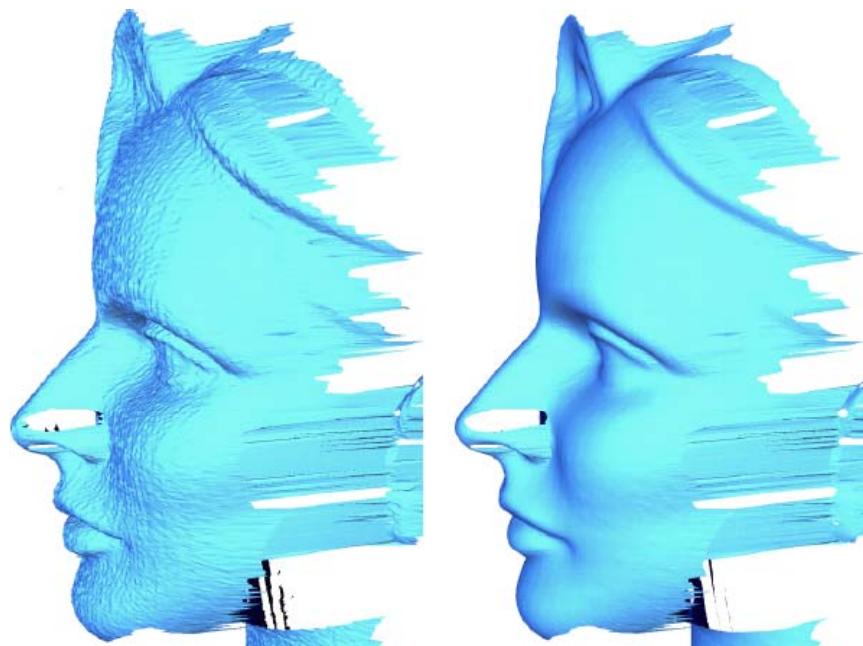


Mesh Smoothing



Mesh Processing Pipeline



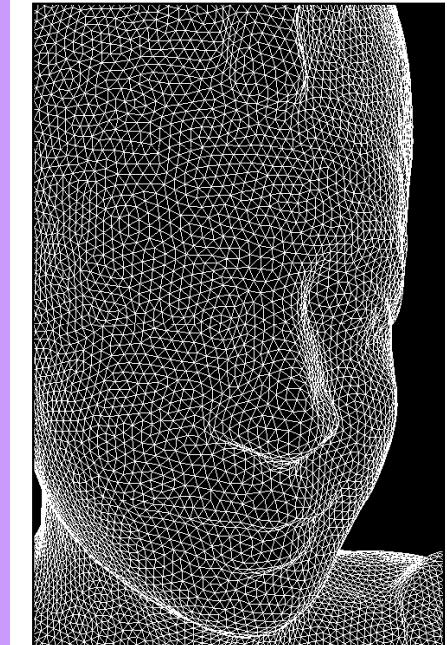
Scan



Reconstruct



Clean



Remesh

...

Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading



Flat
Shading



Gouraud
Shading



Phong
Shading

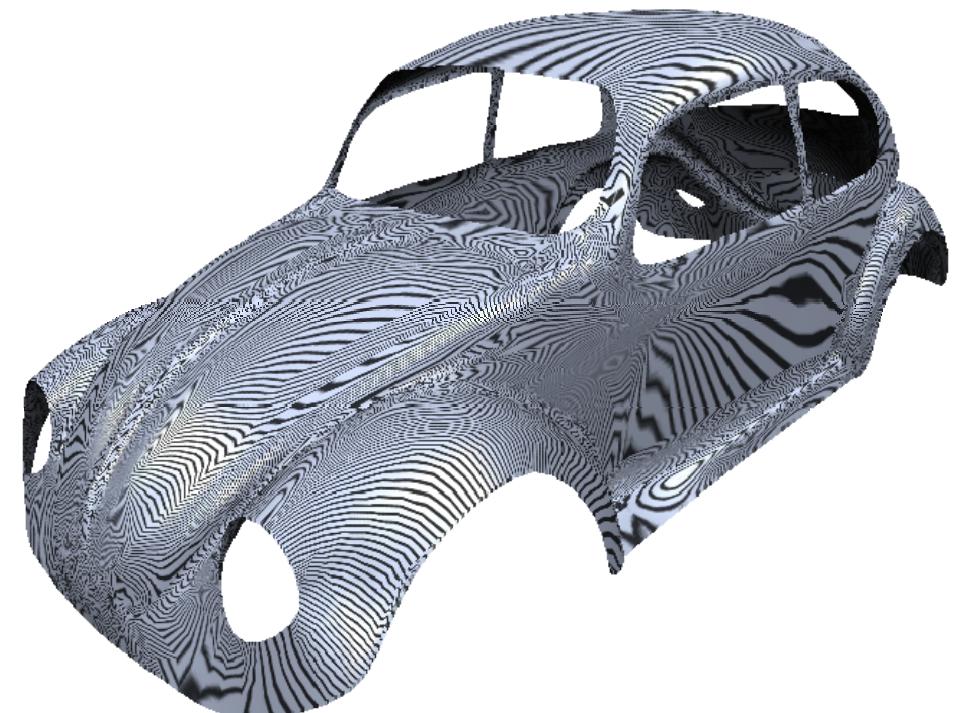
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading



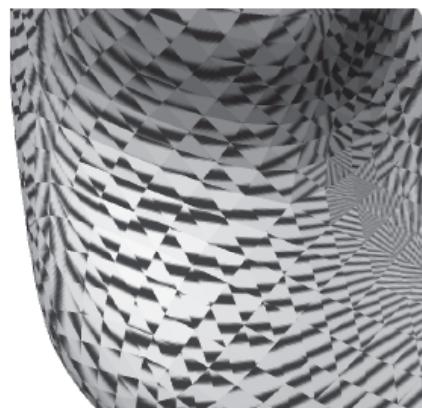
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines



Mesh Quality

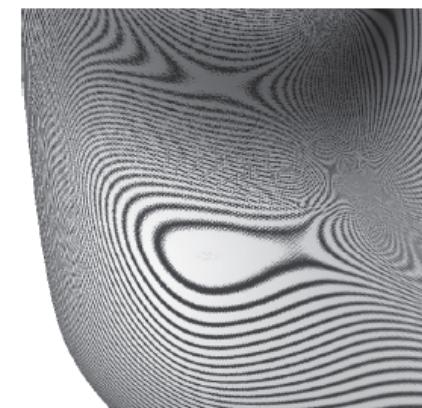
- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - differentiability one order lower than surface
 - can be efficiently computed using graphics hardware



C^0



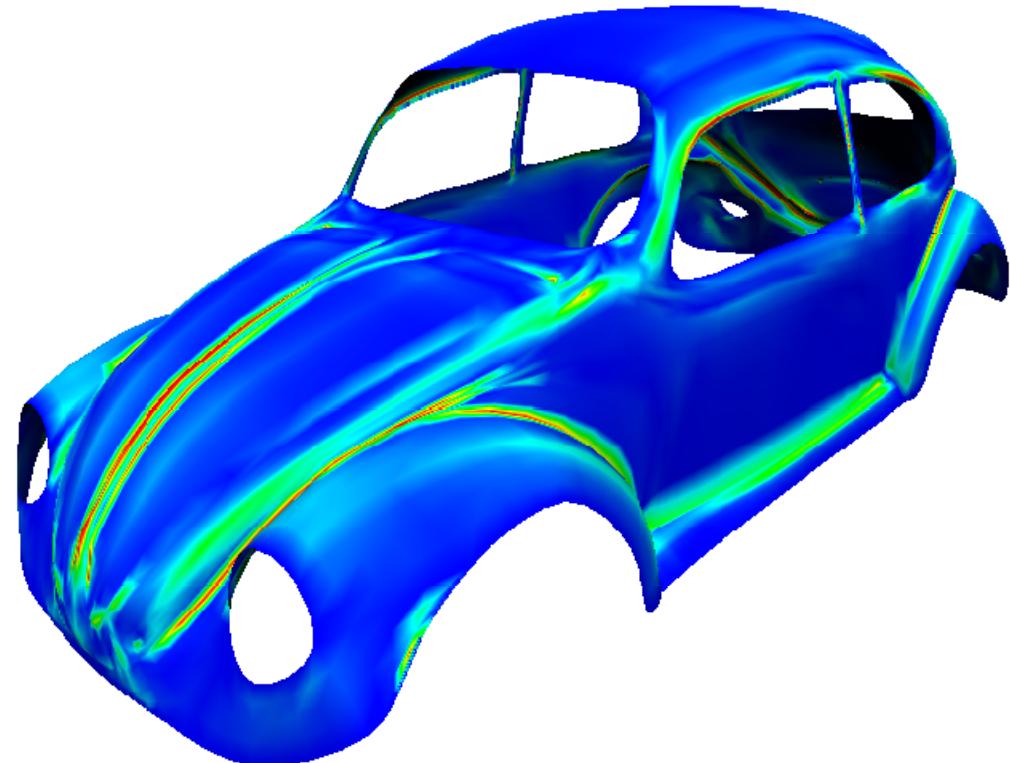
C^1



C^2

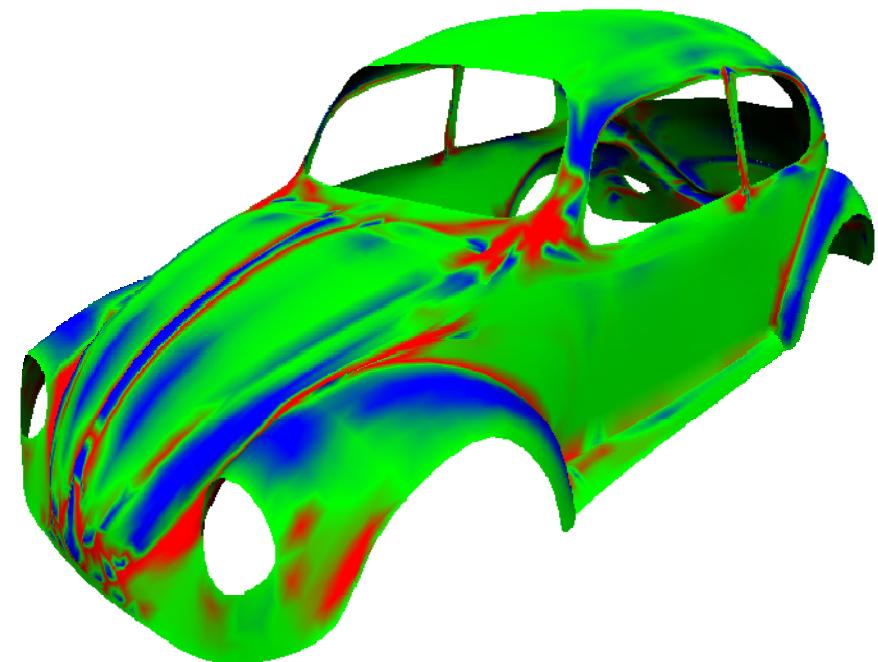
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - Curvature
 - Mean curvature



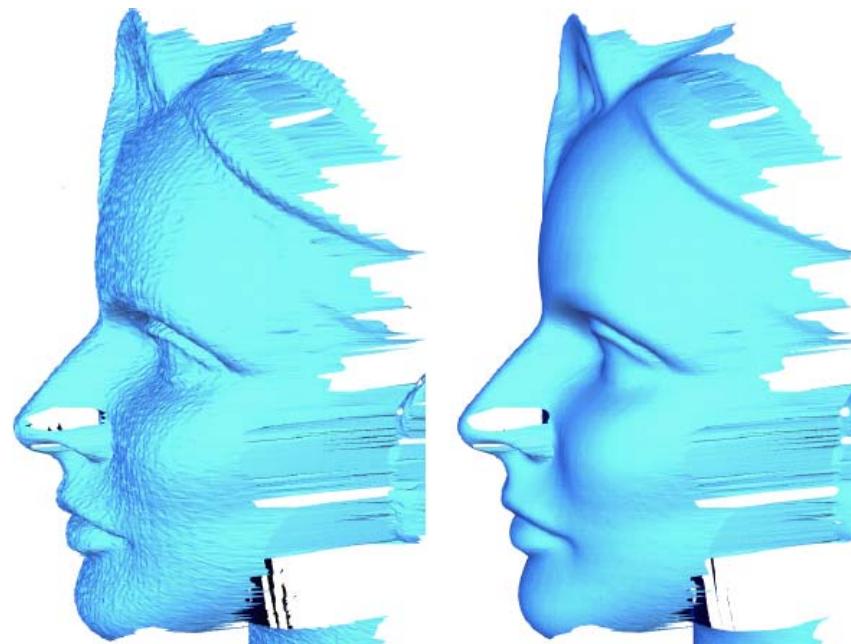
Mesh Quality

- Visual inspection of “sensitive” attributes
 - Specular shading
 - Reflection lines
 - Curvature
 - Mean curvature
 - Gaussian curvature



Motivation

- Filter out high frequency noise



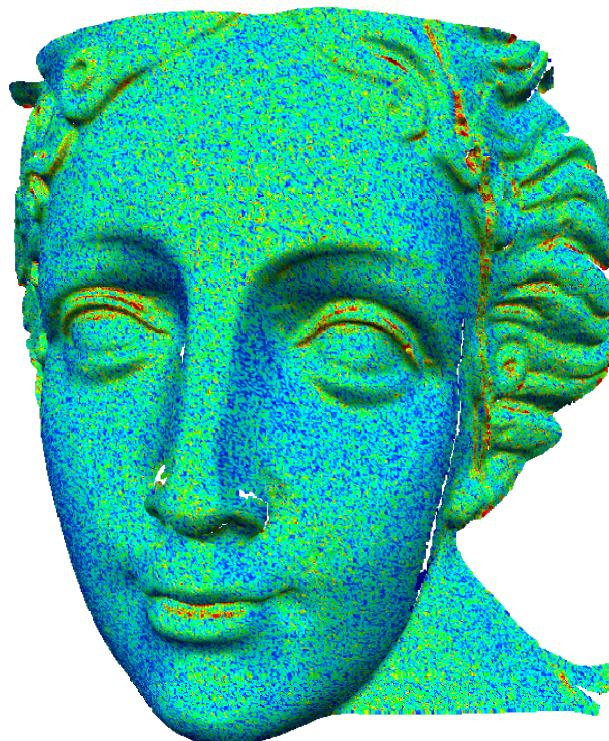
Mesh Smoothing

(aka Denoising, Filtering, Fairing)

Input: Noisy mesh (scanned or other)

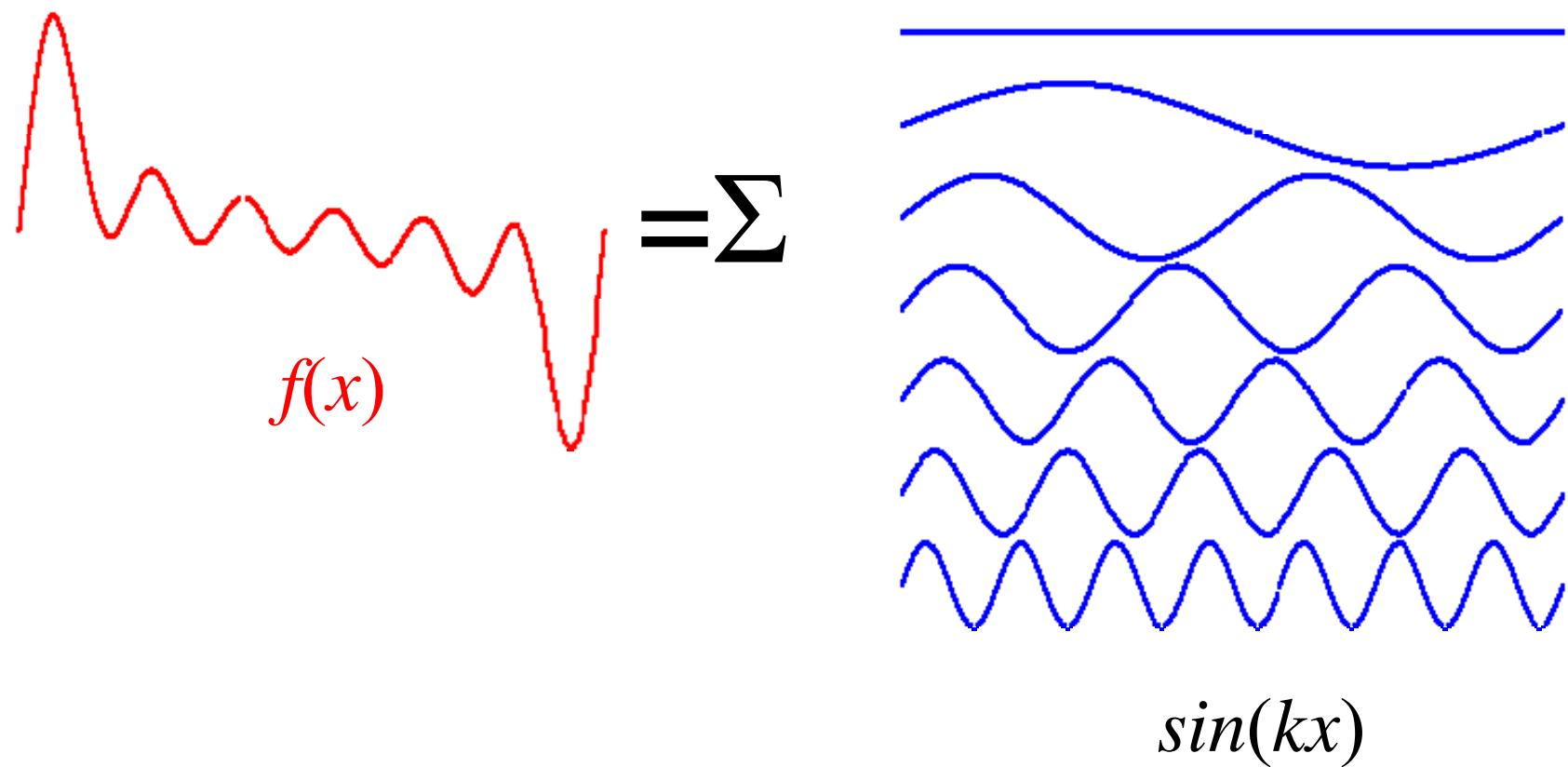
Output: Smooth mesh

How: Filter out high frequency noise



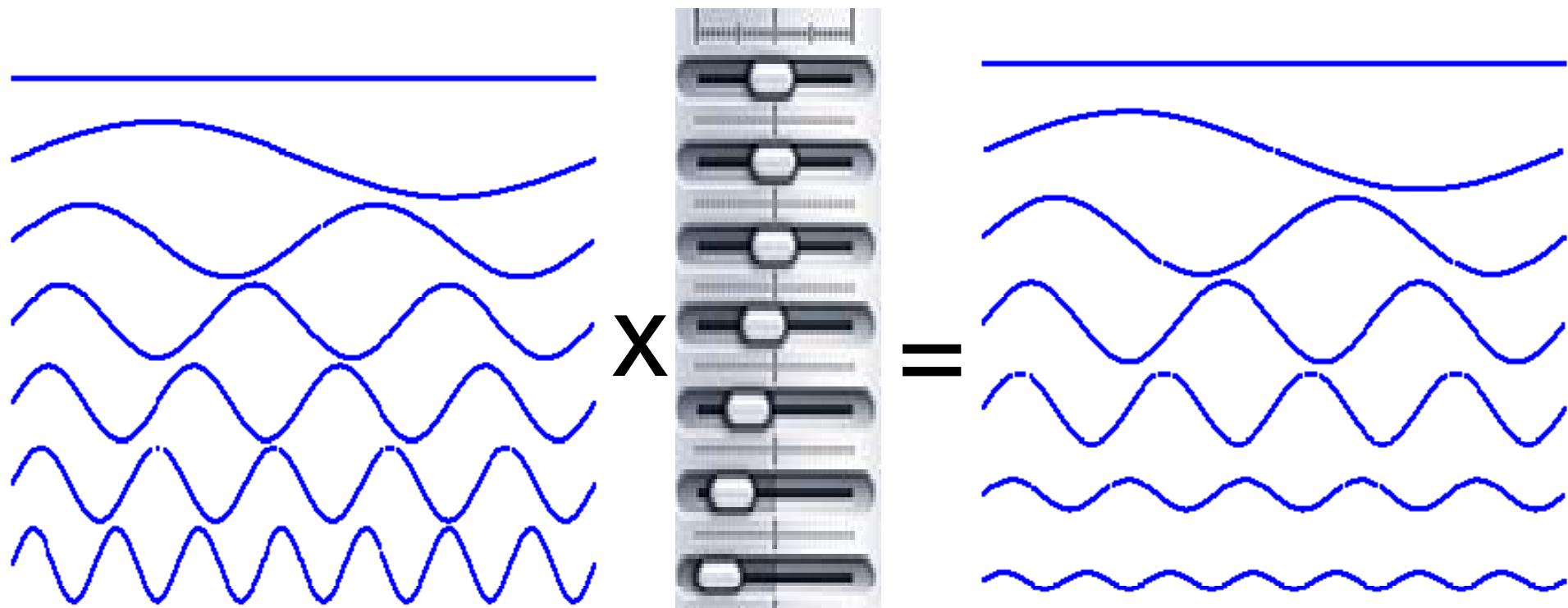
Smoothing by Filtering

Fourier Transform



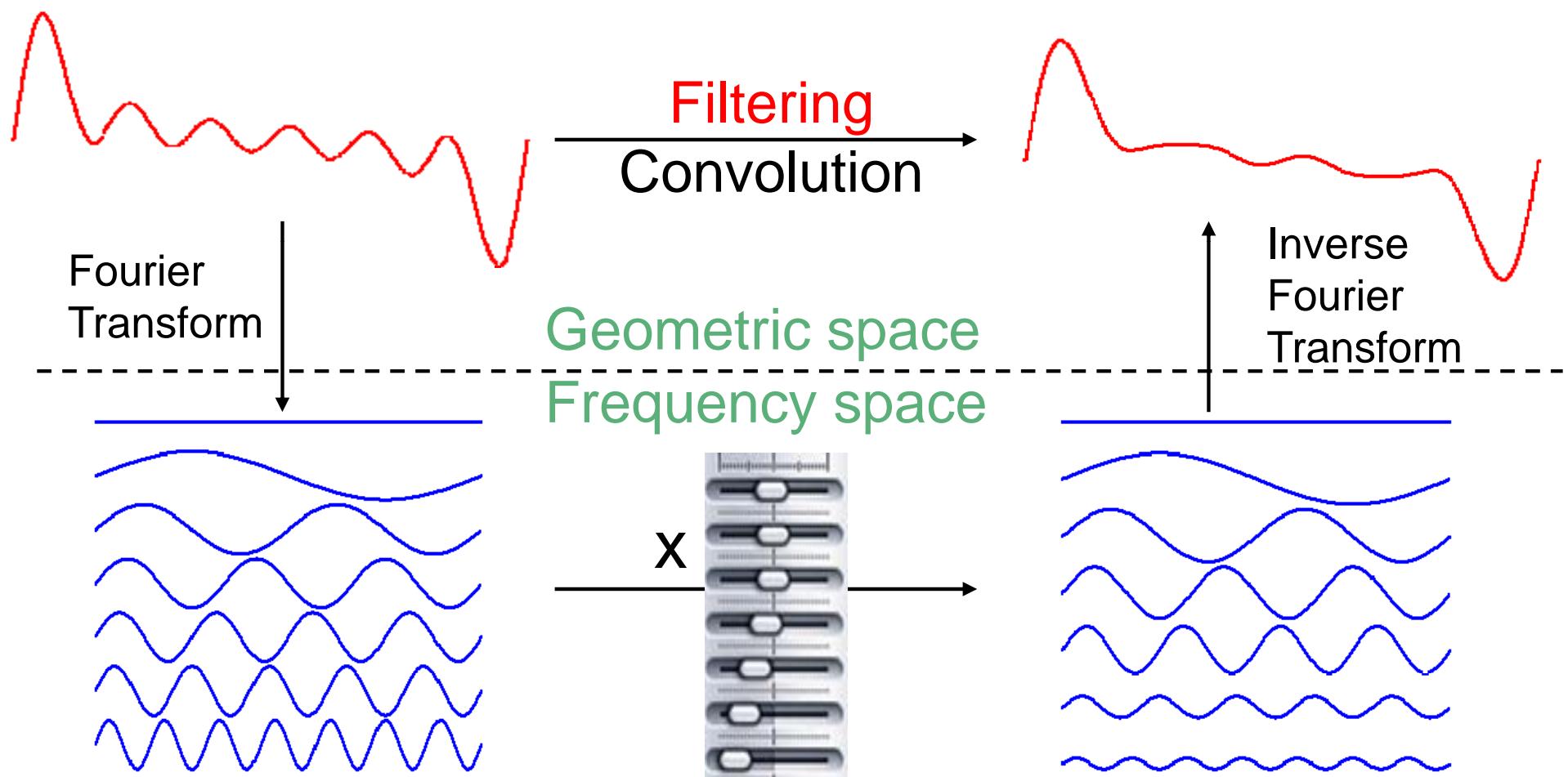
Smoothing by Filtering

Fourier Transform

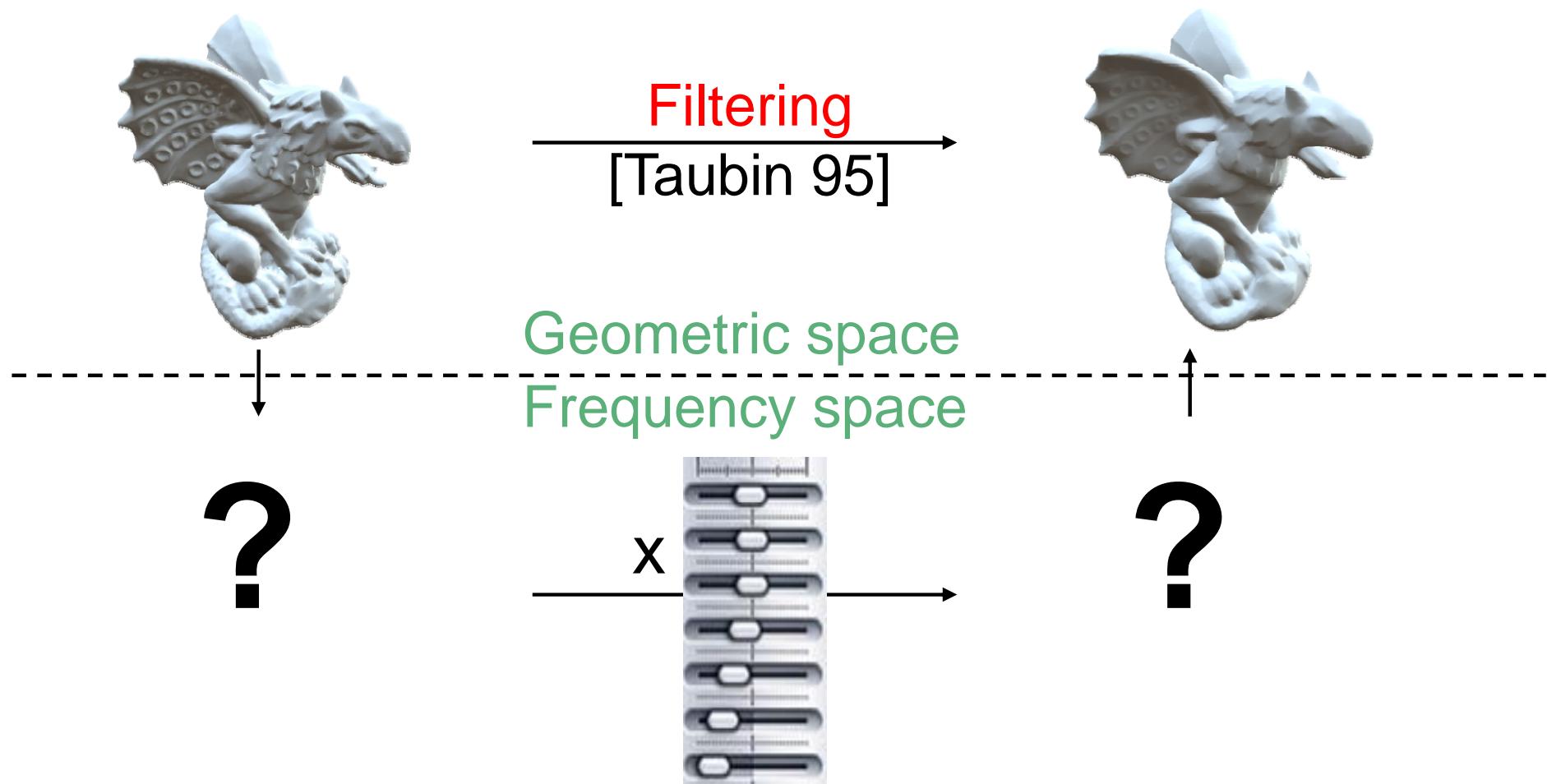


Smoothing by Filtering

Fourier Transform

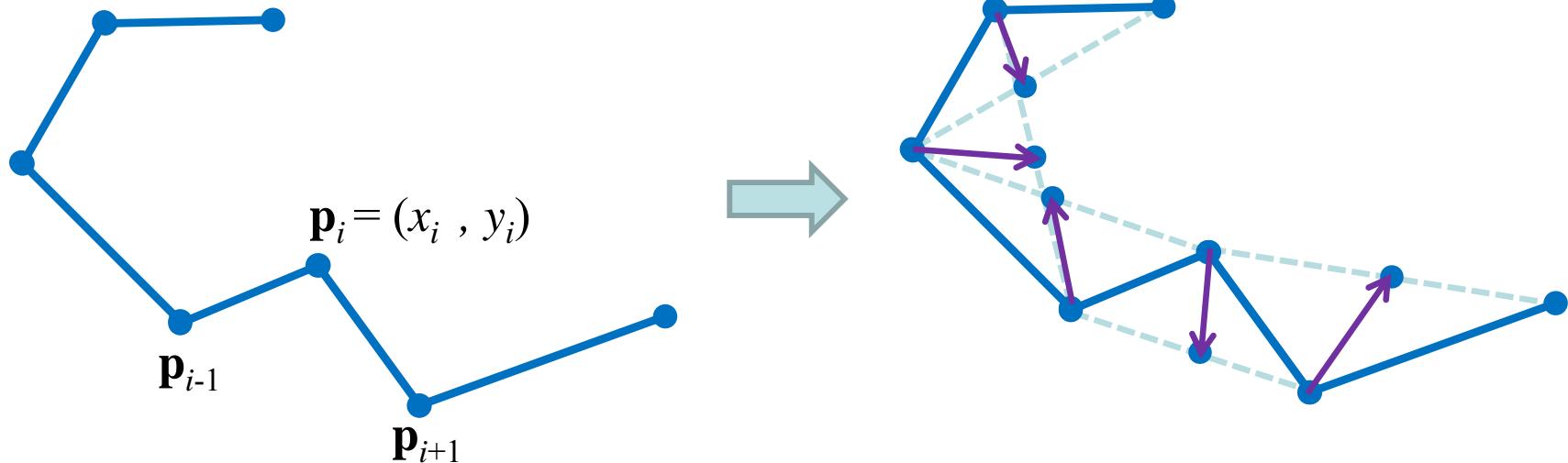


Filtering on a Mesh



Laplacian Smoothing

An easier problem: How to smooth a curve?

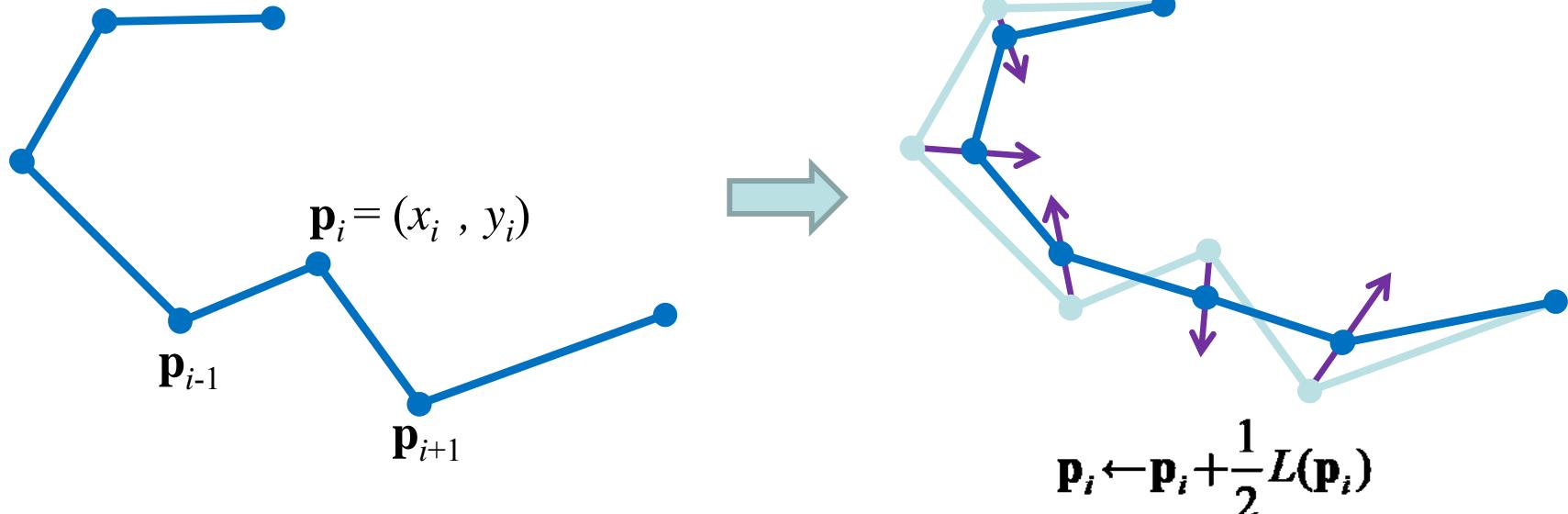


$$(p_{i-1} + p_{i+1})/2 - p_i$$

$$L(p_i) = \frac{1}{2}(p_{i+1} - p_i) + \frac{1}{2}(p_{i-1} - p_i)$$

Laplacian Smoothing

An easier problem: How to smooth a curve?



Finite difference
discretization of second
derivative
= Laplace operator in
one dimension

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

Laplacian Smoothing

Algorithm:

Repeat for m iterations (for non boundary points):

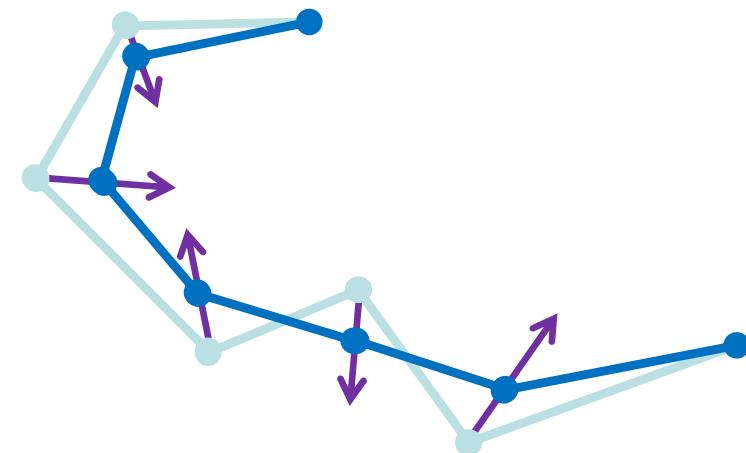
$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda L(\mathbf{p}_i)$$

For which λ ?

$$0 < \lambda < 1$$

Closed curve converges to?

Single point



Spectral Analysis

Closed Curve

Re-write $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda L(\mathbf{p}_i^{(t)})$

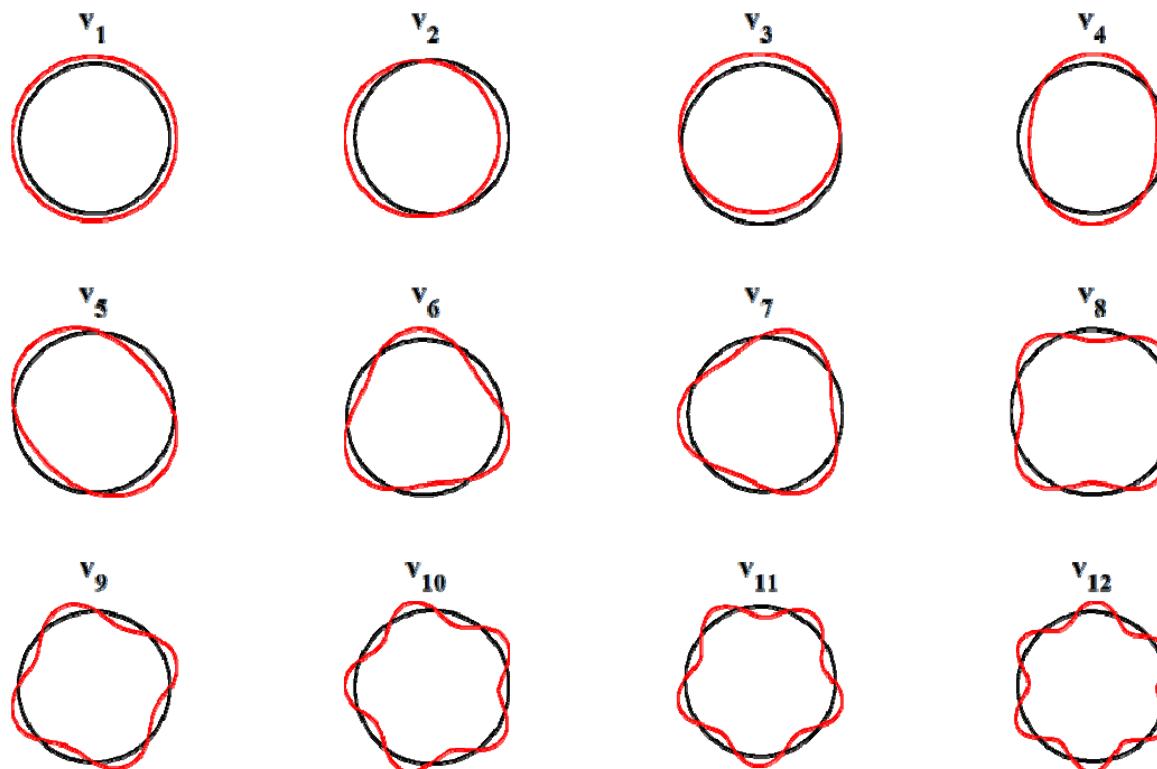
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$$

in matrix notation: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L} \mathbf{P}^{(t)}$

$$\mathbf{P} = \begin{pmatrix} x_1 & y_2 \\ \dots & \dots \\ x_n & y_n \end{pmatrix} \in \mathbb{R}^{n \times 2} \quad \mathbf{L} = \frac{1}{2} \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & & \ddots & & & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

The Eigenvectors of \mathbf{L}

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T \quad \mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \dots & \\ & & & k_n \end{pmatrix}$$



Spectral Analysis

Then: $\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \lambda \mathbf{L} \mathbf{P}^{(t)} = (\mathbf{I} - \lambda \mathbf{L}) \mathbf{P}^{(t)}$

After m iterations: $\mathbf{P}^{(m)} = (\mathbf{I} - \lambda \mathbf{L})^m \mathbf{P}^{(0)}$

Can be described using eigen-decomposition of \mathbf{L}

Filtering high frequencies

$$\mathbf{L} = \mathbf{V} \mathbf{D} \mathbf{V}^T$$

$$\mathbf{V} = \begin{pmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{pmatrix}$$

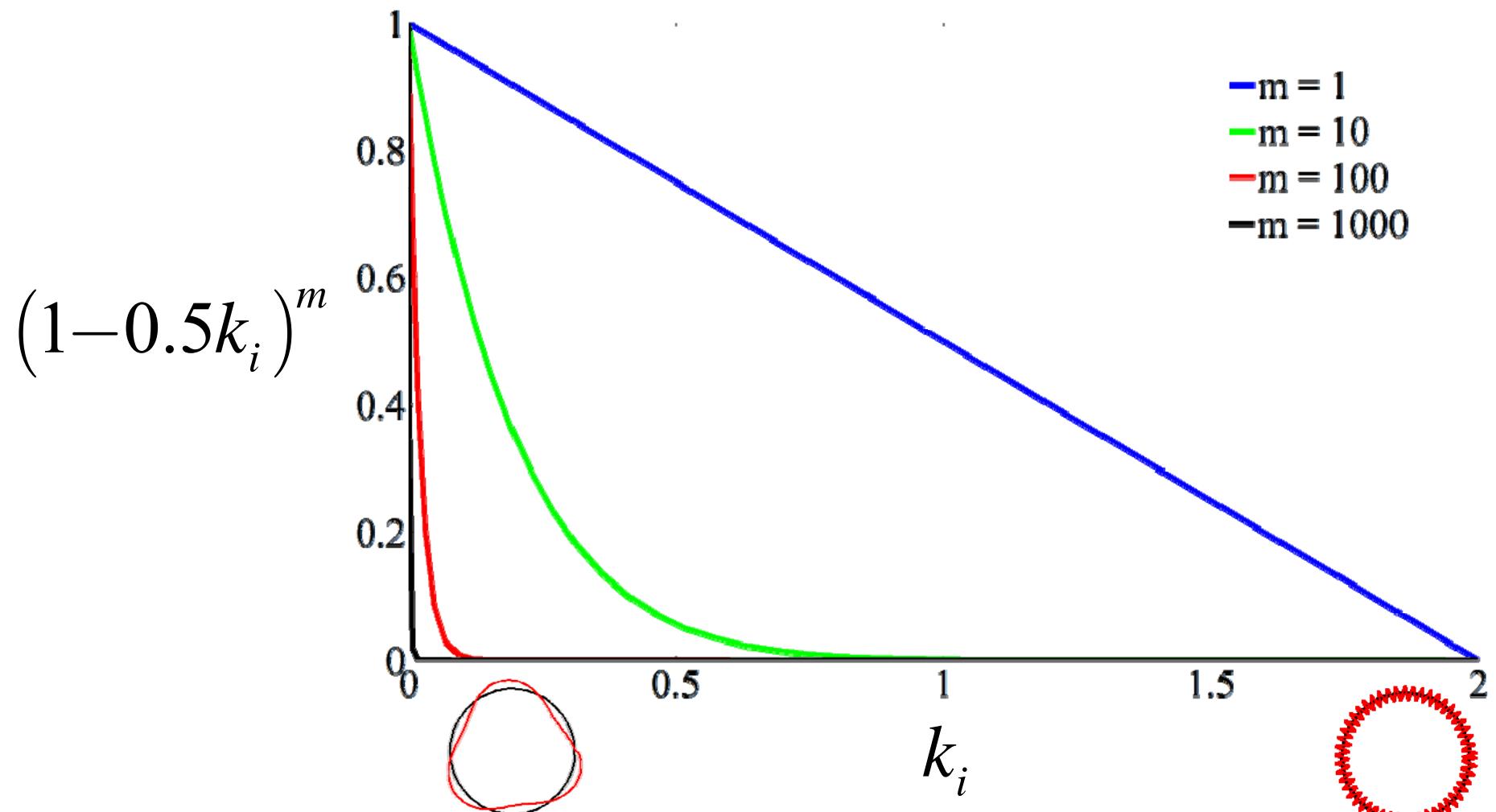


$$\mathbf{P}^{(m)} = \mathbf{V} (\mathbf{I} - \lambda \mathbf{D})^m \mathbf{V}^T \mathbf{P}^{(0)}$$



Spectral Analysis

Laplacian Smoothing



Laplacian Smoothing on Meshes

Same as for curves:

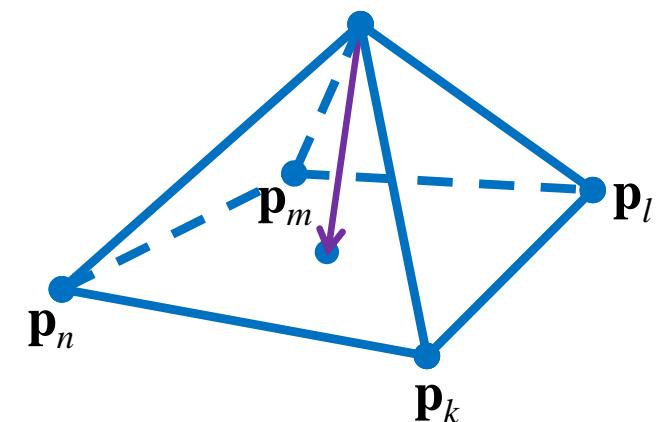
$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$$

$$N_i = \{k, l, m, n\}$$
$$\mathbf{p}_i = (x_i, y_i, z_i)$$

What is $\Delta \mathbf{p}_i$?

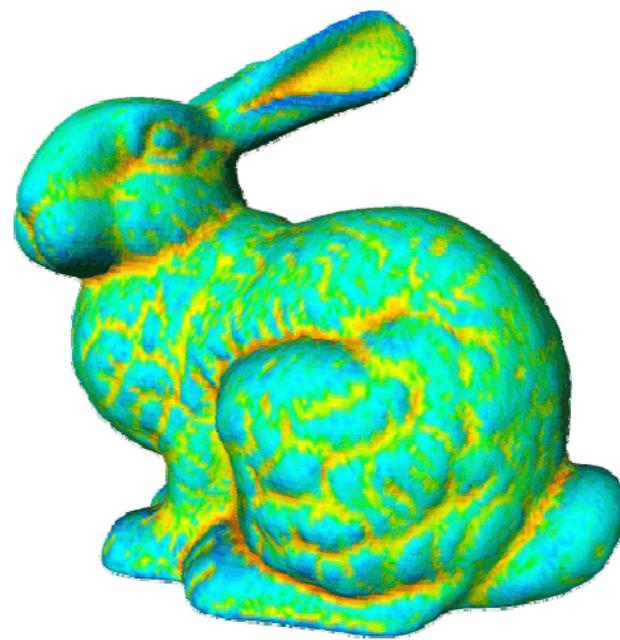


$$\frac{1}{2}(\mathbf{p}_{i+1} + \mathbf{p}_{i-1}) - \mathbf{p}_i$$

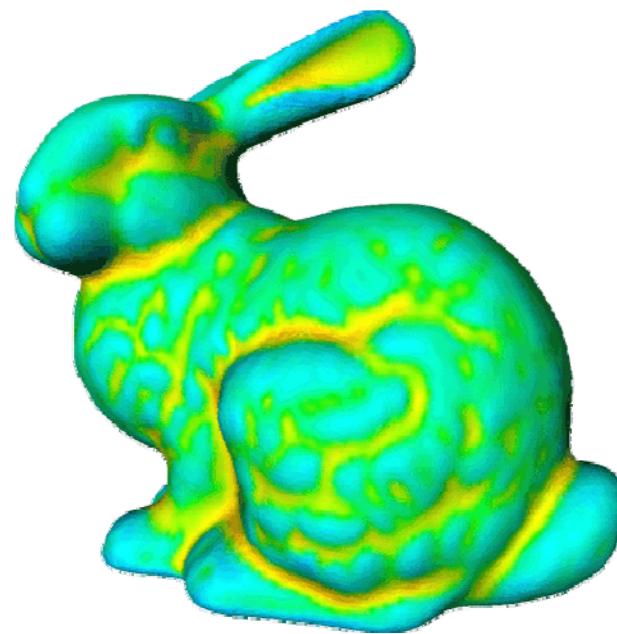


$$\frac{1}{|N_i|} \left(\sum_{j \in N_i} \mathbf{p}_j \right) - \mathbf{p}_i$$

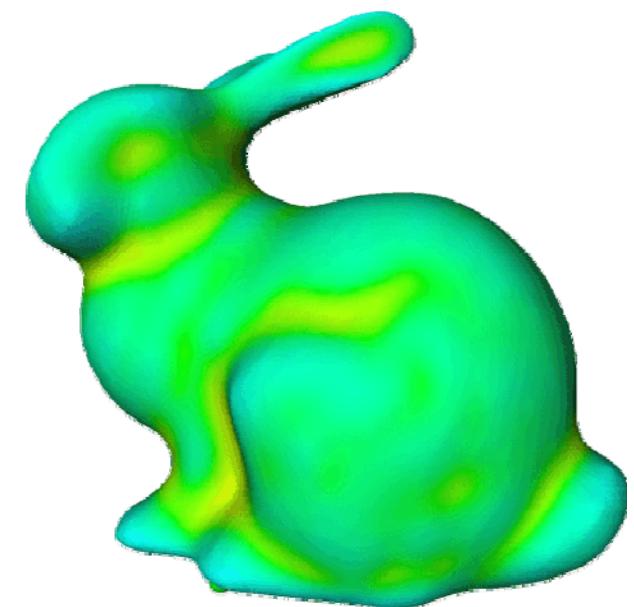
Laplacian Smoothing on Meshes



0 Iterations



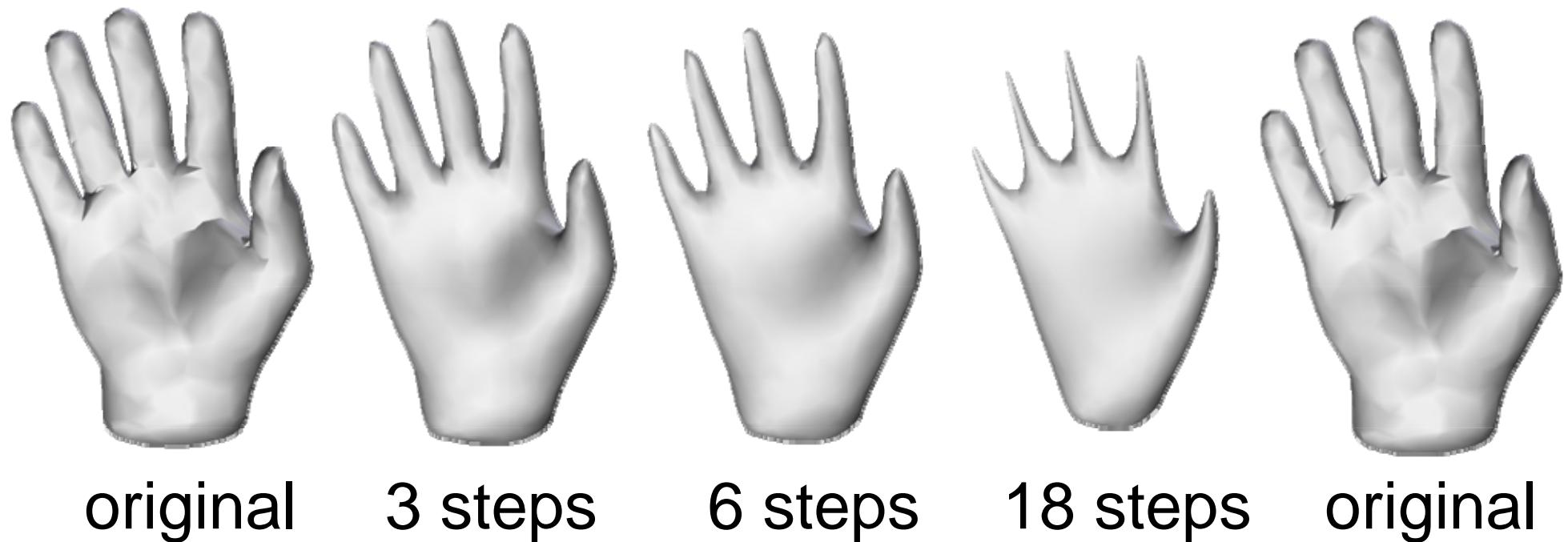
5 Iterations



20 Iterations

Problem - Shrinkage

Repeated iterations of Laplacian smoothing shrinks the mesh



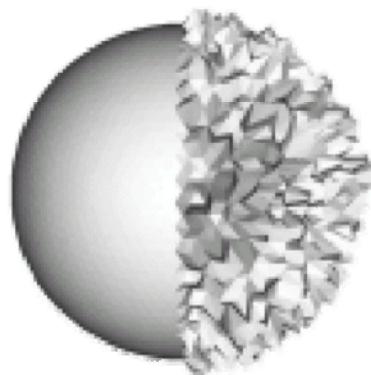
Taubin Smoothing

Iterate:

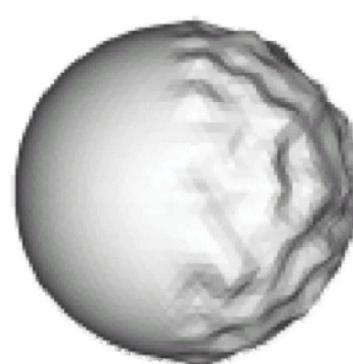
$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i \quad \text{Shrink}$$

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \mu \Delta \mathbf{p}_i \quad \text{Inflate}$$

with $\lambda > 0$ and $\mu < 0$



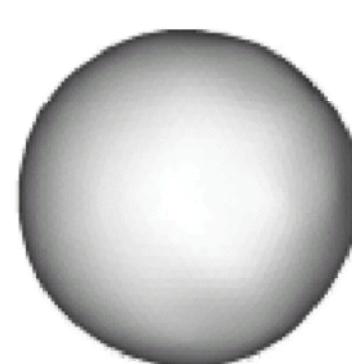
original



10 steps



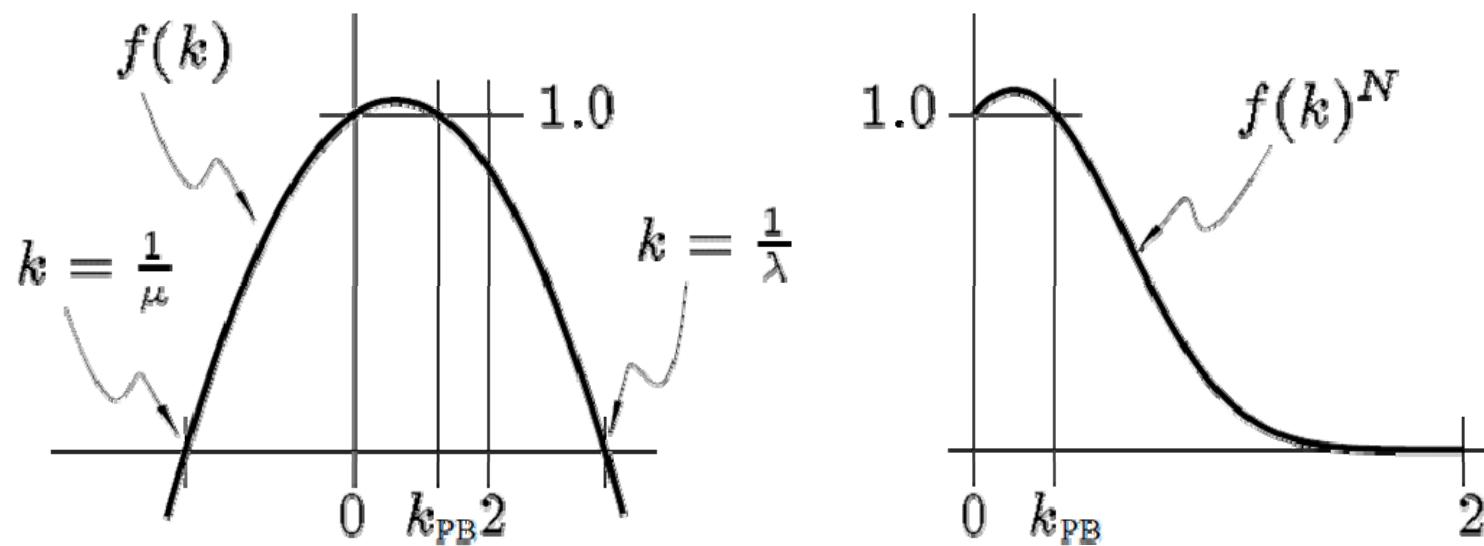
50 steps



200 steps

Spectral Analysis

Taubin Smoothing



$$f(k_i) = (1 - \lambda k_i)(1 - \mu k_i)$$

Laplacian Smoothing

$$\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$$

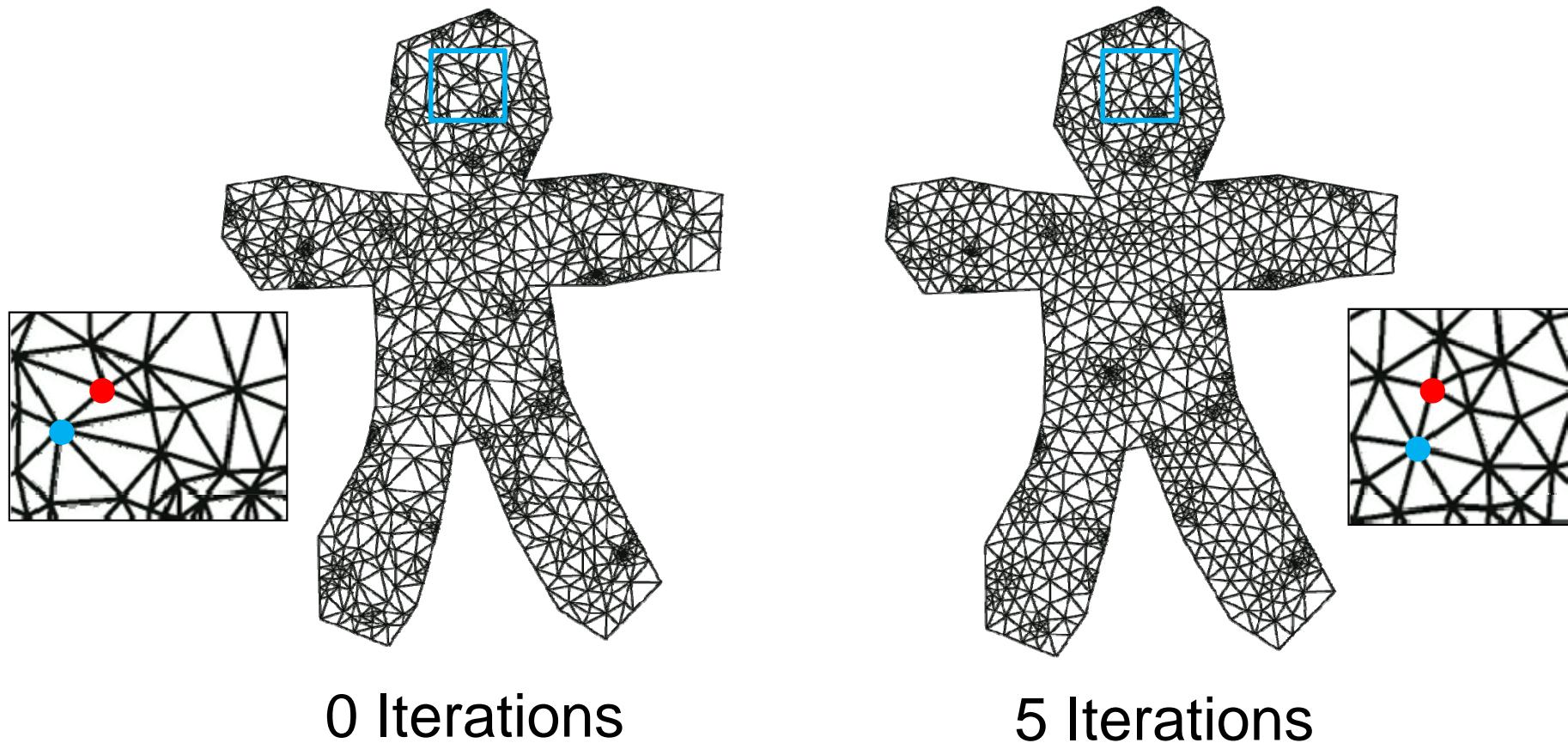
$\Delta \mathbf{p}_i$ = mean curvature normal

 mean curvature flow

Laplace Operator Discretization

The Problem

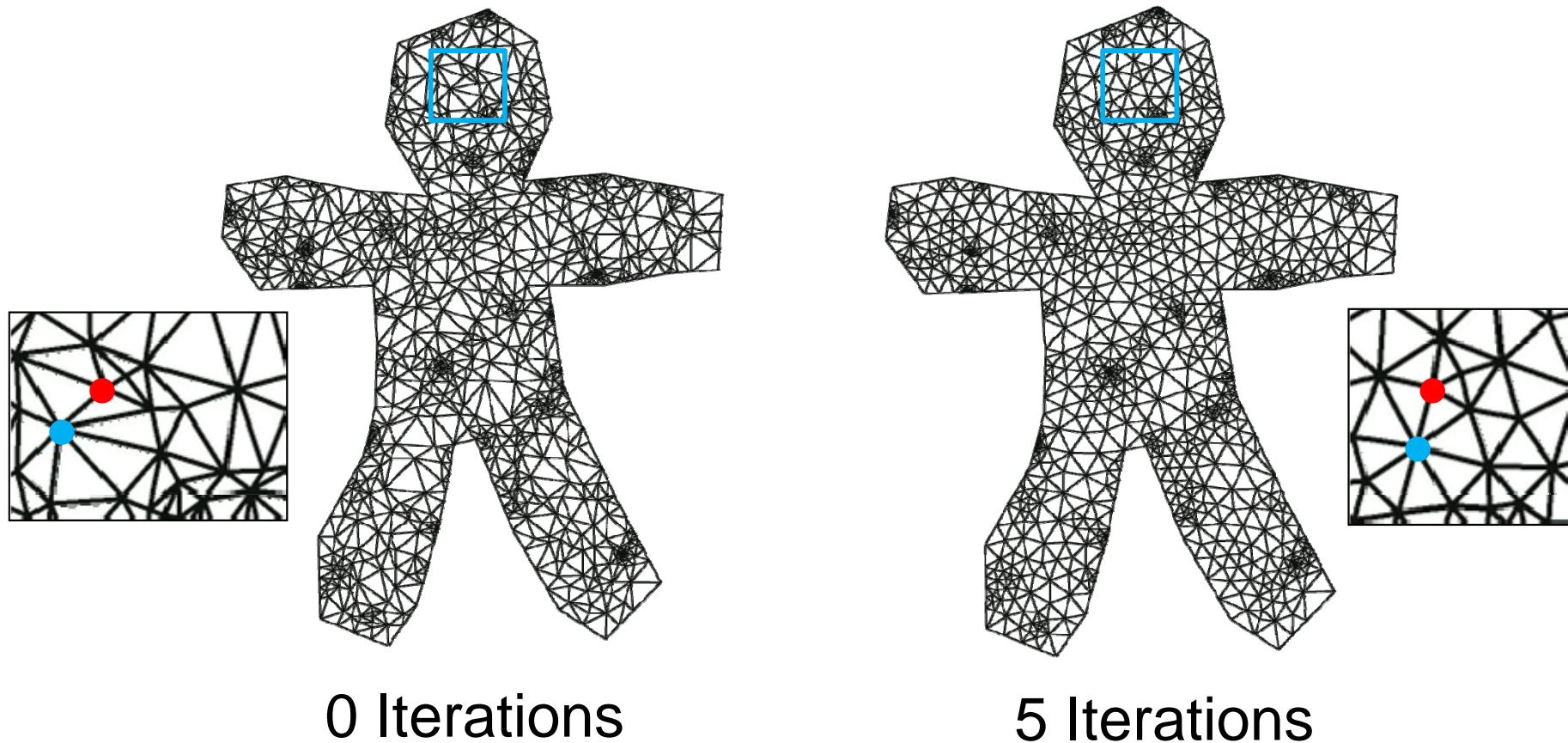
Sanity check – what should happen if the mesh lies in the plane: $\mathbf{p}_i = (x_i, y_i, 0)$?



Laplace Operator Discretization

The Problem

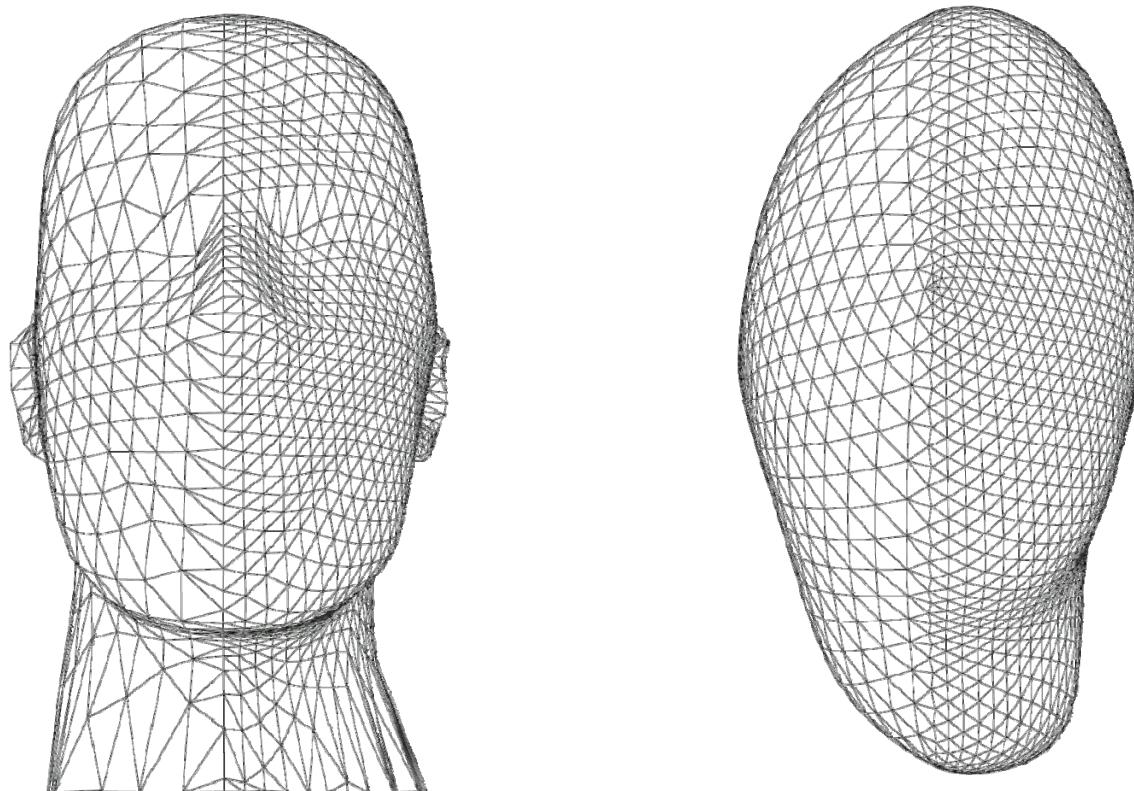
Not good – A flat mesh is smooth, should stay the same after smoothing



Laplace Operator Discretization

The Problem

Not good – The result should not depend on triangle sizes



Laplace Operator Discretization

What Went Wrong?

Back to curves:

$$\frac{1}{2}(\mathbf{p}_{i+1} + \mathbf{p}_{i-1}) - \mathbf{p}_i$$



Same weight for both neighbors,
although one is closer

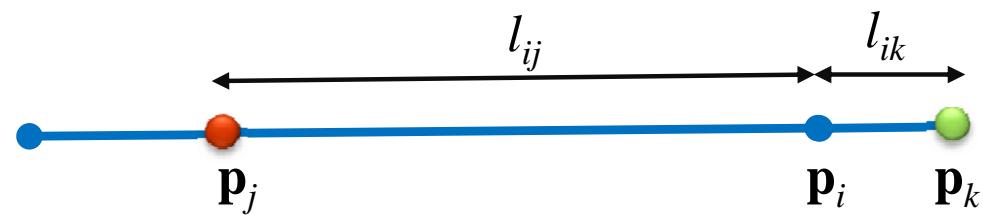
Laplace Operator Discretization

The Solution

Use a weighted average to define Δ

Which weights?

$$w_{ij} = \frac{1}{l_{ij}} \quad w_{ik} = \frac{1}{l_{ik}}$$



$$L(\mathbf{p}_i) = \frac{w_{ij}\mathbf{p}_j + w_{ik}\mathbf{p}_k}{w_{ij} + w_{ik}} - \mathbf{p}_i$$

Straight curves will be invariant to smoothing

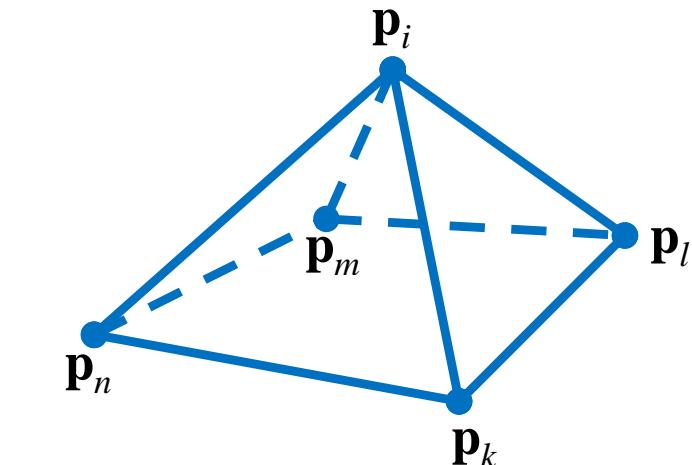
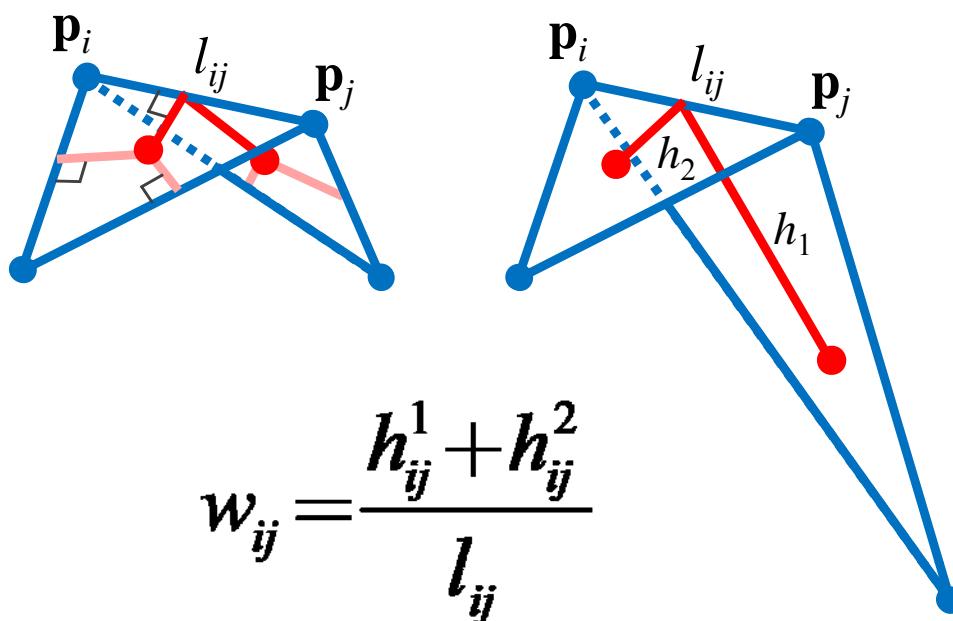
Laplace Operator Discretization

Cotangent Weights

Use a weighted average to define Δ

$$N_i = \{k, l, m, n\}$$

Which weights? $w_{ij} = \frac{1}{l_{ij}}$?



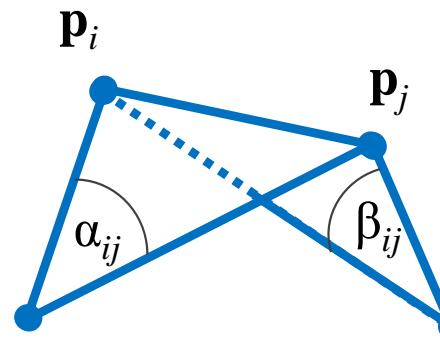
$$L(\mathbf{p}_i) = \frac{1}{\sum_{j \in N_i} w_{ij}} \left(\sum_{j \in N_i} w_{ij} \mathbf{p}_j \right) - \mathbf{p}_i$$

Laplace Operator Discretization

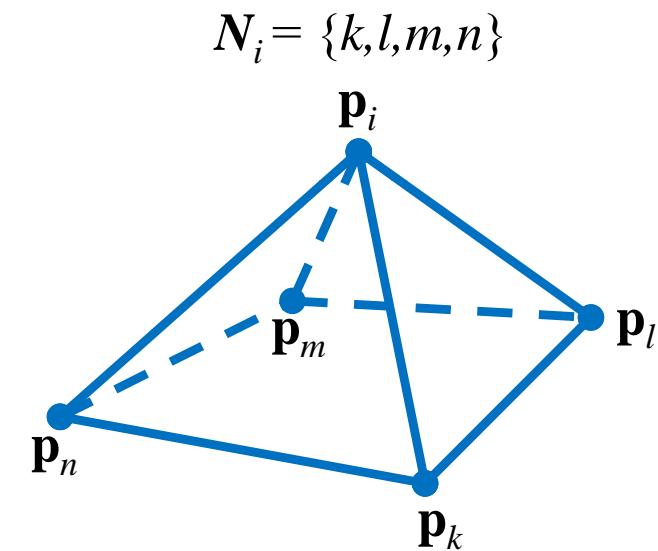
Cotangent Weights

Use a weighted average to define Δ

Which weights?



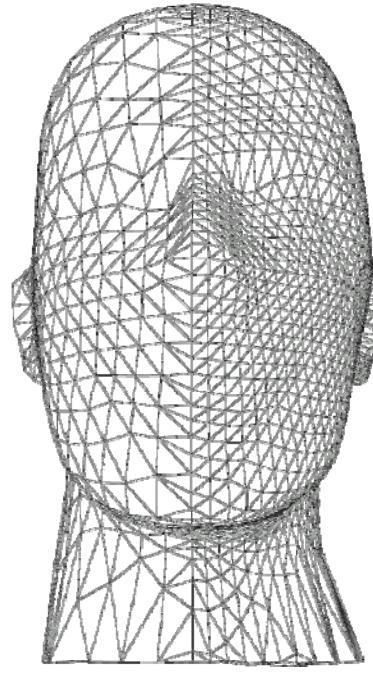
$$w_{ij} = \frac{h_{ij}^1 + h_{ij}^2}{l_{ij}} = \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})$$



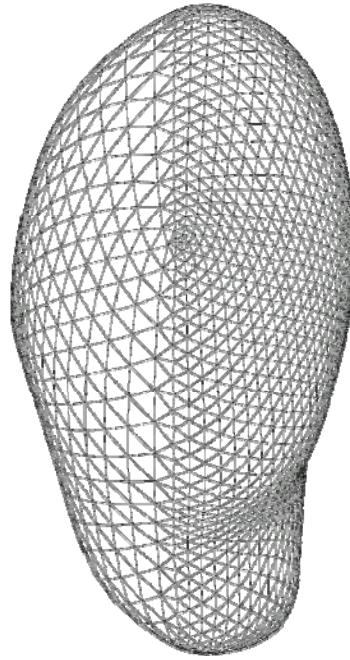
$$L(\mathbf{p}_i) = \frac{1}{\sum w_{ij}} \sum_{j \in N_i} w_{ij} (\mathbf{p}_j - \mathbf{p}_i)$$

Planar meshes will be invariant to smoothing

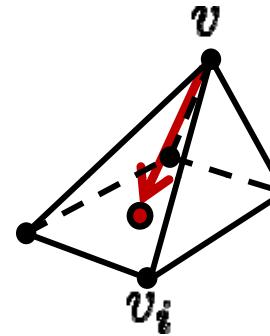
Smoothing with the Cotangent Laplacian



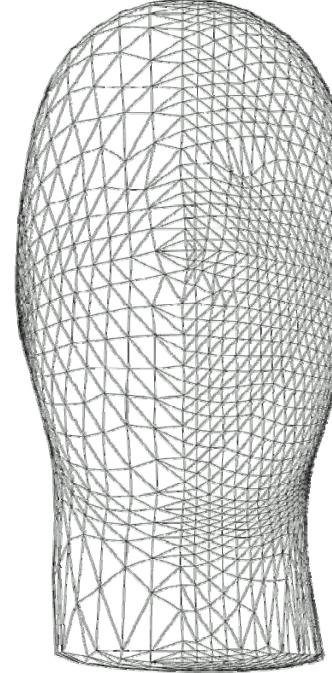
original



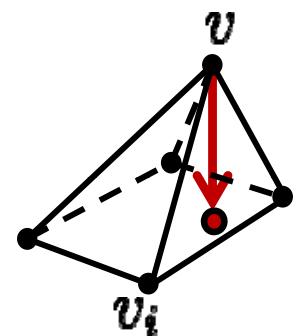
Uniform weights



normal
and
tangential
movement

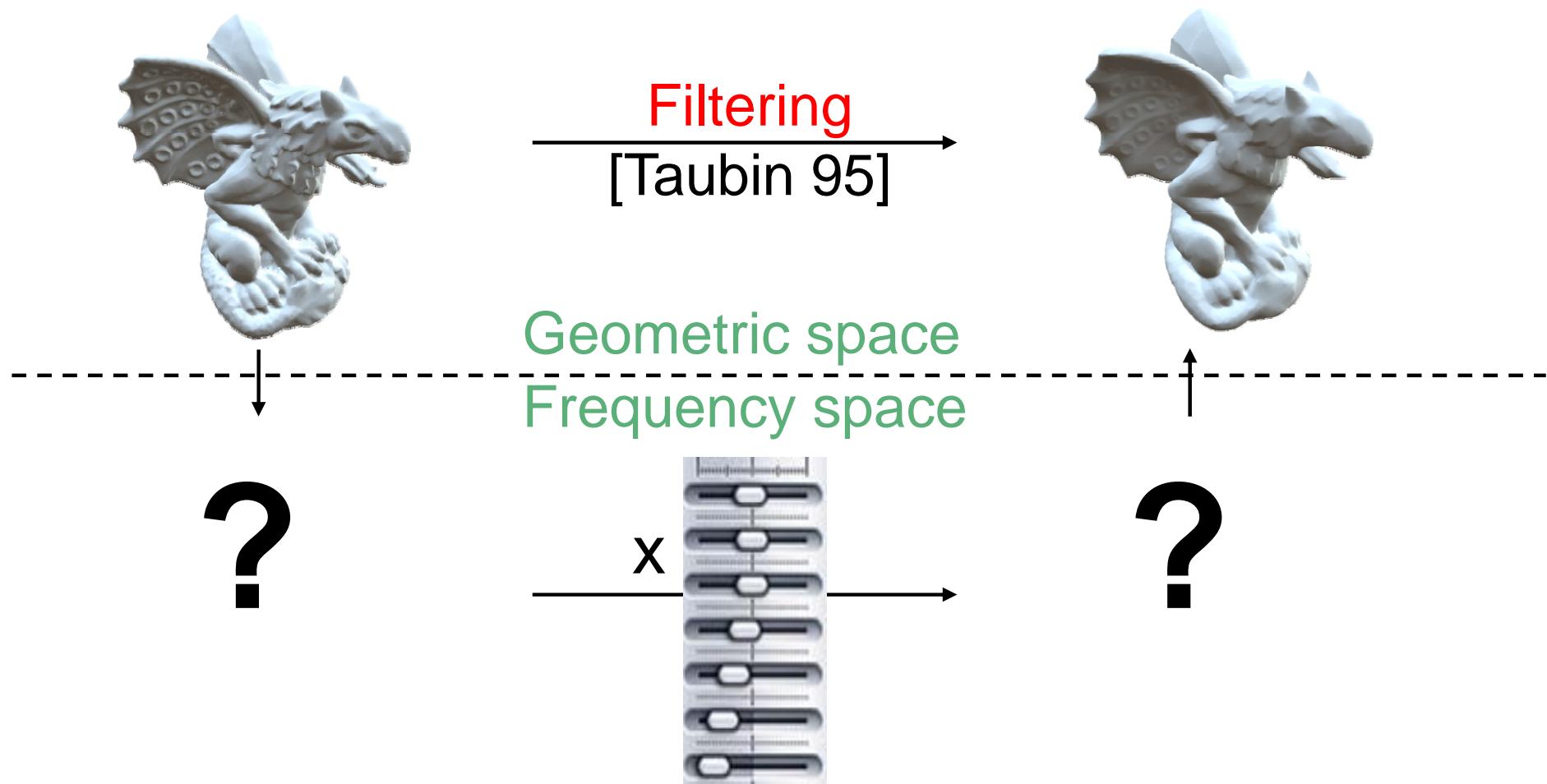


Cotangent weights



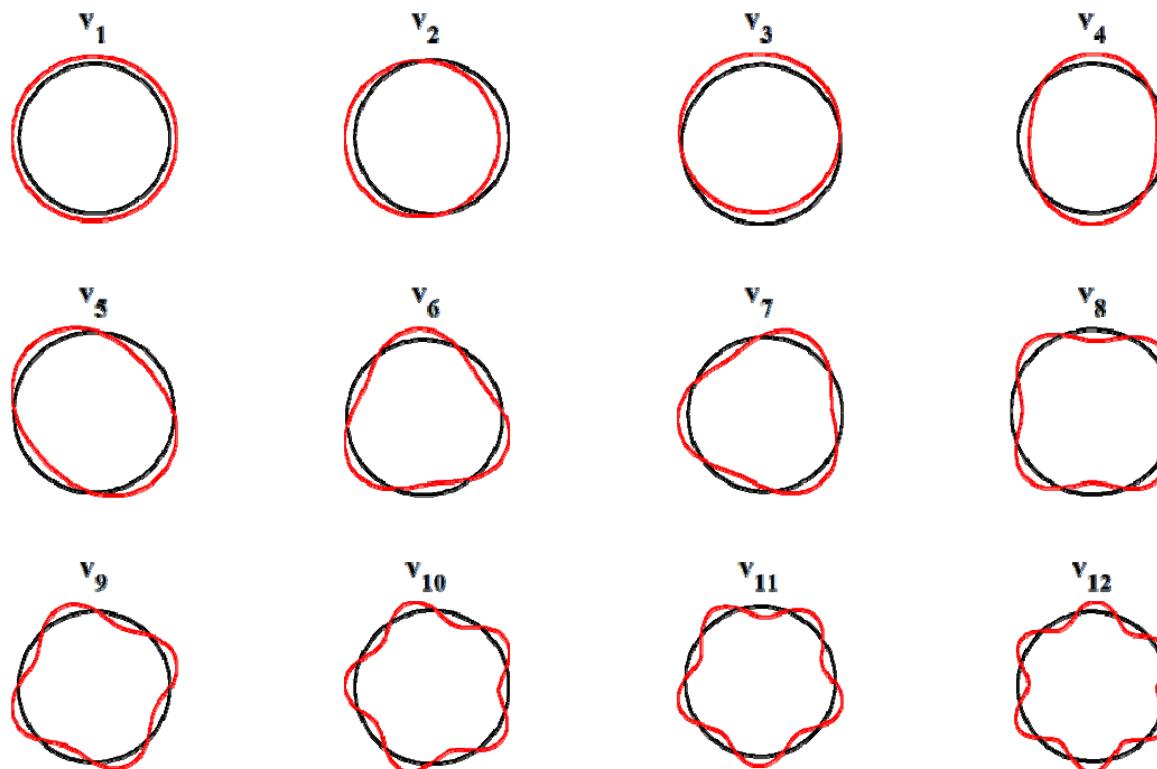
normal
movement

Filtering on a Mesh



The Eigenvectors of \mathbf{L}

$$\mathbf{L} = \mathbf{V}\mathbf{D}\mathbf{V}^T \quad \mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ \dots & & & \\ & & & k_n \end{pmatrix}$$

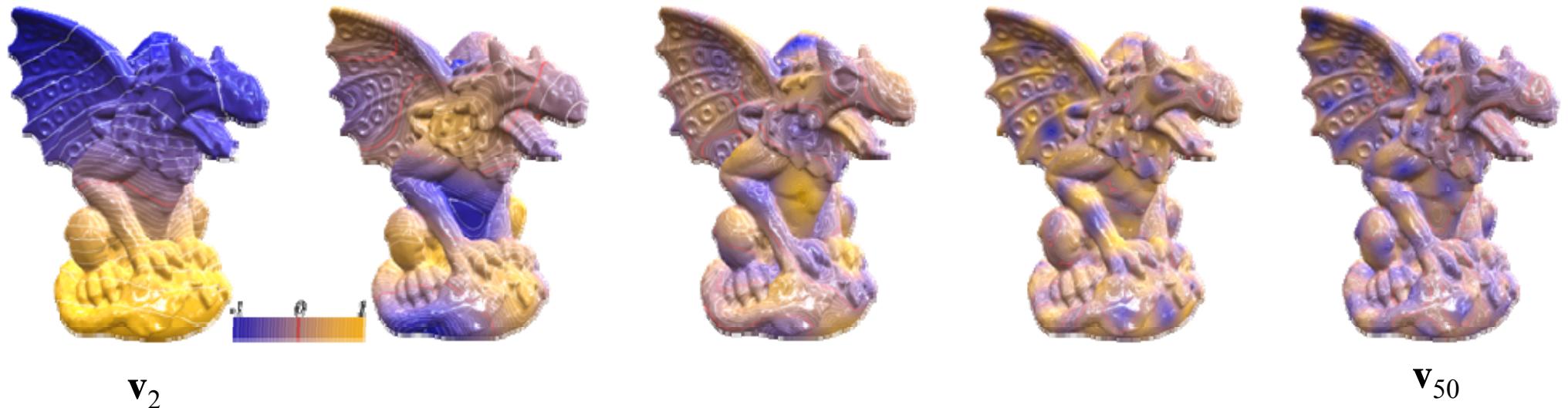


Spectral Analysis

Cotangent Laplacian

$$\mathbf{L} = \mathbf{V} \mathbf{D} \mathbf{V}^T$$

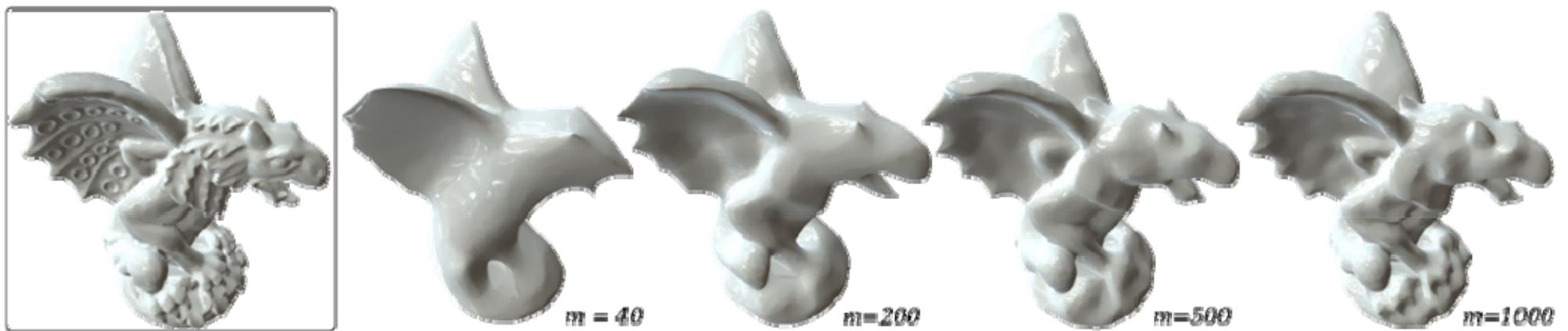
$$\mathbf{V} = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \dots & \\ & & & k_n \end{pmatrix}$$



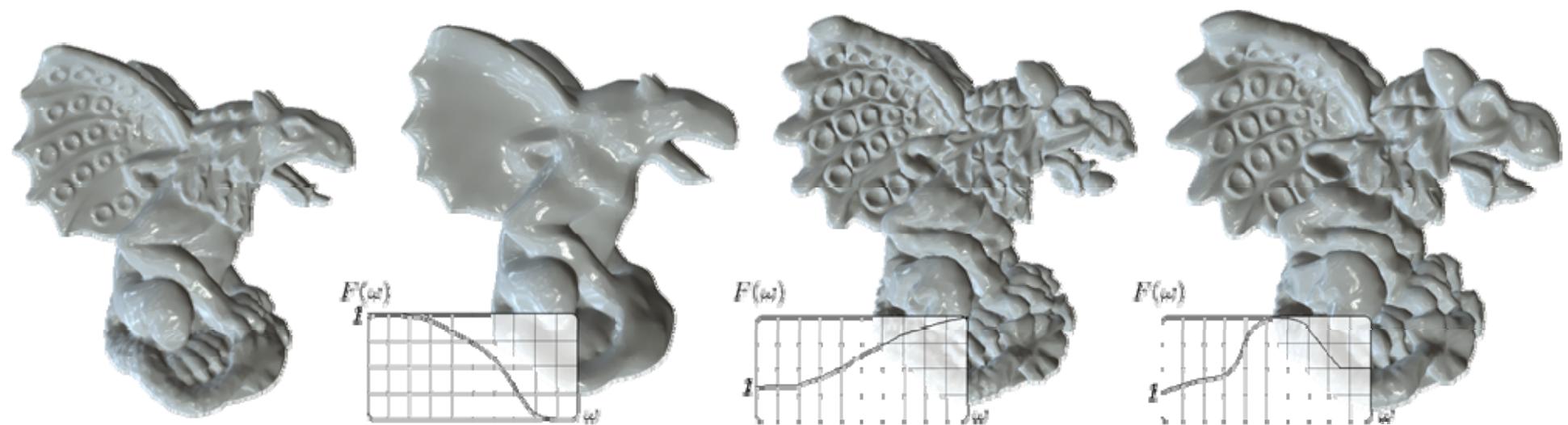
Demo

Smoothing using the Laplacian Eigen-decomposition

$$\mathbf{P}^{smooth} = \mathbf{V}(\mathbf{D}_m) \mathbf{V}^T \mathbf{P} \quad , \quad \mathbf{D}_m = \begin{pmatrix} k_1 & & & \\ & \ddots & & \\ & & k_m & \\ & & & 0 \end{pmatrix}$$



Geometry Filtering



Demo: <http://alice.loria.fr/index.php/software/9-demo/39-manifold-harmonics-demo.html>

Surface Fairing

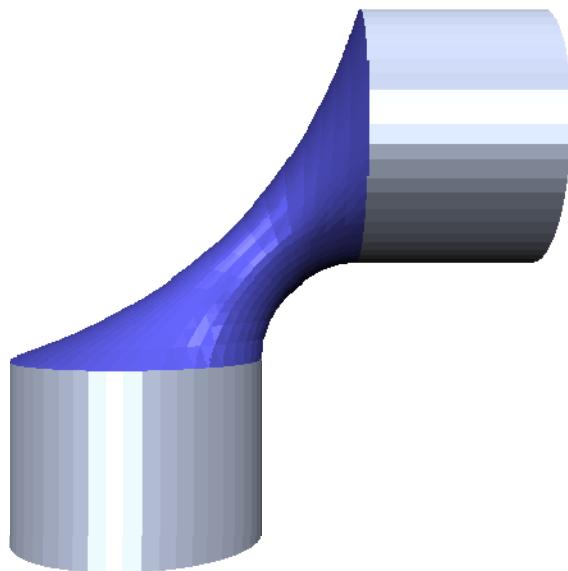
- Find surfaces which are “as smooth as possible”
- Applications
 - Smooth blends
 - Hole filling



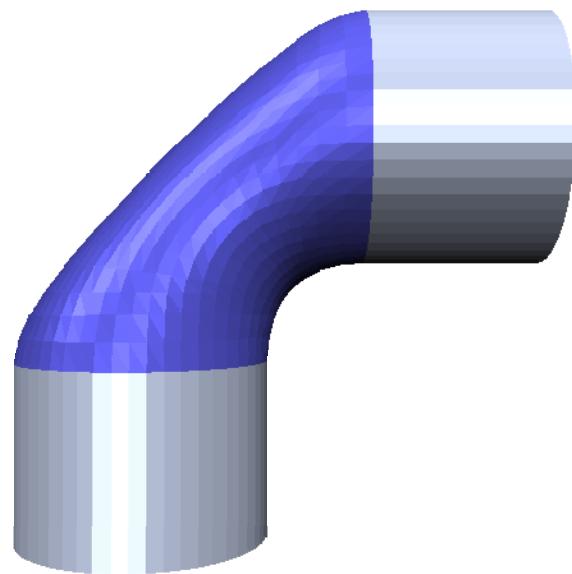
Fairness

- Idea: Penalize “unaesthetic behavior”
- Measure “fairness”
 - Principle of the simplest shape
 - Physical interpretation
- Minimize some fairness functional
 - Surface area, curvature
 - Membrane energy, thin plate energy

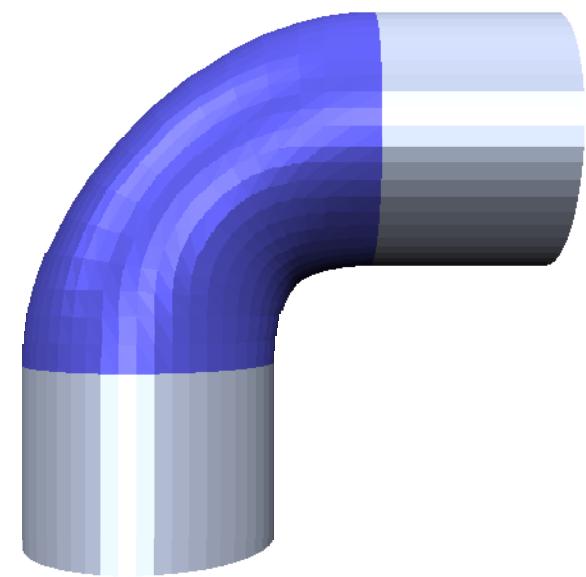
Energy Functionals



Membrane
Surface



Thin Plate
Surface



Minimum Variation
Surface

Non-Linear Energies

- Membrane energy (surface area)

$$\int_S dA \rightarrow \min$$

- Thin-plate energy (curvature)

$$\int_S \kappa_1^2 + \kappa_2^2 dA \rightarrow \min$$

- Too complex... simplify energies

Membrane Surfaces

Linearized Energy

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_{\Omega} \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, dudv \rightarrow \min$$

- Variational calculus

$$\Delta \mathbf{p} = 0$$

Thin-Plate Surfaces

Linearized Energy

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

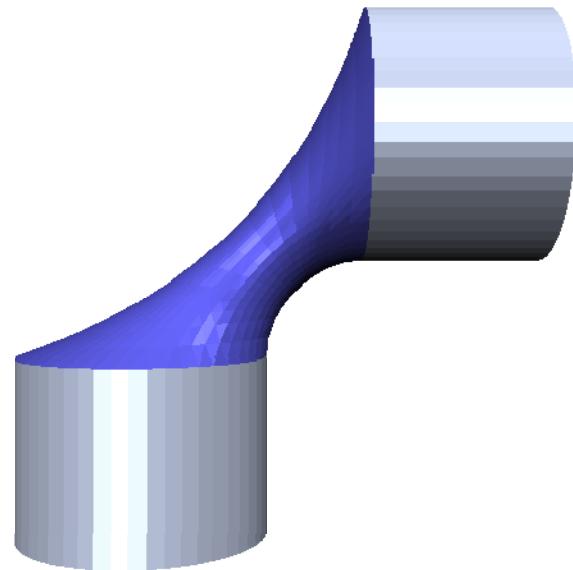
- Thin-plate energy (curvature)

$$\int_{\Omega} \|\mathbf{p}_{uu}\|^2 + 2 \|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \, dudv \rightarrow \min$$

- Variational calculus

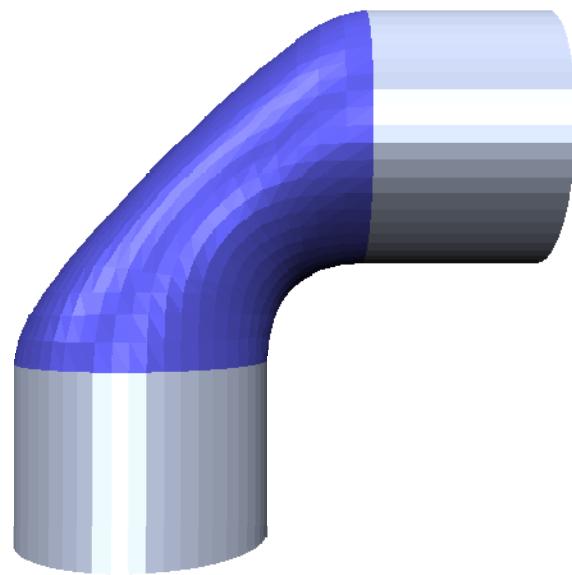
$$\Delta^2 \mathbf{p} = 0$$

Fair Surfaces



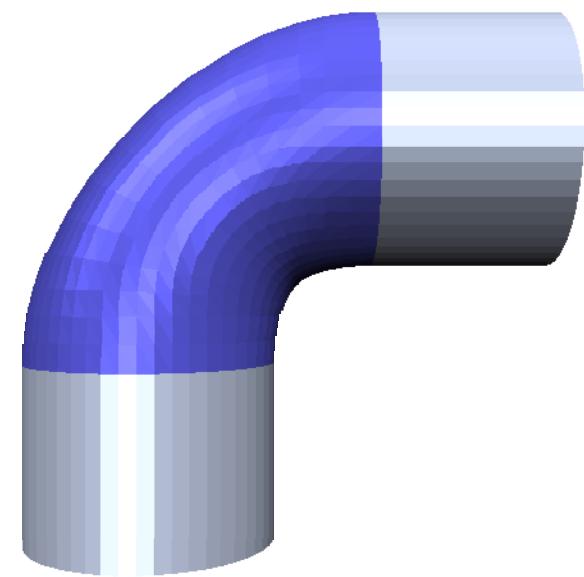
Membrane

$$\Delta s p = 0$$



Thin Plate

$$\Delta^2 s p = 0$$



Minimal Curvature
Variation

$$\Delta^3 s p = 0$$

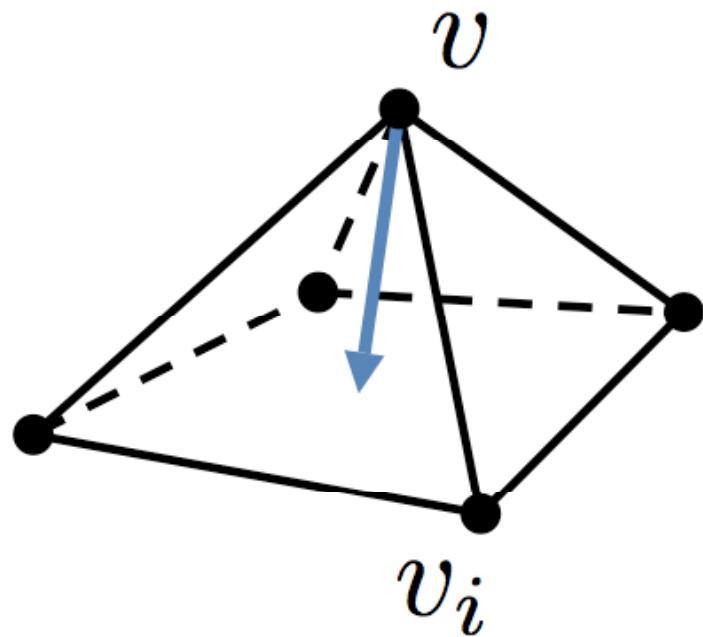
Demo

Exercise

- Smoothing
 - Uniform
 - Cotangent formula
- Smoothness visualization
 - Mean curvature (uniform, weighted)
 - Gaussian curvature
 - Triangle shape

Exercise

- Smoothing
 - Uniform Laplace-Beltrami



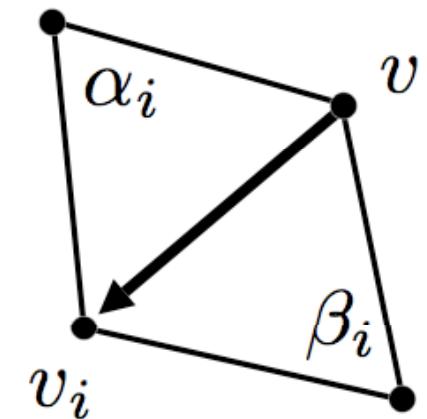
$$L_U(v) = \left(\frac{1}{n} \sum_i v_i \right) - v$$

Exercise

- Smoothing
 - Cotangent Formula (simplified)

$$L_B(v) = \frac{1}{\sum_i w_i} \sum_i w_i (v_i - v)$$

$$w_i = \frac{1}{2} (\cot \alpha_i + \cot \beta_i)$$

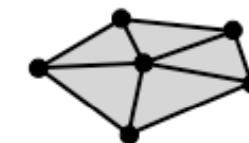
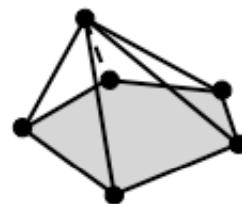


Exercise

- Smoothing

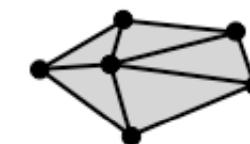
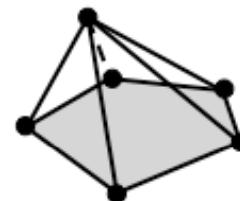
- Uniform Laplace-Beltrami

$$v' = v + \frac{1}{2} \cdot L_U(v)$$



- Cotangent Formula

$$v' = v + \frac{1}{2} \cdot L_B(v)$$



- Move vertices in parallel!

Exercise

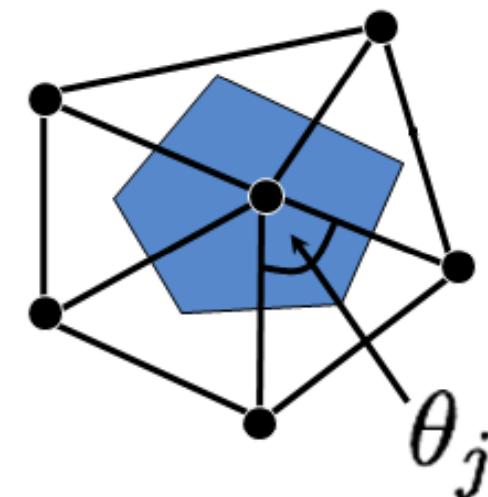
- Curvature

- Mean Curvature

$$H = \|\Delta_S \mathbf{x}\|$$

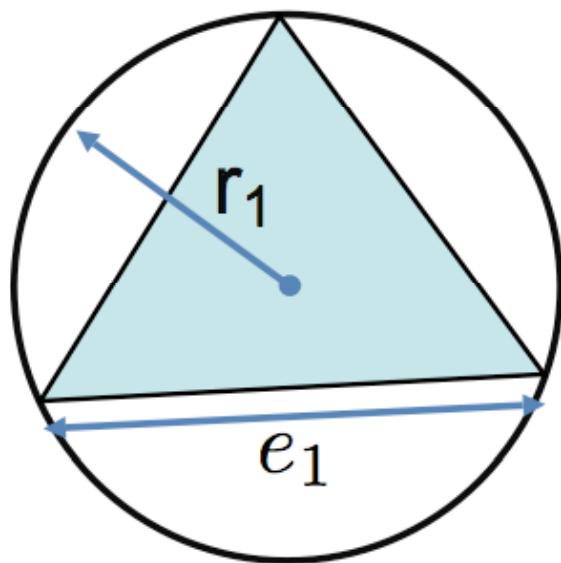
- Gaussian Curvature (simplified)

$$G = (2\pi - \sum_j \theta_j)$$

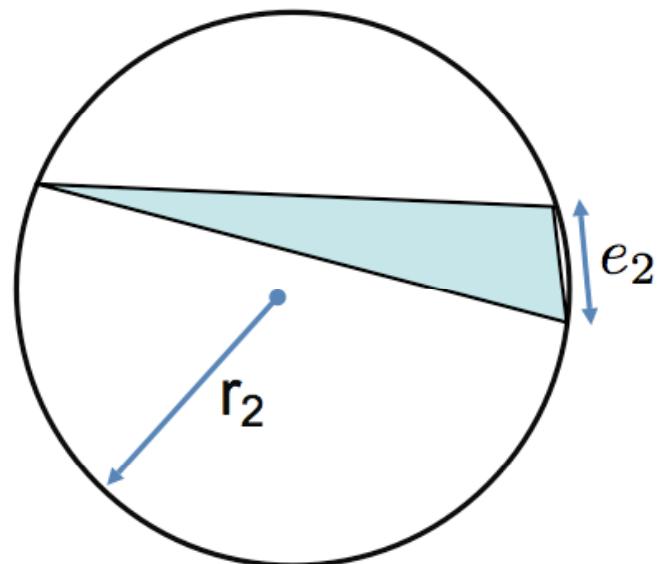


Exercise

- Triangle Shape – circumradius vs. minimal edge length

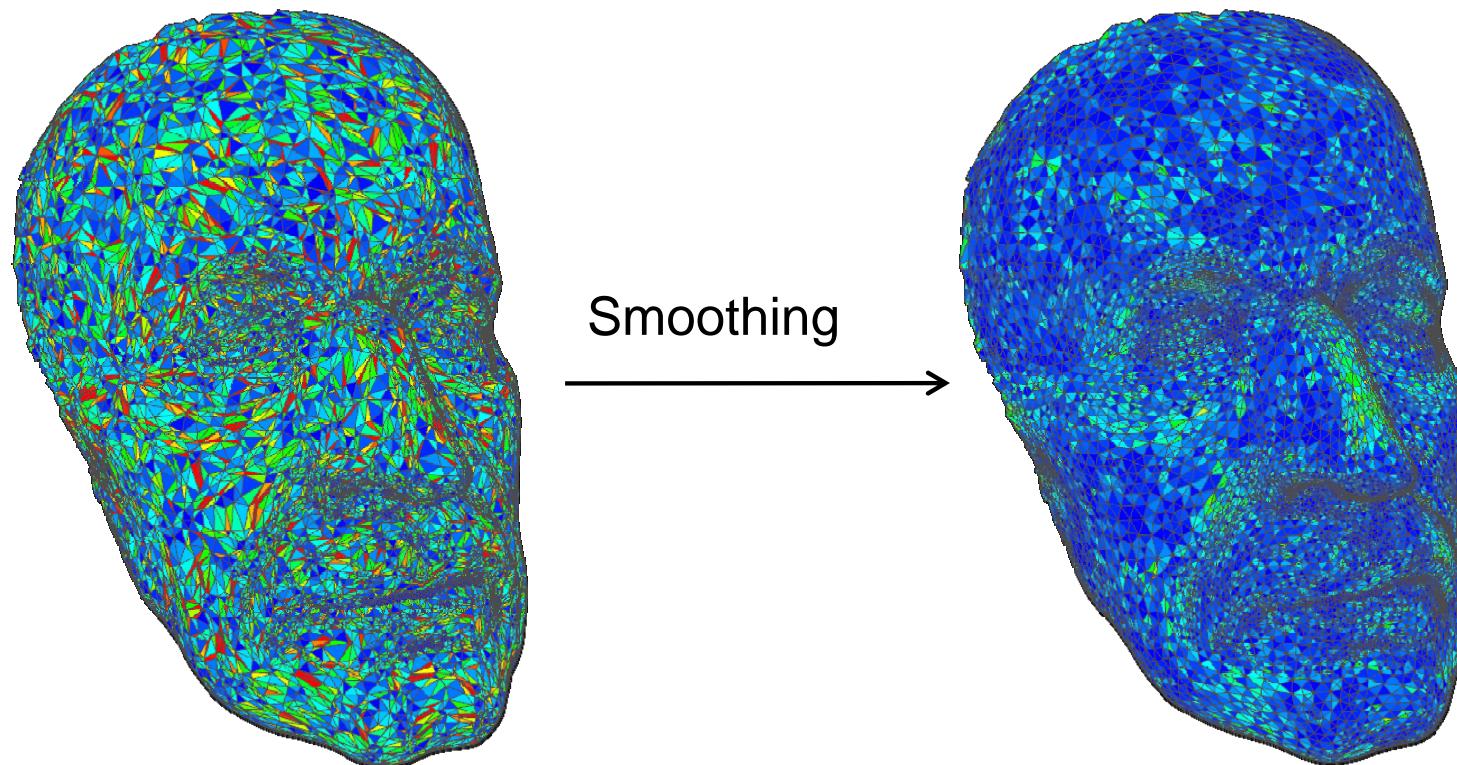


$$\frac{r_1}{e_1} < \frac{r_2}{e_2}$$



Exercise

- Triangle Shape



References

- “A Signal Processing Approach to Fair Surface Design”, Taubin, Siggraph ‘95
- “Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow”, Desbrun et al., Siggraph ’99
- “An Intuitive Framework for Real-Time Freeform Modeling”, Botsch et al., Siggraph ’04
- “Spectral Geometry Processing with Manifold Harmonics”, Vallet et al., Eurographics ‘08