Fuzzy Rules and Fuzzy Reasoning

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computer automation

Fuzzy Rules and Fuzzy Reasoning

- Extension Principle
- Fuzzy Relations
- Linguistic Variables and Values
- Fuzzy Rules
- Fuzzy Reasoning
- Compositional Rule of Inference
- Fuzzy Inference System
  - Automatic control, expert system, pattern recognition, time series prediction, data classification

Extension Principle

- Extension principle
  - Extend crisp domains of mathematical expressions to fuzzy domains
    - Point-to-point mapping → fuzzy set mapping

Suppose $f(.)$ is a function from X to Y and A is a fuzzy set on X,

$$A=\mu_A(x_1)/x_1+\mu_A(x_2)/x_2+\cdots+\mu_A(x_n)/x_n$$

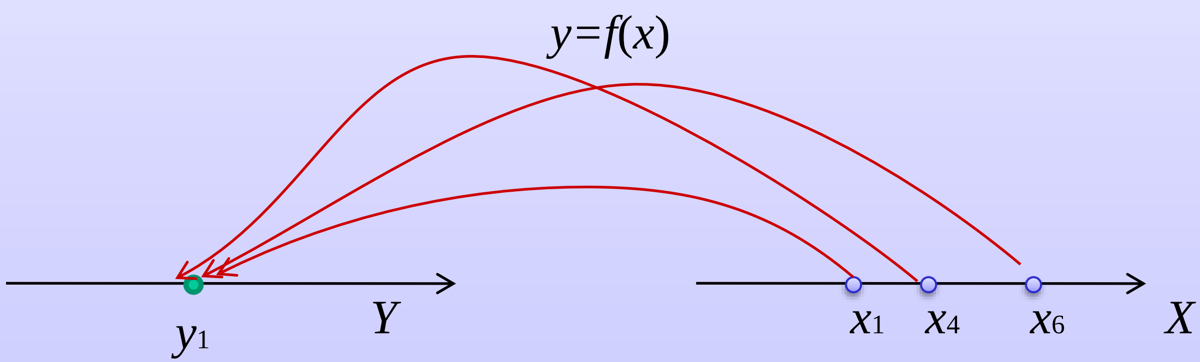
The Extension principle states that the image of fuzzy set A under the mapping $f(.)$ can be expressed as a fuzzy set B

$$B=f(A)=\mu_B(y_1)/y_1+\mu_B(y_2)/y_2+\cdots+\mu_B(y_n)/y_n,$$

$$\text{where } y_i=f(x_i),\ i=1,2,\ldots,n$$

- **If $f(.)$ is a many-to-one mapping, the membership grade is the maximum grade of the points xs that map to the same y.**

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$



Extension Principle

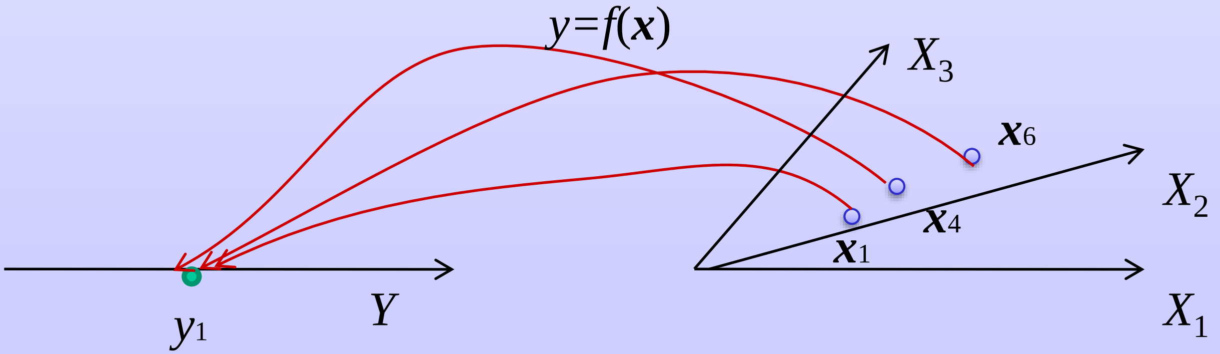
Definition 1: Extension Principle

Suppose that function f is a mapping from an n-dimensional Cartesian product space X_1xX_2x…X_n to a one-dimensional universe Y such that $y=f(x_1,...,x_n)$, and suppose $A_1,\ldots,A_n$ are n fuzzy sets in $X_1,\ldots,X_n$, respectively. Then the extension principle asserts that the fuzzy set B induced by the mapping f is defined by

$$\mu_B(y)=\begin{cases}\max\limits_{\forall(x_i,\cdots,x_n)=f^{-1}(y)}\left(\min\limits_i\left(\mu_{A_i}(x_i)\right)\right), \text{ if } f^{-1}(y)\neq\varnothing \\ 0, \text{ if } f^{-1}(y)=\varnothing\end{cases}$$

Application of Extension

- $f(x_1, \ldots, x_n)$ → **map** → y
- $X_1, \ldots, X_n$ n **discourses** → Y **discourse**
- $X_1, \ldots, X_n + Y$ → $n+1$ **dimensional discourses**
- **An ($n+1$)-D fuzzy set with ($n+1$)MFs → Fuzzy relation**
  - **Support fuzzy inference**
  - **From $x_1, \ldots, x_n$ infer Y**



Fuzzy Relations

- Fuzzy relations are fuzzy sets that map an element to a membership grade
- Unary fuzzy relations are fuzzy sets with one-dimensional MFs that map a one-dimensional point to a membership grade
- Binary fuzzy relations are fuzzy sets on XxY which map each element on XxY to a membership grade between 0 and 1
  - Two-dimensional MFs map an (x,y) point to a membership grade

Definition 2: Binary fuzzy relation

Let X and Y be two universes of discourse. Then

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \middle| (x, y) \in X \times Y \right\}$$

is a binary fuzzy relation in XxY.

- **Relation is a fuzzy set with a membership function**
- **Relate an element in the universe to a membership grade**

Examples of Binary Relations

- y is much greater than x (x, y are numbers)
- x is close to y (x, y are numbers)
- X depends on y (x, y are events)
- X and y look alike (x, y are objects)
- If x is large, then y is small (x is reading and y is action) (simplest fuzzy rule)
  - Used frequently in fuzzy inference system

Relation Compositions

- Relation and relation can be combined by applying composition operations
- Best know composition operation is max-min composition proposed by Zadeh

Definition 3: Max-min Composition

Let R_1 and R_2 be two fuzzy sets defined on XxY and YxZ, respectively. The max-min composition of R_1 and R_2 is a fuzzy set defined by

$$R_1 \circ R_2 = \left\{ \left((x,z), \max_y \ \min\left(\mu_{R_1}(x,y), \mu_{R_2}(y,z) \right) \right) \middle| x \in X, y \in Y, z \in Z \right\}$$

or, equivalently,

$$\mu_{R_1 \circ R_2}(x,\mathrm{z}) = \max_y \ \min\left(\mu_{R_1}(x,y), \mu_{R_2}(y,z) \right) = \vee_{\mathrm{y}} \left(\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z) \right)$$

With the understanding that $\vee$ and $\wedge$ represent max and min.

- For discrete universes, fuzzy relations can be expressed as relation matrices
- Similar to matrix operation can be carried out to compute the composition relation matrix
- Operators × (multiplication) and + (sum) are replaced by ∧ (and) and ∨ (or)

X \ Y				
	0.0	**0.3**	**0.8**	**0.9**
	0.1	**0.2**	**0.7**	**0.8**
	0.0	**0.1**	**0.2**	**0.6**

Y \ z		
	0.6	**0.8**
	0.4	**0.5**
	0.3	**0.3**
	0.1	**0.0**

X \ z		
	0.3	**0.3**
	0.3	**0.3**
	0.2	**0.2**

$$0.0 \wedge 0.6 \vee 0.3 \wedge 0.4 \vee 0.8 \wedge 0.3 \vee 0.9 \wedge 0.1 = 0.0 \vee 0.3 \vee 0.3 \vee 0.1 = 0.3$$

Properties of Composition

$$Associativity: \quad R \circ (S \circ T) = (R \circ S) \circ T$$

$$Distributivity\ over\ union: R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

$$Distributivity\ over\ intersection: R \circ (S \cap T) = (R \circ S) \cap (R \circ T)$$

$$Monotonicity: S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$$

Definition 4: Max product Composition

Let R_1 and R_2 be two fuzzy sets defined on XxY and YxZ, respectively. The max-product composition of R_1 and R_2 is a fuzzy set defined by

$$\mu_{R_1 \circ R_2}(x, z) = \max_y \left(\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z) \right) = \vee_y \left(\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z) \right)$$

Y

X	0.0	0.3	0.8	0.9
	0.1	0.2	0.7	0.8
	0.0	0.1	0.2	0.6

z

Y	0.6	0.8
	0.4	0.5
	0.3	0.3
	0.1	0.0

X	0.24	

0.0x0.6 v 0.3x0.4 v 0.8 x 0.3 v 0.9 x 0.1 = 0.0 v 0.12 v 0.24 v 0.09 = 0.24

- In general max, min (or union, intersection) operations are used in composition computation
- They can be replaced with various T-norm and S-norm computation to have various results

	Y			
X	0.0	0.3	0.8	0.9
	0.1	0.2	0.7	0.8
	0.0	0.1	0.2	0.6

	z	
Y	0.6	0.8
	0.4	0.5
	0.3	0.3
	0.1	0.0

X	?	

(0.0 T 0.6) S (0.3 T 0.4) S (0.8 T 0.3) S (0.9 T 0.1)

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Linguistic Variables and Values

Definition 5: Linguistic variables and other related terminology

A linguistic variable is a quintuple $(x, T(x), X, G, M)$, x is the name of the variable; $T(x)$ is the term set of x—the set of linguistic values (terms); X is the universe of discourse; G is a syntactic rule which generates terms in $T(x)$; and M is a semantic rule which associates with each linguistic value a its meaning $M(a)$, where $M(a)$ denotes a fuzzy set in X

X : age

$T(x)$: {*young, middle aged, old, not young, not very young, …* }

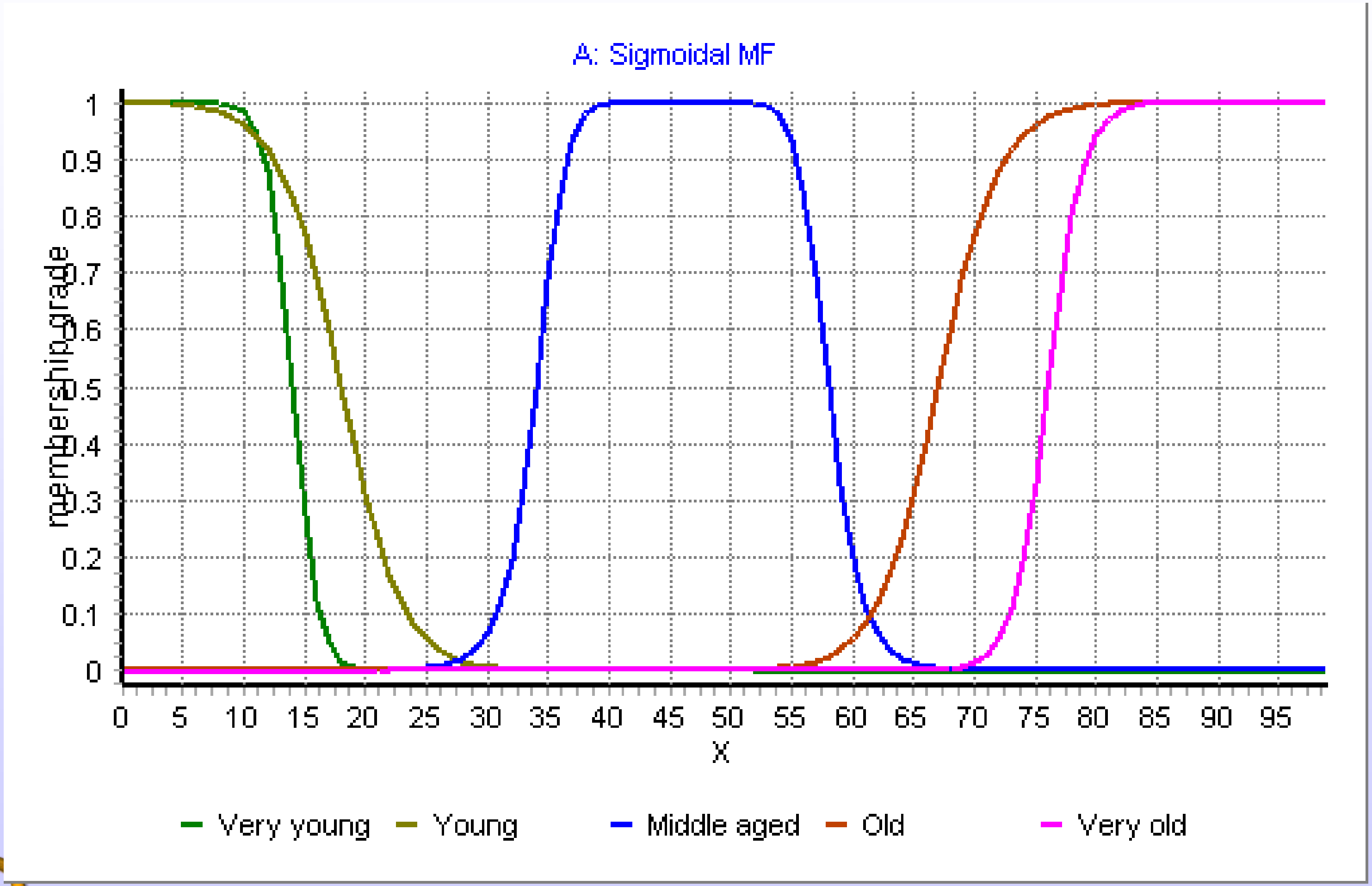
Primary terms: *young, middle aged, old*

Hedges: very, more, less, quite, extremely

Connectors: and, or, either, neither

G: [Hedges] Primary term [connector [Hedges] Primary term]

M: $\mu_{\text{young}}(x), \mu_{\text{middle aged}}(x), \mu_{\text{old}}(x), \mu_{\text{not young}}(x)$



A: Sigmoidal MF
membership grade
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95
X
Very young
Young
Middle aged
Old
Very old

Definition 6: Concentration and dilation of linguistic values

Let A be a linguistic value characterized by a fuzzy set with membership function $\mu(.)$. Then A^k is interpreted as a modified version of the original linguistic value expressed as

$$A^k = \int_X \left(\mu_A(x)\right)^k / x$$

In particular, concentration operation is defined as

$$\mathrm{CON}(A) = A^2$$

dilation operation is defined as

$$\mathrm{DIL}(A) = A^{0.5}$$

- CON(A) **is resulted from applying hedges very**
  - 非常老
- DIL(A) **is resulted from applying hedges more or less**
  - 有些老 **(**多少有些**)**
- **More or less Old, Old, very Old, very very Old, extremely Old**
  - $A^{0.5}, A, A^2, A^4, A^8$

- **Negation operator NOT and connectives AND and OR are used to connect primary linguistic terms to create composite linguistic values (terms)**
  - **Not very young and not very old**
  - **Young but not too young**
- **Linguistic values whose meanings are defined by $M(.)$**
  - **Linguistic value →Fuzzy Set**
  - **$M(.)$→Membership function**

- CON(.), DIL(.), $\cap$, $\cup$, $\neg$ **are used to compose the membership functions of composite linguistic terms**
- **More or less (多少有些)old →**
  DIL(old) $=(\mu_{\text{old}} (age))^{0.5}$
- **Not young and not old →**
  $\neg$young $\cap$ $\neg$old $= (1- \mu_{\text{young}} (age)) \wedge (1- \mu_{\text{old}} (age))$
- **Young but not too young →**
  young $\cap$ $\neg$CON(young)
  $= (\mu_{\text{young}} (age)) \wedge (1- (\mu_{\text{young}} (age))^{2})$
- Extremely = very very very
  Extremely old →
  CON(CON(CON(old))) $= (\mu_{\text{old}} (age))^{8}$

Definition 7: Contrast Intensification

The operation of contrast intensification on a linguistic value A is defined by

$$\mathrm{INT}(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5, \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1.0 \end{cases}$$

$$\mathrm{CON}(A) = A^2 \qquad \mathrm{DIL}(A) = A^{0.5}$$

- **Intensifier the values of which are above 0.5**
- **Diminish those below this point**
- **Reduce the fuzziness of linguistic value *A***

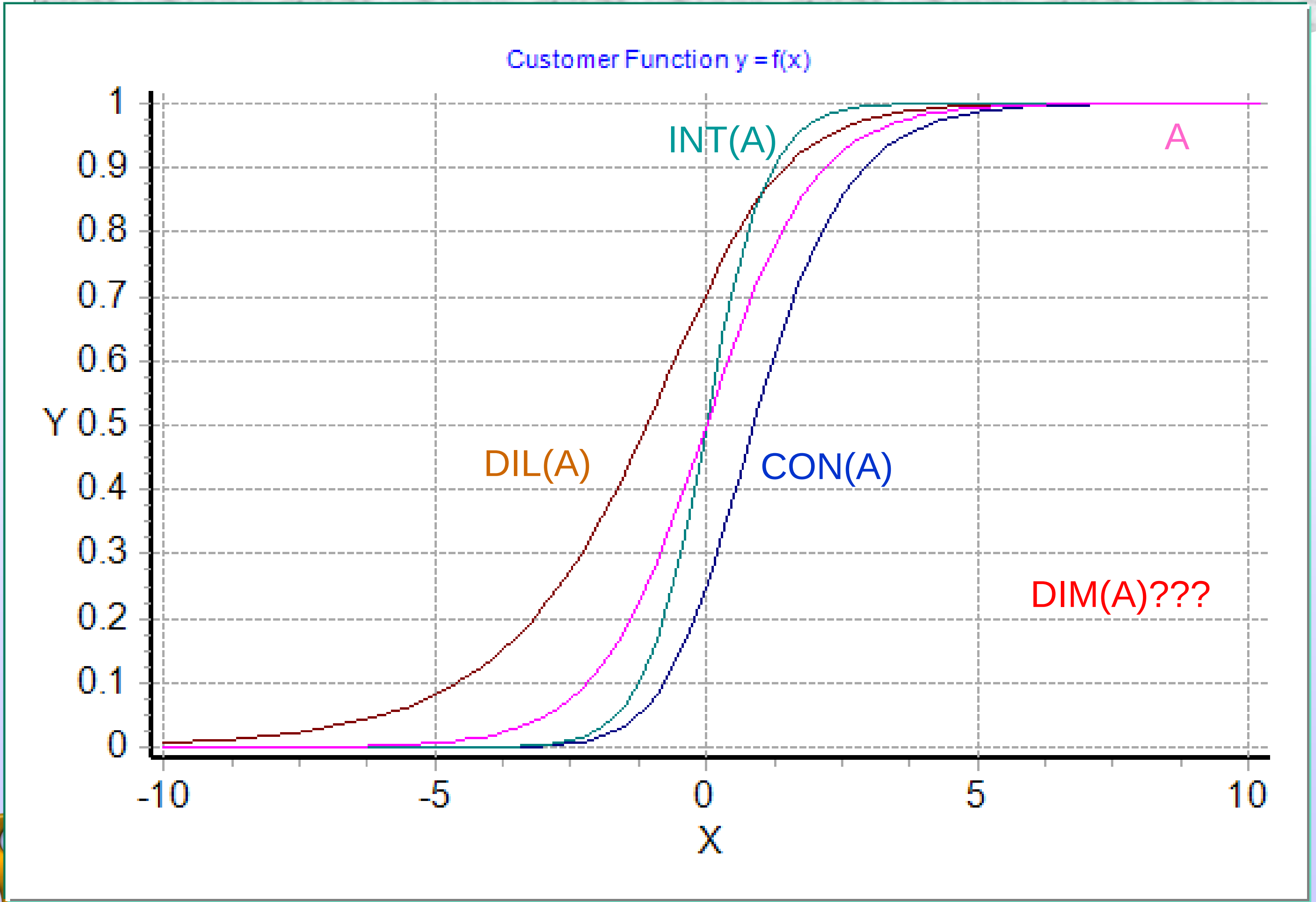
The operation of contrast diminisher on a linguistic value A is defined by

$$DIM(A) = \begin{cases} ??, & for\ 0 \le \mu_A(x) \le 0.5, \\ ??, & for\ 0.5 \le \mu_A(x) \le 1.0 \end{cases},$$

$$such\ that\ DIM(INT(A)) = A$$

$$\mathrm{CON}(A) = A^2$$

$$\mathrm{DIL}(A) = A^{0.5}$$


Customer Function y = f(x)
INT(A)
A
DIL(A)
CON(A)
DIM(A)???
Y
X
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-5
0
5
10

Definition 8: Orthogonality

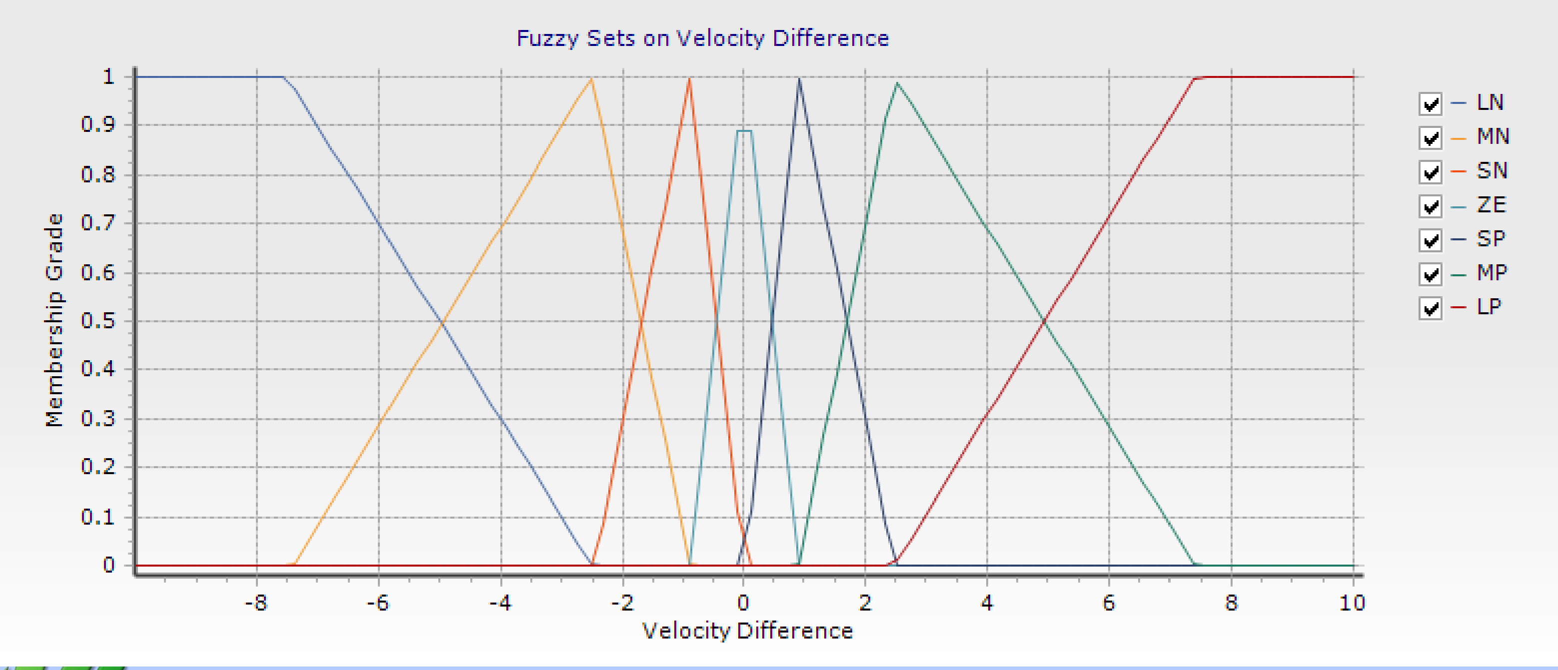
A term set $T = \{t_1, t_2, \ldots, t_n\}$ of a linguistic variable x on the universe X is orthogonal it fulfills the following property:

$$\sum_{i=1}^{n} \mu_{t_i}(x) = 1.0, \ \forall x \in X$$

where t_i's are convex and normal fuzzy sets defined on X and these fuzzy sets make up the term set T.

A fuzzy set is intuitively reasonable:

- The terms in term set must be pertinent
- The terms must cover the universe properly
- The orthogonality must be hold by the membership functions of these terms



Fuzzy If-Then Rules

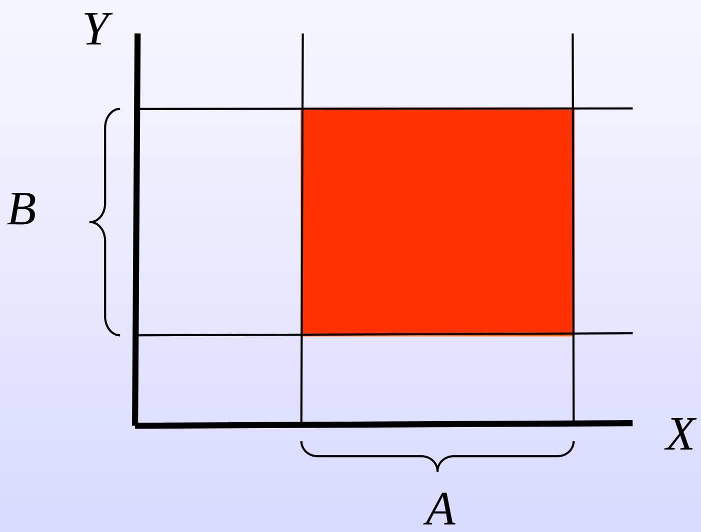
- Fuzzy if-then rule ←→ fuzzy rule ←→ fuzzy implication ←→ fuzzy conditional statement
- Fuzzy if-then rule format:
  - If x is A then y is B
  - A, B are linguistic values defined by fuzzy sets on universes of discourse X and Y, respectively
  - x is A → antecedent or premise
  - y is B → consequence or conclusion

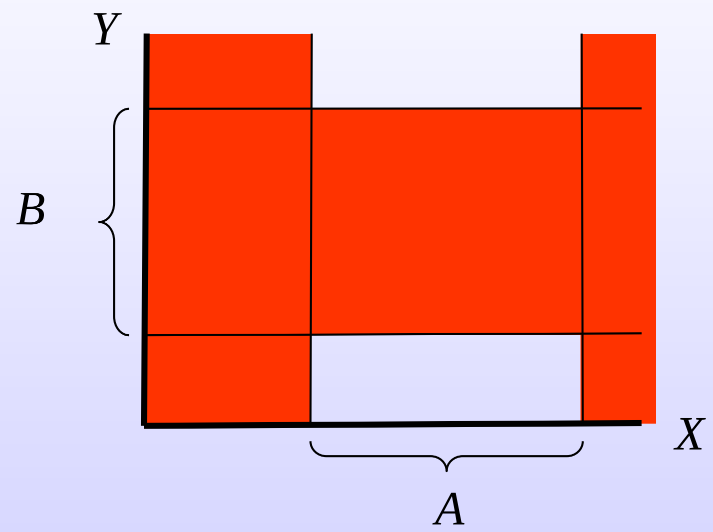
Fuzzy If-then Rules

- If pressure is high, then volume is small
- If the road is slippery, then driving is dangerous
- If a tomato is red, then it is ripe
- If the speed is high, then apply the brake a little

- If x is A then y is B
  - An expression describes a relation between two variables x and y
  - $A \rightarrow B$
- Fuzzy if-then rule
  - A fuzzy relation R between two linguistic variables x and y on the product space AxB
- An if-then fuzzy rule is expressed as a fuzzy relation characterized by a membership function
  - $A \rightarrow B$: A coupled with B
  - $A \rightarrow B$: A entails B

If-then rules:
(1) A **coupled with** B
(2) A **entails** B



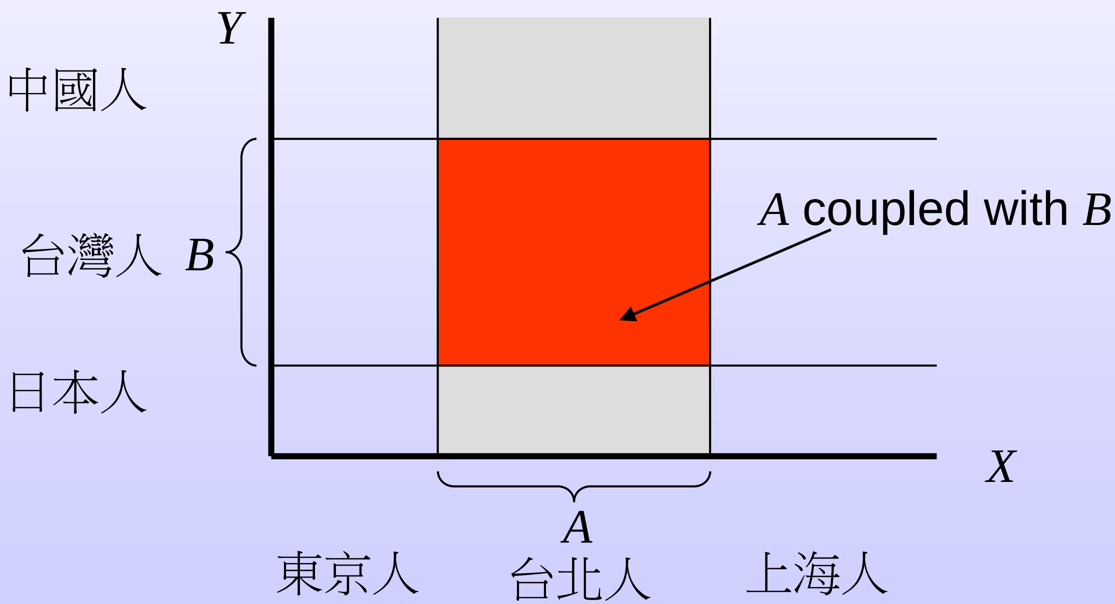


(1) A **coupled with** B **:不care非**A
非 A 部分另有命題(另有法則)

(2) A **entails** B 接受非 A 部分：
非 A 部分沒有命題無其他法則可推論

- A **coupled with** B
  - **T-norm operator**

$$A \to B = R = \int_{X\times Y} \mu_A(x) \tilde{*} \mu_B(x) / (x, y)$$



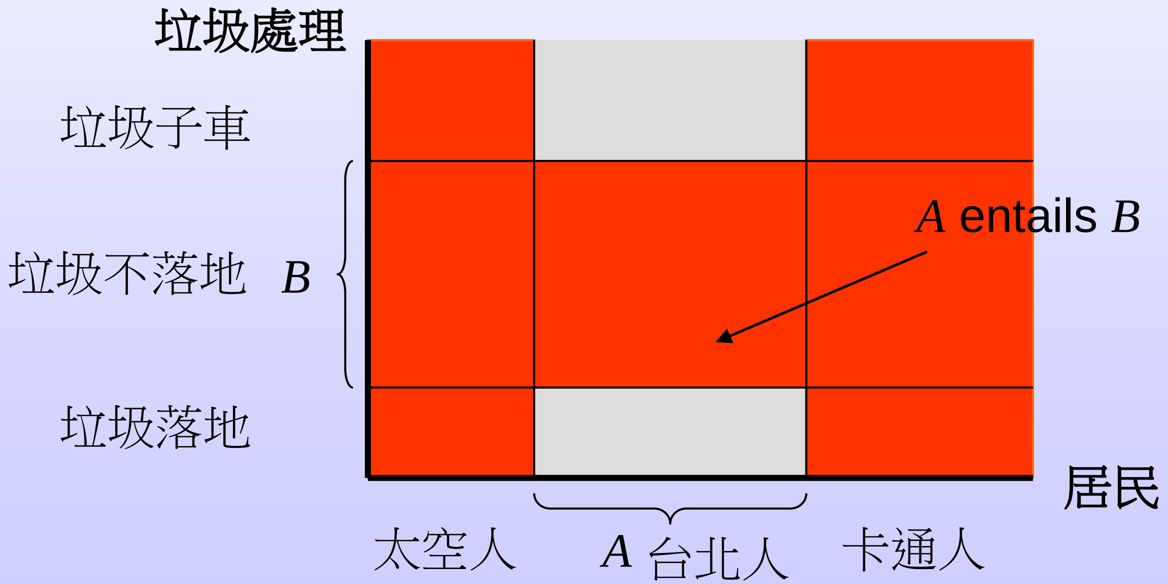
(1) A **coupled with** B 排斥 非 A 部分：上海人東京人是不是台灣人會另有命題，譬如：若是上海人，則是中國人。若是東京人則是日本人。

A entails *B*

- **Is not an S-norm operator for A and B (should be S-norm on Not A and B)**

$$A \to B = R = \int_{X \times Y} \mu_A(x) \tilde{\oplus} \mu_B(y) / (x, y), \tilde{\oplus}\text{: entail relation operator}$$

$$A \to B = R = \int_{X \times Y} (1 - \mu_A(x)) \tilde{+} \mu_B(y) / (x, y), \tilde{+}\text{: S-norm operator}$$

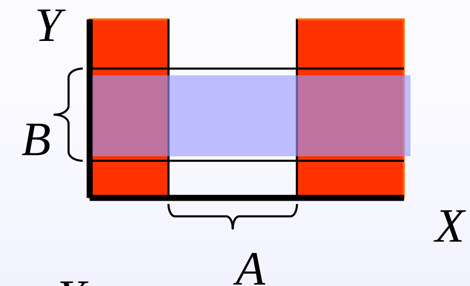


(2) *A* **entails** *B* 接受非 *A* 部分：太空人、卡通人與垃圾落不落地沒有命題。不會有若是卡通人則垃圾該如何的議題。

A entails B

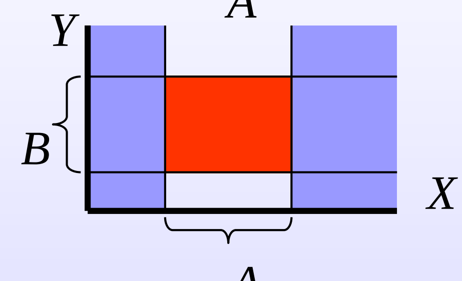
$$R = A \to B = \neg(\mathrm{c})A \cup (\mathrm{c})B$$

- **Material implication (實質蘊涵)**



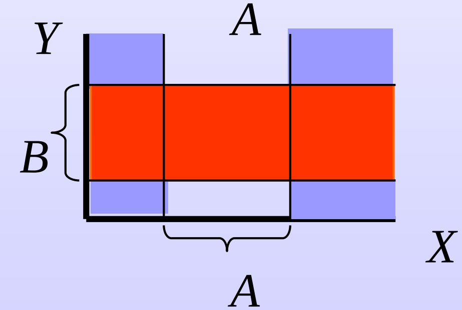
$$R = A \to B = \neg(\mathrm{c})A \cup \big((\mathrm{c})A \cap (\mathrm{c})B\big)$$

- **Propositional Calculus (命題演算) :**



$$R = A \to B = \big(\neg(\mathrm{c})A \cap \neg(\mathrm{c})B\big) \cup (\mathrm{c})B$$

- **Extended Propositional Calculus:**



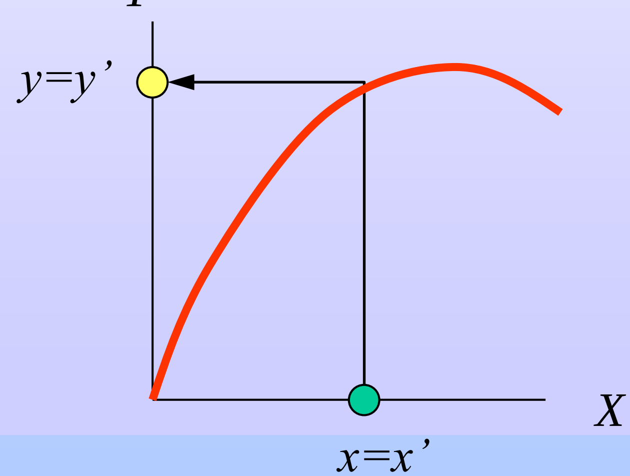
Fuzzy Implication

- **Fuzzy implication function (蘊涵函數)**
  - **Transform the membership grades of x in A and y in *B* into (x, y) in $A \rightarrow B$**
  - **Fuzzy rule function**
  - **Fuzzy set membership function**

$$\mu_R(x,y) = f(\mu_A(x), \mu_B(y)) = f(a,b)$$

$F(x,y)=0$

$(x\text{-}4)\ (x\text{-}4)\text{-}16+y=0$



Fuzzy Implication Function for A coupled with B

- $A \rightarrow B$, A **coupled with** B
- **T-norm operators**
  - **Examples: the four T-norm**

$$R_{\min} = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) / (x, y);\ f_{\min}(a, b) = a \wedge b$$

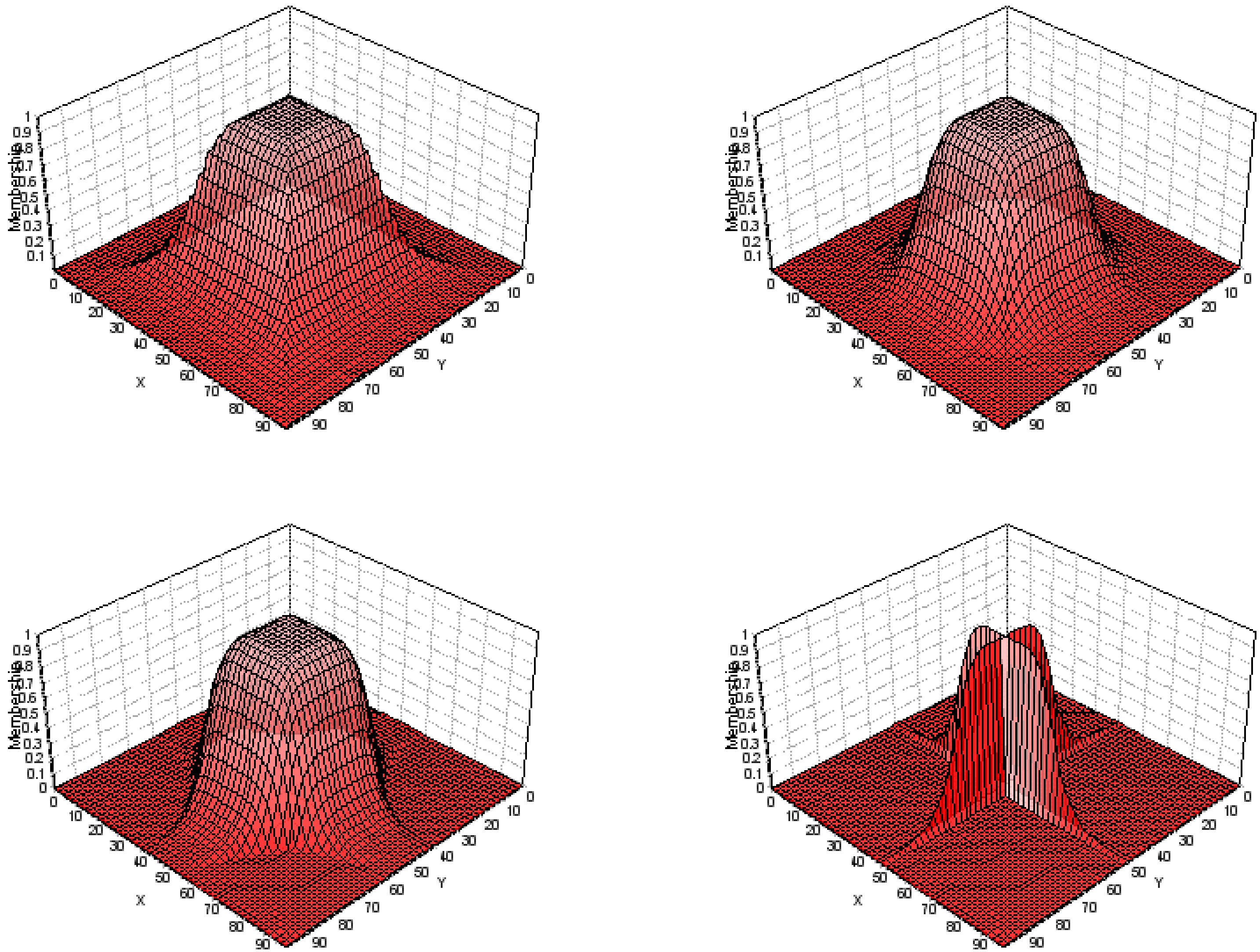
$$R_{ap} = A \times B = \int_{X \times Y} \mu_A(x) \mu_B(y) / (x, y);\ f_{ap}(a, b) = ab$$

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \otimes \mu_B(y) / (x, y);\ f_{bp}(a, b)$$

$$= \int_{X \times Y} 0 \vee \left(\mu_A(x) + \mu_B(y) - 1\right) / (x, y);\ f_{bp}(a, b) = 0 \vee (a + b - 1)$$

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \overset{\wedge}{\cdot} \mu_B(y) / (x, y);$$

$$f_{dp}(a, b) = a \overset{\wedge}{\cdot} b = \begin{cases} a, if\ \ b = 1 \\ b, if\ \ a = 1 \\ 0, otherwise \end{cases}$$


Membership
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
X
0 10 20 30 40 50 60 70 80 90
Y
0 10 20 30 40 50 60 70 80 90

Fuzzy Implication Function for A entails with B

- $A \rightarrow B$, A entails with B
- Not A and B Perform S-norm (T-conorm) operators
  - Examples: the four S-norm

$$R_{\max} = \neg(\mathrm{c})A \cup (\mathrm{c})B = \int_{X\times Y} \mu_{c(A)}(x) \vee \mu_B(y)/(x,y);\ f_{\max}(1-a,b) = (1-a)\vee b = \max(1-a,b)$$

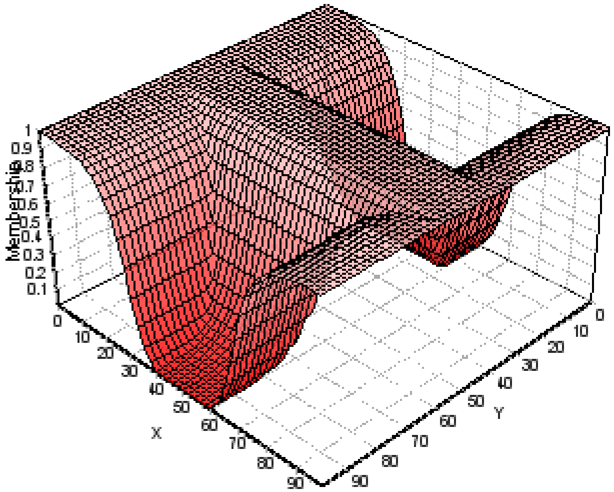
$$R_{as} = \neg(\mathrm{c})A \cup (\mathrm{c})B = \int_{X\times Y} \mu_{c(A)}(x) + \mu_B(y)/(x,y);\ f_{as}(1-a,b) = (1-a)+b-(1-a)b = 1-a+ab$$

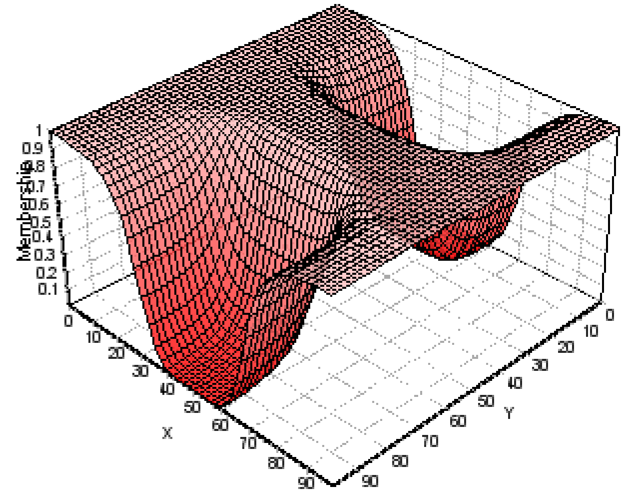
$$R_{bs} = \neg(\mathrm{c})A \cup (\mathrm{c})B = \int_{X\times Y} \mu_{c(A)}(x) \oplus \mu_B(y)/(x,y);\ f_{bs}(1-a,b) =$$

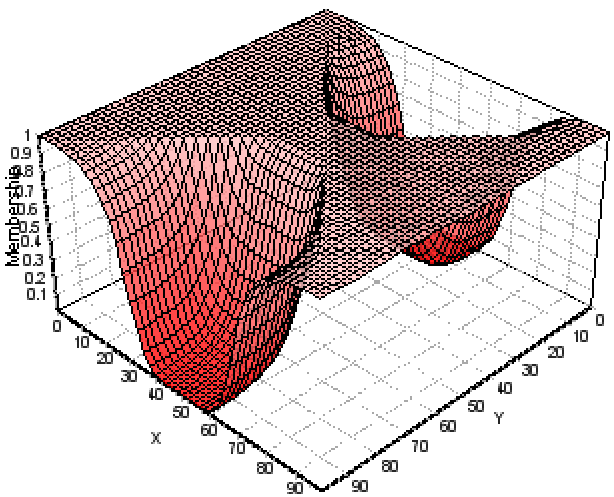
$$= \int_{X\times Y} 1 \wedge \left(\mu_{c(A)}(x) + \mu_B(y)\right)/(x,y);\ f_{bs}(a,b) = 1 \wedge \left((1-a)+b\right)$$

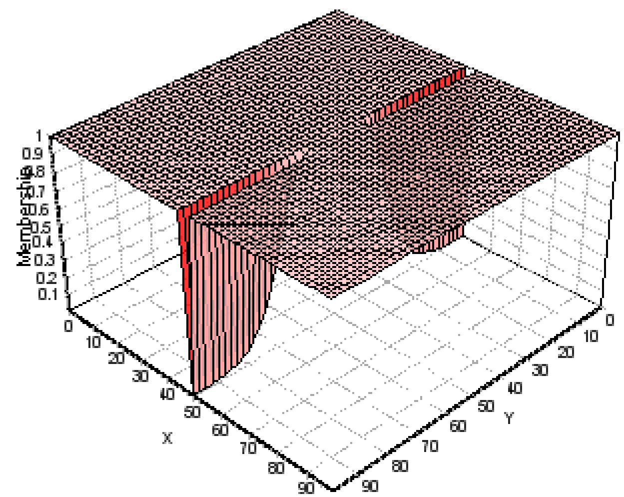
$$R_{ds} = \neg(\mathrm{c})A \cup (\mathrm{c})B = \int_{X\times Y} \mu_{c(A)}(x) \hat{+} \mu_B(y)/(x,y);$$

$$f_{ds}(1-a,b) = (1-a)\hat{+}b = \begin{cases} 1-a, if\ \ b=0 \\ b, if\ 1-a=0 \\ 1, otherwise \end{cases}$$


Membership
X
Y


Membership
X
Y


Membership
X
Y


Membership
X
Y

Another Fuzzy Implication Functions

- Zadeh's arithmetic rule
- Zadeh's max-min rule
- Booean fuzzy implication using max
- Goguen's fuzzy implication

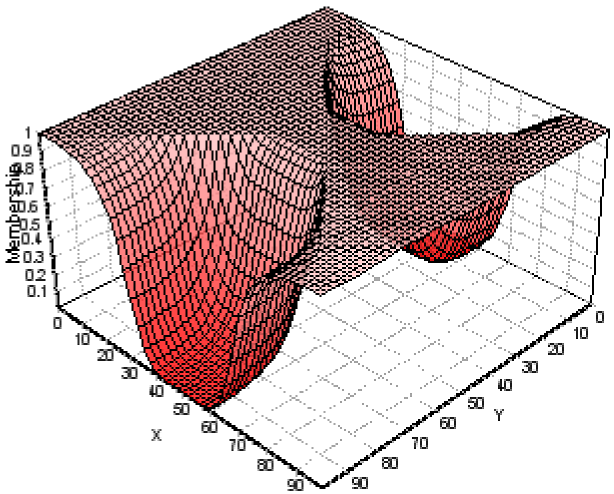
$$R_z = \neg(c)A \cup (c)B = \int_{X\times Y} 1 \wedge \left(1-\mu_A(x)+\mu_B(y)\right)/(x,y) = R_{bs};\ f_z(a,b) = 1\wedge(1-a+b)$$

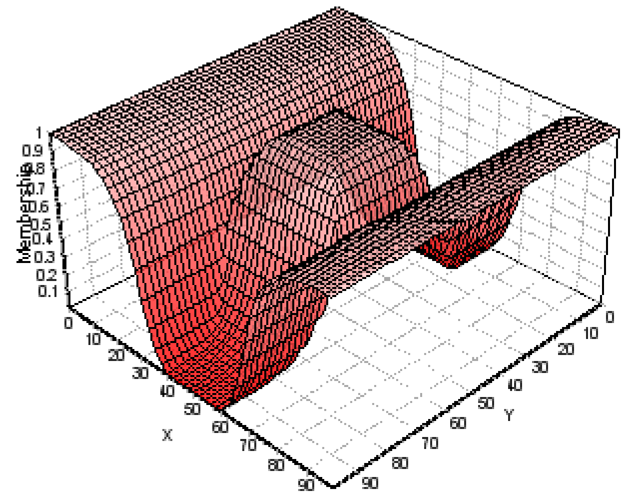
$$R_{mm} = \neg(c)A \cup (c)B = \int_{X\times Y} \left(1-\mu_A(x)\right) \vee \left(\mu_A(x)\wedge\mu_B(y)\right)/(x,y);\ f_{mm}(a,b) = (1-a)\vee(a\wedge b)$$

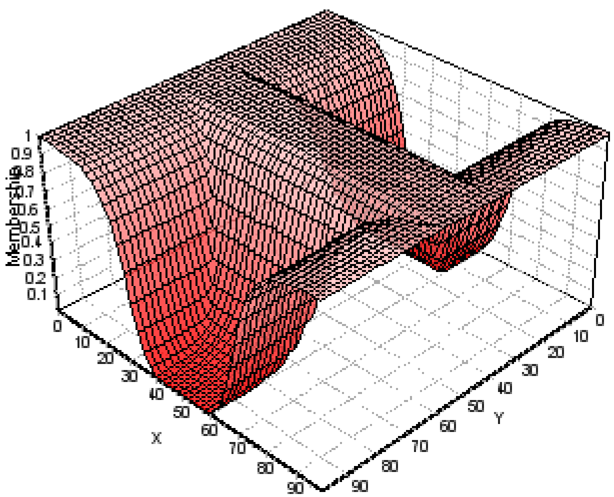
$$R_s = \neg(c)A \cup (c)B = \int_{X\times Y} \left(1-\mu_A(x)\right) \vee \mu_B(y)/(x,y) = R_{\max};\ f_s(a,b) = (1-a)\vee b$$

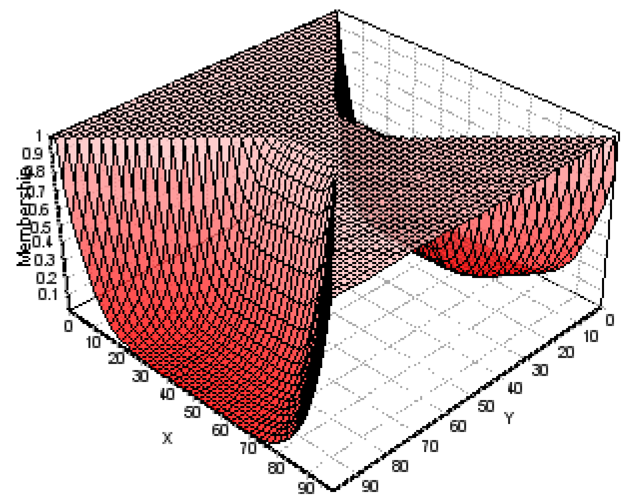
$$R_\Delta = \neg(c)A \cup (c)B = \int_{X\times Y} \mu_A(x) \tilde{<} \mu_B(y)/(x,y);$$

$$f_\Delta(a,b) = a \tilde{<} b = \begin{cases} 1, if\ \ a \le b \\ b/a, if\ \ a > b \end{cases}$$


Membership
X
Y

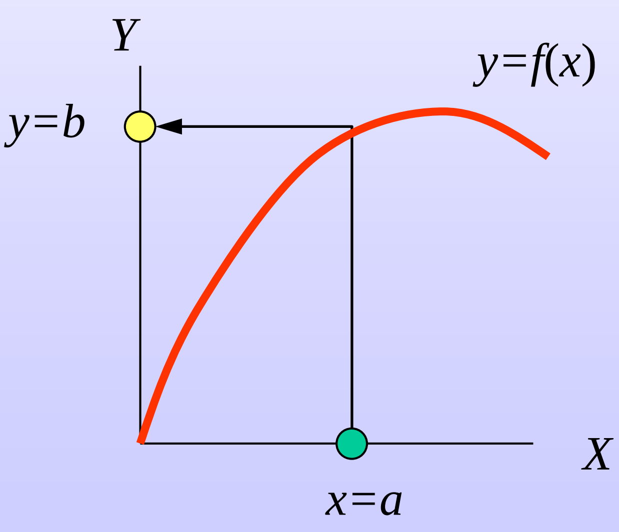

Membership
X
Y

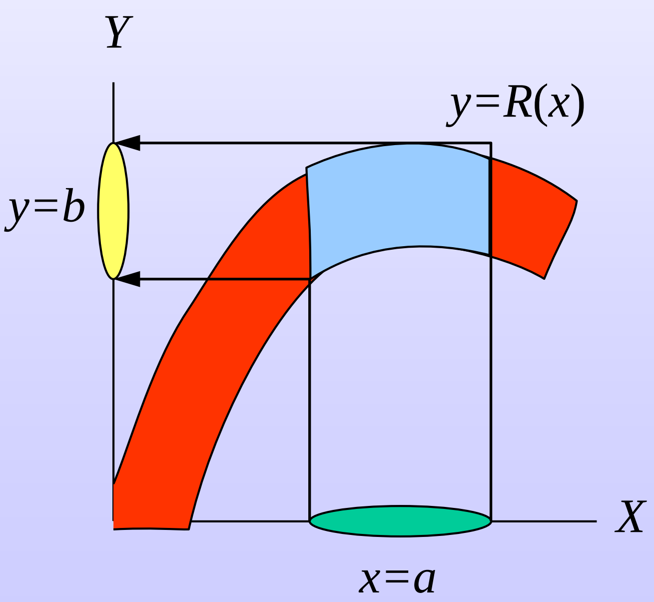

Membership
X
Y


Membership
X
Y

Fuzzy Reasoning

- Fuzzy reasoning ←→ approximate reasoning
  - Inference procedure derives conclusions from a set of fuzzy if-then rules and known facts





Fuzzy Reasoning

- Assume R is a fuzzy rule A→B, defined on XxY
- R: fuzzy relation on XxY; 2-D membership function; 2-D fuzzy set
- A': a fuzzy set of X → fact
- B': the resulting fuzzy set
- Form a cylindrical extension of A': $c(A')$
- Intersect $c(A')$ and R: $c(A') \cap R$
- Project the intersection to Y to form a fuzzy set B'
  - *Usually non-normal fuzzy sets*

$$\mu_{C(A')}(x,y)=\mu_{A'}(x)$$

$$\mu_{C(A')\cap R}(x,y)=\min\left(\mu_{C(A')}(x,y),\mu_R(x,y)\right)$$

$$=\min\left(\mu_{A'}(x),\mu_R(x,y)\right)$$

$$\mu_{B'}(y)=\max_x \min\left(\mu_{A'}(x),\mu_R(x,y)\right)$$

$$=\vee_x\left(\mu_{A'}(x)\wedge\mu_R(x,y)\right)$$

$$B'=A'\circ R$$

Fuzzy Reasoning

- **Basic inference rule in two-valued logic**
  - **Modus ponens (若P則Q的離段邏輯推理)**
    - if P then Q
    - $P \rightarrow Q$ implication
  - **Rule: x is $A \rightarrow y$ is B**
  - **Fact: x is A**
  - **Conclusion: y is B**
- **Fuzzy inference (A' is close to A, B' is close to B)**
  - **Rule: if x is A then y is B**
  - **Fact: x is A'**
  - **Conclusion: y is B'**
- **Approximate reasoning←→Fuzzy reasoning←→Generalized modus ponens (GMP)**

Definition 9: Approximate reasoning (fuzzy reasoning)

Let A, A', and B be fuzzy sets of X, X, and Y, respectively. Assume that the fuzzy implication $A \rightarrow B$ is expressed as a fuzzy relation R on X x Y. Then the fuzzy set B' induced by x is A' and the fuzzy rule if x is A then y is B is defined by

$$\mu_{B'}(y) = \max_x \min\left(\mu_{A'}(x), \mu_R(x,y)\right) = \vee_x \left(\mu_{A'}(x) \wedge \mu_R(x,y)\right)$$

or equivalently,

$$B' = A' \circ R = A' \circ \left(A \rightarrow B\right)$$

Single Rule with Single Antecedent

$$\text{Rule: if } x \text{ is } A \text{ then } y \text{ is } B$$

$$\text{Fact: } x \text{ is } A'$$

$$\text{Conclusion: } y \text{ is } B'$$

$$B' = A' \circ R = A' \circ (A \to B)$$

$$\mu_{B'}(y) = \max_x \min\left(\mu_{A'}(x), \mu_R(x,y)\right)$$

$$= \vee_x \left(\mu_{A'}(x) \wedge \mu_R(x,y)\right)$$

$$= \vee_x \left(\mu_{A'}(x) \wedge \left(\mu_A(x) \wedge \mu_B(y)\right)\right)$$

$$= \vee_x \left(\left(\mu_{A'}(x) \wedge \mu_A(x)\right) \wedge \mu_B(y)\right)$$

$$= \vee_x \left(\mu_{A'}(x) \wedge \mu_A(x)\right) \wedge \mu_B(y)$$

$$= w \wedge \mu_B(y)$$

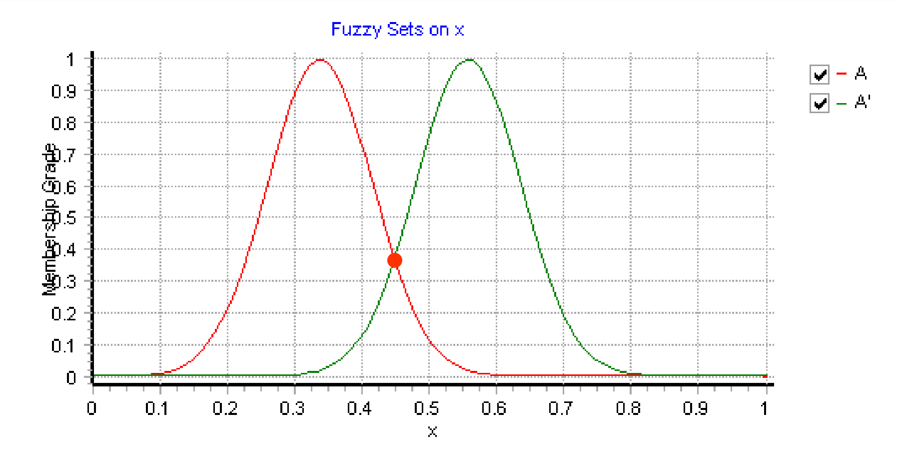
Find the degree of match w as the maximum of then MF of B is clipped by w

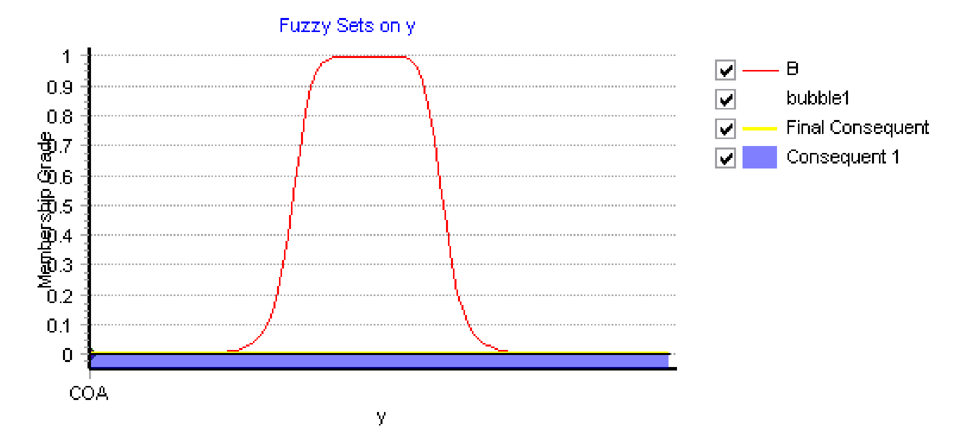
w : a measure of degree of belief for the antecedent part of the rule

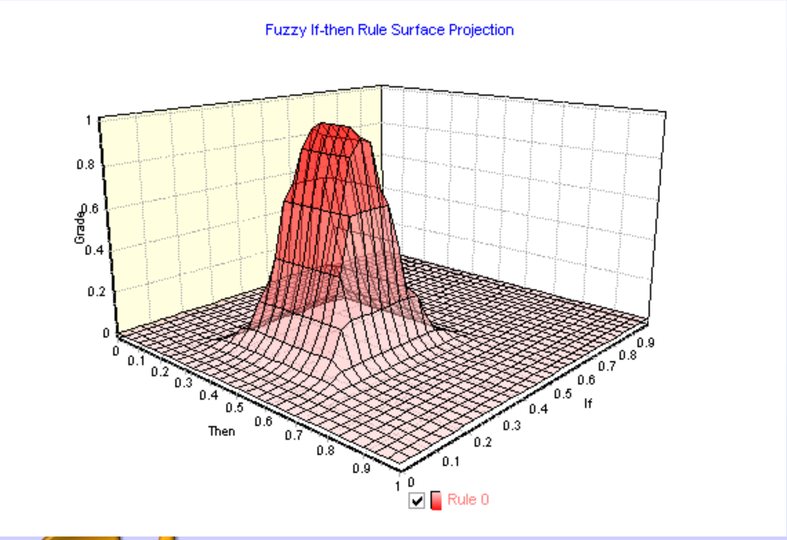
w propagates by the if-then rule

The resulting degree of belief of MF for the consequent part B' should be no greater than w

Single Rule with Single Antecedent

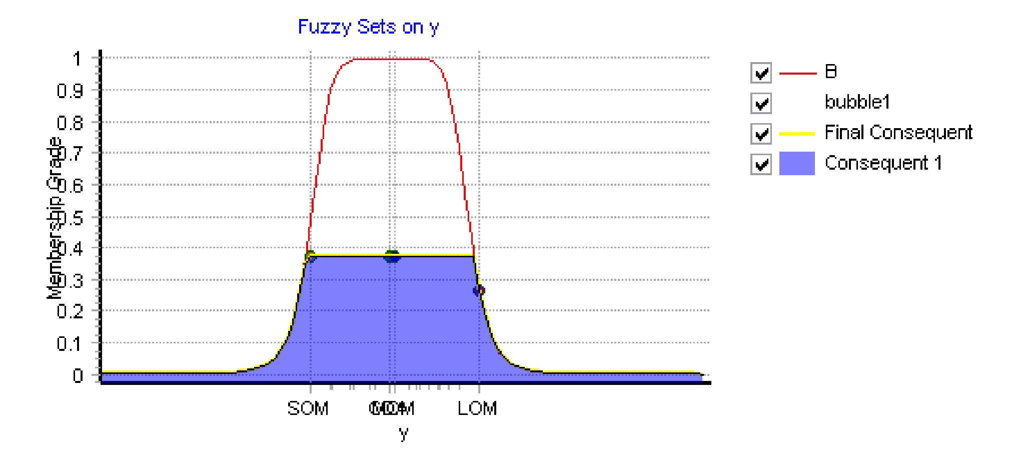






Rule: If x is A then y is B

Now x is A', y is?



Single Rule with Multiple Antecedents

Rule: if x is A and y is B then z is C

Fact: x is A' and y is B'

Conclusion: z is C'

$$A \times B \rightarrow C$$

$$R_m(A,B,C) = (A \times B) \times C$$

$$= \int_{X \times Y \times Z} \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z) / (x,y,z)$$

$$C' = (A' \times B') \circ R_m(A,B,C) = (A' \times B') \circ (A \times B \rightarrow C)$$

$$\mu_{C'}(z) = \left(\vee_{x,y}\left(\mu_{A'}(x) \wedge \mu_{B'}(y)\right)\right) \wedge \left(\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)\right)$$

$$= \vee_{x,y}\left(\left(\mu_{A'}(x) \wedge \mu_{B'}(y)\right) \wedge \left(\mu_A(x) \wedge \mu_B(y)\right)\right) \wedge \mu_C(z)$$

$$= \vee_{x,y}\left(\left(\mu_{A'}(x) \wedge \mu_A(x)\right) \wedge \left(\mu_{B'}(y) \wedge \mu_B(y)\right)\right) \wedge \mu_C(z)$$

$$= \left(\vee_x\left(\mu_{A'}(x) \wedge \mu_A(x)\right)\right) \wedge \left(\vee_y\left(\mu_{B'}(y) \wedge \mu_B(y)\right)\right) \wedge \mu_C(z)$$

$$= (w_1 \wedge w_2) \wedge \mu_C(z)$$

- w_1: **degree of compatibility between** A **and** A'
- w_2: **degree of compatibility between** B **and** B'
- $w_1 \wedge w_2$:
  - **Firing strength of the fuzzy rule**
  - **Degree of fulfillment of the fuzzy rule**

Theorem 1

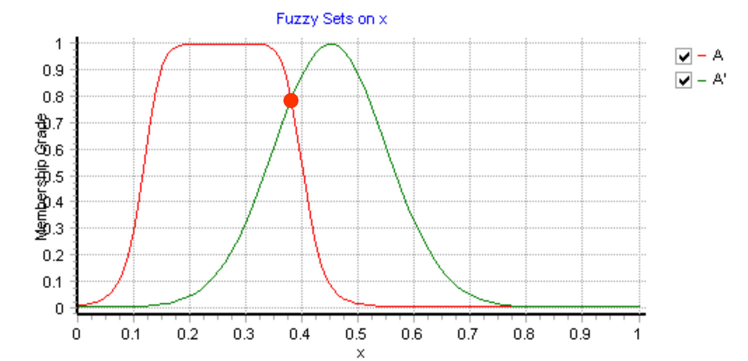
Theorem 1: Decomposition method for calculating C'

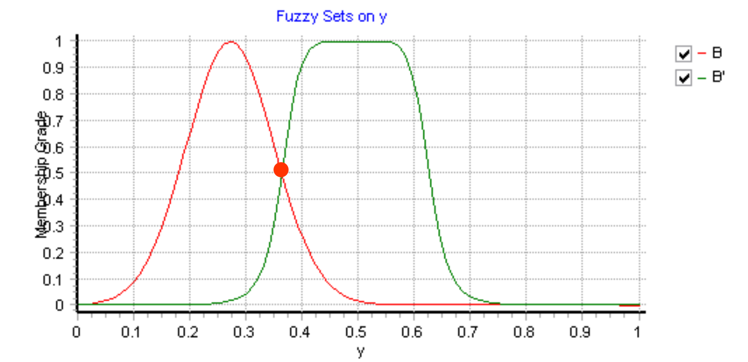
$$C' = \left(A' \times B'\right) \circ \left(A \times B \to C\right)$$
$$= \left(A' \circ \left(A \to C\right)\right) \cap \left(B' \circ \left(B \to C\right)\right)$$

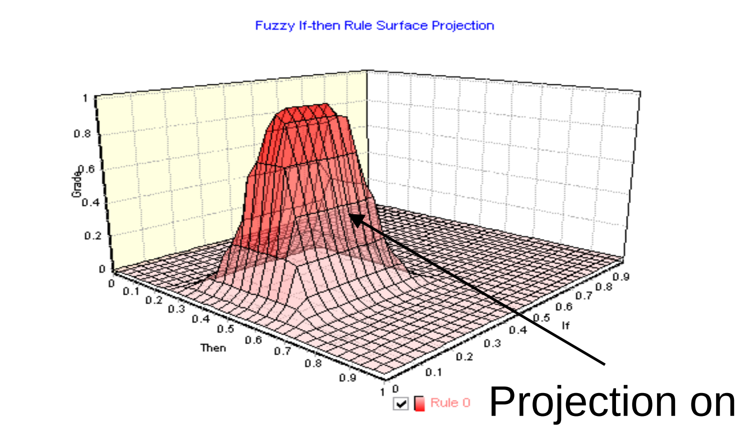
$$\mu_{C'}(z)=\vee_{x,y}\left[\left(\mu_{A'}(x)\wedge\mu_{B'}(y)\right)\wedge\left(\mu_A(x)\wedge\mu_B(y)\wedge\mu_C(z)\right)\right]$$
$$=\left\{\vee_{x,y}\left[\left(\mu_{A'}(x)\wedge\mu_A(x)\right)\wedge\left(\mu_{B'}(y)\wedge\mu_B(y)\right)\right]\right\}\wedge\mu_C(z)$$
$$=\vee_x\left\{\vee_y\left[\left(\mu_{A'}(x)\wedge\mu_A(x)\right)\wedge\left(\mu_{B'}(y)\wedge\mu_B(y)\right)\right]\right\}\wedge\mu_C(z)$$
$$=\vee_x\left\{\left(\mu_{A'}(x)\wedge\mu_A(x)\right)\wedge\left[\vee_y\left(\mu_{B'}(y)\wedge\mu_B(y)\right)\right]\right\}\wedge\mu_C(z)$$
$$=\left\{\vee_x\left[\left(\mu_{A'}(x)\wedge\mu_A(x)\right)\right]\wedge\left[\vee_y\left(\mu_{B'}(y)\wedge\mu_B(y)\right)\right]\right\}\wedge\mu_C(z)$$
$$=\left\{\vee_x\left[\mu_{A'}(x)\wedge\left(\mu_A(x)\wedge\mu_C(z)\right)\right]\right\}\wedge\left\{\vee_y\left[\mu_{B'}(y)\wedge\left(\mu_B(y)\wedge\mu_C(z)\right)\right]\right\}$$
$$=\left\{\vee_x\left[\mu_{A'}(x)\wedge\mu_{C_1}(x,z)\right]\right\}\wedge\left\{\vee_y\left[\mu_{B'}(y)\wedge\mu_{C_2}(y,z)\right]\right\}$$
$$=\mu_{C_1'}(z)\wedge\mu_{C_2'}(z)$$
$$=\mu_{A'\circ(A\to C)}(z)\wedge\mu_{B'\circ(B\to C)}(z)$$

$$C_1'=A'\circ\left(A\to C\right),C_2'=B'\circ\left(B\to C\right)$$

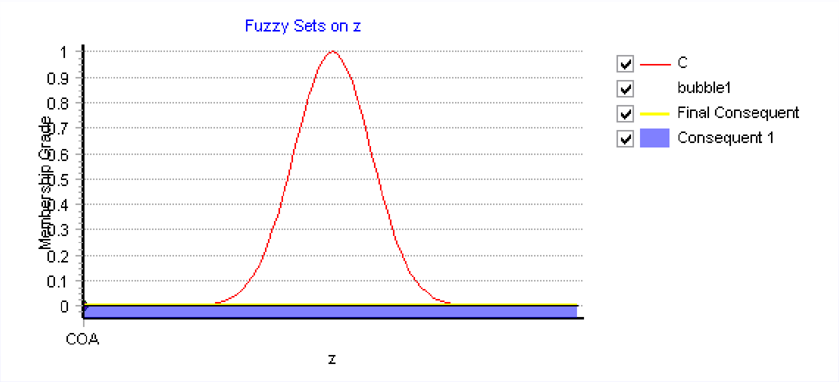
Single Rule with Multiple Antecedents





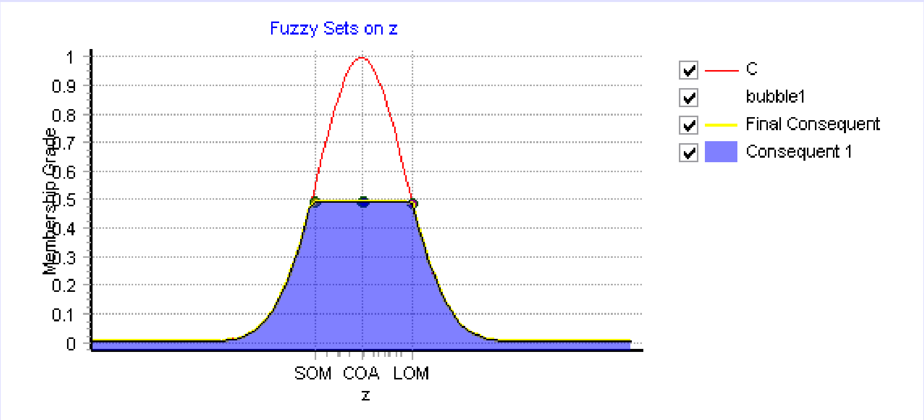


Projection on x-z only



Rule: If x is A and y is B, then z is C

Now x is A' and y is B', z is ?



Multiple Rules with Multiple Antecedents

Rule1: if x is A_1 and y is B_1 then z is C_1

Rule2: if x is A_2 and y is B_2 then z is C_2

Fact: x is A' and y is B'

Conclusion: z is C'

$$R_1 = A_1 \times B_1 \to C_1, R_2 = A_2 \times B_2 \to C_2$$

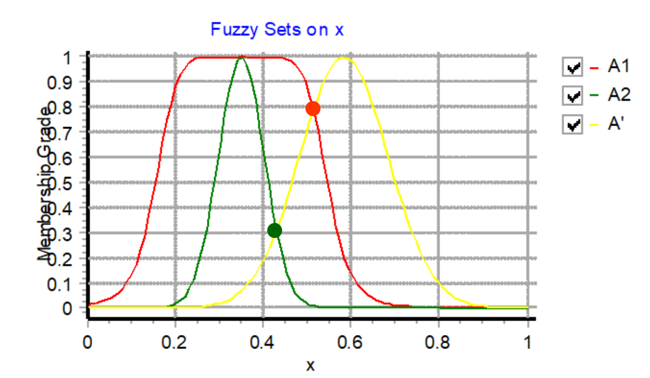
$$C' = \left(A' \times B'\right) \circ \left(R_1 \bigcup R_2\right)$$

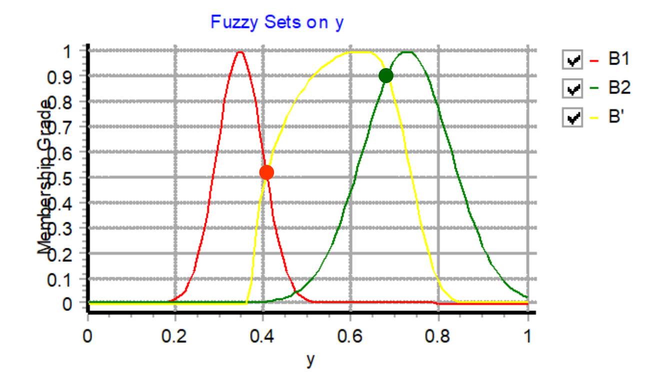
$$= \left(A' \times B'\right) \circ \left(R_1\right) \bigcup \left(A' \times B'\right) \circ \left(R_2\right)$$

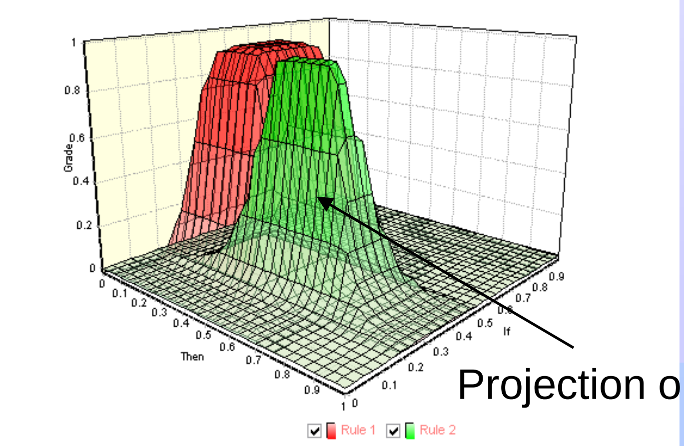
$$= \left(A' \times B'\right) \circ \left(A_1 \times B_1 \to C_1\right) \bigcup \left(A' \times B'\right) \circ \left(A_2 \times B_2 \to C_2\right)$$

$$= C_1' \bigcup C_2'$$

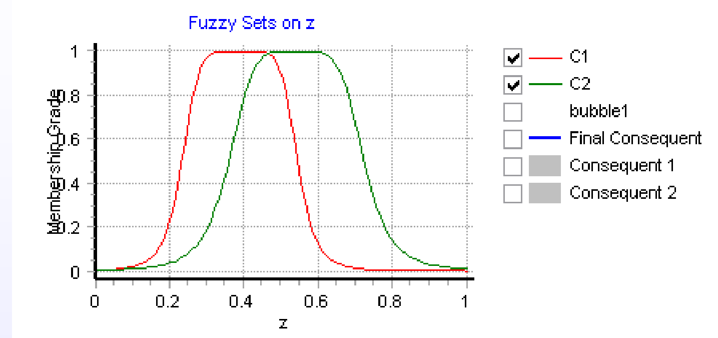
Multiple Rules with Multiple Antecedents



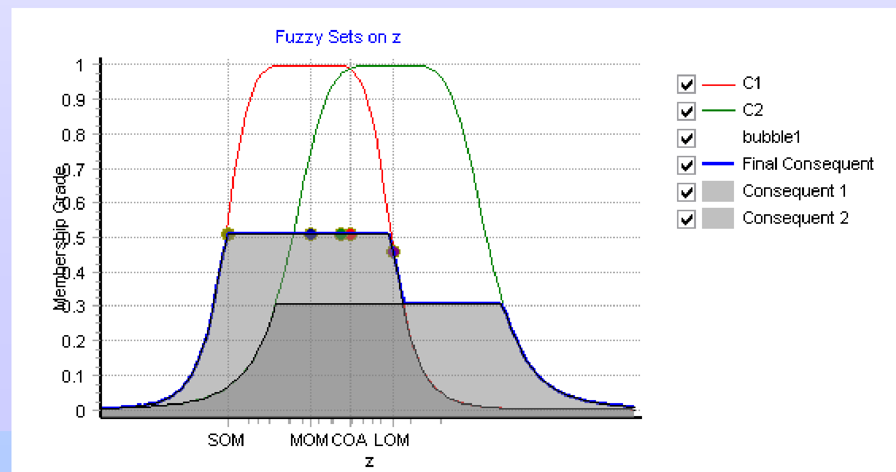




Projection on x-z only



Rule1: If x is A_1 and y is B_1, then z is C_1
Rule2: If x is A_2 and y is B_2, then z is C_2
Now x is A', y is B', z is ?



Fuzzy Reasoning

- **Degrees of compatibility**
  - Compare the facts with the antecedents to find the degree
- **Firing strength**
  - Using fuzzy AND OR operators to form a firing strength for each rule
- **Qualified consequent MFs**
  - Apply the firing strength to the consequent MF of a rule to generate a consequent MF
- **Overall output MF**
  - Aggregate all the qualified consequent MFs

Chapter 3 Exercises

- **Exercises:**
  1 (hint: separate the universe into 3 blocks),
  5 (typos: *new* → *young*; triangular MFs: →
  Gaussian MFs:; $e^{-\left(\frac{x}{20}\right)^2} \rightarrow e^{-\frac{1}{2}\left(\frac{x}{20}\right)^2}$
  and $e^{-\left(\frac{x-100}{30}\right)^2} \rightarrow e^{-\frac{1}{2}\left(\frac{x-100}{30}\right)^2}$ **),**
  6 (typos: *small* → *young*),
  8,
  9
  - **Whenever you are asked to use MATLAB to do something, write your own programming code to generate the required charts**

Chapter 3 Exercises

- Extra: Fully define two linguistic variables x, and y from your daily life that can separately serve as conditional and consequence terms. For each variable, define at least three values (fuzzy sets)
- Due in one week after the announcement (in hard copy)