# Superimposing Voronoi Complexes for Shape Deformation

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**Abstract.** Edelsbrunner et al. defined a framework of shape deformations with shapes bounded by skin manifold. We prove that the infinitely many synthesized shapes in the deformation sequence share finitely many common Voronoi complexes. Therefore, we propose a new algorithm to compute the common Voronoi complexes efficiently for the deformations, and use these common complexes to compute the synthesized shapes in real time. This makes generating, visualizing, and customizing shape deformations feasible.

## 1 Introduction

Edelsbrunner et al. defined a framework of shape space construction [2]. See Figure 1 for an example. Given two reference shapes which are represented by the *skin surfaces*, a smooth deformation can be computed automatically and robustly. This shape space construction provides a new paradigm for geometric modeling, animation designs and molecular motion simulation. However, it is limited to only two shapes, and also, not feasible for real time visualization.

In this work, we extend this blending of shapes to $n$ reference shapes in any dimension. Imagine that given three faces represented by skin surfaces, we can synthesize any new face with features of the reference faces. Moreover, because of the extension to $n$ shapes, we are able to customize the shape deformations by introducing additional reference shapes. In the example of Figure 2, we influence the deformation of the shape $\mathbf{X}$ to $\mathbf{I}$ with a shape $\mathbf{O}$. By applying a positive influence of the shape $\mathbf{O}$, we can create a hole during the deformation. On the contrary, if we apply a negative influence of the shape $\mathbf{O}$, we can compact the deforming shape.

We also make the visualization feasible for real time applications. The bottleneck lies on the generation of the Voronoi complex of each instance of the deforming shapes. The reason is that the shapes are represented by the skin surfaces and they are built on the Delaunay complexes of weighted point sets. It means that for each instance in the morphing, a Voronoi complex (or its Delaunay triangulation) is required. See Figure 1 for an example. The deformation of a shape $\mathbf{X}$ into another shape $\mathbf{I}$ requires a sequence of intermediate shapes $(1-t)\mathbf{X}+t\mathbf{I}$, for $t \in [0,1]$. For each instance in the deformation, it takes $O(m \lg m + m^{\lceil d/2 \rceil})$ time to construct its Voronoi complex in $\mathbb{R}^d$, in which $m$ is

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the cardinality of the weighted point set. Moreover, we usually need a lot of instances in the sequence of deformation to generate a smooth visualization effect. Thus, it is infeasible to compute the intermediate shapes in real time.

**Contribution.** In this paper, we improve the efficiency of the intermediate shapes computation and extend the works of Edelsbrunner:

1. For each Voronoi complex construction, we improve the bound to $O(m^3)$ in any dimension, if the Voronoi complexes of the initial shapes are given.
2. We prove that all the infinite number of intermediate shapes share finitely many Voronoi complexes.
3. We generalize Edelsbrunner's framework to more than two shapes, together with the additional combinatorial structures required in the new type of degenerated Voronoi complexes.

The first two points make it feasible for visualizing the deformation in real time. For example, the Voronoi complexes for the shapes in Figure 1 are identical for $0 < t < 1$. Thus, we generate only one Voronoi complex for all the shapes in the deformation sequence.

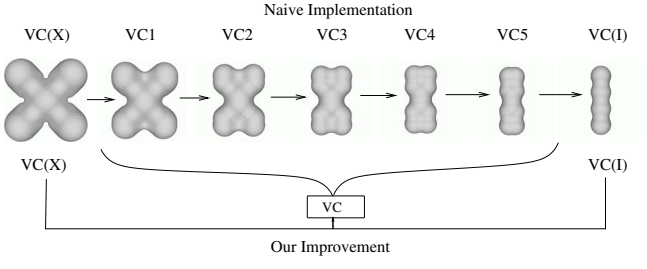



**Fig. 1.** A deformation of the shape **X** into **I** in $\mathbb{R}^3$

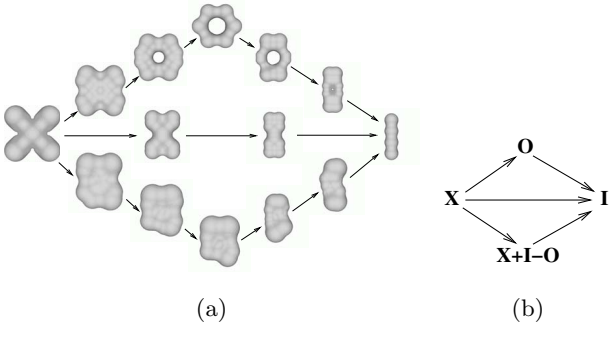



**Fig. 2.** Three deformations of **X** into **I** influenced by **O** in three different ways

**Outline.** In Section 2, we introduce the skin and its construction. In Section 3, we define the the intermediate shape, and present a theorem stating that there are only finitely many Voronoi complexes for all the intermediate shapes. Finally, in Section 4, we design a new algorithm to compute the Voronoi complexes of the intermediate shapes.

## 2 Backgrounds

We introduce the geometric foundation of the skin body that is bounded by the *skin surface*, a compact manifold without boundary in any dimension. It is defined by a weighted point set, its Voronoi and Delaunay complexes. This section serves for the purpose of stating the notations for later sections. Readers who want to know more details about the skin surface should refer to [4]. At the end of this section, we will also introduce the notation of furthest-neighbor Voronoi complex, which is useful in later sections when we extend the morphing with $n$ reference shapes.

**Delaunay and Voronoi Complexes.** The skin surface is built on the structures of Delaunay and Voronoi complexes of a weighted point set. We first define some notations for the Delaunay and Voronoi complexes.

A *weighted point* in $\mathbb{R}^d$ can be written as $b_i = (z_i, w_i) \in \mathbb{R}^d \times \mathbb{R}$, where $z_i \in \mathbb{R}^d$ is its position and $w_i \in \mathbb{R}$ is its weight. We can also view a weighted point, $b_i$, as a ball in $\mathbb{R}^d$ with center $z_i$ and radius $\sqrt{w_i}$. For any $B_0 \subseteq \mathbb{R}^d \times \mathbb{R}$, we denote $z(B_0)$ as the set of the centers of the weighted points in $B_0$. The *weighted distance* of a point $x \in \mathbb{R}^d$ from a weighted point, $b_i$, is defined as

$$\pi_{b_i}(x) = \|xz_i\|^2 - w_i. \tag{1}$$

Given a finite set $B_0$ of $n$ weight points, the *Voronoi region*, $\nu_i$, for each weighted point, $b_i \in B_0$, is

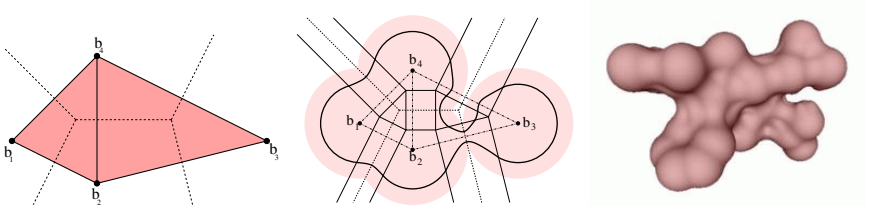
$$\nu_i = \{x \in \mathbb{R}^d \mid \pi_{b_i}(x) \leq \pi_{b_j}(x), b_j \in B_0\}.$$

We define the non-empty intersection of $m$ Voronoi regions as the *Voronoi cell* of a set of weighted points $X \subseteq B_0$, namely, $\nu_X = \bigcap_{b_i \in X} \nu_i$. The collection of all the non-empty Voronoi cells is called the *Voronoi complex* of $B_0$, denoted as , $V_{B_0}$. For each $\nu_X \in V_{B_0}$, its corresponding *Delaunay cell*, $\delta_X$, is the convex hull of the set of centers of $X$, namely, $\text{conv}(z(X))$. The collection of all the Delaunay cells is called the *Delaunay complex* of $B_0$, denoted as $D_{B_0}$.

Given a Delaunay cell $\delta_X$, its *dimension*, $\dim(\delta_X)$, is that of the affine hull of $\delta_X$. If $\dim(\delta_X)$ is no more than $\text{card}(X) - 1$, $\delta_X$ is a simplex. If this is true for all cells in $D_{B_0}$, it is a *simplicial complex*. The Delaunay complex is simplicial under the following general position assumption in $\mathbb{R}^d$.

Assumption 1 (General positions) $\forall \nu_X \in V_{B_0}$, $\text{card}(X) = \dim(\delta_X) + 1$.

The dimension of $\nu_X$, $\dim(\nu_X)$, is $d - \dim(\delta_X)$. We call $\nu_X$ a *simplicial Voronoi cell* if its corresponding Delaunay cell is a simplex.




(a) $V_0$ and $D_0$ of $B_0$. (b) The skin of $B_0$. (c) A skin in $\mathbb{R}^3$.

**Fig. 3.** The left subfigure is the Voronoi and Delaunay complexes of a weighted point set in $\mathbb{R}^2$, $B_0 = \{b_1, b_2, b_3, b_4\}$. The center subfigure is the skin of $B_0$, which is constructed on the complexes of $B_0$. The right subfigure is the skin body of a molecule in $\mathbb{R}^3$

We can define a new Voronoi complex using a different distance function. If we employ $-\pi_{b_i}(x)$ as the distance function, the resulting Voronoi complex is the *furthest-neighbor Voronoi complex*. Each of its elements contains points with the longest weighted distances from the weighted points. The furthest-neighbor Voronoi complex will be used to construct the Voronoi complex of the intermediate shape in the next section.

**Skin.** A skin surface is Specified by a set of weighted points $B_0 = \{b_i \in \mathbb{R}^d \times \mathbb{R} \mid i = 1..n_0\}$. Before defining the skin, three operations on weighted points (or balls) are given. The aim is to establish the operations for linear combinations of weighted points [4] as in points in $\mathbb{R}^d$. Also, these operations are useful in the proof in later sections. For $b_i, b_j \in B_0$ and $\gamma \in \mathbb{R}$, the *addition*, *scalar multiplication* and *square root* of weighted points are defined as

$$\begin{aligned} b_i + b_j &= (z_i + z_j, w_i + w_j + 2\langle z_i, z_j \rangle), \\ \gamma b_i &= (\gamma z_i, \gamma w_i + (\gamma^2 - \gamma)\|z_i\|^2), \quad \text{and} \\ \sqrt{b_i} &= (z_i, w_i/2), \end{aligned}$$

respectively, in which $\langle z_i, z_j \rangle$ is the scalar product of two vectors $z_i$ and $z_j$. A *linear combination* of $B_0$ is $\sum_{i=1}^{n_0} \gamma_i b_i$ with $\gamma_i \in \mathbb{R}$ and $i = 1..n_0$. It is a *convex combination* if $\sum_{i=1}^{n} \gamma_i = 1$ and all $\gamma_i \geq 0$. Thus, we can define the *convex hull* of $B_0$ as $\mathrm{conv}(B_0)$, namely, the set of all the convex combinations of $B_0$.

Now, we are ready to define the skin surface. We first shrink all the balls in $\mathrm{conv}(B_0)$ by the square root operations. Then, the skin surface is the envelop of the underlying space of these shrunken balls, formally,

$$\mathrm{skin}(B_0) = \partial\left(\bigcup \sqrt{\mathrm{conv}(B_0)}\right).$$

For the space bounded by the skin surface, we call this the *skin body*, denoted as $\mathrm{body}(B_0) = \bigcup \sqrt{\mathrm{conv}(B_0)}$. Figure 3(c) shows an example of the skin body in $\mathbb{R}^3$.

**Skin Decomposition.** Here, we discuss the skin decomposition and give the reason why we need the Delaunay and Voronoi complexes for the skin construction. The skin of $B_0$ can be decomposed by *mixed cells*. A mixed cell, $\mu_X$, is a Minkowski sum of a Delaunay cell and its corresponding Voronoi cell, namely, $\mu_X = (\nu_X + \delta_X)/2$. All the mixed cells partition $\mathbb{R}^d$ and decompose the skin surface. Within each mixed cell, $\text{skin}(B_0) \cap \mu_X$ is a quadratic surface in the form of $\sum_{i=1}^{d} \pm x_i^2 = w_X/2$ after translation to the *center* of $\mu_X$. The center and size of $\mu_X$ is defined as

$$z_X = \text{aff}(\delta_X) \cap \text{aff}(\nu_X), \quad \text{and} \tag{2}$$

$$w_X = w_i - \|z_X z_i\|^2, \tag{3}$$

respectively. In $\mathbb{R}^2$, there are three types of mixed cell and the 'patch' is either a portion of a circle or a hyperbola clipped with a mixed cell. See Figure 3(b) for a decomposed skin.

## 3 Intermediate Shapes and Their Voronoi Complexes

In deformation, we start with some reference shapes and generate some mixtures of them, namely, *intermediate shapes*. After discussing how they are generated, we present the theorem of all the intermediate shapes share finitely many Voronoi complexes.

**Intermediate Shapes.** Given a finite set of given shapes, namely, the *reference shapes*, an infinite family of intermediate shapes can be constructed. Each intermediate shape is the mixture of the reference shapes, with a set of weights signaling the similarities between the mixture and the reference shapes.

Let the number of reference shapes be $n$. Let $\mathfrak{B} = \{B_1, \ldots, B_n\}$ be the set of weighted point sets defining the set of reference shapes, $S = \{\text{body}(B_i) \mid B_i \in \mathfrak{B}\}$. To define the intermediate shapes of $S$, we firstly define the affine combinations of $\mathfrak{B}$, namely, $B(c)$.

Given two weighted point sets, $B_0$ and $B_1$, we define the *addition*, and *scalar multiplication* of them as

$$B_0 + B_1 = \{b_i + b_j \mid b_i \in B_0, b_j \in B_1\}, \quad \text{and}$$
$$\gamma B_0 = \{\gamma b_i \mid b_i \in B_0\}, \text{for} \quad \gamma \in \mathbb{R},$$

respectively. With these operations, we can define the *linear combination* of $\mathfrak{B}$ as $B(c) = \sum_{i=1}^{n} \gamma_i B_i$ with the *coefficient vector*, $c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$. If $\sum_{i=1}^{n} \gamma_i = 1$, $B(c)$ is an *affine combination* of $\mathfrak{B}$, namely, the *intermediate weighted point set*. Note that the cardinality of $B(c)$ is no more than $\Pi_{i=1}^{n} \text{card}(B_i)$. We do not limit the weights to be positive for a convex combination because negative weights play an interesting role in shape mixing.

An *intermediate shape* is defined as the skin shape bounded by the skin of $B(c)$, namely, $\text{body}(B(c))$. Each intermediate shape corresponds to a coefficient

vector, $c$, which states the weights of all these reference shapes in this mixture. All the coefficient vectors of the intermediate shapes form a hyperplane $\Gamma \subseteq \mathbb{R}^n$. A *deformation* of one intermediate shape into another is parameterized with a path connecting the two corresponding coefficient vectors in $\Gamma$. Note that each reference shape is also an intermediate shape, whose corresponding coefficient vector is a unit basis vector in $\mathbb{R}^n$.

**Intermediate Voronoi Complexes.** We denote the Voronoi complex of $B(c)$ as $V(c)$, namely, the *intermediate Voronoi complex*. To compute an intermediate shape, body$(B(c))$, the corresponding intermediate Voronoi complex, $V(c)$, is required. However, it is computationally expensive if we compute $V(c)$ for each individual value of $c$, especially in deformation of the intermediate shapes. On the other hand, we will show that $\Gamma$ can be divided into finite number of partitions, and in each partition, the intermediate Voronoi complexes are the same for all the possible values of $c$.

Each weighted point $b(c) \in B(c)$ is an affine combination of $n$ weighted points from the $n$ given $B_i \in \mathfrak{B}$, namely, $b(c) = \sum_{i=1}^{n} \gamma_i b_i$, for some $b_i \in B_i$. In order to define the Voronoi region of $b(c)$ with respect to $B(c)$, we firstly derive the weighted distance of any point $x \in \mathbb{R}^d$ from $b(c)$.

**Lemma 1.** *Let $b(c) = \sum_{i=1}^{n} \gamma_i b_i$, such that $\sum_{i=1}^{n} \gamma_i = 1$. The weighted distance of any point $x \in \mathbb{R}^d$ from $b(c)$ is the affine combination of the weighted distances of $x$ from $b_1, \ldots, b_n$, namely,*

$$\pi_{b(c)}(x) = \sum_{i=1}^{n} \gamma_i \pi_{b_i}(x).$$

*Proof.* We prove this lemma by induction on the number $n$. For $n = 1$, it is true since $\gamma_1 = 1$ and $B(c) = B_1$. When $n = 2$, we have

$$\pi_{b(c)}(x) = \gamma_1 \pi_{b_1}(x) + (1 - \gamma_1)\pi_{b_2}(x), \tag{4}$$

by using Equation (1). The claim is true for $n = 2$.

Assume the claim is true for $n = k$. When $n = k+1$, without loss of generality, we assume $\gamma_{k+1} \neq 1$. Let $\mathfrak{B}' = \mathfrak{B} - \{B_{k+1}\}$ and

$$c' = (\frac{\gamma_1}{1 - \gamma_{k+1}}, \frac{\gamma_2}{1 - \gamma_{k+1}}, .., \frac{\gamma_k}{1 - \gamma_{k+1}}).$$

Let $B'(c')$ be the affine combination of $\mathfrak{B}'$, and $b'(c') \in B'(c')$. Then, $b(c)$ can be expressed as the affine combination of $b'(c')$ and $b_{k+1}$, namely,

$$b(c) = (1 - \gamma_{k+1})b'(c') + \gamma_{k+1} b_{k+1}.$$

Follow Equation (4), the weighted distance of any point $x \in \mathbb{R}^d$ from $b(c)$ is

$$\pi_{b(c)}(x) = (1 - \gamma_{k+1})\pi_{b'(c')}(x) + \gamma_{k+1}\pi_{b_{k+1}}(x) = \sum_{i=1}^{k+1} \gamma_i \pi_{b_i}(x),$$

as required. □

We are now ready to give a new method to determine the Voronoi region of $b(c)$ with respect to $B(c)$. We start constructing $V(c)$ from a simple situation, in which $\gamma_i > 0$ for all $i$.

**Lemma 2.** *For any $b(c) \in B(c)$ with all $\gamma_i > 0$, its Voronoi region with respect to $B(c)$ is the intersection of the Voronoi regions of $b_1, \ldots, b_n$ with respect to $B_1, \ldots, B_n$, respectively, namely,*

$$\nu_{b(c)} = \bigcap_{i=1}^{n} \nu_{b_i}. \tag{5}$$

*Proof.* It is easy to see that $\bigcap_{i=1}^{n} \nu_{b_i} \subseteq \nu_{b(c)}$, because $\gamma_i \pi_{b_i}(x) \leq \gamma_i \pi_{b'_i}(x)$ for all $b'_i \in B_i$. Next, we prove $\nu_{b(c)} \subseteq \bigcap_{i=1}^{n} \nu_{b_i}$. We will show that for $k = 1..n$, and any point $x \in \mathbb{R}^d$, $x \in \nu_{b(c)}$ implies that $x \in \nu_{b_k}$. For any $b'_k \in B_k$, let

$$b'(c) = \sum_{i=1..n}^{i \neq k} \gamma_i b_i + \gamma_k b'_k.$$

We have $\pi_{b(c)}(x) \leq \pi_{b'(c)}(x)$ if $x \in \nu_{b(c)}$, that is,

$$\sum_{i=1}^{n} \gamma_i \pi_{b_i}(x) \leq \sum_{i=1..n}^{i \neq k} \gamma_i \pi_{b_i}(x) + \gamma_k \pi_{b'_k}(x).$$

Simplifying this inequality, we have $\gamma_k \pi_{b_k}(x) \leq \gamma_k \pi_{b'_k}(x)$, which implies $x \in \nu_{b_k}$, as required. □

Next, we generalize this lemma to any possible values of $c$ in $\Gamma$. We assume $\gamma_i \neq 0$ for any $i$, because for any $\gamma_k = 0$,

$$B(c) = \sum_{i=1..n}^{i \neq k} B_i.$$

If $\gamma_i < 0$, the lemma is true if we substitute $\gamma_k$ with $-\gamma_k$, and $\pi_{b_k}(x)$ with $-\pi_{b_k}(x)$. That means, the lemma remains true if we use $-\pi_{b_k}(x)$ as the distance function, and the furthest-neighbor Voronoi region as $\nu_{b_k}$.

For convenience, we assign signs to the Voronoi complex and its elements. For a weighted point set, $B_0$, its Voronoi complex with '+' sign, $V^+_{B_0}$, is the nearest-neighbor Voronoi complex of $B_0$, and $V^-_{B_0}$ is the furthest-neighbor Voronoi complex of $B_0$. The same rule applies to the Voronoi cells $\nu^+_X$ and $\nu^-_X$.

Therefore, the Voronoi cell of $b(c)$ with respect to $B(c)$ is the intersection of the signed Voronoi cells of $b_i$ with respect to $B_i$, whose signs are determined by the signs of the corresponding $\gamma_i$, namely,

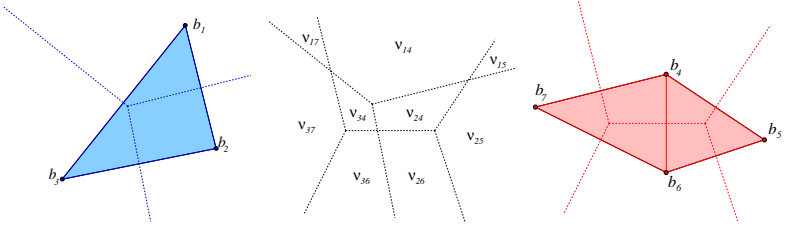
$$\nu_{b(c)} = \bigcap_{i=1}^{n} \nu_{b_i}^{\mathrm{Sign}(\gamma_i)}.$$

This leads to the following Theorem about the intermediate Voronoi complex.

**Theorem 1.** *The intermediate Voronoi complex is the superimposition of all the signed Voronoi complexes,* $V_{B_i}^{Sign(\gamma_i)}$*, that is,*

$$V(c) = \{\nu_{X(c)} = \bigcap_{i=1}^{n} \nu_{X_i}^{Sign(\gamma_i)} \mid \nu_{X(c)} \neq \emptyset, X(c) = \sum_{i=1}^{n} \gamma_i X_i, X_i \subseteq B_i\}.$$

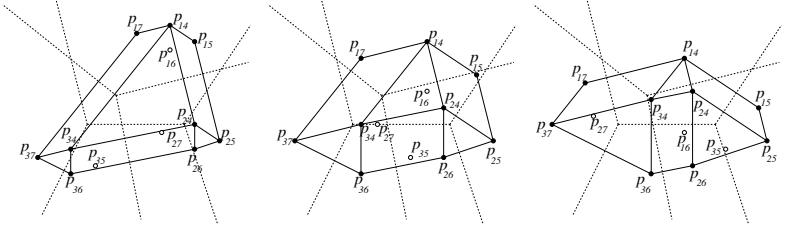
According to this theorem, an intermediate Voronoi cell is the collection of the non-empty intersections of $n$ signed Voronoi cells, from the $n$ signed Voronoi complexes of the reference shapes, respectively. See Figure 4 for an example of superimposing two signed Voronoi complexes when all the coefficients are positive.




(a) The complexes of $B_0 = \{b_1, b_2, b_3\}$.

(b) The Voronoi complex of $tB_0 + (1-t)B_1$ for $t \in (0,1)$.

(c) The complexes of $B_1 = \{b_4, b_5, b_6, b_7\}$.



(d) The complexes of $0.75B_0 + 0.25B_1$.

(e) The complexes of $0.50B_0 + 0.50B_1$.

(f) The complexes of $0.25B_0 + 0.75B_1$.

**Fig. 4.** Deformation of the Delaunay complexes. The Voronoi complex remains the same and it the superimposition of two signed Voronoi complexes

The intermediate Voronoi complexes, $V(c_0)$ and $V(c_1)$, are the same if $\text{Sign}(\gamma_{0,i}) = \text{Sign}(\gamma_{1,i})$ for all $i$, in which $\gamma_{0,i}$, $\gamma_{1,i}$ are the $i$-th coordinates of $c_0$ and $c_1$ respectively. By this, we can divide $\Gamma$ into $2^n - 1$ convex partitions with respect to the signs of the coefficients. The number of partitions is $2^n - 1$ rather than $2^n$, because the coordinates of a coefficient vector $c \in \Gamma$ can not be all negative.

Each partition covers all the values of coefficient vectors, $c = (\gamma_1, \ldots, \gamma_n)$, with the same $(\mathrm{Sign}(\gamma_1), \ldots, \mathrm{Sign}(\gamma_n))$. For any two coefficient vectors in the same partition, their corresponding intermediate Voronoi complexes are the same.

For a deformation of one intermediate shape into another, its parameterizing path is also divided into finitely many pieces. For all the coefficient vectors lie in a same piece, their corresponding intermediate shapes share a same intermediate Voronoi complex. See Figure 4.

**Enumerating Voronoi Cells in $V(c)$.** In $\mathbb{R}^d$, a Voronoi cell, $\nu_X$, is the non-empty intersection of $m$ Voronoi regions. If $m > d + 1 - \dim(\nu_X)$, we call $\nu_X$ *degenerate*. Otherwise, it is simplicial as we defined in the background section. In the intermediate Voronoi complex, the degenerate case is unavoidable. See Figure 4 for an example in $\mathbb{R}^2$. When $n = 2$ and $d = 2$, the intersection of two Voronoi edges is a Voronoi vertex, which is the common intersection of the four Voronoi regions of the four weighted points in the intermediate weighted point set. It is degenerate because it is the intersection of four Voronoi regions while its dimension, $\dim(\nu_{X(c)}) = 0$. There are three such degenerate Voronoi cells in the intermediate Voronoi complex in Figure 4.

We classify all the Voronoi cells in $V(c)$, each of which is either simplicial or degenerate, with regard to the dimensions of their contributors. Let $\tau_j$ be the number of contributors whose dimensions are $j$. We define the *type* of a Voronoi cell as a tuple, $(\tau_0, \ldots, \tau_{d-1})$. The variable $\tau_d$ is omitted, because it is dependent, namely, $\tau_d = n - \sum_{j=0}^{d-1} \tau_j$. Enumerating all the types of $\nu_{X(c)}$ is equivalent to enumerating all the tuples, $(\tau_0, \ldots, \tau_{d-1})$, with the constraint

$$\sum_{j=0}^{d-1}((d-j)\tau_j) \leq d. \tag{6}$$

Next, we are going to explain why this constraint is sufficient. For convenience, we denote the dimension of $\nu_{X(c)}$ as $d(c)$, and the dimension of the contributor, $\nu_{X_i}$, as $d_i$, namely, $d_i = \dim(\nu_{X_i})$. Generally, the dimension of the intersection of two Voronoi cells, $\nu_X$ and $\nu_Y$, is $\dim(\nu_X \cap \nu_Y) = \dim(\nu_X) + \dim(\nu_Y) - d$. The Voronoi cell, $\nu_{X(c)}$, which is the intersection of $n$ Voronoi cells, has the dimension

$$d(c) = \sum_{i=1}^{n} d_i - (n-1)d. \tag{7}$$

We can get the Constraint (6) by substituting $d(c) \geq 0$, $\sum_{i=1}^{n} d_i = \sum_{j=0}^{d}(j\tau_j)$, and $n = \sum_{j=0}^{d} \tau_j$ into the Inequality (7).

To list out all the possible tuples in $\mathbb{R}^d$, we firstly list out all the possible values of $\tau_0$ under the Constraint (6). We get all the possible values of $\tau_1$ for each value of $\tau_0$. Repeating this progress from $\tau_1$ to $\tau_{d-1}$ under the Constraint (6), we can enumerate all the possible tuples. For example, there are seven types of Voronoi cells in $\mathbb{R}^3$, under the constraint $3\tau_0 + 2\tau_1 + \tau_2 \leq 2$. The tuples,

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(0,0,0)$, represent the four simplicial types, Voronoi vertex, edge, polygon, and polytope, respectively. The tuples, $(0,0,2)$, $(0,0,3)$, and $(0,1,1)$ represent three degenerate types.

## 4 Algorithm for Computing Intermediate Complexes

An intermediate shape, body$(B(c))$, is immediate once the intermediate Voronoi complex, $V(c)$, is determined. In this section, we introduce the algorithm to compute $V(c)$.

An intuitive yet time consuming approach is to compute the Voronoi complex of $B(c)$. Assuming $\text{card}(B_i) \leq m$, $i = 1..n$, the cardinality of $B(c)$ is no more than $m^n$. Therefore, the time for computing the Voronoi complex of $B(c)$ directly is

$$O(nm^n \log m + m^{n \times \lceil d/2 \rceil}) \tag{8}$$

[1,3,6].

On the other hand, according to Theorem 1, we can compute $V(c)$ by superimposing the signed Voronoi complexes of the reference shapes. We prove that when $d > 2$, computing $V(c)$ by superimposing is faster than computing it from $B(c)$ directly.

Assuming the value of $c$ is fixed, we simplify the notations $V_{B_i}^{\text{Sign}(\gamma_i)}$ to $V_{B_i}$, and $\nu_{X_i}^{\text{Sign}(\gamma_i)}$ to $\nu_{X_i}$ in the rest of the paper.

**Superimposing Two Voronoi Complexes.** Let $V_0$ and $V_1$ be two intermediate Voronoi complexes. We construct their superimposition, $V_{0,1}$ by computing each of its Voronoi region $\nu_{i,j} = \nu_i \cap \nu_j$, for $\nu_i \in V_0$ and $\nu_j \in V_1$. We can compute $V_{0,1}$ by brute force, namely, testing all the possible pairs of $\nu_i$ and $\nu_j$ for non-empty intersection. This can be achieved by linear programming algorithms.

However, we can improve the construction by a breath-first search manner. Given one Voronoi region $\nu_{i,j} \in V_{0,1}$, we can compute the neighbors of $\nu_{i,j}$ by considering all neighbors of $\nu_i$ and $\nu_j$ in $V_0$ and $V_1$ respectively. The time for this breadth-first search algorithm is output sensitive. The worst case is that we tested all the possible pairs of Voronoi cells, namely $m^2$ pairs. Testing whether two Voronoi regions have non-empty intersection needs $O(d!m)$ time by the linear programming method [5]. Thus, the total time complexity of superimposing $V_0$ and $V_1$ is $O(d!m^3)$.

**Superimposing $n$ Voronoi Complexes.** Next we introduce the algorithm of computing $V(c)$ by superimposing $n$ Voronoi complexes, which is based on the algorithm of superimposing two Voronoi complexes.

For convenience, let $k = \lceil \log_2(n) \rceil$ and $\eta = 2^{k-1}$ and we have $\eta < n \leq 2\eta$. We divide the way of superimposing $n$ Voronoi complexes into three cases, namely, $\eta < n \leq 9\eta/8$, $9\eta/8 < n \leq 3\eta/2$, and $3\eta/2 < n \leq 2\eta$. We prove that when $d > 2$, our algorithm can achieve better efficiency than computing the Voronoi complex of $B(c)$ directly.

We will prove the case for $3\eta/2 < n \leq 2\eta$ first. We superimpose the $n$ Voronoi complexes according to a binary superimposing tree. See Figure 5. In the first

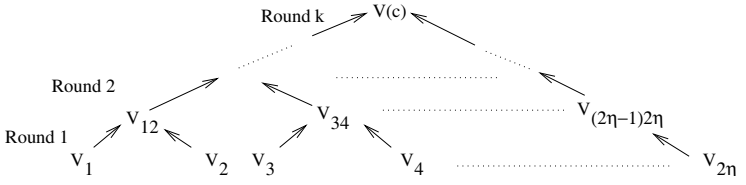



**Fig. 5.** Superimposing $n$ Voronoi complexes when $3\eta/2 < n \leq 2\eta$

round, we superimpose $V_{2j-1}$ and $V_{2j}$ for $j = 1..\eta$. The time complexity for this round is $O(d!2^{k-1}m^3)$. In the second round, we superimpose $V_{(4j-3)(4j-2)}$ and $V_{(4j-1)(4j)}$ for $j = 1..\eta/2$. In either $V_{(4j-3)(4j-2)}$ or $V_{(4j-1)(4j)}$, there are no more than $m^2$ Voronoi regions. Thus, the time complexity for the second round is $O(d!2^{k-2}m^{3\times 2})$. Generally, for the $i$-th round, the time complexity is $O(d!2^{k-i}m^{3\times 2^{i-1}})$, $i = 1..k$. After $k$ rounds of such superimposition, we can get $V(c)$. Therefore, the total time complexity to superimpose the $n$ Voronoi complexes, $V_1$, $V_2$, ... and $V_n$, is

$$O(\sum_{i=1}^{k}(d!2^{k-i}m^{3\times 2^{i-1}})).$$

Since $m$ is greater than 2 in practice, we have $(d!2^{k-i_1}m^{3\times 2^{i_1-1}}) < (d!2^{k-i_2}m^{3\times 2^{i_2-1}})$ for any $i_1 < i_2$. Thus, the total time complexity is equivalent to $O(kd!m^{3\times 2^{k-1}})$. For a fixed $d$, we can ignore the $d!$ and only consider the time complexity as $O(km^{3\times 2^{k-1}})$. Adding in the time of computing $V_1, .., V_n$, the overall time complexity to compute $V(c)$ is

$$O(km^{3\times 2^{k-1}} + nm\log m + nm^{\lceil d/2\rceil}). \tag{9}$$

When $d > 2$ and $n > 1$, the time complexity of the intuitive way in Equation 8 is dominated by $O(m^{n\times\lceil d/2\rceil})$, and the time complexity of our algorithm in Equation 9, is dominated by $O(km^{3\times 2^{k-1}} + nm^{\lceil d/2\rceil})$. In practice, we can assume that $n$, $k$, and $d$ are smaller than $m$. Under such assumption, our algorithm has better efficiency if we can show that $m^{3\times 2^{k-1}} < m^{n\lceil d/2\rceil}$. Since $n > 3\eta/2$ and $d > 2$, we have

$$m^{3\times 2^{k-1}} = m^{3\times\eta} < m^{2n} \leq m^{n\lceil d/2\rceil},$$

as required.

When $9\eta/8 < n \leq 3\eta/2$ and $\eta < n \leq 9\eta/8$, we can build the superimposing tree in two similar ways, both of which has better efficiency than the intuitive way.

## 5 Conclusion

In this paper, we designed a new algorithm to compute the Voronoi complex of an intermediate shape basing on Theorem 1. We prove that when $d > 2$, our algorithm is faster than computing the Voronoi complex from $B(c)$ directly. Moreover, since all the intermediate shapes share finitely many common Voronoi complexes, we are able to compute intermediate shapes in real time, by reusing the generated common Voronoi complexes. This makes it possible to generate, visualize, and customize shape deformations.

About the future direction, we want to save the time complexity of superimposing $n$ Voronoi complexes by rearranging the order of the pairwise superimpositions. We may solve this problem with dynamic programming. The goal is to change the superimposing order so that the cardinalities of the intermediate superimposing results are as small as possible.

## References

[1] Chazelle, B. An optimal convex hull algorithm in any fixed dimension. In *Discrete Comput. Geom.* (1993), pp. 10:377–409.

[2] Cheng, H., Edelsbrunner, H., and Fu, P. Shape Space from Deformation. *Proc. 6th Pacific Conf.. Comput. Graphics Appl.* (1998), 104–113.

[3] Clarkson, K. L., and Shor, P. W. Applications of random sampling in computational geometry, II. *Discrete and Computational Geometry 4*, 1 (1989), 387–421.

[4] Edelsbrunner, H. Deformable Smooth Surface Design. *Discrete Comput. Geom 21* (1999), 87–115.

[5] Seidel, R. Linear programming and convex hulls made easy. In *Proc. 6th Annu ACM Sympos. Coput. Geom.* (1990), pp. 211–215.

[6] Seidel, R. Small-dimensional linear programming and convex hulls made easy. In *Discrete Comput. Geom.* (1991), pp. 6:423–434.