

Kings in Generalized Tournaments

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Thesis Defense

Background

tournaments

Definition (Maurer 1980)

An **tournament** consists of a set of **vertices** V , and a set of **edges** E which consists of ordered pairs of vertices, such that

- ▶ for all vertex a , $(a, a) \notin E$.
- ▶ for all vertices a, b , $(a, b) \in E$ or $(b, a) \in E$, but not both.

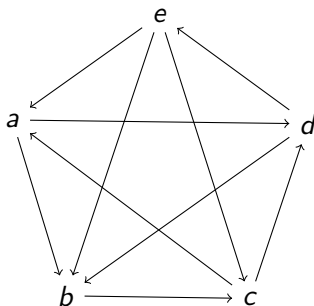


Figure: example of a tournament.

Background

tournaments

If (a, b) in edge set E , we write it as $a \rightarrow b$ or a beats b .

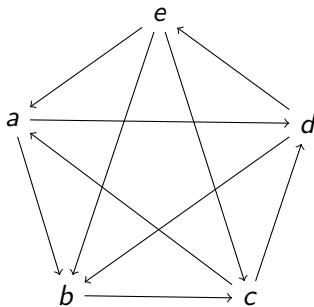


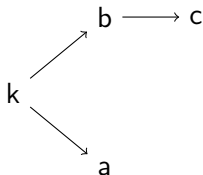
Figure: example of a tournament.

Background

kings

Definition (Maurer 1980)

A **king** (or **2-king**) in a oriented graph is a vertex that can beat every vertex by at most 2 steps.



Background

kings

The kings in this graph are a, d, e .

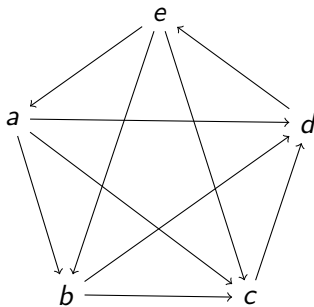
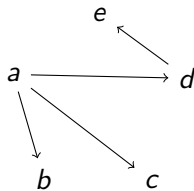


Figure: example of a tournament.

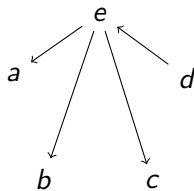
Background

kings

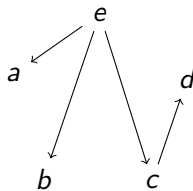
The kings in this graph are a, d, e :



(a) a is a king.



(b) d is a king.



(c) e is a king.

Generalization of Tournaments

relation between three generalizations

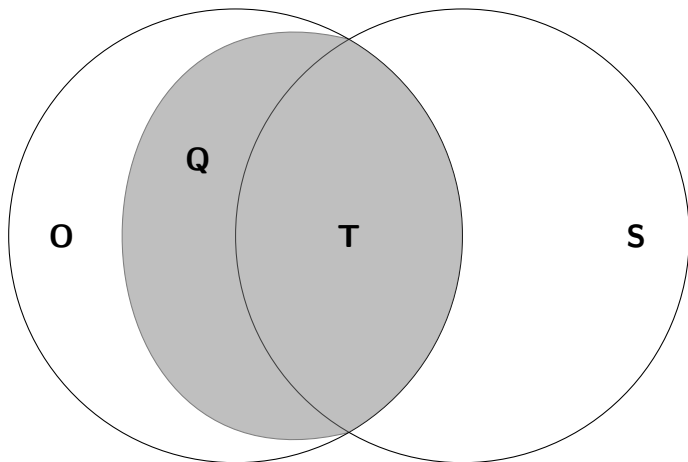


Figure: the relationship between tournaments, semi-complete digraphs, oriented graphs, and quasi-transitive oriented graphs.

Quasi-transitive Oriented Graph

oriented graph

Definition

An **oriented graph** consists of a set of **vertices** V , and a set of **edges** E which consists of ordered pairs of vertices, such that

- ▶ for all vertex a , $(a, a) \notin E$.
- ▶ for all vertices $\{a, b\}$, if a beats b , then b does not beat a .

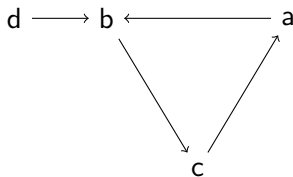


Figure: example of an oriented graph.

Quasi-transitive Oriented Graph

oriented graph

Definition

A vertex a is **adjacent** to vertex b if $a \rightarrow b$ or $b \rightarrow a$.

Definition

There exists a **tie** between a and c , if a is not adjacent to c .

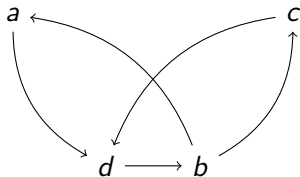


Figure: another example of an oriented graph.

Quasi-transitive Oriented Graph

oriented graph

King in this graph is d, e :

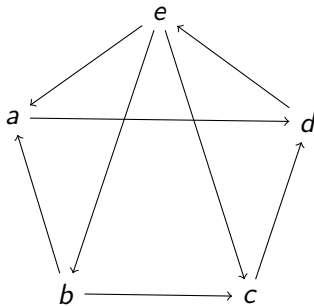


Figure: an example of kings.

Quasi-transitive Oriented Graph

definition

Definition (Bang-Jensen and Huang 1995)

Quasi-transitive oriented graph is an oriented graph such that if $a \rightarrow b$ and $b \rightarrow c$ then a and c are adjacent.

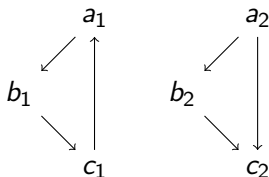


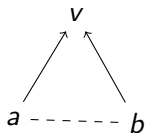
Figure: examples of quasi-transitive oriented graphs.

Quasi-transitive Oriented Graph

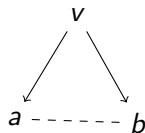
ties

Lemma

In Quasi-transitive oriented graphs, If vertex a ties with vertex b , then for any other vertex v that does not tie with a or b , $v \rightarrow \{a, b\}$ or $\{a, b\} \rightarrow v$



(a) $v \rightarrow \{a, b\}$.

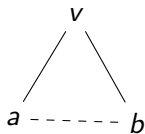


(b) $\{a, b\} \rightarrow v$.

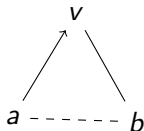
Figure: an example where a ties with b .

Quasi-transitive Oriented Graph

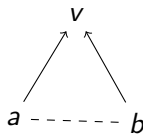
tie transmits arrows



(a) both a and b are adjacent to v .



(b) let $a \rightarrow v$.



(c) then b has to beat a .

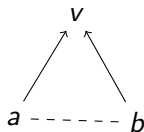
Figure: arrow direction transmitted from a to b .

Quasi-transitive Oriented Graph

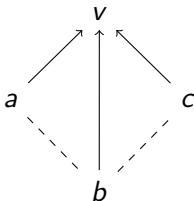
tie transmits arrows



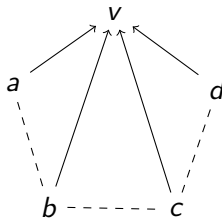
(a) starts with vertex a .



(b) add a vertex b that ties a .



(c) add another vertex c ties b .



(d) add another vertex d ties c .

Figure: tie is transmitting the direction of arrow.

Quasi-transitive Oriented Graph

tie paths

Definition

In a digraph, a **tie path** from vertex a_0 to vertex a_n , or a tie path between vertices a_0 and a_n , is a sequence of vertices $[a_0, a_1, a_2, \dots, a_{n-1}, a_n]$, such that for all $0 \leq k < n$, a_k ties a_{k+1} . Note that all the a_i 's do not have to be distinct.

$$a \text{ --- } b \text{ --- } c \text{ --- } d$$

Figure: a tie path from a to b .

Quasi-transitive Oriented Graph

tie paths

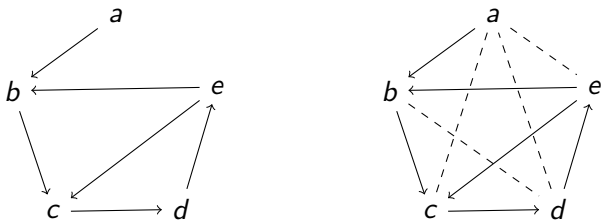


Figure: tie path example.

Quasi-transitive Oriented Graph

tie paths

Lemma

In Quasi-transitive oriented graphs, If vertex a ties with vertex b , then for any other vertex v that does not tie with a or b , $v \rightarrow \{a, b\}$ or $\{a, b\} \rightarrow v$

Lemma

For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.

Quasi-transitive Oriented Graph

tie paths

Lemma

For every tie path in a quasi-transitive oriented graph, if vertex v is adjacent to all the vertices on the tie path, then v beats every vertex on the tie path, or v is beaten by every vertex on the tie path.

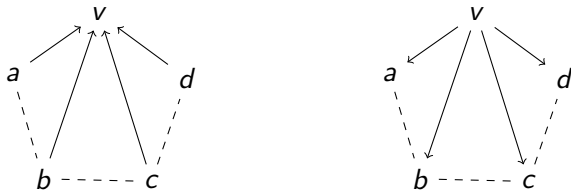
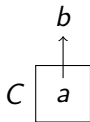


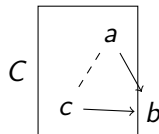
Figure: a tie path from a to d .

Quasi-transitive Oriented Graph

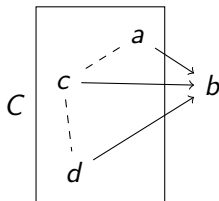
tie components



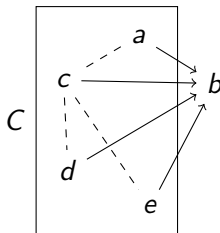
(a) a beats b .



(b) direction transmitted from a to c .



(c) transmitted from a to c to d .



(d) transmitted from a to c to e .

Figure: the arrow direction to b was transmitted in tie component C .

Quasi-transitive Oriented Graph

tie components

Definition

In a digraph G , a **tie component of vertex** a : $C(a)$, all vertices v such that there exists a tie path between a and v .

Definition

In a digraph G , a tie component $C(a)$ is a **trivial tie component** if $C(a)$ only contains a itself and no other vertex.

Corollary

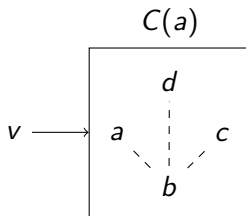
In a digraph, if vertex v is not in tie component $C(a)$, then v is adjacent to $C(a)$.

Quasi-transitive Oriented Graph

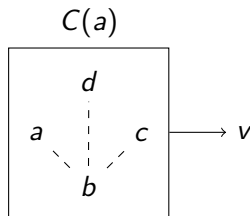
tie components

Lemma

In a quasi-transitive oriented graph, for any tie component $C(a)$ and any vertex v such that $v \notin C(a)$, then either $C(a) \rightarrow v$ or $v \rightarrow C(a)$.



(a) $v \rightarrow C(a)$.

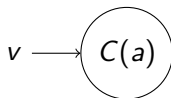


(b) $C(a) \rightarrow v$.

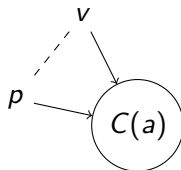
Quasi-transitive Oriented Graph

tie components

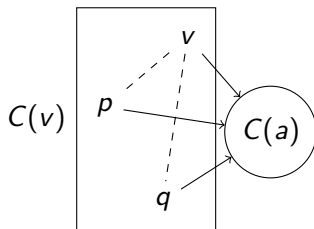
Consider tie component $C(v)$. Without loss of generosity, assume $v \rightarrow C(a)$



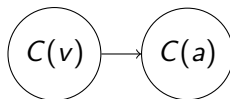
(a) direction is $v \rightarrow C(a)$.



(b) transmitted from v to p .



(c) transmitted from v to q .



(d) $C(v) \rightarrow C(a)$.

Quasi-transitive Oriented Graph

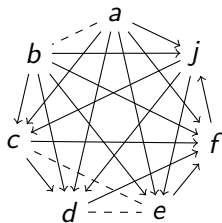
condensation to tournaments

Theorem

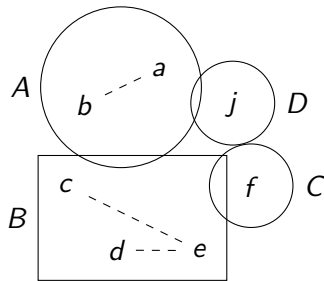
For any two distinct tie components $C(v)$ and $C(a)$ in a quasi-transitive oriented graph, $C(v) \rightarrow C(a)$ or $C(a) \rightarrow C(v)$.

Quasi-transitive Oriented Graph

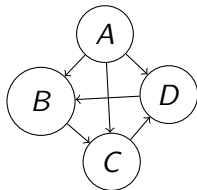
condensation to tournaments



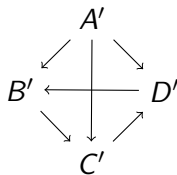
(a) a quasi-transitive oriented graph.



(b) find its tie components.



(c) components beat each other.



(d) components are just like vertices.

Quasi-transitive Oriented Graph

condensation to tournaments

Definition

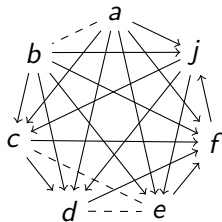
A **tie component condensation** is a function $f : Q \rightarrow T$, where Q is a quasi-transitive oriented graph, T is a directed graph. And f maps $V(Q)$ to $V(T)$ surjectively, such that:

- ▶ if $a, b \in V(Q)$ are in the same tie component, then $f(a) = f(b)$.
- ▶ if $a, b \in V(Q)$ are in 2 distinct tie components A, B respectively, then
 - ▶ $f(a) \rightarrow f(b)$ if $A \rightarrow B$.
 - ▶ $f(b) \rightarrow f(a)$ if $B \rightarrow A$.
 - ▶ otherwise, tie condensation do not exists.

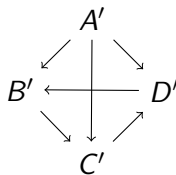
Quasi-transitive Oriented Graph

condensation to tournaments

- ▶ if $a, b \in V(Q)$ are in the same tie component, then $f(a) = f(b)$.
- ▶ if $a, b \in V(Q)$ are in 2 distinct tie components A, B respectively, then
 - ▶ $f(a) \rightarrow f(b)$ if $A \rightarrow B$.
 - ▶ $f(b) \rightarrow f(a)$ if $B \rightarrow A$.
 - ▶ otherwise, tie condensation do not exists.



(a) before condensation.



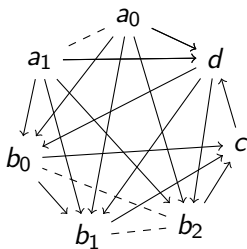
(b) after condensation.

Quasi-transitive Oriented Graph

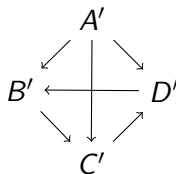
condensation to tournaments

Theorem

The component condensation of given quasi-transitive oriented graph Q will always result in a unique tournament T , where T is called the **underlying tournament of Q** .



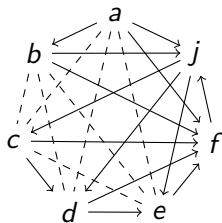
(a) quasi-transitive oriented graph Q .



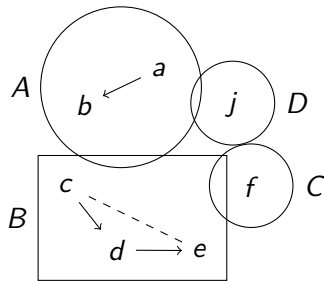
(b) underlying tournament of Q .

Quasi-transitive Oriented Graph

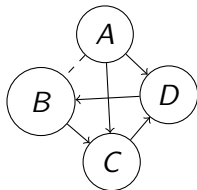
graph condensation



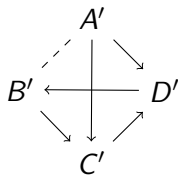
(a) an oriented graph G .



(b) components of G .



(c) relationship between components.



(d) condensed graph H .

Quasi-transitive Oriented Graph

graph condensation

Theorem

Given a quasi-transitive graph Q and its tie component condensation f , consider the set of all the condensations $f_k : G_k \rightarrow T_k$, where G_k has the same tie structure as Q and T_k is a tournament: $F = \{f_0, f_1, \dots, f_{n-1}, f_n\}$. f is an efficient condensation in F .

Quasi-transitive Oriented Graph

property of kings

Theorem

If vertex k is a king in quasi-transitive oriented graph G , then k is adjacent to every other vertex in G .

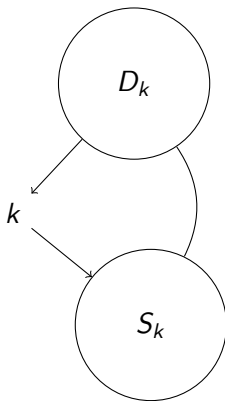


Figure: the rich structure of a king in a quasi-transitive oriented graph.

Quasi-transitive Oriented Graph

property of kings

Theorem

A vertex k is a king in a quasi-transitive oriented graph if and only if

- ▶ *k is in a trivial tie component.*
- ▶ *the result of k after tie component condensation is a king in the underlying tournament.*

Thank You

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References



Bang-Jensen, Jørgen and Jing Huang (Oct. 1, 1995).
“Quasi-transitive digraphs”. In: *Journal of Graph Theory* 20.2,
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In: *Mathematics Magazine* 53.2, p. 67. ISSN: 0025570X. DOI:
10.2307/2689952. URL: <http://www.jstor.org/stable/10.2307/2689952?origin=crossref> (visited on 01/27/2018).