Knowledge-Based Systems 109 (2016) 71–83

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Efficient updating of probabilistic approximations with incremental objects

Chuan Luo[a,*], Tianrui Li[b], Hongmei Chen[b], Hamido Fujita[c], Zhang Yi[a]

[a] College of Computer Science, Sichuan University, Chengdu 610065, China
[b] School of Information Science and Technology, Southwest Jiaotong University, Chengdu 611756, China
[c] Faculty of Software and Information Science, Iwate Prefectural University, 020–0693, Iwate, Japan

ARTICLE INFO

Article history:
Received 13 April 2016
Revised 19 June 2016
Accepted 20 June 2016
Available online 23 June 2016

Keywords:
Probabilistic rough sets
Dynamic data
Incremental learning
Concept approximation

ABSTRACT

Probabilistic rough set model, which is established by incorporating the theory of probability into rough set theory, aims to model imprecise data with the tolerance of decision errors in terms of conditional probability and probabilistic parameters. The volume of data is frequently varied dynamically. It is very time consuming to analyze the updates of data incessantly from computation perspective. Incremental learning technique is desired to improve computational efficiency, which poses an incremental capability for adaptive knowledge maintenance to accommodate varied data. In this paper, we focus on efficient updating of probabilistic approximations with incremental objects in a dynamic information table. The dynamic characteristics of conditional partition and decision classification on the universe are analyzed when the insertion or deletion of objects occurs. Different updating patterns of conditional probability are presented for different combinatorial varieties of the conditional and decision classes. Meanwhile, incremental algorithms for updating probabilistic approximations are proposed, which are proficient to efficiently classify the incremental objects into decision regions by avoiding re-computation efforts. Experiments on benchmark data sets indicate that the proposed algorithms outperform the static algorithm in the presence of dynamic variation of the universe.

1. Introduction

Concept learning refers to distinguishing exemplars from non-exemplars of various categories according to available relevant features [10,15,26]. Granular computing provides a granule-based computational paradigm for concept learning via information granulation and concept approximation, which can be considered as a label of theories by using granules in problem solving [1,20]. Rough sets is a concrete theory of granular computing, where the use of indiscernibility relation results in the information granulation, and the uncertain concepts are approximated by rough approximations, which can divide the universe into three pair-wise disjoint decision regions, namely, the positive, boundary and negative regions [18,22,27–29].

Successful applications of rough sets in different real-world applications have demonstrated its usefulness in recent years. Several generalizations of rough set models have been developed for the practical needs in concept learning, such as the dominance-based rough set model [4], the tolerance relation-based rough set model [8], the graded rough set model [32], the composite rough set model [36], the fuzzy rough set model and the rough fuzzy set model [2,35]. As we know, the rough approximations of a concept in classical rough set model is constructed by a strict inclusion relation, i.e., the determination of objects into decision regions should be fully certain. However, probabilistic patterns by allowing acceptable level of errors are often required in practical, due to its generality, flexibility and insensitivity to noises [30]. To address this issue, the probabilistic generalization of rough sets has been paid close attentions [31,37]. By introducing a pair of parameters α and β, Yao proposed a decision-theoretic rough set model based on bayesian decision theory, in which the parameters can be systematically determined by loss function during decision making [33]. Wong and Ziarko proposed the 0.5-probabilistic rough set model by using a parameter of 0.5 on the probability of an object belonging to a given concept [25]. Ziarko introduced a variable precision rough set model based on the notion of graded set inclusion [38]. By using prior probability as a reference for the absence of parameters, Ślęzak and Ziarko proposed a parameter-free extension of the variable precision rough set model, called Bayesian

* Corresponding author.
E-mail addresses: cluo@scu.edu.cn (C. Luo), trli@swjtu.edu.cn (T. Li), hmchen@swjtu.edu.cn (H. Chen), HFujita-799@acm.org (H. Fujita), zhangyi@scu.edu.cn (Z. Yi).

http://dx.doi.org/10.1016/j.knosys.2016.06.025
0950-7051/

rough set model [24]. To adopt the Bayesian decision theory and the naive independence assumption in probability estimation, Yao and Zhou proposed a naive Bayesian rough set model [34]. Greco et al. proposed a parameterized rough set model to model data relationships expressed by frequency distribution instead of the strict inclusion relation [5]. Considering that the parameter optimization problem is actually a competitive game, Herbert and Yao proposed a game-theoretic rough set model by incorporating the game theory into the decision-theoretic rough set model [6].

Recent data-intensive applications lead to the data updates continuously. A naive solution for ongoing data analysis in the presence of updates occur is to re-run the static algorithm on the entire updated data, which is obviously, not an efficient solution. Consider that the size of updated portion is usually much smaller than the entire data, incremental learning is an alternative manner for improving efficiency since it can utilize the previously learned results to analyze the newly updated data [3,17]. The variation of objects is the most common situation of data updating in information systems, such as the transactions in electronic commerce system, the records in Web log system, the accidents in public traffic system, and so on. Many valuable studies have been devoted for incremental data analysis based on rough set theory with incremental objects. In order to make the concept approximation has the incremental learning ability, Liu et al. investigated incremental algorithms for computing approximations in the incomplete information systems with missing values under four different binary relations [12]. Zhang et al. developed matrix-based incremental algorithms for updating approximations in the composite information system with multiple types of attributes [36]. Li et al. proposed incremental method for updating approximations based on dominance-based rough set model to deal with dynamic preference-ordered data [11]. Furthermore, considering the application of rough sets to incremental feature selection and rule extraction, Zheng and Wang incorporated the notion of rule tree to rough set theory, and developed an incremental algorithm for rule extraction [39]. Huang et al. developed an incremental algorithm for updating alternative rules by modifying partial previously induced rule sets with the insertion of an object [7]. Qian et al. introduced a general feature selection framework, namely, positive approximation, to accelerate the heuristic search process with incremental change of universe [21]. Liang et al. introduced the updating mechanisms of information entropy when multiple objects are inserted into a decision table. An incremental feature selection algorithm was proposed based on information entropy [14]. Shu and Qian proposed incremental positive region-based feature selection algorithm when multiple objects varied in the incomplete decision system [23]. Lang et al. proposed incremental approaches for computing type-1 and type-2 characteristic matrices of coverings, which also be integrated into feature selection in the dynamic covering decision system [9]. Qian et al. investigated incremental feature selection approach based on the compact discernibility matrix in the incomplete decision system [19].

Most probabilistic rough set approaches so far assumed that the data in an information system is static, and fail to take into account the dynamic data environment. To address this issue, Liu et al. discussed the incremental approaches for updating probabilistic approximations with the variation of attributes in an dynamic information system. According to analyzing the uncertainties of decision regions, incremental updating propositions were investigated based on the concept of boundary sets, which are divided from boundary region [13]. In this study, we focus on the updating scenario of an information system with incremental objects. The dynamic updating of the conditional partition and the decision classification of the universe are formalized with the insertion and deletion of objects. Different updating patterns of the conditional probability in probabilistic rough set model are analyzed. Incremental algorithms for updating probabilistic approximations are proposed when the objects are inserted into or deleted from an information system, respectively. To facilitate our discussions, the rest of the paper is organized as follows. We briefly review basic notions used in probabilistic rough sets in Section 2. Section 3 introduces the incremental updating strategies for probabilistic approximations when objects vary in an information system. In Section 4, incremental algorithms are presented for updating probabilistic approximations. Section 5 describes our experimental results and Section 6 concludes the paper.

2. Preliminary knowledge on probabilistic rough sets

In this section, we will review several basic notions and concepts of probabilistic rough sets [30,31,33,37].

Rough sets based data analysis is initiated from the definability of concepts and the approximations of concepts, which are defined with respect to an information table that describes all available information of data. The rows and columns of an information table are labelled with objects and attributes respectively, and the entries of an information table are attribute values.

Definition 1. An information table is the following tuple:

$$S = (U, AT = C \cup D, \{V_a | a \in AT\}, \{I_a | a \in AT\}), \tag{1}$$

where U is a finite nonempty set of objects; AT is a finite nonempty set of attributes, where C is the condition attribute set, D is referred to as the decision attribute set and $C \cap D = \emptyset$; V_a is a nonempty set of values of $a \in AT$, and I_a: $U \to V_a$ is an information function that maps an object in U to a value of V_a for an attribute $a \in AT$, i.e., $I_a(x) \in V_a$.

An information table is simply denoted by $S = (U, C \cup D)$ throughout this paper.

The indiscernibility of objects in a universe is a fundamental issue of rough sets, which is used for constructing a granulated view of the universe, and is modeled by an equivalence relation.

Definition 2. Given a subset of attributes $A \subseteq C$, let R_A denote an equivalence relation on U, which can be defined as follows:

$$R_A = \{(x, y) \in U \times U | I_a(x) = I_a(y), \forall a \in A\}, \tag{2}$$

The equivalence relation R_A partitions U into equivalence classes given by: $U/R_A = \{[x]_A | x \in U\}$, where $[x]_A = \{y \in U | (x, y) \in R_A\}$ denotes the equivalence class determined by x with respect to A. Each equivalence class contains a set of indistinguishable objects, which may be viewed as a granule of the universe. Based on the granulated view of the universe, rough approximations can be obtained for interpreting and representing concepts in the granulated universe.

Definition 3. Given an equivalence relation R_A and a subset of universe $X \subseteq U$, the lower and upper approximations of X can be defined as follows.

$$\begin{aligned} \underline{R}(X) &= \{x \in U | [x]_A \subseteq X\}, \\ \overline{R}(X) &= \{x \in U | [x]_A \cap X \neq \emptyset\}. \end{aligned} \tag{3}$$

According to the lower and upper approximations, the positive, negative and boundary regions of X are defined by:

$$\begin{aligned} \mathrm{POS}(X) &= \underline{R}(X) = \{x \in U | [x]_A \subseteq X\}, \\ \mathrm{BND}(X) &= \overline{R}(X) - \underline{R}(X) = \{x \in U | [x]_A \cap X \neq \emptyset, [x]_A \not\subseteq X\}, \\ \mathrm{NEG}(X) &= U - \overline{R}(X) = \{x \in U | [x]_A \cap X = \emptyset\}. \end{aligned} \tag{4}$$

The definition of rough approximations by set inclusion in the Pawlak's rough sets only considers full inclusion, i.e., the classification must be completely certain, which is too sensitive to data accuracy and hence restricts its practical application [18]. In order to allow some tolerance of errors for defining rough approximations based on probabilistic information, conditional probability is used to state the degree of overlap between the equivalence class and the target concept.

Definition 4. Given a subset of universe $X \subseteq U$, $\forall x \in U$, the conditional probability of x belonging to X can be simply estimated as follows:

$$Pr(X|[x]) = |X \cap [x]|/|[x]|. \tag{5}$$

where $|\bullet|$ denotes the cardinality of a set.

Based on the conditional probability, Pawlak's rough approximations in Eq. (3) can be equivalently defined by:

$$\begin{aligned} \underline{R}(X) &= \{x \in U | Pr(X|[x]) = 1\}, \\ \overline{R}(X) &= \{x \in U | Pr(X|[x]) > 0\}. \end{aligned} \tag{6}$$

And, the three regions defined in Eq. (4) can be equivalently defined by:

$$\begin{aligned} \mathrm{POS}(X) &= \{x \in U | Pr(X|[x]) = 1\}, \\ \mathrm{BND}(X) &= \{x \in U | 0 < Pr(X|[x]) < 1\}, \\ \mathrm{NEG}(X) &= \{x \in U | Pr(X|[x]) = 0\}. \end{aligned} \tag{7}$$

Obviously, in Pawlak's rough sets, the magnitude of the probability value is not taken into account for defining rough approximations, while using two extreme probability values, i.e., 0 and 1. By using a pair of flexible probabilistic threshold values α and β, Yao et al. introduced a general probabilistic rough set model, called a decision-theoretic rough sets [33].

Definition 5. Given a subset of universe $X \subseteq U$ and a pair of thresholds α and β, the (α, β)-probabilistic lower and upper approximations are defined by:

$$\begin{aligned} \underline{R}_{(\alpha,\beta)}(X) &= \{x \in U | Pr(X|[x]) \geq \alpha\}, \\ \overline{R}_{(\alpha,\beta)}(X) &= \{x \in U | Pr(X|[x]) > \beta\}. \end{aligned} \tag{8}$$

And, the (α, β)-probabilistic positive, negative and boundary regions of X are defined by:

$$\begin{aligned} \mathrm{POS}_{(\alpha,\beta)}(X) &= \{x \in U | Pr(X|[x]) \geq \alpha\}, \\ \mathrm{BND}_{(\alpha,\beta)}(X) &= \{x \in U | \beta < Pr(X|[x]) < \alpha\}, \\ \mathrm{NEG}_{(\alpha,\beta)}(X) &= \{x \in U | Pr(X|[x]) \leq \beta\}. \end{aligned} \tag{9}$$

According to different choices of α and β values, many existing rough set models can be explicitly derived from the decision-theoretic rough set model, such as the Pawlak's rough set model, the 0.5 probabilistic rough set model, the symmetric and asymmetric variable precision rough set models. Furthermore, by using Bayesian decision theory, decision-theoretic rough sets provide a systematic method for determining the probabilistic thresholds α and β based on the minimization of overall decision cost, which provides a solid decision-theoretic foundation of probabilistic rough set model [30].

3. Dynamic updating probabilistic approximations with incremental objects

Given a dynamic information table, incremental objects occur in that new objects become available and old objects become unwanted continuously over time. Incremental learning approaches are desirable to accelerate knowledge acquisition from such dynamic data environment. In this section, we introduce incremental mechanisms for the updating of probabilistic approximations with incremental objects in a dynamic information table.

3.1. Updating probabilistic approximations when inserting objects

This section introduces incremental updating mechanisms of the probabilistic approximations with the insertion of new objects in a dynamic information table. Firstly, we discuss a special case of the insertion of objects, i.e., a single object is inserted into an information table. Given an information table $S = (U, C \cup D)$, $A \subseteq C$. We assume that $U/A = \{E_1, E_2, \ldots, E_m\}$, $U/D = \{D_1, D_2, \ldots, D_n\}$. For any conditional class $E_i \in U/A$, there are two updating situations will occur when a single object $\tilde{x}$ inserts into U, i.e., if the inserted object $\tilde{x}$ satisfies that $\exists x \in E_i$, $(x, \tilde{x}) \in R_A$, then we have $E_i' = E_i \cup \{\tilde{x}\}$, else the conditional class will keep constant, $E_i' = E_i$. Similarly, for any decision class $D_j \in U/D$, if $\exists x \in D_j$, $\tilde{x}$ satisfies that $\forall d \in D$, $I_d(x) = I_d(\tilde{x})$, then $D_j' = D_j \cup \{\tilde{x}\}$, else D_j keeps constant, $D_j' = D_j$.

Different updating situations of the conditional class and the decision class will lead to the different updating patterns of the conditional probability of an object $x \in E_i$ belonging to D_j. Table 1 enumerates the detailed variation tendencies of conditional probability when a single object is inserted into an information table, where "↑" denotes the increasing tendency, "↓" denotes the decreasing tendency, and "-" denotes the invariable tendency.

Based on the above four updating patterns of the conditional probability, we discuss the updating principles of the probabilistic approximations of a decision class $D_j \in U/D$ from two updating situations: (1) the inserted object $\tilde{x}$ belongs to D_j; (2) the inserted object $\tilde{x}$ does not belong to D_j. Propositions 1–3 introduce the incremental mechanisms of the probabilistic approximations with the insertion of a single object.

Proposition 1. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When a single object $\tilde{x}$ is inserted into U, the (α, β)-probabilistic positive region of D_j can be updated as:*

(1) *If the inserted object $\tilde{x}$ belongs to D_j, i.e., $\tilde{x} \in D_j$, then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j); \\ \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup E_i \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j) \\ & \wedge Pr(D_i'|E_i') \geq \alpha; \\ \mathrm{POS}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the inserted object $\tilde{x}$ does not belong to D_j, i.e., $\tilde{x} \notin D_j$, then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j) \\ & \wedge Pr(D_i'|E_i') \geq \alpha; \\ \mathrm{POS}_{(\alpha,\beta)}(D_j) - E_i, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j) \\ & \wedge Pr(D_i'|E_i') < \alpha; \\ \mathrm{POS}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. (1) Assume that the object $\tilde{x} \in D_j$ is inserted into U. Based on the updating patterns of the conditional probability in Table 1, we have $\forall E_i \in U/A$, if $\tilde{x} \in E_i$, then $Pr(D_j'|E_i') > Pr(D_j|E_i)$. That is, $\forall E_i \in U/A$, if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E_i' = E_i \cup \{\tilde{x}\} \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}$; else if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, with increase of the conditional probability, we have if $Pr(D_j'|E_i') \geq \alpha > Pr(D_j|E_i)$, then $E_i' = E_i \cup \{\tilde{x}\} \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup E_i' = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup E_i \cup \{\tilde{x}\}$; else if $\alpha > Pr(D_j'|E_i') > Pr(D_j|E_i)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j)$. On the other hand, if $\tilde{x} \notin E_i$, we can obtain that the conditional probability keeps constant from Table 1, i.e., $Pr(D_j'|E_i') = Pr(D_j|E_i)$. It follows that $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j)$ holds.

(2) Assume that the object $\tilde{x} \notin D_j$ is inserted into U. $\forall E_i \in U/A$, if $\tilde{x} \in E_i$, then we can obtain that $Pr(D_j'|E_i') < Pr(D_j|E_i)$ from Table 1. Thus, if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$ and $Pr(D_j|E_i) > Pr(D_j'|E_i') \geq \alpha$, then we have $E_i' = E_i \cup \{\tilde{x}\} \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}$; else if $Pr(D_j|E_i) \geq \alpha > Pr(D_j'|E_i')$, then $E_i' = E_i \cup \{\tilde{x}\} \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) - E_i$ holds.

Table 1
Updating patterns of the conditional probability with the insertion of a single object.

	Patterns	E'_i	D'_j	$Pr(D'_j\|E'_i)$	Variation
1.	$\tilde{x} \in E_i \wedge \tilde{x} \in D_j$	$E_i \cup \{\tilde{x}\}$	$D_j \cup \{\tilde{x}\}$	$(\|D_j \cap E_i\| + 1)/(\|E_i\| + 1)$	↑
2.	$\tilde{x} \notin E_i \wedge \tilde{x} \in D_j$	E_i	$D_j \cup \{\tilde{x}\}$	$\|D_j \cap E_i\|/\|E_i\|$	–
3.	$\tilde{x} \in E_i \wedge \tilde{x} \notin D_j$	$E_i \cup \{\tilde{x}\}$	D_j	$\|D_j \cap E_i\|/(\|E_i\| + 1)$	↓
4.	$\tilde{x} \notin E_i \wedge \tilde{x} \notin D_j$	E_i	D_j	$\|D_j \cap E_i\|/\|E_i\|$	–

Otherwise, if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, we have $Pr(D_j|E_i) < \alpha$. Hence, with decrease of the conditional probability, $Pr(D'_j|E'_i) < \alpha$ always hold, that is, $E'_i \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, then $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j)$. On the other hand, if $\tilde{x} \notin E_i$, we have $Pr(D'_j|E'_i) = Pr(D_j|E_i)$ from Table 1, then $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j)$ holds. □

Proposition 2. *Let $S = (U, AT = C \cup D)$ be an information system, $A \subseteq C$. When a single object $\tilde{x}$ is inserted into U, the (α, β)-probabilistic boundary region of D_j can be updated as:*

(1) *If the inserted object $\tilde{x}$ belongs to D_j, i.e., $\tilde{x} \in D_j$, then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j) - E_i, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \geq \alpha; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup E_i \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the inserted object $\tilde{x}$ does not belong to D_j, i.e., $\tilde{x} \notin D_j$, then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup E_i \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j) - E_i, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \leq \beta; \\ \mathrm{BND}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. The proof is similar to Proposition 1. □

Proposition 3. *Let $S = (U, AT = C \cup D)$ be an information system, $A \subseteq C$. When a single object $\tilde{x}$ is inserted into U, the (α, β)-probabilistic negative region of D_j can be updated as:*

(1) *If the inserted object $\tilde{x}$ belongs to D_j, i.e., $\tilde{x} \in D_j$, then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \leq \beta; \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j) - E_i, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) > \beta; \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the inserted object $\tilde{x}$ does not belong to D_j, i.e., $\tilde{x} \notin D_j$, then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j); \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup E_i \cup \{\tilde{x}\}, & \tilde{x} \in E_i \wedge E_i \not\subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \leq \beta; \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. The proof is similar to Proposition 1. □

Real-life data often varies with batches of data, i.e., the variation of multiple objects. Based on the incremental mechanisms with respect to the insertion of a single object, Propositions 4–6 introduce the incremental mechanisms for the updating of probabilistic approximations with the insertion of multiple objects. For the convenience of method derivations, let $S = (U, C \cup D)$. We assume that $\triangle U$ is an incremental object set which will be added into the information table, i.e., $U' = U \cup \triangle U$. With the addition of the new objects, the conditional partition and the decision classification of the universe will changed dynamically, i.e., some new objects will be merged into the existing classes while some new objects will form new classes. Assume that the conditional partition of the inserted object set $\triangle U$ is $\triangle U/A = \{M_1, M_2, \ldots, M_k, M_{k+1}, \ldots, M_p\}$, where $M_i (1 \leq i \leq k)$ denotes the incremental conditional classes which will be merged into the existing classes, and $M_i (k+1 \leq i \leq p)$ denotes the incremental conditional classes which will not be merged into any existing classes, i.e., form new conditional classes. Hence, the dynamic updating of the conditional partition on the updated universe $U' = U \cup \triangle U$ can be formalized as:

$$U'/A = \{E'_1, E'_2, \ldots, E'_k, E'_{k+1}, E'_{k+2}, \ldots, E'_m, E'_{m+1}, E'_{m+2}, \ldots, E'_{m+p-k}\}$$

where $E'_i = E_i \cup M_i (i = 1, 2, \cdots, k)$ denote the merged conditional classes, $E'_i = E_i (i = k+1, k+2, \cdots, m)$ denote the unchanged conditional classes, and $E'_i = M_{i-m+k} (i = m+1, m+2, \cdots, m+p-k)$ denote the newly inserted conditional classes.

Similarly, we assume that $\triangle U/D = \{Z_1, Z_2, \ldots, Z_{k'}, Z_{k'+1}, \ldots, Z_q\}$, and the dynamic updating of the decision classification on the updated universe $U' = U \cup \triangle U$ can be formalized as:

$$U'/D = \{D'_1, D'_2, \ldots, D'_{k'}, D'_{k'+1}, D'_{k'+2}, \ldots, D'_n, D'_{n+1}, D'_{n+2}, \ldots, D'_{n+q-k'}\}$$

where $D'_j = D_j \cup Z_j (j = 1, 2, \cdots, k')$ denote the merged decision classes, $D'_j = D_j (j = k'+1, k'+2, \cdots, n)$ denote the unchanged decision classes, and $D'_j = Z_{j-n+k'} (j = n+1, n+2, \cdots, n+q-k')$ denote the newly inserted decision classes.

Based on the conditional partition and the decision classification of the universe, Table 2 enumerates different updating patterns of the conditional probability of an object $x \in E_i$ belonging to D_j. In the first updating pattern, the variation tendency of the conditional probability can not be determined directly based on the updating of conditional class and decision class, "?" is used to denote the uncertainty of the variation tendency. Furthermore, the following updating patterns are discussed with respect to the existed conditional classes and decision classes in the original universe U. For the new generated classes in U', i.e., $E'_i (i = m+1, m+2, \cdots, m+p-k)$ and $D'_j (j = n+1, n+2, \cdots, n+q-k')$, the conditional probability should be estimated by definition since no prior information can be incorporated.

In the following, Propositions 4–6 introduce the updating mechanisms of the probabilistic approximations, when multiple objects are inserted into an information table.

Proposition 4. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When an object set $\triangle U$ is inserted into U, the (α, β)-probabilistic positive region of D_j can be updated as:*

(1) *If $D'_j = D_j \cup Z_j$, i.e., $j = 1, 2, \cdots, k'$, then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

Table 2
Updating patterns of the conditional probability with the insertion of multiple objects.

	Patterns	E_i'	D_j'	$Pr(D_j'\|E_i')$	Variation
1.	$i = 1, 2, \cdots, k \wedge j = 1, 2, \cdots, k'$	$E_i \cup M_i$	$D_j \cup Z_j$	$(\|D_j \cap E_i\| + \|M_i \cap Z_j\|)/(\|E_i\| + \|M_i\|)$	?
2.	$i = k+1, k+2, \cdots, m \wedge j = 1, 2, \cdots, k'$	E_i	$D_j \cup Z_j$	$\|D_j \cap E_i\|/\|E_i\|$	–
3.	$i = 1, 2, \cdots, k \wedge j = k'+1, k'+2, \cdots, n$	$E_i \cup M_i$	D_j	$\|D_j \cap E_i\|/(\|E_i\| + \|M_i\|)$	↓
4.	$i = k+1, k+2, \cdots, m \wedge j = k'+1, k'+2, \cdots, n$	E_i	D_j	$\|D_j \cap E_i\|/\|E_i\|$	-

Table 3
Updating patterns of the conditional probability with the deletion of a single object.

	Patterns	E_i'	D_j'	$Pr(D_j'\|E_i')$	Variation
1.	$\bar{x} \in E_i \wedge \bar{x} \in D_j$	$E_i - \{\bar{x}\}$	$D_j - \{\bar{x}\}$	$(\|D_j \cap E_i\| - 1)/(\|E_i\| - 1)$	↓
2.	$\bar{x} \notin E_i \wedge \bar{x} \in D_j$	E_i	$D_j - \{\bar{x}\}$	$\|D_j \cap E_i\|/\|E_i\|$	–
3.	$\bar{x} \in E_i \wedge \bar{x} \notin D_j$	$E_i - \{\bar{x}\}$	D_j	$\|D_j \cap E_i\|/(\|E_i\| - 1)$	↑
4.	$\bar{x} \notin E_i \wedge \bar{x} \notin D_j$	E_i	D_j	$\|D_j \cap E_i\|/\|E_i\|$	–

where $\Delta = \{E_i' \in U'/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D_j'|E_i') \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$, $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$;

(2) *Else if* $D_j' = D_j$, *i.e.,* $j = k'+1, k'+2, \cdots, n$, *then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) - \Theta,$$

where $\Theta = \{E_i \in U/A | Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$.

Proof. (1) Assume that the decision class D_j is updated as $D_j' = D_j \cup Z_j$. $\forall E_i (i = 1, 2, \cdots, k)$, according to the updating patterns of the conditional probability in Table 2, we have $Pr(D_j'|E_i') = (|D_j \cap E_i| + |M_i \cap Z_j|)/(|E_i| + |M_i|)$, which follows that if $Pr(Z_j|M_i) > Pr(D_j|E_i)$, then $Pr(D_j'|E_i') > Pr(D_j|E_i)$. With increase of the conditional probability, according to the Definition 5, we know that if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E_i' \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; else if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur, if $Pr(D_j'|E_i') \geq \alpha > Pr(D_j|E_i)$, then $E_i' \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, else if $\alpha > Pr(D_j'|E_i') > Pr(D_j|E_i)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$. Hence, we have $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Delta$, where $\Delta = \{E_i' \in U'/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D_j'|E_i') \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$. On the other hand, $\forall E_i (i = 1, 2, \cdots, k)$, if $Pr(Z_j|M_i) > Pr(D_j|E_i)$, we have $Pr(D_j'|E_i') < Pr(D_j|E_i)$. With decrease of the conditional probability, according to the Definition 5, we know that if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; otherwise, if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur, if $\alpha \leq Pr(D_j'|E_i') < Pr(D_j|E_i)$, then $E_i' \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, else if $Pr(D_j'|E_i') < \alpha \leq Pr(D_j|E_i)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) - \Delta'$, where $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$. Furthermore, $\forall E_i (i = k+1, k+2, \cdots, m)$, since the conditional probability will keep constant from Table 2, the positive region will keep constant with respect to the updating of E_i.

(2) Assume that the decision class D_j keep constant, i.e., $D_j' = D_j$. According to the updating patterns of the conditional probability in Table 2, $\forall E_i (i = 1, 2, \cdots, k)$, we know that $Pr(D_j'|E_i') < Pr(D_j|E_i)$. With decrease of conditional probability, according to the Definition 5, we know that if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; otherwise, if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur: if $\alpha \leq Pr(D_j'|E_i') < Pr(D_j|E_i)$, then $E_i' \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; else if $Pr(D_j'|E_i') < \alpha \leq Pr(D_j|E_i)$, then $E_i' \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) - \Theta$, where $\Theta = \{E_i \in U/A | Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$. $\forall E_i (i = k+1, k+2, \cdots, m)$, since the conditional probability will keep constant from Table 2, the positive region will keep constant with respect to the updating of E_i. □

Proposition 5. *Let* $S = (U, C \cup D)$ *be an information system,* $A \subseteq C$. *When an object set* $\triangle U$ *is inserted into* U, *the* (α, β)*-probabilistic boundary region of* D_j *can be updated as:*

(1) *If* $D_j' = D_j \cup Z_j$, *i.e.,* $1 \leq j \leq k'$, *then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

where if $Pr(Z_j|M_i) > Pr(D_j|E_i)$, *then* $\Delta = \{E_i' \in U'/A | \beta < Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$, $\Delta' = \{E_i \in U/A | Pr(D_j'|E_i') \geq \alpha, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$; *else if* $Pr(Z_j|M_i) < Pr(D_j|E_i)$, *then* $\Delta = \{E_i' \in U'/A | \beta < Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$, $\Delta' = \{E_i \in U/A | Pr(D_j'|E_i') \leq \beta, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$.

(2) *Else if* $D_j' = D_j$, *i.e.,* $j = k'+1, k'+2, \cdots, n$, *then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \Theta - \Theta',$$

where $\Theta = \{E_i' \in U'/A | \beta < Pr(D_j'|E_i') < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$, $\Theta' = \{E_i \in U/A | Pr(D_j'|E_i') \leq \beta, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$.

Proof. The proof is similar to Proposition 4. □

Proposition 6. *Let* $S = (U, C \cup D)$ *be an information system,* $A \subseteq C$. *When an object set* $\triangle U$ *is inserted into* U, *the* (α, β)*-probabilistic negative region of* D_j *can be updated as:*

(1) *If* $D_j' = D_j \cup Z_j$, *i.e.,* $j = 1, 2, \cdots, k'$, *then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

where $\Delta = \{E_i' \in U'/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D_j'|E_i') \leq \beta, E_i \not\subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$, $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D_j'|E_i') > \beta, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$;

Table 4
Updating patterns of the conditional probability with the deletion of multiple objects.

	Patterns	E_i'	D_j'	$Pr(D_j'\|E_i')$	Variation
1.	$i = 1, 2, \cdots, k \wedge j = 1, 2, \cdots, k'$	$E_i - M_i$	$D_j - Z_j$	$(\|D_j \cap E_i\| - \|M_i \cap Z_j\|)/(\|E_i\| - \|M_i\|)$	?
2.	$i = k+1, k+2, \ldots, m-p+k \wedge j = 1, 2, \ldots, k'$	E_i	$D_j - Z_j$	$\|D_j \cap E_i\|/\|E_i\|$	–
3.	$i = 1, 2, \ldots, k \wedge j = k'+1, k'+2, \ldots, n$	$E_i - M_i$	D_j	$\|D_j \cap E_i\|/(\|E_i\| - \|M_i\|)$	↑
4.	$i = k+1, k+2, \ldots, m \wedge j = k'+1, k'+2, \ldots, n-p+k'$	E_i	D_j	$\|D_j \cap E_i\|/\|E_i\|$	–

Table 5
The description of data sets.

No.	Data sets	Abbreviation	Samples	Attributes	Classes	Source
1	Solar flare	Solar Flare	1389	10	7	UCI
2	Seismic-bumps	Seismic-bumps	2584	19	2	UCI
3	Molecular biology (Splice-junction Gene Sequences)	Molecular Biology	3190	61	3	UCI
4	Nursery	Nursery	12,960	8	5	UCI
5	Letter recognition	Letter Recognition	20,000	16	26	UCI
6	Connect-4	Connect-4	67,557	42	3	UCI

(2) *Else if $D'_j = D_j$, i.e., $j = k'+1, k'+2, \cdots, n$, then*

$NEG'_{(\alpha,\beta)}(D_j) = NEG_{(\alpha,\beta)}(D_j) \cup \Theta$,

where $\Theta = \{E'_i \in U'/A | Pr(D'_j|E'_i) \le \beta, E_i \not\subseteq NEG_{(\alpha,\beta)}(D_j), i = 1, 2, \cdots, k\}$.

Proof. The proof is similar to Proposition 4. □

3.2. Updating probabilistic approximations when deleting objects

This section introduces incremental updating mechanisms of the probabilistic approximations with the deletion of objects in a dynamic information table. We also discuss a special case of the deletion of objects firstly, i.e., a single object is deleted from an information table. Given an information table $S = (U, C \cup D)$, $A \subseteq C$. We assume that $U/A = \{E_1, E_2, \ldots, E_m\}$, $U/D = \{D_1, D_2, \ldots, D_n\}$. For any conditional class $E_i \in U/A$, there are two updating situations will occur when a single object $\tilde{x}$ is deleted from U, i.e., if the deleted object $\bar{x}$ satisfies that $\exists x \in E_i$, $(x, \tilde{x}) \in R_A$, then we have $E'_i = E_i - \{\bar{x}\}$, else the conditional class will keeps constant, $E'_i = E_i$. Similarly, for any decision class $D_j \in U/D$, if $\exists x \in D_j$, $\bar{x}$ satisfies that $\forall d \in D$, $I_d(x) = I_d(\tilde{x})$, then $D'_j = D_j - \{\bar{x}\}$, else D_j keep constant, $D'_j = D_j$. Based on the different updating situations of the conditional class and the decision class, Table 1 enumerates the variation tendencies of conditional probability when a single object is deleted from an information table.

Based on the above four updating patterns of the conditional probability with the decrease of universe, Propositions 7–9 introduce the incremental mechanisms for the updating of probabilistic approximations with the deletion of a single object.

Proposition 7. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When a single object $\bar{x}$ is deleted from U, the (α, β)-probabilistic positive region of D_j can be updated as:*

(1) *If the deleted object $\bar{x}$ belongs to D_j, i.e., $\bar{x} \in D_j$, then*

$$POS'_{(\alpha,\beta)}(D_j) = \begin{cases} POS_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq POS_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \ge \alpha; \\ POS_{(\alpha,\beta)}(D_j) - E_i, & \bar{x} \in E_i \wedge E_i \subseteq POS_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) < \alpha; \\ POS_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the deleted object $\bar{x}$ does not belong to D_j, i.e., $\bar{x} \notin D_j$, then*

$$POS'_{(\alpha,\beta)}(D_j) = \begin{cases} POS_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq POS_{(\alpha,\beta)}(D_j); \\ POS_{(\alpha,\beta)}(D_j) \cup E_i - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \not\subseteq POS_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \ge \alpha; \\ POS_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. (1) Assume that the object $\bar{x} \in D_j$ is deleted from U. Based on the updating patterns of the conditional probability in Table 3, we have $\forall E_i \in U/A$, if $\bar{x} \in E_i$, then $Pr(D'_j|E'_i) < Pr(D_j|E_i)$. Then, $\forall E_i \subseteq POS_{(\alpha,\beta)}(D_j)$, two updating situations may occur, that is, if $\alpha \le Pr(D'_j|E'_i) < Pr(D_j|E_i)$, then $E'_i = E_i - \{\bar{x}\} \subseteq POS'_{(\alpha,\beta)}(D_j)$, i.e., $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j) - \{\bar{x}\}$; else if $Pr(D'_j|E'_i) < \alpha \le Pr(D_j|E_i)$, then we have $E'_i = E_i - \{\bar{x}\} \not\subseteq POS'_{(\alpha,\beta)}(D_j)$, i.e., $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j) - E_i$. $\forall E_i \in U/A$, if $\bar{x} \in E_i$, and $E_i \not\subseteq POS_{(\alpha,\beta)}(D_j)$, then according to the Definition 5, with decrease of the conditional probability, $E'_i \not\subseteq POS'_{(\alpha,\beta)}(D_j)$ holds. Furthermore, $\forall E_i \in U/A$, if $\bar{x} \notin E_i$, then we know that $Pr(D'_j|E'_i) = Pr(D_j|E_i)$ from Table 3. Hence, $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j)$ always hold.

(2) Assume that the object $\bar{x} \notin D_j$ is deleted from U. $\forall E_i \in U/A$, if $\bar{x} \in E_i$, then we can obtain that $Pr(D'_j|E'_i) > Pr(D_j|E_i)$ from Table 3. Then, based on Definition 5, with increase of the conditional probability, if $E_i \subseteq POS_{(\alpha,\beta)}(D_j)$, then $E'_i = E_i - \{\bar{x}\} \subseteq POS_{(\alpha,\beta)}(D_j)$, which implies $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j) - \{\bar{x}\}$; else if $E_i \not\subseteq POS_{(\alpha,\beta)}(D_j)$, since $Pr(D'_j|E'_i) > Pr(D_j|E_i)$, two updating situations may occur: if $Pr(D'_j|E'_i) \ge \alpha > Pr(D_j|E_i)$, then $E'_i = E_i - \{\bar{x}\} \subseteq POS'_{(\alpha,\beta)}(D_j)$, i.e., $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j) \cup E_i - \{\bar{x}\}$; else if $\alpha > Pr(D'_j|E'_i) > Pr(D_j|E_i)$, we have $E'_i \not\subseteq POS'_{(\alpha,\beta)}(D_j)$ is true. On the other hand, $\forall E_i \in U/A$, if $\bar{x} \notin E_i$, then $Pr(D'_j|E'_i) = Pr(D_j|E_i)$ holds from Table 1. Thus $POS'_{(\alpha,\beta)}(D_j) = POS_{(\alpha,\beta)}(D_j)$ always hold. □

Proposition 8. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When a single object $\bar{x}$ is deleted from U, the (α, β)-probabilistic boundary region of D_j can be updated as:*

(1) *If the deleted object $\bar{x}$ belongs to D_j, i.e., $\bar{x} \in D_j$, then*

$$BND'_{(\alpha,\beta)}(D_j) = \begin{cases} BND_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq BND_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ BND_{(\alpha,\beta)}(D_j) - E_i, & \bar{x} \in E_i \wedge E_i \subseteq BND_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \le \beta; \\ BND_{(\alpha,\beta)}(D_j) \cup E_i - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq POS_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ BND_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the deleted object $\bar{x}$ does not belong to D_j, i.e., $\bar{x} \notin D_j$, then*

$$BND'_{(\alpha,\beta)}(D_j) = \begin{cases} BND_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq BND_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ BND_{(\alpha,\beta)}(D_j) - E_i, & \bar{x} \in E_i \wedge E_i \subseteq BND_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \ge \alpha; \\ BND_{(\alpha,\beta)}(D_j) \cup E_i - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq NEG_{(\alpha,\beta)}(D_j) \\ & \quad \wedge \beta < Pr(D'_j|E'_i) < \alpha; \\ BND_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. The proof is similar to Proposition 7. □

Proposition 9. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When a single object $\bar{x}$ is deleted from U, the (α, β)-probabilistic negative region of D_j can be updated as:*

(1) *If the deleted object $\bar{x}$ belongs to D_j, i.e., $\bar{x} \in D_j$, then*

$$NEG'_{(\alpha,\beta)}(D_j) = \begin{cases} NEG_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq NEG_{(\alpha,\beta)}(D_j); \\ NEG_{(\alpha,\beta)}(D_j) \cup E_i - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \not\subseteq NEG_{(\alpha,\beta)}(D_j) \\ & \quad \wedge Pr(D'_j|E'_i) \le \beta; \\ NEG_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

(2) *Else if the deleted object $\bar{x}$ does not belong to D_j, i.e., $\bar{x} \notin D_j$, then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \begin{cases} \mathrm{NEG}_{(\alpha,\beta)}(D_j) - \{\bar{x}\}, & \bar{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \wedge Pr(D'_j|E'_i) \leq \beta; \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j) - E_i, & \bar{x} \in E_i \wedge E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j) \\ & \wedge Pr(D'_j|E'_i) > \beta; \\ \mathrm{NEG}_{(\alpha,\beta)}(D_j), & Otherwise. \end{cases}$$

Proof. The proof is similar to Proposition 7. □

In the following, we discuss the incremental solution for updating probabilistic approximations when multiple objects are deleted from an information table. Assume that $\triangle U$ is an incremental object set which will be deleted from the universe, i.e., $U' = U - \triangle U$. And, the conditional partition on the deleted object set $\triangle U$ is $\triangle U/A = \{M_1, M_2, \ldots, M_k, M_{k+1}, \ldots, M_p\}$, where $M_i(1 \leq i \leq k)$ denotes the partial objects in the existed conditional classes, i.e., $M_i \subset E_i$, which will be deleted from E_i; $M_i(k+1 \leq i \leq p)$ denotes the existed conditional class which will be removed entirely, i.e., $M_i = E_i$. Then, the dynamic updating of the conditional partition on the updated universe $U' = U - \triangle U$ can be formalized as:

$$U'/A = \{E'_1, E'_2, \ldots, E'_k, E'_{k+1}, E'_{k+2}, \ldots, E'_{m-p+k}\}$$

where $E'_i = E_i - M_i (i = 1, 2, \cdots, k)$ denote the divided conditional classes, and $E'_i = E_i (i = k+1, k+2, \cdots, m-p+k)$ denote the unchanged conditional classes.

Similarly, for the updating of decision classification, assume that $\triangle U/D = \{Z_1, Z_2, \ldots, Z_{k'}, Z_{k'+1}, \ldots, Z_q\}$, the dynamic updating of the decision classification on the updated universe $U' = U - \triangle U$ can be also formalized as:

$$U'/D = \{D'_1, D'_2, \ldots, D'_{k'}, D'_{k'+1}, D'_{k'+2}, \ldots, D'_{n-q+k'}\}$$

where $D'_j = D_j - Z_j (j = 1, 2, \cdots, k')$ denote the divided decision classes, and $D'_j = D_j (j = k'+1, k'+2, \cdots, n-q+k')$ denote the unchanged decision classes.

Based on the updating of the conditional partition and the decision classification of the universe, all possible variation tendencies of the conditional probability of any object $x \in E_i$ belonging to D_i can be summarized in Table 4.

In the following, Propositions 10–12 introduce the updating mechanisms of the probabilistic approximations, when multiple objects are deleted from an information table.

Proposition 10. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When an object set $\triangle U$ is deleted from U, the (α, β)-probabilistic positive region of D_j can be updated as:*

(1) *If $D'_j = D_j - Z_j$, i.e., $j = 1, 2, \ldots, k'$, then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

where $\Delta = \{E'_i \in U'/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D'_j|E'_i) \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$, $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D'_j|E'_i) < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$;

(2) *Else if $D'_j = D_j$, i.e., $j = k'+1, k'+2, \ldots, n-p+k'$, then*

$$\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Theta,$$

where $\Theta = \{E'_i \in U'/A | Pr(D'_j|E'_i) \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$.

Proof. (1) Assume that the decision class D_j is updated as $D'_j = D_j \cup Z_j$. $\forall E_i (i = 1, 2, \ldots, k)$, according to the updating patterns of the conditional probability in Table 4, we have $Pr(D'_j|E'_i) = (|D_j \cap E_i| - |M_i \cap Z_j|)/(|E_i| - |M_i|)$, which follows that if $Pr(Z_j|M_i) < Pr(D_j|E_i)$, then $Pr(D'_j|E'_i) > Pr(D_j|E_i)$. With increase of the conditional probability, according to the Definition 5, we know that if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E'_i \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$ always hold; else if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur: if $Pr(D'_j|E'_i) \geq \alpha > Pr(D_j|E_i)$, then $E'_i \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; else if $\alpha > Pr(D'_j|E'_i) > Pr(D_j|E_i)$, then $E'_i \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$. Hence, we have $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Delta$, where $\Delta = \{E'_i \in U'/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D'_j|E'_i) \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$. On the other hand, $\forall E_i (i = 1, 2, \ldots, k)$, if $Pr(Z_j|M_i) > Pr(D_j|E_i)$, we have $Pr(D'_j|E'_i) < Pr(D_j|E_i)$. With decrease of the conditional probability, according to the Definition 5, we know that if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E'_i \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$ always hold; otherwise, if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur: if $\alpha \leq Pr(D'_j|E'_i) < Pr(D_j|E_i)$, then $E'_i \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$; else if $Pr(D'_j|E'_i) < \alpha \leq Pr(D_j|E_i)$, then $E'_i \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$, which implies that $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) - \Delta'$, where $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D'_j|E'_i) < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$. Furthermore, $\forall E_i (i = k+1, k+2, \ldots, m)$, since the conditional probability will keep constant from Table 4, the positive region will keep constant with respect to the updating of E_i.

(2) Assume that the decision class D_j keep constant, i.e., $D'_j = D_j$. According to the updating patterns of the conditional probability in Table 4, $\forall E_i (i = 1, 2, \ldots, k)$, we know that $Pr(D'_j|E'_i) > Pr(D_j|E_i)$. With increase of conditional probability, according to the Definition 5, we know that if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, then $E'_i \subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$ always hold; otherwise, if $E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$, two situations may occur: if $\alpha > Pr(D'_j|E'_i) > Pr(D_j|E_i)$, then $E'_i \not\subseteq \mathrm{POS}'_{(\alpha,\beta)}(D_j)$ is true; else if $Pr(D'_j|E'_i) \geq \alpha > Pr(D_j|E_i)$, then E'_i should be taken as positive, i.e., $\mathrm{POS}'_{(\alpha,\beta)}(D_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup \Theta$, where $\Theta = \{E'_i \in U'/A | Pr(D'_j|E'_i) \geq \alpha, E_i \not\subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$. $\forall E_i (i = k+1, k+2, \ldots, m)$, since the conditional probability will keep constant from Table 4, the positive region will keep constant with respect to the updating of E_i. □

Proposition 11. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When an object set $\triangle U$ is deleted from U, the (α, β)-probabilistic boundary region of D_j can be updated as:*

(1) *If $D'_j = D_j \cup Z_j$, i.e., $1 \leq j \leq k'$, then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

where if $Pr(Z_j|M_i) < Pr(D_j|E_i)$, then $\Delta = \{E'_i \in U'/A | \beta < Pr(D'_j|E'_i) < \alpha, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$, $\Delta' = \{E_i \in U/A | Pr(D'_j|E'_i) \geq \alpha, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$; else if $Pr(Z_j|M_i) > Pr(D_j|E_i)$, then $\Delta = \{E'_i \in U'/A | \beta < Pr(D'_j|E'_i) < \alpha, E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$, $\Delta' = \{E_i \in U/A | Pr(D'_j|E'_i) \leq \beta, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$.

(2) *Else if $D'_j = D_j$, i.e., $j = k'+1, k'+2, \ldots, n-p+k'$, then*

$$\mathrm{BND}'_{(\alpha,\beta)}(D_j) = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup \Theta - \Theta',$$

where $\Theta = \{E'_i \in U'/A | \beta < Pr(D'_j|E'_i) < \alpha, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$, $\Theta' = \{E_i \in U/A | Pr(D'_j|E'_i) \geq \alpha, E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$.

Proof. The proof is similar to Proposition 10. □

Proposition 12. *Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$. When an object set $\triangle U$ is deleted from U, the (α, β)-probabilistic negative region of D_j can be updated as:*

(1) *If $D'_j = D_j \cup Z_j$, i.e., $j = 1, 2, \ldots, k'$, then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup \Delta - \Delta',$$

Algorithm 1: Updating probabilistic approximations with increase of conditional probability.

Input:
(1) The probabilistic approximations: $\mathrm{POS}_{(\alpha,\beta)}(D_j)$, $\mathrm{BND}_{(\alpha,\beta)}(D_j)$, $\mathrm{NEG}_{(\alpha,\beta)}(D_j)$.
(2) The updated conditional class and decision class: $E_i' = E_i \cup M_i$ or $E_i - M_i$, $D_j' = D_j \cup Z_j$ or $D_j - Z_j$.
Output:
The updated probabilistic approximations: $\mathrm{POS}'_{(\alpha,\beta)}(D_j')$, $\mathrm{BND}'_{(\alpha,\beta)}(D_j')$, $\mathrm{NEG}'_{(\alpha,\beta)}(D_j')$.

begin
  if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$ **then**
    $\mathrm{POS}'_{(\alpha,\beta)}(D_j') = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup M_i$;
  else if $E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j)$ **then**
    if $Pr(D_j'|E_i') \ge \alpha$ **then** $\mathrm{BND}'_{(\alpha,\beta)}(D_j') = \mathrm{BND}_{(\alpha,\beta)}(D_j') - E_i$, $\mathrm{POS}'_{(\alpha,\beta)}(D_j') = \mathrm{POS}_{(\alpha,\beta)}(D_j') \cup E_i'$;
    else $\mathrm{BND}'_{(\alpha,\beta)}(D_j') = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup M_i$;
  else
    if $Pr(D_j'|E_i') \ge \alpha$ **then** $\mathrm{NEG}'_{(\alpha,\beta)}(D_j') = \mathrm{NEG}_{(\alpha,\beta)}(D_j') - E_i$, $\mathrm{POS}'_{(\alpha,\beta)}(D_j') = \mathrm{POS}_{(\alpha,\beta)}(D_j') \cup E_i'$;
    else if $\beta < Pr(D_j'|E_i') < \alpha$ **then** $\mathrm{NEG}'_{(\alpha,\beta)}(D_j') = \mathrm{NEG}_{(\alpha,\beta)}(D_j') - E_i$, $\mathrm{BND}'_{(\alpha,\beta)}(D_j') = \mathrm{BND}_{(\alpha,\beta)}(D_j') \cup E_i'$;
    else $\mathrm{NEG}'_{(\alpha,\beta)}(D_j') = \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup M_i$;
  end
  Output probabilistic approximations $\mathrm{POS}'_{(\alpha,\beta)}(D_j')$, $\mathrm{BND}'_{(\alpha,\beta)}(D_j')$, $\mathrm{NEG}'_{(\alpha,\beta)}(D_j')$, where $D_j' \in U \cup \Delta U/D$.
end

where $\Delta = \{E_i' \in U'/A | Pr(Z_j|M_i) > Pr(D_j|E_i) \wedge Pr(D_j'|E_i') \le \beta, E_i \not\subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$, $\Delta' = \{E_i \in U/A | Pr(Z_j|M_i) < Pr(D_j|E_i) \wedge Pr(D_j'|E_i') > \beta, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$;

(2) *Else if* $D_j' = D_j$, *i.e.*, $j = k'+1, k'+2, \ldots, n-p+k'$, *then*

$$\mathrm{NEG}'_{(\alpha,\beta)}(D_j) = \mathrm{NEG}_{(\alpha,\beta)}(D_j) - \Theta,$$

where $\Theta = \{E_i \in U/A | Pr(D_j'|E_i') > \beta, E_i \subseteq \mathrm{NEG}_{(\alpha,\beta)}(D_j), i = 1, 2, \ldots, k\}$.

Proof. The proof is similar to Proposition 10. □

4. Algorithms for updating probabilistic approximations with incremental objects

Based on the incremental mechanisms presented in Section 3, we know that the key issue of the dynamic change of probabilistic approximations is the variation of conditional probability. Different updating patterns of conditional probability will lead to different maintenance mechanisms. In accordance with the incremental mechanisms, we proposed two incremental algorithms for updating probabilistic approximations with the variation of conditional probability in terms of the update of a decision class and a conditional class. The detailed algorithm procedures are outlined in Algorithms 1 and 2.

Considering the dynamic updating of the conditional partition and the decision classification with incremental objects, two incremental algorithms for updating probabilistic approximations are developed with the insertion and deletion of incremental objects by incorporating Algorithms 1 and 2. The detailed pseudo-code of the incremental algorithms are given in Algorithms 3 and 4.

Algorithm 4 is an incremental algorithm for updating probabilistic approximations when inserting objects. Step 2 is computing the new conditional partition and decision classification on the incremental objects ΔU, whose time complexity is $O(|\Delta U|^2)$; Steps 3–6 are updating the original conditional partition and decision classification on U with the combination of new results on ΔU, whose time complexity is $O(|U||\Delta U|)$. Steps 7–18 are updating probabilistic approximations *w.r.t.*, the merged decision classes, in which Steps 8–14 are updating probabilistic approximations with Algorithms 1 and 2 according to the incremental mechanisms presented in Propositions 4–6, whose time complexity is $O(|U|kk')$, where k and k' are the number of the merged decision and conditional classes, respectively; Steps 15–17 are computing the probabilistic approximations w.r.t., the newly inserted conditional classes based on the definition since no prior results can be used, whose time complexity is $O(|U|(p-k))$, where $p-k$ is the number of the newly inserted conditional classes; Steps 19–23 are updating probabilistic approximations w.r.t., the unchanged decision classes and the merged conditional classes by Algorithm 3, whose time complexity is $O(|U|(n-k')k)$, and $n-k'$ is the number of the unchanged decision classes; Steps 24–26 are computing probabilistic approximations w.r.t., the newly inserted decision classes based on the definition since no prior results can be used, whose time complexity is $O(|U|(q-k'))$, and $q-k'$ is the number of the newly inserted decision classes. Considering that the number of the conditional and decision classes are often much smaller than the size of universe, and the size of incremental objects is smaller than the size of original objects, the time complexity of Algorithm 4 is $O(|U||\Delta U|)$.

Algorithm 4 is an incremental algorithm for updating probabilistic approximations when deleting objects. Step 2 is computing the conditional partition and decision classification on the incremental objects ΔU, whose time complexity is $O(|\Delta U|^2)$; Step 3 is removing the wholly deleted conditional classes from probabilistic approximations, whose time complexity is $O(|\Delta U|)$; Steps 4–7 are updating the original conditional partition and decision classification on U with the removal of computational results on ΔU, whose time complexity is $O(|U-\Delta U||\Delta U|)$. Steps 8–16 are updating probabilistic approximations with Algorithms 1 and 2 according to the incremental mechanisms presented in Propositions 10–12, whose time complexity is $O(|U|kk')$, where k and k' are the number of the divided decision and conditional classes, respectively; Steps 17–21 are updating probabilistic approximations w.r.t., the unchanged decision classes and the divided conditional classes with Algorithm 1, whose time complexity is $O(|U|(n-q)k$, where $n-q$ is the number of the unchanged decision classes. Hence, the time complexity of Algorithm 4 is $O(|U-\Delta U||\Delta U|)$.

Algorithm 2: Updating probabilistic approximations with decrease of conditional probability.

Input:
(1) The probabilistic approximations: $\mathrm{POS}_{(\alpha,\beta)}(D_j)$, $\mathrm{BND}_{(\alpha,\beta)}(D_j)$, $\mathrm{NEG}_{(\alpha,\beta)}(D_j)$.
(2) The updated conditional class and decision class: $E'_i = E_i \cup M_i$ or $E_i - M_i$, $D'_j = D_j \cup Z_j$ or $D_j - Z_j$.
Output:
The updated probabilistic approximations: $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$.

begin
  if $E_i \subseteq \mathrm{POS}_{(\alpha,\beta)}(D_j)$ **then**
    if $Pr(D'_j|E'_i) \geq \alpha$ **then** $\mathrm{POS}'_{(\alpha,\beta)}(D'_j) = \mathrm{POS}_{(\alpha,\beta)}(D_j) \cup M_i$;
    else if $\beta < Pr(D'_j|E'_i) < \alpha$ **then** $\mathrm{POS}'_{(\alpha,\beta)}(D'_j) = \mathrm{POS}_{(\alpha,\beta)}(D'_j) - E_i$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j) = \mathrm{BND}_{(\alpha,\beta)}(D'_j) \cup E'_i$;
    else $\mathrm{POS}'_{(\alpha,\beta)}(D'_j) = \mathrm{POS}_{(\alpha,\beta)}(D'_j) - E_i$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j) = \mathrm{NEG}_{(\alpha,\beta)}(D'_j) \cup E'_i$;
  else if $E_i \subseteq \mathrm{BND}_{(\alpha,\beta)}(D_j)$ **then**
    if $\beta < Pr(D'_j|E'_i) < \alpha$ **then** $\mathrm{BND}'_{(\alpha,\beta)}(D'_j) = \mathrm{BND}_{(\alpha,\beta)}(D_j) \cup M_i$;
    else $\mathrm{BND}'_{(\alpha,\beta)}(D'_j) = \mathrm{BND}_{(\alpha,\beta)}(D'_j) - E_i$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j) = \mathrm{NEG}_{(\alpha,\beta)}(D'_j) \cup E'_i$;
  else
    $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j) = \mathrm{NEG}_{(\alpha,\beta)}(D_j) \cup M_i$;
  end
  Output probabilistic approximations $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$, where $D'_j \in U \cup \Delta U/D$.
end

Algorithm 3: Incremental algorithm for updating probabilistic approximations when inserting objects.

Input:
(1) An information table $S = (U, C \cup D)$, where $A \subseteq C$;
(2) The conditional partition $U/A = \{E_1, E_2, \ldots, E_m\}$, and the decision classification $U/D = \{D_1, D_2, \ldots, D_n\}$;
(3) The probabilistic approximations: $\mathrm{POS}_{(\alpha,\beta)}(D_j)$, $\mathrm{BND}_{(\alpha,\beta)}(D_j)$, $\mathrm{NEG}_{(\alpha,\beta)}(D_j)$.
(4) The newly object set ΔU that will be inserted into S.
Output:
The updated probabilistic approximations: $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$.

begin
  Compute $\triangle U/A = \{M_1, \ldots, M_k, \ldots, M_p\}$, $\triangle U/D = \{Z_1, \ldots, Z_{k'}, \ldots, Z_q\}$;
  Update $U \cup \triangle U/A = \{E'_1, \ldots, E'_k, \ldots, E'_m, \ldots, E'_{m+p-k}\}$,
  where $E'_i = E_i \cup M_i (i = 1, \ldots, k)$, $E'_i = E_i (i = k+1, \ldots, m)$, $E'_i = M_{i-m+k} (i = m+1, \ldots, m+p-k)$;
    $U \cup \triangle U/D = \{D'_1, \ldots, D'_{k'}, \ldots, D'_n, \ldots, D'_{n+q-k'}\}$,
  where $D'_j = D_j \cup Z_j (j = 1, \ldots, k')$, $D'_j = D_j (j = k'+1, \ldots, n)$, $D'_j = Z_{j-n+k'} (j = n+1, \ldots, n+q-k')$;
  for $1 \leq j \leq k'$ **do** // The merged decision classes: $D'_j = D_j \cup Z_j$;
    for $1 \leq i \leq k$ **do** // The merged conditional classes: $E'_i = E_i \cup M_i$;
      if $Pr(Z_j|M_i) > Pr(D_j|E_i)$ **then**
        Update probabilistic approximations by Algorithm 1;
      else if $Pr(Z_j|M_i) < Pr(D_j|E_i)$ **then**
        Update probabilistic approximations by Algorithm 2;
      end
    end
    for $m+1 \leq i \leq m+p-k$ **do** // The newly inserted conditional classes: $E'_i = M_{i-m+k}$;
      Compute probabilistic approximations according to Definition 5
    end
  end
  for $k'+1 \leq j \leq n$ **do** // The unchanged decision classes: $D'_j = D_j$;
    for $1 \leq i \leq k$ **do** // The merged conditional classes: $E'_i = E_i \cup M_i$;
      Update probabilistic approximations by Algorithm 2;
    end
  end
  for $n+1 \leq j \leq n+q-k'$ **do** // The newly inserted decision classes: $D'_j = D_{j-n+k'}$;
    Compute probabilistic approximations according to Definition 5
  end
  Output probabilistic approximations $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$ for $\forall D'_j \in U \cup \Delta U/D$.
end

Algorithm 4: Incremental algorithm for updating probabilistic approximations when deleting objects.

Input:
(1) An information table $S = (U, C \cup D)$, where $A \subseteq C$;
(2) The conditional partition $U/A = \{E_1, E_2, \dots, E_m\}$, and the decision classification $U/D = \{D_1, D_2, \dots, D_n\}$;
(3) The probabilistic approximations: $\mathrm{POS}_{(\alpha,\beta)}(D_j)$, $\mathrm{BND}_{(\alpha,\beta)}(D_j)$, $\mathrm{NEG}_{(\alpha,\beta)}(D_j)$.
(4) The old object set ΔU that will be deleted from S.
Output:
The updated probabilistic approximations: $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$.

begin
  Compute $\triangle U/A = \{M_1, \dots, M_k, \dots, M_p\}$, $\triangle U/D = \{Z_1, \dots, Z_{k'}, \dots, Z_q\}$;
  Remove $M_i (i = k+1, \dots, p)$ from $\mathrm{POS}(D_j)$, $\mathrm{BND}(D_j)$, $\mathrm{NEG}(D_j) (j = k'+1, \dots, q)$;
  Update $U - \triangle U/A = \{E'_1, \dots, E'_k, \dots, E'_{m-p+k}\}$,
    where $E'_i = E_i - M_i (i = 1, 2, \dots, k)$, $E'_i = E_i (i = k+1, k+2, \dots, m-p+k)$;
    $U - \triangle U/D = \{D'_1, \dots, D'_{k'}, \dots, D'_{n-q+k'}\}$,
    where $D'_j = D_j - Z_j (j = 1, 2, \dots, k')$, $D'_j = D_j (j = k'+1, k'+2, \dots, n-q+k')$;
  for $1 \le j \le k'$ **do** // The divided decision classes: $D'_j = D_j - Z_j$;
    for $1 \le i \le k$ **do** // The divided conditional classes: $E'_i = E_i - M_i$;
      if $Pr(Z_j|M_i) < Pr(D_j|E_i)$ **then**
        Update probabilistic approximations by Algorithm 1;
      else if $Pr(Z_j|M_i) > Pr(D_j|E_i)$ **then**
        Update probabilistic approximations by Algorithm 2;
      end
    end
  end
  for $k'+1 \le j \le n-q+k'$ **do** // The unchanged decision classes: $D'_j = D_j$;
    for $1 \le i \le k$ **do** // The divided conditional classes: $E'_i = E_i - M_i$;
      Update probabilistic approximations by Algorithm 1;
    end
  end
  Output probabilistic approximations $\mathrm{POS}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{BND}'_{(\alpha,\beta)}(D'_j)$, $\mathrm{NEG}'_{(\alpha,\beta)}(D'_j)$ for $\forall D'_j \in U \cup \Delta U/D$.
end

5. Experimental analysis

In this section, we evaluate the efficiency of the incremental algorithms for updating probabilistic approximations with the dynamic variation of incremental objects. We present an experimental evaluation using six data sets from the machine learning data repository, University of California at Irvine [16]. The detailed information of experimental data sets is outlined in Table 5. We implemented algorithms in Java on Eclipse Kepler platform with JDK 1.7.0, which carried out on a PC with the operation system Windows 8, Intel(R) Core(TM) i3-3227U CPU GHz and 8 GB memory. Furthermore, we present an analytical comparison of the proposed incremental algorithms, and derive the incremental speedup factors to show the performance more clearly, where the incremental speedup is defined as $\frac{T_n}{T_i}$, where T_n is the execution time of the non-incremental algorithm and T_i is the execution time of the incremental algorithms.

To compare the computational efficiency of the incremental and non-incremental algorithms with the insertion of incremental objects, we performed experiments on each data set in Table 5. Let U denote its universe. We selected 50% objects as the basic data set, then R_i^p objects from the remaining universe are randomly selected as the incremental objects which will be inserted into the basic data set, and $R_i^p (p = 1, 2, \dots, 10)$ is the ratio of the size of inserting objects and the size of original objects, varying from 10% up to 100%. Fig. 1 displays the detailed change trend of computation time *versus* different inserting ratios. As shown in each sub-figure of Fig. 1, it is easy to see that the proposed incremental algorithm is always faster than the non-incremental counterpart. The Wilcoxon signed-rank test is conducted at a 5% significance level to establish that the difference in computation time is statistically significant. It appears that the p-value of the observed results in Fig. 1 is $8.345e-12$, which guaranteed that the performance of incremental algorithm is significantly better than the non-incremental algorithm.

Table 6 further presents the incremental speedup *versus* different inserting ratios of the objects. It can be observed that the incremental algorithm yields $2.3 - 14.1\times$ speedup over the non-incremental algorithm with the inserting ratio 10%. A further observation is that the speedup is becoming smaller with the increase of the inserting ratio. However, the speedup still achieves $1.3 - 2.6\times$ when the inserting ratio is 100%.

To compare the computational efficiency of the incremental and non-incremental algorithms with the deletion of incremental objects, we randomly draw R_d^q objects from the whole universe U as the deleting objects, and $R_d^q (q = 1, 2, \dots, 9)$ is the ratio of the size of deleting objects and the size of original objects, varying from 10% up to 90%. Fig. 2 displays the detailed change trend of computation time *versus* different deleting ratios. From Fig. 2, we can observe that the incremental algorithm usually performs faster than the non-incremental algorithm until a threshold value of the deleting ratio is reached, such as 85% for the data set Solar Flare, 80% for the data sets Nursery and Connect-4. And the advantage of the incremental algorithm is gradually narrowing with the increase

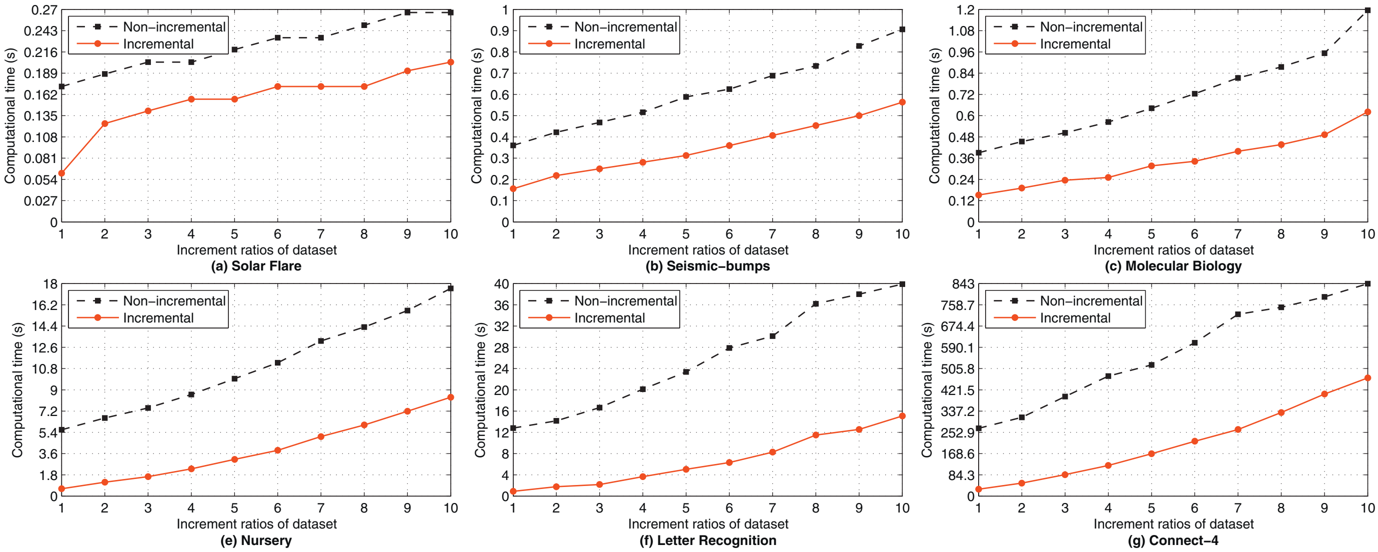

Fig. 1. Computation times of incremental and non-incremental algorithms *versus* different inserting ratios.

Table 6
The incremental speedup *versus* different inserting ratios of the objects.

Data sets	R_d(Increment ratios)									
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
Solar Flare	2.774	1.504	1.440	1.301	1.404	1.361	1.361	1.454	1.385	1.310
Seismic-bumps	2.308	1.936	1.872	1.836	1.879	1.741	1.690	1.617	1.656	1.609
Molecular biology	2.572	2.377	2.131	2.247	2.025	2.117	2.038	2.002	1.937	1.921
Nursery	9.008	5.615	4.521	3.729	3.189	2.909	2.601	2.377	2.186	2.100
Letter recognition	14.079	8.014	7.608	5.476	4.630	4.403	3.641	3.149	3.027	2.645
Connect-4	9.755	6.043	4.621	3.910	3.094	2.798	2.729	2.258	1.953	1.797

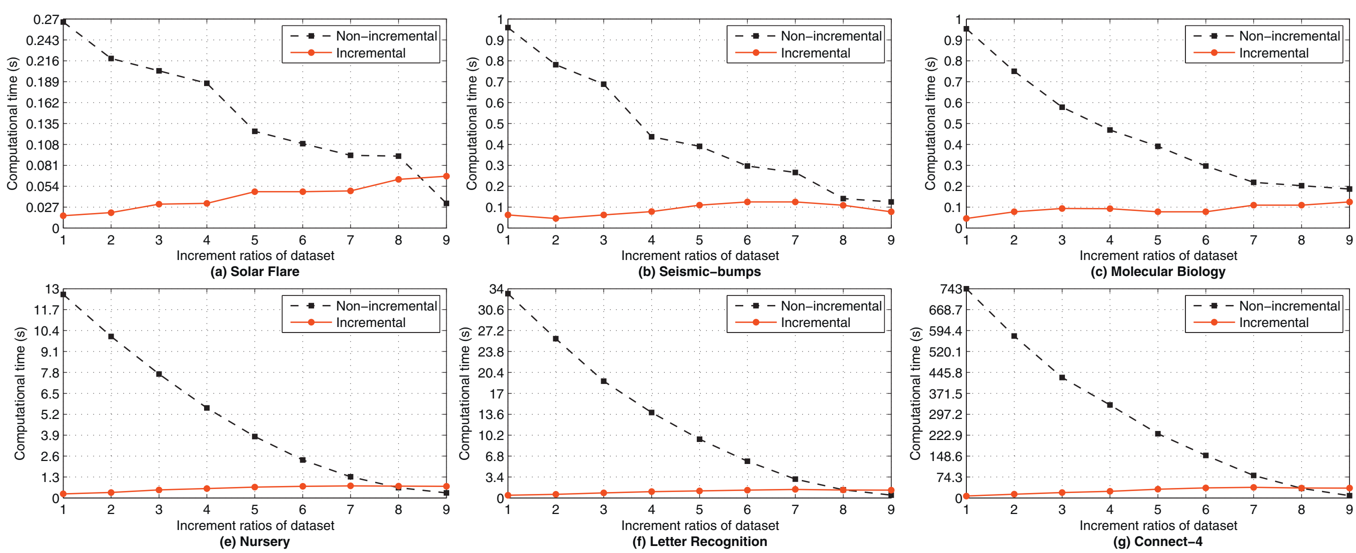

Fig. 2. Computation times of incremental and non-incremental algorithms *versus* different deleting ratios.

of deleting ratio. The Wilcoxon signed-rank test is conducted at a 5% significance level to establish that the difference in computation time is statistically significant. The *p*-value of the observed results in Fig. 2 is $1.436e-07$, which guaranteed that the performance of incremental algorithm is significantly better than the non-incremental algorithm with different deleting ratios.

The incremental speedups *versus* different deleting ratios of the objects are recorded in Table 6. We can find that the incremental algorithm yields $15.2-109.7\times$ speedup over the non-incremental algorithm with the deleting ratio 10%. The advantage of computational efficiency is more obvious than the insertion of objects. On the other hand, the speedup is also becoming smaller with the increase of the deleting ratio. Furthermore, the speedup will be less than 1 when the deleting ratio reaches 80% for Nursery and Connect-4, 90% for Nursery and Connect-4, which means that the non-incremental algorithm will outperform incremental algorithm with a threshold value of the deleting ratio, and the threshold value differs depending on the data sets. (Table 7)

Table 7
The incremental speedup *versus* different deleting ratios of the objects.

Data sets	R_d(Increment ratios)								
	10%	20%	30%	40%	50%	60%	70%	80%	90%
Solar flare	16.625	10.950	6.548	5.844	2.660	2.320	1.958	1.476	0.478
Seismic-bumps	15.222	16.978	10.921	5.532	3.555	2.376	2.128	1.294	1.603
Molecular biology	20.717	9.615	6.149	5.043	5.013	3.808	1.991	1.846	1.496
Nursery	50.588	29.166	15.400	9.613	5.684	3.282	1.751	0.875	0.434
Letter recognition	75.461	43.423	23.341	13.664	8.306	4.663	2.224	1.028	0.354
Connect-4	109.729	43.693	22.112	14.005	7.401	4.244	2.120	0.964	0.227

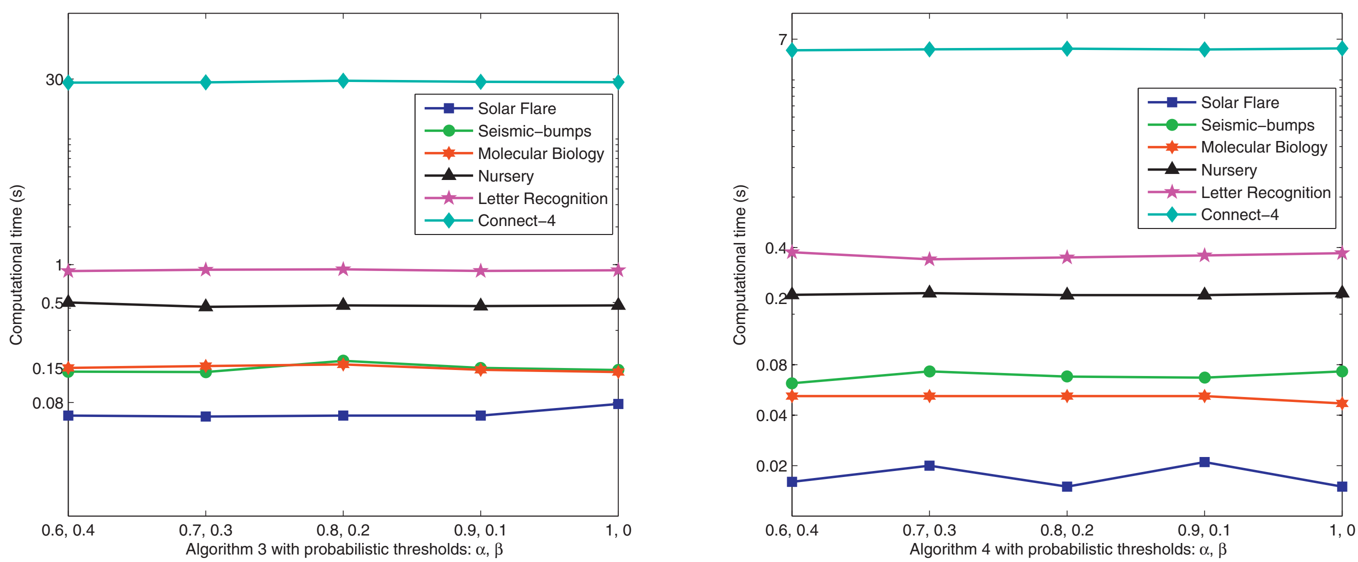

Fig. 3. Computation times of incremental algorithms (Algorithms 3 and 4) *versus* different thresholds α and β.

As probabilistic rough sets is a typical parameterized rough set model, we further conduct a series of experiments to verify the performance impact of the proposed incremental algorithm with different probabilistic thresholds α and β. The detailed change trend of computation time *versus* different thresholds are displayed in Fig. 3. It is easy to note from Fig. 3 that the computational times fluctuate a little with different thresholds α and β. It further validated that the time complexity of computation depends mostly on the data scale, which is consistent with the analysis of algorithm's time complexity.

6. Conclusions

Updates are collected and applied to the information system periodically in an incremental manner. Knowledge derived from the dynamic information system by data mining algorithm has to be updated as well. Aimed at improving the computational efficiency of probabilistic rough sets based concept approximation with dynamic variation of objects, we successfully developed in this paper an incremental solution that enables updating probabilistic approximations efficiently, which based on the incremental estimation of the variation tendency of conditional probability. Our theoretic analysis guarantees the correctness of the proposed incremental mechanisms for updating approximations when the insertion or addition of objects occurs, which yields the same computational results as the non-incremental way. Experimental results on different UCI data sets demonstrated the feasibility of the incremental algorithms for the improvements in computing performance. In this study, sets of updates are processed once at a time without considering the continuous, rapid updates of data, i.e., data streams. In the future, the insertion and deletion of the ordered sequence of objects will be considered to improve the efficiency of concept approximation based on rough set theory. Real practical applications will be exploited to evaluate the usability of our method furthermore.

Acknowledgments

This work is supported by the Key Program of National Natural Science Foundation of China (Nos. 61432012, U1435213) and the National Natural Science Foundation of China (Nos. 61573292, 61572406).

References

[1] A. Bargiela, W. Pedrycz, Granular Computing: An Introduction, Kluwer Academic Publishers, Dordrecht, 2003.
[2] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (2–3) (1990) 191–209.
[3] V. Ganti, J. Gehrke, R. Ramakrishnan, DEMON: Mining and monitoring evolving data, IEEE Trans. Knowl. Data Eng. 13 (1) (2001) 50–63.
[4] S. Greco, B. Matarazzo, R. Slowinski, Rough set theory for multicriteria decision analysis, Eur. J. Oper. Res. 129 (2001) 1–47.
[5] S. Greco, B. Matarazzo, R. Slowinski, Parameterized rough set model using rough membership and bayesian confirmation measures, Int. J. Approx. Reason. 49 (2008) 285–300.
[6] J.P. Herbert, J.T. Yao, Game-theoretic rough set, Fundamenta Informaticae 108 (2011) 267–286.
[7] C.C. Huang, T.L. Tseng, Y.N. Fan, C.H. Hsu, Alternative rule induction methods based on incremental object using rough set theory, Appl. Soft Comput. 13 (2013) 372–389.
[8] M. Kryszkiewicz, Rough set approach to incomplete information systems, Inf. Sci. 112 (1998) 39–49.
[9] G.M. Lang, Q.G. Li, M.J. Cai, T. Yang, Characteristic matrixes-based knowledge reduction in dynamic covering decision information systems, Knowl. Based Syst. 85 (2015) 1–26.
[10] J.H. Li, C.L. Mei, W.H. Xu, Y.H. Qian, Concept learning via granular computing: A cognitive viewpoint, Inf. Sci. 298 (2015) 447–467.

[11] S.Y. Li, T.R. Li, D. Liu, Dynamic maintenance of approximations in dominance-based rough set approach under the variation of the object set, Int. J. Intel. Syst. 28 (8) (2013) 729–751.

[12] D. Liu, T.R. Li, J.B. Zhang, A rough set-based incremental approach for learning knowledge in dynamic incomplete information systems, Int. J. Approx. Reason. 55 (8) (2014) 1764–1786.

[13] D. Liu, T.R. Li, J. Zhang, Incremental updating approximations in probabilistic rough sets under the variation of attributes, Knowl. Based Syst. 73 (2015) 81–96.

[14] J.Y. Liang, F. Wang, C.Y. Dang, Y.H. Qian, A group incremental approach to feature selection applying rough set technique, IEEE Trans. Knowl. Data Eng. 26 (2) (2014) 294–308.

[15] J.M. Ma, Y.Y. Yao, Rough set approximations in multi-granulation fuzzy approximation spaces, Fundamenta Informaticae 142 (1–4) (2015) 145–160.

[16] D. Newman, S. Hettich, C. Blake, C. Merz, UCI repository of machine learning databases, university of california, Department of Information and Computer Science, Irvine, CA, 1998. http://archive.ics.uci.edu/ml/

[17] S. Ozawa, S.N. Pang, N. Kasabov, Incremental learning of chunk data for online pattern classification systems, IEEE Trans. Neural Netw. 19 (6) (2008) 1–14.

[18] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.

[19] W.B. Qian, W.H. Shu, Y.H. Xie, B.R. Yang, J. Yang, Feature selection using compact discernibility matrix-based approach in dynamic incomplete decision system, J. Inf. Sci. Eng. 31 (2) (2015) 509–527.

[20] Y.H. Qian, Y.B. Li, J.Y. Liang, G.P. Lin, C.Y. Dang, Fuzzy granular structure distance, IEEE Trans. Fuzzy Syst. 23 (6) (2015) 2245–2259.

[21] Y.H. Qian, J.Y. Liang, W. Pedrycz, C.Y. Dang, Positive approximation: An accelerator for attribute reduction in rough set theory, Artif. Intel. 174 (9–10) (2010) 597–618.

[22] Y.H. She, On the rough consistency measures of logic theories and approximate reasoning in rough logic, Int. J. Approx. Reason. 55 (1) (2014) 486–499.

[23] W.H. Shu, W.B. Qian, An incremental approach to attribute reduction from dynamic incomplete decision systems in rough set theory, Data Knowl. Eng. 100 (2015) 116–132.

[24] D. Slezak, W. Ziarko, Bayesian rough set model, in: Proceedings of FDM 2002, 2002, pp. 131–135.

[25] S.K.M. Wong, W. Ziarko, Comparison of the probabilistic approximate classification and the fuzzy set model, Fuzzy Sets Syst. 21 (1987) 357–362.

[26] Y.Y. Yao, Three-way decisions and cognitive computing, Cognit. Comput. (2016a) 1–12.

[27] Y.Y. Yao, A triarchic theory of granular computing, Granular Comput. (2016b) 1–13.

[28] Y.Y. Yao, B. Zhou, Two bayesian approaches to rough sets, Eur. J. Oper. Res. 251 (3) (2015) 904–917.

[29] Y.Y. Yao, Information granulation and rough set approximation, Int. J. Intel. Syst. 16 (1) (2001) 87–104.

[30] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (2010) 341–353.

[31] Y.Y. Yao, S. Greco, R. Slowinski, Probabilistic rough sets, in: J. Kacprzyk, W. Pedrycz (Eds.), Springer Handbook of Computational Intelligence, Springer-Verlag, Berlin, 2015, pp. 387–411.

[32] Y.Y. Yao, T.Y. Lin, Graded rough set approximations based on nested neighborhood systems, in: Proceedings of 5th European Congress on intelligent techniques and Soft computing, 1997, pp. 196–200.

[33] Y.Y. Yao, S.K.M. Wong, P. Lingras, A decision-theoretic rough set model, in: Proceedings of ISMIS90, 1990, pp. 17–25.

[34] Y.Y. Yao, B. Zhou, Naive bayesian rough sets, in: Proceedings of RSKT2010, 2010, pp. 713–720.

[35] H.Y. Zhang, S.Y. Yang, Inclusion measure for typical hesitant fuzzy sets, the relative similarity measure and fuzzy entropy, Soft Comput. 20 (4) (2016) 1277–1287.

[36] J.B. Zhang, T.R. Li, H.M. Chen, Composite rough sets for dynamic data mining, Inf. Sci. 257 (2014) 81–100.

[37] W. Ziarko, Probabilistic approach to rough sets, Int. J. Approx. Reason. 49 (2) (2008) 272–284.

[38] W. Ziarko, Variable precision rough sets model, J. Comput. Syst. Sci. 46 (1993) 39–59.

[39] Z. Zheng, G.Y. Wang, RRIA: A rough set and rule tree based incremental knowledge acquisition algorithm, Fundamenta Informaticae 59 (2004) 299–313.