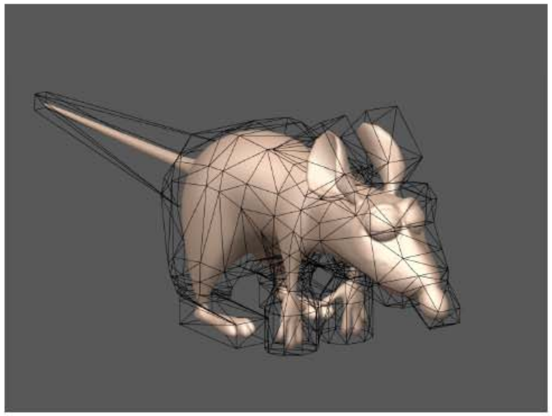
Deformation II



Space Deformation

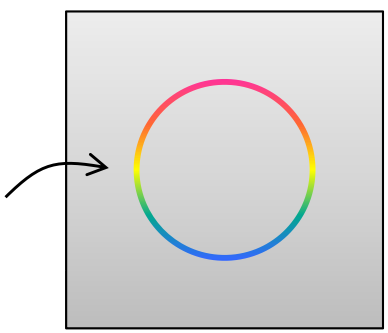
- Deformation function on ambient space

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

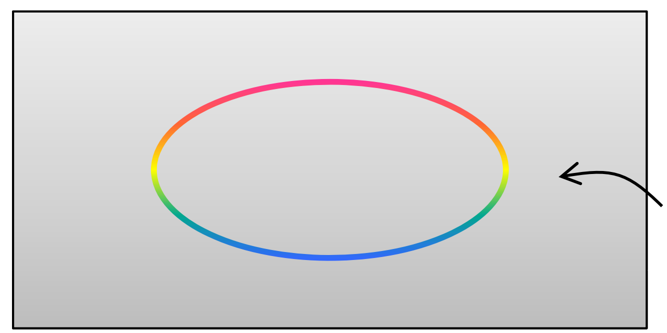
- Shape S deformed by applying f to points of S

$$S' = f\,(S)$$

S



$$f\,(x,y)=(2x,y)$$



S'

Motivation

- Can be applied to any geometry
  - Meshes (= non-manifold, **multiple components**)
  - Polygon soups
  - Point clouds
  - Volumetric data

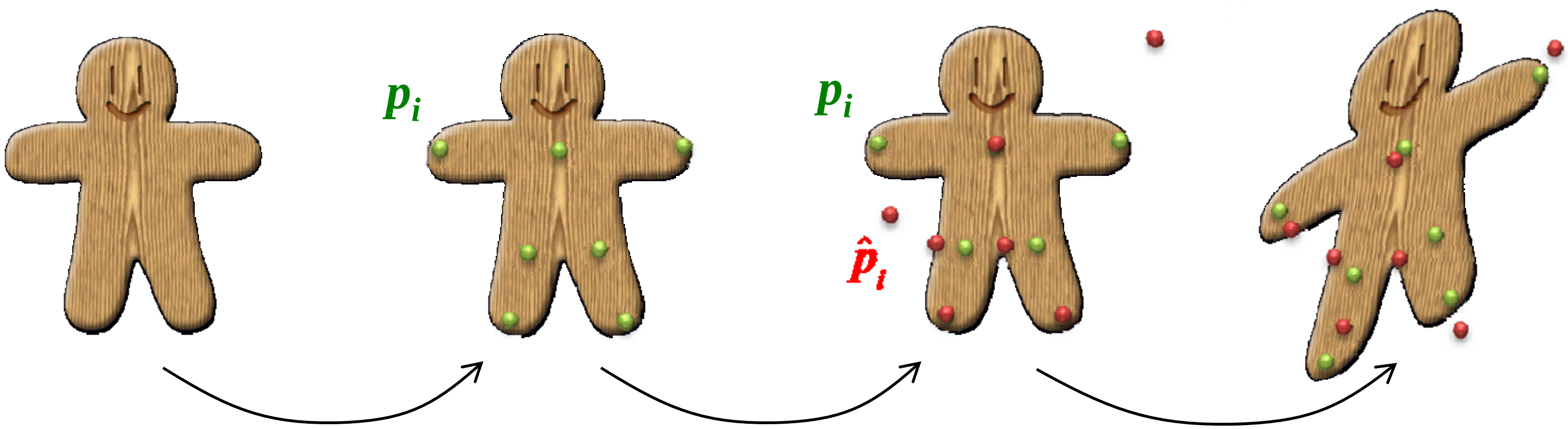
- Complexity decoupled from geometry complexity
  - Can pick the best complexity for required deformation

Required Properties

- Invariant to global operators
  - Global translation
  - Global rotation
- Smooth
- Efficient to compute
- “Intuitive deformation” ?
  - Can pose constraints as in surface deformation

MLS Deformation

[Schaeffer et al. ‘06]



1. Handles p_i

2. Target locations $\hat{p}_i$

3. Find best affine transformation that maps p_i *to* $\hat{p}_i$

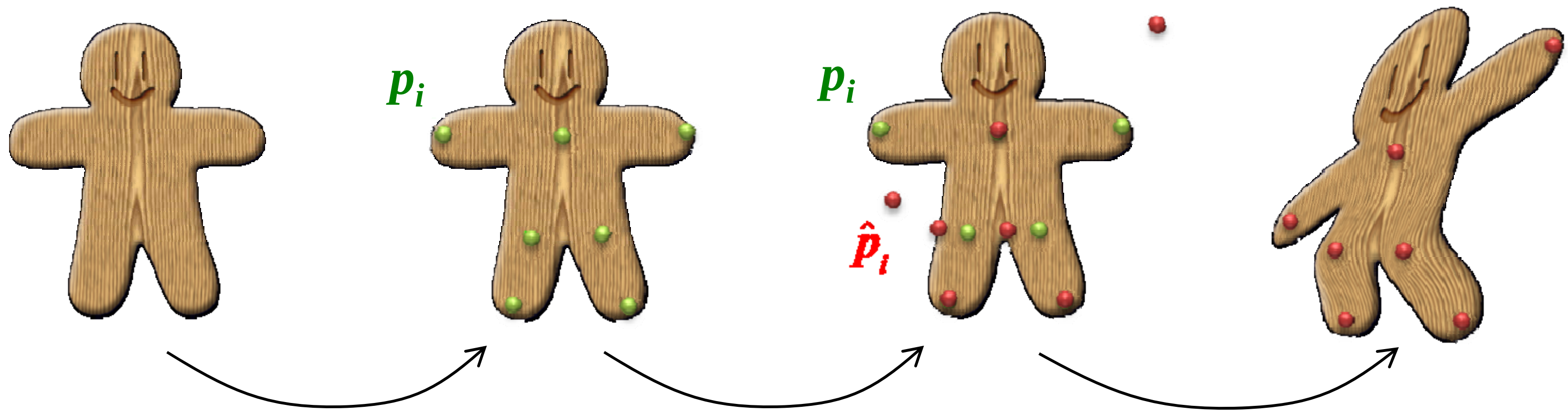
$$\min_{M,T} \sum_i \left| \left(M p_i + T \right) - \hat{p}_i \right|^2$$

4. Deform

$$f(v) = M\mathbf{v} + T$$

MLS Deformation

[Schaeffer et al. '06]



1. Handles p_i
2. Target locations $\hat{p}_i$

3. Find best affine transformation that maps p_i *to* $\hat{p}_i$

$$\min_{M,T} \sum_i \left| \frac{1}{|p_i - v|} \left(M p_i + T \right) - \hat{p}_i \right|^2$$

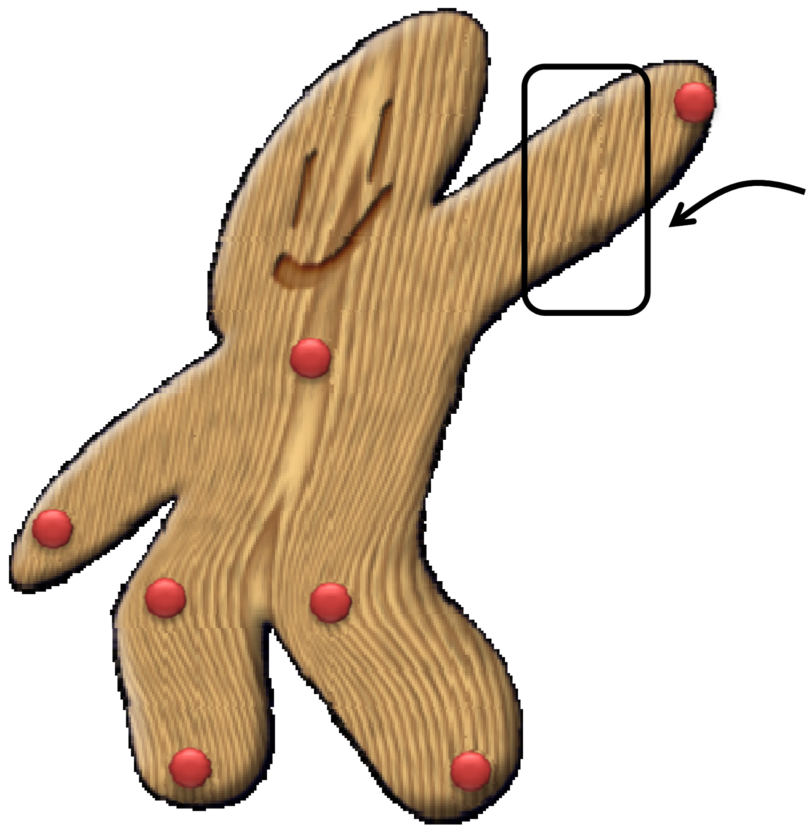
4. Deform

$$f(v) = Mv + T$$

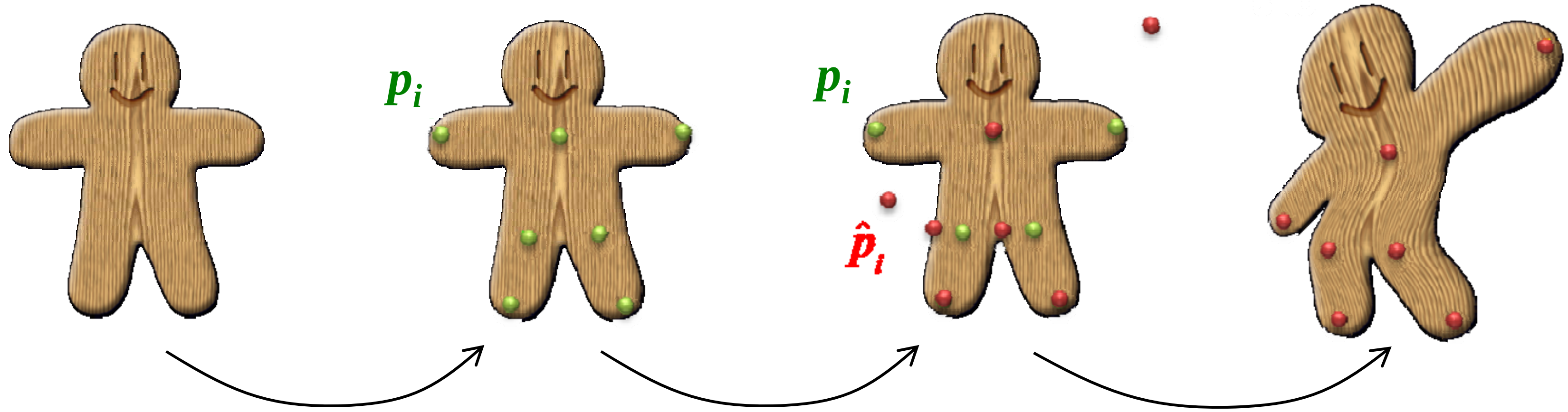
Closed form solution

Similarity

~~Affine~~ Transformations?



Similarity Transformations



1. Handles p_i

2. Target locations $\hat{p}_i$

$$M = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

3. Find best similarity transformation that maps p_i *to* $\hat{p}_i$

$$\min_{c,s,T} \sum_i \left| \frac{1}{\left| p_i - v \right|} \left(\begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} p_{x,i} & p_{y,i} \\ p_{y,i} & -p_{x,i} \end{pmatrix} + T \right) - \hat{p}_i \right|^2$$

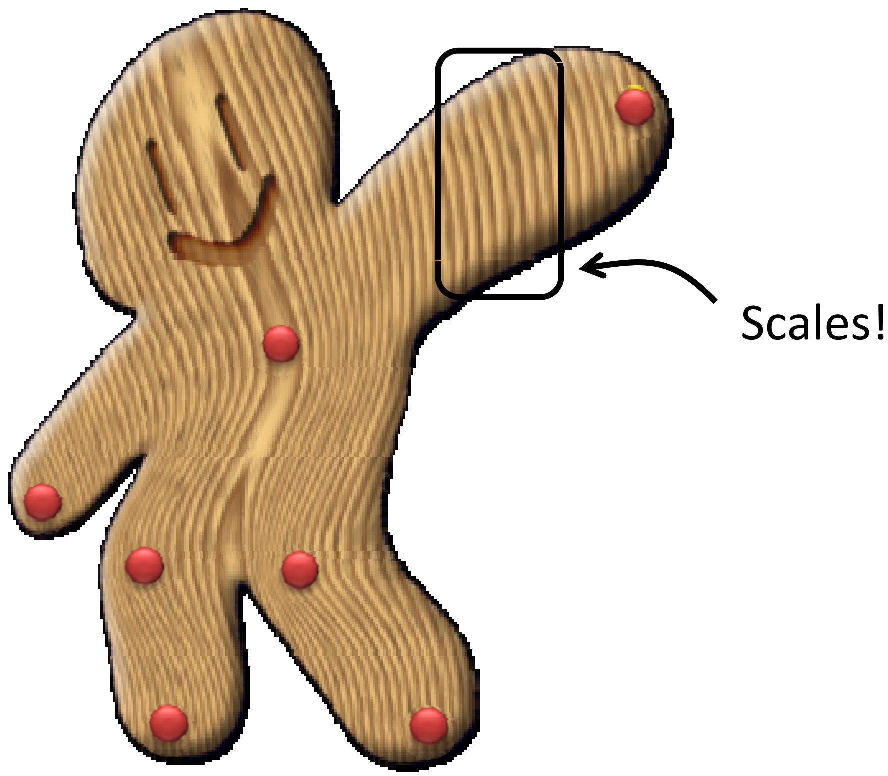
4. Deform

$$f(v) = Mv + T$$

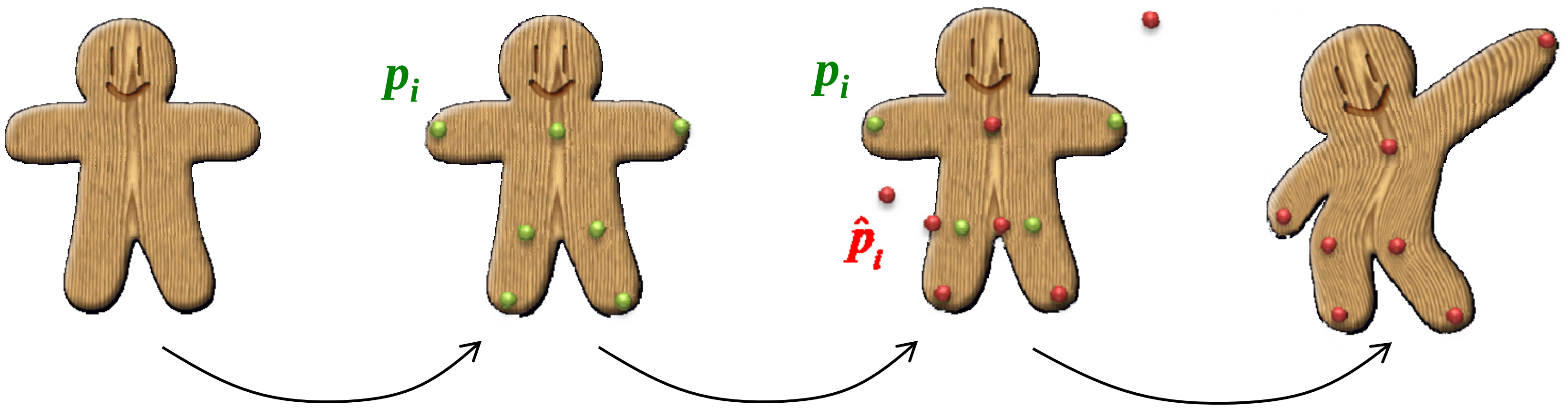
Closed form solution

Rigid

~~Similarity~~ Transformations?



Rigid Transformations



1. Handles p_i 2. Target locations $\hat{p}_i$

$$c^2 + s^2 = 1$$

3. Find best rigid transformation that maps p_i *to* $\hat{p}_i$

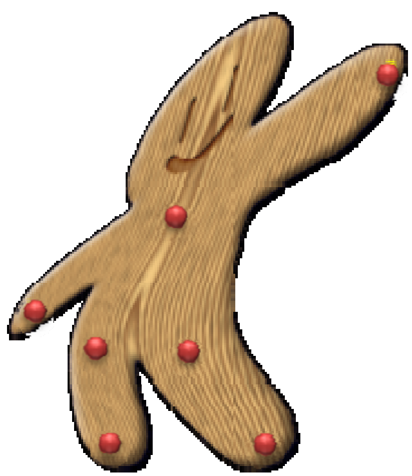
$$\min_{c,s,T} \sum_i \left| \frac{1}{\left| p_i - v \right|} \left(\begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} p_{x,i} & p_{y,i} \\ p_{y,i} & -p_{x,i} \end{pmatrix} + T \right) - \hat{p}_i \right|^2$$

4. Deform

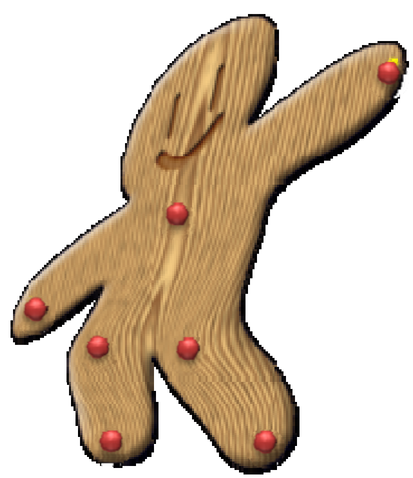
$$f(v) = Mv + T$$

Closed form solution given best similarity

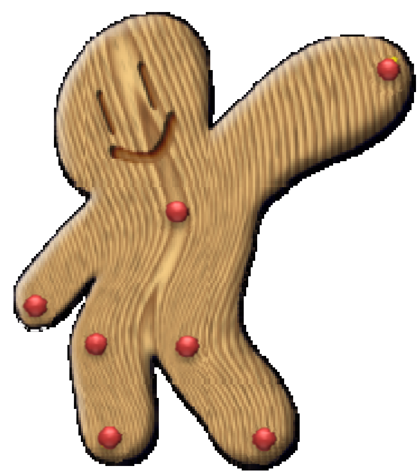
Comparison



Thin-Plate
[Bookstein ’89]



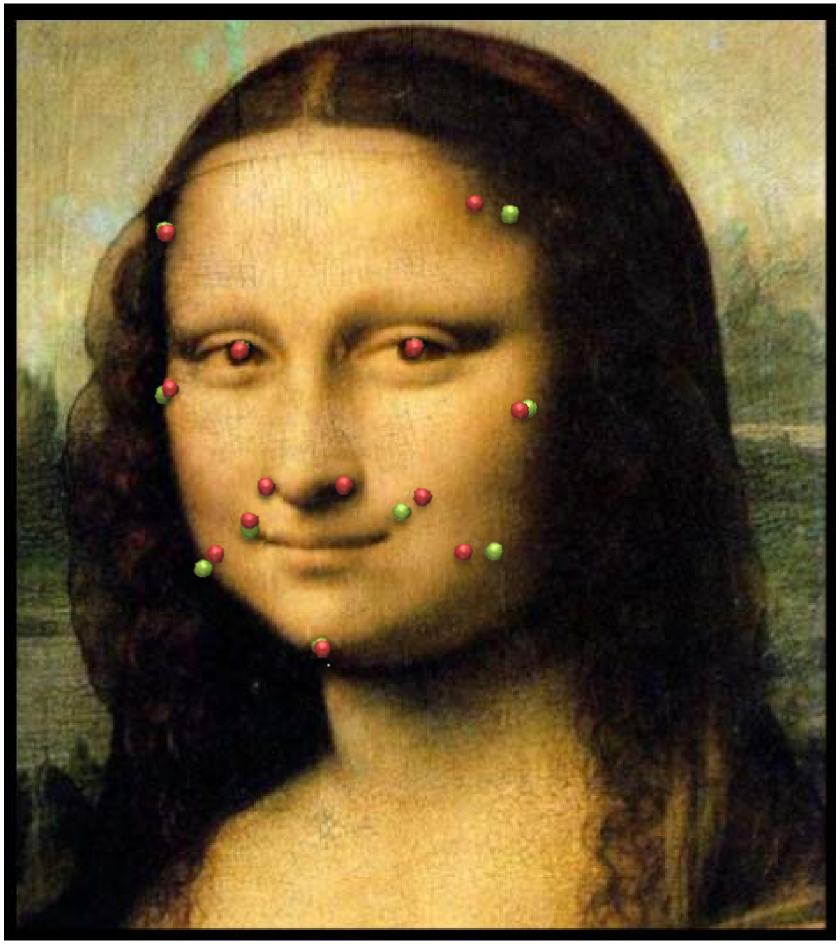
Affine MLS



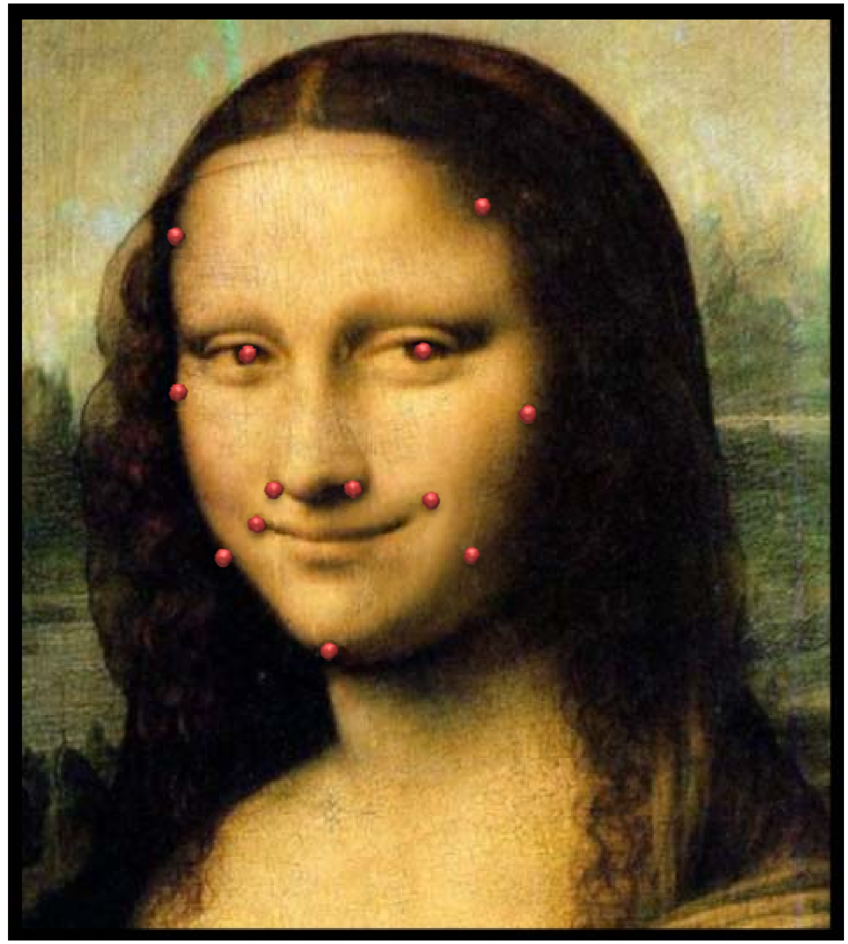
Similarity MLS

Rigid MLS

Examples



Before



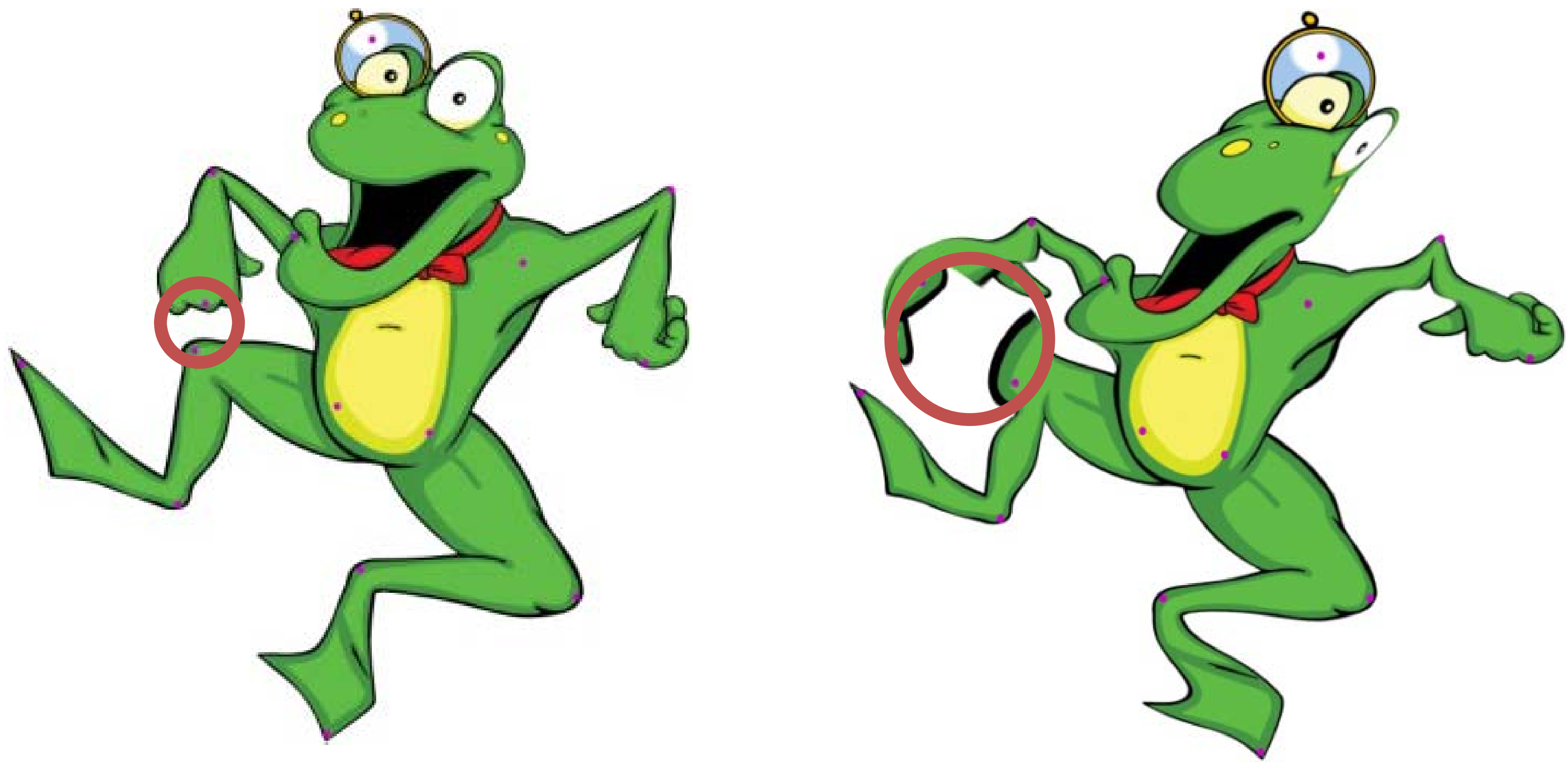
After

Examples

Horse Giraffe

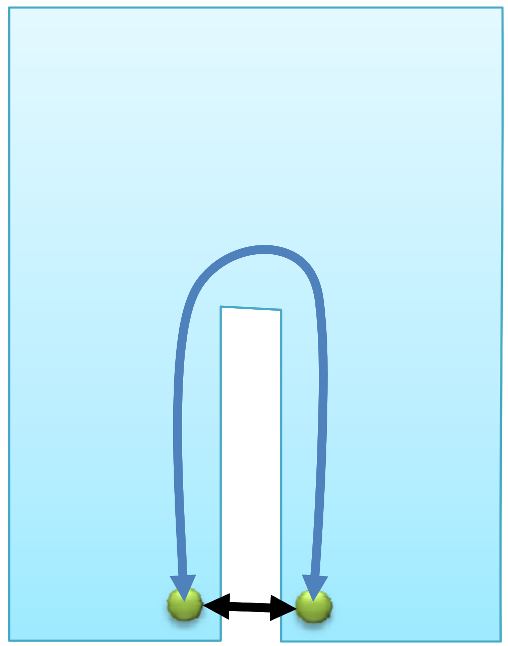
Limitations

Deforms all space - is not “shape aware”



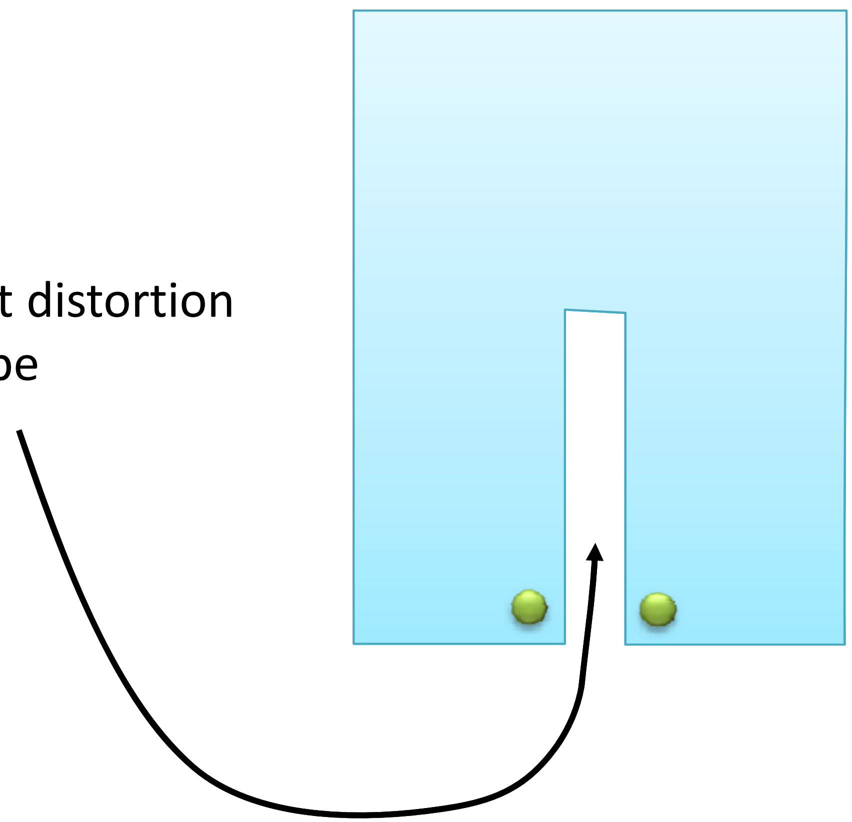
The "Pants" Problem

Small Euclidean distance
Large **geodesic** distance



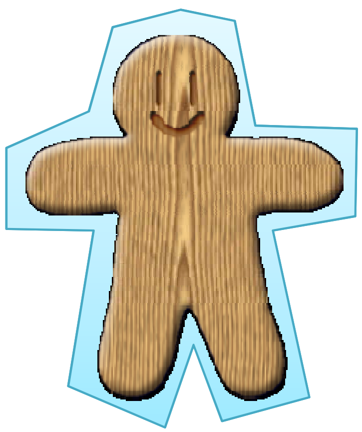
The “Pants” Problem

Don’t care about distortion
outside the shape



Solution: Cages

- Enclose the shape in a "cage" $\Omega \subset \mathbb{R}^n$
- Deformation function defined only on cage

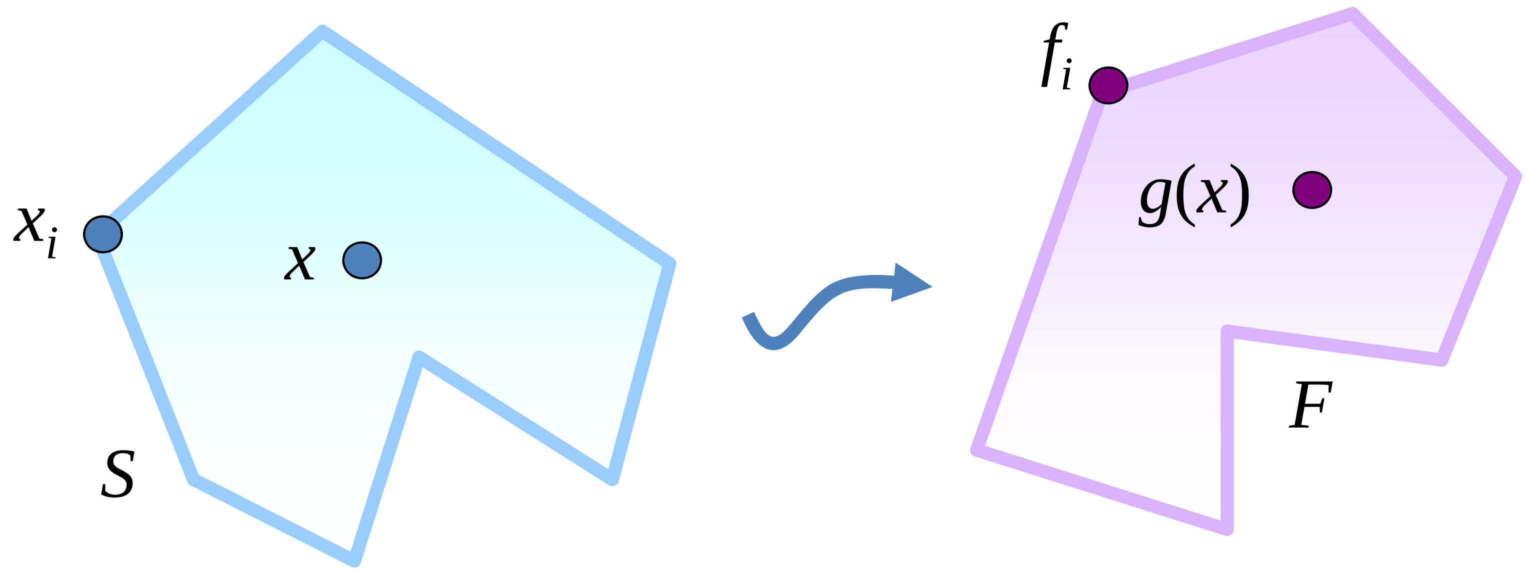


$$f : \Omega \to \mathbb{R}^n$$

- New problem: how to build the cage?

Deformation with a Cage

or: Rules of the Game

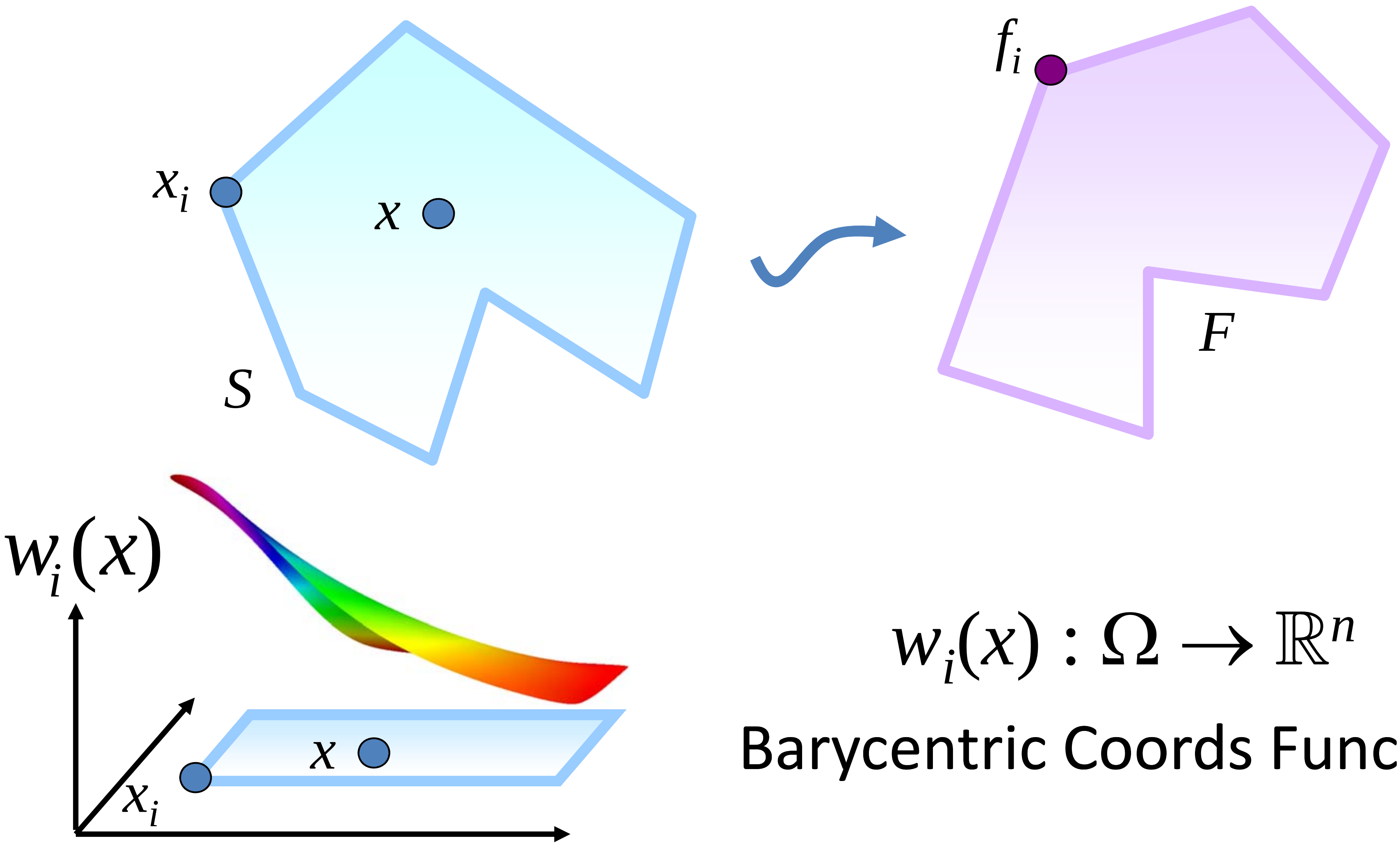


$$S = \{x_1, x_2, \ldots, x_n\} \qquad x_i \rightarrow f_i$$

Source polygon — Target polygon

$g(x) = ?$ Interior?

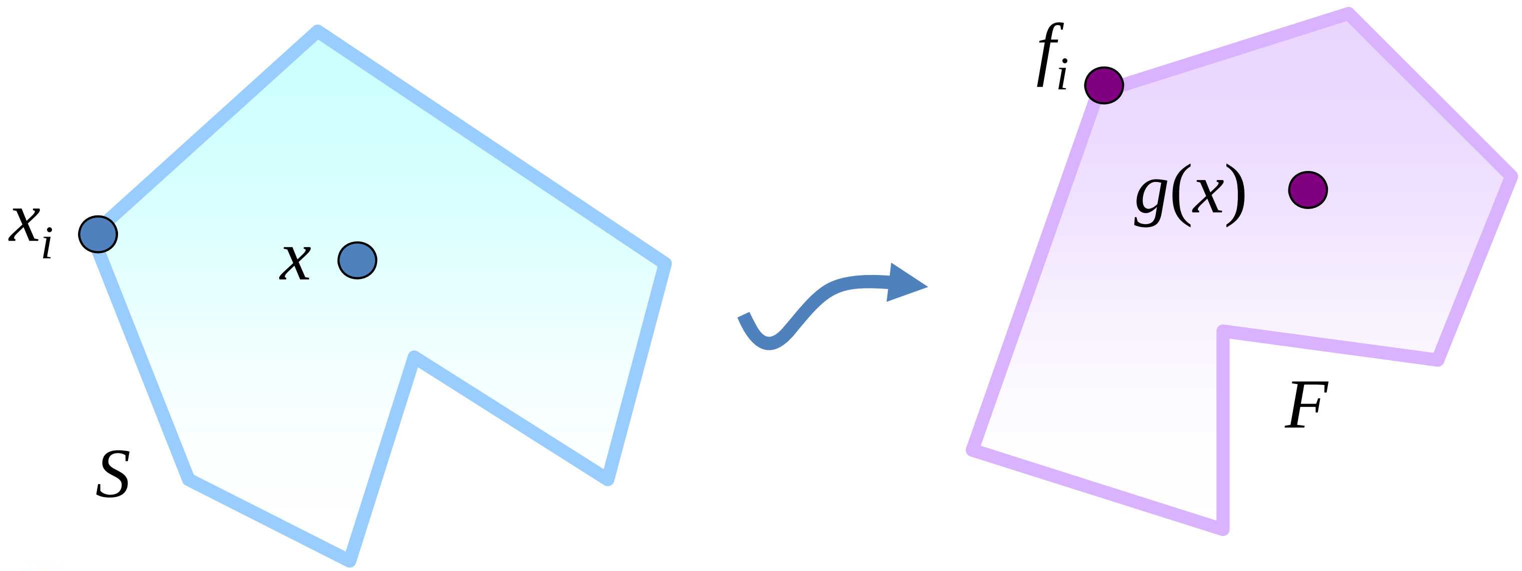
Barycentric Coordinates

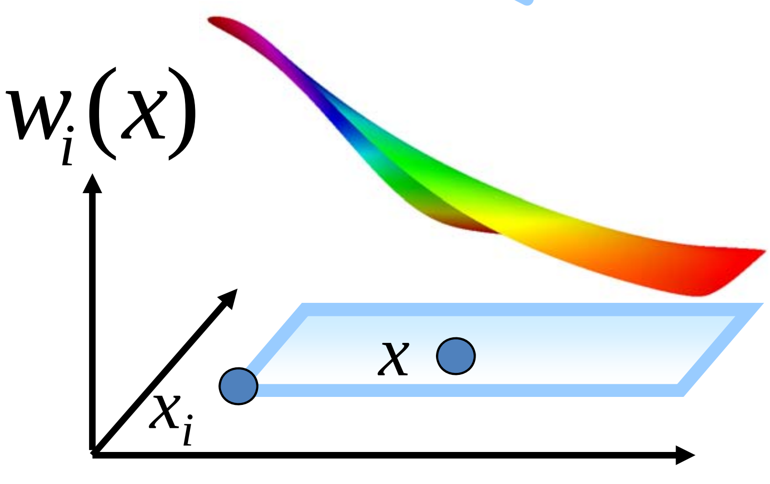


$$w_i(x) : \Omega \to \mathbb{R}^n$$

Barycentric Coords Function

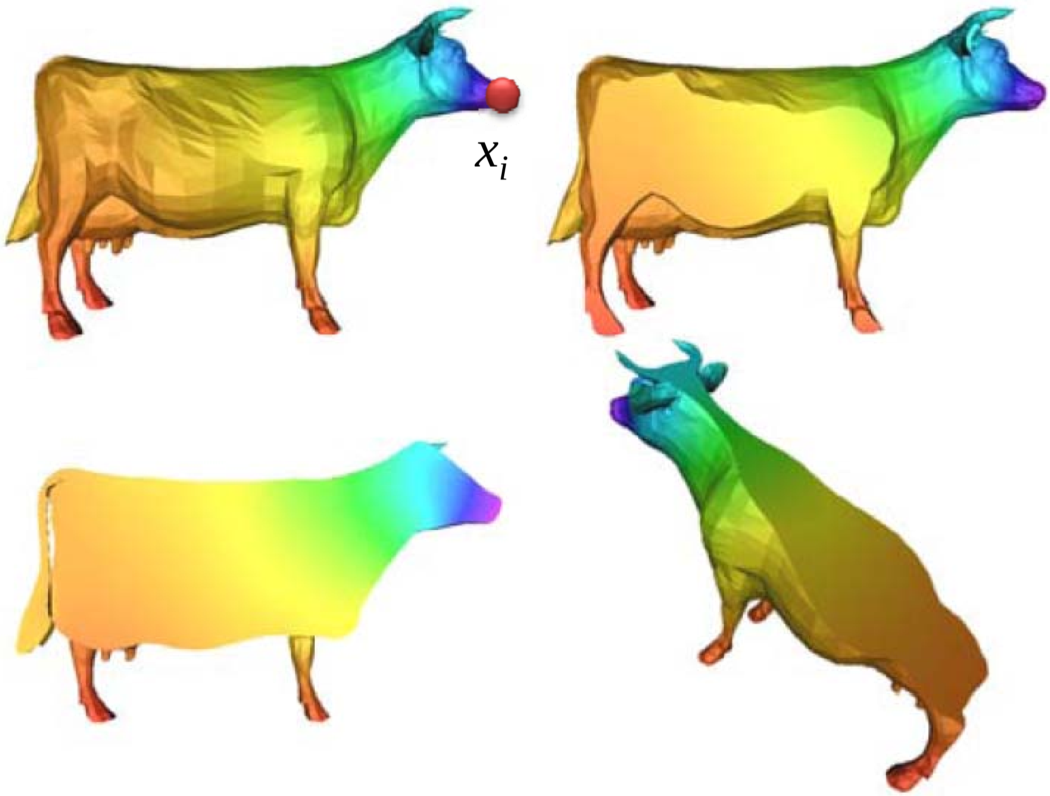
Barycentric Coordinates





$$g_F(x) = \sum_{i=1}^{n} w_i(x) f_i$$

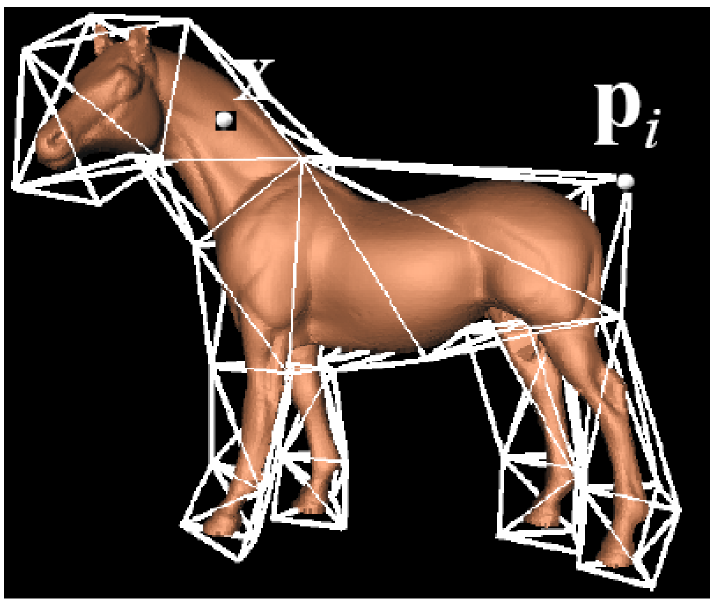
Example

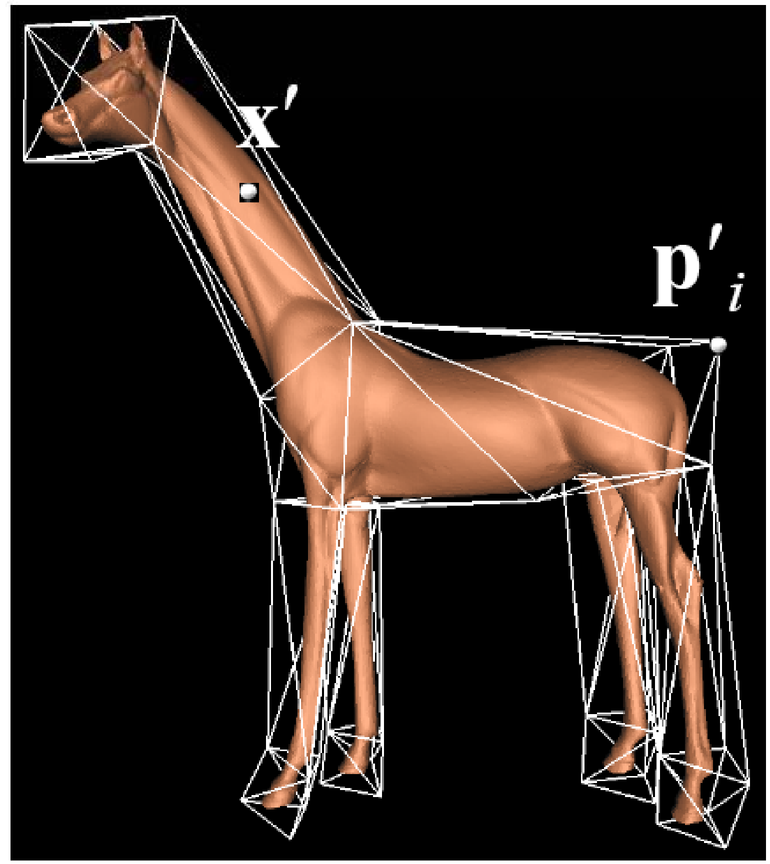


$w_i(x)$

Example

$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\mathbf{p}'_i$$





Barycentric Coordinates

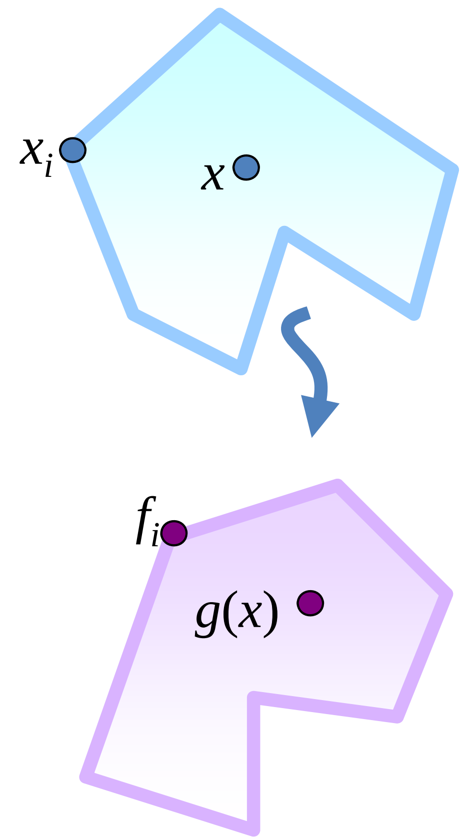
Required properties

- Translation invariance (constant precision)

$$\sum_{i=1}^{n} w_i(x) = 1$$

- Reproduction of identity (linear precision)

$$\sum_{i=1}^{n} w_i(x) x_i = x$$



$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Barycentric Coordinates

Constant + linear precision = affine invariance

$$
\begin{aligned}
g_{Ax_i+T}(x) &= \sum_i w_i(x)(Ax_i + T) \\
&= A\underbrace{\sum_i w_i(x)x_i}_{x} + T\underbrace{\sum_i w_i(x)}_{1} = \\
&= Ax + T
\end{aligned}
$$

Barycentric Coordinates

Required properties

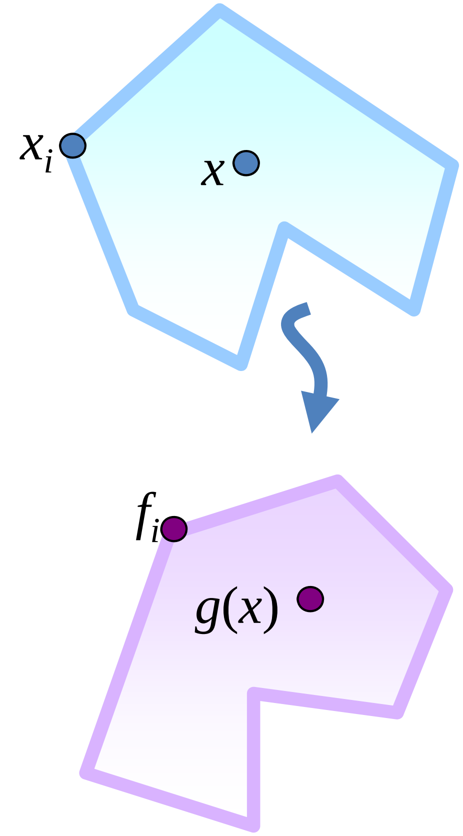
- Smoothness – at least C1

- Interpolation (Lagrange property)

$$f(x_j) = f_j$$

$$\Updownarrow$$

$$w_i(x_j) = \delta_{ij}$$

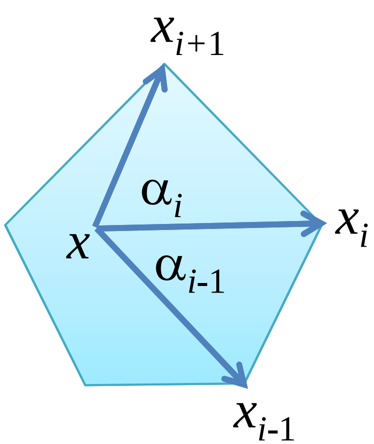


$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Example: Mean Value Coords

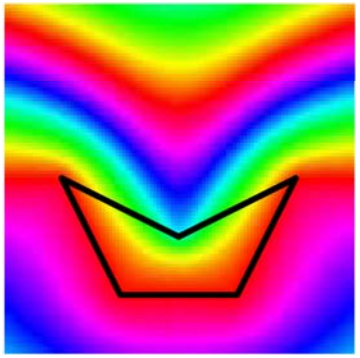
[3D: Ju et al ’05]

$$k_i(x) = \frac{\tan\left(\frac{\alpha_{i-1}}{2}\right) + \tan\left(\frac{\alpha_i}{2}\right)}{|x_i - x|}$$

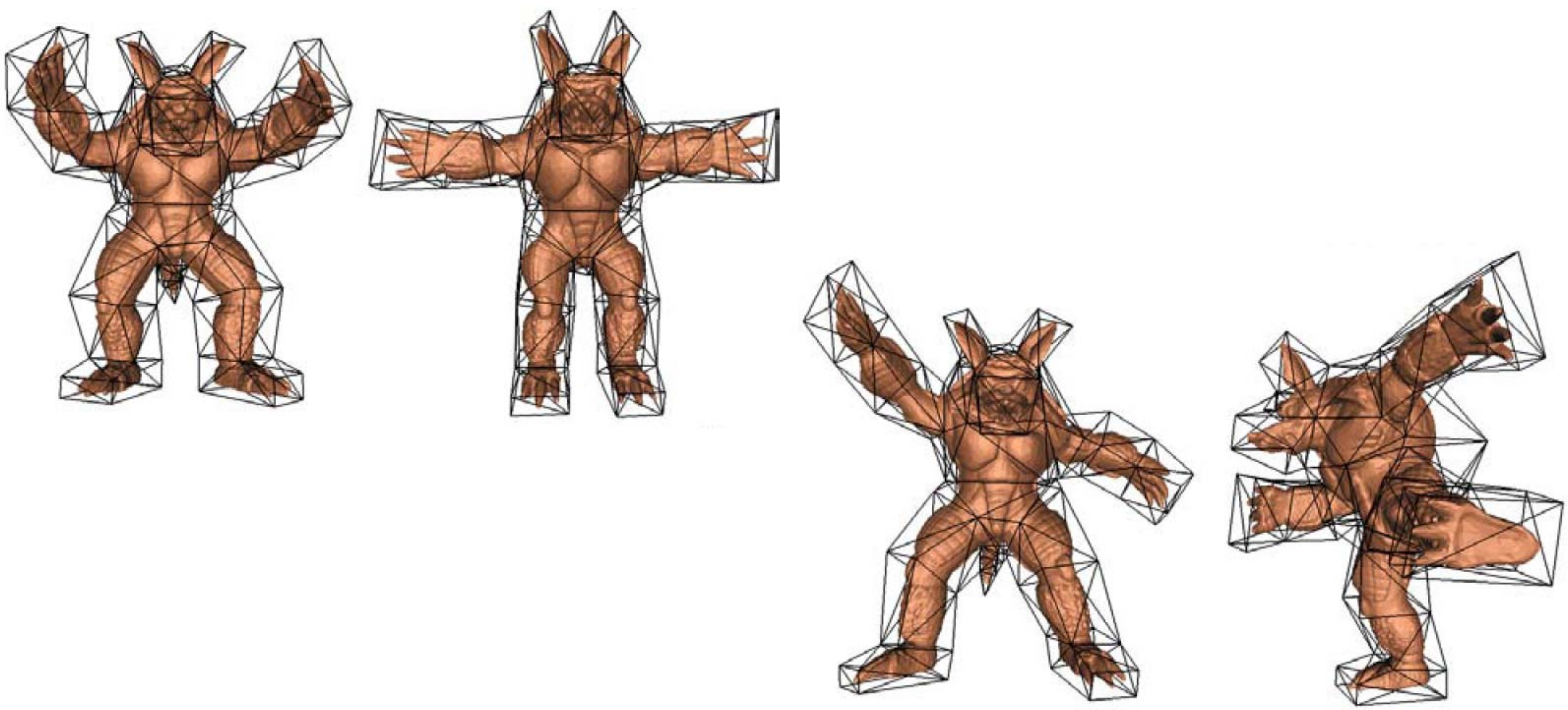


$$w_i(x) = \frac{k_i(x)}{\sum_i k_i(x)}$$

Closed form!



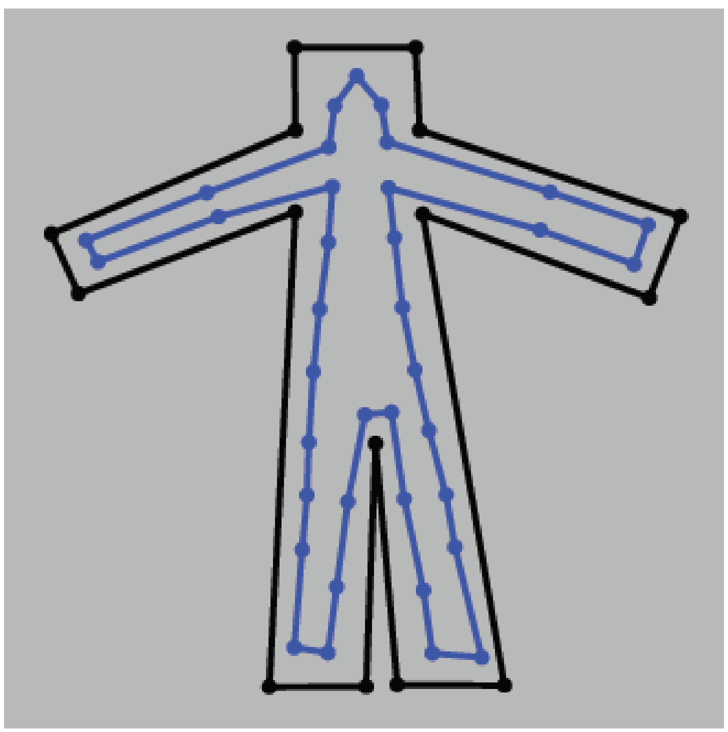
Example: Mean Value Coords

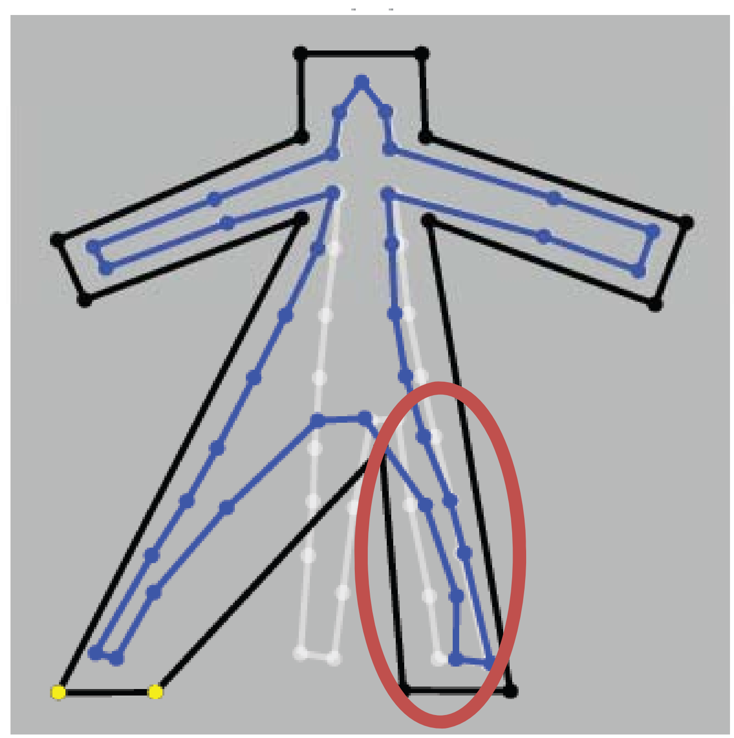


MV - Limitations

Back to the pants problem

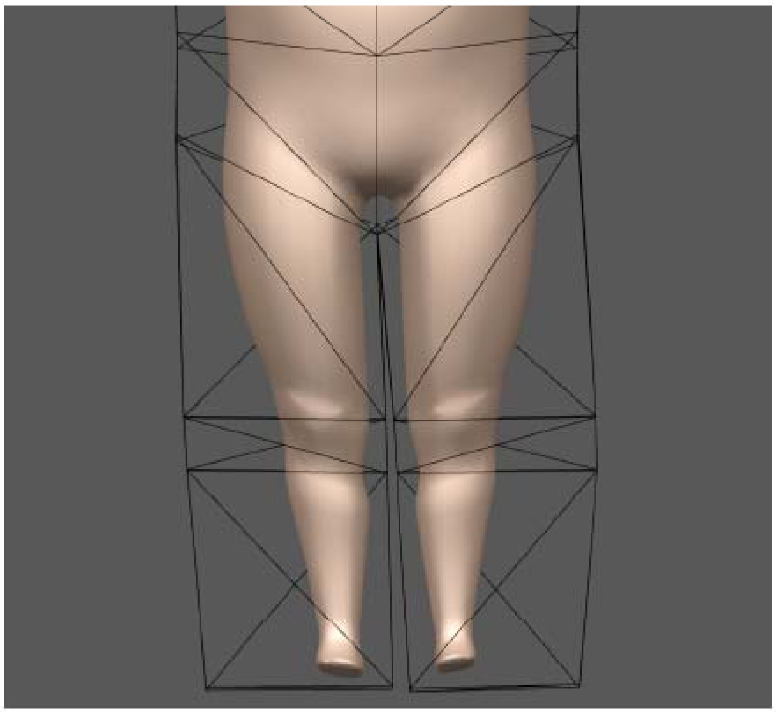
- MV **negative** on concave polygons

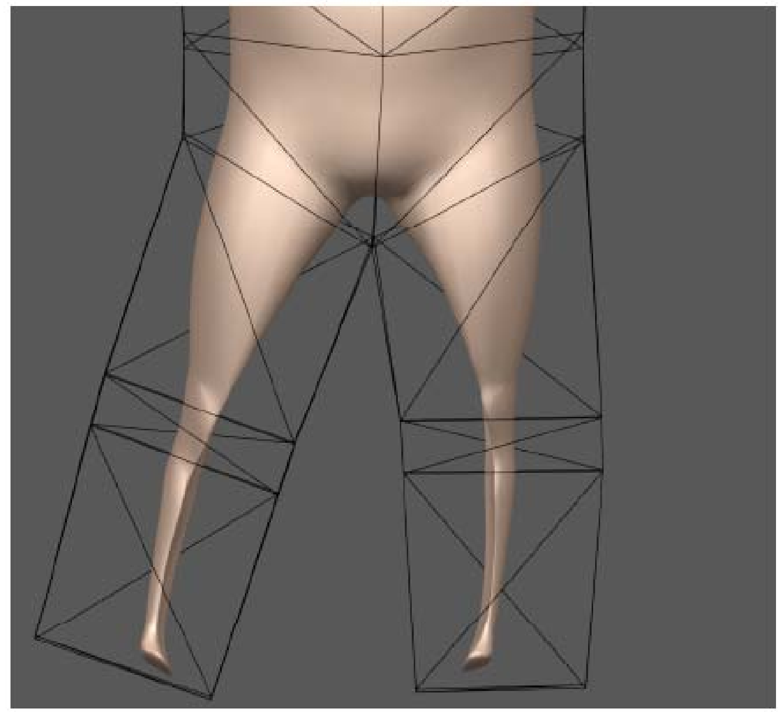




MV - Limitations

Other leg moves in opposite (!) direction





Barycentric Coords

Additional property required:

$$w_i(x) \geq 0$$

Mean value coords only positive
on convex polygons

Harmonic Coordinates

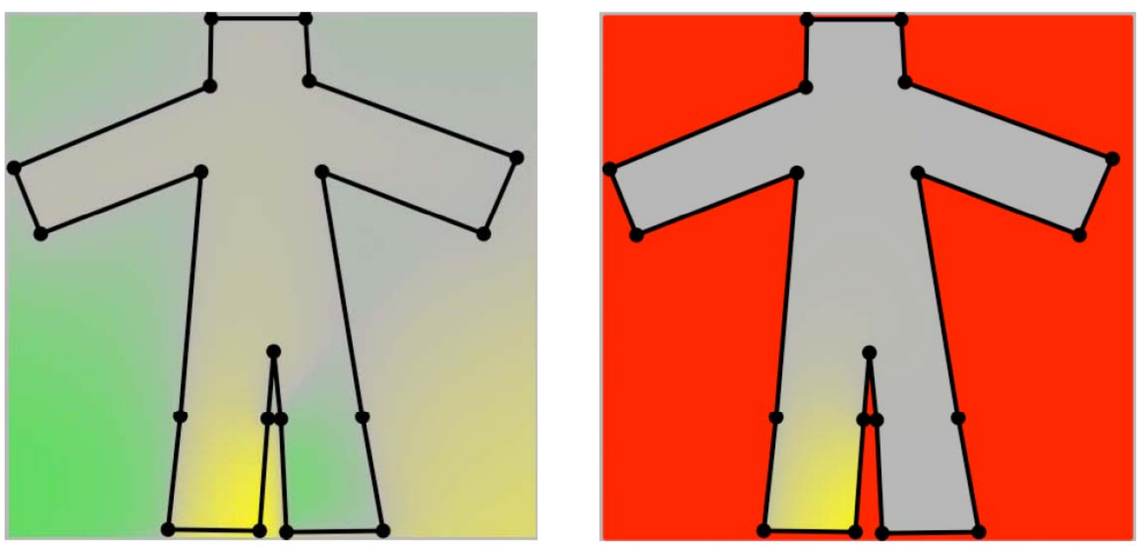
[Joshi et al '07]

Solve for $w_i(x)$:

$$\nabla^2 w_i(x) = 0$$

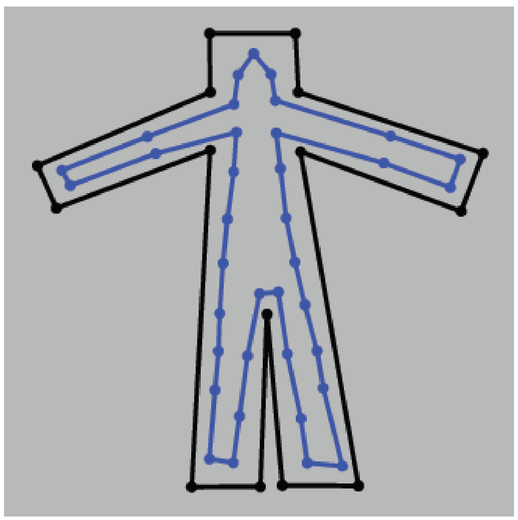
subject to: w_i linear on the boundary and

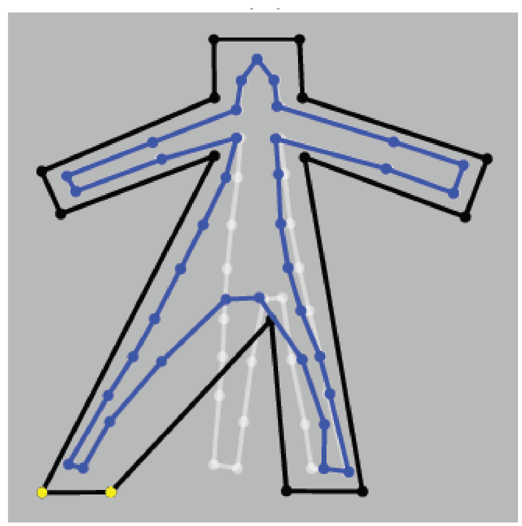
$$w_i(x_j) = \delta_{ij}$$



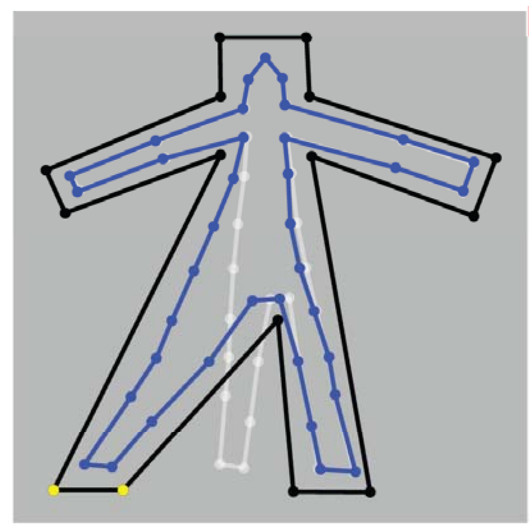
MVC HC

Harmonic Coordinates



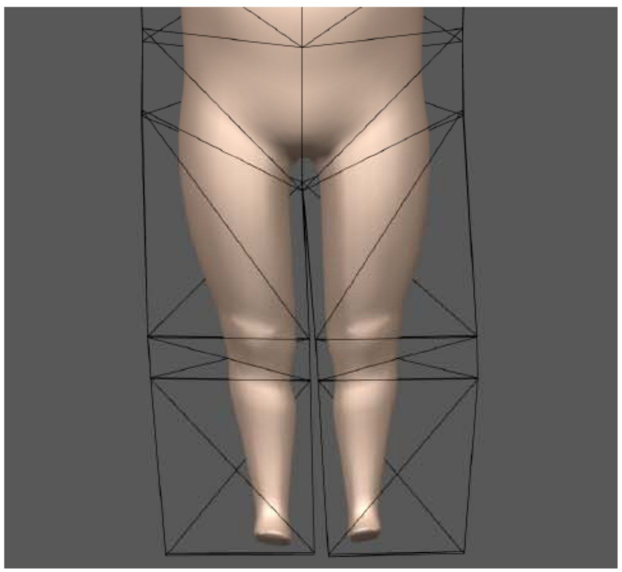


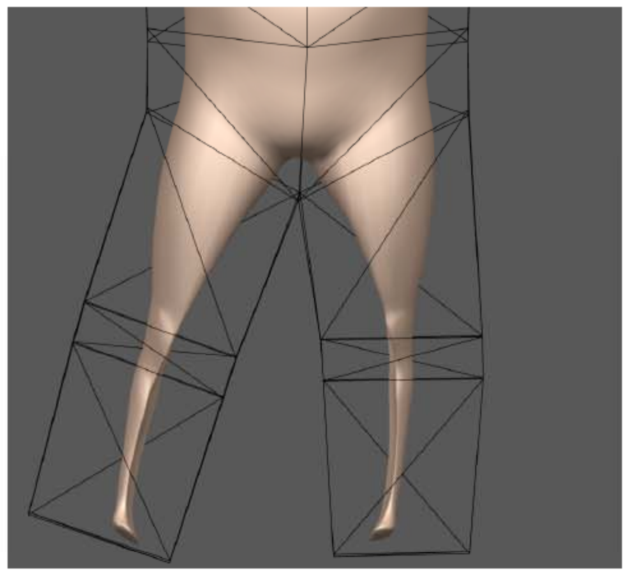
MVC



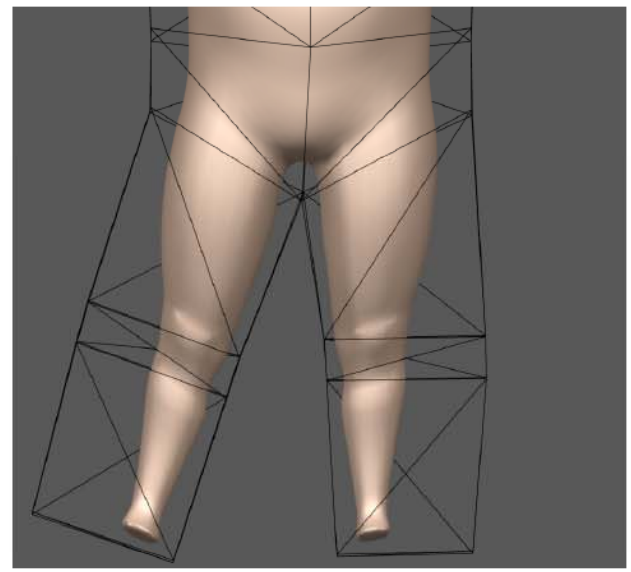
HC

Harmonic Coordinates





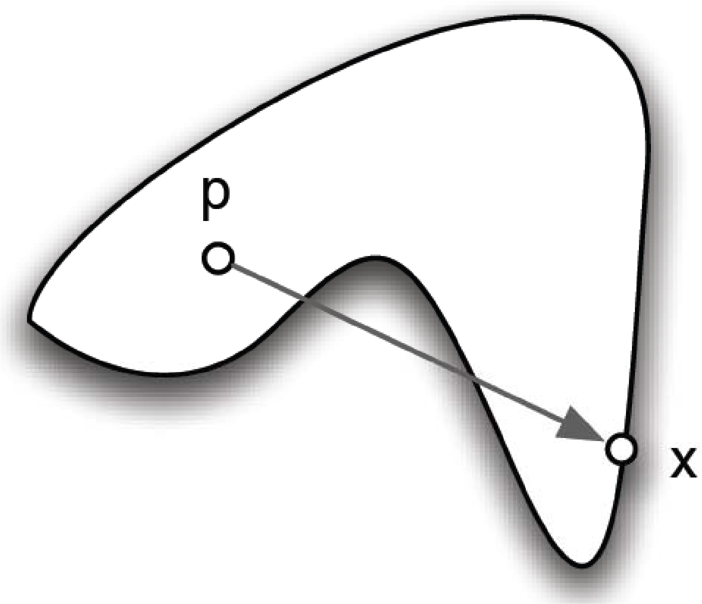
MVC



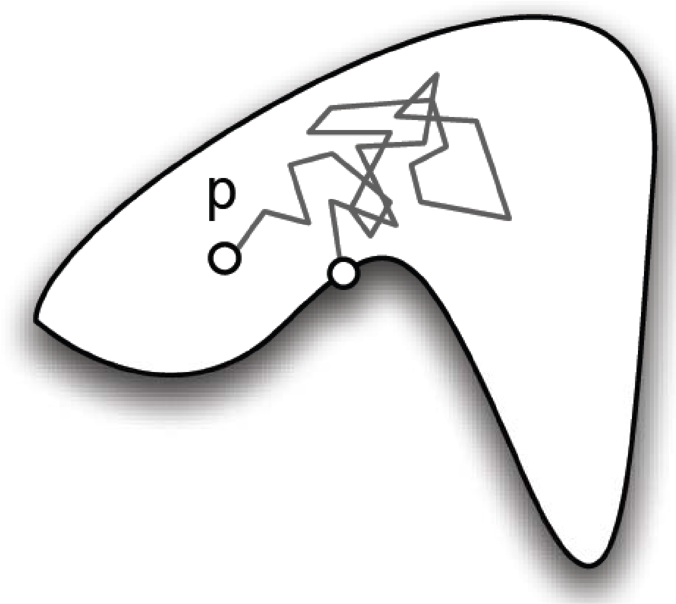
HC

Harmonic Coordinates

Why does it work?



MVC use
Euclidean distances



HC use
resistance distances

Harmonic Coordinates

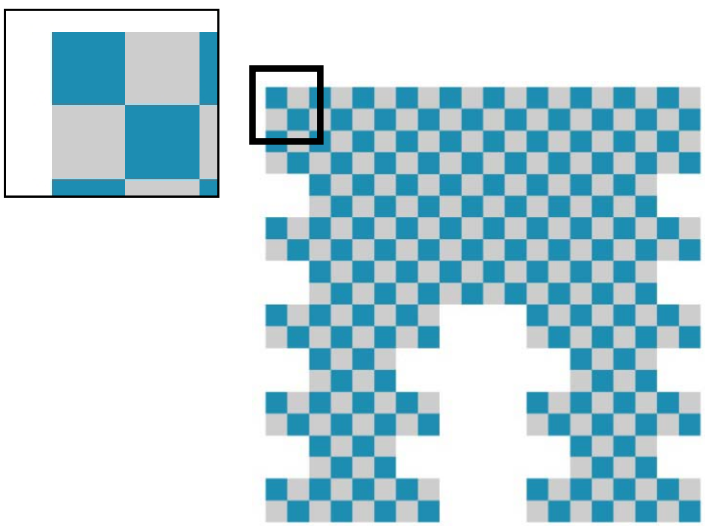
Properties:

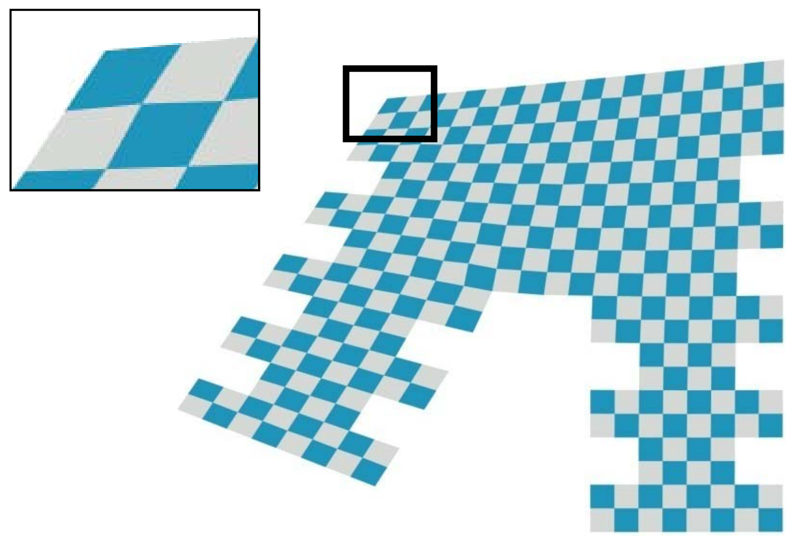
- All required properties
  - Smooth, translation + rotation invariant
- Positive everywhere
- No closed form, need to solve a PDE

Limitations

- Affine invariant
- Wasn't that a required property??

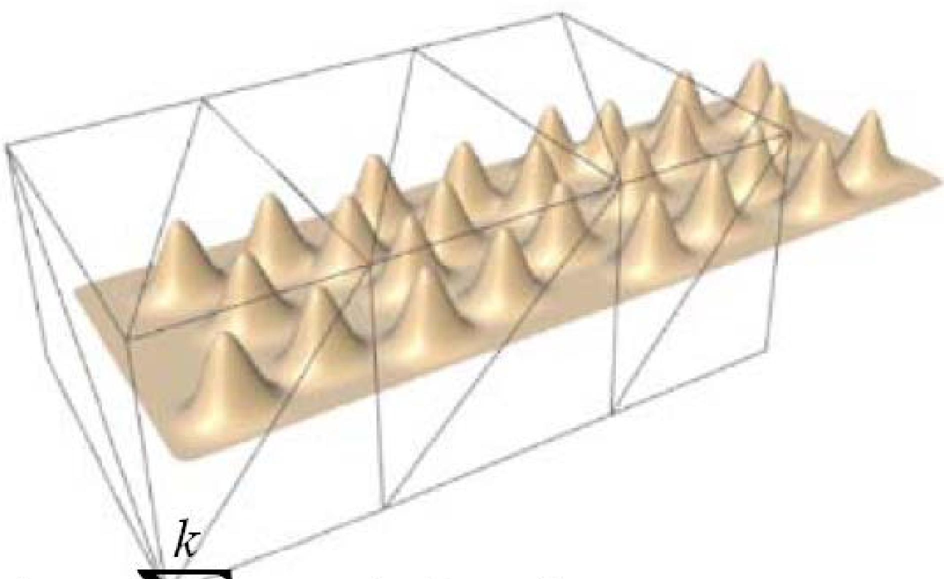
$$g_F(x) = \sum_{i=1}^{n} w_i(x) f_i \quad , \quad g_{A(F)}(x) = A\left(g(x)\right)$$

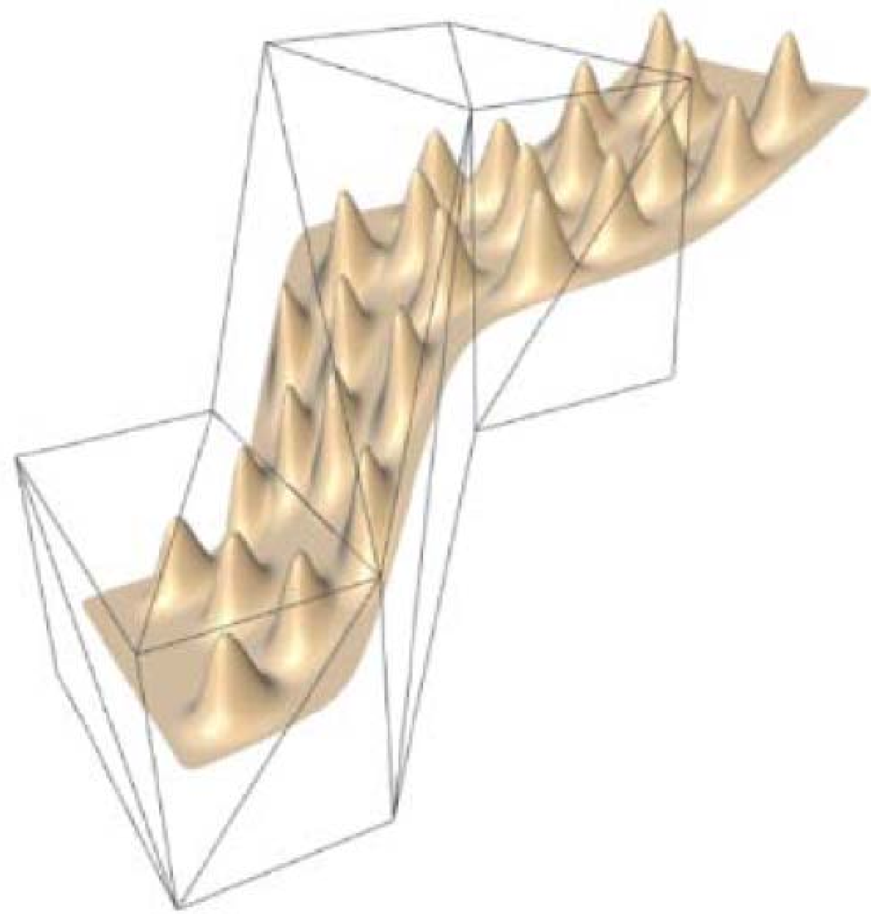




Affine Invariance is Evil

(for deformation anyway)

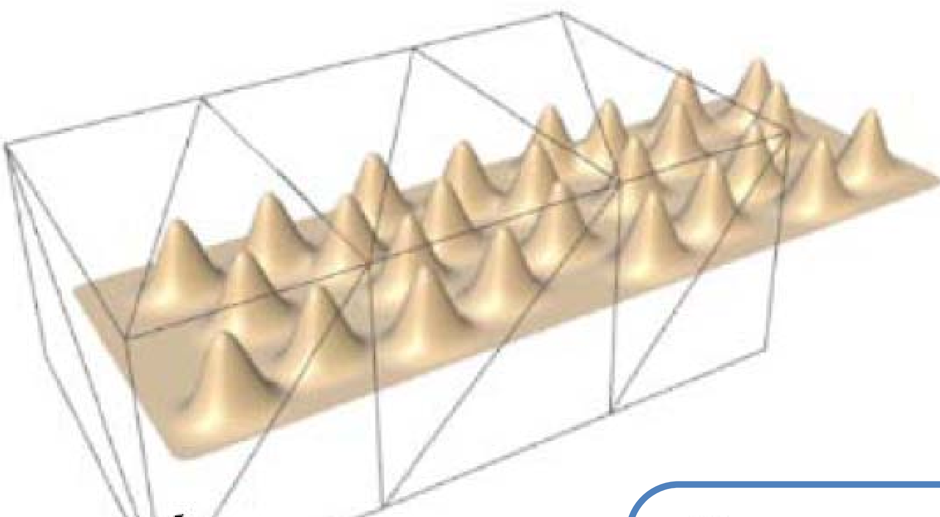


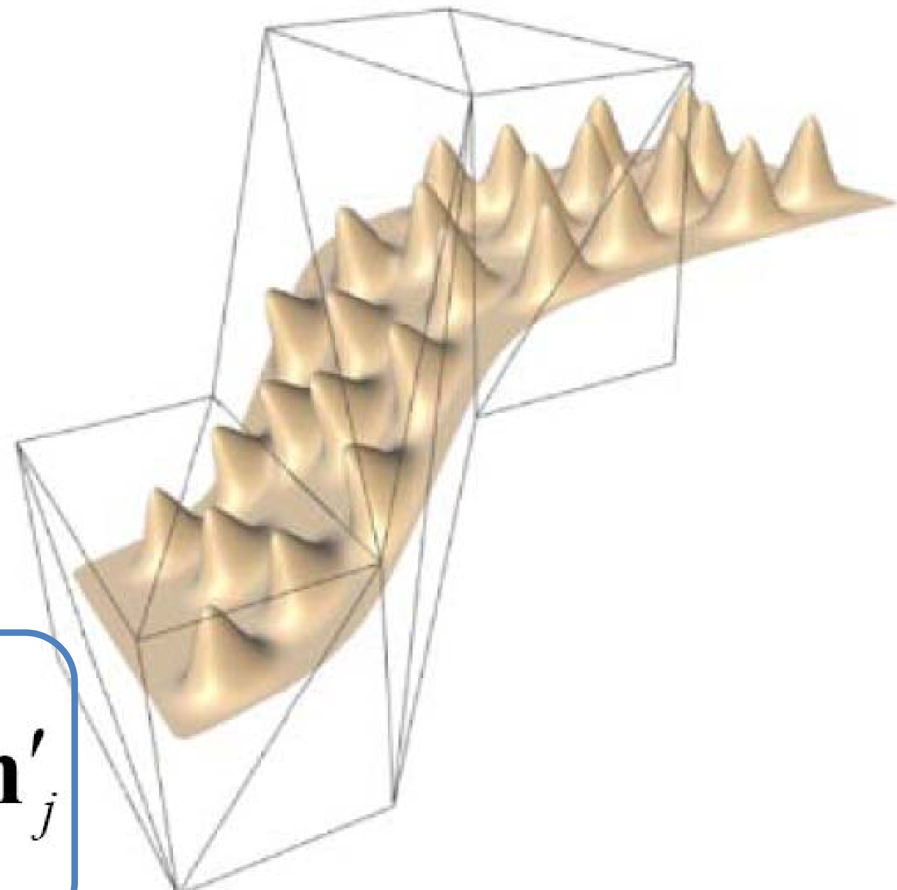


$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\mathbf{p}'_i$$

Green Coordinates [Lipman et al '08]

Coordinates also depend on faces!

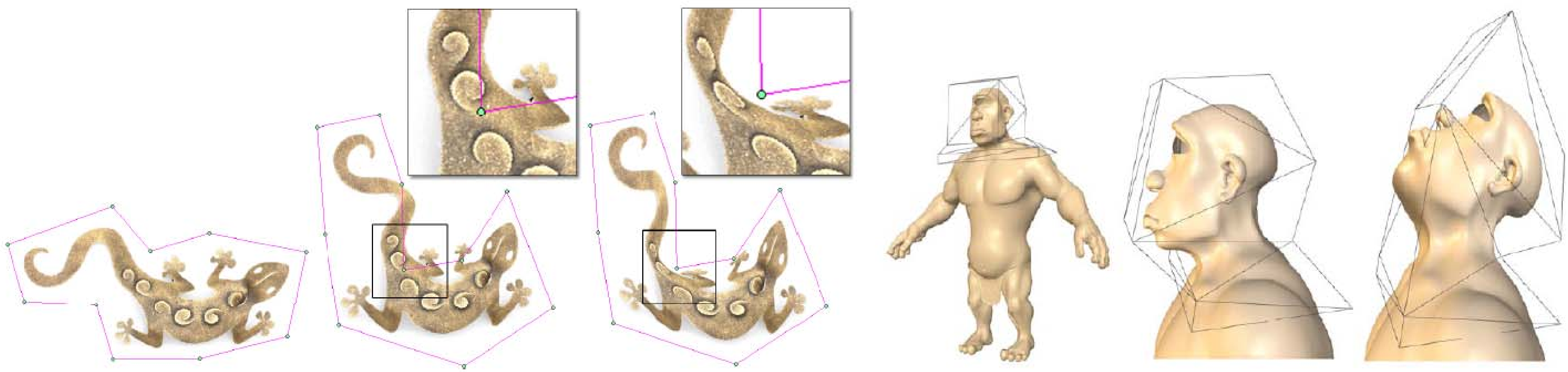




$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\mathbf{p}'_i + \boxed{\sum_{j=1}^{m} \psi_j(\mathbf{x})\mathbf{n}'_j}$$

Green Coordinates

- Closed form
- Harmonic
- Conformal in 2D, quasi-conformal in 3D
- Not interpolating



GC MVC GC

Cauchy-Green Coordinates

[Weber et al. '09]

Work with **complex** coordinates instead of real

Can derive Green coords much easier

Can find other families of coords

Complex Bary Coords

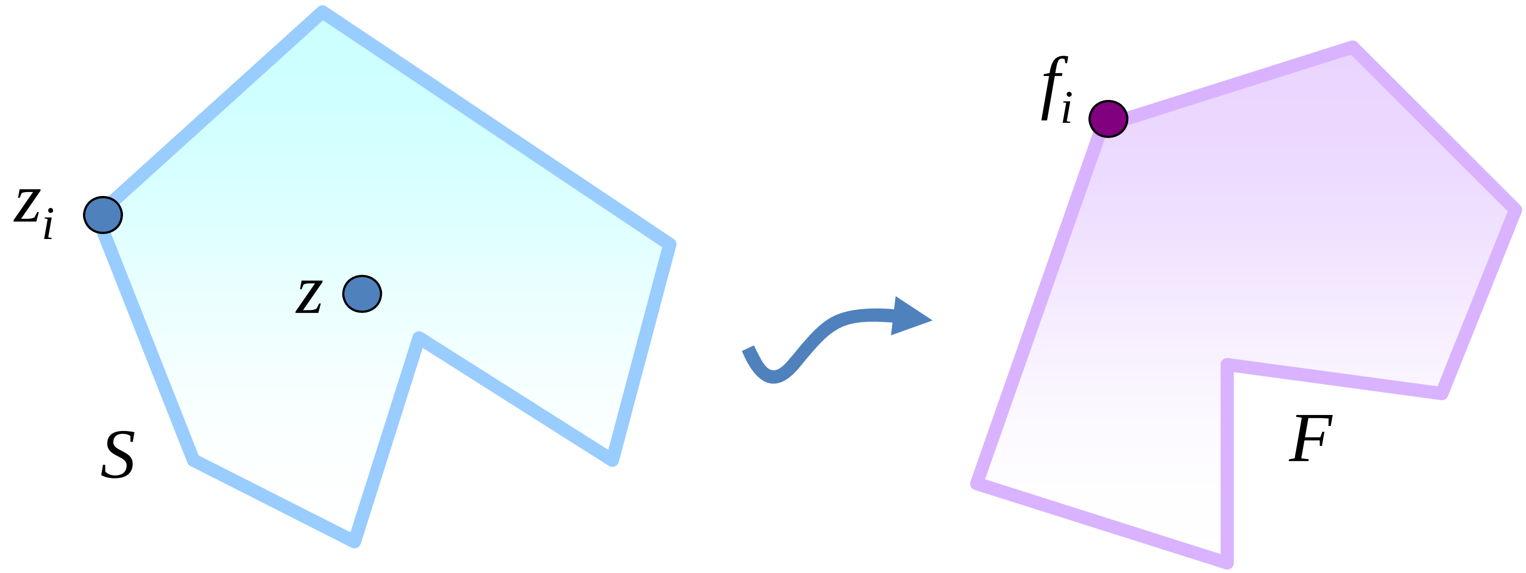
- All real bary coords schemes are affine invariant

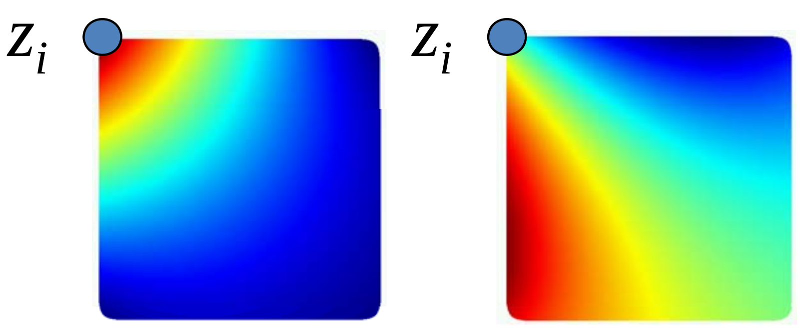
$$g(x) = \sum_{i=1}^{n} w_i(x) f_i \quad , \quad \begin{matrix} x_i, f_i, x \in R^2 \\ w_i(x) \in R \end{matrix}$$

- Define complex bary coords

$$g(z) = \sum_{i=1}^{n} w_i(z) f_i \quad , \quad \begin{matrix} z_i, f_i, z \in C \\ w_i(z) \in C \end{matrix}$$

Deformation with Complex Bary Coords



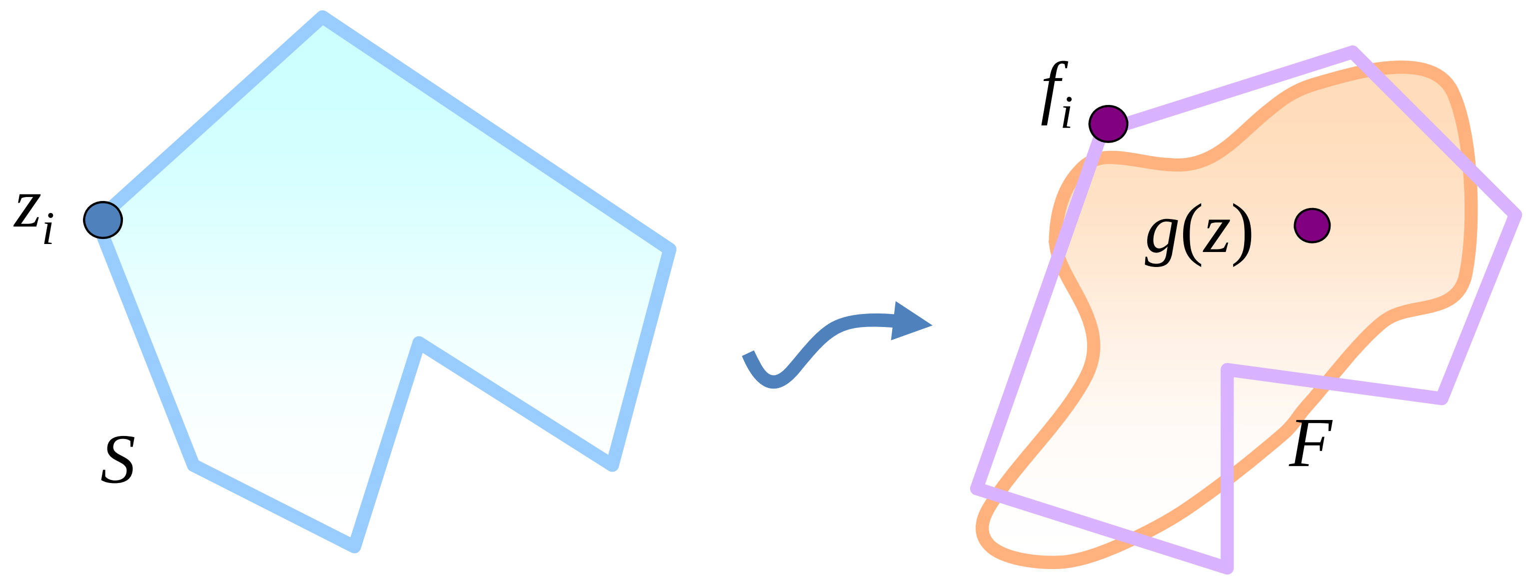


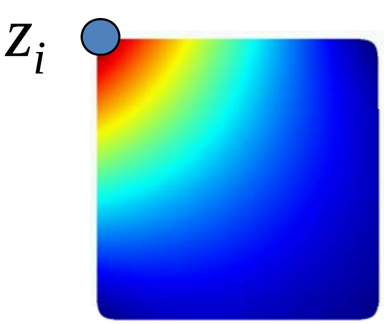
$real(w_i(z))$ $imag(w_i(z))$

$$w_i(z)$$

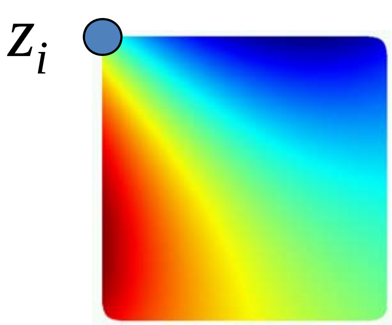
Complex Bary Coords Function

Deformation with Complex Bary Coords





$real(w_i(z))$



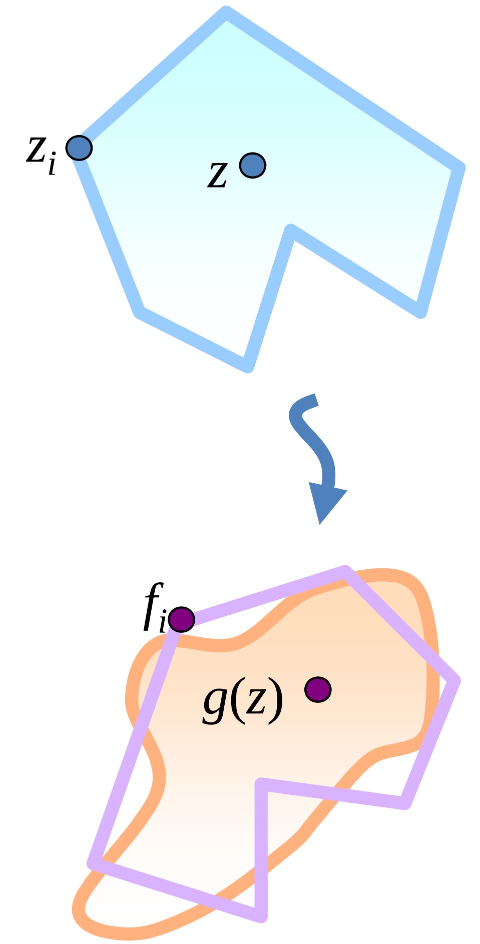
$imag(w_i(z))$

$$g(z) = \sum_{i=1}^{n} w_i(z) f_i$$

Discrete Cauchy-Green Coordinates

Smooth case:

$$g(z) = \frac{1}{2\pi i} \oint_S \frac{1}{w - z} f(w)\,dw$$



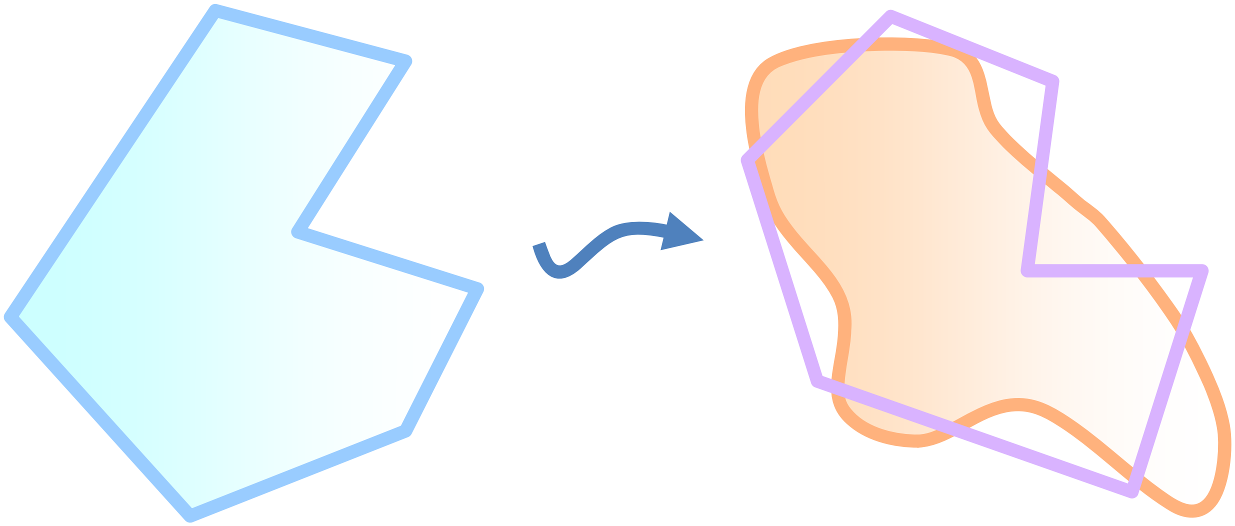
S is polygon → integrate over edges:

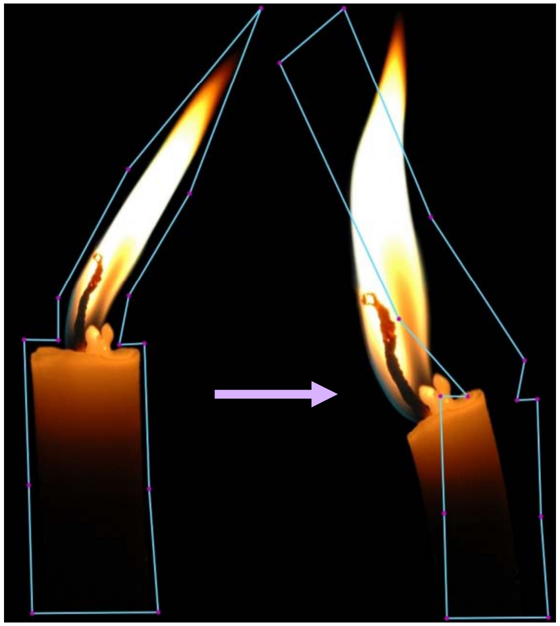
$$g(z) = \frac{1}{2\pi i} \sum_{i=1}^{n} \boxed{\oint_{e_i} \frac{1}{w - z} f(w)\,dw}$$

Closed form expression

$$g(z) = \sum_{i=1}^{n} w_i(z) f_i$$

Better Coordinates?





Given source & target polygon

→ A conformal map of interior

"Best" conformal map?

Variational Cauchy Coordinates

Find best $u_1,..,u_n$ such that

$$g_u(z) = \sum_{j=1}^{n} C_j(z) u_j$$

Is optimal

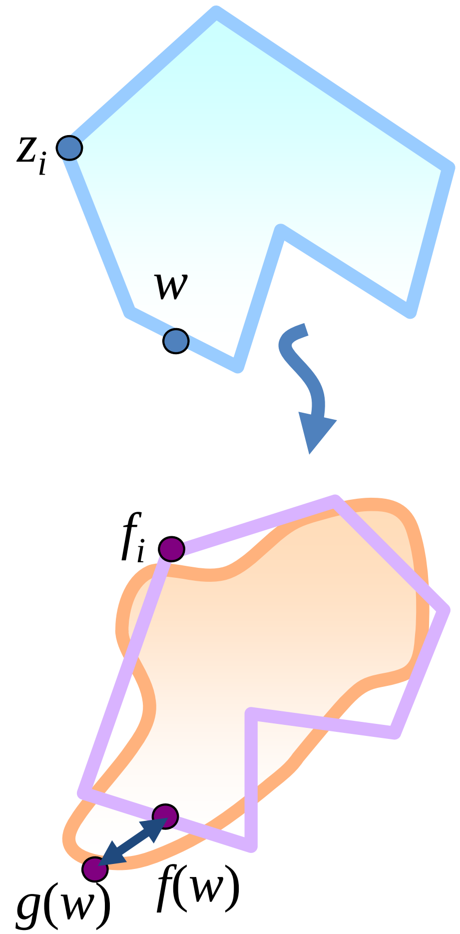
→ Minimize energy functional on g

Szegö Coordinates

Goal: Best fit to target polygon

$$E(g) = \oint_S \left| \boxed{g(w)} - \boxed{f(w)} \right|^2 ds$$

$$= \oint_S \left| \sum_{j=1}^{n} C_j(w) u_j - f(w) \right|^2 ds$$

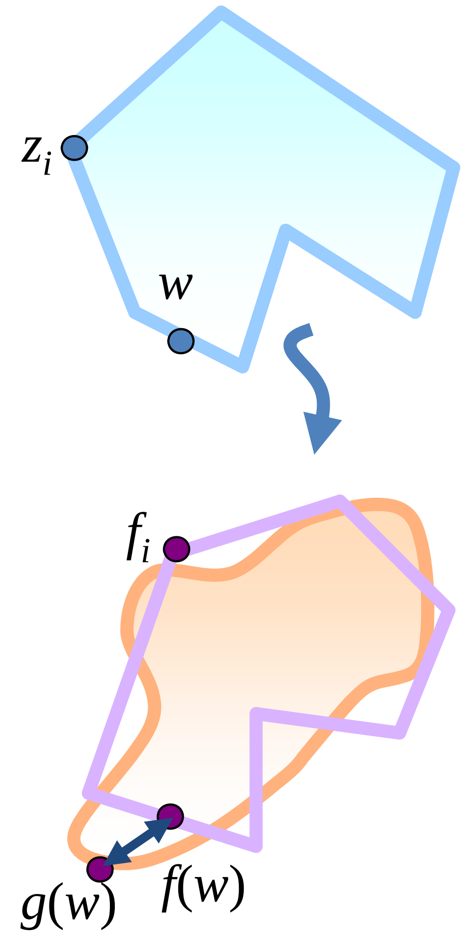


Szegö Coordinates

Goal: Best fit to target polygon

$$E(g) = \oint_S \left| \; g(w) \; - \; f(w) \; \right|^2 ds$$

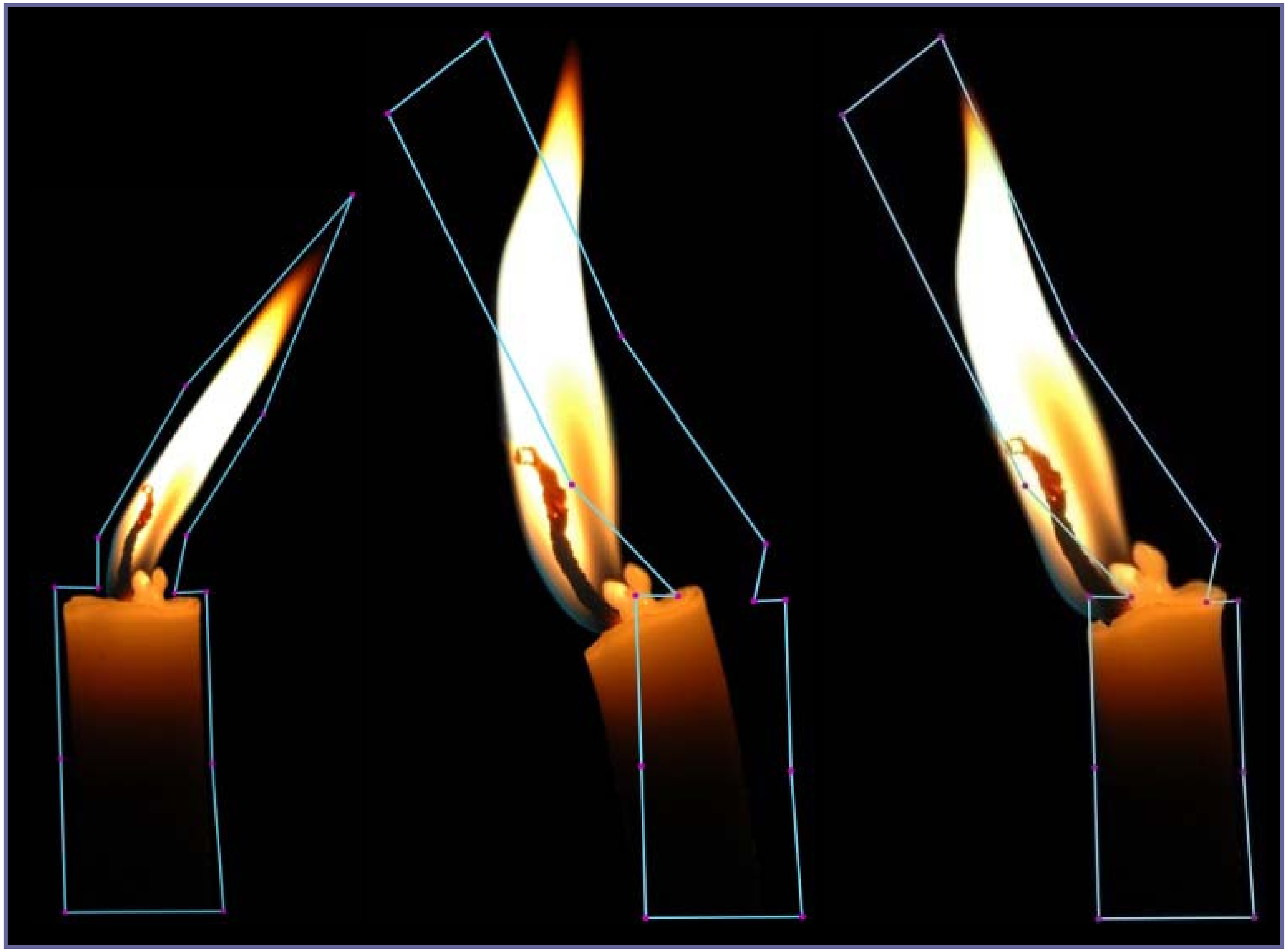
$$= \oint_S \left| \sum_{j=1}^{n} C_j(w) u_j - f(w) \right|^2 ds$$



Solve: Sample boundary →

Linear equations in u_j

Szegö Coordinates - Comparison



Source Cauchy-Green Szegö

Point-to-Point Coordinates

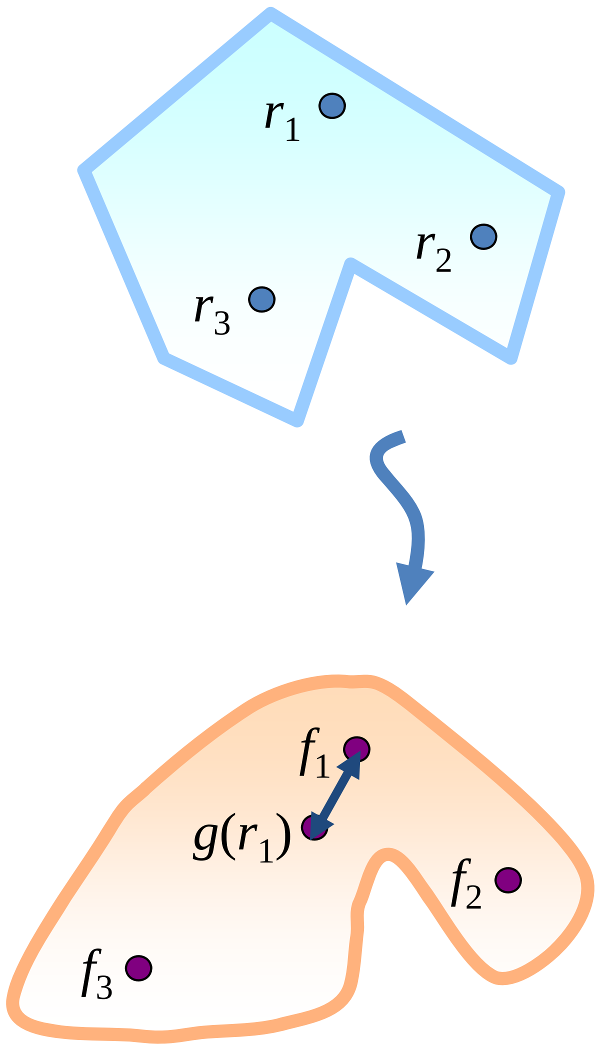
Goal: Free user from the cage

$$E(g) = \sum_{k=1}^{p} \left| \boxed{g(r_k)} - \boxed{f_k} \right|^2$$

$$+\lambda^2 \int_S \left| g''(w) \right|^2 ds$$

Solve: Sample boundary

→ Linear equations in u_j



Limitations

- Conformal not always best – large scales
  - Isometric?

- Constructing a cage hard in 3D
  - Cage-less?

References

- “On Linear Variational Surface Deformation Methods” [Botsch & Sorkine ‘08]
- Tutorial: “Interactive Shape Modeling and Deformation” [Sorkine & Botsch ’09]
- “Image deformation using moving least squares” [Schaefer et al ’06]
- “Mean Value Coordinates for Closed Triangular Meshes” [Ju et al ’05]
- “Harmonic coordinates for character articulation” [Joshi et al ’07]
- “Green Coordinates” [Lipman et al ’08]
- “Complex Barycentric Coordinates with Applications to Planar Shape Deformation” [Weber et al. ’09]
- Excellent webpage on barycentric coordinates: http://www.inf.usi.ch/hormann/barycentric/

Thank you!