Fuzzy Sets

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computer automation

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Fuzzy Set

- **Classical set**
  - **A crisp boundary, e.g.** $A=\{x|x>170\}$
  - **Sharp transition between inclusion and exclusion**
  - **Belong to vs. not belong to**
- **Human's concepts and thoughts**
  - **Abstract, imprecise,**
    - e.g. Tall Persons?
    - $A=\{\text{height} \mid \text{height} > 170\text{cm}\}$?
  - **Transition should be smooth and gradual**
  - **Degrees can be characterized by a member function**

Fuzzy Sets

- Hot Temperatures, Cool Temperatures
- Smart Persons, Dumb Persons
- Rich People, Poor People
- Secure Systems, Insecure Systems
- Fat People, Lean People

Fuzziness

- Imprecisely defined sets of classes play an important role in
  - Human thinking
  - Domains of pattern recognition
  - Information Communication
  - Abstraction
- Fuzziness
  - Does not come from the randomness of the constituent members of the sets
  - Come from the uncertain and imprecise nature of abstract thoughts and concepts

Basic Definitions and Terminology

Definition 1: Fuzzy sets and membership function

If X is a collection of objects denoted generically by x, then a fuzzy set A in X is defined as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X\} \quad 0 \le \mu_A(x) \le 1$$

where $\mu_A(\cdot)$ is called the membership function for the fuzzy set A.

$\mu_A(x)$ is the membership value of x.

X is the universe of discourse, or universe

Crisp Sets Examples:

Children Set $A = \{1, 2, 3, 4, 5, 6\}$

Elder Set $B = \{x \mid x > 60\}$

Fuzzy Sets Examples:

Children Set $$A = \left\{ \begin{matrix} (1, 1.0), (2, 0.9), (3, 0.8), (4, 0.7), (5, 0.6), (6, 0.5), \\ ...., (16, 0.0), (17, 0.0), ..., (120, 0.0) \end{matrix} \right\}$$

Elder Set $B = \{(Age, \mu_B(Age)) \mid Age > 0\}$

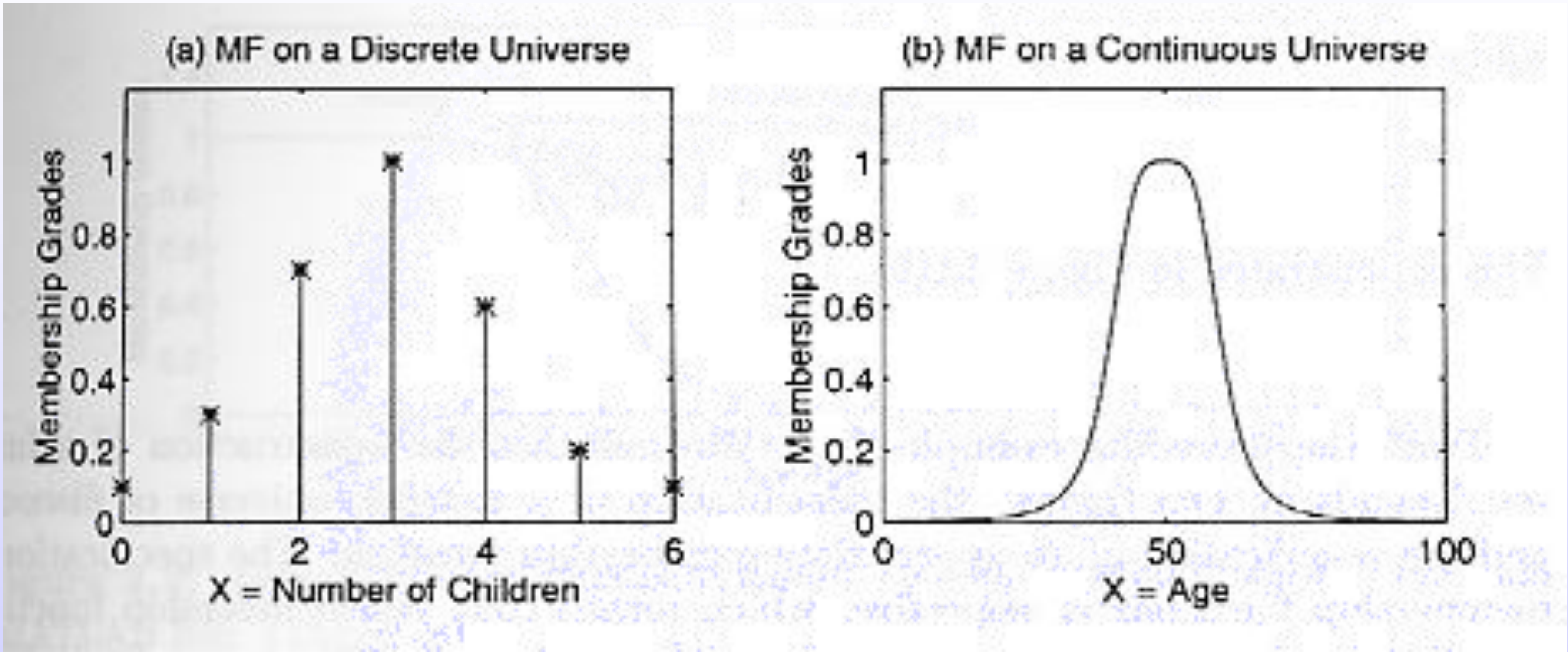
Sets vs. Fuzzy Sets

- Classical Sets
  - Sets
  - Nonfuzzy sets
  - Crisp sets
  - Ordinary sets
  - Membership value is either 1 (in the set) or 0 (not in the set)
- Fuzzy set is an extension of the classical set
- The membership value is permitted to have any values between 0 and 1

Discrete Universe vs. Continuous Universe

- Continuous Universe
  - Characteristics of the set elements is continuous
  - E.g., age, height, weight, real numbers, date time, length
- Discrete Universe
  - Characteristics of the set elements is either symbolic or valued
  - E.g. Cities, Individual names, shapes

Discrete vs. Continuous



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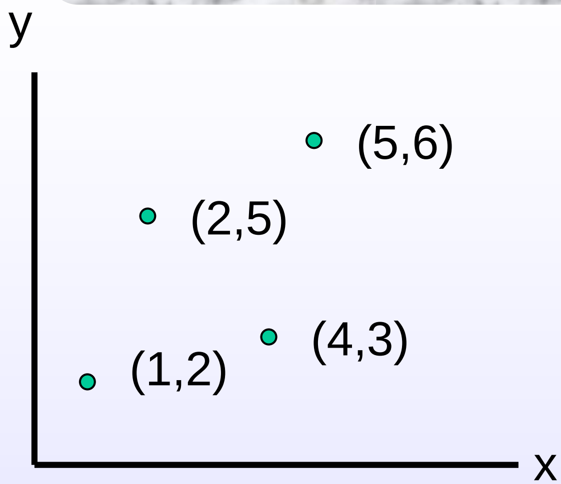
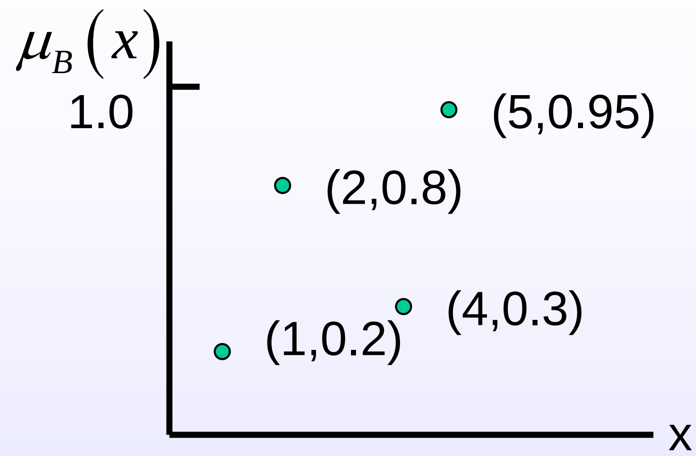
Crisp Set Examples vs. Fuzzy Set Examples

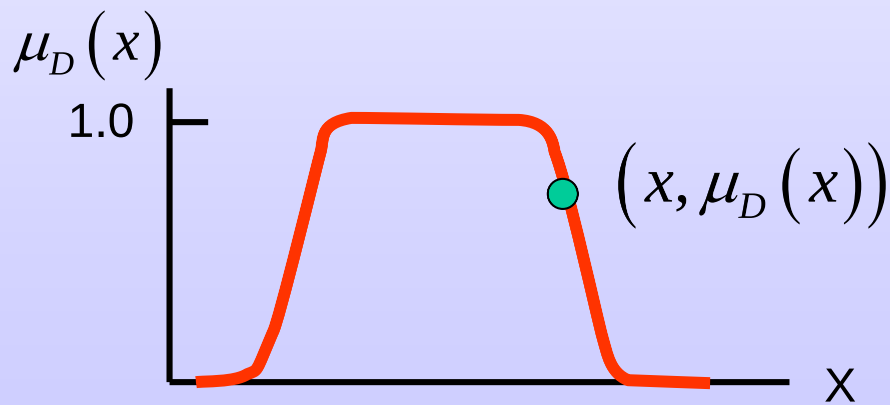


$$A=\{(1,2),(2,5),(4,3),(5,6)\}$$



$$B=\{(1,0.2),(2,0.8),(4,0.4),(5,0.95)\}$$

$$C=\{x \mid -2\le x\le 5\}$$



$$D=\{(x,\mu_D(x)) \mid x\in X\}$$

Fuzzy Set Construction

- A suitable universe of discourse
- Membership function
  - Specification is subjective
  - Nonrandomness
- X is a collection of discrete objects

$$A = \sum_{x_i \in X} \mu_A(x_i)/x_i$$

$$A = 0.9/Taipei + 0.9/Kaohsuing + 0.6/Taichung$$

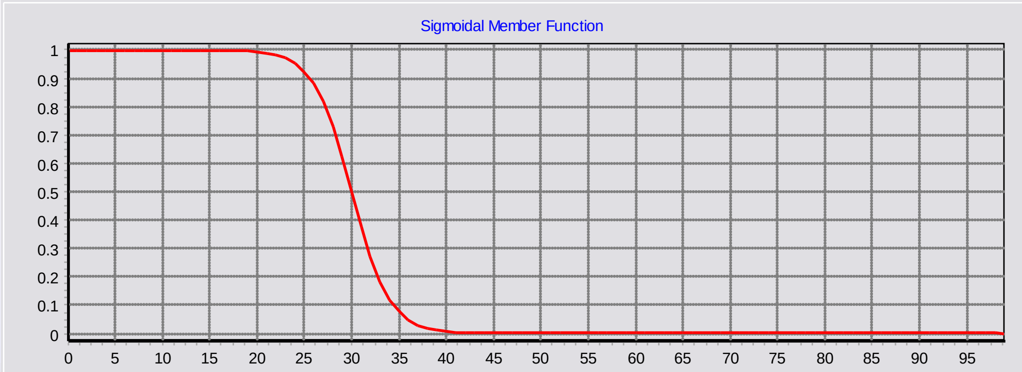
$$A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3$$

- X is a continuous space

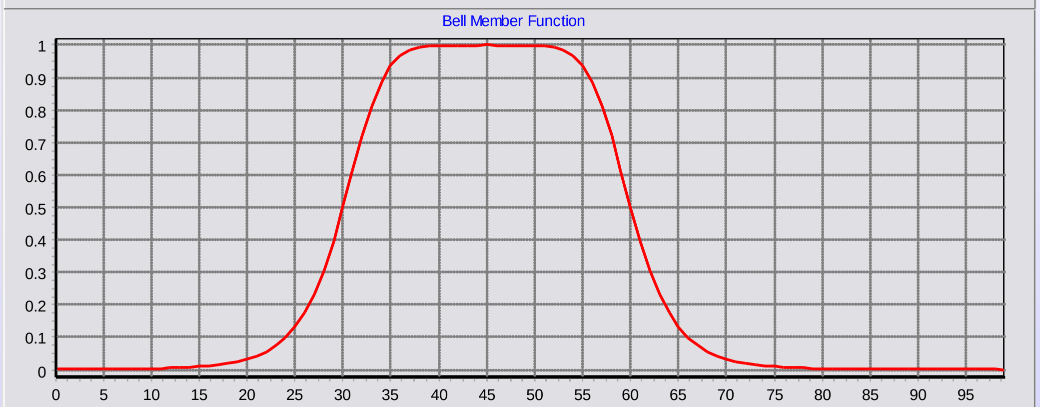
$$A = \int_X \mu_A(x)/x$$

$$A = \int_X \frac{1}{1+\left(\frac{x-50}{10}\right)^4} \Bigg/ x$$

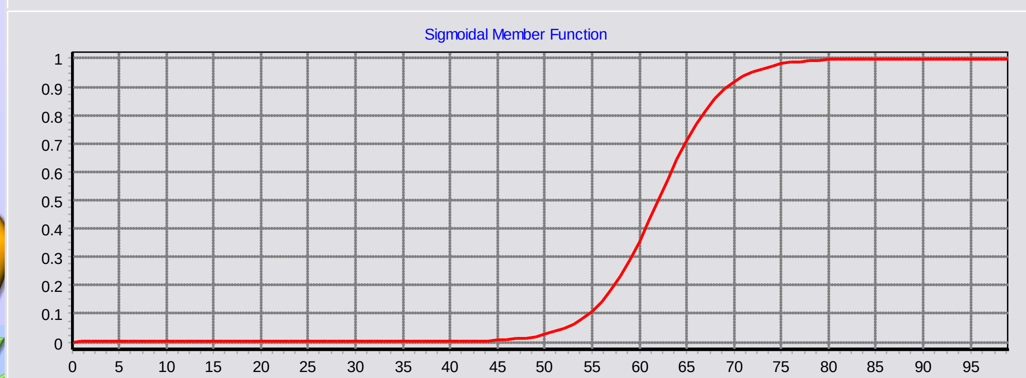
Age Universe Fuzzy Sets



Young Fuzzy Set



Middle aged
Fuzzy Set



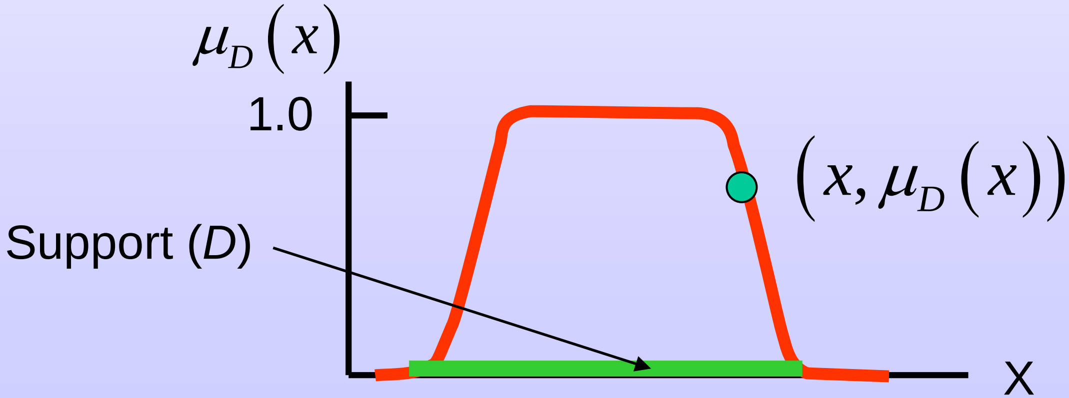
Old Fuzzy Set

Support

Definition 2: Support

The support of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$:

$$\text{support}(A) = \{x \mid \mu_A(x) > 0\}$$



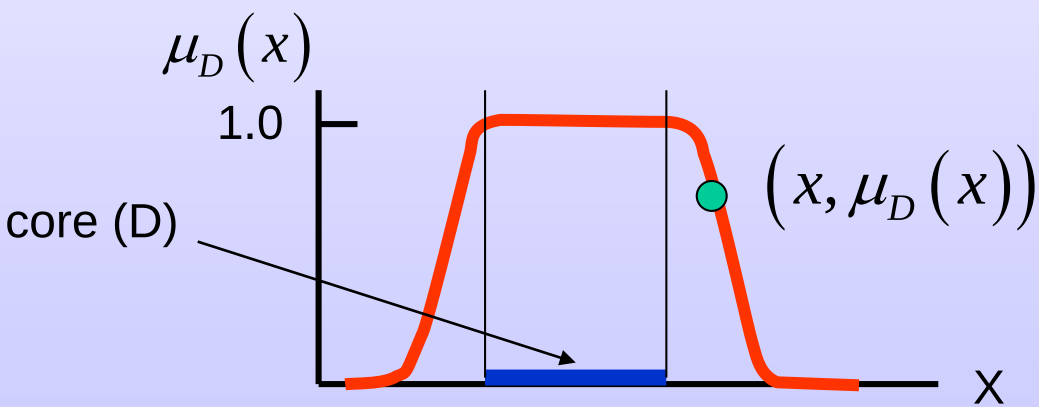
$$D = \{(x, \mu_D(x)) \mid x \in X\}$$

Core

Definition 3: Core

The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x)=1$:

$$\text{core}(A)=\left\{x \middle| \mu_A(x)=1\right\}$$



$$D=\left\{\left(x,\mu_D(x)\right)\middle| x\in X\right\}$$

Normality and Crossover Points

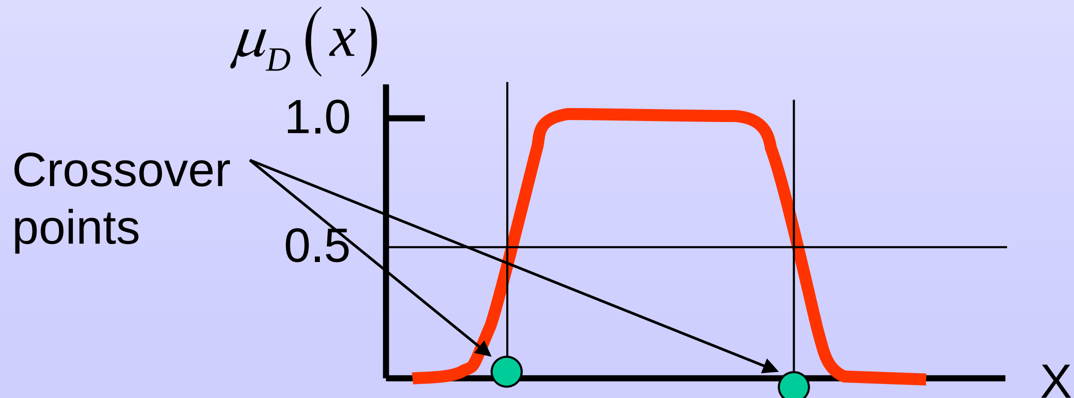
Definition 4: Normality

A fuzzy set A is normal if its core is nonempty.

Definition 5: Crossover points

A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$.

$$\text{crossover}(A) = \{x \mid \mu_A(x) = 0.5\}$$

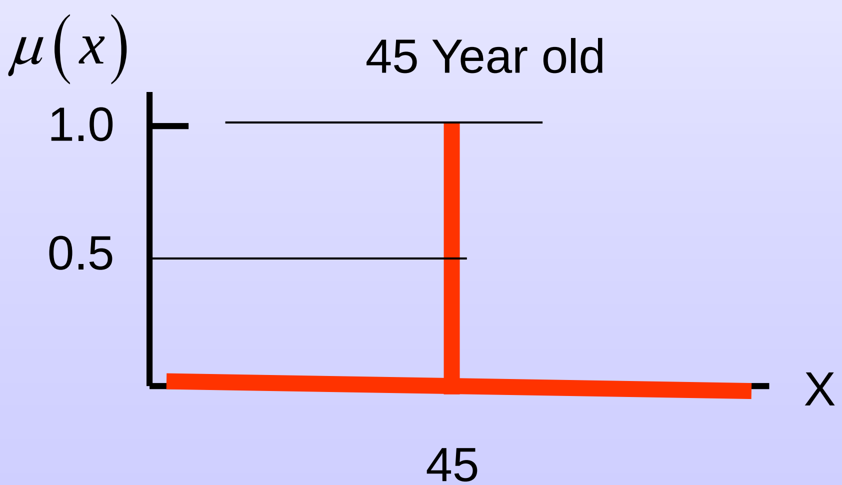


$$D = \{(x, \mu_D(x)) \mid x \in X\}$$

Fuzzy Singleton

Definition 6: Fuzzy Singleton

A fuzzy set whose support is a single point x in X with $\mu_A(x) = 1 \geq 0$ is called a fuzzy singleton.



Definition 7: α-cut, strong α-cut

The α–cut set or α–level set of a fuzzy set A is a crisp set defined by

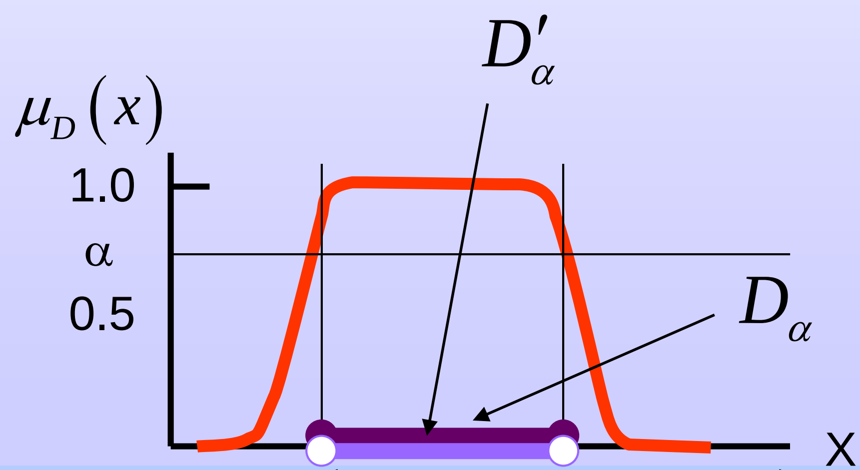
$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$

Strong α–cut set or strong α–level set are

$$A'_\alpha = \{x \mid \mu_A(x) > \alpha\}$$

$$A'_0 = \mathrm{support}(A)$$

$$A_1 = \mathrm{core}(A)$$



$$D = \{(x, \mu_D(x)) \mid x \in X\}$$

Convexity

Definition 8: Convexity

A fuzzy set A is convex if and only if for any x_1 and $x_2 \in X$ and any $\lambda \in [0,1]$,

$$\mu_A\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left\{\mu_A\left(x_1\right), \mu_A\left(x_2\right)\right\}$$

Shape convexity of function $f(x)$

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)$$

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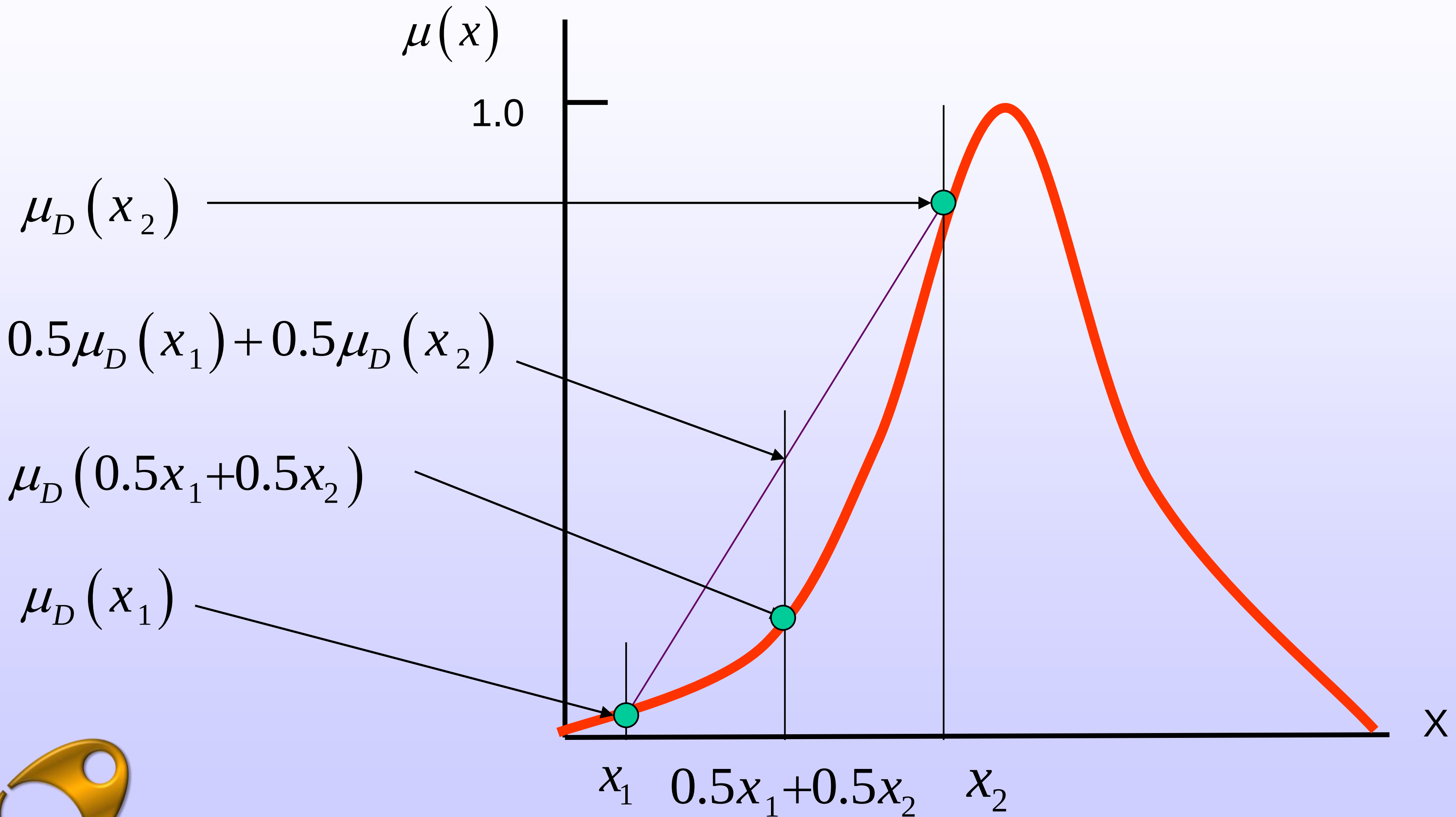
A Convex Fuzzy Set

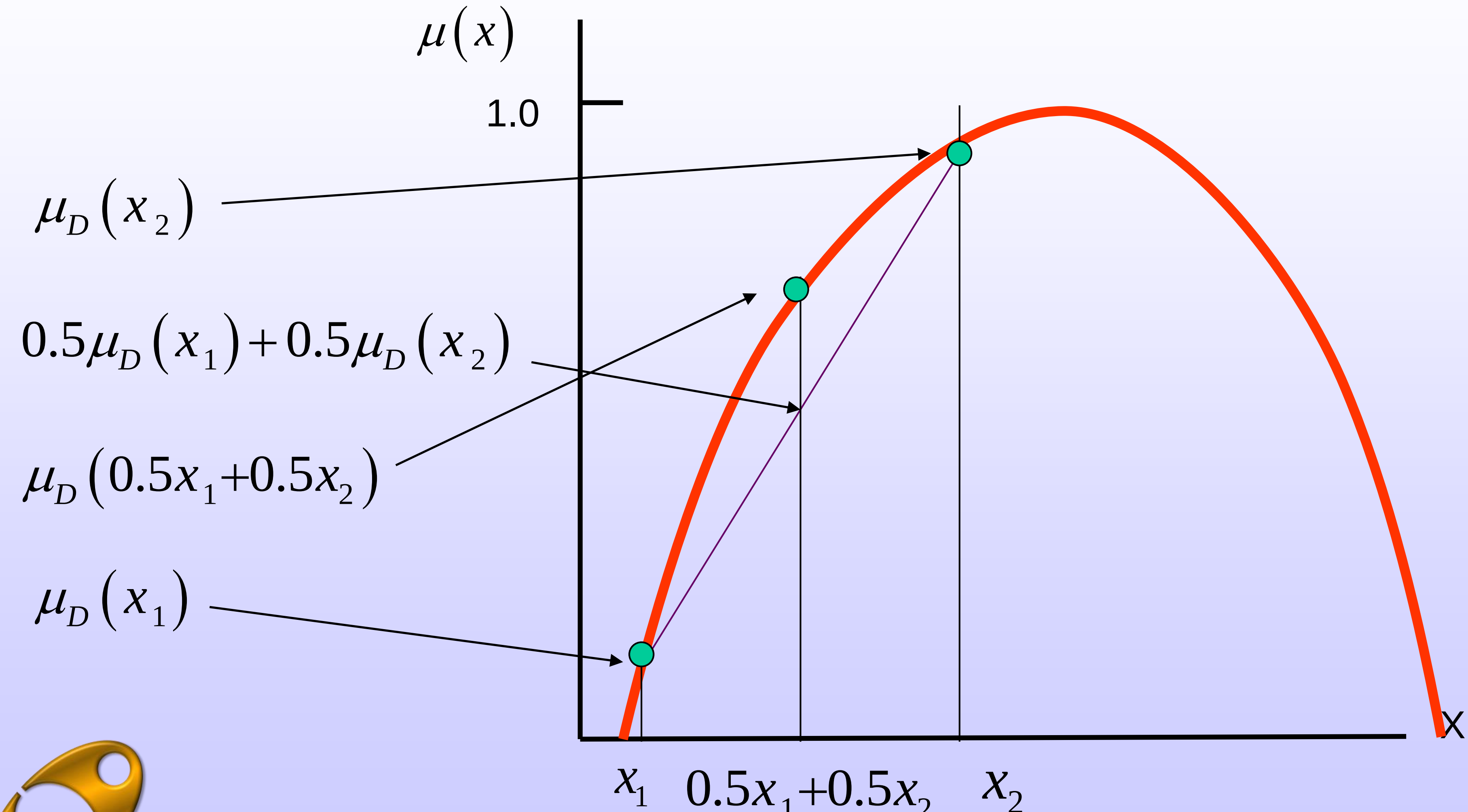
Shape of the Membership function is nonconvex



$$D=\{(x,\mu_D(x))|x\in X\}$$

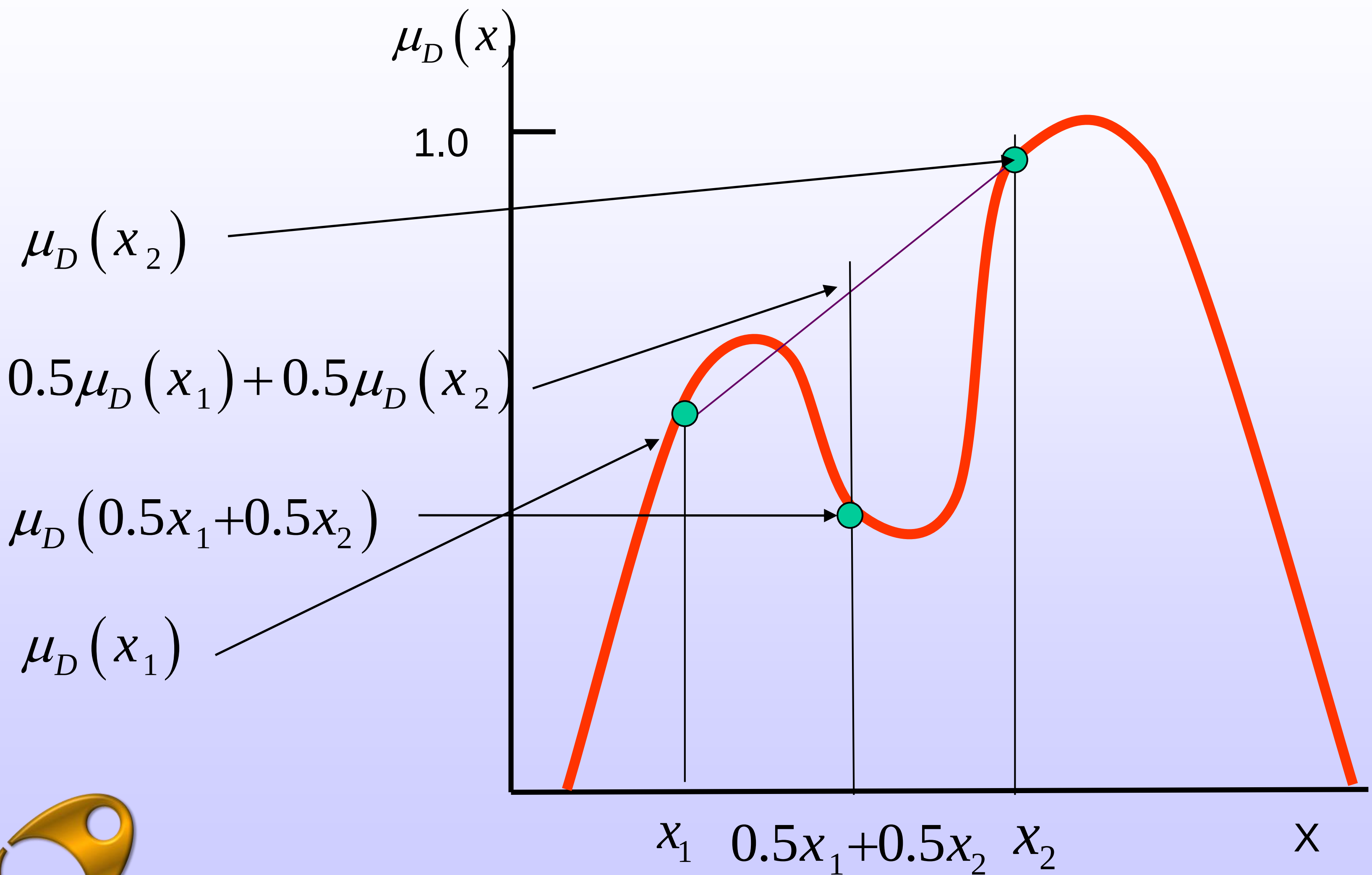
A Convex Fuzzy Set

Shape of the Membership function is convex



$$D=\{(x,\mu_D(x))|x\in X\}$$

A Nonconvex Fuzzy set



$$D=\{(x,\mu_D(x))|x\in X\}$$

Fuzzy Numbers and Bandwidth

Definition 9: Fuzzy Numbers

A fuzzy number A is a fuzzy set in the real line (R) that satisfies the conditions for normality and convexity.

Definition 10: Bandwidths of normal and convex fuzzy sets

For a normal and convex fuzzy set, the bandwidth or width is defined as the distance between the tow unique crossover points:

$$\text{width}(A) = |x_2 - x_1|, \mu_A(x_1) = \mu_A(x_2) = 0.5$$

Symmetry, Open left, open right, closed

Definition 11: Symmetry

A fuzzy set A is symmetric if its member function is symmetric around a certain point $x = c$

$$\mu_A(c+x) = \mu_A(c-x), \text{for all } x \in X$$

Definition 12: Open left, open right, closed

A fuzzy set A is open left if

$$\lim_{x \to -\infty} \mu_A(x) = 1 \text{ and } \lim_{x \to +\infty} \mu_A(x) = 0$$

Open right if

$$\lim_{x \to -\infty} \mu_A(x) = 0 \text{ and } \lim_{x \to +\infty} \mu_A(x) = 1$$

Closed if

$$\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$$

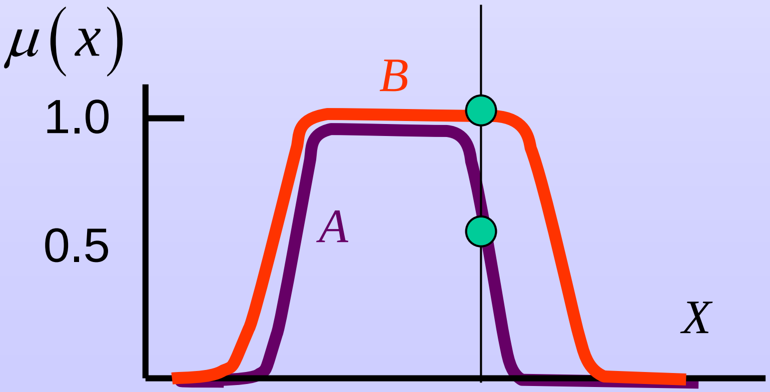
Set-Theoretic Operations

- **Union, intersection, and complement operations**

Definition 13: Containment or Subset

Fuzzy set A is contained in fuzzy set B if and only if $\mu_A(x) \le \mu_B(x)$ for all x.

$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$$



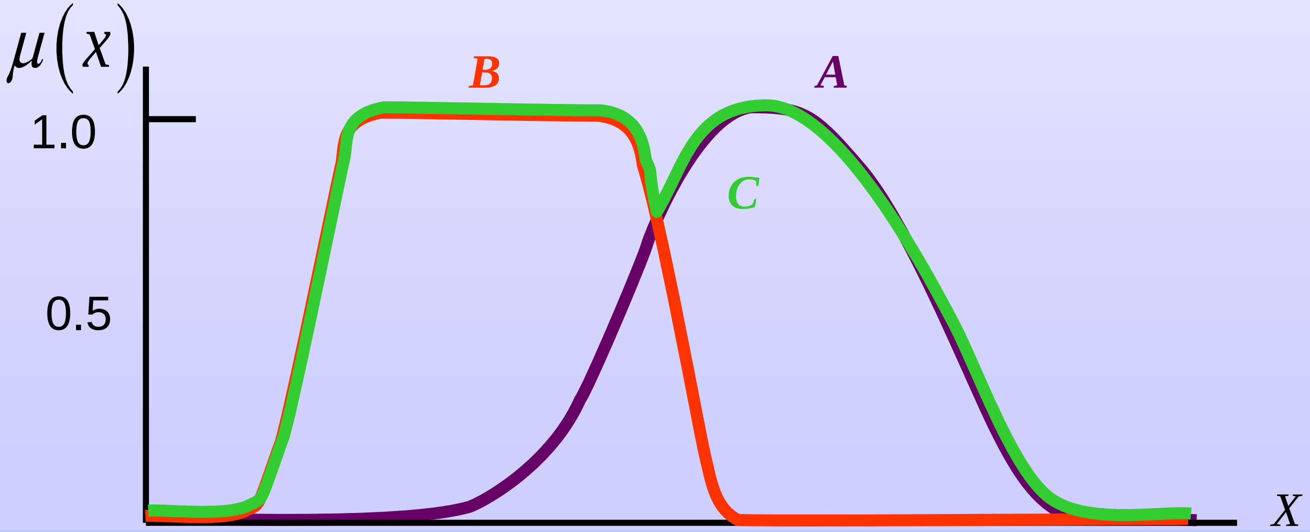
A is a subset of B

A is smaller than or equal to B

Definition 14: Union (disjunction)

The union of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cup B$ or $C = A$ OR B, whose member function is related to those of A and B by

$$\mu_C(x) = \max\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \vee \mu_B(x)$$

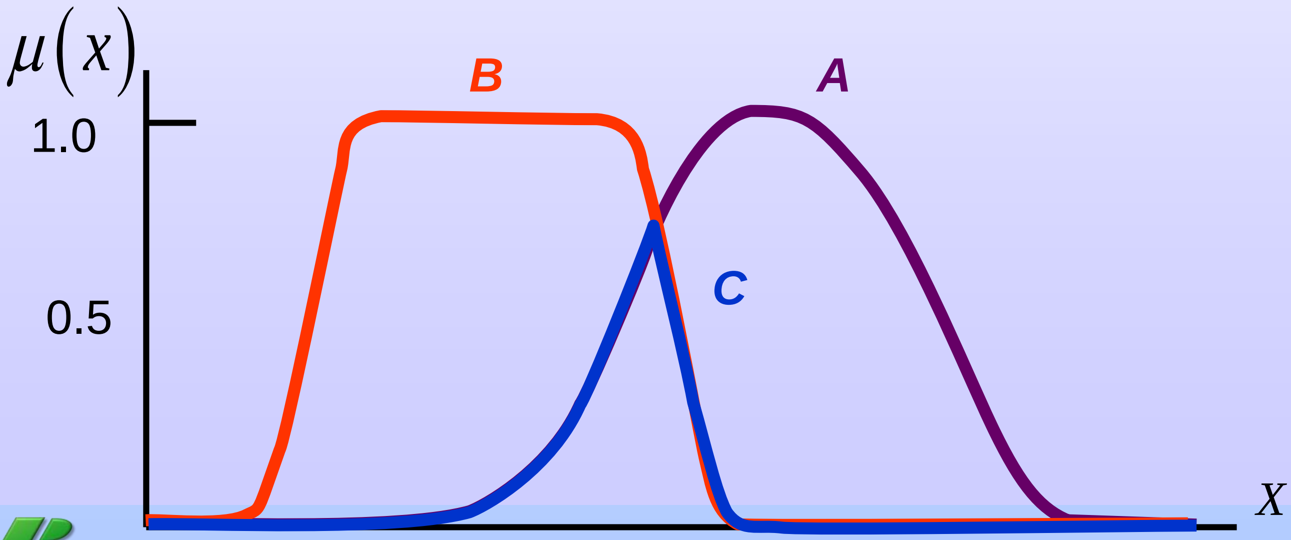


Intersection (conjunction)

Definition 15: Intersection (conjunction)

The intersection of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cap B$ or $C = A \text{ AND } B$, whose member function is related to those of A and B by

$$\mu_C(x) = \min\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \wedge \mu_B(x)$$

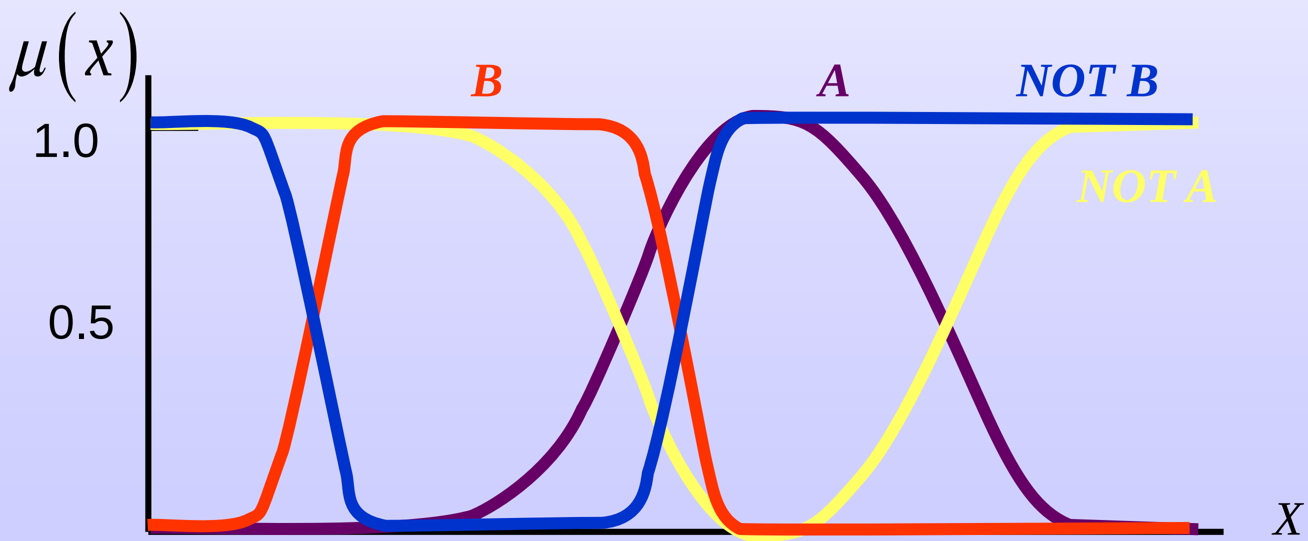


Complement (negation)

Definition 16: Complement (negation)

The complement of fuzzy set A, denoted by $\bar{A}$ or $\neg A$ or NOT A is defined as

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$



Cartesian product and co-product

Definition 17: Cartesian Product and Co-product

Let A and B be fuzzy sets in X and Y, respectively. The Cartesian product of A and B, denoted by AxB, is a fuzzy set in the product space XxY with the membership function

$$\mu_{A\times B}(x,y)=\min\left(\mu_A(x),\mu_B(y)\right)$$

Cartesian co-product of A and B, denoted by A+B, is a fuzzy set in the product space XxY with the membership function

$$\mu_{A+B}(x,y)=\max\left(\mu_A(x),\mu_B(y)\right)$$

Member Function Formulation and Parameterization

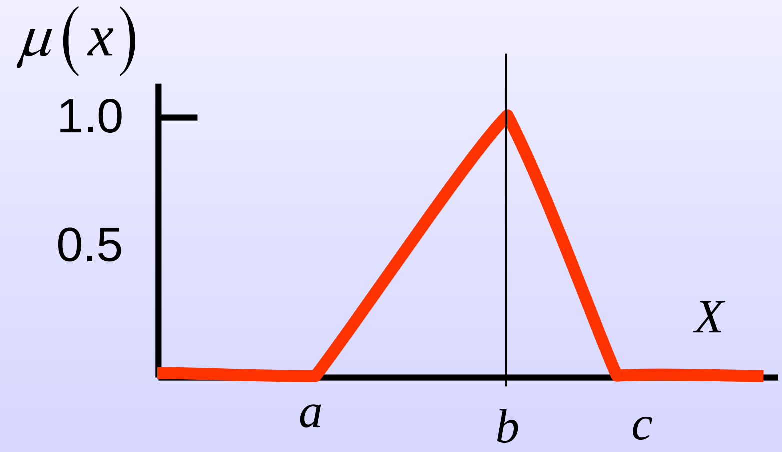
- A fuzzy set is completely characterized by its MF
- One-dimensional MF
  - Triangular MF
  - Trapezoidal MF
  - Gaussian MF
  - Generalized bell MF
  - Sigmoidal MF
  - Left-right MF
- Two-dimensional MF
  - Cylindrical extensions of 1-D MF
  - Composite and noncomposite MF
  - Projections to 1-D MF

Triangular MF

Definition 18: Triangular MFs

A triangular MF is specified by three parameters $\{a,b,c\}$ as follows:

$$\mathrm{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a. \\ (x-a)/(b-a), & a \le x \le b. \\ (c-x)/(c-b), & b \le x \le c. \\ 0, & c \le x. \end{cases}$$



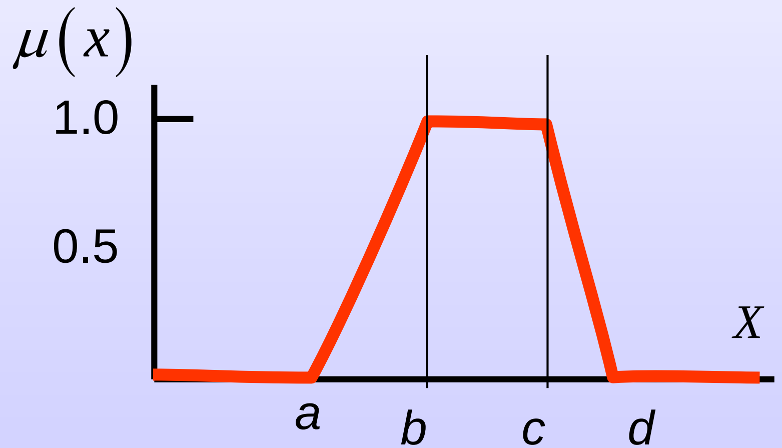
$$\mathrm{triangle}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF

Definition 19: Trapezoidal MFs

A trapezoidal MF is specified by four parameters $\{a,b,c,d\}$ as follows:

$$\text{trapezoidal}(x;a,b,c,d) = \begin{cases} 0, & x \le a. \\ (x-a)/(b-a), & a \le x \le b. \\ 1, & b \le x \le c. \\ (d-x)/(d-c), & c \le x \le d. \\ 0, & d \le x \end{cases}$$



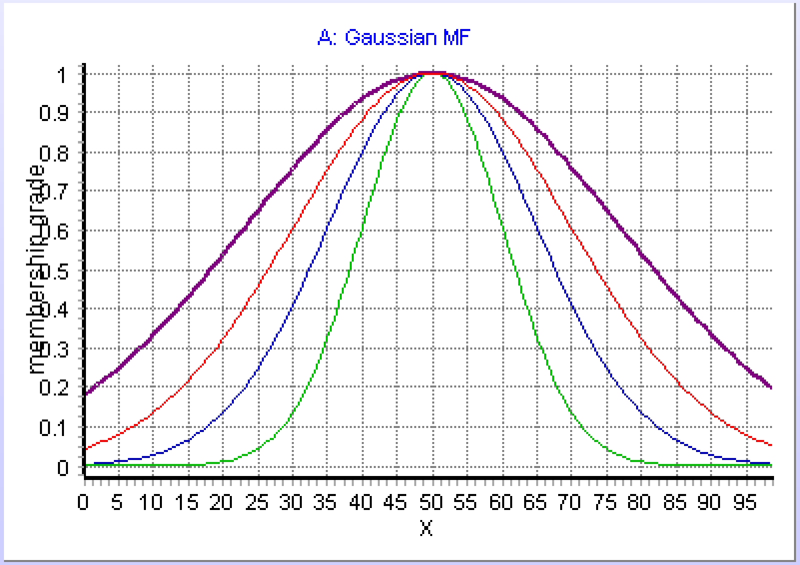
$$\text{trapezoidal}(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Gaussian MF

Definition 20: Gaussian MFs

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ as follows:

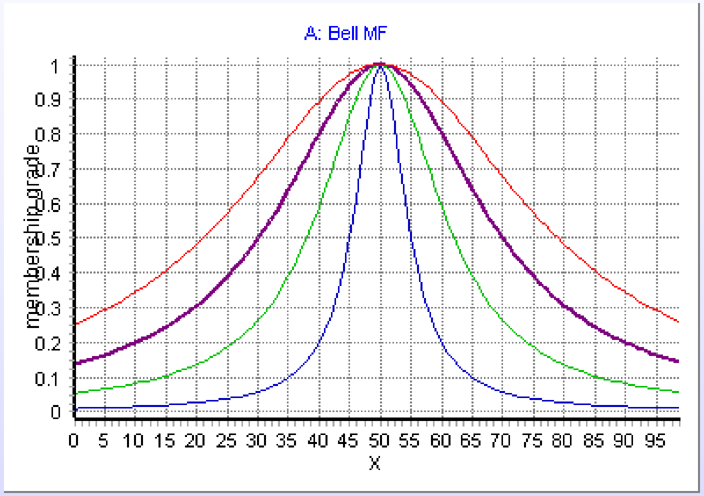
$$\text{Gaussian}(x; c, \sigma) = e^{-\frac{(x-c)^2}{2\sigma^2}}$$



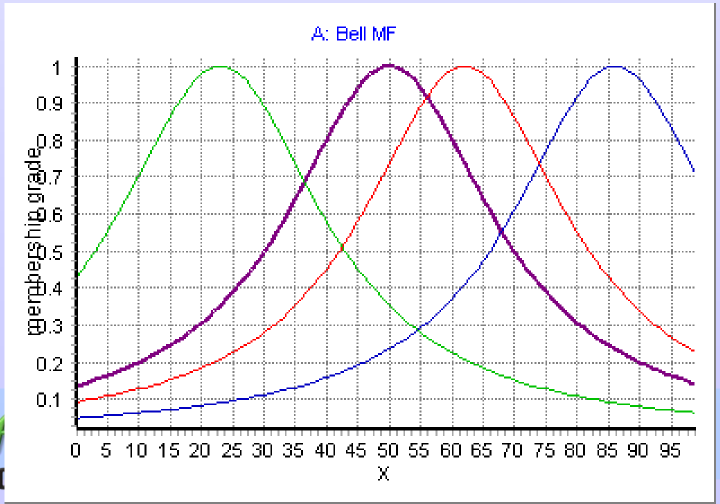
Bell MFs

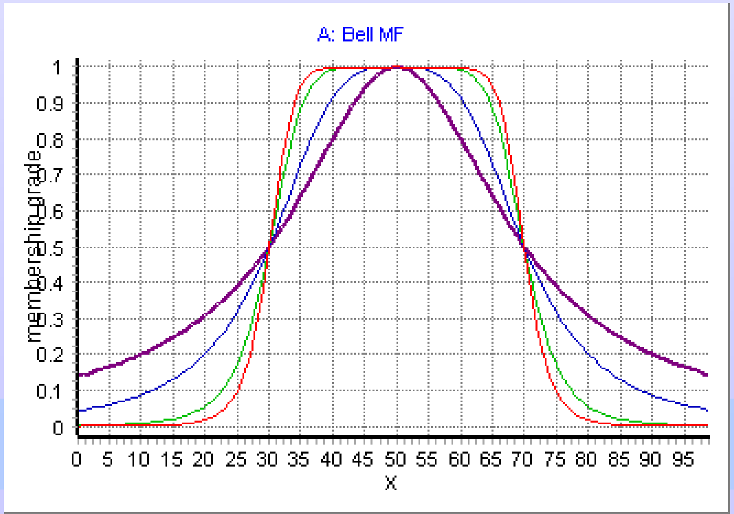
Definition 21: Generalized Bell MF

A generalized bell MF is specified by three parameters $\{a,b,c\}$ as follows:



$$\mathrm{bell}(x;a,b,c)=\frac{1}{1+\left|\dfrac{x-c}{a}\right|^{2b}}$$





- **Gaussian and bell functions are popular**
  - Smooth and Concise
  - Gaussian is well known in probability and statistics
  - Bell functions have one more degree of freedom to adjust the steepness at the crossover points
- **Not able to specify asymmetric MF**

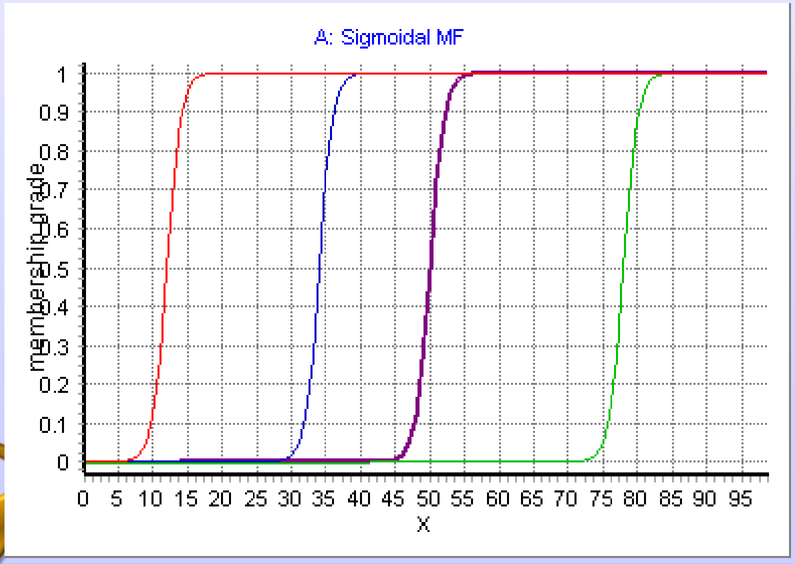
Sigmoidal MFs

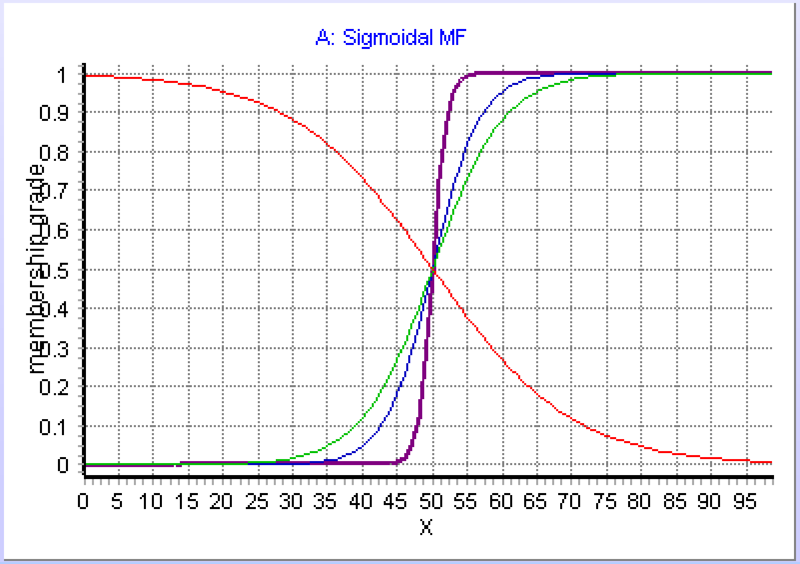
Definition 22: Sigmoidal MF

A sigmoidal MF is defined by

$$\operatorname{sig}(x;a,c)=\frac{1}{1+e^{-a(x-c)}}$$

where a controls the slope at the crossover point $x=c$.





Sigmoidal MF

- Can be open left, open right
  - Suitable for specifying “very large”, “very young”
- Frequently used as activation function of neural networks

Closed Nonsymmetric MFs

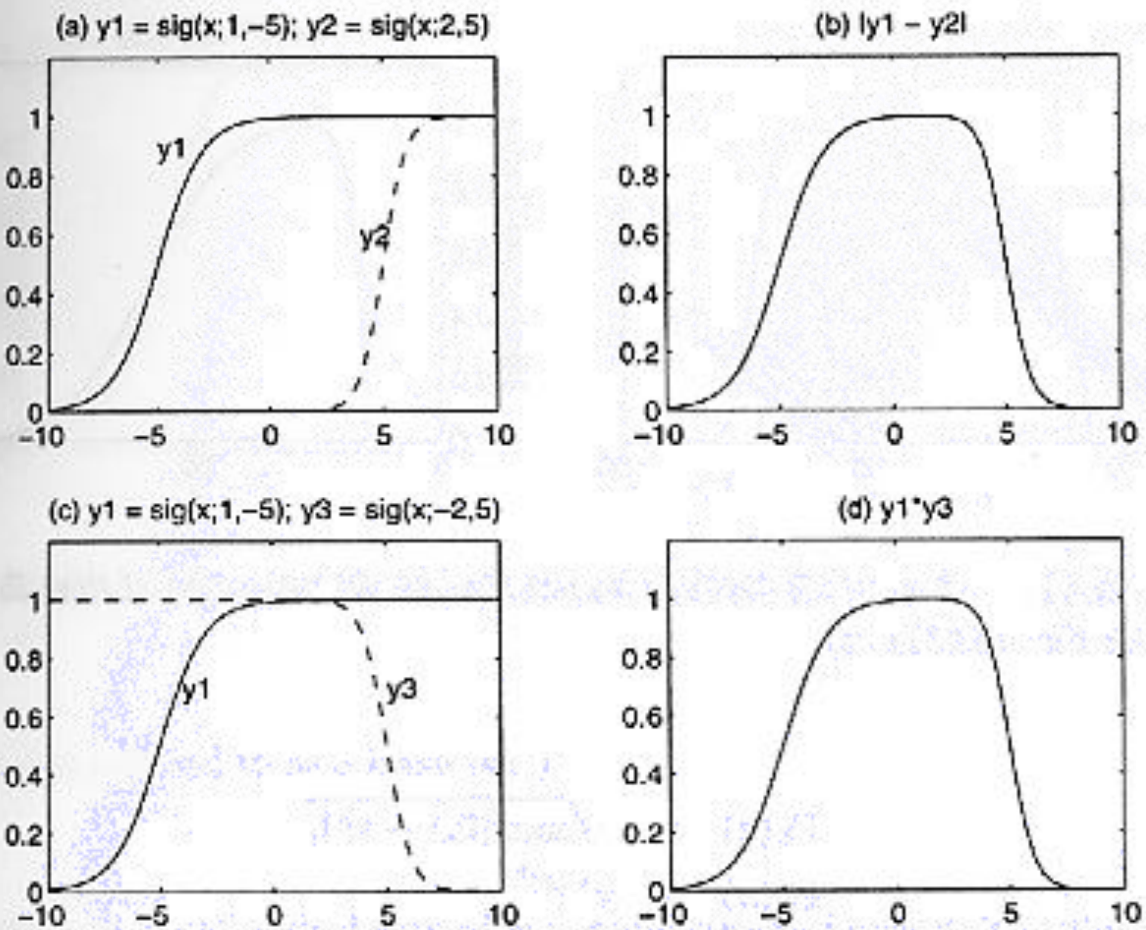
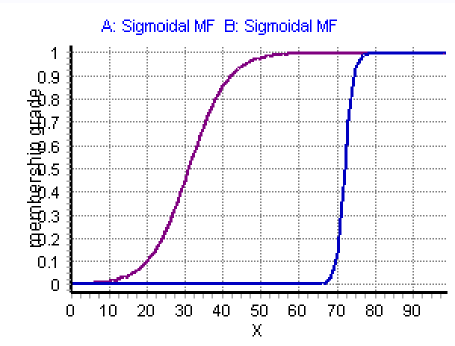
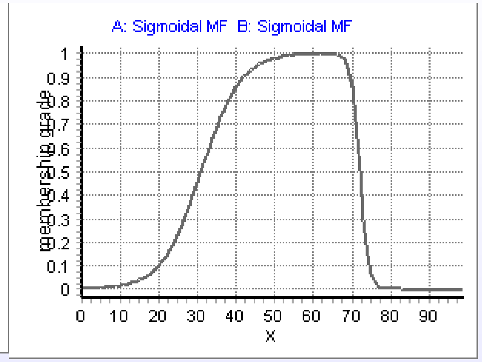


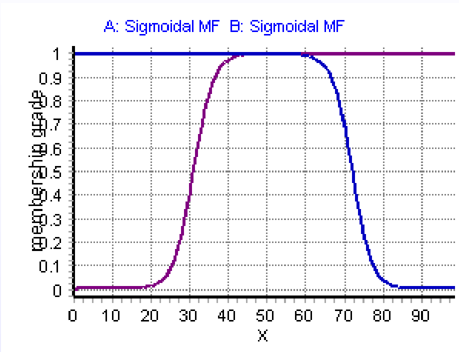
Figure 2.10. *(a) Two sigmoidal functions y_1 and y_2; (b) a close MF obtained from $|y_1 - y_2|$; (c) two sigmoidal functions y_1 and y_3; (d) a close MF obtained from $y_1 y_3$.* (MATLAB file: `disp_sig.m`)



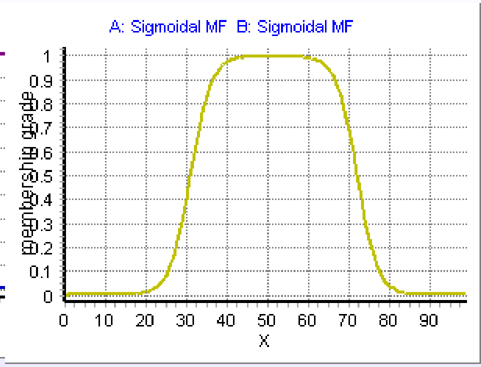
A, B



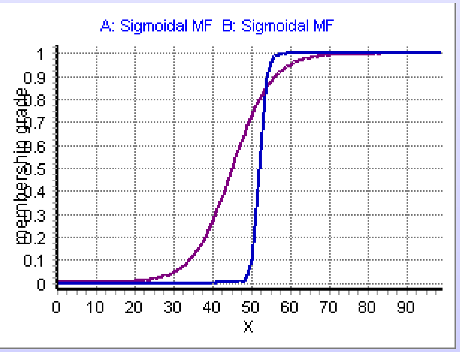
$|A - B|$



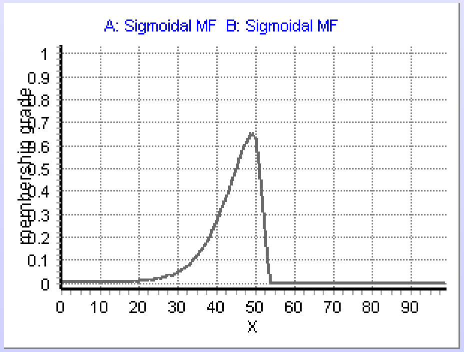
A, B



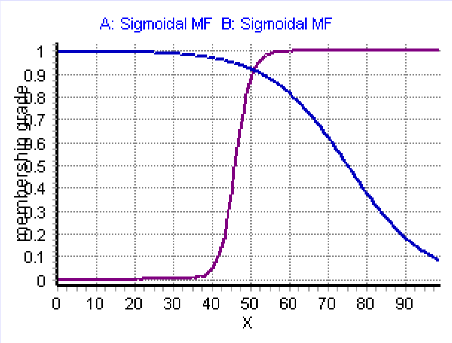
AB



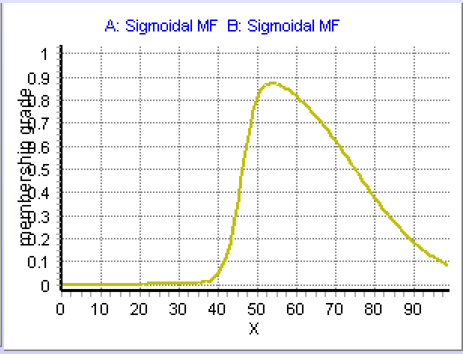
A, B



$|A - B|$



A, B



AB

Left-Right MFs

Definition 23: Left-right MF

A left-right MF or L-R MF is specified by three parameters $\{c,\alpha,\beta\}$:

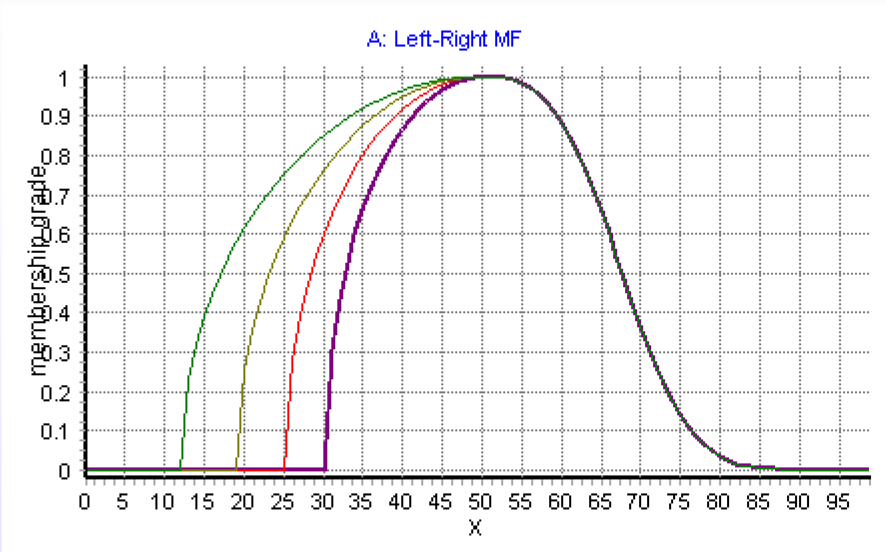
$$\mathrm{LR}\left(x;c,\alpha,\beta\right)=\begin{cases} F_L\left(\dfrac{c-x}{\alpha}\right), & x\le c. \\ F_R\left(\dfrac{x-c}{\beta}\right), & x\ge c, \end{cases}$$

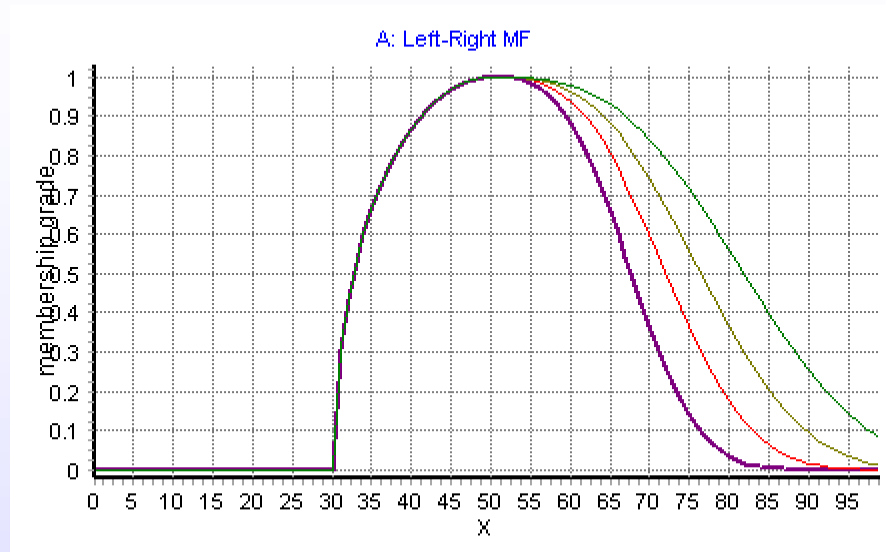
$F_L(z), F_R(z)$ are monotonically decreasing function defined on $[0,\infty)$ with

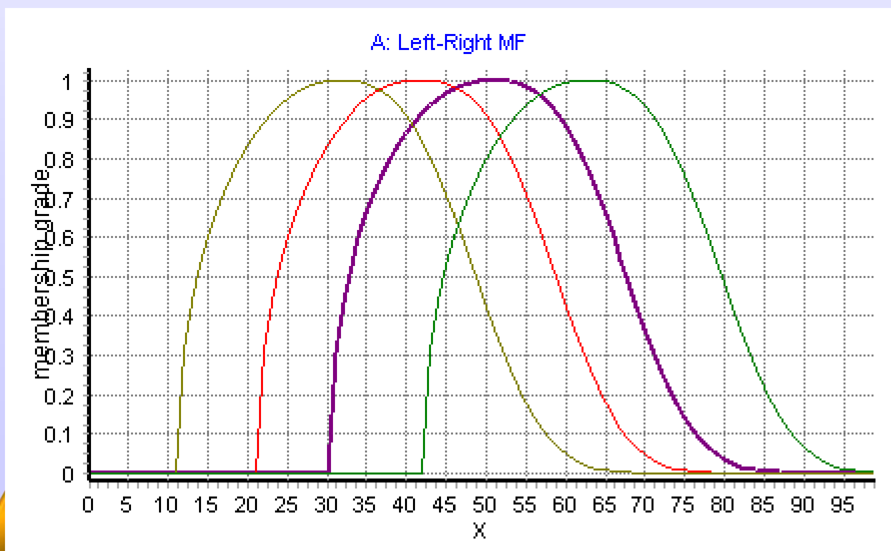
$$F_L(0)=F_R(0)=1 \, and \lim_{z\to\infty}F_L(z)=\lim_{z\to\infty}F_R(z)=0$$

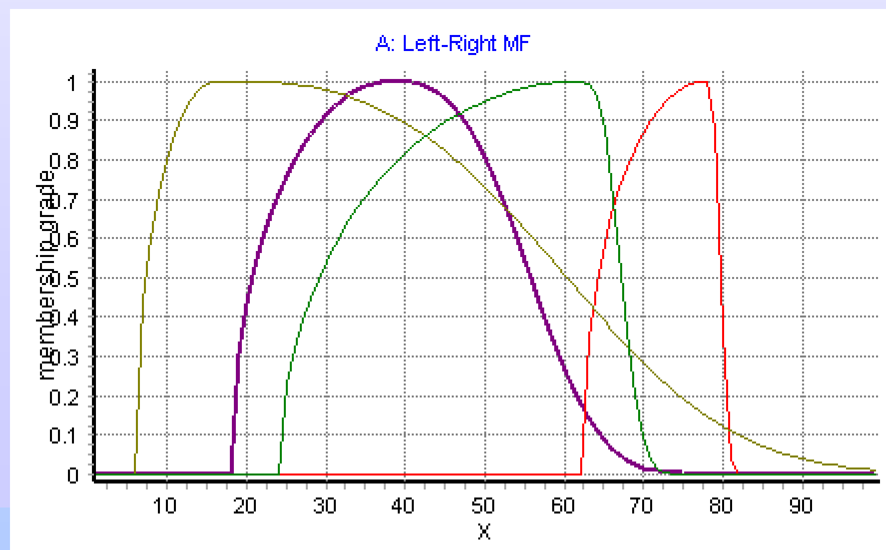
For example: $F_L(z)=\sqrt{\max\left(0,1-z^2\right)}$

$$F_R(z)=e^{-|z|^3}$$


A: Left-Right MF
membership grade
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
X


A: Left-Right MF
membership grade
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
X


A: Left-Right MF
membership grade
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
X


A: Left-Right MF
membership grade
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
10 20 30 40 50 60 70 80 90
X

Two-Dimensional Membership Functions

- Two inputs in different universe of discourse
  - Height and weight
  - Age and income

$$B = \sum_{\mathbf{v}_i \in X \times \mathrm{Y}} \mu_B(\mathbf{v}_i)/\mathbf{v}_i \qquad B = \sum_{x_i \in X,\, y_i \in Y,} \mu_B(x_i, y_i)/(x_i, y_i)$$

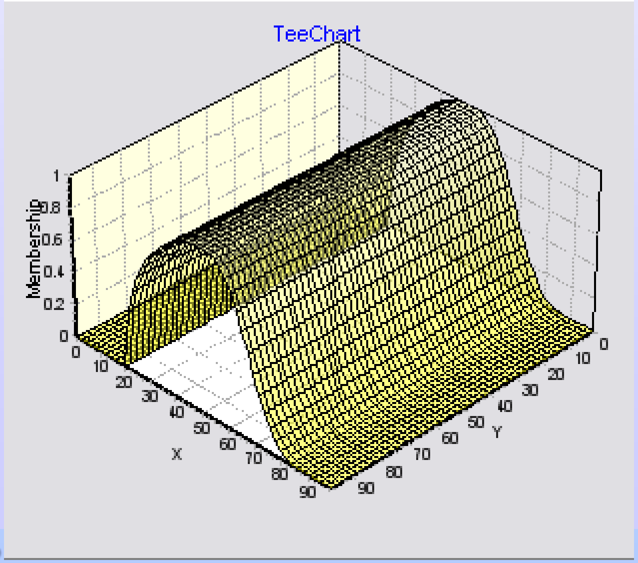
$$B = \int_{X \times Y} \mu_B(\mathbf{v})/\mathbf{v} \qquad B = \int_{X \times Y} \mu_B(x, y)/(x, y)$$

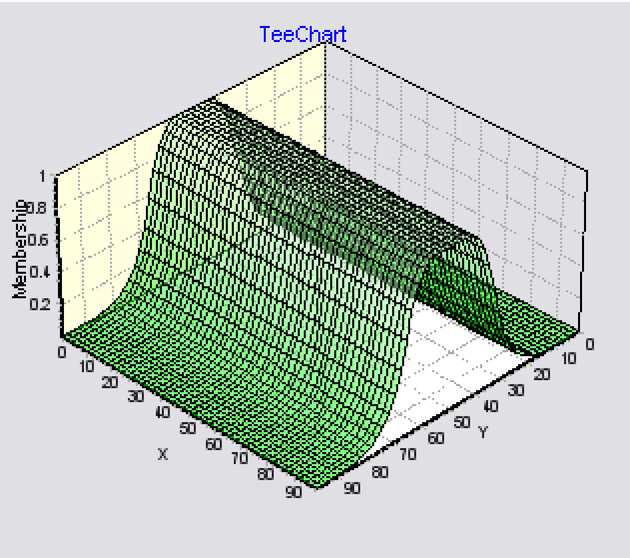
Cylindrical Extensions

Definition 24: Cylindrical extensions of one-dimensional fuzzy sets

If A is a fuzzy set in X, its cylindrical extension in X x Y is a Fuzzy set $B=c(A)$ defined by

$$B = c(A) = \int_{X \times Y} \mu_A(x)/(x, y)$$





Projection

Definition 25: Projections of fuzzy sets

Let R be a two-dimensional fuzzy set on X x Y. The projections of R onto X and Y are defined as

$$R_x = \int_X \left(\max_y \mu_R(x, y) \right) / x$$

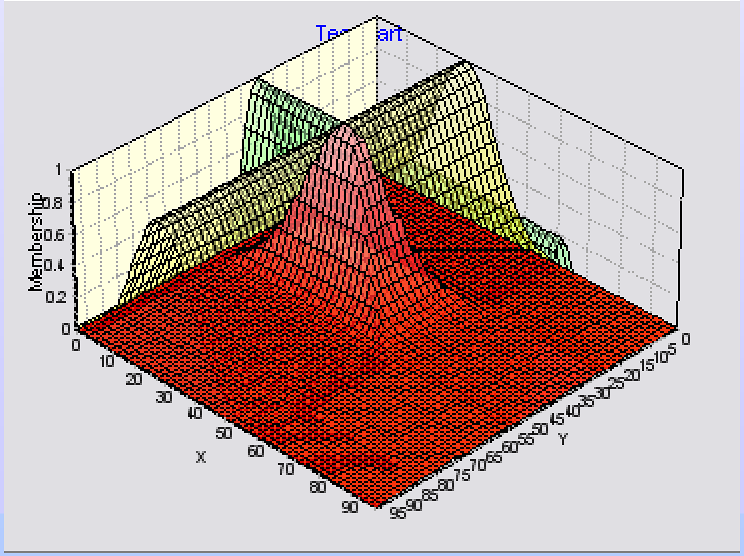
$$R_y = \int_Y \left(\max_x \mu_R(x, y) \right) / y$$

Composite vs. Noncomposite 2-D MFs

- **Composite**
  - **An MF of two dimensions can be expressed as an analytic expression of two MFs of one dimension**

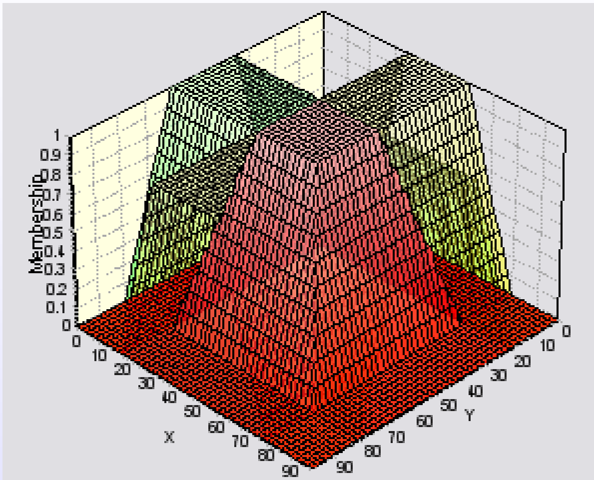
$$\mu_R(x,y) = e^{\left(-\left(\frac{x-3}{2}\right)^2 - (y-4)^2\right)} = e^{-\left(\frac{x-3}{2}\right)^2} e^{-(y-4)^2}$$

$$= \text{Gaussian}(x;3,2)\ \text{Gaussian}(y;4,1)$$

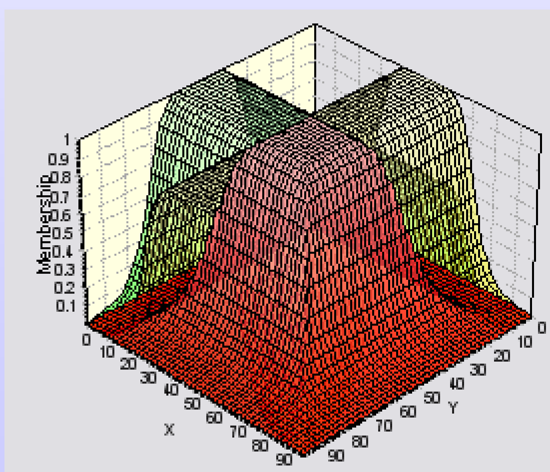


Composite 2D MFs based on min and max operators

Cartesian Product

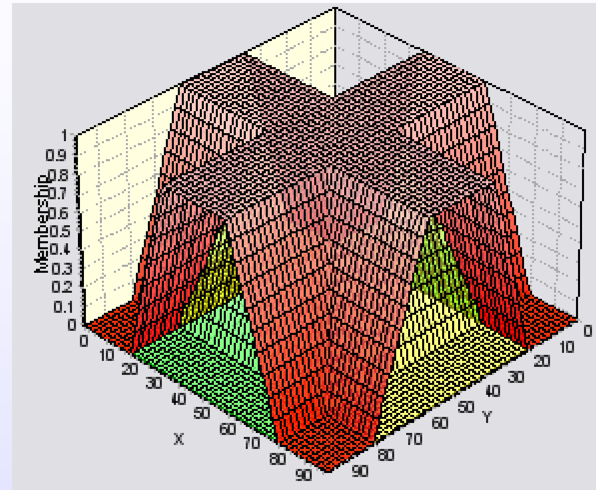


Z=min(trap(x),trap(y))

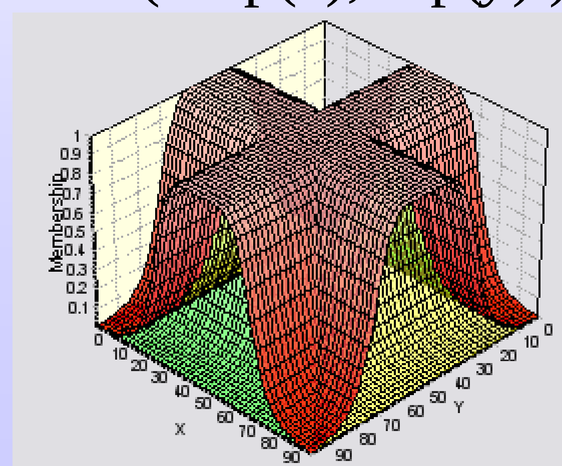


Z=min(bell(x),bell(y))

Cartesian Co-Product

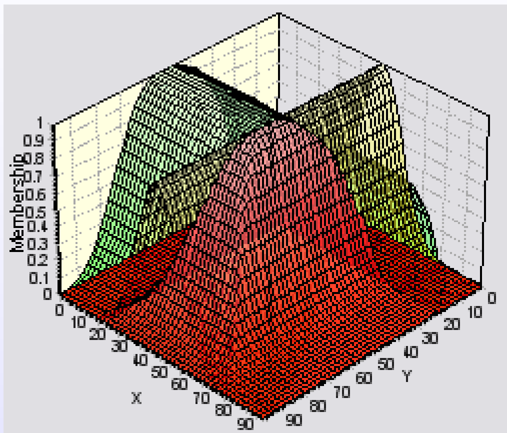


Z=max(trap(x),trap(y))



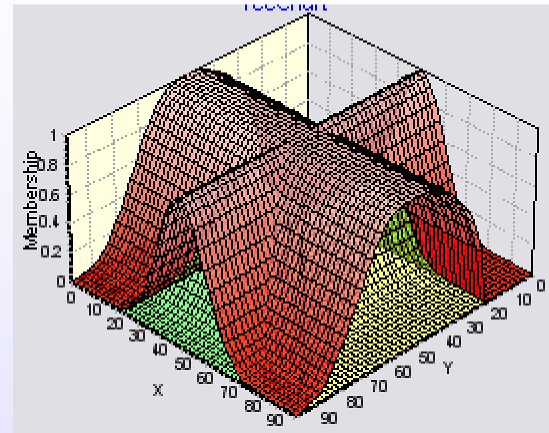
Z=max(bell(x),bell(y))

Cartesian Product

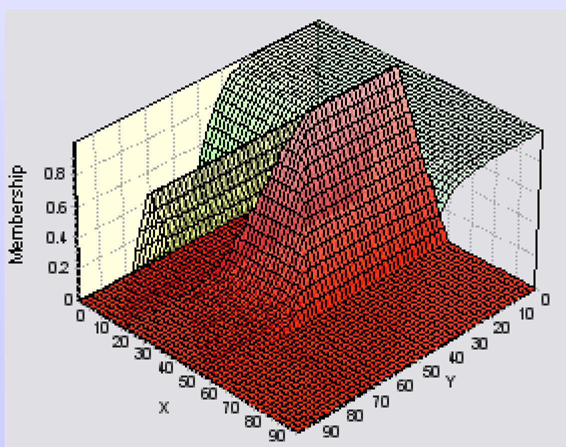


Z=min(Gaussian(x),LR(y))

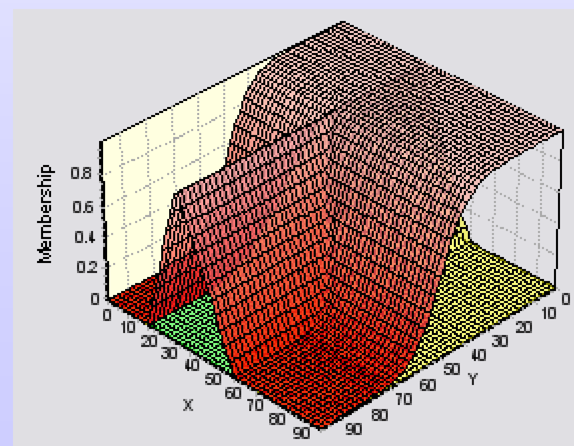
Cartesian Co-Product



Z=max(Gaussian(x),LR(y))



Z=min(tri(x),sig(y))



Z=max(tri(x),sig(y))

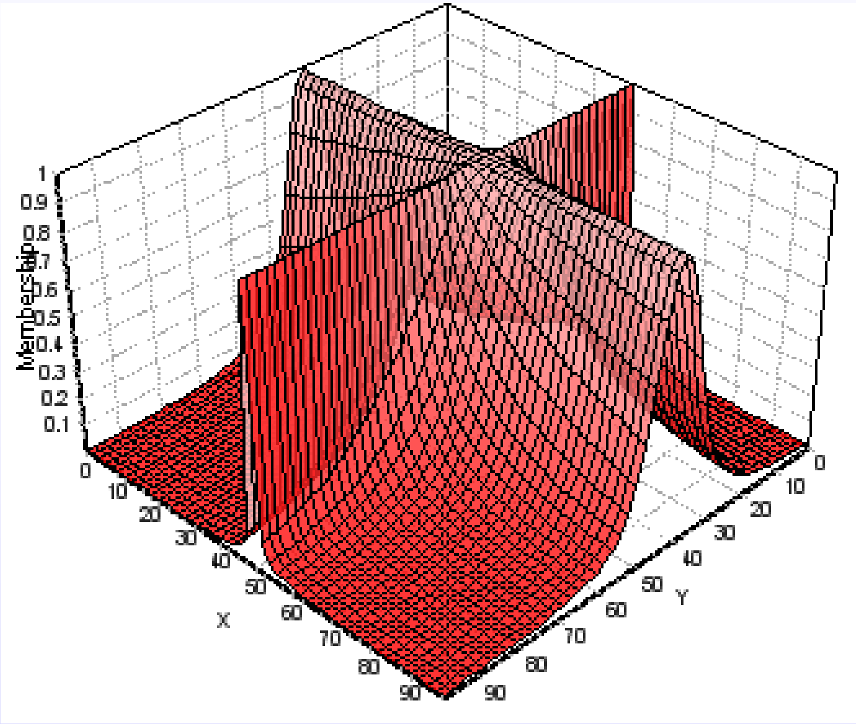
- **Min operator applied to aggregate 1D MFs to result a 2D MF**
  - **Is an intersection of two 1D MFs (no exactly correct)**
    - $A \wedge B \rightarrow$ 1D MF
  - **Is an intersection of two 2D MFs (Correct)**
    - $c(A) \wedge c(B) \rightarrow$ 2D MF
  - **Is a Cartesian product (Correct)**
    - AxB

- **Max operator applied to aggregate 1D MFs to result a 2D MF**
  - **Is a union of two 1D MFs (no exactly correct)**
    - $A \vee B$ → 1D MF
  - **Is a union of two 2D MFs (Correct)**
    - $c(A) \vee c(B)$ → 2D MF
  - **Is a Cartesian co-product (Correct)**
    - $A+B$

Noncomposite

- Otherwise

$$\mu_S(x,y) = \frac{1}{1+\left|\frac{x-50}{20}\right|\left|\frac{y-40}{10}\right|^{2.5}}$$



Derivative of Parameterized MFs

- Derivatives of an MF with respect to argument and parameters
  - Necessary for adaptive fuzzy system

$$y = Gaussian(x;\sigma,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

$$\frac{\partial y}{\partial x} = -\frac{x-c}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2} = -\frac{x-c}{\sigma^2} y$$

$$\frac{\partial y}{\partial \sigma} = \frac{(x-c)^2}{\sigma^3} e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2} = \frac{(x-c)^2}{\sigma^3} y$$

$$\frac{\partial y}{\partial c} = \frac{x-c}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2} = \frac{x-c}{\sigma^2} y$$

$$y = bell(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

$$\frac{\partial y}{\partial x} = \frac{-\frac{2b}{a}\left|\frac{x-c}{a}\right|^{2b-1}}{\left(1 + \left|\frac{x-c}{a}\right|^{2b}\right)^2} = -\frac{2b}{a}\left(\frac{a}{x-c}\right)\left|\frac{x-c}{a}\right|^{2b} y^2$$

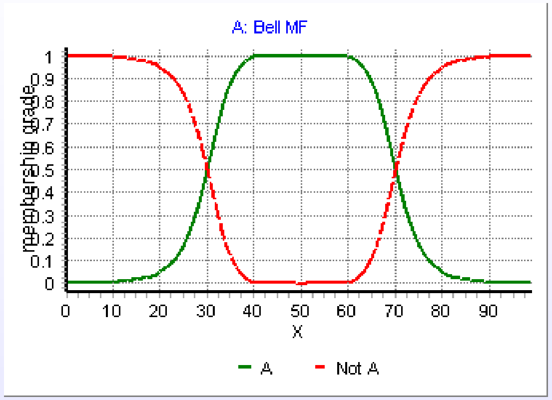
$$= -\frac{2b}{x-c}\left|\frac{x-c}{a}\right|^{2b} y^2 = -\frac{2b}{x-c}\left(\frac{1}{y} - 1\right) y^2$$

$$= -\frac{2b}{x-c}(1-y)y, \ if \ \ x \neq c$$

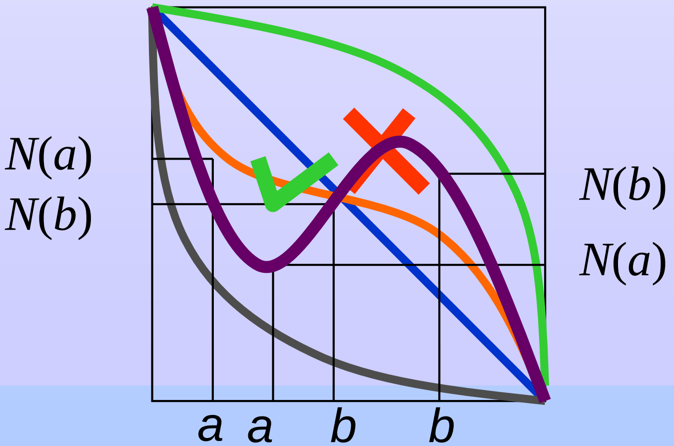
General Fuzzy Complement Operator

- **Formal Definition**

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

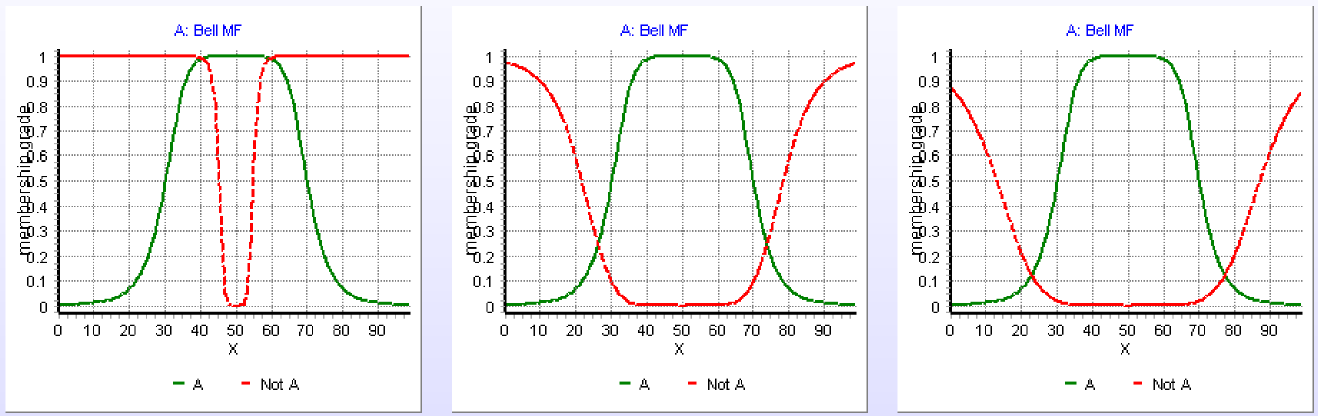


- **General Fuzzy complement operator**
  - **A continuous function** $N:[0,1] \rightarrow [0,1]$:
    - $N(0) = 1$ and $N(1) = 0$
    - $N(a) >= N(b)$ if $a <= b$
- **Involution**
  - $N(N(a)) = a$



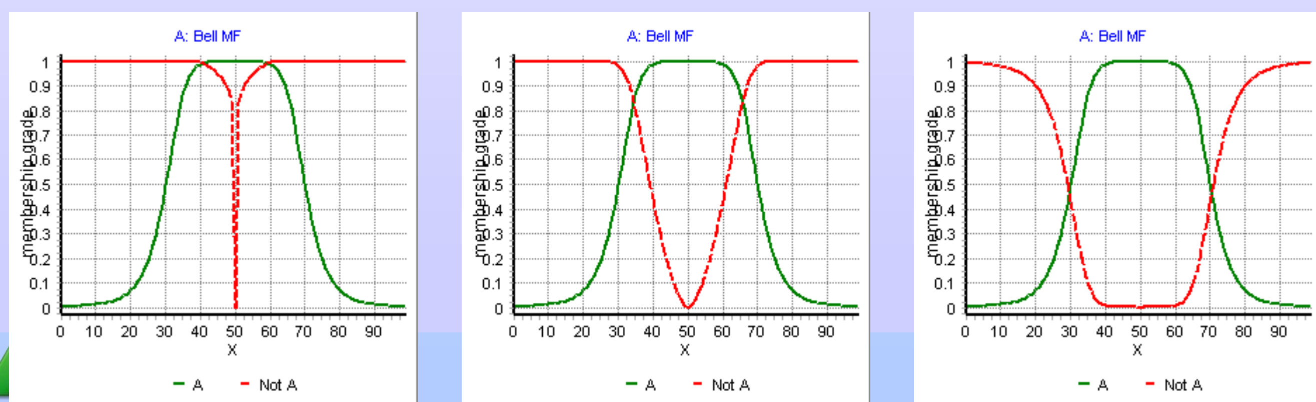
Sugeno's complement

$$N_s(a) = \frac{1-a}{1+sa}, s > -1$$



Yager's complement

$$N_w(a) = (1-a^w)^{1/w}, w > 0$$



General Intersection Operator

- **Formal Definition**

$$\mu_C(x) = \min\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \wedge \mu_B(x)$$

- **General intersection operator**
  - **A continuous function** T:[0,1]x[0,1]→[0,1]:
  - **General formula**

$$\mu_{A\cap B}(x) = T\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \tilde{*} \mu_B(x)$$

  - $\tilde{*}$ **binary fuzzy intersection operator, T-norm operator**

Definition 26: T-norm **(triangular norm)**

A T-norm operator is a two-place function $T(.,.)$ satisfying

$T(0,0)=0,\ T(a,1) = T(1,a) = a$	(boundary)
$T(a,b) \le T(c,d)$ if $a \le c$ and $b \le d$	(monotonicity)
$T(a,b) = T(b,a)$	(commutativity)
$T(a, T(b,c)) = T(T(a,b), c)$	(associativity)

T-norm operators

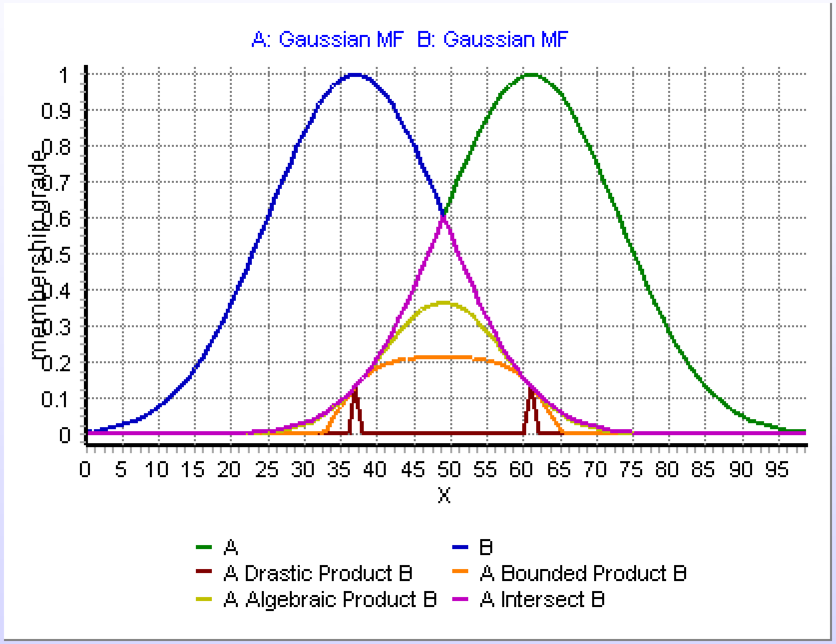
Minimum T-norm: $T_{\min}(a,b) = \min(a,b) = a \wedge b.$

Algebraic product: $T_{ap}(a,b) = ab.$

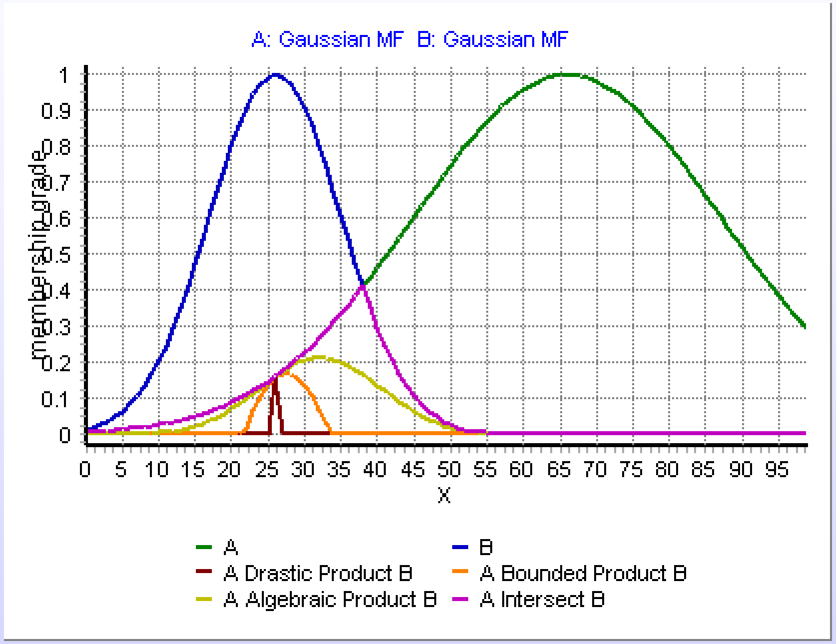
Bounded product: $T_{bp}(a,b) = \max\{0, (a+b-1)\} = 0 \vee (a+b-1)$

Drastic product: $T_{dp}(a,b) = \begin{cases} a, \text{if } b = 1 \\ b, \text{if } a = 1 \\ 0, \text{if } a, b < 1 \end{cases}$

$$T_{dp}(a,b) \le T_{bp}(a,b) \le T_{ap}(a,b) \le T_{\min}(a,b)$$



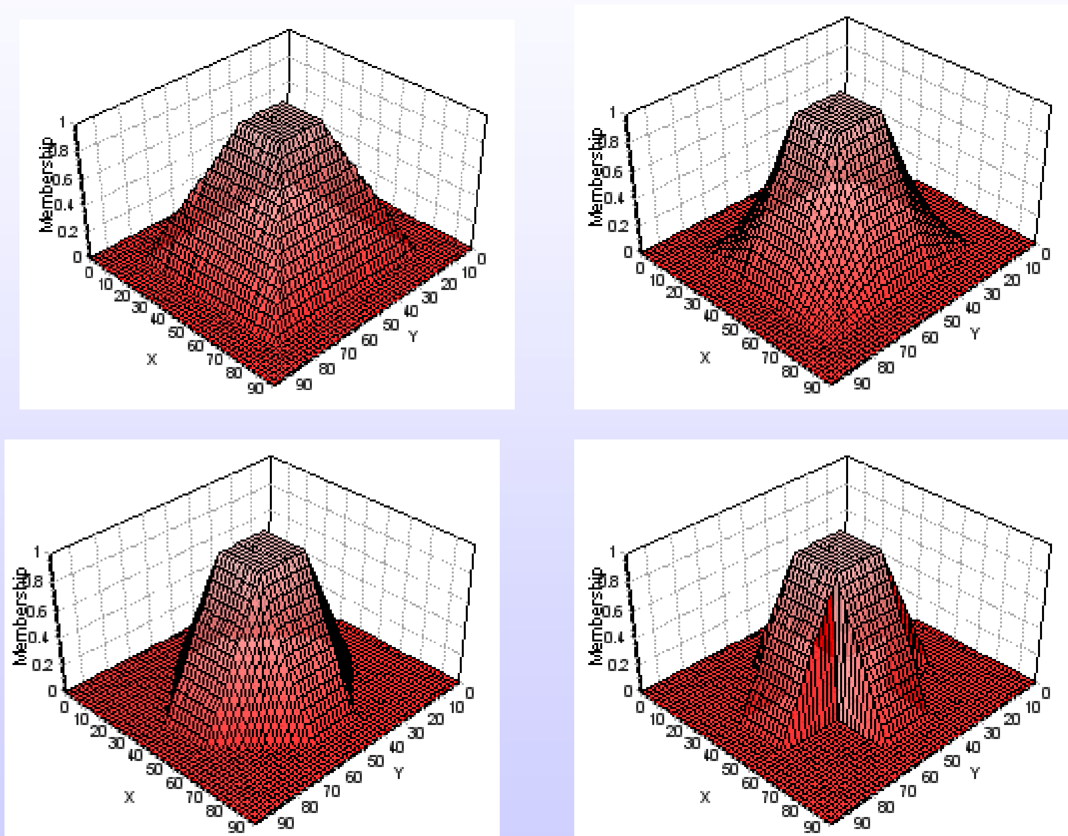
A: Gaussian MF B: Gaussian MF
membership grade
X
A
B
A Drastic Product B
A Bounded Product B
A Algebraic Product B
A Intersect B



A: Gaussian MF B: Gaussian MF
membership grade
X
A
B
A Drastic Product B
A Bounded Product B
A Algebraic Product B
A Intersect B

T-norm intersection operator samples

Used to compose 2D MFs



General Union Operator

- **Formal Definition**

$$\mu_C(x) = \max\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \vee \mu_B(x)$$

- **General union operator**
  - **A continuous function** S:[0,1]x[0,1]→[0,1]:
  - **General formula**

$$\mu_{A \cup B}(x) = S\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) \tilde{+} \mu_B(x)$$

  - $\tilde{+}$ **binary fuzzy union operator, S-norm operator or T-conorm**

Definition 27: S-norm (T-conorm)

An S-norm (T-conorm) operator is a two-place function S(.,.) satisfying

$S(1,1)=1,\ S(a,0) = S(0,a) = a$	(boundary)
$S(a,b) \leq S(c,d)$ if $a \leq c$ and $b \leq d$	(monotonicity)
$S(a,b) = S(b,a)$	(commutativity)
$S(a, S(b,c)) = S(S(a,b), c)$	(associativity)

S-norm (T-conorm) operators

Maximum S-norm : $S_{\max}(a,b) = \max(a,b) = a \vee b.$

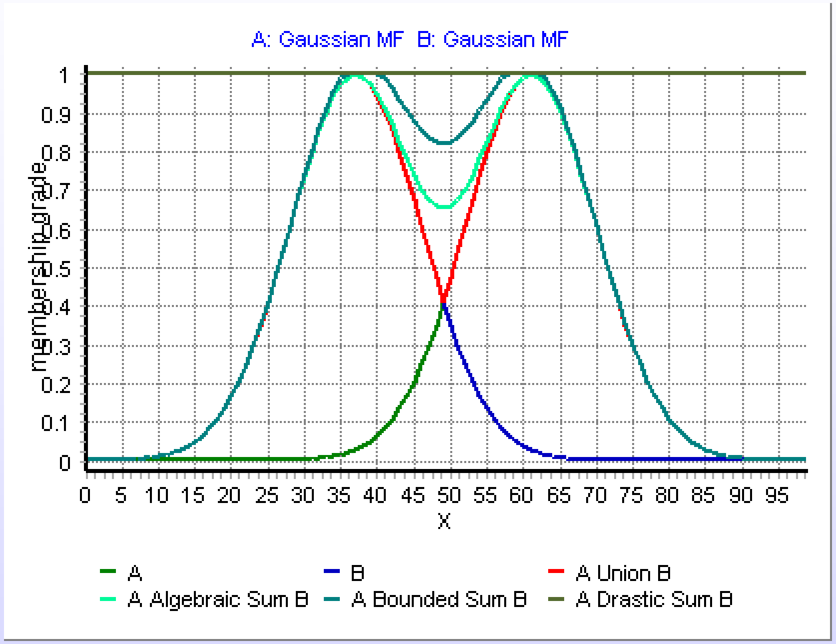
Algebraic sum S-norm: $S_{as}(a,b) = a + b - ab.$

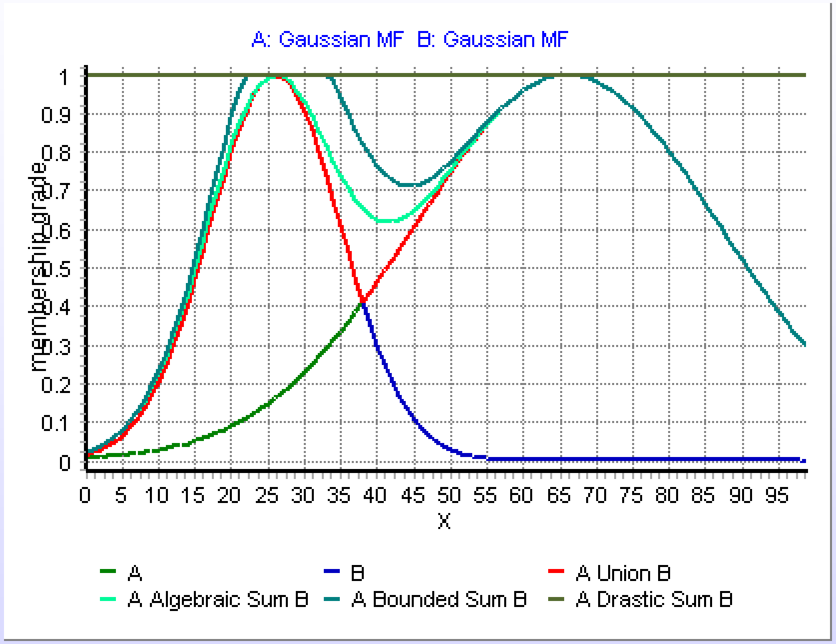
Bounded sum S-norm: $S_{bs}(a,b) = \min\{1, (a+b)\} = 1 \wedge (a+b)$

Drastic sum S-norm: $S_{ds}(a,b) = \begin{cases} a, \text{if } b = 0 \\ b, \text{if } a = 0 \\ 1, \text{if } a, b > 0 \end{cases}$

$$S_{\max}(a,b) \le S_{as}(a,b) \le S_{bs}(a,b) \le S_{ds}(a,b)$$

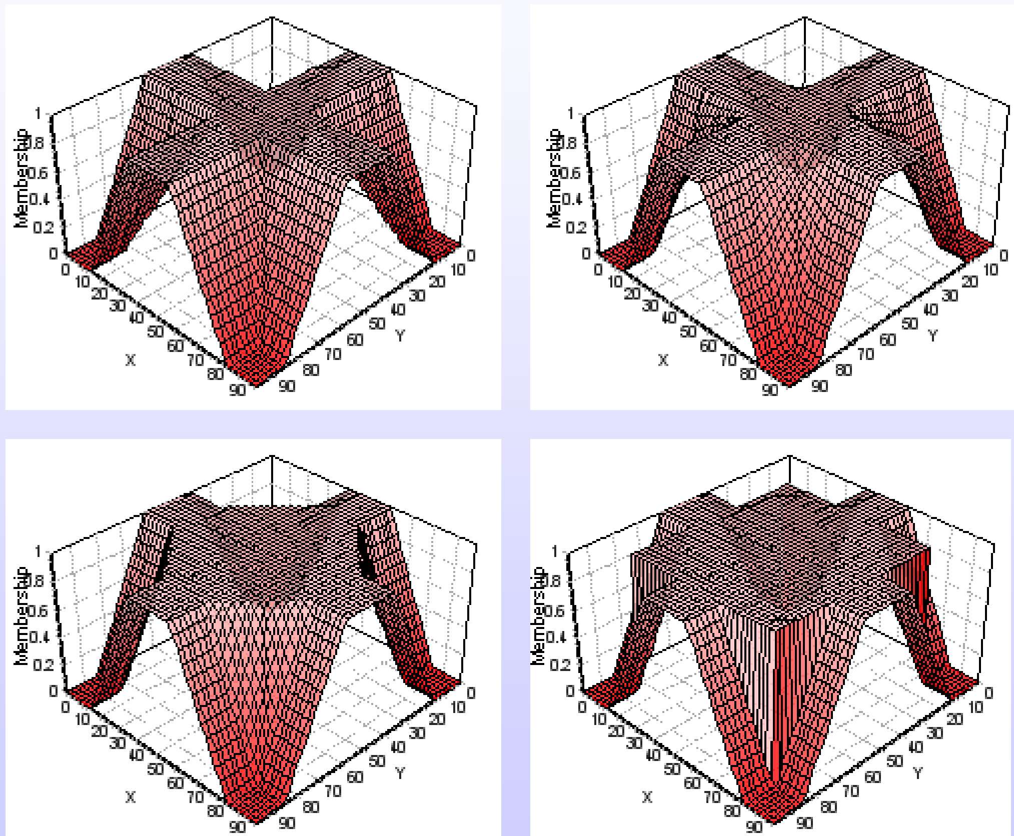
computer automation


A: Gaussian MF B: Gaussian MF
membership grade
X
A
B
A Union B
A Algebraic Sum B
A Bounded Sum B
A Drastic Sum B


A: Gaussian MF B: Gaussian MF
membership grade
X
A
B
A Union B
A Algebraic Sum B
A Bounded Sum B
A Drastic Sum B

S-norm intersection operator samples

Used to compose 2D MFs



Theorem 1: DeMorgan's law

T-norm T(.,.) and S-norms S(.,.) are duals which support the generalization of DeMorgan's law:

$$T(a,b) = N(\ S(\ N(a), N(b)\ )\ )$$
$$S(a,b) = N(\ T(\ N(a), N(b)\ )\ ).$$

$$a \,\tilde{*}\, b = N\left(N(a) \,\tilde{+}\, N(b)\right)$$
$$a \,\tilde{+}\, b = N\left(N(a) \,\tilde{*}\, N(b)\right)$$

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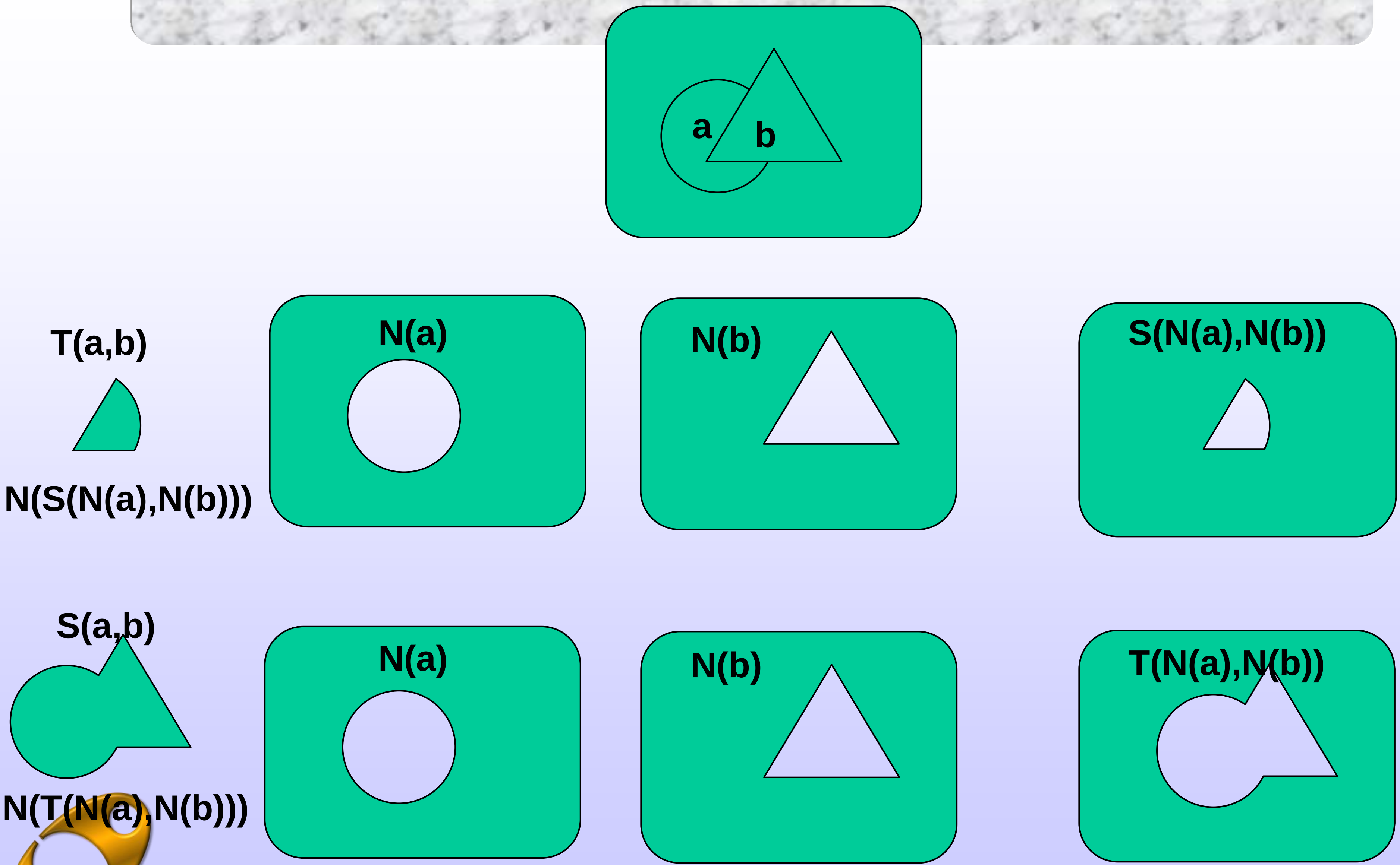
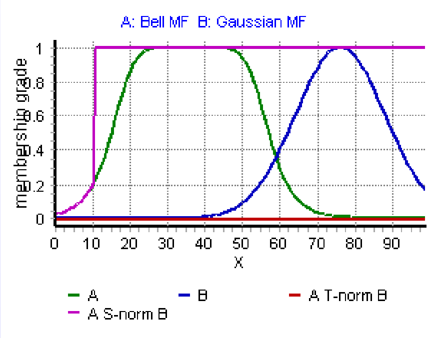
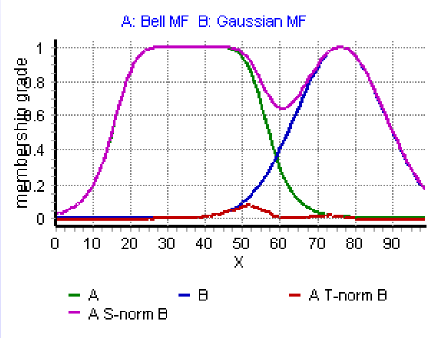
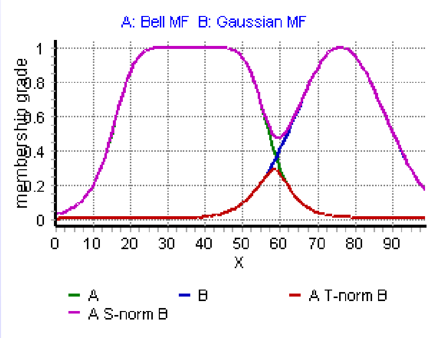

a
b
T(a,b)
N(S(N(a),N(b)))
N(a)
N(b)
S(N(a),N(b))
S(a,b)
N(T(N(a),N(b)))
N(a)
N(b)
T(N(a),N(b))

Parameterized T-norm and S-norms

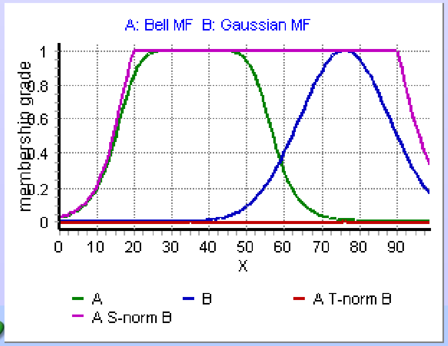
- Several parameterized T-norms and dual T-conorms (S-norms) are proposed
  - Schweizer and Sklar

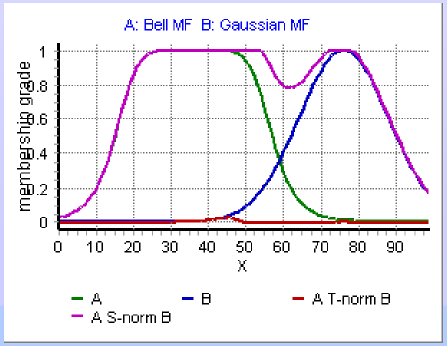


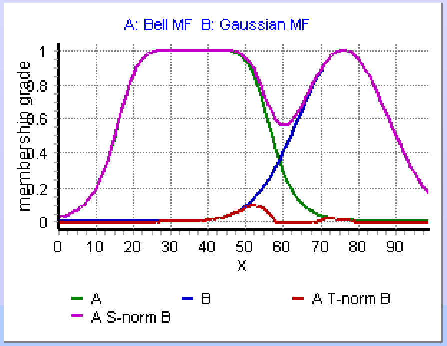




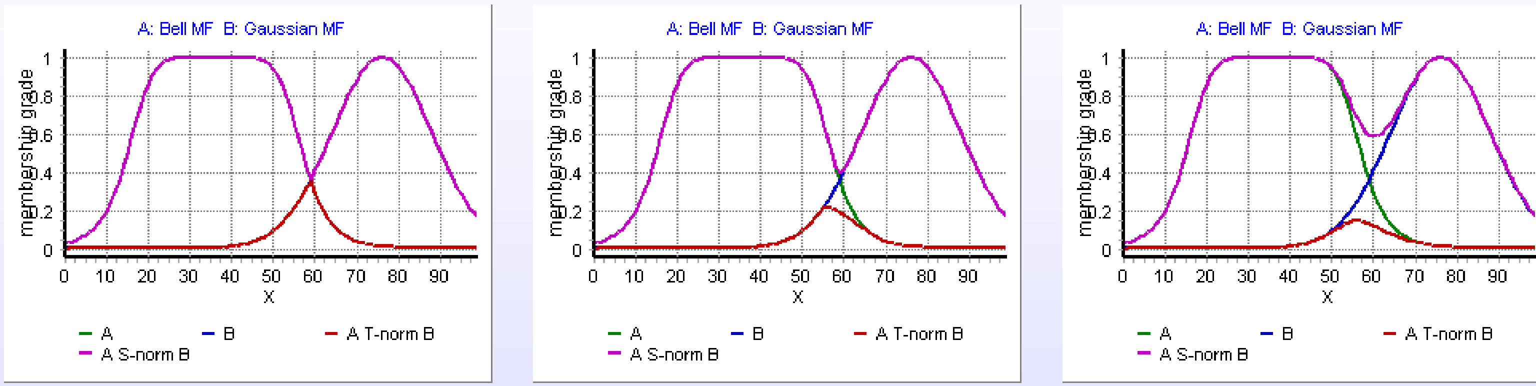
  - Yager



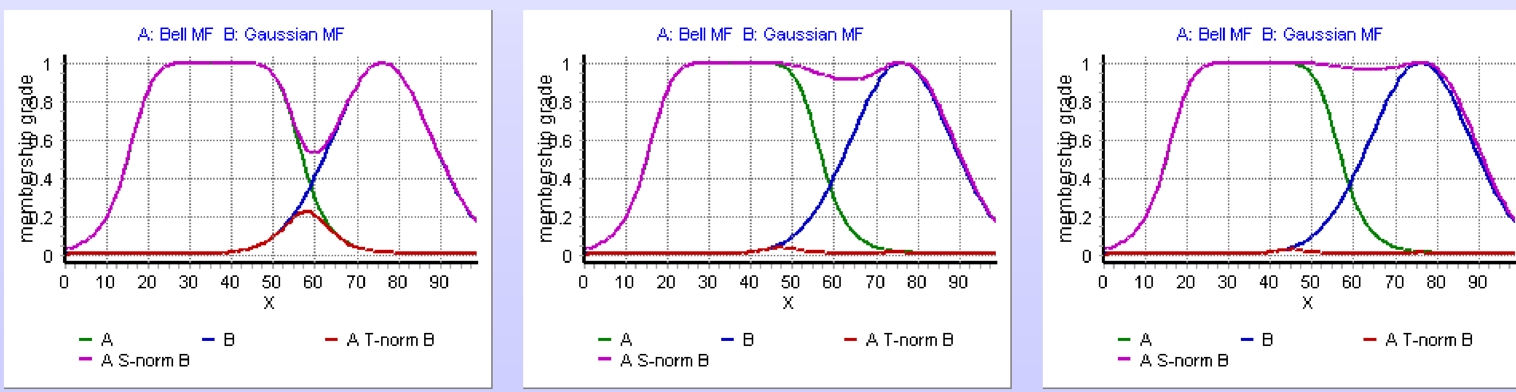




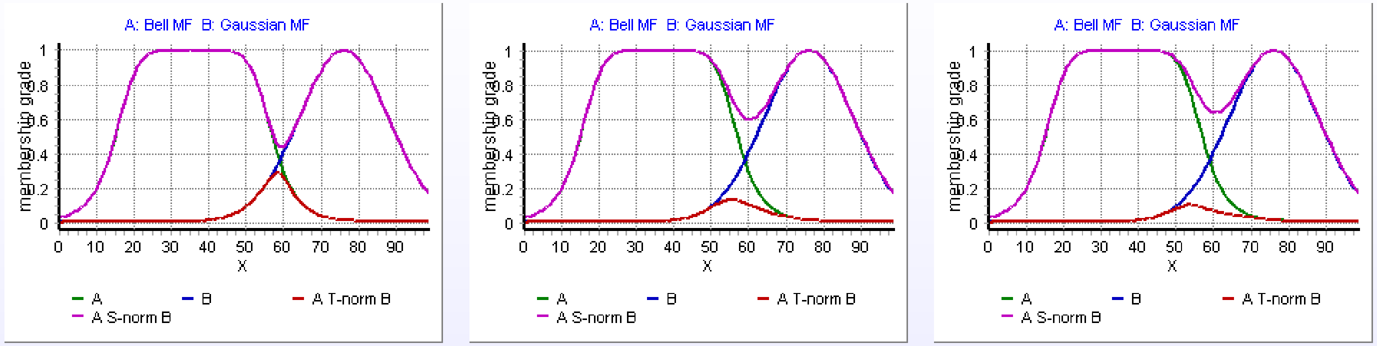
Dubois and Prade



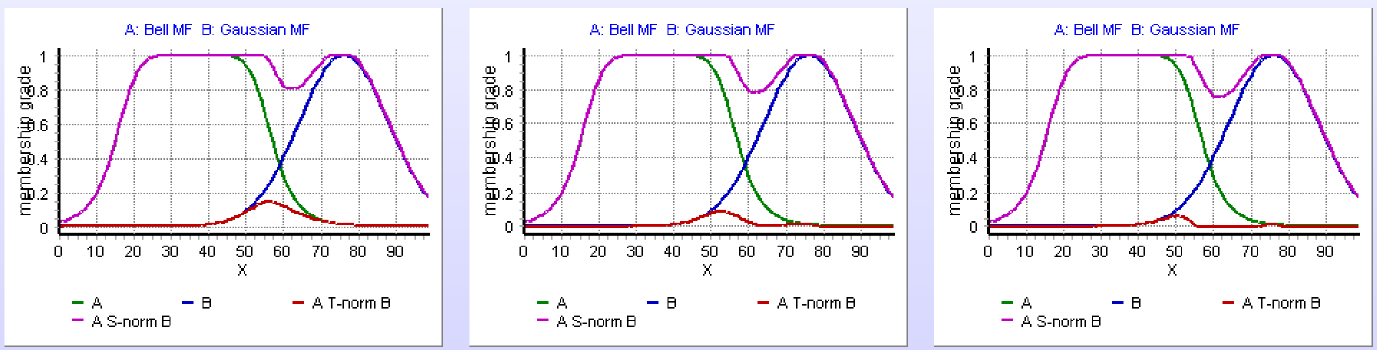
Hamacher



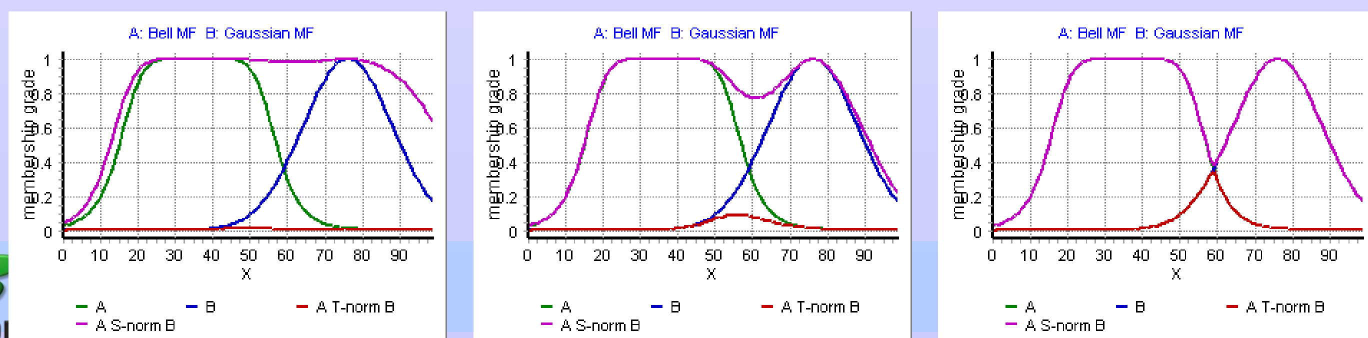
Frank



Sugeno



Dombi



computer automation

Homework for Chapter 2

- **Exercises 5, 6, 8, 12, and 19**
  - **Whenever you are asked to use MATLAB to do something, write your own programming code to generate the required charts**