Fuzzy Inference Systems

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Enthusiasm • Leadership • Innovation • Teamwork • Excellence

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Fuzzy Inference Systems

- **Fuzzy systems**
  - **Computing framework based on fuzzy set theory, fuzzy if-then rule, fuzzy reasoning**
  - **Applications**
    - Automatic control, data classification, decision analysis, expert systems, time series prediction, robotics, pattern recognition
- **Fuzzy system=fuzzy-rule-based systems=fuzzy expert system=fuzzy model=fuzzy associative memory=fuzzy logic controller**

- **Three components**
  - **Rule base**
    - A selection of fuzzy rules
  - **Database**
    - Define the membership functions used in fuzzy rules
  - **Reasoning mechanism**
    - Perform the inference procedure
- **Inputs**
  - **Fuzzy or crisp**
- **Outputs**
  - **Usually fuzzy sets**
  - **Sometimes crisp**
    - In control systems
    - Need a defuzzificaion mechanism
      - Extract a crisp value that best represents the fuzzy set

Fuzzy inference system

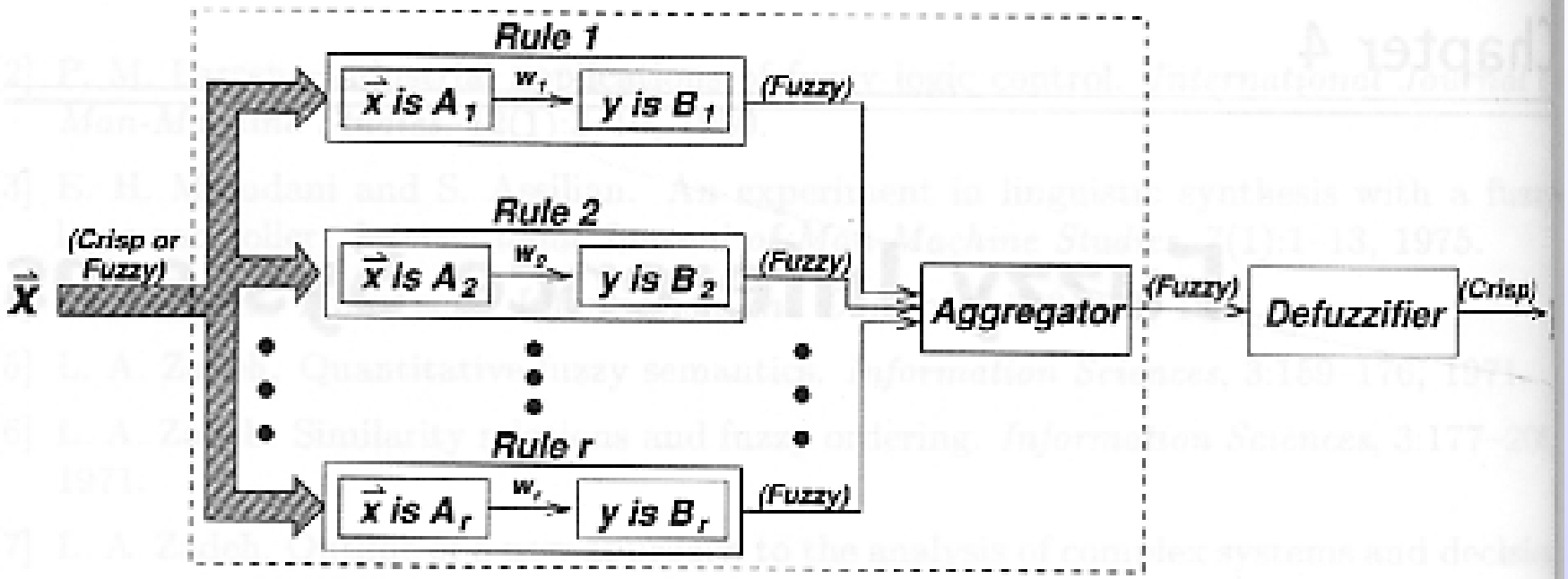


Figure 4.1. *Block diagram for a fuzzy inference system.*

- **A fuzzy inference system with crisp inputs and outputs**
  - **A nonlinear mapping**
    - A number of fuzzy if-then rules
    - Antecedents define fuzzy regions in the input space
    - Consequent specifies the output fuzzy regions

Topics in This Chapter

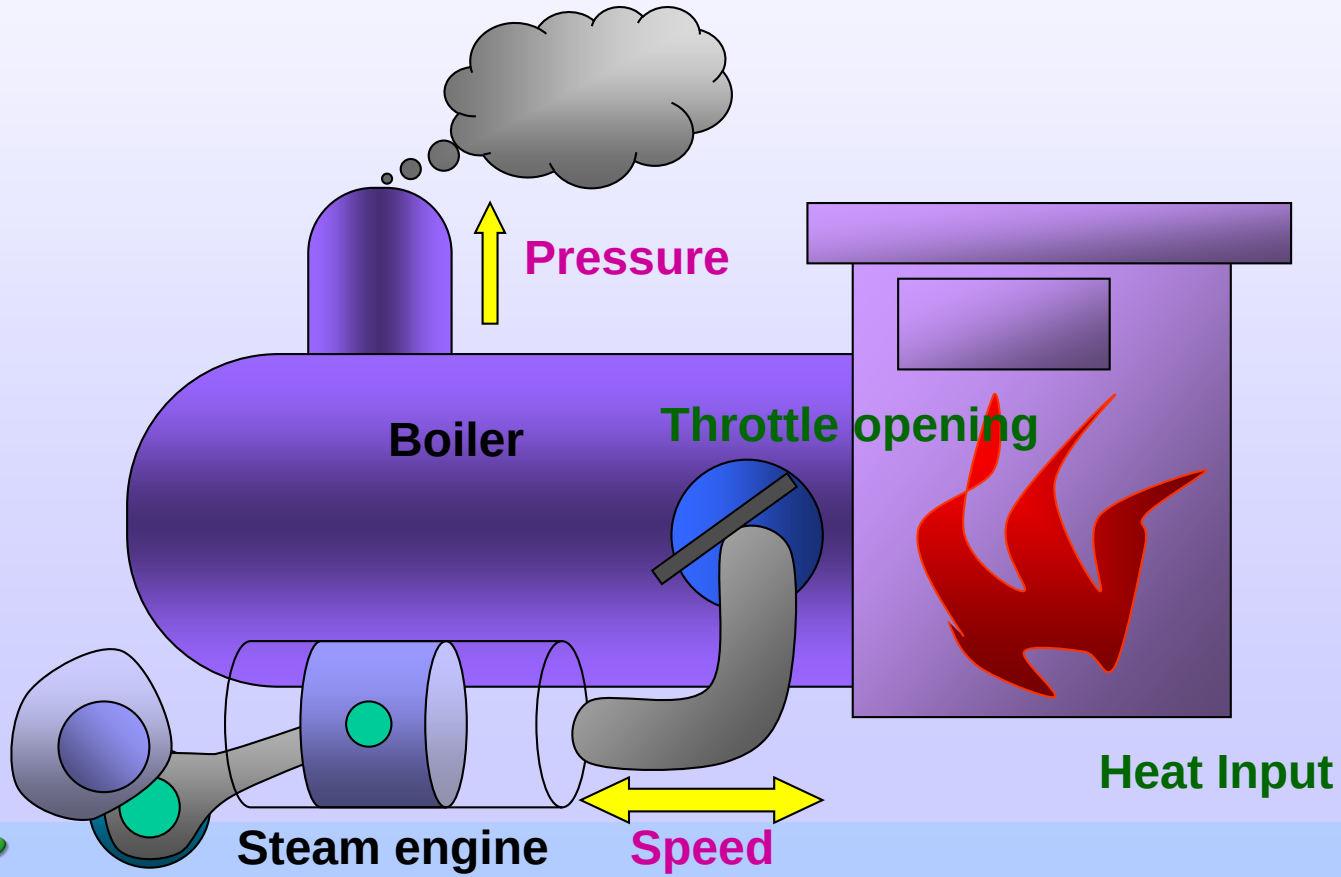
- 1. Three fuzzy inference systems (models)
  - Mamdani fuzzy models
  - Sugeno Fuzzy models
  - Tsukamoto fuzzy models
- 2. Input space partition
- 3. Features and problems in fuzzy modeling

Mamdani Fuzzy Inference System (1975)

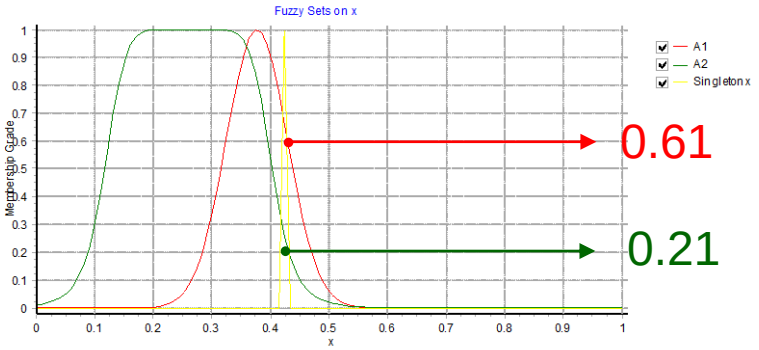
- **A steam engine and boiler controller**
  - **Two control factors (heat input, throttle open)**
    - Two crisp inputs: Pressure, Speed
    - Two crisp outputs: Heat input, Throttle open

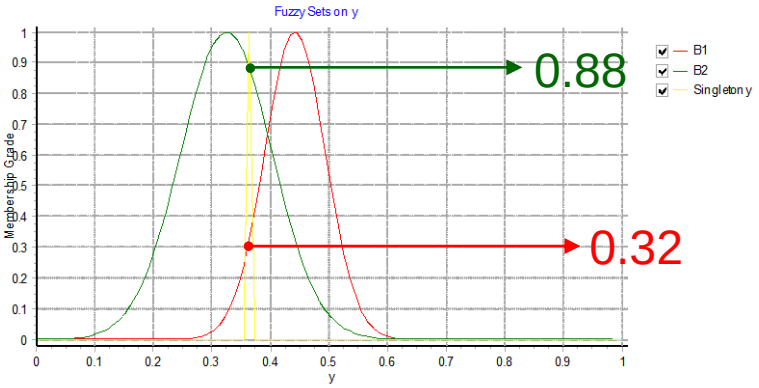
Mamdani Fuzzy Models

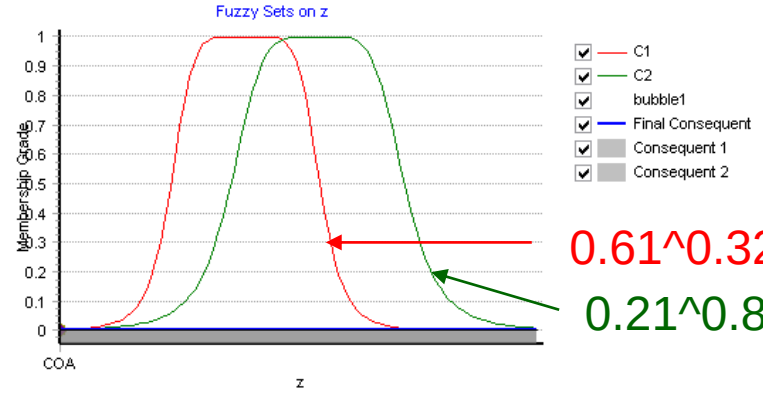
- **Mamdani fuzzy inference system (1975)**
  - **Two controllers of a steam engine**
    - Speed & Pressure → Heat input
    - Speed & Pressure → Throttle Opening

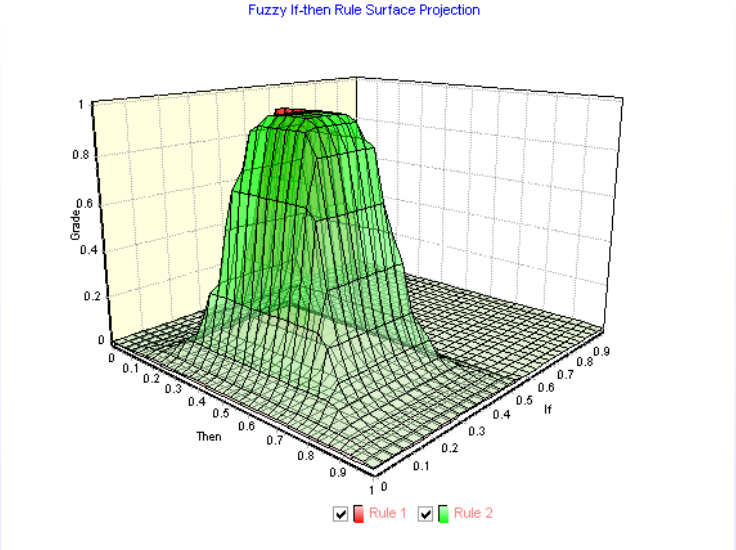


Cut Mamdani Consequents





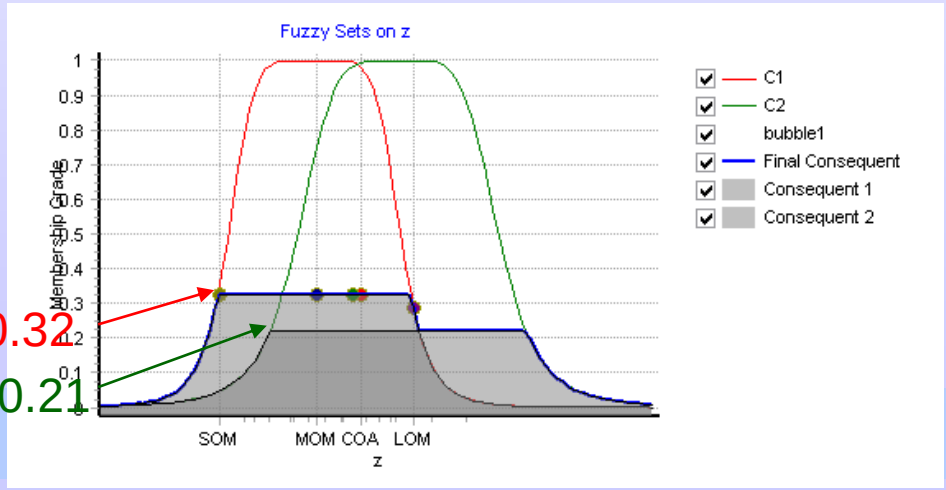




Rule 1: If x is A_1 and y is B_1 then z is C_1

Rule 2: If x is A_2 and y is B_2 then z is C_2

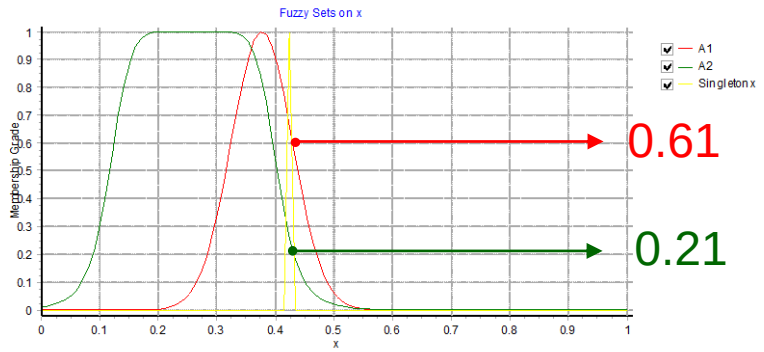
Now x is singleton , y is singleton , then ?

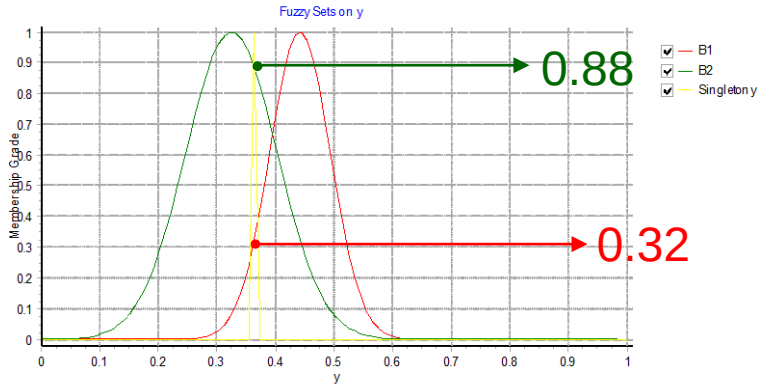


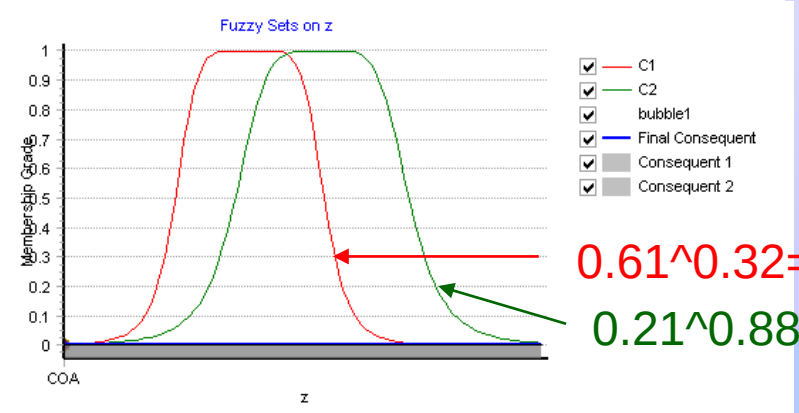
Modified Mamdani system

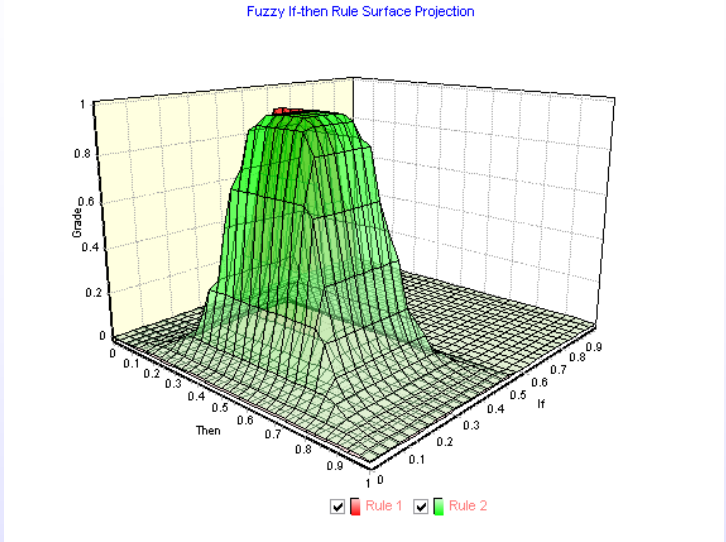
- **Operation Variations**
  - **T-norm: intersect, min, algebraic product**
  - **S-norm: union, max, algebraic sum**
  - **Composition: max-product**
    - (original max-min)
- **Scaled system (not cut)**
  - **The inferred outputs are scaled fuzzy sets scaled down via algebraic product**

Scaled Mamdani Consequents





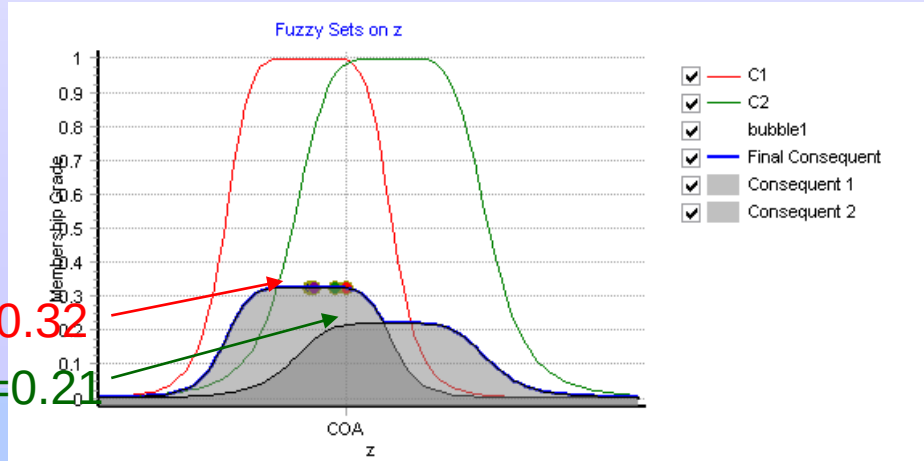




Rule 1: If x is A_1 and y is B_1 then z is C_1

Rule 2: If x is A_2 and y is B_2 then z is C_2

Now x is singleton , y is singleton , then ?



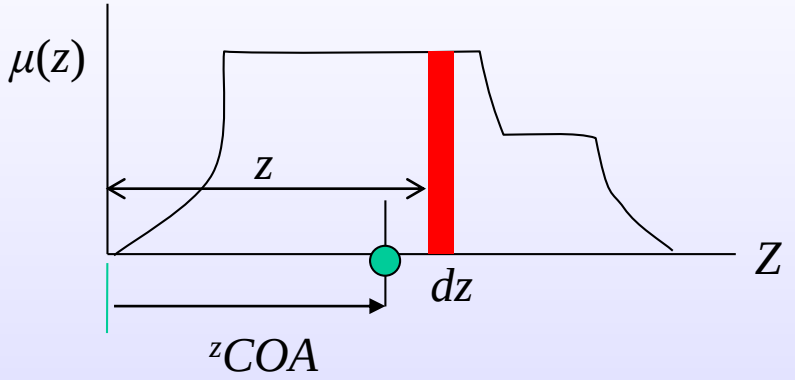
Fuzzification and Defuzzification

- If a crisp input is used
  - Treated as a fuzzy singleton
  - Fuzzified
- If a crisp output is required
  - The resulting fuzzy set must be defuzzified to generate a crisp output
  - A crisp value is extracted to represent the fuzzy set

Difuzzification

Centroid of area

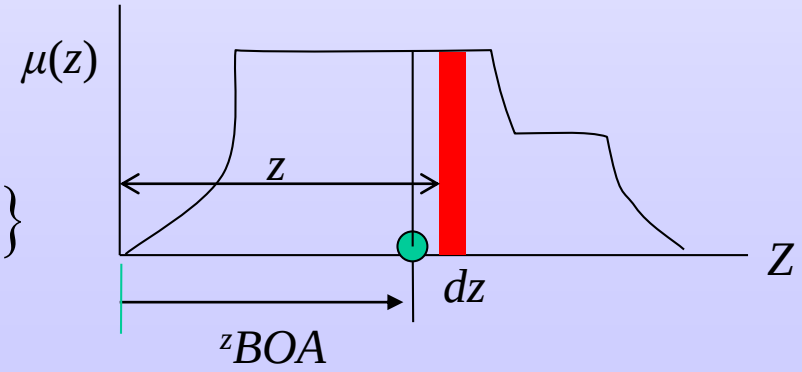
$$z_{COA} = \frac{\int_Z \mu_A(z)\, z\, dz}{\int_Z \mu_A(z)\, dz}$$



Bisector of area

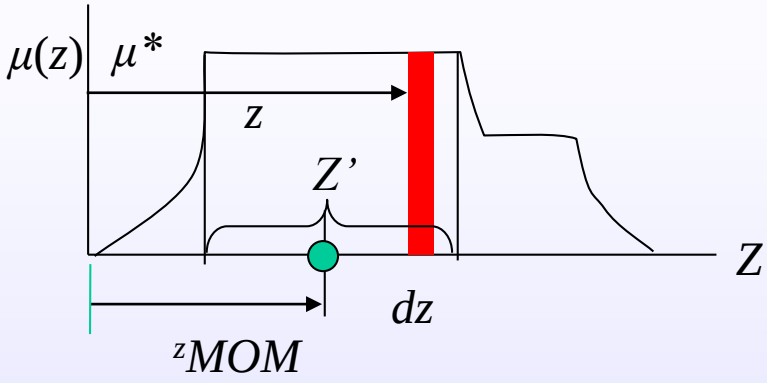
$$\int_{\alpha}^{z_{BOA}} \mu_A(z)\, dz = \int_{z_{BOA}}^{\beta} \mu_A(z)\, dz,$$

$$\alpha = \min\{z \mid z \in Z\},\ \beta = \max\{z \mid z \in Z\}$$

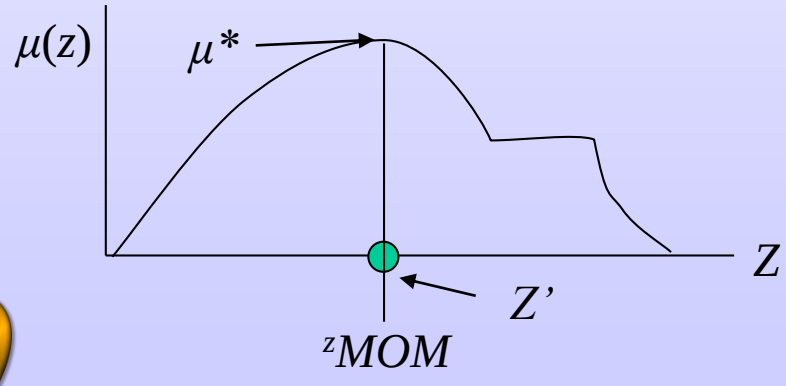


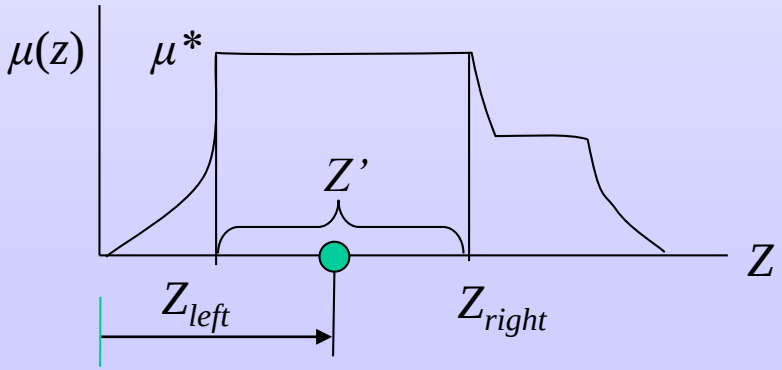
Mean of Maximum

$$z^{MOM} = \frac{\int_{Z'} z dz}{\int_{Z'} dz},$$



$$Z' = \{ z \mid \mu_A(z) = \mu^* \}, \ \mu^* = \max\{ \mu_A(z) \}$$



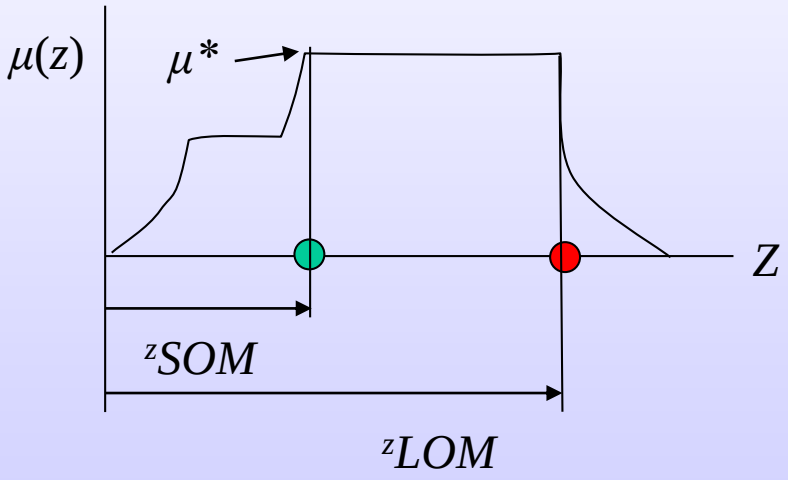


$$z^{MOM} = 0.5(Z_{left} + Z_{right})$$

Smallest of maximum

$$z_{SOM} = \min\left\{z \mid \mu_A(z) = \mu^*\right\},$$

$$\mu^* = \max\left\{\mu_A(z)\right\}$$

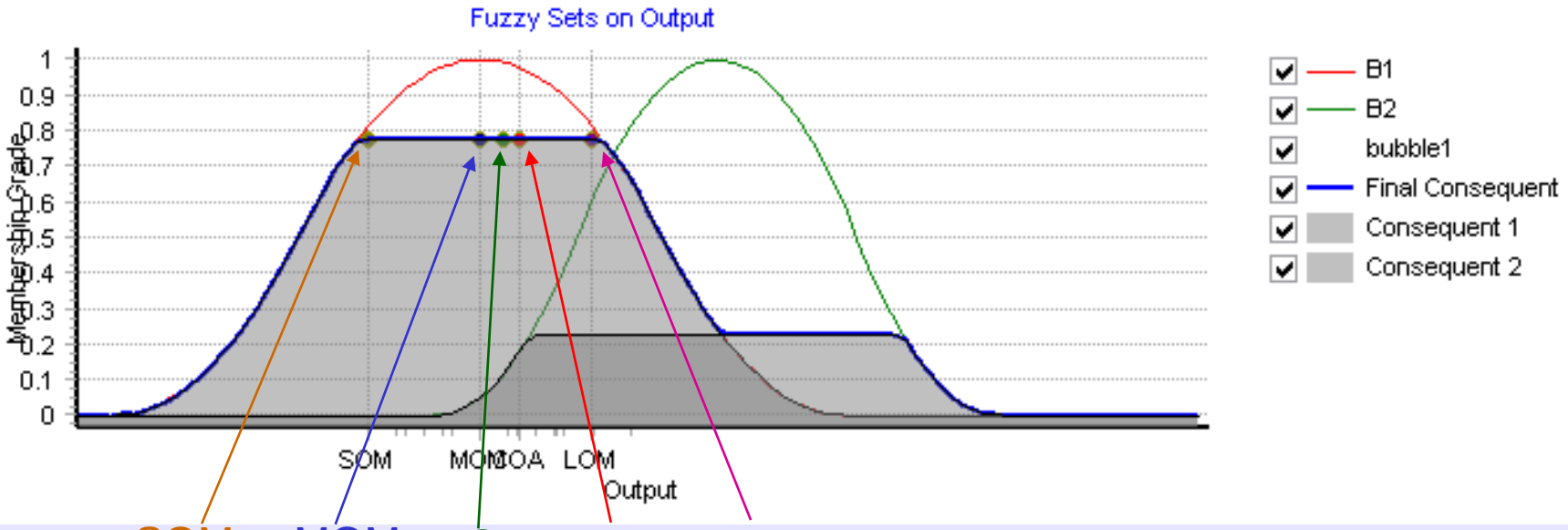


Largest of maximum

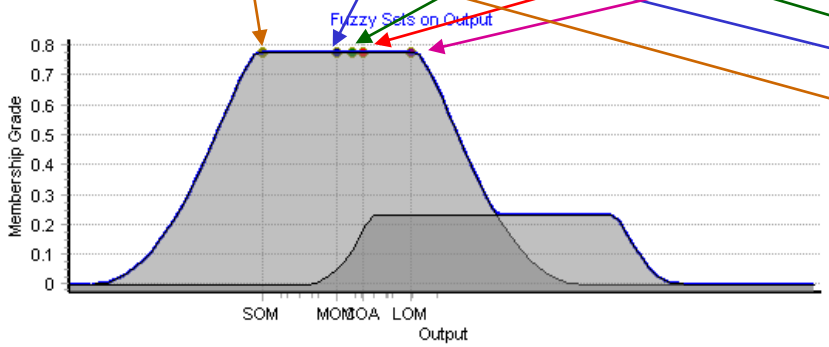
$$z_{LOM} = \max\left\{z \mid \mu_A(z) = \mu^*\right\},$$

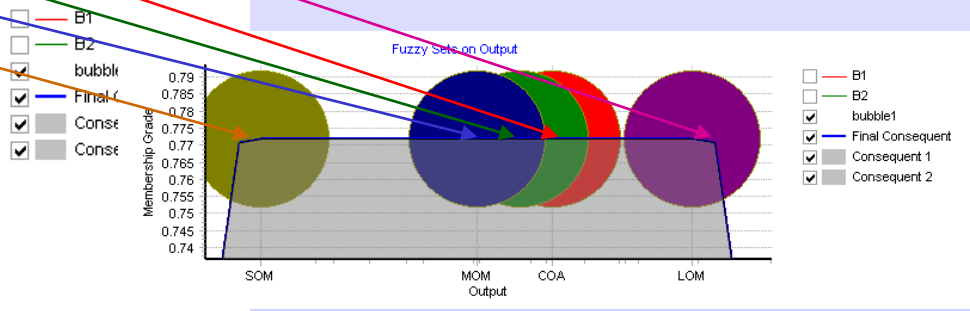
$$\mu^* = \max\left\{\mu_A(z)\right\}$$

Defuzzification Sample



SOM MOM BOA COA LOM

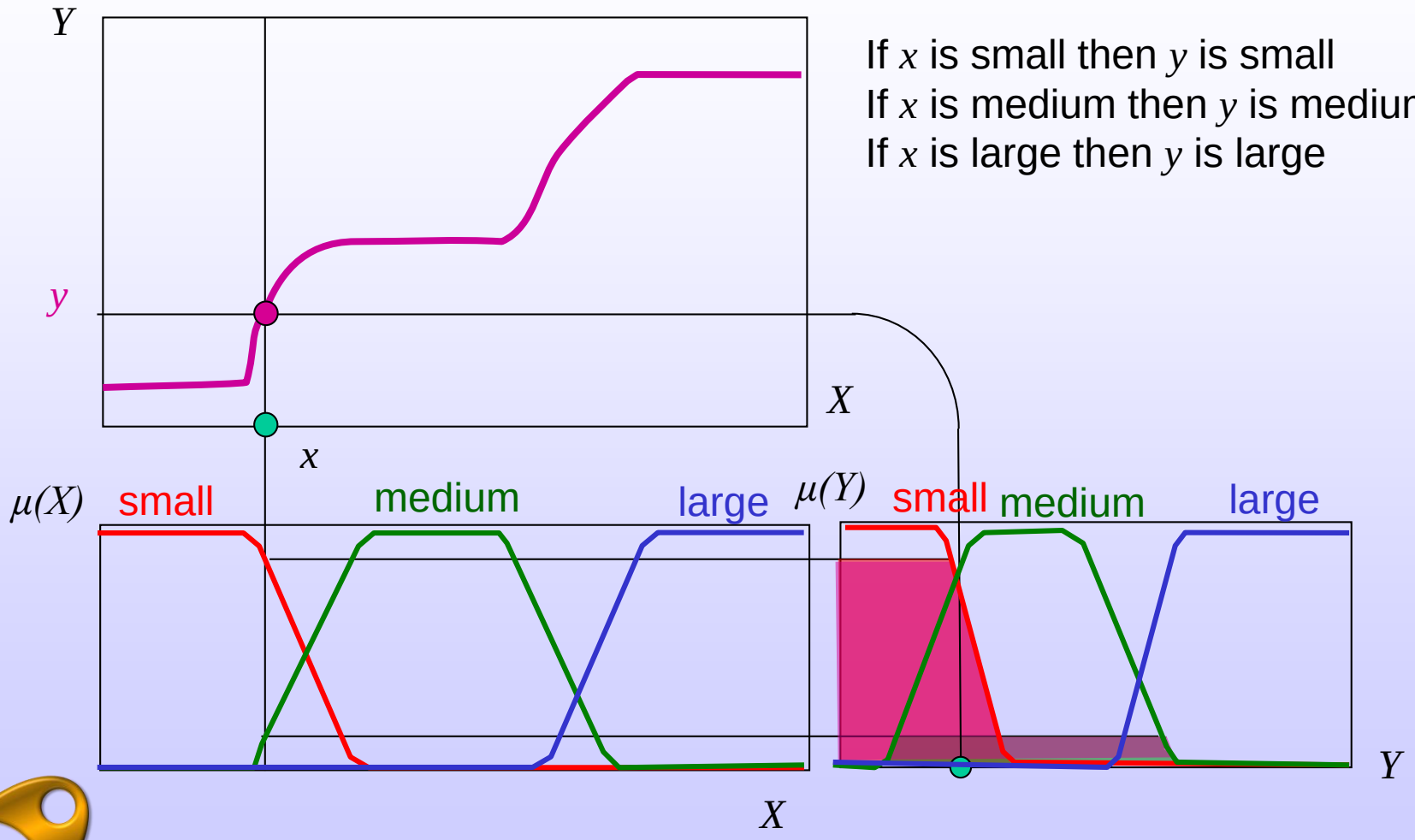




Comments about Defuzzification

- Integration computation, maximum and minimum analysis are costly
  - Usually the output fuzzy set does not have closed-form formulation
  - Numerical calculation for integration is costly
- Fuzzy inference systems without defuzzification are preferred

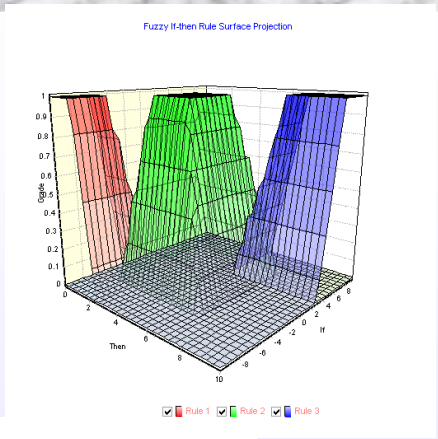
Single-input Single-output Mamdani Fuzzy Model

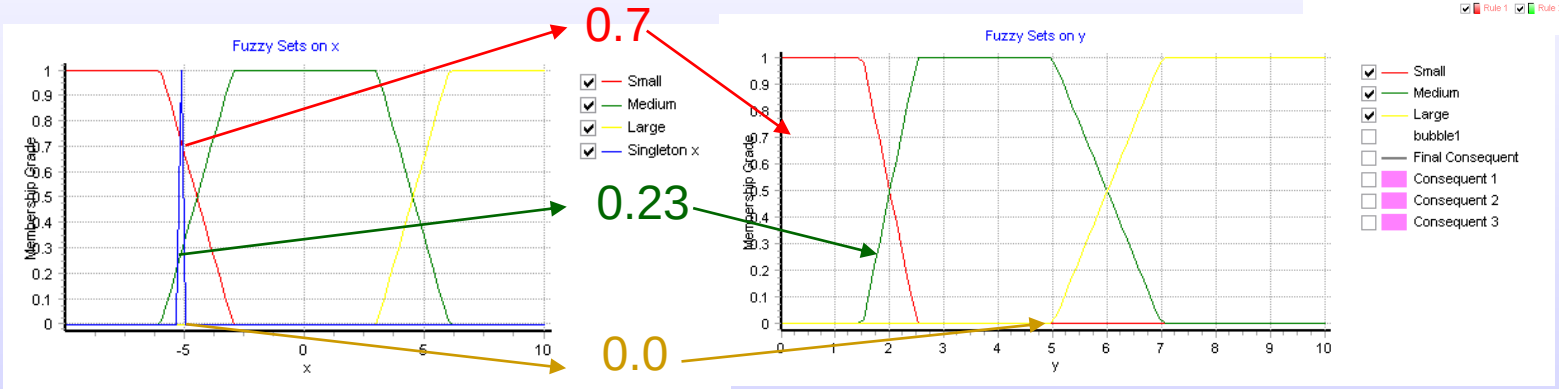


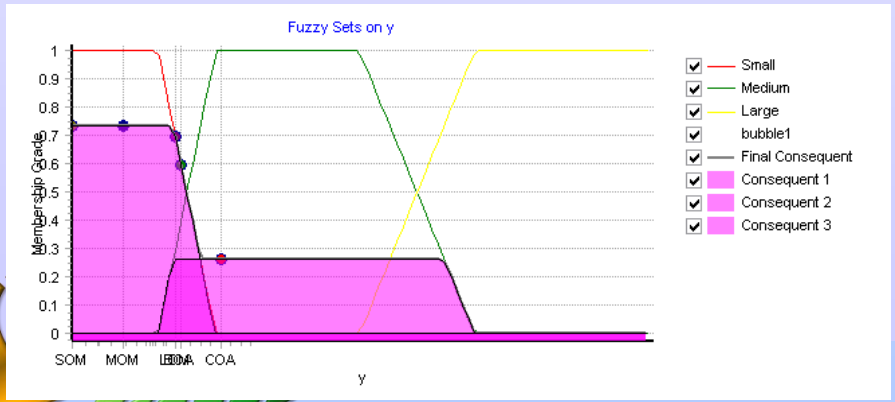
If x is small then y is small
If x is medium then y is medium
If x is large then y is large

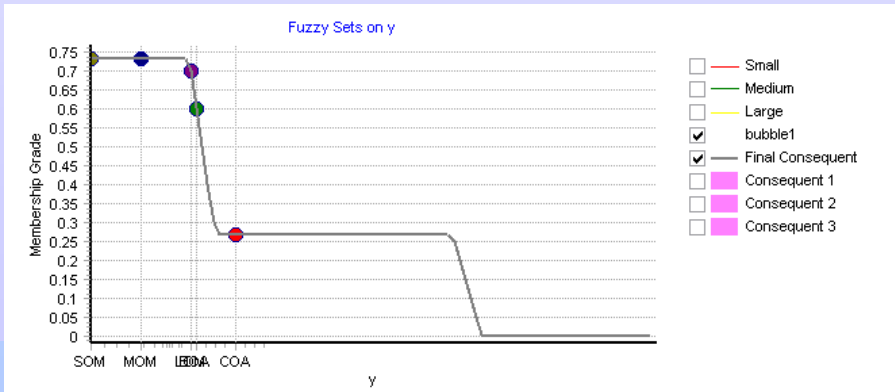
Single Input Single Output Example

Rule 1: If x is Small then y is Small
Rule 2: If x is Medium then y is Medium
Rule 3: if x is Large then y is Large
Now x is singleton x = -5.3, then y=?

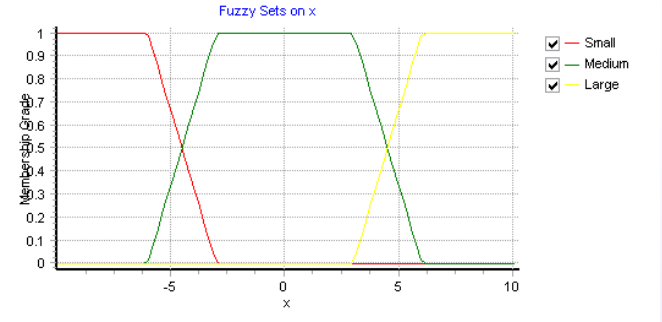


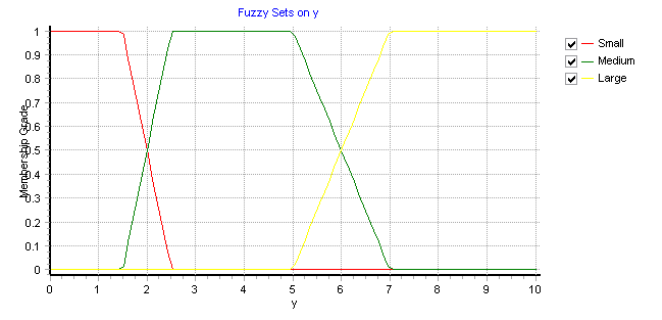






Input-Output Maps

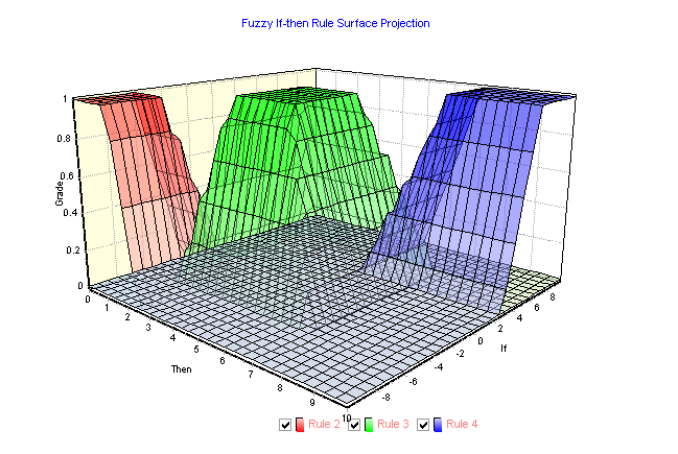


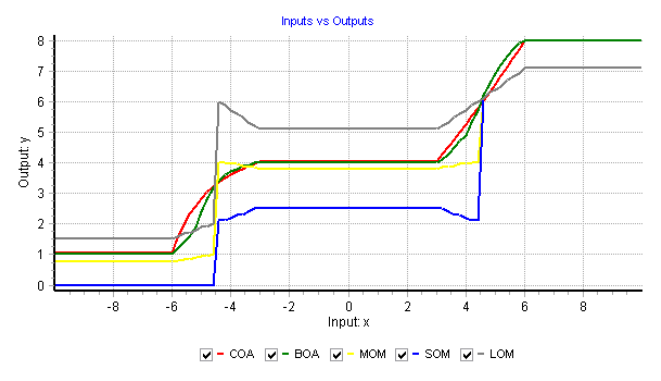


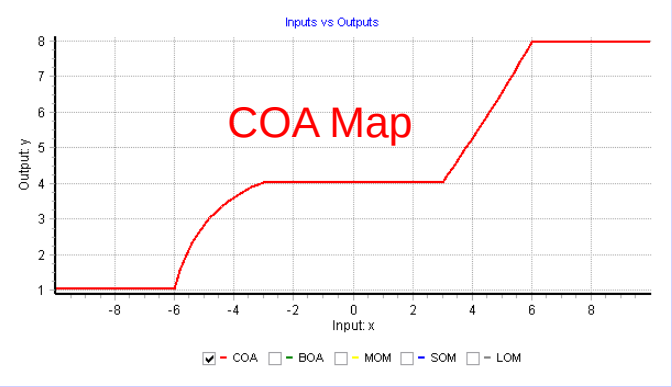
Rule 1: If x is Small then y is Small

Rule 2: If x is Medium then y is Medium

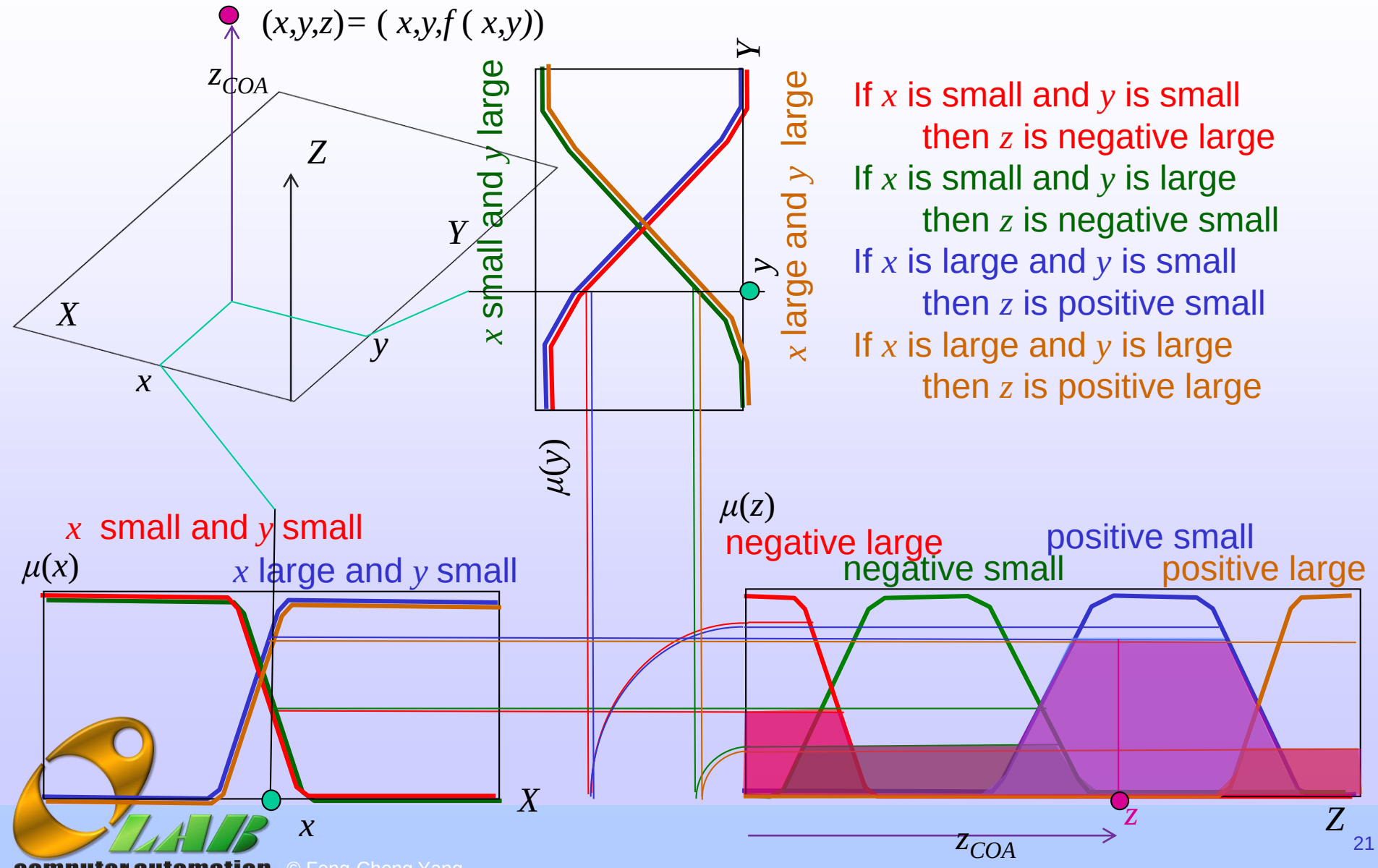
Rule 3: if x is Large then y is Large







Two-Input Single-Output Mandani Fuzzy Model



If x is small and y is small
then z is negative large

If x is small and y is large
then z is negative small

If x is large and y is small
then z is positive small

If x is large and y is large
then z is positive large

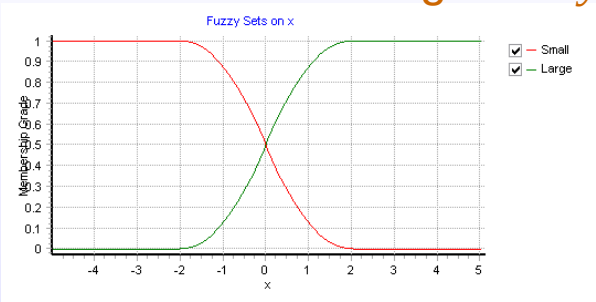
Two Inputs Single Output Example

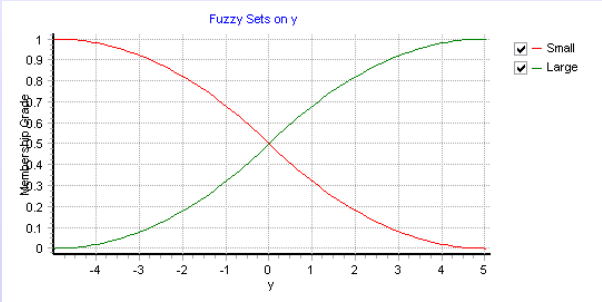
If x is small and y is small then z is negative large

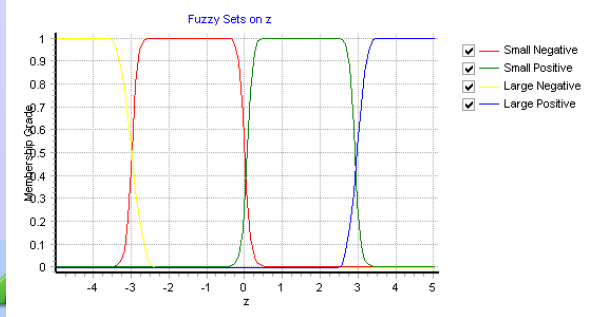
If x is small and y is large then x is negative small

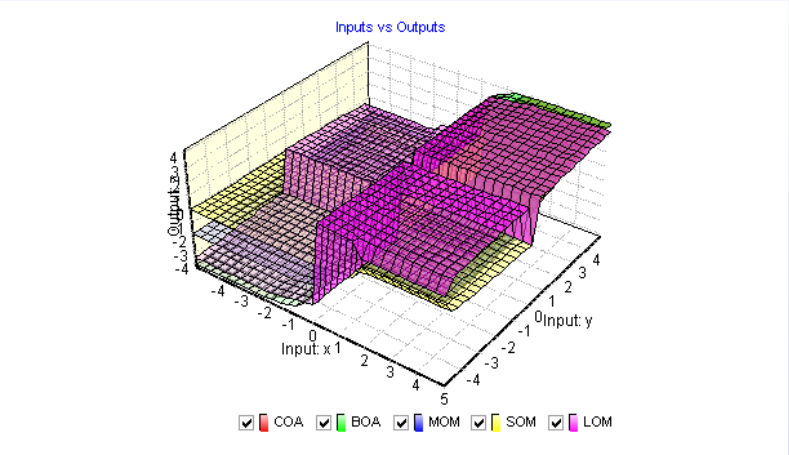
If x is large and y is small then z is positive small

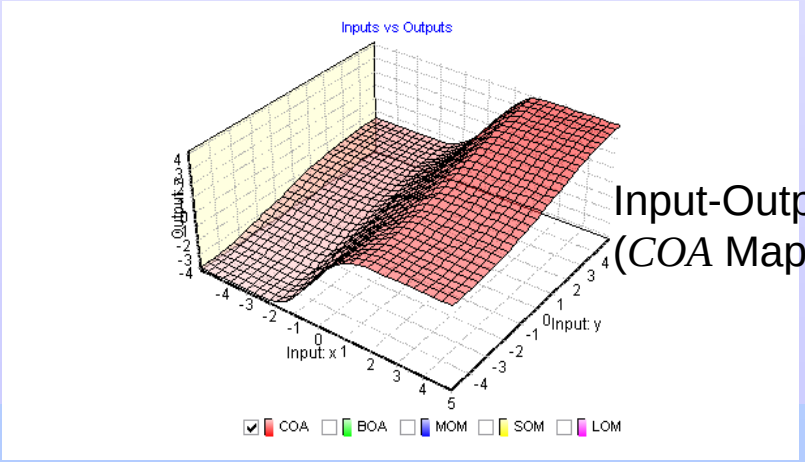
If x is large and y is large then z is positive large











Input-Output Maps (*COA* Map)

Assignment chap4-1

- Write a computer program with graphics displays to show the membership functions and input-output curve or surface of the Mamdani fuzzy model shown in Example 4.2 and Fig. 4.6.

Variants of Fuzzy Inference system

- For computation efficiency or mathematical tractability
- Firing strength computation for AND'ed antecedent:
  - Product → min; i.e., ab → min(a,b)
- Qualified consequent MFs
  - ??→ min (cut-out)
- Aggregate qualified consequent
  - ??→max

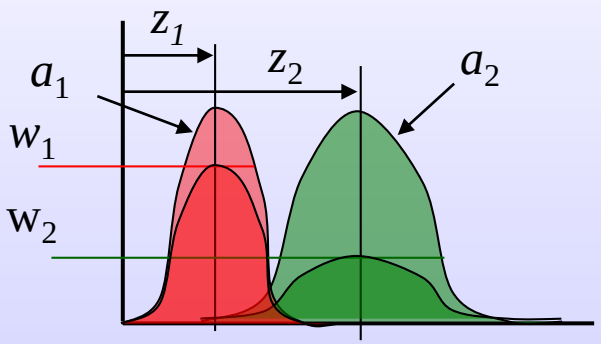
Mamdani Fuzzy Inference System

- One must specify the function for these operations clearly
  - AND operator for calculating the firing strength of a rule with AND'ed antecedents
    - $\min(a,b,c)$, $0.33(a+b+c)$, abc, ...
  - OR operator for calculating the firing strength of a rule with OR'ed antecedents
    - $\max(a,b,c)$, …
  - Implication operator for calculating qualified consequent MFs based on given firing strength
    - $w.\mu(z)$ (product), $w\wedge\mu(z)$ (cut out by w) ,...
  - Aggregate operator for aggregating qualified consequent MFs to generate an overall MF
    - union, point-wise summation,…
  - Defuzzification operator for transforming an output MF to a crisp single output value
    - *COA*, *BOA*,…

Theorem 1. Computation shortcut for Mamdani fuzzy inference systems

Under sum-product composition, the output of a Mamdani fuzzy inference system with centroid defuzzification is equal to the weighted average of the centroids of consequent MFs, where each of the weighting factors is equal to the product of a firing strength and the consequent MF's area.

Qualified consequent MF ← product
Aggregation of qualified MF's ← Sum



$$\mu_{C'}(z) = w_1\mu_{C_1}(z) + w_2\mu_{C_2}(z)$$

$$z_{COA} = \frac{\int_Z \mu_{C'}(z)\,z dz}{\int_Z \mu_{C'}(z)\,dz} = \frac{w_1\int_Z \mu_{C_1}(z)\,z dz + w_2\int_Z \mu_{C_2}(z)\,z dz}{w_1\int_Z \mu_{C_1}(z)\,dz + w_2\int_Z \mu_{C_2}(z)\,dz} = \frac{w_1a_1z_1 + w_2a_2z_2}{w_1a_1 + w_2a_2}$$

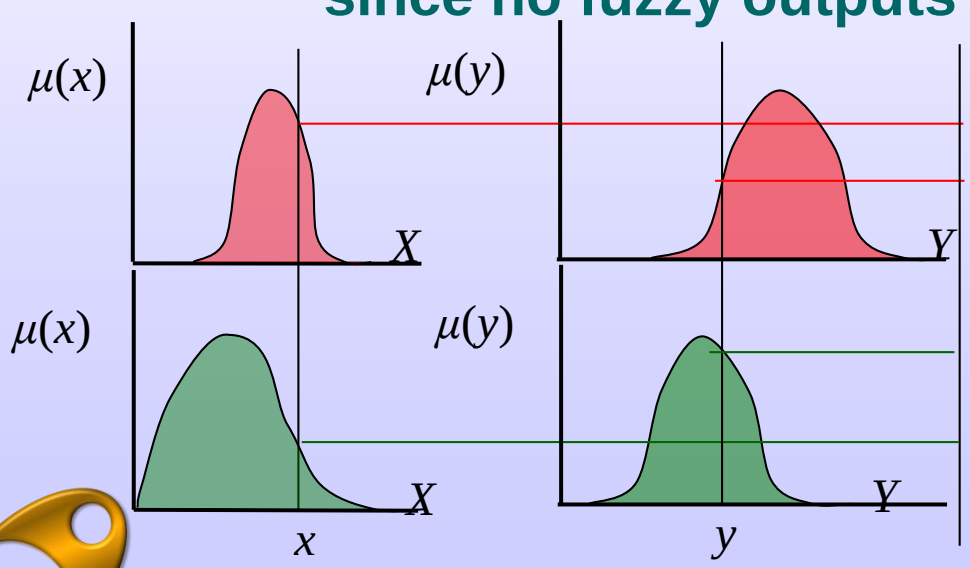
$$a_i = \int_Z \mu_{C_i}(z)\,dz : \text{Area of the MF};\ z_i = \frac{\int_Z \mu_{C_i}(z)\,z dz}{\int_Z \mu_{C_i}(z)\,dz} : \text{centroid of the MF}$$

Sugeno Fuzzy Models

- **Rules defined: if x is A and y is B then $z = f(x,y)$**
  - **Outputs are not fuzzy sets**
  - **Consequents are crisp functions**
    - Are usually polynomial in x, y
  - **First-order Sugeno fuzzy model**
    - $f(x,y)$ is a first-order polynomial
  - **Zero-order Sugeno fuzzy model**
    - $f(x,y)$ is a constant
    - A special Mamdani model with output fuzzy singletons
    - A special Tsukamoto model with step MF

First-order Sugeno Fuzzy Model

- Crisp output
- Overall output is a weighted average of each crisp output
  - Avoid time-consuming process of defuzzification
  - Can be more simplified as a weighted sum
  - Firing strengths are not used to infer outputs, since no fuzzy outputs



min or product

w_1

$z_1 = p_1 x + q_1 y + r_1$

w_2

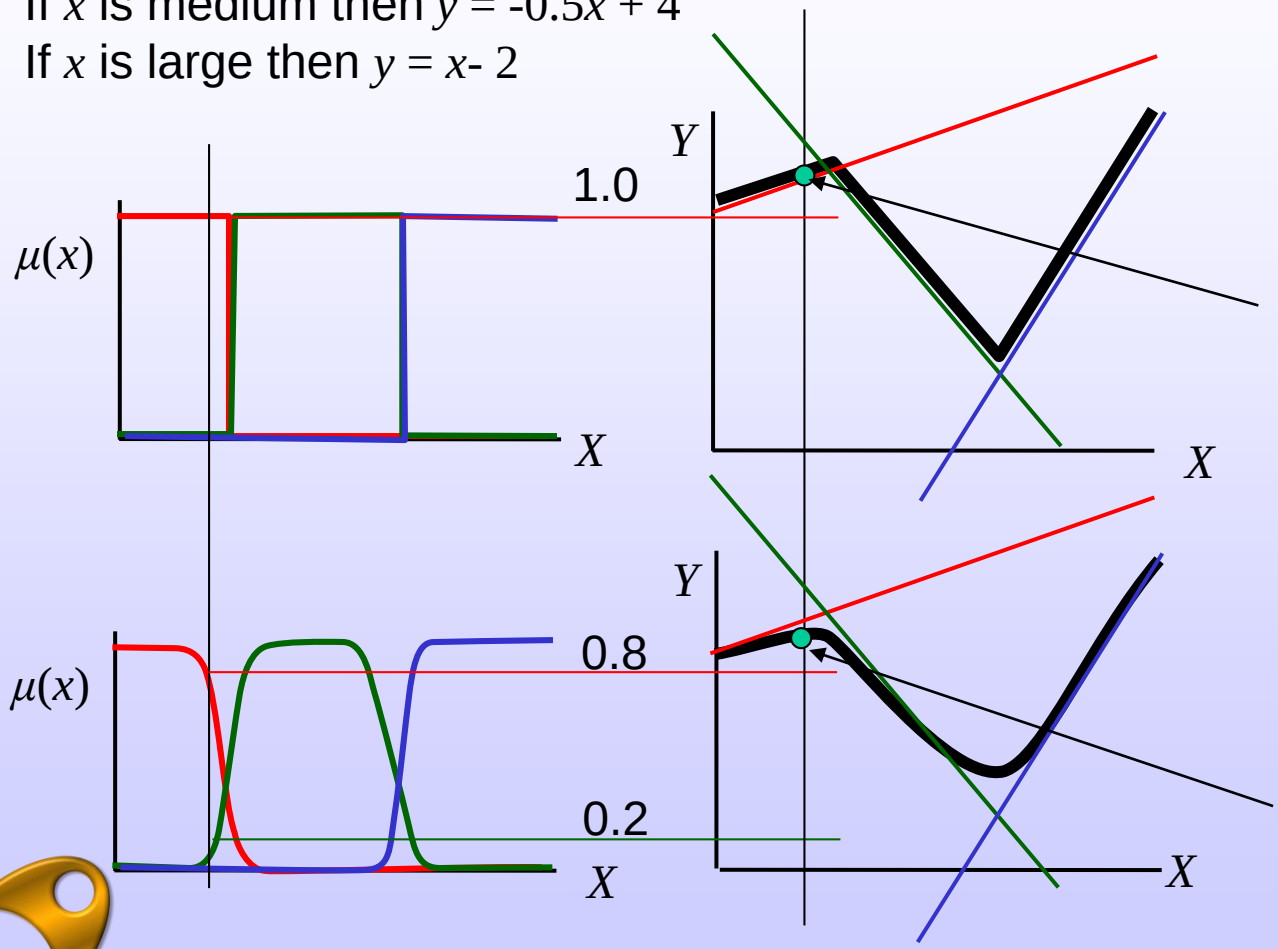
$z_2 = p_2 x + q_2 y + r_2$

$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}; \text{weighted avarage}$$

$$z = w_1 z_1 + w_2 z_2; \text{weighted sum}$$

Fuzzy and Nonfuzzy Rule Set

If x is small then $y = 0.1x + 6.4$
If x is medium then $y = -0.5x + 4$
If x is large then $y = x - 2$



$$\frac{1y_1 + 0y_2 + 0y_3}{1+0+0} = y_1$$

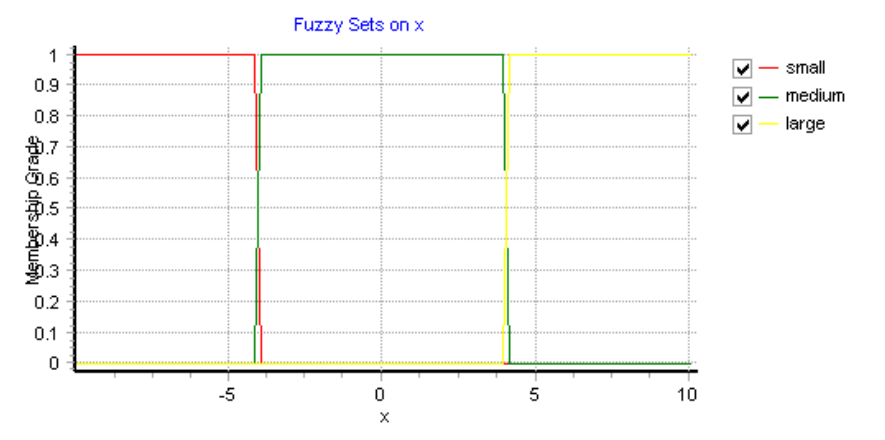
$$y = \frac{w_1y_1 + w_2y_2 + w_3y_3}{w_1 + w_2 + w_3}$$

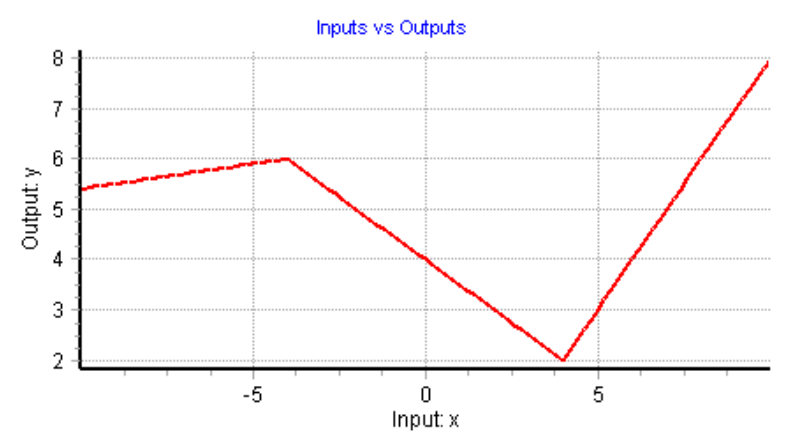
$$\frac{0.8y_1 + 0.2y_2 + 0y_3}{0.8+0.2+0} = 0.8y_1 + 0.2y_2$$

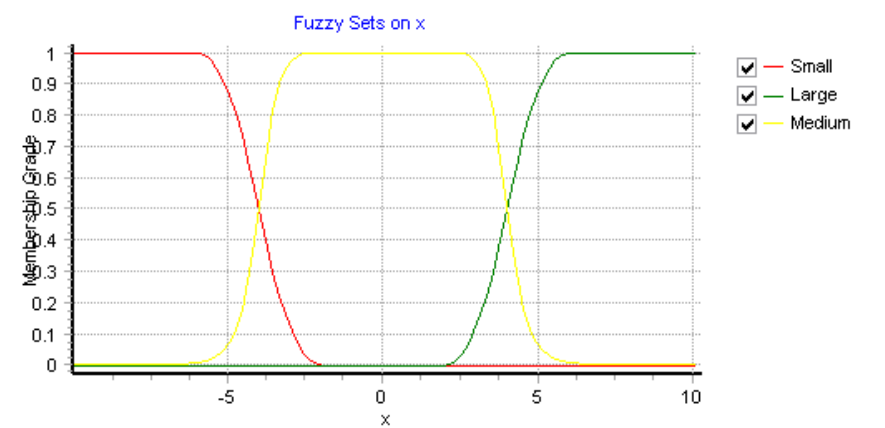
Nonfuzzy vs. Fuzzy Sets

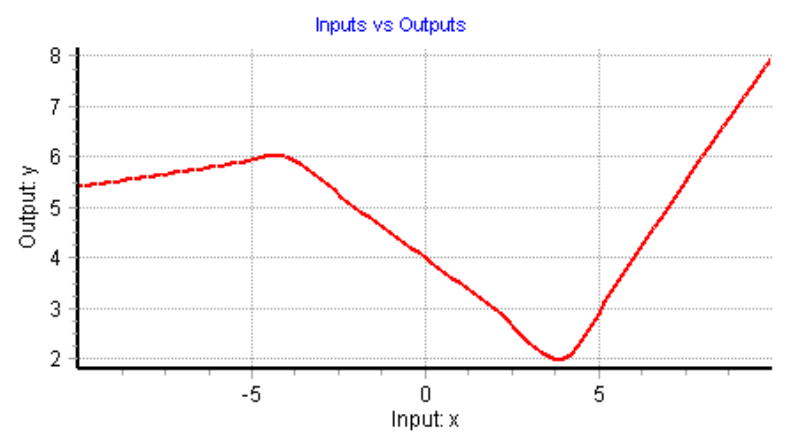
If x<-4.3 then y =0.1x+6.4
If -4.3< x <4.3 then y = -0.5x + 4
If 4.3< x then y = x- 2

If x is small then y =0.1x+6.4
If x is medium then y = -0.5x + 4
If x is large then y = x- 2



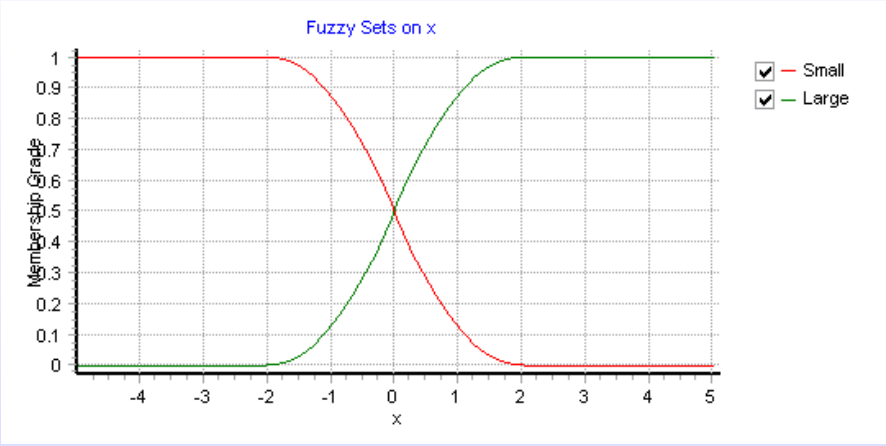


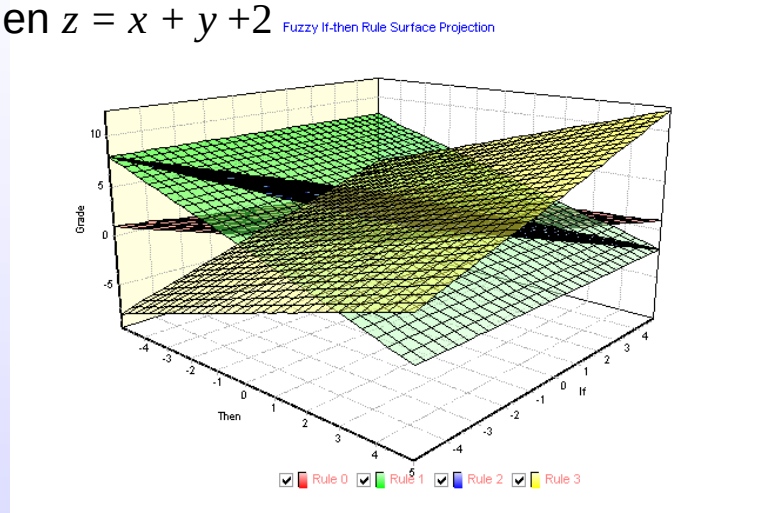


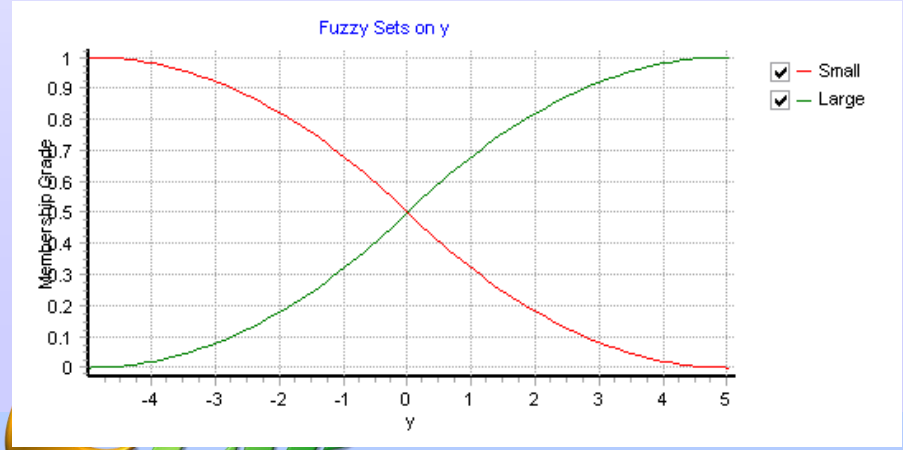


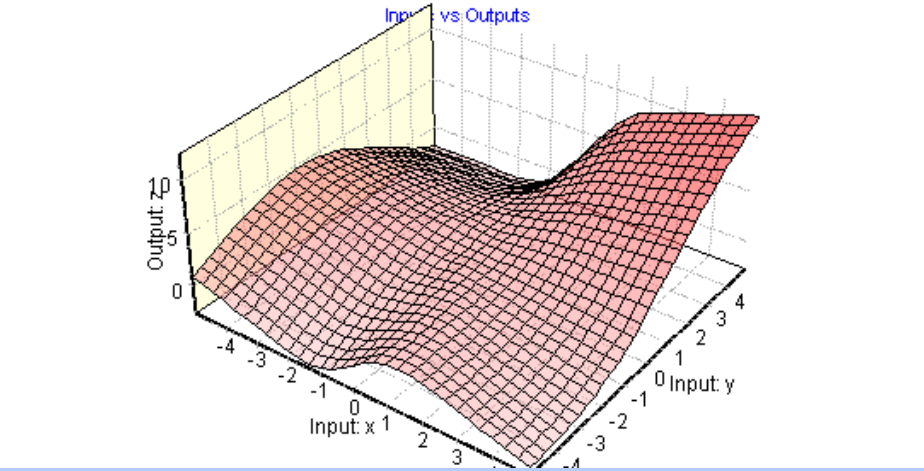
Two-inputs Example

If x is small and y is small, then $z=-x+y+1$
If x is small and y is large, then $z = -y + 3$
If x is large and y is small, then $z = -x + 3$
If x is large and y is large, then $z = x + y + 2$









Summary for Sugeno Fuzzy model

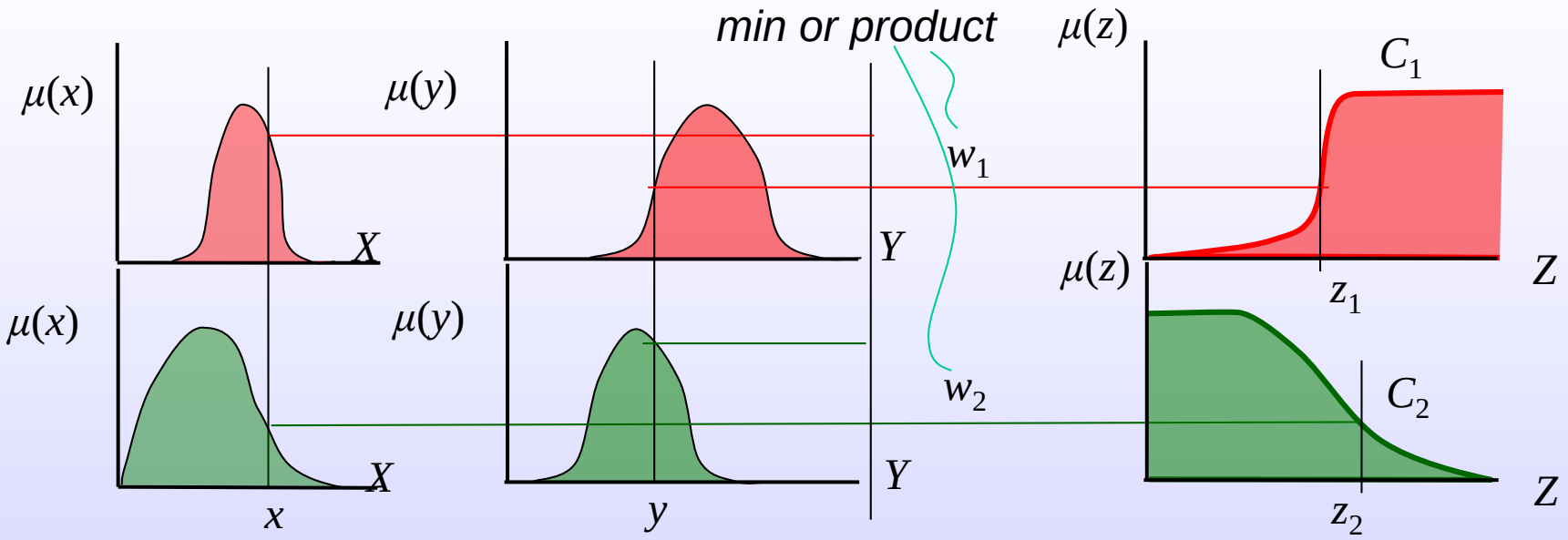
- **No composition of inference**
  - **Since output are not fuzzy sets**
  - **If inputs are fuzzy sets**
    - Firing strength can be calculated
    - Input fuzziness can not propagate to fuzzy outputs
      - No fuzzy outputs
    - Fuzzy characteristics are missing
- **Advantages**
  - **Avoid time-consuming defuzzification**
  - **Quick computation for overall outputs**

Assignment chap4-2

- Write a computer program with graphics displays to show the membership functions and input-output curve or surface of the Sugeno fuzzy model shown in Example 4.4 and Fig. 4.10.

Tsukamoto Fuzzy Model

- The consequents of each fuzzy if-then rule is represented by a fuzzy set with a monotonical MF
  - No time-consuming defuzzification
  - A fuzzy output set generate crisp output from the firing strength
  - Overall output is calculated by weighted average

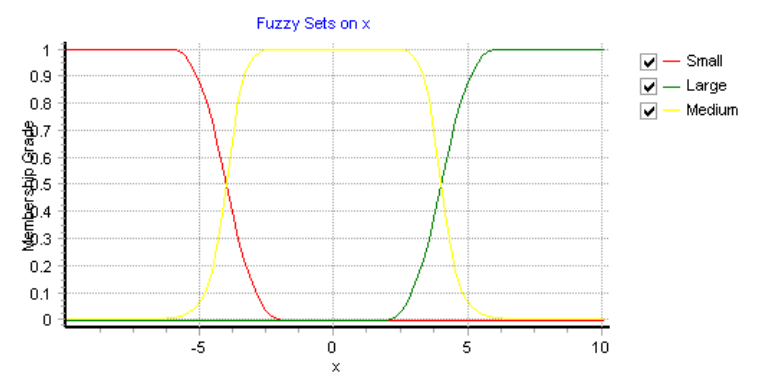


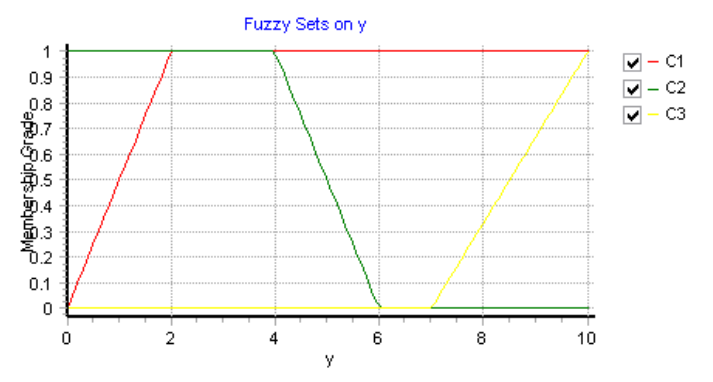
$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}; \text{weighted avarage}$$

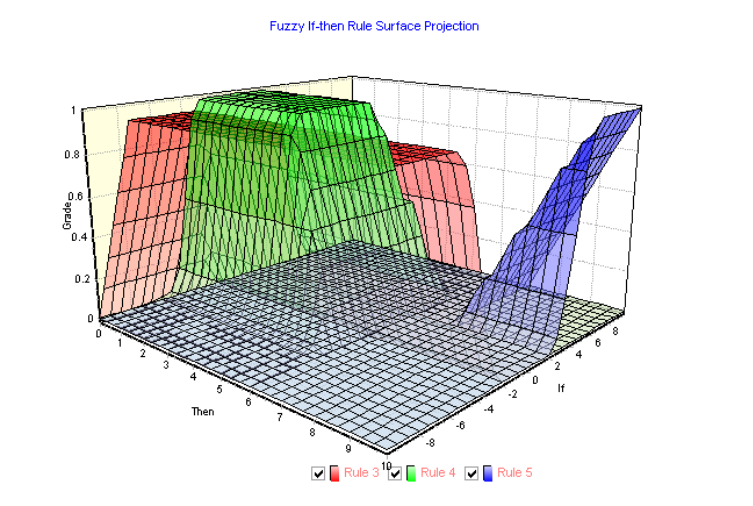
$$z = w_1 z_1 + w_2 z_2; \text{weighted sum}$$

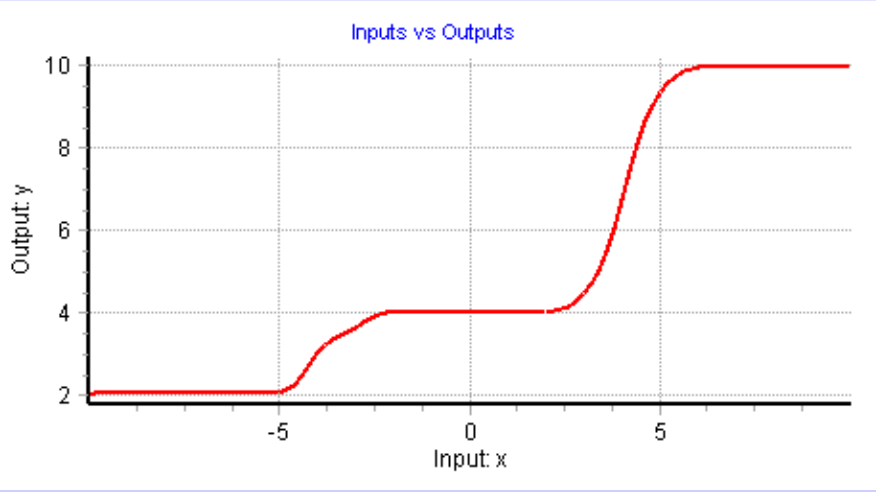
Example

If x is small then y is c_1
If x is medium then y is c_2
If x is large then y is c_3









Assignment chap4-3

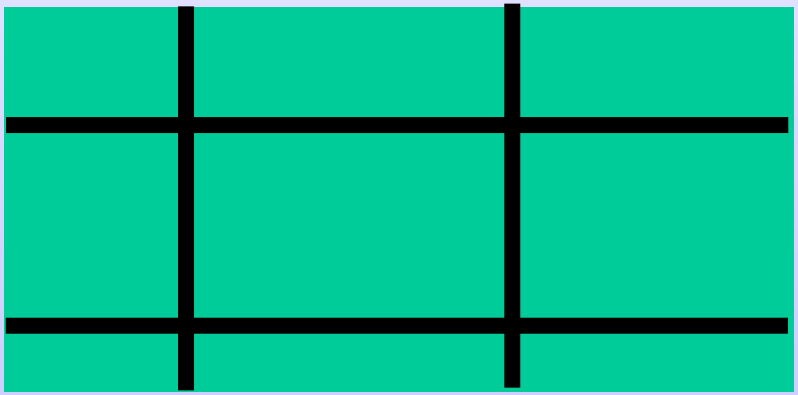
- Write a computer program with graphics displays to show the membership functions and input-output curve or surface of the Tsukamoto fuzzy model shown in Example 4.5 and Fig. 4.12.

How to partition an input space

- **Divide and conquer**
  - **The antecedent**
    - Defines a local fuzzy region
  - **Consequent**
    - Describe the behavior within the region
      - A consequent MF
      - A constant value
      - A linear equation
  - **Various inference systems share the same antecedent structure and various consequents**
- **Three types of input partitions**
  - **Grid, Tree, Scatter Partitions**

Grid Partition

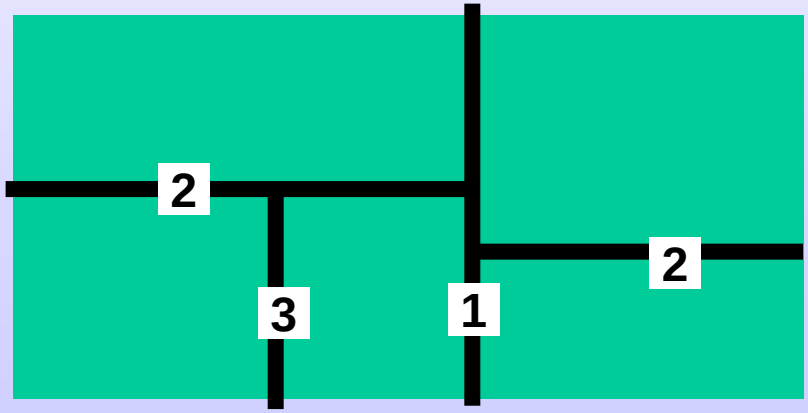
- Frequently used for fuzzy controller
- Involves only several input state variables
- A small number of MFs
  - 10 inputs, 2MFs/input
  - 20MFs
  - 2^{10}=1024 fuzzy if-then rules are required ← curse of dimensionality
    - I_1=1 and I_2 = 1 and … I_{10} = 1 then xxx



2 inputs (X, Y), 6 MFs, 3^2=9 rules

Tree Partition

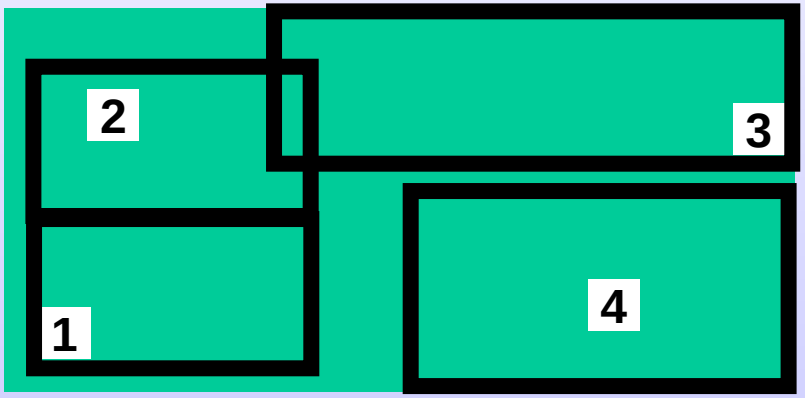
- Reduce rules for exponentially increase problems
- More MFs are required
- MFs have no clear linguistic meanings
- In general, orthogonality is roughly hold in XxY, but not for X or Y only.



2 inputs (X, Y), 8 MFs, 5 rules

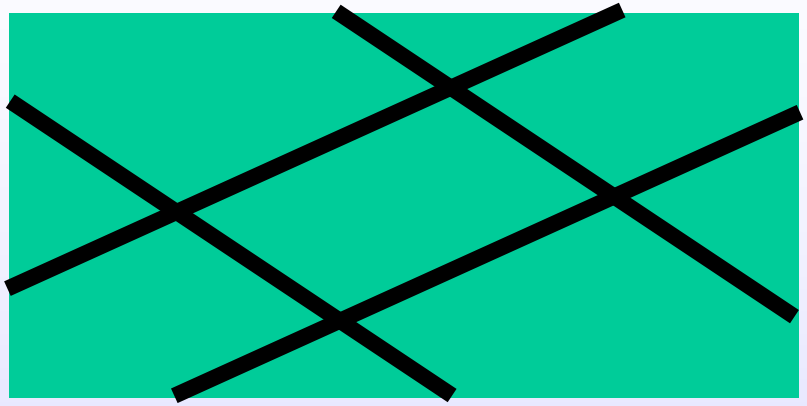
Scatter Partition

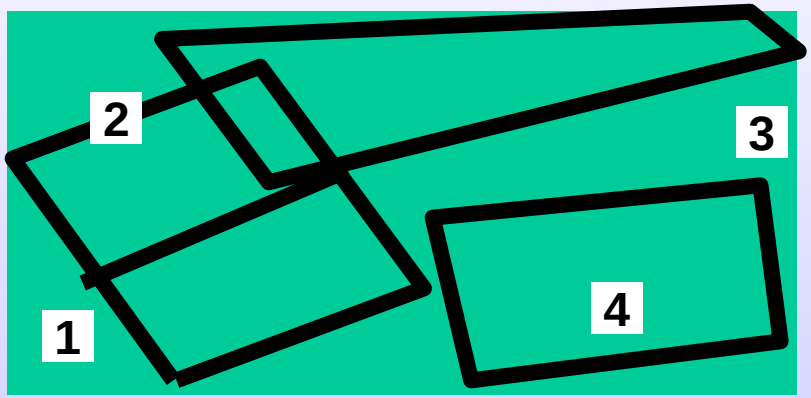
- Cover only a subset of the input space
- The subset characterize a region of possible occurrence of input vectors
- In general, orthogonality does not hold in neither X, Y, nor XxY

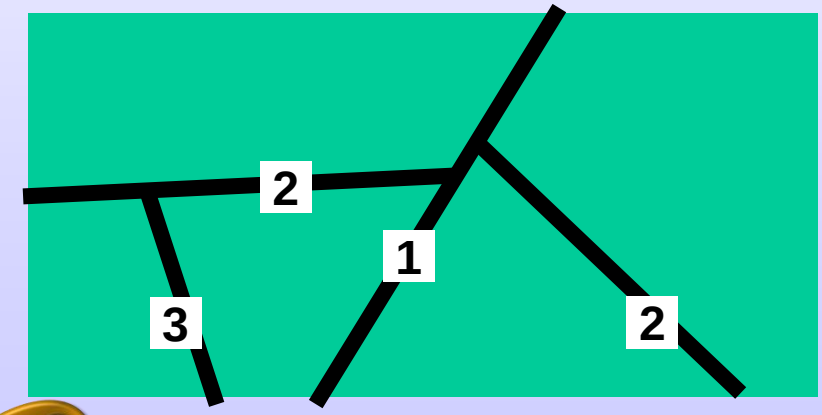


2 inputs, 7 MFs, 4 rules

Inputs are transformed from input variables







Fuzzy Modeling

- **Design of a fuzzy inference system**
  - **Based on the past known behavior of a target system**
  - **Examples:**
    - Human operator
      - Fuzzy controller
    - Medical doctor
      - Fuzzy expert system

- **Fuzzy modeling**
  - **Easily to incorporate human expertise about the target system directly**
    - Domain knowledge
  - **When input-output data of a target system is available**
    - Conventional system identification techniques can be used for fuzzy modeling
    - Numerical data can be derived and extracted

Fuzzy modeling guidelines

- **(Stage 1) Identification of surface structure**
  - **Select relevant input and output variables**
  - **Choose a specific type of fuzzy inference system**
  - **Determine the number of linguistic terms associated with each input and output variables**
  - **Design a collection of fuzzy if-then rules. How?**
    - (1) own knowledge of the target system,
    - (2) information from human expert,
    - (3) trial and error

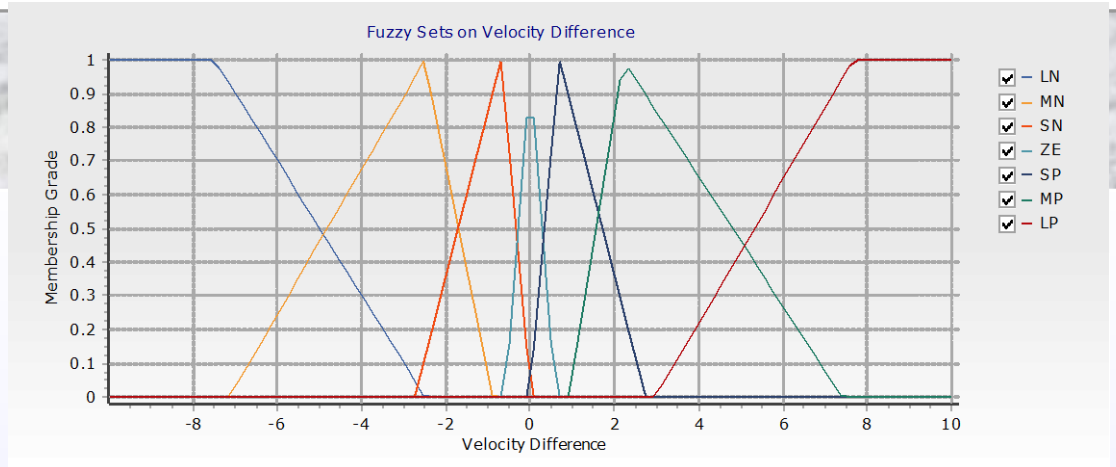
Fuzzy modeling guidelines

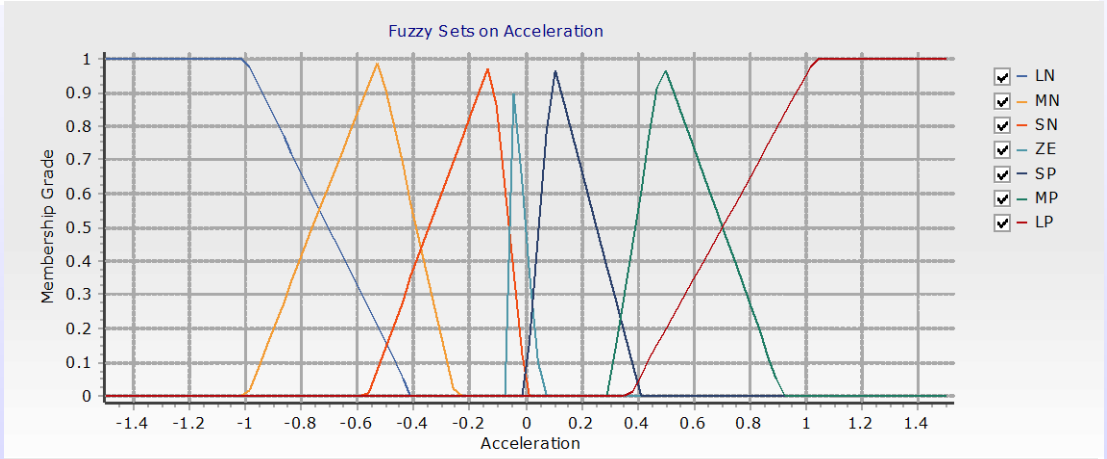
- **(Stage 2) Identification of deep structure**
  - **Choose an appropriate family of parameterized MFs**
    - Human experts
  - **Interview human experts to determine the parameters of the MFs**
    - Human experts
  - **Refine the parameters of the MFs using regression and optimization techniques**
    - Available input-output data set
    - Training

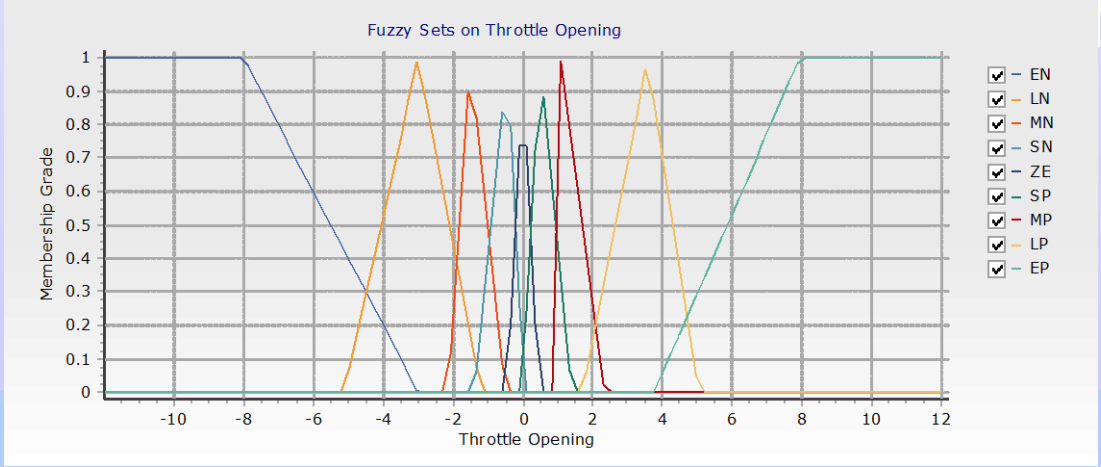
Sample Application

- **Platoons of cars**
  - Karaaslan et al.(1991) and Hsu et al.(1991)
  - High-speed groups of smart cars on freeways of the future
  - No input-output math model
  - Use common sense rules—like a human driver
- **Fuzzy Throttle Controller**
  - Nonlinear dynamics of driving a highspeed car
  - True math models are too complex

- **Inputs velocity difference (with respect to the platoon velocity) and acceleration**
- **Output the throttle**
  - **Velocity difference**
    - 7 MFs: LN, MN, SN, ZE, SP, MP, LP
  - **Acceleration**
    - 7 MFs: LN, MN, SN, ZE, SP, MP, LP
  - **If-then rules**
    - Grid Partition→7x7 = 49 Rules
  - **Output**
    - 9 MFs: EN, LN, MN, SN, ZE, SP, MP, LP, EP





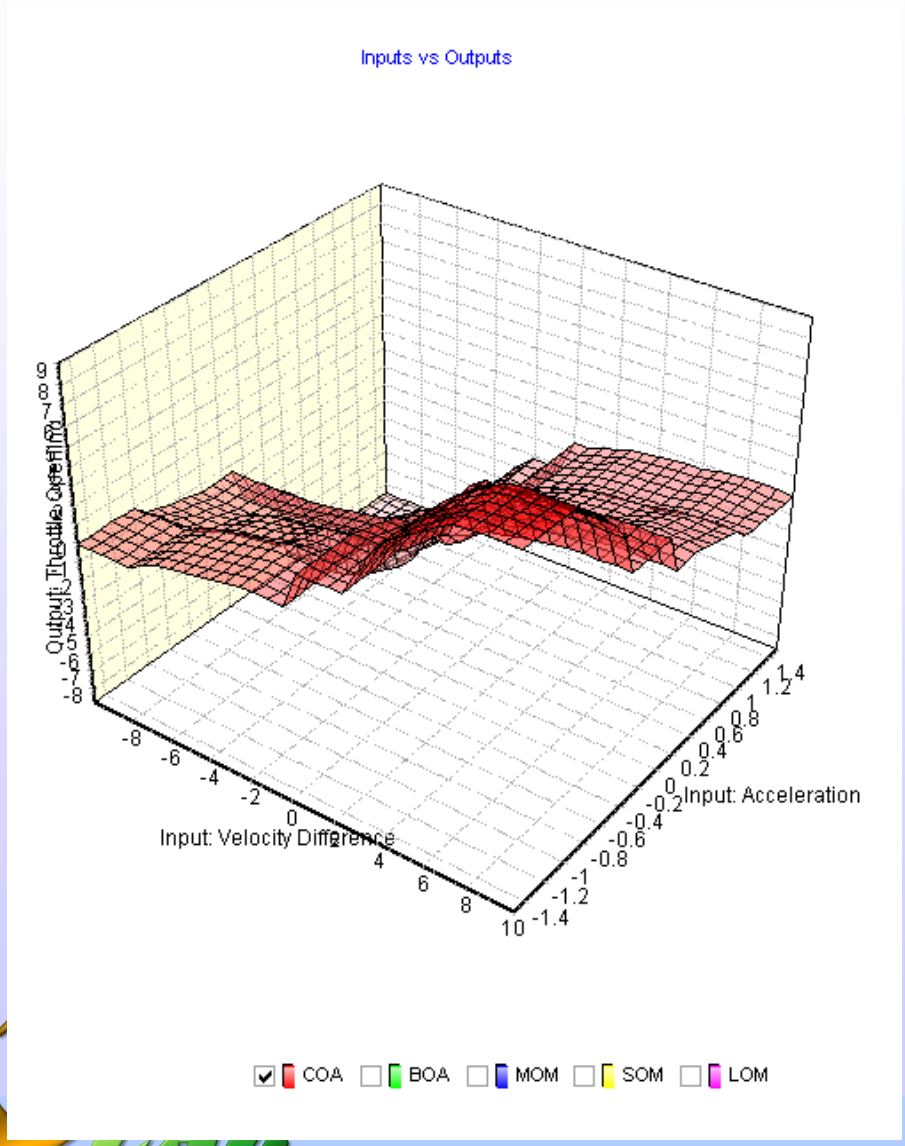


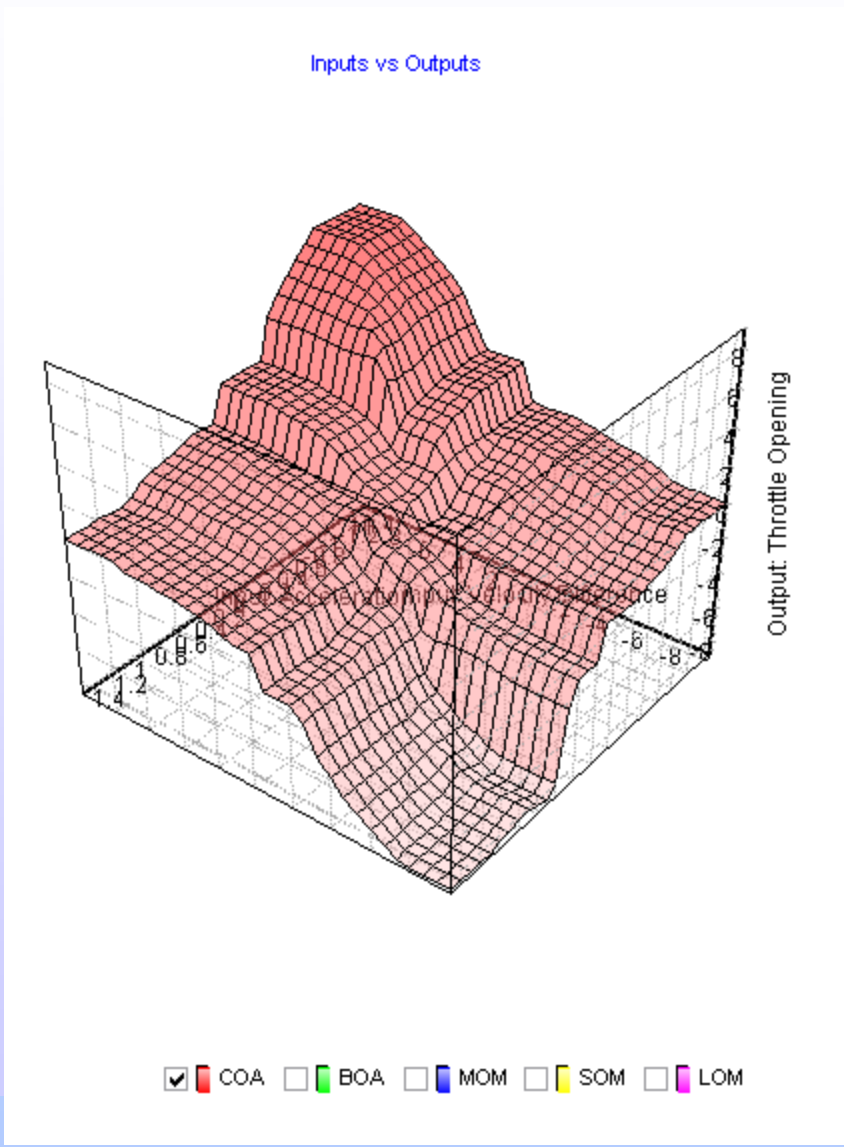
49 If-then Rules

R1: If velocity difference is LN and acceleration is LN then throttle is ZE

Throttle	Velocity Difference: LN	MN	SN	ZE	SP	MP	LP
Acceleration: LN	ZE	MP	MP	MP	LP	LP	EP
MN	MN	ZE	ZE	MP	MP	MP	LP
SN	MN	SN	SN	SP	SP	MP	LP
ZE	LN	MN	SN	ZE	SP	MP	LP
SP	LN	MN	SN	SN	SP	SP	MP
MP	EN	LN	MN	MN	SN	SP	MP
LP	EN	LN	LN	MN	MN	SN	ZE

Resulting Input-Output (COA)





Chapter 4 Exercises

- 3, 4, 6 (write a program is recommended), 7, 8 (write a program is recommended)
  - You may use your software system developed in Assignment -Chap4 to solve these problems
- 6,7,8, and 9. Use your FIS.

Assignment

- Write a software system with graphics displays to be able to model fuzzy inference systems
- Can model Mamdani systems and generate input-output maps
  - Capable of obtaining COA defuzzification
- You can incorporate Assignment chap4-1, chap4-2, and chap4-3 into an integrated Assignment or develop three subsystems separately