Homework 3
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1. $\alpha = \frac{1}{L}$, $\beta = \frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}$.

   Assume u is eigenvlue, then we can get:

   $u^2 - (1+\beta)(1-\alpha\lambda_i)u + \beta(1-\alpha\lambda_i)$,

   thus $u = -\frac{1}{2}[(1+\beta)(1-\alpha\lambda_i) \pm \sqrt{(1+\beta)^2(1-\alpha\lambda_i)^2 - 4\beta(1-\alpha\lambda_i)}]$

   We need to prove that $(1+\beta)^2(1-\alpha\lambda_i)^2 - 4\beta(1-\alpha\lambda_i) < 0$

   Put $\alpha = \frac{1}{L}$, $\beta = \frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}$ into formula, we can get:

   $(1-\alpha\lambda_i)[(1+\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}})^2(1-\frac{1}{L}\lambda_i) - 4\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}]$, and $1-\alpha\lambda_i > 0$.

   $\Longrightarrow (1+\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}})^2(1-\frac{\lambda_i}{L}) - 4\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}} = \frac{4L}{(\sqrt{L}+\sqrt{m})^2}\frac{L-\lambda_i}{L} - \frac{4(L-m)}{(\sqrt{L}+\sqrt{m})^2} = \frac{4(m-\lambda_i)}{(\sqrt{L}+\sqrt{m})^2} < 0$

   So the roots are distinct complex numbers.

2. $f(x) = \frac{1}{2}x^TQx - b^Tx + c$, $\bigtriangledown f(x) = Qx - bx$,

   And $\bigtriangledown f(x^*) = 0 \to Qx^* - b = 0 \to x^* = Q^{-1}b$

   Then we can write $\bigtriangledown f(x) = Qx - bx = Q(x - x^*)$

   For heavy ball:

   $x^{k+1} = x^k - \alpha \bigtriangledown (x^k) + \beta(x^k - x^{k-1})$, where $\alpha = \frac{4}{(\sqrt{L}+\sqrt{m})^2}$ and $\beta = \frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}$

   $\Longrightarrow x^{k+1} - x^* = x^k - x^* - \alpha Q(x^k - x^*) + \beta((x^k - x^*) - (x^{k-1} - x^*))$

   $$\begin{bmatrix} x^{k+1} - x^* \\ x^k - x^* \end{bmatrix} = \begin{bmatrix} I - \alpha Q + \beta & -\beta \\ I & 0 \end{bmatrix} \begin{bmatrix} x^k - x^* \\ x^{k-1} - x^* \end{bmatrix}$$

   By defining $w^k = \begin{bmatrix} x^{k+1} - x^* \\ x^k - x^* \end{bmatrix}$, $T = \begin{bmatrix} I - \alpha Q + \beta & -\beta \\ I & 0 \end{bmatrix}$, we can have $w^k = Tw^{k-1}$.

   After rearranging the T matrix, we can get $\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_n \end{bmatrix}$, where $T_i = \begin{bmatrix} 1 - \alpha\lambda_i + \beta & -\beta \\ I & 0 \end{bmatrix}$

   And the eigenvalues of T_i is solution to $u^2 - (1 - \alpha\lambda_i + \beta)u + \beta = 0$.

   $u = \frac{(1-\alpha\lambda_i+\beta) \pm \sqrt{(1-\alpha\lambda_i+\beta)^2 - 4\beta}}{2}$

   And then we can get:

   $$\begin{aligned} (1-\alpha\lambda_i+\beta)^2 - 4\beta &= (1 + \frac{L-m}{(\sqrt{L}+\sqrt{m})^2} - \frac{4\lambda_i}{(\sqrt{L}+\sqrt{m})^2})^2 - 4 \cdot \frac{L-m}{(\sqrt{L}+\sqrt{m})^2} \\ &= (\frac{2L+2\sqrt{Lm}}{(\sqrt{L}+\sqrt{m})^2} - \frac{4\lambda_i}{(\sqrt{L}+\sqrt{m})^2})^2 - 4 \cdot \frac{L-m}{(\sqrt{L}+\sqrt{m})^2} \\ &= \frac{4}{(\sqrt{L}+\sqrt{m})^2} \cdot [(L+m-2\lambda_i)^2 - (L-m)(L+m+2\sqrt{LM})] \\ &= \frac{4}{(\sqrt{L}+\sqrt{m})^2} \cdot [m^2 + Lm + 2m\sqrt{Lm} + 4\lambda_i^2 - 4L\lambda_i - 4\lambda_i\sqrt{Lm}] \end{aligned}$$

Then we set $\phi(\lambda) = 4\lambda^2 - 4(L+\sqrt{Lm})\lambda + m^2 + Lm + 2m\sqrt{Lm}$, and can easily get $\phi(L) < 0$, and $\phi(m) < 0$

Combined with the fact that $\phi(\lambda)$ is quadratic, $\phi(\lambda) < 0$, $\forall \lambda \in [m, L]$, so $(1-\alpha\lambda_i+\beta)^2 - 4\beta < 0$, which means the $u_{i,1}$ and $u_{i,2}$ are two complex numbers.

$u_{i,1} = \frac{1}{2}[(1-\alpha\lambda_i+\beta) + i\sqrt{4\beta-(1-\alpha\lambda_i+\beta)^2}]$

$u_{i,2} = \frac{1}{2}[(1-\alpha\lambda_i+\beta) - i\sqrt{4\beta-(1-\alpha\lambda_i+\beta)^2}]$

$|u_{i,1}| = |u_{i,2}| = \frac{1}{2}\sqrt{(1-\alpha\lambda_i+\beta)^2+4\beta-(1-\alpha\lambda_i+\beta)^2} = \sqrt{\beta}$

So $\rho(T) = \sqrt{\beta} = \sqrt{\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}}$

And according to the Gelfand's Formula, we can get

$\rho(T) = (\lim_{k\to\infty} \|T^k\|)^{1/k}$

A consequence is that $\forall \epsilon > 0$, there is $\|T^k\| \le c(\rho(T)+\epsilon)^k$

$\implies \|w^k\| = \|T^k w^0\| \le \|T^k\|\|w^0\| \le c\|w^0\|(\rho(T)+\epsilon)^k$

And because $\rho(T) = \sqrt{\frac{\sqrt{L}-\sqrt{m}}{\sqrt{L}+\sqrt{m}}} < 1$, so $\|w^k\|$ converges linearly to zero.

3. According to problem, we have $x^0 = 0$, and $x_i^* = 1 - i/(n+1)$,

$\|x^0 - x^*\|_2^2 = \sum_{i=1}^n (1-\frac{i}{n+1})^2 = \frac{1}{(n+1)^2} \cdot \frac{n(n+1)(2n+1)}{6} \le \frac{n\cdot 2(n+1)}{6(n+1)} = \frac{n}{3}$

$\|x^k - x^*\|_2^2 = \sum_{i=k+1}^n (1-\frac{i}{n+1})^2 = \frac{1}{(n+1)^2} \cdot \frac{(n-k)(n-k+1)(2n-2k+1)}{6} \ge \frac{(n-k)\cdot(n-k)\cdot 2(n-k)}{6(n+1)^2} = \frac{(n-k)^3}{3(n+1)^2}$

And because $\frac{1}{3} \ge \frac{\|x^0-x^*\|_2^2}{n}$, we get $\|x^k - x^*\|_2^2 \ge \frac{(n-k)^3}{3(n+1)^2} \ge \frac{(n-k)^3}{n(n+1)^2}\|x^0 - x^*\|_2^2$