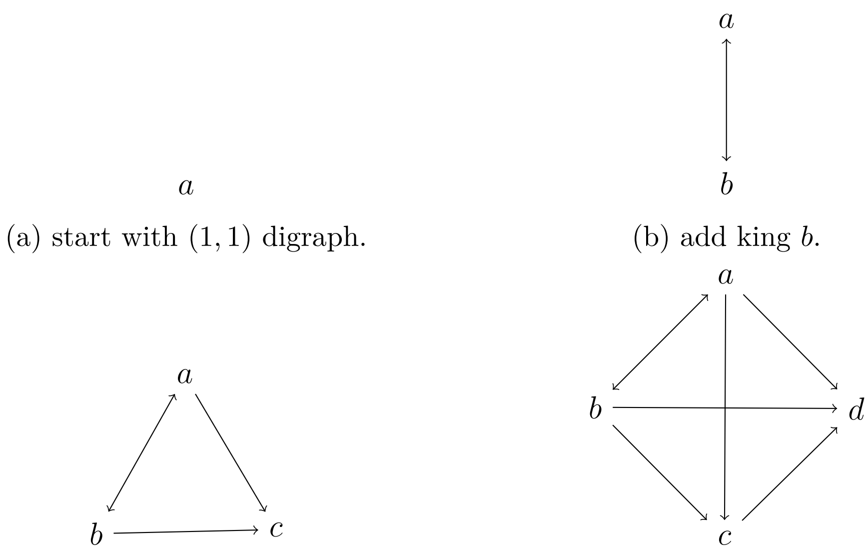


Figure 5.6: an $(n+1, k+1)$ semi-complete digraph by adding a new king a.



(a) start with $(1, 1)$ digraph.

(b) add king b.

(c) add non-king c.

(d) add non-king d.

Figure 5.7: to construct a $(4, 2)$ semi-complete digraph.

- See Fig. 5.6. We can add one king by adding one vertex that double ties all the vertices in the original graph. In other words, we can construct an $(n+1, k+1)$ semi-complete digraph from any (n, k) semi-complete digraph.

Therefore, we can obtain any (n, k) flock by: start with a $(1, 1)$ semi-complete digraph first add $k-1$ kings to get a (k, k) semi-complete digraph; then add $n-k$ non-king vertices to get an (n, k) semi-complete digraph. $\square$

In Fig. 5.7, we give an example of how to construct a $(4, 2)$ semi-complete digraph using the inductive algorithm we introduced in Theorem 5.6:

1. Start with a single vertex, which is a $(1, 1)$ semi-complete digraph.
2. Add a king b, by letting it double tie with a which gives us a $(2, 2)$ semi-complete digraph.

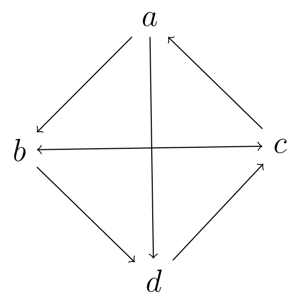


Figure 5.8: a $(4,4)$ semi-complete digraph with only one double tie

3. Add a non-king c, by letting every vertex in the $(2,2)$ semi-complete digraph beat c. which gives us a $(3,2)$ semi-complete digraph.
4. Add a non-king d, by letting every vertex in the $(3,2)$ semi-complete digraph beat d, which gives us a $(4,2)$ semi-complete digraph.

Using the method in Theorem 5.6, when we add the i_{th} king, we are creating $(i-1)$ ties, so adding k kings would require a total of $1+2+\cdots+(k-1)$ double ties. The summation can be simplified to $\frac{k(k-1)}{2}$. Thus, the number of double ties blows up with the increase of k (number of kings).

Therefore we want to investigate how to use double ties more "efficiently": what is the minimum number of double ties needed to get an (n,k) semi-complete digraph.

Lemma 5.7. *All $(n,2)$ semi-complete digraphs with $n \geq 2$ can be constructed with only one double tie.*

Proof. use the method described in Theorem 5.6.

First add 1 king to a $(1,1)$ semi-complete digraph to get a $(2,2)$ semi-complete digraph (one double tie added). Then add $(n-2)$ non-king vertices to get an $(n,2)$ semi-complete digraph (no double tie added).

Therefore we add only one double tie to construct any $(n,2)$ semi-complete digraph.

□

Lemma 5.8. *there exists a $(4,4)$ semi-complete digraph with only one double tie.*

Proof. Fig. 5.8 gives an example of a $(4,4)$ semi-complete digraph with only one double tie. The only double tie is between b and c. □

Theorem 5.9. *for all $n \geq k \geq 1$, there exists an (n, k) semi-complete digraph with at most one double tie.*

Proof. Recall Theorem 4.5, there exists an (n, k) tournament with the exception of $(n, 2)$ and $(4, 4)$. Because tournaments are also semi-complete digraphs, therefore we can use Theorem 4.5 along with Lemma 5.7 and Lemma 5.8 to get the result.

The proofs of Theorem 4.5, Lemma 5.7, and Lemma 5.8 all provide constructions for these semi-complete digraphs. Therefore we can construct an (n, k) semi-complete digraph with at most one double tie, for all $n \geq k \geq 1$. $\square$

Chapter 6

Oriented Graph

The properties of semi-complete digraphs basically inherit the properties of tournaments, and most of the proofs are almost the same.

We now move on to another family of graphs that are less similar to tournaments: oriented graphs. We will investigate general properties of oriented graphs in this chapter. In later Chapter 7, we will focus on specific families of oriented graphs.

Recall Definition 4.14, a oriented graph is a digraph that doesn't have self-loops (a vertex beats itself) and double ties.

Theorem 6.1. *If there is an emperor k in an oriented graph G, then k is the only king in the graph.*

Proof. The emperor k beats every vertex in one step, therefore it is a king.

Because there is no double ties in oriented graph, if k is an emperor, then no vertex in G can beats k, hence no vertex can beat k in one or two steps. Therefore, k is the only king in the graph. □

Corollary 6.2. *In an oriented graph with n vertices, if there exists a vertex with out-degree $n-1$, then there is only one king in the graph.*

Proof. A vertex with out-degree $n-1$ needs to beat every vertex in the graph except itself. Therefore that vertex is an emperor, and because of Theorem 6.1, there will be only one king in this graph. □

$a \quad b$

(a) 2 vertices with no edge.

$a \longrightarrow b$

(b) 2 vertices with 1 edge.

Figure 6.1: all the oriented graph with 2 vertices.

Lemma 6.3. *For a oriented graph G, if we add a new edge into G to get G', then G' will not have less kings than G.*

Proof. Need to show that for every king k in G, k is also a king in G'. We can see that for every pair of vertices a, b if $a \to b$ in G, then $a \to b$ in G', since we are not removing any edges.

- If k beats a vertex v by one step in G, then $k \to v$ in G, therefore $k \to v$ in G'
- If k beats a vertex v by 2 steps in G, then exists vertex a such that $k \to a \to v$ in G, then $k \to a \to v$ in G' and k beats v by 2 steps in G'.

Therefore, k is also a king in G', then every king in G is preserved in G'. Therefore, G' have less king than G. $\square$

Lemma 6.4. *There do not exists a $(1, 0)$ oriented graph.*

Proof. If any oriented graph only have 1 vertex, then that vertex by definition is a king. Therefore, there does not exist a $(1, 0)$ oriented graph. $\square$

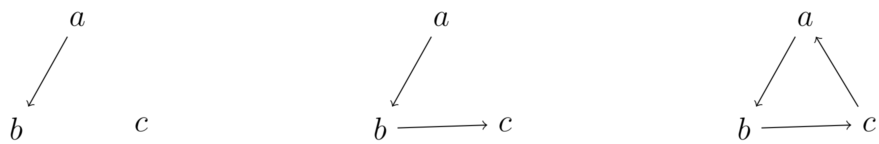
Lemma 6.5. *There does not exist a $(2, 2)$ oriented graph*

Proof. Fig. 6.1 shows all the possible oriented graphs with 2 vertices. Notice, the oriented graph in Fig. 6.1a has 0 king, and the oriented graph in Fig. 6.1b has only 1 king. Therefore, there does not exists a $(2, 2)$ oriented graph. $\square$

Lemma 6.6. *There does not exist a $(3, 2)$ oriented graph*

Proof. We consider all the oriented graph with 3 vertices.

First, consider the graph with maximum out-degree 0. The graph with maximum out-degree 0 will have no edge, therefore there cannot be a king, since no vertex can beats other vertices by 2 steps.



(a) 3 vertices with no edge. (b) 3 vertices with 1 edge. (c) 3 vertices with 3 edges.

Figure 6.2: all the oriented graph with 3 vertices and maximum out-degree 1.

Then, consider the graph with maximum out-degree 1. In Fig. 6.2, We show every oriented graph with 3 vertices and maximum out-degree 1. In Fig. 6.2a, there is no king; in Fig. 6.2b, there is 1 king; in Fig. 6.2c, every vertex is a king, therefore it has 3 kings.

Finally, we consider the oriented graph with maximum out-degree 2. By Corollary 6.2, because we have 3 vertex in the graph, and at least one vertex need to have degree 2, there there can only be 1 king.

Therefore, there is no $(3, 2)$ oriented graph. □

Lemma 6.7. *There does not exist a* $(4, 4)$ *oriented graph.*

Proof. Assume there exists a $(4, 4)$ oriented graph. When we add edges to this $(4, 4)$ oriented graph:

- the number of kings cannot increase, since a graph cannot have more kings than vertices.
- the number of kings cannot decrease, because of Lemma 6.3

Then we can keep adding edges until every pair of vertices are adjacent and get a $(4, 4)$ tournament. By Theorem 4.5, there do not exists $(4, 4)$ tournaments.

Therefore, there cannot exists $(4, 4)$ oriented graph. □

After investigating what kinds of oriented graphs do not exist, we then move on to all the possible oriented graphs that we can construct.

Lemma 6.8. *There exists an* $(n, 2)$ *oriented graph for* $n \geq 4$.

Proof. We can see in Fig. 6.3 that c_1 cannot dominate c_2 by 1 or 2 steps, and other $c_i (i \neq 1)$ cannot dominate any other vertex, because the out-degree of these vertices are 0.

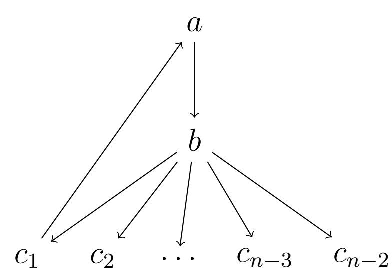


Figure 6.3: only a and b are kings for $n \geq 4$.

a is a king because $a \to b$ and b beats every c_i, therefore a beats b by one step, and a beats every c_i by 2 steps. b beats every c_i by 2 steps, and b beats a by 2 steps: $b \to c_1 \to a$. □

Lemma 6.9. *There exists $(n, 0)$ oriented graph for $n \geq 0$ except $n = 1$*

Proof. For a oriented graph with n vertices with no edge, every vertex cannot beat any other vertex. Therefore, the graph has 0 kings and the graph is a $(n, 0)$ oriented graph. □

Theorem 6.10. *There exists an (n, k) oriented graph for all $n \geq k \geq 0$, with the exception of $(1, 0)$, $(2, 2)$, $(3, 2)$, and $(4, 4)$ oriented graph.*

Proof. Theorem 4.5 shows that there exists an (n, k) tournament for all $n \geq k \geq 1$ with the exception of $(n, 2)$, and $(4, 4)$.

Because tournaments are also oriented graphs and by Lemma 6.5, Lemma 6.6, Lemma 6.4, Lemma 6.7, Lemma 6.8, and Lemma 6.9, we can conclude that the theorem is correct. □

We generalize the result from [6] on tournaments to oriented graphs and show that there are only 4 (n, k) oriented graphs that do not exists.

Following the idea from Chapter 5, one of the questions to ask is how can we use ties more "efficiently". The construction method in the proof of Lemma 6.8 is very inefficient.

Here we present a "better" way to construct these oriented graphs that only uses one tie.

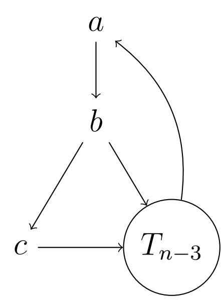


Figure 6.4: the constructive proof for Lemma 6.11

a b

Figure 6.5: $(2,0)$ oriented graph.

Lemma 6.11. *There exists an* $(n, 2)$ *oriented graph with only one tie, for* $n \geq 4$.

Proof. See Fig. 6.4, T_{n-3} is a tournament of $n-3$ vertices. In this graph, the only tie is between a and c and the only kings are a and b.

a is a king because, $a \to b \to c$ and $a \to b \to T_{n-3}$, therefore a beats b by 1 step and a beats c and T_{n-3} by 2 steps. b is a king because, $b \to c$ and $b \to T_{n-3} \to a$ (because T_{n-3} is not empty), therefore b beats c and T_{n-3}, and b beats a by 2 steps.

c is not a king, there is no path from c to b. Any vertex v in T_{n-3} cannot be a king, because the closest path between v and c is $v \to a \to b \to c$ which has length 3. □

Lemma 6.12. *There exists* $(n, 0)$ *oriented graph where* $n \neq 1$ *with at most 1 tie.*

Proof. First, we can see that $(0, 0)$ oriented graph exists, it is just an empty graph with no vertex and edge.

Then, in Fig. 6.5 we show that $(2, 0)$ oriented graph with one tie exists: it is just 2 vertices and no edge between them.

In Fig. 6.6, we give a way to construct an $(n+1, 0)$ oriented graph from an $(n, 0)$ oriented graph. We denote the $(n, 0)$ oriented graph as G.

We need to show the resulting graph in Fig. 6.6 has no king: first, s cannot be a king, because it does not beat any vertex. We then show any vertex $v \in V(G)$ is not a king. Because v is not a king in G, there exists a vertex $v' \in V(G)$ such that v cannot beat it in 1 or 2 steps in G. Then v still cannot beats v' in this graph, because s is the only added vertex, and the path $v \to s \to v'$ do not exists. □

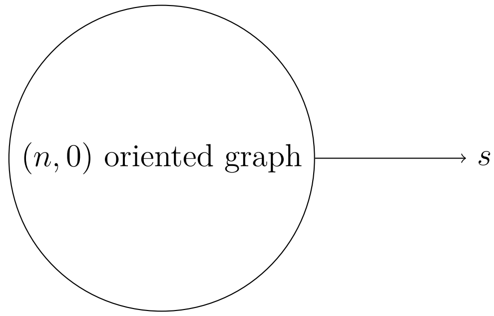


Figure 6.6: construct $(n+1, 0)$ oriented graph from $(n, 0)$ oriented graph.

With Lemma 6.11 and Lemma 6.12, we can prove the following theorem.

Theorem 6.13. *There exists an* (n, k) *oriented graph with at most one tie for all* $n \geq k \geq 0$*, with the exception of* $(1, 0)$*,* $(2, 2)$*,* $(3, 2)$*, and* $(4, 4)$ *oriented graphs.*

Proof. Almost the same proof as Theorem 6.10, just substitute Lemma 6.8, Lemma 6.9 with Lemma 6.11 and Lemma 6.12, respectively. □

in other words, Theorem 6.13 states that for all (n, k) oriented graphs that exists, they can be constructed with only one tie.

Chapter 7

Quasi-transitive Oriented Graph

7.1 Definitions

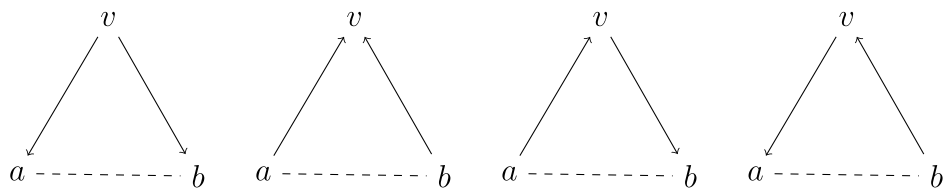
As mentioned in Chapter 6 In an oriented graph, for any pair of vertices a, b, either a beats b, b beats a or a ties b. So, there are $3^{\binom{n}{2}}$ different oriented graphs on n vertices. with no other restrictions, it is difficult to produce theorems about them. Therefore we decided to focus on another special case of oriented graph, which is called "quasi-transitive oriented graph":

Definition 7.1. *a **quasi-transitive oriented graph*** $\boldsymbol{G}$ *is an oriented graph such that, for all vertices* $\boldsymbol{a, b, c} \in \boldsymbol{V(G)}$*, if* $\boldsymbol{a} \rightarrow \boldsymbol{b} \rightarrow \boldsymbol{c}$*, then* $\boldsymbol{a}$ *is adjacent to* $\boldsymbol{c}$*. See Fig. 7.1*

Because every vertex is adjacent to every other vertex in a tournament, tournaments is a special case of quasi-transitive oriented graph.

Although tournaments are just a small subset of quasi-transitive oriented graphs,

Figure 7.1: if $a \rightarrow b \rightarrow c$ then a is adjacent to c.



(a) $v \to \{a, b\}$. (b) $\{a, b\} \to v$. (c) $a \to v \to b$. (d) $b \to v \to a$.

Figure 7.2: all the possible beating relations between vertex v and vertices a, b

and many quasi-transitive oriented graph are very different from tournaments (for example, every graph with no edge is a quasi-transitive oriented graph), quasi-transitive oriented graphs inherits many properties of tournaments [3].

7.2 Ties and Tie Paths

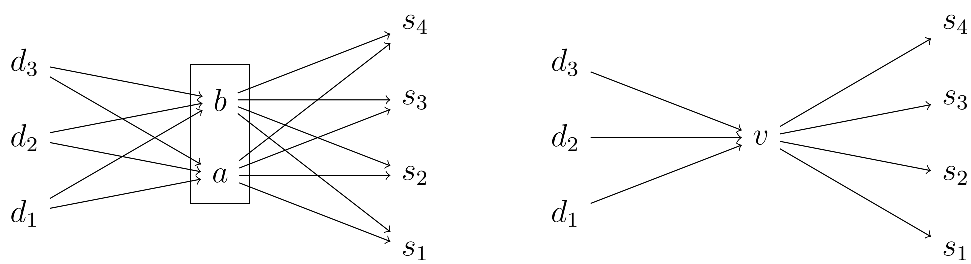
Properties of ties in quasi-transitive digraphs are vastly important in this chapter.

Lemma 7.1. *In a quasi-transitive oriented graph G, if vertex a ties vertex b, then for each vertex v that is adjacent to both a and b, either v beats both a and b, or v is beaten by both a and b ($\{a, b\} \to v$ or $v \to \{a, b\}$).*

Proof. Because v is adjacent to both a and b, we show all the possible beating relationships between v and a, b in Fig. 7.2. Because G is a quasi-transitive oriented graph, if $a \to v \to b$ or $b \to v \to a$ then a, b have to be adjacent. Therefore, only the possible beating relationships are $\{a, b\} \to v$ or $v \to \{a, b\}$. □

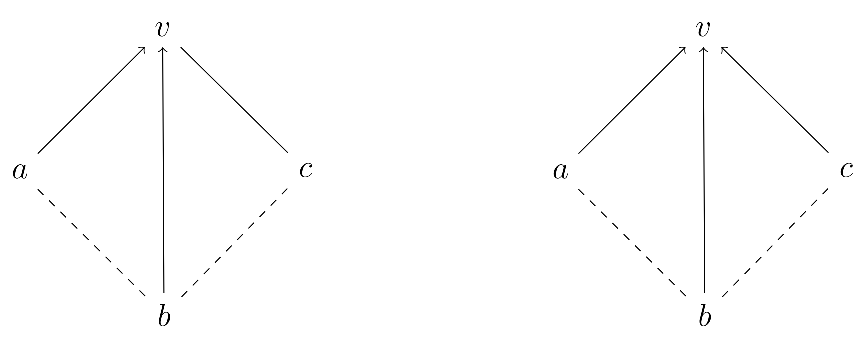
Fig. 7.3a shows a nice visualization of Lemma 7.1. Consider the set of vertices that are adjacent to both a and b. In this figure the set is $\{d_1, d_2, d_3, s_1, s_2, s_3, s_4\}$. This set is partitioned into 2 parts: those vertices that dominates both a and b $\{d_1, d_2, d_3\}$, and those vertices that are beaten by both a and b $\{s_1, s_2, s_3, s_4\}$ The behavior of $\{a, b\}$ is somewhat like a vertex, see Fig. 7.3b.

However, this nice visualization in Fig. 7.3 has a very strong prerequisite, that is, all the vertices need to be adjacent to both a and b. What will happen when we add another vertex that ties either a or b and is adjacent to v?



(a) all the vertices are adjacent to a, b. (b) the set $\{a, b\}$ behaves like v.

Figure 7.3: only look at all the vertices adjacent to both a and b, then $\{a, b\}$ behaves like a vertex.



(a) add a vertex that c ties b, but not v. (b) by Lemma 7.1, $c \to v$.

Figure 7.4: ties are “transmitting” arrow directions.

(a) a path from a to d. (b) a tie path from a to d.

Figure 7.5: tie paths are just like paths.

In Fig. 7.4a we add a vertex c that ties b, but needs to be adjacent to v (we use an edge without arrow to denote adjacency). Because b, c are both adjacent to v, and c ties b, therefore either $\{b, c\} \to v$ or $v \to \{b, c\}$. However, in the graph, $b \to v$, therefore $c \to v$. Thus, we get Fig. 7.4b.

We find out that because of the ties between a, b, c, the arrow direction between v, a and v, b, got "transmitted" to v, c via the Lemma 7.1. We formalize this "arrow transmission" idea:

Definition 7.2. *In a digraph, a* ***tie path*** *from vertex* $\boldsymbol{a_0}$ *to vertex* $\boldsymbol{a_n}$*, or a tie path between vertices* $\boldsymbol{a_0}$ *and* $\boldsymbol{a_n}$*, is a sequence of vertices* $[\boldsymbol{a_0, a_1, a_2, \ldots, a_{n-1}, a_n}]$*, such that for all* $\boldsymbol{0 \leq k < n}$*,* $\boldsymbol{a_k}$ *ties* $\boldsymbol{a_{k+1}}$*. If there is* $\boldsymbol{n+1}$ *vertices in that sequence, we will say* ***the length of the tie path*** *is* $\boldsymbol{n}$*. Note that all the* $\boldsymbol{a_i}$*'s do not have to be distinct.*

A tie path is similar to a path (called a "walk" in [8]) in an undirected graph; just substitute all the edges in a path with ties. See Fig. 7.5.

Tie path are useful in analyzing quasi-transitive oriented graphs:

Lemma 7.2. *In a digraph, if there exists a tie path from vertex a to vertex b and there exists a tie path from vertex b to vertex c, then there exists a tie path from a to c.*

Proof. Write the tie path from a to b as: $[a, a_0, a_1, \ldots, a_n, b]$, and the tie path from b to c as: $[b, b_0, b_1, \ldots, b_m, c]$. Then there exists a tie path $[a, a_0, a_1, \ldots, a_n, b, b_0, b_1, \ldots, b_m, c]$ from a to c. See Fig. 7.6. $\square$

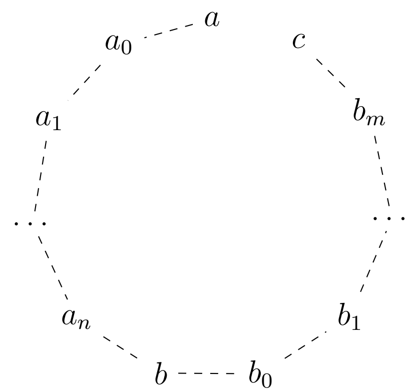


Figure 7.6: tie path from a to b, and from b to c.

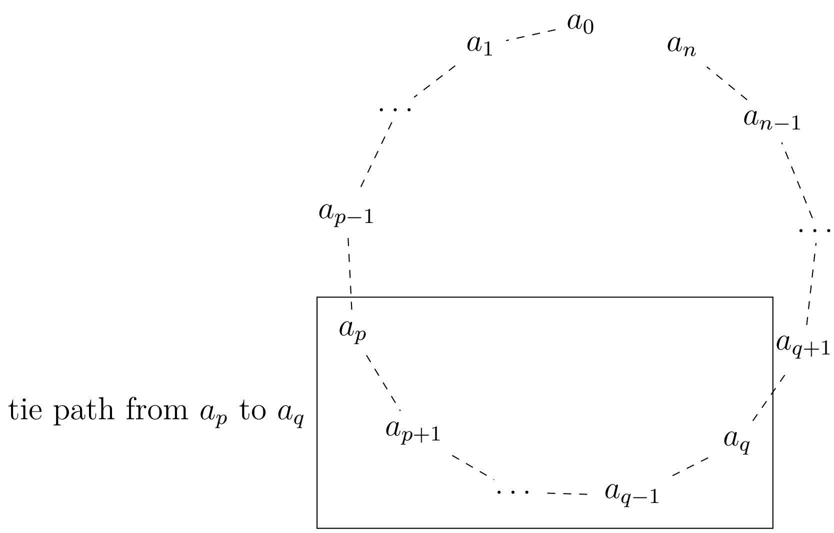


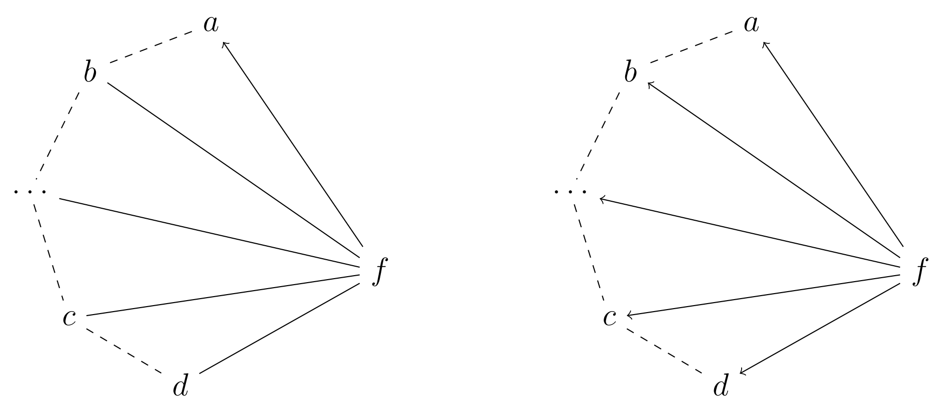
Figure 7.7: the tie path from a_0 to a_n.

Lemma 7.3. *In a digraph, if there exists a tie path* $[a_0, a_1, \dots, a_{n-1}, a_n]$*, then for all* $p \neq q$ $0 \leq p \leq n$ *and* $0 \leq q \leq n$*, there exists a tie path between* p *and* q*.*

Proof. Without loss of generality, assume $p < q$. Then we can find a tie path $[a_p, a_{p+1}, \dots, a_{q-1}, a_q]$, see Fig. 7.7. □

After looking at Fig. 7.4, we can hypothesize that tie paths can transmit arrow directions across the whole path in quasi-transitive oriented graph. See Fig. 7.8.

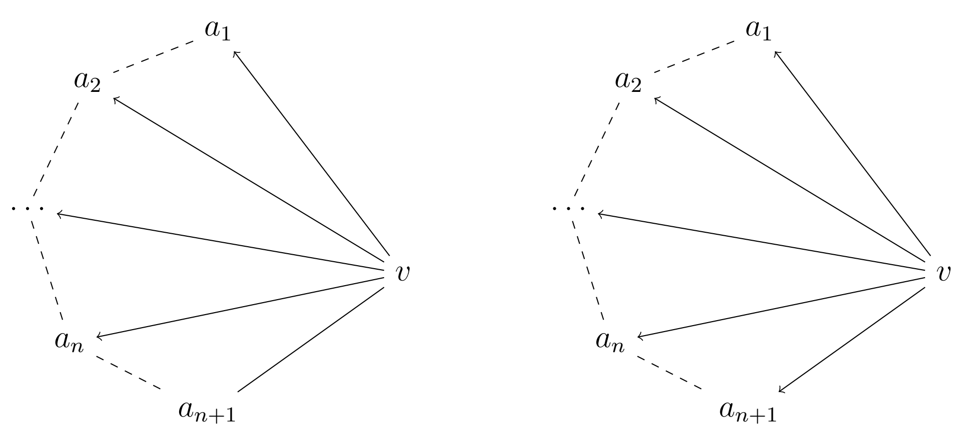
Lemma 7.4. *For every tie path in a quasi-transitive oriented graph, if vertex* v *is adjacent to all the vertices on the tie path, then* v *beats every vertex on the tie path, or* v *is beaten by every vertex on the tie path.*



(a) a is beaten by f, and other vertices adjacent to f

(b) the arrow direction is "transmitted" through the tie path

Figure 7.8: tie path transmits the arrow direction.



(a) a_{n+1} is adjacent to v.

(b) then a_{n+1} has to be beaten by v.

Figure 7.9: the induction step of the proof of Lemma 7.4

Proof. We can prove this lemma via induction, follow the intuition in Fig. 7.4.

Start with a tie path of length 1, then we have a_0 ties a_1 in this tie path, and v is adjacent to both of them. Lemma 7.1 proves the result for path of length 1.

Assume the property holds for any tie path of length n and vertex v, then we need to prove this property holds for any tie path of length $n+1$. See Fig. 7.9, take a tie path $[a_0, a_1, \ldots, a_n, a_{n+1}]$. we can apply the induction hypothesis to the tie path $[a_0, a_1, \ldots, a_n]$. Because a_n ties a_{n+1}, and v is adjacent to both of them. By Lemma 7.1

- Case 1, $v \to \{a_0, \ldots, a_n\}$: then $v \to a_{n+1}$, then v beats the whole tie path;
- Case 2, $\{a_0, \ldots, a_n\} \to v$, then $a_{n+1} \to v$, then v is beaten by the whole tie path.

Therefore this lemma still holds for tie path of length $n+1$. $\square$

Lemma 7.4 generalizes Lemma 7.1, however, it is still far from elegant, because we still requires the vertex v to be adjacent to the whole tie path.

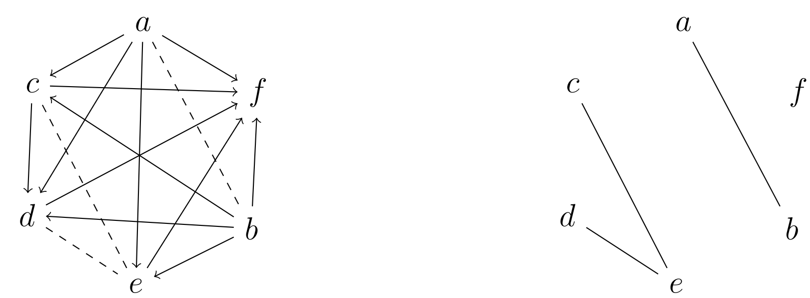
7.3 Tie Components

We continue to explore the relationship between tie paths and arrow direction transmitted by ties. We to define the following structure with the intuition of "path" and "connected component" in undirected graph[8].

Definition 7.3. *In a digraph G, a **tie component of vertex a: $C(a)$**, all vertices v such that there exists a tie path between a and v. a is called a **representitive** of $C(a)$*

In Fig. 7.10, we first change every tie in Fig. 7.10a to an undirected edge and then remove all the directed edges to obtain Fig. 7.10b, and tie components in Fig. 7.10a exactly correspond to connected components in Fig. 7.10b. Therefore, just like a tie path is similar to a path in an undirected graph, a tie component is very similar to a connected component (called a "component" in [8]) in an undirected graph, and we can bring all the properties and intuitions of connected components to tie components.

We prove results about tie components in digraphs. Notice that theses results work in all digraphs, not just in quasi-transitive oriented graphs.



(a) distinct tie components are $\{a, b\}, \{c, d, e\}, \{f\}$

(b) connected components are $\{a, b\}, \{c, d, e\}, \{f\}$

Figure 7.10: the tie components in (a) are the connected components in (b).

Theorem 7.5. *For a digraph G, we define a relation R on $V(G)$ by for vertices $a, b \in V(G)$, $a\ R\ b$ if there exists a tie path between a, b. R is an equivalence relation.*

Proof. Reflexive: given a vertex a, $[a]$ is a tie path of length 0. therefore there exists a tie path from a to itself.

Symmetry: if there exists a tie path from a_0 to a_n: $[a_0, a_1, a_2, \ldots, a_n]$, then because tie do not have directions, $[a_n, a_{n-1}, \ldots, a_1, a_0]$ is a tie path from a_n to a_0.

Transitive: by Lemma 7.2. □

Corollary 7.6. *For a digraph G, tie component is an equivalence class on $V(G)$.*

Proof. By Theorem 7.5, and the definition of tie component. □

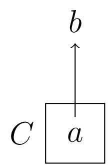
Corollary 7.7. *For digraph G, tie components form a unique partition of $V(G)$.*

Proof. By Corollary 7.6, tie components form an equivalence class on $V(G)$, and equivalence classes form a unique partition of the set $V(G)$ [4]. □

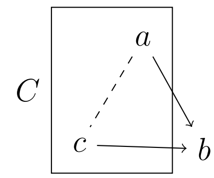
Corollary 7.8. *In a digraph, if vertex v is not in tie component $C(a)$, then v is adjacent to $C(a)$.*

Proof. Because $C(a)$ is an equivalence class on the relation R such that $a\ R\ b$ if there exists a tie path between a and b. Because v not in the equivalence class $C(a)$, for every vertex $a' \in C(a)$, $v\ R\ a'$ is false [4]. Therefore there is no tie path between v and a, thus v does not tie a'.

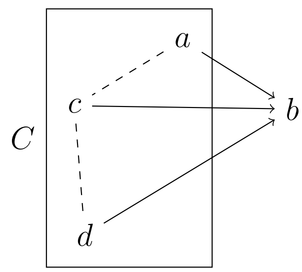
Therefore, v is adjacent to every vertex in $C(a)$. □



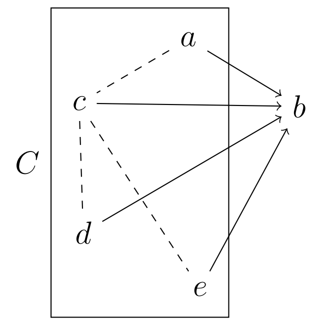
(a) a beats b.



(b) direction transmitted from a to c.



(c) transmitted from a to c to d.



(d) transmitted from a to c to e.

Figure 7.11: the arrow direction to b was transmitted in tie component C.

Our previous results that ties "transmit" arrow directions in quasi-transitive oriented graph, had us see that for a tie component C and a vertex $v \notin C$, $C \to v$ or $v \to C$. See Fig. 7.11, for a visualization.

We then formalize and generalize this idea into two theorems:

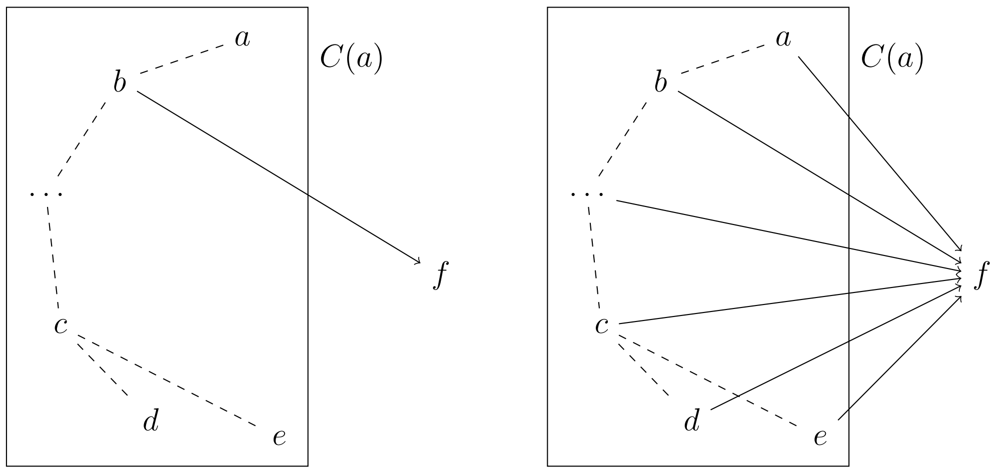
Lemma 7.9. *In a quasi-transitive oriented graph, for any tie component $C(a)$ and any vertex v such that $v \notin C(a)$, then either $C(a) \to v$ or $v \to C(a)$.*

Proof. By Corollary 7.8, we know that v is adjacent to $C(a)$. Let x, y be two vertices in $C(a)$. There exists a tie path between them, so both x and y beats v or v beats both x, y. Since this is true for all pairs of vertices on $C(a)$, we get $v \to C(a)$ or $C(a) \to v$. $\square$

Corollary 7.10. *In a quasi-transitive oriented graph, for any tie component $C(a)$, If there exists one vertex $v \in C(a)$ that beats (be beaten) a vertex $v' \notin C(a)$. Then $C(a) \to v'$ ($v' \to C(a)$). See Fig. 7.12*

Proof. Because $v' \notin C(a)$, then either $v' \to C(a)$ or $C(a) \to v'$ by Lemma 7.9.

Case 1, $v \to v'$: because $v \in C(a)$, then v' cannot beat all of $C(a)$. Therefore $C(a) \to v'$.



(a) a single vertex in $C(a)$ beats f.

(b) component $C(a)$ has to beat f.

Figure 7.12: a vertex b beats f will force the component $C(a)$ to beats f.

Case 2, $v' \to v$: because $v \in C(a)$, then v' cannot be beaten by all of $C(a)$. Therefore $v' \to C(a)$. □

To further generalize Lemma 7.9, we prove the following theorem:

Theorem 7.11. *For any two distinct tie components $C(a)$ and $C(b)$ in a quasi-transitive oriented graph, $C(a) \to C(b)$ or $C(b) \to C(a)$.*

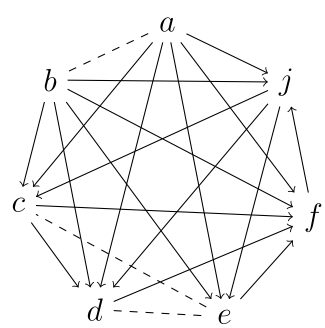
Proof. Take any vertex a' in $C(a)$. By Corollary 7.7, a' is not in $C(b)$. Then because of Lemma 7.9, $a' \to C(b)$ or $C(b) \to a'$.

- Case 1, $a' \to C(b)$: take any element $b' \in C(b)$. By Corollary 7.10, $C(a) \to b'$. Therefore $C(a)$ beats every vertex in $C(b)$, then $C(a) \to C(b)$.
- Case 2, $C(b) \to a'$: by the same reasoning, $C(b) \to C(a)$.

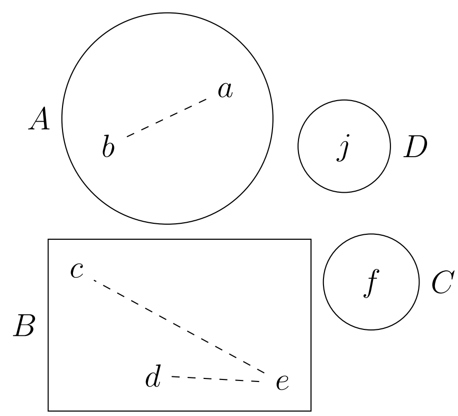
□

After several pages of theorems, we can finally take a pause and understand what we are saying here. We combine Corollary 7.7 and Theorem 7.11, and try to understand it visually.

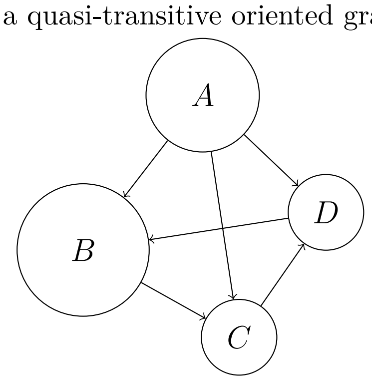
See Fig. 7.13. For every quasi-transitive digraph, we can split it into tie components, and every tie component either beats another tie component or is beaten by another tie component. Then, tie components behave like vertices. Because there are no ties



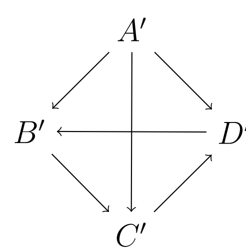
(a) a quasi-transitive oriented graph.



(b) find its tie components.



(c) components beat each other.



(d) components are just like vertices.

Figure 7.13: condense the tie components and get a tournament.

between tie components, The graph formed by tie components is a tournament, which is the "most well understood class of directed graphs" [1].

7.4 Graph Condensations

Definition 7.4. *A **condensation** is a function $\boldsymbol{f} : \boldsymbol{G} \to \boldsymbol{H}$, where $\boldsymbol{G}$ and $\boldsymbol{H}$ are oriented graphs, and $\boldsymbol{f}$ maps $\boldsymbol{V(G)}$ to $\boldsymbol{V(H)}$ surjectively, such that*

- *for any two vertices $\boldsymbol{a, b} \in \boldsymbol{V(G)}$, if $\boldsymbol{f(a)} \to \boldsymbol{f(b)}$ in $\boldsymbol{H}$ then $\boldsymbol{a} \to \boldsymbol{b}$ in $\boldsymbol{G}$.*
- *for any two vertices $\boldsymbol{a, b} \in \boldsymbol{V(G)}$, if $\boldsymbol{f(a)}$ ties $\boldsymbol{f(b)}$ in $\boldsymbol{H}$ then $\boldsymbol{a}$ ties $\boldsymbol{b}$ in $\boldsymbol{G}$.*

*We call $\boldsymbol{G}$ the **uncondensed graph**, and $\boldsymbol{H}$ the **condensed graph**.*

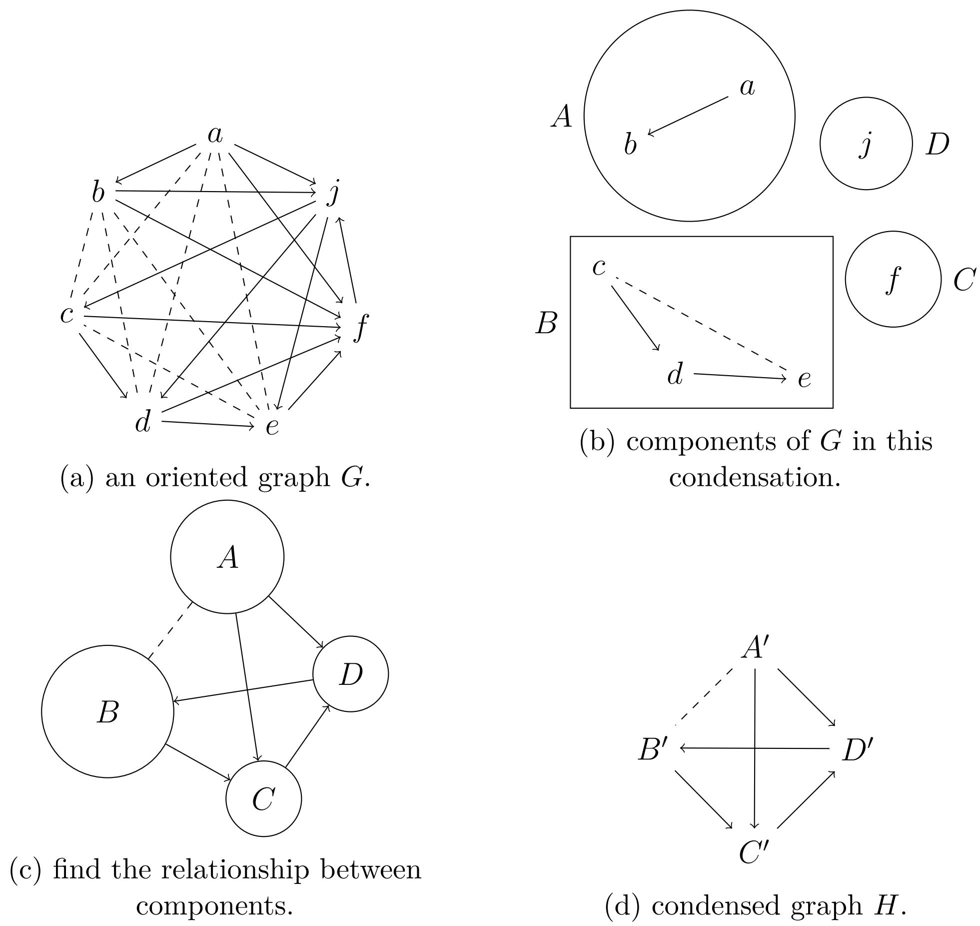
Two examples of condensations are identity condensation and trivial condensation.

Identity condensation i maps an oriented graph G to itself, such that i maps every vertex to itself, and $f(a) \to f(b)$ if and only if $a \to b$. The identity condensation preserves all the edges and ties, and do not change the graph at all. We can see that this map is a condensation by definition.

Another example is trivial condensation t, which maps any oriented graph G into a single point. Because for any two vertices $a, b \in G$, $t(a) = t(b)$, therefore by definition, $t(a)$ does not tie or beat $t(b)$. Therefore, this mapping also satisfies the definition of condensation.

Definition 7.5. *Given oriented graph G as the uncondensed graph of condensation f, A **component** of vertex $\boldsymbol{a} \in \boldsymbol{V(G)}$ is the pre-image of $\boldsymbol{f(a)}$, in other words, the component of $\boldsymbol{a}$ is $\boldsymbol{\{x \in V(G) \mid f(x) = f(a)\}}$.*

A component of vertex a is a set of vertices that are condensed into the same vertex as a. For example, the component of vertex a in oriented graph G with identity condensation is a set that only contains a itself, and the component of a vertex a in oriented graph G with the trivial condensation is set of all the vertices, because all vertices in G are mapped onto a single vertex in H, therefore all the vertices in G are condensed into the same vertex as a.



(a) an oriented graph G.

(b) components of G in this condensation.

(c) find the relationship between components.

(d) condensed graph H.

Figure 7.14: an example of graph condensation.

In Fig. 7.14, we show a general example of condensation. In this example, the distinct components in G are A, B, C and D, and they are condensed into vertices A', B', C' and D' respectively. The component of a is A, the component of d is B, and the component of c is also B, because both c and d are mapped into vertex B' in oriented graph H.

We proof several theorems involving condensation and components to help us with our goal of identifying kings in quasi-transitive oriented graphs.

Theorem 7.12. *Given any oriented graph G and condensation f, distinct components in G partition $V(G)$.*

Proof. Prove distinct component disjoint: Assume there exists two vertex a and b, such that the component of a is A; component of b is B, and $A \neq B$; $A \cap B \neq \emptyset$. Therefore, there exists $v \in A \cap B$. By definition of components, $f(v) = f(a)$, and $f(v) = f(b)$, therefore $f(b) = f(a)$, and $B = \{x \in V(G) \mid f(x) = f(b)\} = \{x \in V(G) \mid f(x) = f(a)\} = A$. $A = B$, contradiction.

Prove every vertex is in a component: for every vertex $v \in V(G)$, v is in the component of v because $f(v) = f(v)$. □

Lemma 7.13. *Given any oriented graph G and condensation f, for any two distinct components A and B,*

- *for all $a \in A, b \in B$, if a ties b, then $f(a)$ ties $f(b)$.*
- *for all $a \in A, b \in B$, if $a \to b$, then $f(a) \to f(b)$*

Proof. Because A and B are distinct components, then by Theorem 7.12, A and B are disjoint, therefore for any vertex $a \in A$, and any vertex $b \in B$, $f(a) \neq f(b)$.

Given vertex $a \in A$ ties vertex $b \in B$. Then $f(a)$ cannot beat $f(b)$, otherwise a should beat b, by definition of condensation. Also, $f(b)$ cannot beat $f(a)$, otherwise b should beat a, therefore $f(a)$ ties $f(b)$.

Given vertex $a \in A$ beats vertex $b \in B$. Then $f(a)$ cannot tie $f(b)$, otherwise a should tie b Also $f(b)$ cannot beat $f(a)$, otherwise b should beat a. Therefore $f(a) \to f(b)$. □

Theorem 7.14. *Given any oriented graph G and condensation f, for any two distinct components A and B,*

- *if any vertex $a \in A$ ties any vertex $b \in B$, then A ties B.*
- *if any vertex $a \in A$ beats any vertex $b \in B$, then $A \to B$.*

Proof. Because A and B are distinct components, then by Theorem 7.12, A and B are disjoint, therefore for any vertex $a \in A$, and any vertex $b \in B$, $f(a) \neq f(b)$.

Given vertex $a \in A$ ties vertex $b \in B$, by Lemma 7.13, $f(a)$ ties $f(b)$. Thus, for every vertex $a' \in A$, and every vertex $b' \in B$, because $f(a') = f(a)$ and $f(b') = f(b)$, and $f(a)$ ties $f(b)$, therefore $f(a')$ ties $f(b')$. By the definition of condensation, a' ties b'. Therefore, A ties B.

Given vertex $a \in A$ beats vertex $b \in B$. by Lemma 7.13, $f(a) \to f(b)$. Thus for every vertex $a' \in A$, and every vertex $b' \in B$, because $f(a') = f(a)$ and $f(b') = f(b)$, and $f(a) \to f(b)$, therefore $f(a') \to f(b')$. By the definition of condensation, $a' \to b'$. Therefore, A ties B. ☐

Corollary 7.15. *Given any oriented graph G and condensation f, for any two distinct components A and B, either $A \to B$, or $B \to A$, or A ties B.*

Proof. Because components are not empty (component of v always contains v itself), we can take $a \in A$, and $b \in B$. Because G is an oriented graph, then either $a \to b$, $b \to a$, or a ties b.

According to Theorem 7.14, if $a \to b$, then $A \to B$; if $b \to a$, then $B \to A$; if a ties b, then A ties B. ☐

In Fig. 7.15, we show an example of a set of vertices $\{b, c, d\}$ that behaves like the vertex v. We then try to visualize Corollary 7.15, with the help of Fig. 7.14, we find out that graph condensation just takes the idea in Fig. 7.15 to the next level: all the components behave like vertices. A condensation looks for those sets of vertices that behave like a single vertex (components), and then condenses them into a single vertex.

Notice that all oriented graph have the identity condensation and trivial condensation, but some oriented graphs may not have other condensations defined on them.

(a) every vertex either beats $\{b, c, d\}$ or is beaten by them.

(b) vertices $\{b, c, d\}$ in Fig. 7.15a behaves exactly like v in this graph.

Figure 7.15: $\{b, c, d\}$ behaves like a vertex.

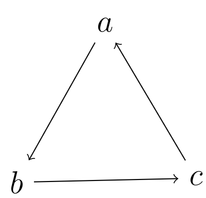


Figure 7.16: an oriented graph with only identity condensation and trivial condensation.

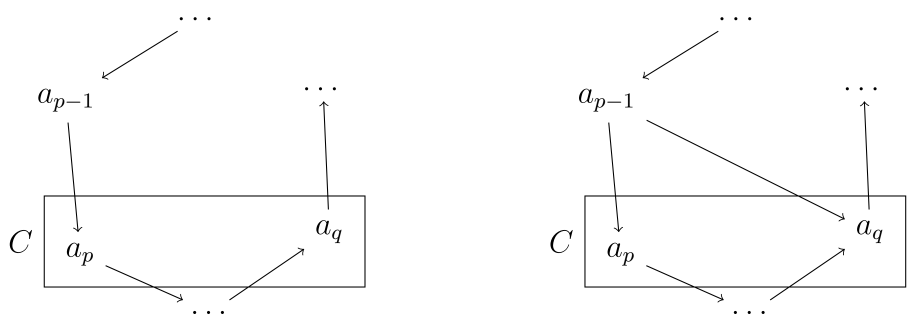
For example, the graph in Fig. 7.16 cannot have any condensation defined on it except for identity condensation and trivial condensation, because we cannot find any component that is not the whole graph or a single vertex. For example, let's look at set $\{a, c\}$. Vertex b beats a vertex in the set (b beats a), and is beaten by a vertex in the set (c is beaten by b). Therefore the set $\{a, c\}$ is not a component.

Condensation is a relatively strong transformation, because it preserves many properties of the graph. We will prove some theorems that will be useful in later sections, when looking for kings

Lemma 7.16. *Let $f : G \to H$ be a condensation of oriented graph G and let a_0, a_n be two vertices in G from different components. If $P : a_0 \to a_1 \to \cdots \to a_n$ is a shortest path from a_0 to a_n, then vertices $a_0, a_1, \ldots, a_n$ are all in different components.*

Proof. Suppose not. Take the the smallest p such that there exists a_p and $a_q (p < q \leq n)$ are in the same component C.

Since a_p is the first vertex in the P that belongs to the same component as another vertex in P. a_{p-1} cannot be in the same component as a_p, then $a_{p-1} \notin C$. Because $a_{p-1} \to a_p$ and Theorem 7.14, $a_{p-1} \to C$. Thus $a_{p-1} \to a_q$, and path $a_0 \to a_1 \to \cdots \to$



(a) a_p and a_q are in the same component. (b) by Theorem 7.14, $a_{p-1} \to a_q$.

Figure 7.17: the path took a "detour" at a_{p-1}.

$a_{p-2} \to a_{p-1} \to a_q \to a_{q+1} \to \cdots \to a_{n-1} \to a_n$ exists and shorter than P. See Fig. 7.17. Contradiction. $\square$

Theorem 7.17. *Given a condensation $f : G \to H$, for any $a_0, a_n \in V(G)$, such that a_0 and a_n are in two distinct components. A shortest path from a_0 to a_n is $P_G : a_0 \to a_1 \to \cdots \to a_{n-1} \to a_n$ in G if and only if a shortest path from $P_H : f(a_0)$ to $f(a_n)$ is $f(a_0) \to f(a_1) \to \cdots \to f(a_{n-1}) \to f(a_n)$ in H.*

Proof. Prove ($\Rightarrow$): Suppose P_G is a shortest path but P_H is not a shortest path in H. Then there exists a shorter path P'_H in H from $f(a_0)$ to $f(a_n)$: $f(a_0) \to f(b_1) \to f(b_2) \to \cdots \to f(b_{m-1}) \to f(b_m) \to f(a_n)$ $m < n - 1$. Because a_0 and a_n are in distinct components, the path P'_H does not have length 0. The path $a_0 \to b_1 \to b_2 \to \cdots \to b_{m-1} \to b_m \to a_n$ exists (by definition of condensation) and is shorter than P_G, a contradiction.

Prove ($\Leftarrow$): Suppose P_H is a shortest path in H but P_G is not a shortest path in G Then there exists a shorter path P'_G in G from a_0 to a_n: $a_0 \to b_1 \to b_2 \to \cdots \to b_{m-1} \to b_m \to a_n$, $m < n - 1$. Because of Lemma 7.16, $a_0, b_1, b_2, \ldots, b_m, a_n$ are all in different components. Then by Lemma 7.13, $f(a_0) \to f(b_1) \to f(b_2) \to \cdots \to f(b_m) \to f(a_n)$ exists and is shorter than P_H, contradiction. $\square$

Corollary 7.18. *Let $f : G \to H$ be a condensation, for all $a_0, a_n \in G$, such that a_0 and a_n are in two distinct components, a_0 beats a_n by n steps in G if and only if $f(a_0)$ beats $f(a_n)$ by n steps in H.*

Proof. Result of Theorem 7.17 and definition of "beats by n steps". $\square$

Corollary 7.19. *Given a condensation $f : G \to H$, vertex k is a king in G if and only if*

- *k is a king in the induced subgraph of the component of k.*
- *$f(k)$ is a king in H.*

Proof. Denote component of k as C. Denote the induced subgraph of C as G_c.

($\Rightarrow$): Suppose k is a king in G. Then k beats every vertex in $V(G)$ by 1 or 2 steps. In particular, k beats every vertex outside of C by 1 or 2 steps. Hence, by Corollary 7.18, $f(k)$ beats every vertex in H by 1 or 2 steps, Therefore $f(k)$ is a king in H.

We know k beats every vertex in C by 1 or 2 steps in G. Let $v \in C$ and $v \neq k$. We need to show k beats every vertex in G by 1 or 2 steps in G_C.

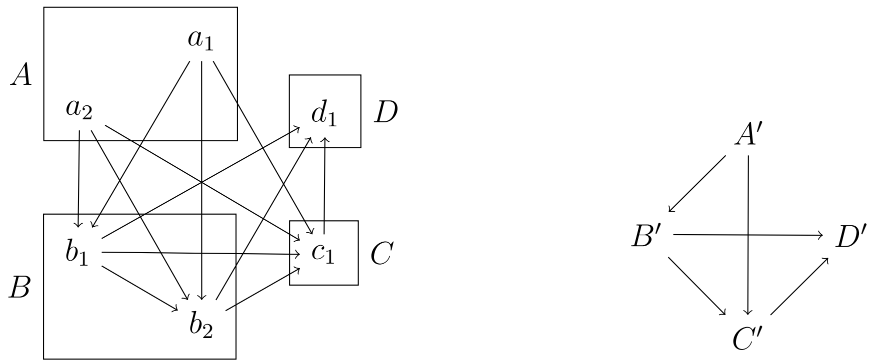
- Case 1: k beats v by 1 step, then k beats v by 1 step in G_c.
- Case 2: k beats v by 2 steps, then there exists $a \in V(G)$ such that $k \to a \to v$ in G. because $v, k \in C$ and Theorem 7.14, $a \in C$ (if $a \notin C$ will cause a contradiction). Therefore, k beats v by 2 steps in G_c.

Thus, k is a king in G_c, and $f(k)$ is a king in H.

Prove ($\Leftarrow$): Suppose $f(k)$ is a king in H, and k is a king in G_C. Since $f(k)$ is a king in H by 1 or 2 steps, $f(k)$ beats every other vertex $f(v)$ in H. Thus, k beats every other vertex that is not in C by 1 or 2 steps, by Corollary 7.18. Since k is a king in G_c, k beats every vertex in C by 1 or 2 steps. Therefore, k beats every vertex in $V(G)$ by 1 or 2 steps, so k is a king in G. $\square$

Although condensation is a very strong transformation, some information in the graph does get lost via this transformation. For example, the beating relationships between all the vertices in the same components are lost.

In Fig. 7.18, we show an example of a condensation, where the vertices a_1, a_2 are condensed into A'; b_1, b_2 are condensed into B'; c_1 is condensed into C'; and d_1 is condensed into D. If we just look at the condensed graph, we notice that we cannot recreate the beating relationship between a_1 and a_2, therefore the information about the beating relationship is lost. This fact is true for all the vertices in the same component.



(a) the uncondensed oriented graph G. (b) the condensed graph H.

Figure 7.18: information lost during condensation.

Another example in this graph is that we cannot know the beating relationship between b_1 and b_2 just by the condensed graph.

Notice that the beating relationship between components is not lost. For example, if we want to know the beating relationship between b_1 and a_2, we first observe that $A' \to B'$ in the condensed graph, then by definition of a condensation, a_2 has to beat b_1 in the uncondensed graph. Another example is that we can know that a_1 ties d_1 because A' ties D' in the condensed graph.

Therefore, an "efficient condensation" should keep as many vertices as possible. One of the most "efficient" condensation is the identity condensation, because it does not lose any information about this graph. However, identity condensation also do not mutate the graph at all, therefore it is not very practical.

Definition 7.6. *Given a set of condensations $F = \{f_0, f_1, \ldots, f_n\}$ where $f_k : G_k \to H_k$ and all the $V(G_k)$ are equal for $0 \le k \le n$, an* ***efficient condensation $\boldsymbol{f : G \to H}$ in $\boldsymbol{F}$*** *is a condensation such that $\boldsymbol{H}$ has largest vertex set in $\boldsymbol{\{H_0, H_1, H_2, \ldots H_n\}}$*

Notice in this definition, all the G_k's where $0 \le k \le n$ are not necessarily distinct. One example of efficient condensation is given a graph G and all the condensation defined on G, that is $\{f_0, f_1, \ldots, f_n\}$ where $f_k : G \to H_k$. Then the identity condensation, is the only efficient condensation in this set, since it keeps all the vertices.

Later, we will restrict the types of graphs H can be. In so doing, the identity condensation will no longer be allowed.

7.5 Tie Component Condensations

In the previous section we talked about the general property of graph condensations. In this section, we will focus on a specific type of condensation mentioned in previous sections and Fig. 7.13. Recall that a tie component is a set of vertices

Definition 7.7. *A **tie component condensation** is a function* $\boldsymbol{f} : \boldsymbol{Q} \to \boldsymbol{T}$*, where* $\boldsymbol{Q}$ *is a quasi-transitive oriented graph,* $\boldsymbol{T}$ *is a directed graph. And* $\boldsymbol{f}$ *maps* $\boldsymbol{V(Q)}$ *to* $\boldsymbol{V(T)}$ *surjectively, maps* $\boldsymbol{E(Q)}$ *to* $\boldsymbol{E(T)}$ *surjectively, such that:*

- *if* $\boldsymbol{a, b} \in \boldsymbol{V(Q)}$ *are in the same tie component, then* $\boldsymbol{f(a) = f(b)}$.
- *if* $\boldsymbol{a, b} \in \boldsymbol{V(Q)}$ *are in 2 distinct tie components* $\boldsymbol{A, B}$ *respectively, then*
  - $\boldsymbol{f(a) \to f(b)}$ *if* $\boldsymbol{A \to B}$.
  - $\boldsymbol{f(b) \to f(a)}$ *if* $\boldsymbol{B \to A}$.
  - *otherwise, tie condensation do not exists.*

Corollary 7.20. *All tie component condensations are graph condensation, where the components of the uncondensed graphs are the tie components of the uncondensed graphs.*

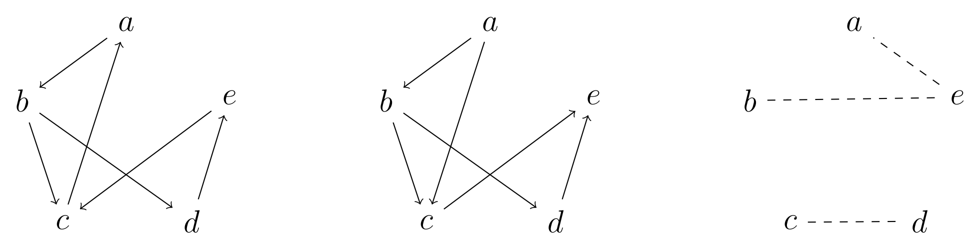
Corollary 7.21. *For any quasi transitive orientated graph* Q*, tie component condensation* $f : Q \to T$ *will always exists and* T *will always be a tournament.*

Proof. By Theorem 7.11 and the definition of tie component condensation, tie component condensation always exists.

Also by Theorem 7.11 and the definition of tie component condensation, for any two distinct vertices $f(a), f(b) \in T$, either $f(a) \to f(b)$ or $f(b) \to f(a)$. Therefore, T is a tournament. □

Fig. 7.13 is a great example of tie component condensation. In this condensation, the function g maps a, b to A', maps c, d, e to B', maps f to C', and maps j to D'.

It is also helpful to note that the tie components C and D are trivial tie components, since they only contain vertex g and j respectively.



(a) graph G. (b) graph H. (c) "tie structures" of G, H.

Figure 7.19: G and H has the same tie.

Corollary 7.22. *Given a quasi-transitive digraph Q, there only exists a unique tie component condensation $f : Q \to T$.*

Proof. Tie component condensation exists by Corollary 7.21; tie component condensation is unique by Corollary 7.7. □

Definition 7.8. *For a given quasi-transitive oriented graph Q, we call the result of tie component condensation of Q the **underlying tournament** of $\boldsymbol{Q}$.*

It is a nice result that every quasi-transitive oriented graph can be condensed into a tournament, which is one of the most understood family of digraphs. But just to say that a condensation to tournament exists is not helpful enough, since for every graph, there exists a trivial condensation, which also produce a tournament with only one vertex (a single vertex is, by definition, a tournament). Therefore, we need to investigate what we call the "efficiency" of the tie component condensation.

Definition 7.9. *For two orientated graphs G, H, such that*

- *$V(G) = V(H)$,*
- *a ties b in H if and only if a ties b in G,*

*then we say G, H has the same **tie structure**.*

Graphs with the same tie structure means that if we draw out all the ties from two graph, the graph formed by the ties are the same. In other words, these two graphs are differ by the orientation of their edges, but not their ties.

In Fig. 7.19a and Fig. 7.19b, we show two graph G and H with the same tie structure, In Fig. 7.19c, we draw out all the ties from the previous two graphs, and discover that

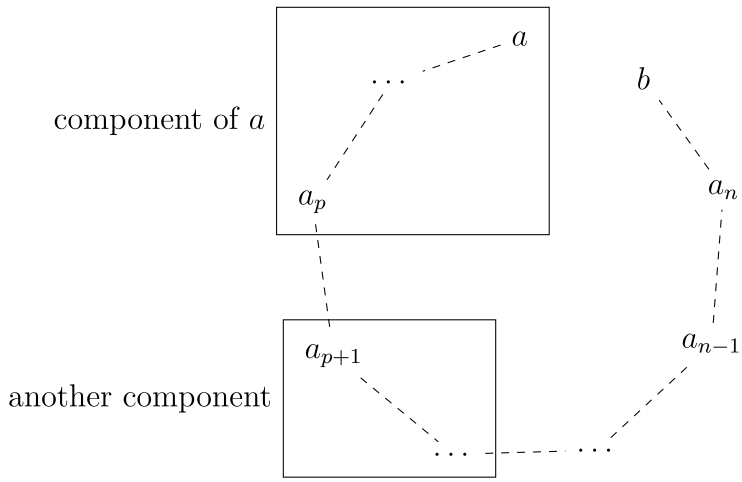


Figure 7.20: vertices a_p and a_{p+1} crosses components.

they are the same. the only differences between these two graphs are that $c \to a$, $e \to c$ in G, however $a \to c$, $c \to e$ in H, that is, the only differences between G and H are the orientations of these two edges.

Corollary 7.23. *For G and H with the same tie structure, if there exists a tie path between a, b in G, then there exists a tie path between a, b in H.*

Proof. since given any two vertices $p, q \in G$, if p ties q in G then p ties q in H. Therefore if there exists a tie path $[a, a_0, a_1, \dots, a_n, b]$ in G, then the same tie path exists in H □

Theorem 7.24. *Given a quasi-transitive graph Q and its tie component condensation f, consider the set of all the condensations $f_k : G_k \to T_k$, where G_k has the same tie structure as Q and T_k is a tournament: $F = \{f_0, f_1, \dots, f_{n-1}, f_n\}$. f is an efficient condensation in F.*

Proof. Because G_k are has the same tie structure as Q, all G_k's have the same vertices.

If there exists another condensation $f' : G' \to T'$ such that T' has more vertices than the underlying tournament T of Q, then there exists two vertices a, b in the same tie component in Q, and in different components in G', because the number of components in the uncondensed graph is the same as the number of vertices in the condensed graph.

Because a, b are in the same component in Q, there exists a tie path between a and b in Q. By Corollary 7.23, there exists a tie path between a and b in G'. Because a and

b are in different components in G', there exists a point on the tie path between a and b that "crosses components" in G'.

Formally, denote the tie path between a, b as: $[a, a_0, a_1, \ldots, a_n, b]$, then there exists a_p such that a_p and a_{p+1} are not in the same component. Because a_p ties a_{p+1} and they are in distinct components, then by Lemma 7.13, $f'(a)$ ties $f'(a')$. Therefore T' is not a tournament. Contradiction. □

Theorem 7.24 states that tie component condensation is not only a most efficient condensation to tournaments on any quasi-transitive orientated graph, tie component condensation is a most efficient in all the condensations to tournaments defined on all orientated graphs with the same tie structure.

To put it in other way, for all the orientated graphs with the same tie structure, quasi-transitive orientated graphs are the ones that can be condensed into tournaments most efficiently.

7.6 Kings

The definition of king states that it can beat every vertex by one or two steps, and the definition of quasi-transitive oriented graph states that if there exists a path of length two (implies beats by 1 or 2 steps) from one vertex to another vertex, then they are adjacent.

Both definitions are related to "beats by 1 or 2 steps", therefore kings in quasi-transitive oriented graphs have nice properties.

Lemma 7.25. *In a quasi-transitive oriented graph, a king is adjacent to every vertex.*

Proof. Given a king k, and another vertex v, then there are two possibilities:

- Case 1, k beats v by one step: therefore $k \to v$, k and v are adjacent.
- Case 2, k beats v by two steps: there exists vertex a, such that $k \to a \to v$. Then by definition of quasi-transitive oriented graph, k needs to be adjacent to v

□

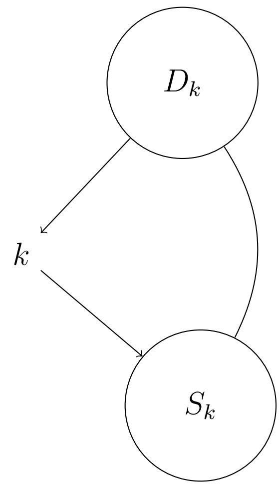


Figure 7.21: the rich structure of a king in a quasi-transitive oriented graph.

Lemma 7.26. *In a quasi-transitive oriented graph, for any king k, $\{D_k, S_k, \{k\}\}$ partitions the graph.*

Proof. Proof disjoint: pretty obvious. D_k, S_k disjoint because we are working in an oriented graph. D_k, S_k disjoint with $\{k\}$ because a vertex cannot beat itself.

Proof the union is the whole vertex set: by Lemma 7.25 □

Lemma 7.27. *In a quasi-transitive oriented graph, for any king k, D_k, S_k are adjacent.*

Proof. By definition of quasi-transitive oriented graph, because every vertex in D_k beats k and then beats every vertex in S_k, every vertex in D_k is adjacent to every vertex in S_k. □

Theorem 7.28. *In a quasi-transitive oriented graph, if we have a king k, then*

- *$D_k, S_k, \{k\}$ partitions the vertex set.*
- *D_k and S_k is adjacent.*

See Fig. 7.21, this figure shows the rich structure of a king in quasi-transitive oriented graph. We were able to see a similar partition structure mentioned in Chapter 5, and Lemma 5.1 (the partition structure in Fig. 7.21 is the same as $\{\{v\}, D_v, S_v\}$ for any vertex v in a tournament [6]).

D_k cannot tie with anything outside of D_k, and similarly for S_k. If there is a king in a quasi-transitive oriented graph, then this graph becomes "very connected", the only places ties can appear are inside the induced subgraphs of D_k and S_k.

Since we have deduced nice structures about ties in quasi-transitive oriented graphs, it is only logical to combine the property of king with the property of tie (tie component).

Theorem 7.29. *A vertex k is a king in a quasi-transitive oriented graph if and only if*

- *k is in a trivial tie component.*
- *the result of k after tie component condensation is a king in the underlying tournament.*

Proof. By Corollary 7.19, k is a king if and only if

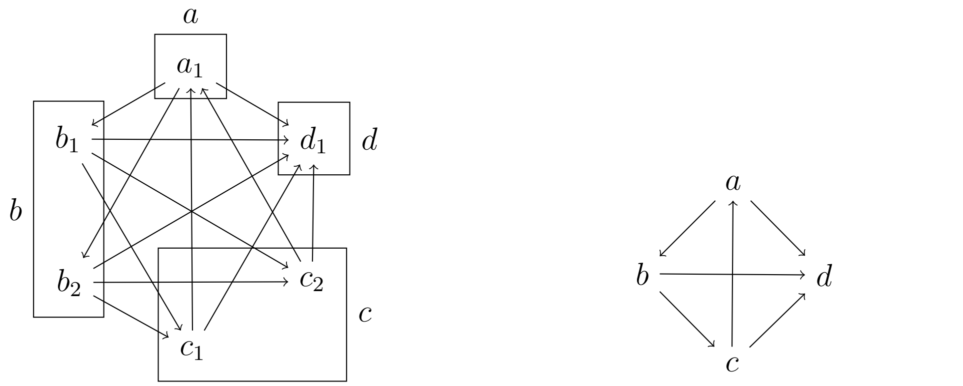
- k is a king in the induced subgraph of its tie component.
- the result of k after tie component condensation is a king in the underlying tournament.

So we need to show k is a king in induced subgraph of its tie component if and only if k is in a trivial tie component.

Suppose k is a king in the induced graph of its tie component, and the fact that the induced subgraph is a quasi-transitive oriented graph. Because of Lemma 7.25, k is a king in its tie component implies k is adjacent to all vertices in the tie component. There can be no tie path in this tie component. Therefore k has to be in a trivial tie component.

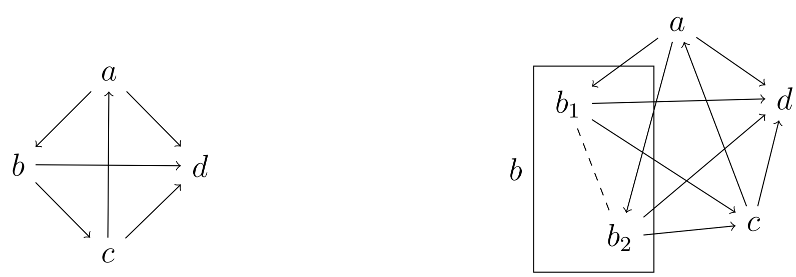
Suppose k is is in a trivial tie component, then k is a king in the induced graph of its tie component. Recall that the definition of king says that a digraph with one vertex, that vertex is a king in the digraph. □

In Fig. 7.22, we show a quasi-transitive oriented graph G and its underlying tournament. In the underlying tournament the kings are a, b, c, but only a_1 can be a king of G, since vertices b and c do not correspond to trivial tie components. (tie component $\{b_1, b_2\}$ is condensed into b, and tie component $\{c_1, c_2\}$ is condensed into c). Notice, although vertex d_1 is in a trivial tie component, d_1 is not a king in G, because in the underlying tournament, d (the image of d_1 under tie condensation) is not a king.



(a) quasi-transitive oriented graph G. (b) the underlying tournament of G.

Figure 7.22: a quasi-transitive digraph and its underlying tournament.



(a) start with a tournament with 3 kings. (b) then change king b to $\{b_1, b_2\}$.

Figure 7.23: construct a quasi-transitive oriented graph with 2 kings.

This theorem not only gives us a way to identify kings in a quasi-transitive oriented graph, but it helps us to construct a quasi-transitive oriented graph with a certain number of kings.

For example, if we want to construct a quasi-transitive oriented graph with k kings, we first start with a quasi-transitive graph with k' kings, where $k' > k$, and then we change $k' - k$ kings into "non-trivial tie components". This method enables us to construct a quasi-transitive oriented graphs with arbitrary number of kings. See Fig. 7.23, we start with a tournament with 3 kings $\{a, b, c\}$, then we change the king b into a non-trivial tie component $\{b_1, b_2\}$. Then the only kings left are a and c, and we get a quasi-transitive oriented graph with 2 kings.

Another significant usage of Theorem 7.29 is to use the properties of kings in tournaments to prove properties of kings in quasi-transitive oriented graphs.

Definition 7.10. *k is a **great king** in a digraph $\boldsymbol{G}$, if and only if $\boldsymbol{k}$ is the only king* *in digraph* $\boldsymbol{G}$*, and* $\boldsymbol{k}$ *is not an emperor.*

By Theorem 4.1, Corollary 5.5, a great king does not exists in semi-complete digraphs and tournaments.

Corollary 7.30. *If there exists a great king in a quasi-transitive oriented graph* G*, then there exists at least two ties in* G*.*

Proof. Assume k is the great king in G. Then there exists a vertex v such that $v \to k$. Denote the underlying tournament of G as T, and the tie component condensation of G as f.

By Lemma 7.13, $f(v) \to f(k)$. Therefore, by Theorem 7.29 $f(k)$ is a king, but not an emperor in T. Because a tournament never has exactly 2 kings and exactly 1 king must be an emperor, there exists at least 3 kings in T. Because k is the only king in G, there has to exist at least 2 non-trivial tie components. Therefore, G have at least 2 ties. □

Theorem 7.29 provides us a way to detect, construct, and understand kings in quasi-transitive oriented graph using the properties and constructions of kings in tournaments.

Chapter 8

Further Problems

In Chapter 5 and Chapter 6, we see that because of the addition of two types of ties, we can construct many more (n, k) semi-complete digraphs or (n, k) oriented graphs than (n, k) tournaments. Then what can we construct if we limit the number of ties or double ties?

Definition 8.1. *a (n, k, t) digraph is a digraph with n vertices, k kings and t ties (or double ties).*

In Chapter 5, and Chapter 6, we explore the construction of (n, k) semi-complete digraphs and (n, k) oriented graphs with 1 tie. In other word, we have solved the problem of constructing $(n, k, 1)$ oriented graph and semi-complete digraphs. What happens with t gets larger?

One of the useful thing to note when approaching the problem of the existence of (n, k, t) digraphs (or oriented graphs, semi-complete digraphs, etc.) in the end of Section 7.6, we give a way to construct a quasi-transitive oriented graph with given number of kings. This way of constructing quasi-transitive oriented graphs may be very useful, since we can manipulate the number of ties in the graph without changing the number of vertices and kings. For a tie components with n vertices, the number of ties in the component can be any number between $n - 1$ (ties forms a spanning tree of the component) and $\frac{n(n+1)}{2}$ (there exists a tie between every two vertices).

In Section 4.4, we showed that there exists king with very low out-degrees. What is the property of a king with very low out-degree? What is the property of a king with only out-degree 1? what is the property of a king that has the lowest out-degree in the graph? What is the property of a king that has the lowest out-degrees among all of the kings in the graph?

In Section 7.6, we have showed that king has very rich structures. However, Lemma 7.26 and Lemma 7.27 are not used in this thesis. Can we use these 2 property to deduce any property of quasi-transitive oriented graph with kings? A interesting path is that Lemma 7.27 implies in a quasi-transitive oriented graph with king k, all the tie components are either the subset of D_k or S_k.

Definition 8.2. *A infinite graph is a graph with infinite number of vertices.*

Can we extend these result to infinite graph? Can we keep the definition of tie components when we expand the context to infinite graph? How will the property of infinite path and infinite tie path change the results in this thesis?

In [2] and [3], the author uses the idea of strong digraph and strong components (see the definition of strong and strong components in [8]) to deduce the property of quasi-transitive oriented graph. We can show that given any *non-trivial* condensation, the uncondensed graph is strong if and only if its underlying tournament is strong (*spoiler alert!*: Lemma 7.16 will be useful). Can we combine our results with the result in [2] and [3]?

In Section 7.4, we showed that graph condensation is a very useful transformation. and many graph only have identity condensation and trivial condensation defined on it. How hard is a condensation? How many graph only have identity condensation and trivial condensation on them?

How useful is graph condensations? we have showed that they preserves the shortest path (Theorem 7.17), and we can show that they preserves the strong property of the graph. What are other property that graph condensations preserves?

List of Figures

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