Article

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Contents

$\boxed{1}$ Suggestions

- Learn math in a targeted manner according to your own motivation, rather than learning it extensively. No one has so much time to systematically study mathematics in one direction.

$\boxed{2}$ Algebraic Geometry Prerequisites

1. Algebraic Varieties
2. Schemes
3. Cohomology
4. Curves
5. Surfaces
6.

- algebraic variety
  An algebraic variety is a generalization to n dimensions of algebraic curves.
- algebraic curve
  An algebraic curve over a field $\mathbb{K}$ is an equation $f(X, Y) = 0$, where $f(X, Y)$ is a polynomial in X and Y with coefficients in $\mathbb{K}$.
- algebraically closed field
  A field $\mathbb{K}$ is said to be algebraically closed if every polynomial with coefficients in $\mathbb{K}$ has a root in $\mathbb{K}$.
- algebraic closure
  The field $\overline{\mathbb{F}}$ is called an algebraic closure of $\mathbb{F}$ if $\overline{\mathbb{F}}$ is algebraic over $\mathbb{F}$ and if every polynomial $f(x) \in \mathbb{F}[x]$ splits completely over $\overline{\mathbb{F}}$, so that $\overline{\mathbb{F}}$ can be said to contain all the elements that are algebraic over $\mathbb{F}$.

  Example.
  $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$.

- algebraic extension
  An extension $\mathbb{F}$ of a field $\mathbb{K}$ is said to be algebraic if every element of $\mathbb{F}$ is algebraic over $\mathbb{K}$ (i.e., is the root of a nonzero polynomial with coefficients in $\mathbb{K}$).
- field extension
  A field $\mathbb{K}$ is said to be an extension field (*or* field extension, *or* extension) of a field $\mathbb{F}$ if $\mathbb{F}$ is a subfield of $\mathbb{K}$, denoted by $\mathbb{K}/\mathbb{F}$.

  Example.
  $\mathbb{C}$ is an extension field of $\mathbb{R}$, and $\mathbb{R}$ is an extension field of $\mathbb{Q}$.

- field
- ring
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3 Isbell duality

- A general abstract adjunction.

$$(\mathcal{O} \dashv \mathrm{Spec}) : \mathrm{CoPresheaves} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} \mathrm{Presheaves}$$

4 Sheaf Theory

4.1 Definition of sheaf and presheaf

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5 Rings

- A **unit** (*or* **invertible element**) in a ring R is an element u s.t. there is an element $v \in R$ with $uv = 1$. Such a v is unique. It is denoted u^{-1}, and called the **inverse** of u.
- A **field** is a ring in which every nonzero element in invertible.
  Remark: Both "$+$" and "$\times$" are Abelian group.
- A **zerodivisor** in R is a nonzero element $r \in R$ s.t. there is a nonzero element $s \in R$ with $rs = 0$.
- A nonzero element that is not a zerodivisor is a **nonzerodivisor**.
- An **ideal** in a commutative ring R is an additive subgroup I s.t. if $r \in R$ and $s \in I$, then $rs \in I$.
  An ideal I is said to be generated by a subset $S \subseteq R$ if every element $t \in I$ can be written in the form $t = \sum_{1}^{n} r_i s_i$ with r_i in R and s_i in S.
  Remark: We can think of S as a basis, and $\sum_{1}^{n} r_i s_i$ as combination $r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$.
  We shall write (S) for the ideal generated by a subset $S \subseteq R$; if S consists of finitely many elements $s_1, \ldots, s_n$, then we usually write $(s_1, \ldots, s_n)$ in place of (S).
  By convention, the ideal generated by the empty set is 0.
- An ideal is **principal** if it can be generated by one element.
- An ideal I of a commutative ring R is **prime** if $I \neq R$ (we usually say that I is a **proper ideal** in this case) and if $f, g \in R$ and $fg \in I$ implies $f \in I$ or $g \in I$.
  Remark: f, g are element.
- Equivalently, I is prime if for any ideals J, K with $JK \subseteq I$ we have $J \subseteq I$ or $K \subseteq I$. It follows by induction on n that if I is prime and contains a product of ideals (or even a product of sets) $J_1 J_2 \ldots J_n$, then I contains one of the J_i.
- The ring R is called a **domain** if 0 is prime.
- A **maximal ideal** of R is a proper ideal P not contained in any other proper ideal.
- If $P \subseteq R$ is a maximal ideal, then R/P is a field, so P is prime.
- R is called a **local ring** if P is the unique maximal ideal. We sometimes saying that (R, P) is a local ring.

- An element $h \in R$ is **prime** if it generates a prime ideal – equivalently, h is prime if h is not a unit, and whenever h divides a product fg, then h divides f or h divides g.
  Remark: $h \mid fg \implies h \mid f$ or $h \mid g$.
- If R is a commutative ring, then a **commutative algebra** over R (*or* **commutative R-algebra**) is commutative ring S together with a homomorphism $\alpha\colon R \to S$ of rings.
- Let R be a ring. An element $r \in R$ is **irreducible** if it is not a unit and if whenever $r = st$ with $s, t \in R$, then one of s and t is a unit.
- A ring R is **factorial** (*or* a **unique factorization domain**, sometimes abbreviated U.F.D.) if R is an integral domain and elements of R can be factored uniquely into irreducible elements, the uniqueness being up to factors which are units (this is the same sense in which factorization in $\mathbb{Z}$ is unique).
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$\boxed{6}$ Modules

- If R is a ring, then an R-module M is an Abelian group with an action of R, that is, a map $R \times M \to M$, written $(r, m) \mapsto rm$, satisfying for all $r, s \in R$ and $m, n \in M$,
  - $r(sm) = (rs)m$ (associativity)
  - $r(m + n) = rm + rn$
  - $(r + s)m = rm + sm$ (distributivity, *or* bilinearity)
  - $1m = m$ (identity)
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