THESIS

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Electromagnetic Scattering Problem with Higher Order Impedance Boundary Conditions and Integral Methods

under the supervision of

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Contents

Chapter 0

Introduction

Les concepteurs de systèmes radars et d'antenne se sont intéressés à l'étude théorique de la diffusion des ondes électromagnétiques. L'intérêt pour ce sujet a suscité des recherches intensives dans ce domaine depuis longtemps. Les innovations technologiques et informatiques ont permis un développement de la modélisation et notamment pour la modélisation de la diffusion des ondes électromagnétiques. Mais les méthodes numériques utilisent en général un grand nombre d'inconnues pour la description des champs électromagnétiques, notamment à haute fréquence.

Dans les études de problème de dispersion on considere souvent un objet parfaitment conducteur recouvert ou non d'une couche. Cette couche mince peut être une surface homogène, chirale ou une surface sélective en fréquence (FSS). Les FSS sont très utilisées dans la modélisation d'antennes.

Il existe deux méthodes importantes pour résoudre les équations de Maxwell en régime harmonique. La première méthode est la méthode des volumes. Elle permet de calculer les champs sur tout le volume. Si le domaine de calcul est non borné, on peut par exemple utiliser des conditions aux limites sur une frontière artificielle. Elle tient compte des caractéristiques physiques des matériaux étudiés et des effets d'anisotropie, mais elle nécessite un grand nombre d'inconnues. La seconde méthode est la méthode intégrale. Elles consistent à calculer les courants électrique et magnétique sur la surface de l'objet rayonnant. Ici, nous allons utiliser la méthode intégrale pour la résolution du problème de diffraction d'une onde électromagnétique en régime harmonique.

Il est nécessaire de considérer des conditions aux limites sur la surface de l'objet rayonnant pour que le problème soit bien posé. Cette condition approchée aux limites sur l'objet fait intervenir un opérateur intégral appelée opérateur d'impédance Z dont la forme la plus simple est Z est constant appelée condition de Léontovitch. On sait que cette condition trouve ses limites quand le matériau a une épaisseur proche de la longueur d'onde. C'est pourquoi, dans cette thèse nous développons une nouvelle méthode dans laquelle l'opérateur d'impédance tient compte de l'angle d'incidence appelée conditions aux limites d'impédance d'ordre supérieur (HOIBC).

Dans [R-Sb, R-Sa], les auteurs ont proposé une méthode d'approxiamtion de l'opérateur d'impédance qui tient compte de l'angle d'incidence en dimension 2 et 3. Des résultats numériques sont donnés notamment dans le cas d'un cylindre recouvert d'un diélectrique et pour la sphère.

D'autres travaux dans ce domaine existent, voir [BSb, BSc, BS, BSa].

L'objectif principal de cette thèse est donc de proposer une nouvelle formulation variationnelle du problème de diffraction électromagnétique en utilisant une méthode purement intégrale couplée avec des conditions d'impédance d'ordre élevée. Nous résolvons ici les équations de Maxwell en régime harmonique.

Dans le chapitre 1, le problème de diffusion en regime harmonique des ondes électromagnétiques par un obstacle est décrit comme un système d'équations avec des conditions au bord. L'unicité de la solution de ce problème est montré. Nous présentons la formulation intégrale des équations de Maxwell pour utiliser la méthode intégrale. Dans le chapitre 2, nous introduisons des espaces fonctionnels de base et les opérateurs qui sont utilisés dans la suite.

Le but de cette thèse est d'écrire une formulation variationnelle. On est donc obligé d'écrire l'opérateur d'impédance Z en fonction d'opérateurs intégraux pour pouvoir l'introduire dans la formulation. Les chapitres 3-4 sont consacrés à cette étude.

Dans le chapitre 5, on propose une formulation variationnelle en 2D et on étudie l'existence et l'unicité. Dans le chapitre 6, on discrétise la formulation et on donne la forme explicite de toutes les matrices et on donne de nombreux résultats numériques qui valident la méthode.

En dimension 3, on se heurte à la définition des opérateurs intégraux. Deux approches sont proposées. Dans le chapitre 7, nous formulons le problème en utilisant des multiplicateurs de Lagrange discrétisés avec les fonctions de Bendali. Nous étudions l'existence et l'unicité de la solution pour cette formulation. Dans le chapitre suivant, la discrétisation du problème est étudiée. Dans le chapitre 9, nous proposons une autre méthode qui utilise les fonctions de base de Buffa-Christiansen. Quelques résultats numériques du cas tridimensionnel sont présentés dans le chapitre 10 pour la formulation étudiée dans le chapitre 7.

Chapter 0

Introduction

Radar and antenna system designers are interested in the theoretical study of the scattering of electromagnetic waves. Interest in this topic has prompted intensive research in this area long time ago. However rigorous analysis was not performed until recently. The development of the computing technology improves modeling possibility and it increases the interest in the scattering problem of electromagnetic waves. The difficulties of numerical methods include the necessity of using a large number of unknowns in the description of high frequency electromagnetic fields.

The scattering problem is being studied for conducting bodies and for a perfect conducting body covered by a complex layer. The complex layer is considered as a homogeneous surface, as a chiral surface or as a frequency selective surface. Presently, the frequency selective surface is important for design artificial coatings of antenna.

There are two important methods for solving the Maxwell's equations in harmonic regime. The first method is the volume method. It locates their computations all over the volume internal and external objects. It uses a domain containing the obstacles bounded by an artificial border. It considers the physical characteristics of the media, in particular the effects of anisotropy, but it requires a large number of unknowns and the management of explicit boundary conditions. The second one is the integral method. It places unknowns on the boundaries of the object and it takes into account the boundary conditions. It allows reducing the external problem to a system of integral equations defined on the surface of the obstacle. However, they can only be applied to homogeneous bodies. We can use this method for a three dimensional domain and for a two dimensional domain. Here, we are interested in solving time-harmonic scattering problem for a coated body by the integral method.

In order to ensure a unique solution to the boundary value problem it is necessary to apply boundary condition. Generally, we take the constant impedance operator, known as standard or Leontovich impedance boundary conditions. This approximation does not depend on incident angle at all. In this thesis we deal with higher order impedance boundary conditions.

Recently, the higher order impedance boundary conditions have been studied in [R-Sb, R-Sa]. These conditions take into account the incident angle at each point of the surface and include derivatives of tangential components of the fields that are equivalent to transverse wave numbers. Y. Rahmat-Samii and D.J. Hoppe present the numerical results for two dimensional cylinders with lossy and lossless dielectric coatings and for three dimensional body of revolution. The system was solved for tangential electric **E** and magnetic **H** fields.

Later, the higher order impedance boundary condition is applied to study the scattering problem from a finite planar or curved infinitesimally thin frequency selective surface embedded in a dielectric layer [BSb, BSc, BS, BSa]. B.Stupfel introduces differential operators to express higher order impedance boundary conditions. He solves combined field integral equation for unknown current density **J**.

The main subject of this thesis is to propose a new variational formulation of electromagnetic scattering problem with approximate impedance boundary conditions. The impedance depends on the coating layer thickness, the dielectric characteristics and the incident angle. In practice the incident angle is unknown. Therefore our purpose was to write the formulation that is valid in a large angular range. We proposed the differential forms of the impedance boundary condition. The formulation is presented for both two and three dimensional models. It includes both the electric and magnetic current densities **J** and **M** as unknowns. We studied existence and uniqueness for the variational problems in both models.

In chapter 1, the time-harmonic scattering problem of electromagnetic waves by an obstacle is described as a system of equations with boundary conditions. The uniqueness of the solution of this problem is shown. We present integral formulation of Maxwell's equations to use the integral method. In chapter 2, we introduce some basic functional spaces and operators that is used in this thesis.

Chapters 3-4 are devoted to impedance boundary conditions. We consider thin dielectric layer in chapter 3. The impedance operator is approximated as a ratio of polynomials of differential operators, so that the boundary conditions are presented as an equation of these polynomials. We call this condition as higher order IBC (HOIBC). HOIBC of first and second order for two dimensional case are described. Using differential operators first order IBC is proposed for three dimensional case. We propose sufficient uniqueness conditions that imply restrictions on the coefficients of HOIBC. In chapter 4, different ways to find the coefficients for HOIBC using the exact IBC are described.

Chapters 5-6 of the thesis deal with the two dimensional case. In chapter 5, we assemble the variational formulation with the first or the second order HOIBC. The existence and uniqueness of the solution of this problem are shown. The discretization of two dimensional problem is proposed in chapter 6 and the calculation of matrices are described to solve this problem. We present numerical results.

In chapter 7, we formulate three dimensional problem and we study the existence and uniqueness of the solution. Chapter 8 is concerned with the discretization of three dimensional problem. The problem with auxiliary unknowns is discretized with help of Rao-Wilton-Glisson basis functions for mesh triangulation. In the chapter 9, we introduce Buffa-Christiansen basis functions and propose a method to apply these functions for three dimensional HOIBC formulation. Some numerical results of three dimensional case are presented in chapter 10.

Chapter 1

Scattering problem

Contents

1.1 Introduction

We present the scattering problem as the system of equations. The system includes boundary conditions on the surface of an object and radiation conditions far from an object. Here, we prove the uniqueness theorem and we introduce an integral formulation of the problem as well.

1.2 Mathematical model of physical problem

We consider the scattering problem of electromagnetic waves ($\mathbf{E}$, $\mathbf{H}$) by a perfect conducting body with a complex coating, as depicted in figure 1.1. We denote Ω an open domain in $\mathbb{R}^n$ (with $n = 2, 3$) with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$, which can be equipped with an exterior unit normal vector field $\mathbf{n}$. Electromagnetic waves propagate in $\Omega^+ = \mathbb{R}^n \backslash \overline{\Omega}$. We illuminate this system by incident electromagnetic waves. Scattering waves occur when incident waves bounce off an object in a variety of directions. The amount of scattering waves that take place depends on the wavelength of the incident waves and structure of the object. We determine total electromagnetic fields $(\mathbf{E}, \mathbf{H})$ in Ω^+ as:

$$\begin{cases} \mathbf{E} = \mathbf{E}^{inc} + \mathbf{E}^{sc} \\ \mathbf{H} = \mathbf{H}^{inc} + \mathbf{H}^{sc} \end{cases} \tag{1.1}$$

Superscripts *inc* and *sc* characterize incident and scattered fields, respectively. Waves propagation medium is described by two values ε (electrical permittivity) and μ (magnetic permeability), where we have $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ for free space.

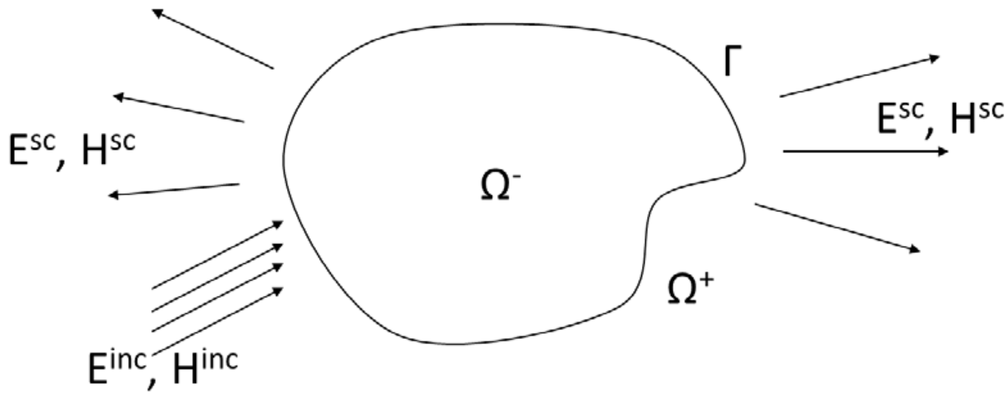

Figure 1.1: Scattering Problem

We are interested in the time-harmonic electromagnetic fields that are defined as

$$\begin{cases}\mathbf{E}(x,t) = \Re(\mathbf{E}(x)e^{i\omega t}) \\ \mathbf{H}(x,t) = \Re(\mathbf{H}(x)e^{i\omega t})\end{cases} \tag{1.2}$$

where ω denotes the pulsation. The fields outside the body are governed by Maxwell's equations for a free space. The harmonic solution verifies following equations:

$$\begin{cases}\mathbf{rotE} + i\omega\mu\mathbf{H} = 0 \\ \mathbf{rotH} - i\omega\varepsilon\mathbf{E} = 0\end{cases} \tag{1.3}$$

The fields inside the coating are governed by a set of equations that take into account the detailed electromagnetic properties of the coating.

We consider boundary condition that binds the tangent electric and magnetic fields. The medium characteristics give an impedance at each point of the surface Γ.

$$\mathbf{E}_{tg} - Z(\mathbf{n} \times \mathbf{H}) = 0 \quad on\ \Gamma \tag{1.4}$$

where Z is impedance operator that depends on incident angle, medium thickness and characteristics ε and μ. Subscript tg denotes tangent component on the surface Γ defined as:

$$\mathbf{E}_{tg} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n}).$$

The boundary condition (1.4) is called impedance boundary condition (IBC). The simplest form of which is known as Leontovich IBC or standard IBC (SIBC), where $Z = constant$. [R-Sb, SV, TL] The IBC can be partially constant (if the object is formed by different materials) or more different.

For the correct formulation of the problem, we should introduce asymptotic behavior of the fields $(\mathbf{E}, \mathbf{H})$, the Silver-Müller radiation condition:

$$\lim_{r\to\infty} r(\mathbf{E} \times \mathbf{n}_r + \mathbf{H}) = 0, \tag{1.5}$$

where $r = |\mathbf{x}|$ and $\mathbf{n}_r = \dfrac{\mathbf{x}}{|\mathbf{x}|}$, $\mathbf{x} \in \mathbb{R}^3$.

So, we have next problem:

Problem 1.2.1 *Find* $(\boldsymbol{E}, \boldsymbol{H})$ *such that*

$$\begin{cases} \mathbf{rot}\boldsymbol{E} + ik_0\mu\boldsymbol{H} = 0 \quad in\ \Omega^+ \\ \mathbf{rot}\boldsymbol{H} - ik_0\varepsilon\boldsymbol{E} = 0 \quad in\ \Omega^+ \\ \boldsymbol{E}_{tg} - Z(\boldsymbol{n} \times \boldsymbol{H}) = 0 \quad on\ \Gamma \\ \lim_{r\to\infty} r(\boldsymbol{E} \times \boldsymbol{n}_r + \boldsymbol{H}) = 0 \end{cases} \tag{1.6}$$

1.3 Uniqueness of the scattering problem

Let $\mathbf{E}$ be a solution of the problem (1.6) therefore it verifies Helmholtz equation

$$\Delta\mathbf{E} + k^2\mathbf{E} = 0. \tag{1.7}$$

We recall the Rellich lemma.

Lemma 1.3.1 ***(Rellich)****: Let* Ω *be the open complement of a closed domain and* $u \in L^2(\mathbb{R}^3 - \Omega)$ *is a solution of the Helmholtz equation, satisfying*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 dx = 0. \tag{1.8}$$

Then $u = 0$*, in* $\mathbb{R}^3 - \Omega$*.*

Below we propose the uniqueness theorem [BSa, FL]. It gives us sufficient condition on characteristics of medium to get unique solution of the problem 1.2.1. The existence will be shown later in this thesis.

Theorem 1.3.1 *The problem 1.2.1 admits a unique solution, if following relations are verified:*

$$\begin{cases} \Im(\mu) \leq 0, \\ \Im(\varepsilon) \leq 0, \\ \Re(k_0 \int_\Gamma \mathbf{E}^* \cdot (\mathbf{n} \times \mathbf{H}) ds) \geq 0. \end{cases} \tag{1.9}$$

Proof. : Let $\mathbf{E}$ is a non-zero solution of scattering problem. Let B_R is a ball of radius R that is big enough for $\Omega \subseteq B_R$ and $\mathbf{n}_R$ be an exterior unit normal vector of B_R. We remember that $\mathbf{n}$ is an exterior unit vector of $\Gamma = \partial\Omega$. From Maxwell's equations we have

$$\mathrm{rot}(\mu^{-1}\mathrm{rot}\mathbf{E}) - k_0^2\varepsilon\mathbf{E} = 0 \quad in\ B_R\backslash\Omega.$$

Then we write this equation in variational form. We take $\mathbf{E}^*$ (conjugate to $\mathbf{E}$) as a test function and we integrate over volume $B_R\backslash\Omega$:

$$\int_{B_R\backslash\Omega^-} \mathrm{rot}(\mu^{-1}\mathrm{rot}\mathbf{E}) \cdot \mathbf{E}^* - k_0^2\varepsilon\mathbf{E} \cdot \mathbf{E}^* d\omega = 0. \tag{1.10}$$

We use integration by parts

$$\int_{B_R\setminus\Omega}\mu^{-1}\mathrm{rot}\mathbf{E}\cdot\mathrm{rot}\mathbf{E}^*d\omega+\int_{\partial B_R}(\mathbf{n}_R\times\mu^{-1}\mathrm{rot}\mathbf{E})\cdot\mathbf{E}^*ds$$
$$-\int_\Gamma(\mathbf{n}\times\mu^{-1}\mathrm{rot}\mathbf{E})\cdot\mathbf{E}^*ds-\int_{B_R\setminus\Omega^-}k_0^2\varepsilon\mathbf{E}\cdot\mathbf{E}^*d\omega=0 \tag{1.11}$$

According to Maxwell's equations (1.6) we replace rot$\mathbf{E}$ by next relation

$$\mathrm{rot}\mathbf{E}=-ik_0\mu\mathbf{H}\ \ in\ B_R\setminus\Omega.$$

So we get next two equations

$$-\int_\Gamma(\mathbf{n}\times\mu^{-1}\mathrm{rot}\mathbf{E})\cdot\mathbf{E}^*ds=ik_0\int_\Gamma\mathbf{E}^*\cdot(\mathbf{n}\times\mathbf{H})ds$$

$$\int_{\partial B_R}(\mathbf{n}_R\times\mu^{-1}\mathrm{rot}\mathbf{E})\cdot\mathbf{E}^*ds=-ik_0\int_{\partial B_R}\mathbf{E}^*\cdot(\mathbf{n}_R\times\mathbf{H})ds=-ik_0\int_{\partial B_R}\mathbf{H}\cdot(\mathbf{E}^*\times\mathbf{n}_R)ds$$

Since

$$\left|\int_{\partial B_R}-\mathbf{H}\cdot(\mathbf{E}^*\times\mathbf{n}_R)-|\mathbf{H}|^2ds\right|=\left|\int_{\partial B_R}-\mathbf{H}\cdot(\mathbf{E}^*\times\mathbf{n}_R+\mathbf{H}^*)ds\right|$$
$$\leq\int_{\partial B_R}|\mathbf{H}||\mathbf{E}^*\times\mathbf{n}_R+\mathbf{H}^*|ds$$

and for sufficiently big R, because of the behavior of the fields at infinity (Silver-Müller radiation condition):

$$-ik_0\lim_{R\to\infty}\int_{\partial B_R}\mathbf{H}\cdot(\mathbf{E}^*\times\mathbf{n}_R)ds=-ik_0\lim_{R\to\infty}\int_{\partial B_R}|\mathbf{H}|^2ds$$

Finally

$$\lim_{R\to\infty}\int_{\partial B_R}ik_0|\mathbf{H}|^2ds+\int_\Gamma ik_0\mathbf{E}^*\cdot(\mathbf{n}\times\mathbf{H})ds+\int_{B_R\setminus\Omega^-}\mu^{-1}|\mathrm{rot}\mathbf{E}|^2-k_0\varepsilon|\mathbf{E}|^2d\omega=0$$

We take imaginary part

$$\lim_{R\to\infty}\int_{\partial B_R}k_0|\mathbf{H}|^2ds+\int_\Gamma k_0\Re(\mathbf{E}^*\cdot(\mathbf{n}\times\mathbf{H}))ds+\int_{B_R\setminus\Omega^-}\Im(\mu^{-1})|\mathrm{rot}\mathbf{E}|^2-k_0\Im(\varepsilon)|\mathbf{E}|^2d\omega=0$$

According to the theorem hypothesis every integral of the last equation is not less than zero, so each should be equal to zero. Particularly the first integral is zero. The Rellich lemma gives that $\mathbf{H}=0$, which implies that $\mathbf{E}=0$ and we have the uniqueness of a solution.

□

That is how we get sufficient uniqueness condition (SUC) in general form

$$\boxed{\Re(k_0\int_\Gamma(\mathbf{n}\times\mathbf{H})\cdot\mathbf{E}^*)\geq 0} \tag{1.12}$$

From a physical point of view, this condition has its own meaning. In some studies described the condition necessary for conservative or dissipative system.

1.4 Integral formulation of Maxwell's equations

We will use this integral method to solve problem. We express the electromagnetic field as a function of potentials defined on Γ. The Stratton-Chu formulation (B.20) helps us to characterize the electromagnetic fields in terms of surface current densities. These current densities are uknowns in the integral formulation of the problem.

We introduce current densities $\mathbf{J}$ and $\mathbf{M}$ on the boundary Γ as follows

$$\mathbf{M} = [\mathbf{E} \times \mathbf{n}]^{+}_{-} \quad \mathbf{J} = [\mathbf{n} \times \mathbf{H}]^{+}_{-}$$

where $[\ ]^{+}_{-}$ denotes difference between upper $(+)$ and lower (-) values of interface, $\mathbf{n}$ is the exterior normal vector to the surface.

We introduce the variational form of the operators $(B - S)$, $(P + Q)$ and I as follows

$$\langle (B-S)\mathbf{A}, \boldsymbol{\psi} \rangle = i \iint_\Gamma kG\mathbf{A} \cdot \boldsymbol{\psi} - \frac{1}{k} G \nabla_y \cdot \mathbf{A} \nabla_x \cdot \boldsymbol{\psi} dy dx \tag{1.13}$$

$$\langle (P+Q)\mathbf{A}, \boldsymbol{\psi} \rangle = \frac{1}{2} \int_\Gamma \boldsymbol{\psi} \cdot (\mathbf{n} \times \mathbf{A}) dx + \iint_\Gamma (\boldsymbol{\psi} \times \mathbf{A}) \cdot \nabla_x G dy dx \tag{1.14}$$

$$\langle I\mathbf{A}, \boldsymbol{\psi} \rangle = \int_\Gamma \mathbf{A} \cdot \boldsymbol{\psi} dx \tag{1.15}$$

Therefore we can rewrite the Stratton-Chu formulation (B.20) in terms of these operators:

$$< Z_0 (B-S)\mathbf{J}, \boldsymbol{\Psi}_J > + < (P+Q)\mathbf{M}, \boldsymbol{\Psi}_J > = < IE^{inc}, \boldsymbol{\Psi}_J > \tag{1.16}$$

$$- < (P+Q)\mathbf{J}, \boldsymbol{\Psi}_M > + < \frac{1}{Z_0}(B-S)\mathbf{M}, \boldsymbol{\Psi}_M > = < IH^{inc}, \boldsymbol{\Psi}_M > \tag{1.17}$$

Note however that these two equations are completely equivalent. Later, we will use the crossing relations between the different fields to complete the system, such as impedance boundary conditions. It is also clear from (B.19) that the knowledge of $\mathbf{M}$ and $\mathbf{J}$ on the contour of the volume is sufficient to determine the field throughout the space.

1.5 Conclusion

We have established a system of equations (1.6) describing the scattering problem of a coated object with the boundary condition on Γ. We have introduced the impedance boundary conditions (IBC). IBC is essential to our work. Radiation conditions helped us to establish the conditions (1.9) that give the uniqueness of solution.

The Stratton-Chu formulation indicates how the volume problem reduced to the problem on a surface of an object. This constitutes the main concept of integral method. Moreover numerical results derived with help of the code of Dassault Aviation (Thanks a lot), that is based on these equations.

Chapter 2

Functional framework

Contents

2.1 Introduction

In this chapter we give some definitions for differential operators in Ω^+ and in $\Gamma = \partial\Omega$. We define Sobolev spaces with their norms that necessary to establish the variational formulations. We introduce scalar and vectorial Laplacian operators on the surface. We recall Green formula and trace results.

Integral operators $(B - S)$ and $(P + Q)$ with Green kernel will be defined. We saw them in the previous chapter. Here, we will propose their properties. At the end we present two Fredholm alternatives. They are interesting for existence and uniqueness theorem.

2.2 Definitions of differential operators

2.2.1 Differential operators in Ω

We define well-known differential operators such as gradient, divergence and curl:

$$\nabla : D'(\Omega) \to D'(\Omega)^3$$

$$v \to (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3})$$

$$\text{div} : D'(\Omega)^3 \to D'(\Omega)$$

$$\mathbf{v} = (v_1, v_2, v_3) \to \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

$$\mathbf{rot} : D'(\Omega)^3 \to D'(\Omega)^3$$

$$\mathbf{v} = (v_1, v_2, v_3) \to (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})$$

These operators have next properties:

$$\begin{cases} \mathbf{rot}\nabla v = 0 \quad \forall v \in D'(\Omega) \\ \mathrm{div}\ \mathbf{rot}\mathbf{v} = 0 \quad \forall \mathbf{v} \in D'(\Omega)^3 \end{cases}$$

It means that the kernel of **rot** includes image of ∇ and kernel of div includes image of **rot**:

$$\begin{cases} Im\nabla \subset ker\ \mathbf{rot} \\ Im\ \mathbf{rot} \subset ker\ \mathrm{div} \end{cases}$$

Let Ω be an open set in $\mathbb{R}^3$. We briefly recall the definition and the main properties of the Sobolev spaces on Ω.

First, we define following space:

$$H^1(\Omega) = \{v \in L^2(\Omega), \nabla\mathbf{v} \in L^2(\Omega)^3\},$$

equipped with the norm

$$\|v\|_{H^1(\Omega)} = (\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\nabla(\mathbf{v})\|^2_{L^2(\Omega)})^{1/2}.$$

Then, we introduce

$$H(\mathrm{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3,\ \mathrm{div}(\mathbf{v}) \in L^2(\Omega)\}$$

with the norm

$$\|\mathbf{v}\|_{H(\mathrm{div},\Omega)} = (\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}(\mathbf{v})\|^2_{L^2(\Omega)})^{1/2}$$

and the space

$$H(\mathrm{rot}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3,\ \mathbf{rot}(\mathbf{v}) \in L^2(\Omega)^3\}$$

equipped with the norm

$$\|\mathbf{v}\|_{H(\mathrm{rot},\Omega)} = (\|\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathbf{rot}(\mathbf{v})\|^2_{L^2(\Omega)})^{1/2}.$$

We assume next hypothesis:

- Γ is a bounded Lipschitz boundary.
- Ω is placed locally on one side of the Γ.

We recall the following results

Lemma 2.2.1 *The space $D(\overline{\Omega})^3$ is dense in $H(\mathrm{div}, \Omega)$ and in $H(\mathrm{rot}, \Omega)$, as $D(\overline{\Omega})$ is dense in $H^1(\Omega)$. [MC]*

2.2.2 Differential operators on Γ

Now we are going to define differential operators on the surface Γ, with help of scalar and vector functions defined on the same surface:

$$f \in D(\Gamma) \to \tilde{f} \in D(\mathbb{R}^3) \textit{ such that } \tilde{f}|_\Gamma = f$$

and

$$\mathbf{g} \in D(\Gamma)^3 \to \tilde{\mathbf{g}} \in D(\mathbb{R}^3)^3 \textit{ such that } \tilde{\mathbf{g}}|_\Gamma = \mathbf{g}$$

where $\mathbf{n} \cdot \mathbf{g} = 0$.

Definition 1 *On the surface Γ we have the tangential rotational vector for $f \in D(\Gamma)$:*

$$\mathbf{rot}_\Gamma f(x) = \nabla \tilde{f}(x) \times \mathbf{n} = \mathbf{rot}(\mathbf{n} f(x)). \tag{2.1}$$

And the scalar surface rotational vector for $\mathbf{g} \in D(\Gamma)^3$ such that $\mathbf{n} \cdot \mathbf{g} = 0$:

$$\mathrm{rot}_\Gamma \mathbf{g}(x) = \mathbf{n} \cdot \mathbf{rot} \tilde{\mathbf{g}}(x). \tag{2.2}$$

Definition 2 *On the surface Γ we have the scalar Laplacian or Laplace-Beltrami operator acting on a function $f \in D(\Gamma)$ is*

$$\Delta_\Gamma f(x) = \mathrm{div}_\Gamma \nabla_\Gamma f(x) = -\mathrm{rot}_\Gamma \mathbf{rot}_\Gamma f(x). \tag{2.3}$$

The vectorial Laplacian or Hodge operator acting on a tangent vector field $\mathbf{g} \in D(\Gamma)^3$ is

$$\Delta_\Gamma \mathbf{g} = \nabla_\Gamma \mathrm{div}_\Gamma \mathbf{g} - \mathbf{rot}_\Gamma \mathrm{rot}_\Gamma \mathbf{g}. \tag{2.4}$$

The surface divergence for $\mathbf{g} \in D(\Gamma)^3$ such that $\mathbf{n} \cdot \mathbf{g} = 0$:

$$\mathrm{div}_\Gamma \mathbf{g}(x) = -\mathrm{rot}_\Gamma (\mathbf{g}(x) \times \mathbf{n}) \tag{2.5}$$

The tangential gradient for $f \in D(\Gamma)$:

$$\nabla_\Gamma f(x) = -\mathbf{n} \times (\mathbf{n} \times \nabla \tilde{f}(x)) \tag{2.6}$$

Proposition 2.2.1 *The operators $\mathbf{rot}_\Gamma$ and rot_Γ extend in unique way to $D'(\Gamma)^3$ and $D'(\Gamma)$ by relations:*

$$\begin{cases} < \mathbf{rot}_\Gamma f, \boldsymbol{\psi} >= - < f, \mathrm{rot}_\Gamma \boldsymbol{\psi} > \;\; \forall f \in D'(\Gamma), \forall \boldsymbol{\psi} \in D(\Gamma)^3 \textit{ such that } \boldsymbol{\psi} \cdot \mathbf{n} = 0 \\ < \mathrm{rot}_\Gamma \mathbf{g}, \phi >= - < \mathbf{g}, \mathbf{rot}_\Gamma \phi > \;\; \forall \mathbf{g} \in D'(\Gamma)^3 \textit{ such that } \mathbf{g} \cdot \mathbf{n} = 0, \forall \phi \in D(\Gamma) \end{cases}$$

The operators ∇_Γ and div_Γ extend in unique way to $D'(\Gamma)$ and $D'(\Gamma)^3$ by relations:

$$\begin{cases} < \nabla_\Gamma f, \boldsymbol{\psi} >= - < f, \mathrm{div}_\Gamma \boldsymbol{\psi} > \;\; \forall f \in D'(\Gamma), \forall \boldsymbol{\psi} \in D(\Gamma)^3 \textit{ such that } \boldsymbol{\psi} \cdot \mathbf{n} = 0 \\ < \mathrm{div}_\Gamma \mathbf{g}, \phi >= - < \mathbf{g}, \nabla_\Gamma \phi > \;\; \forall \mathbf{g} \in D'(\Gamma)^3 \textit{ such that } \mathbf{g} \cdot \mathbf{n} = 0, \forall \phi \in D(\Gamma) \end{cases}$$

Proposition 2.2.2 *From the definition of surface divergence (2.5), we give next equations for all* $\mathbf{g} \in D'(\Gamma)^3$ *such that* $\mathbf{g} \cdot \mathbf{n} = 0$*:*

$$\mathrm{div}_\Gamma \mathbf{g} = -\mathrm{div}_\Gamma(\mathbf{n} \times (\mathbf{n} \times \mathbf{g}))$$

$$\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{g}) = -\mathrm{rot}_\Gamma \mathbf{g}$$

Now we are going to define Sobolev spaces on the surface Γ. The spaces $H^{-1/2}(\mathrm{div},\ \Gamma)$ and $H^{-1/2}(\mathrm{rot},\ \Gamma)$ are defined as follows:

$$H^{-1/2}(\mathrm{div},\ \Gamma) = \{\mathbf{g} \in H^{-1/2}(\Gamma)^3,\ \mathbf{g} \cdot \mathbf{n} = 0,\ \mathrm{div}_\Gamma(\mathbf{g}) \in H^{-1/2}(\Gamma)\}$$

$$\|\mathbf{g}\|_{-1/2,\mathrm{div}_\Gamma} = \Big(\|\mathbf{g}\|^2_{-1/2,H(\Gamma)} + \|\mathrm{div}_\Gamma(\mathbf{g})\|^2_{-1/2,H(\Gamma)}\Big)^{1/2}$$

and

$$H^{-1/2}(\mathrm{rot},\ \Gamma) = \{\mathbf{g} \in H^{-1/2}(\Gamma)^3,\ \mathbf{g} \cdot \mathbf{n} = 0,\ \mathbf{rot}_\Gamma(\mathbf{g}) \in H^{-1/2}(\Gamma)^3\}$$

$$\|\mathbf{g}\|_{-1/2,\mathrm{rot}_\Gamma} = \Big(\|\mathbf{g}\|^2_{-1/2,H(\Gamma)} + \|\mathbf{rot}_\Gamma(\mathbf{g})\|^2_{-1/2,H(\Gamma)}\Big)^{1/2}$$

Proposition 2.2.3 *We have duality relations:*

$$\Big(H^{-1/2}(\mathrm{rot},\ \Gamma)\Big)' = H^{-1/2}(\mathrm{div},\ \Gamma) \quad \textit{and} \quad \Big(H^{-1/2}(\mathrm{div},\ \Gamma)\Big)' = H^{-1/2}(\mathrm{rot},\ \Gamma)$$

and $\forall \mathbf{f} \in H^{-1/2}(\mathrm{div},\ \Gamma);\quad \forall \mathbf{g} \in H^{-1/2}(\mathrm{rot},\ \Gamma)$ *we have*

$$\|\mathbf{f}\|_{(-1/2,\mathrm{rot}_\Gamma)'} = \sup_{\mathbf{g} \neq 0 \in H^{-1/2}(\mathrm{rot},\Gamma)} \frac{<\mathbf{f},\mathbf{g}>}{\|\mathbf{g}\|_{-1/2,\mathrm{rot}_\Gamma}} = \|\mathbf{f}\|_{-1/2,\mathrm{div}_\Gamma}$$

$$\|\mathbf{g}\|_{(-1/2,\mathrm{div}_\Gamma)'} = \sup_{\mathbf{f} \neq 0 \in H^{-1/2}(\mathrm{div},\Gamma)} \frac{<\mathbf{g},\mathbf{f}>}{\|\mathbf{f}\|_{-1/2,\mathrm{div}_\Gamma}} = \|\mathbf{g}\|_{-1/2,\mathrm{rot}_\Gamma}$$

Green's Formula: For all $\mathbf{u}, \mathbf{v} \in H(\mathrm{rot}, \Omega)$

$$\int_\Omega (\mathbf{u} \cdot \mathbf{rot}(\mathbf{v}^*) - \mathbf{rot}(\mathbf{u}) \cdot \mathbf{v}^*) dx = <\mathbf{n} \times \mathbf{u}_{|\Gamma}, \mathbf{n} \times (\mathbf{n} \times \mathbf{v}_{|\Gamma})> \tag{2.7}$$

where $< .,. >$ denotes antiduality product $H^{-1/2}(\mathrm{div},\Gamma) - H^{-1/2}(\mathrm{rot},\Gamma)$.

2.2.3 Trace results

The following trace theorems are basic in fundamental analysis for electromagnetism

Theorem 2.2.1 *We define a linear application* γ_0 *by:*

$$\forall \mathbf{u} \in H^1(\Omega), \quad \gamma_0 \mathbf{u} = \mathbf{u}_{|\Gamma}$$

Then γ_0 *is continuous from* $H^1(\Omega)$ *to* $H^{1/2}(\Gamma)$ *equipped with the norm and also we have:*

$$\forall \mathbf{u} \in H^1(\Omega), \|\gamma_0 \mathbf{u}\|_{1/2,H(\Gamma)} \leq C(\Gamma)\|\mathbf{u}\|_{H^1(\Omega)}$$

$$\forall \mathbf{f} \in H^{1/2}(\Gamma), \quad \exists \mathbf{u} \in H^1(\Omega) \quad \textit{such that}$$

$$\mathbf{f} = \gamma_0 \mathbf{u} \quad \textit{and} \quad \|\mathbf{u}\|_{H^1(\Omega)} \leq C(\Gamma)\|\mathbf{f}\|_{1/2,H(\Gamma)}$$

Proof. : See [TE] pp.110-113. □

Theorem 2.2.2 *We define linear application* γ_n *as:*

$$\forall \mathbf{u} \in [H^1(\Omega)]^3, \quad \gamma_n \mathbf{u} = -\mathbf{n} \times \mathbf{u}_{|\Gamma}$$

Then, γ_n *extends in unique form to a continuous linear application from* $H(\mathrm{rot}, \Omega)$ *to* $H^{-1/2}(\mathrm{div}, \Gamma)$ *and also we have:*

$$\forall \mathbf{u} \in H(\mathrm{rot}, \Omega), \quad \|\gamma_n \mathbf{u}\|_{-1/2,\mathrm{div}_\Gamma} \leq C(\Gamma)\|\mathbf{u}\|_{H(\mathrm{rot},\Omega)}$$

$$\forall \mathbf{f} \in H^{-1/2}(\mathrm{div}, \Gamma), \; \exists \mathbf{u} \in H(\mathrm{rot}, \Omega)$$

$$\textit{such that} \quad \mathbf{f} = \gamma_n \mathbf{u} \quad \textit{and} \quad \|\mathbf{u}\|_{H(\mathrm{rot},\Omega)} \leq C(\Gamma)\|\mathbf{f}\|_{-1/2,\mathrm{div}_\Gamma}$$

Proof. : See [TE] pp.124-125 □

Theorem 2.2.3 *We define linear application* Π_Γ *as:*

$$\forall \mathbf{u} \in [H^1(\Omega)]^3, \quad \Pi_\Gamma \mathbf{u} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{u}_{|\Gamma})$$

Then, Π_Γ *extends in unique form to a continuous linear application from* $H(\mathrm{rot}, \Omega)$ *to* $H^{-1/2}(\mathrm{rot}, \Gamma)$ *and also we have:*

$$\forall \mathbf{u} \in H(\mathrm{rot}, \Omega), \quad \|\Pi_\Gamma \mathbf{u}\|_{-1/2,H(\mathrm{rot},\Gamma)} \leq C(\Gamma)\|\mathbf{u}\|_{H(\mathrm{rot},\Omega)}$$

$$\forall \mathbf{g} \in H^{-1/2}(\mathrm{rot}, \Gamma), \; \exists \mathbf{u} \in H(\mathrm{rot}, \Omega)$$

$$\textit{such that} \quad \mathbf{g} = \Pi_\Gamma \mathbf{u} \quad \textit{and} \quad \|\mathbf{u}\|_{H(\mathrm{rot},\Omega)} \leq C(\Gamma)\|\mathbf{g}\|_{-1/2,\mathrm{rot}_\Gamma}$$

Proof. : See [TE] pp.125-126 □

2.3 Properties of integral operators

Some solutions of the problems of diffraction of electromagnetic waves can be expressed with the help of potential and in particular the potential of single and dual layer defined on the surface of the obstacle. We present here the properties of the harmonic potential.

Definition 3 *We introduce the potentials $(B - S)$ and Q, that are defined by*

$$(B - S)\mathbf{J} := \int_\Gamma \left(G(x, y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x, y)\mathrm{div}_\Gamma \mathbf{J} \right) d\Gamma(y) \tag{2.8}$$

$$Q\mathbf{M} := \int_\Gamma \nabla_y G(x, y) \times \mathbf{M} d\Gamma(y) \tag{2.9}$$

and $G(x, y)$ is the Green kernel giving the outgoing solutions to the scalar Helmholtz equation:

- *in 2D case*

$$G(x, y) := \frac{\pi}{i} H_0^{(2)}(k|x - y|), \quad \nabla_x G(x, y) := -\frac{\pi k}{i|x - y|} H_1^{(2)}(k_j|x - y|)(\mathbf{x} - \mathbf{y}) \tag{2.10}$$

- *in 3D case*

$$G(x, y) := \frac{e^{-ik|x-y|}}{4\pi|x - y|}, \quad \nabla_x G(x, y) := -\frac{(1 + ik|x - y|)}{4\pi|x - y|^3} e^{-ik|x-y|}(\mathbf{x} - \mathbf{y}) \tag{2.11}$$

According to the theorem 4.6 in [TL](Chapter I, p.43) we have:

Theorem 2.3.1 *The operator Q is continuous from $H^{-1/2}(\mathrm{div}, \Gamma)$ to $H^{-1/2}(\mathrm{rot}, \Gamma)$ and we have that:*

$$|(\mathbf{n} \times Q + \frac{I}{2})\mathbf{M}|_{-1/2,\mathrm{div}_\Gamma} \leq C|\mathbf{M}|_{-1/2,\mathrm{div}_\Gamma} \quad \forall \mathbf{M} \in H^{-1/2}(\mathrm{div}, \Gamma) \tag{2.12}$$

And according to theorem 2.2 in [TL](Chapter II, p.61) we have:

Theorem 2.3.2 *The operator $(B - S)$ is an isomorphisme from $H^{-1/2}(\mathrm{div}, \Gamma)$ to $H^{-1/2}(\mathrm{rot}, \Gamma)$ and it verifies the inequality:*

$$\|(B - S)\boldsymbol{\phi}\|_{-1/2,\mathrm{rot}_\Gamma} \leq C\|\boldsymbol{\phi}\|_{-1/2,\mathrm{div}_\Gamma} \tag{2.13}$$

and the coercivity relation $\forall \boldsymbol{\phi} \in H^{-1/2}(\mathrm{div}, \Gamma)$:

$$\Re(< \boldsymbol{\phi}, (B - S)\boldsymbol{\phi} >) \geq C\|\boldsymbol{\phi}\|^2_{-1/2,\mathrm{div}_\Gamma} \tag{2.14}$$

Let us mention the abstract theorems known as Fredholm alternative [JCN].

2.3.1 Fredholm alternative

Theorem 2.3.3 *: Let V be a Hilbert space. Let H be a Hilbert space which contains V. Let $a(u,v)$ be continuous bilinear form on $V \times V$ which satisfies:*

$$\Re[a(u,v)] \geq \alpha \|u\|_V^2 - c\|u\|_H^2, \alpha > 0, \forall u \in V. \tag{2.15}$$

We consider the variational problem

$$a(u,v) = (g,v); \quad \forall v \in V; g \in V^*.$$

Suppose that the injection of V into H is compact. Then the variational problem (VP) satisfies the Fredholm alternative i.e.

- either it admits a unique solution in V,

- or it has a finite dimension kernel, and a unique solution upto any element in this kernel, when the duality product of the right-hand side g vanishes on every element in this kernel.

We take existence and uniqueness theorem from [JCN](p.245, Theorem 5.6.1).

Theorem 2.3.4 *(Existence and Uniqueness): Let V and W be Hilbert spaces. Let $A(.,.)$ be a bilinear form continuous on $V \times V$ which satisfies*

$$\Re[A(u,\bar{u})] \geq \alpha \|u\|_V^2 - C\|u\|_H^2, \quad \alpha > 0, \ \forall u \in V \tag{2.16}$$

where H is Hilbert space containing V. Let $B(q,v)$ be a bilinear form continuous on $W \times V$ which satisfies:

$$\sup_{\|u\|_V=1} |B(q,u)| \geq \beta \|q\|_W - C\|q\|_L, \quad \beta > 0, \ \forall q \in W \tag{2.17}$$

where L is a Hilbert space containing W.

Consider the following variational problem, with $g_1 \in V^*$ and $g_2 \in W^*$:

$$\begin{cases} A(u,v) & +B(p,v) = (g_1,v) \quad \forall v \in V \\ B(q,u) & = (g_2,q) \quad \forall q \in W \end{cases} \tag{2.18}$$

Denote by V_0 the kernel of the bilinear form B in V, i.e.

$$V_0 = \{u \in V, \ B(q,u) = 0, \ \forall q \in W\}$$

Suppose that the injection form V_0 into H is compact and the injection from W into L is compact. Suppose that there exists an element $u_{g_2} \in V$ such that:

$$B(q,u_{g_2}) = (g_2,q), \quad \forall q \in V.$$

Then the variational problem (2.18) satisfies the Fredholm alternative, i.e.

-either it admits a unique solution in $V \times W$,

-or it admits a finite dimension kernel, and a solution defined up to any element in this kernel, when the right-hand side (g_1,g_2) vanishes on any element in this kernel.

2.4 Conclusion

In this chapter, we defined the vectorial Laplacian or Hodge opertor (2.4). Stupfel uses this operator to define higher order impedance boundary conditions [BSb]. We defined some Sobolev spaces such as $H^{-1/2}(div, \Gamma)$ and $H^{-1/2}(rot, \Gamma)$, that are usually taken as a spaces of unknowns. In the previous chapter we introduced integral equations (1.16) and (1.17) with operators $(B - S)$ and Q. Here we defined these operators more precisely. We will use their inequalities to show the continuity and the coercivity of a bilinear form of the problem. And last two Fredholm alternative theorems are very important to prove the existence of a solution to the scattering problem in two and three dimensional cases.

Chapter 3

Approximation of impedance operator

Contents

3.1 Introduction

In this chapter, we introduce exact impedance operators Z^{ex} on infinitesimally thin tangent plane [R-Sb]. Stupfel derives the same impedance operators in [BS]. In Appendix C, we propose his calculation of exact impedance operators. Rahmat-Samii introduced the same tensor of exact impedance. He approximated Z^{ex} as a ratio of polynomial of differential operators in [R-Sb]. We use decomposition of Hodge operator to approximate the impedance operator in the three dimentional case. Whence we reduce to two dimentional impedance boundary conditions, as in [R-Sb, BSb]. We will determine sufficient uniqueness conditions based on uniqueness theorem (1.3.1) from chapter 1.

3.2 Differential forms of impedance boundary conditions

We approximate impedance at each point of the surface as impedance for infinite plane layer. An incident wave has different incident angles at different points of the surface. We take the boundary conditions as invariant to incident angle, herewith it takes into consideration that incident angle varies. We consider a plane isotropic medium with a local orthogonal basis (x,y,z) on a tangent plane, where a normal vector $\mathbf{n}$ is in z-direction and (x,y) is a tangent plane. The following exact impedances are obtained for the dielectric plane layer [R-Sb, BS]

$$Z_{xy}(k_x,k_y) = Z_{yx}(k_x,k_y) = i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_xk_y}{kk_z}\tan[k_zd] \tag{3.1}$$

$$Z_{xx}(k_x,k_y) = -i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_x^2k_z^2+k_y^2k^2}{kk_z(k_x^2+k_y^2)}\tan[k_zd] \tag{3.2}$$

and

$$Z_{yy}(k_x,k_y) = -i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_y^2k_z^2+k_x^2k^2}{kk_z(k_y^2+k_x^2)}\tan[k_zd]. \tag{3.3}$$

Z is the impedance tensor of wave numbers (k_x, k_y), wave frequency and coating at each point of a surface.

We assume that the plane-wave fields are written in the following forms:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}_1E_0e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{e}_2H_0e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are two constant real unit vectors; E_0, H_0 are complex amplitudes which are constant in space and time. [JDJ]

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0e^{-i(k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}})\cdot\mathbf{r}}$$

$$\partial_x\mathbf{E}(\mathbf{r}) = -ik_x\mathbf{E}(\mathbf{r})$$

$$\partial_x^2\mathbf{E}(\mathbf{r}) = -k_x^2\mathbf{E}(\mathbf{r})$$

So we can replace partial derivatives by k_x and k_y components

$$\partial_x = -ik_x \quad and \quad \partial_y = -ik_y \tag{3.4}$$

or

$$\partial_x^2 = -k_x^2,\ \partial_{xy}^2 = -k_xk_y \quad and \quad \partial_y^2 = -k_y^2. \tag{3.5}$$

In [R-Sb] the impedance boundary conditions are written using the spectral domain approach and are approximated as a ratio of second order polynomials for a coating, invariant under rotation. Those approximation equations could be written as

$$(1+b_1\partial_x^2+b_2\partial_y^2)E_x+(b_1-b_2)\partial_{xy}^2E_y = (a_1-a_2)\partial_{xy}^2H_x-(a_0+a_1\partial_x^2+a_2\partial_y^2)H_y \tag{3.6}$$

and

$$(b_1-b_2)\partial_{xy}^2E_x+(1+b_2\partial_x^2+b_1\partial_y^2)E_y = (a_0+a_2\partial_x^2+a_1\partial_y^2)H_x+(a_2-a_1)\partial_{xy}^2H_y \tag{3.7}$$

Note that $\mathbf{n}\times\mathbf{H} = -H_y\mathbf{x}+H_x\mathbf{y}$. And HOIBC is written in matrix form

$$\begin{bmatrix} 1+b_1\partial_x^2+b_2\partial_y^2 & (b_1-b_2)\partial_{xy}^2 \\ (b_1-b_2)\partial_{xy}^2 & 1+b_2\partial_x^2+b_1\partial_y^2 \end{bmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} =$$

$$\begin{bmatrix} a_0+a_1\partial_x^2+a_2\partial_y^2 & (a_1-a_2)\partial_{xy}^2 \\ (a_1-a_2)\partial_{xy}^2 & a_0+a_2\partial_x^2+a_1\partial_y^2 \end{bmatrix}\begin{pmatrix} -H_y \\ H_x \end{pmatrix} \tag{3.8}$$

3.2.1 3D case form

Now we are going to replace partial derivatives by differential operators, the components of Hodge operator L_D and L_R [BSb]. We define the operators L_D and L_R for all vector function $\mathbf{A}$ sufficiently smooth, such that $\mathbf{A} \cdot \mathbf{n} = 0$

$$L_D(\mathbf{A}) = \nabla_{tg}(\mathrm{div}_{tg}\mathbf{A})$$

$$L_R(\mathbf{A}) = \mathbf{rot}_{tg}(\mathrm{rot}_{tg}\mathbf{A})$$

and we remember Hodge operator that was defined earlier in (2.4) chapter 2

$$L = L_D - L_R = \nabla_{tg}\mathrm{div}_{tg} - \mathbf{rot}_{tg}\mathrm{rot}_{tg}.$$

We know that:

$$\begin{aligned} L_D\mathbf{A} &= \nabla_{tg}(\mathrm{div}_{tg}\mathbf{A}) = \nabla_{tg}(\partial_x A_x + \partial_y A_y) = \\ &= (\partial_x^2 A_x + \partial_{xy}^2 A_y)\mathbf{x} + (\partial_{xy}^2 A_x + \partial_y^2 A_y)\mathbf{y} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} L_R\mathbf{A} = \mathbf{rot}_{tg}(\mathrm{rot}_{tg}\mathbf{A}) = \mathbf{rot}\{\mathbf{n}(\mathbf{rot}\mathbf{A})_n\} = \mathbf{rot}\{\mathbf{n}(\partial_x A_y - \partial_y A_x)\} = \\ (\partial_{xy}^2 A_y - \partial_y^2 A_x)\mathbf{x} + (\partial_{xy}^2 A_x - \partial_x^2 A_y)\mathbf{y}. \end{aligned} \tag{3.10}$$

In equations (3.6) and (3.7) we can replace tangential derivatives by these operators L_D and L_R.

$$\mathbf{x}\cdot(\mathbf{E}_{tg}+b_1 L_D\mathbf{E}_{tg}-b_2 L_R\mathbf{E}_{tg}) = \mathbf{x}\cdot(a_0(\mathbf{n}\times\mathbf{H})+a_1 L_D(\mathbf{n}\times\mathbf{H})-a_2 L_R(\mathbf{n}\times\mathbf{H})) \tag{3.11}$$

and

$$\mathbf{y}\cdot(\mathbf{E}_{tg}+b_1 L_D\mathbf{E}_{tg}-b_2 L_R\mathbf{E}_{tg}) = \mathbf{y}\cdot(a_0(\mathbf{n}\times\mathbf{H})+a_1 L_D(\mathbf{n}\times\mathbf{H})-a_2 L_R(\mathbf{n}\times\mathbf{H})). \tag{3.12}$$

So, we propose three dimensional HOIBC approximation as

$$\boxed{Z_{3D}: \quad (I + b_1 L_D - b_2 L_R)\mathbf{E}_{tg} = (a_0 I + a_1 L_D - a_2 L_R)(\mathbf{n}\times\mathbf{H})}. \tag{3.13}$$

3.2.2 First order form in 2D case

Here we need to consider two different situations. In the first one, we consider case when the electric field is perpendicular to the incident plane, as shown on figure (3.1 a). Incident, scattered and transverse electric fields are directed toward the viewer. The direction of magnetic field was chosen such that energy current has positive direction, i.e. direction of wave propagation. We call this case, transverse-electric (TE) polarization. In the second case electric fields are parallel to incident plane, as shown in figure (3.1 b). In this case we call it transverse-magnetic (TM) polarization.

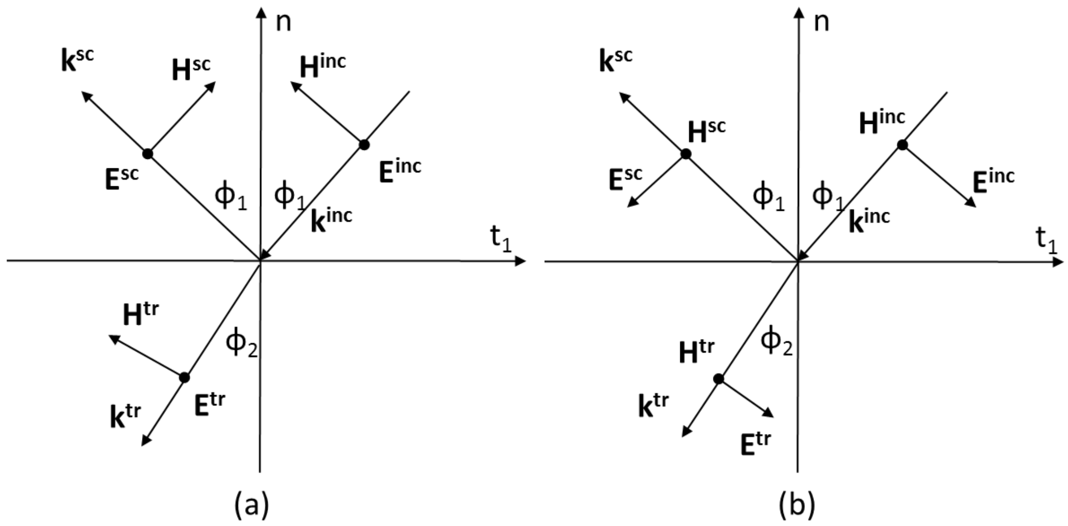

Figure 3.1: Reflection and refraction with (a) TE and (b) TM polarizations.

We assume that the incident fields propagate perpendicular to the cylinder axis, so that $\partial/\partial y = 0$. And fields are polarized either with the electric field in the y direction (E polarization or TM), or with the magnetic field in the y direction (H polarization or TE). In two dimensional case we have that $\partial_y \equiv 0$, so

$$\begin{pmatrix} 1+b_1\partial_x^2 & 0 \\ 0 & 1+b_2\partial_x^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a_0+a_1\partial_x^2 & 0 \\ 0 & a_0+a_2\partial_x^2 \end{pmatrix}\begin{pmatrix} -H_y \\ H_x \end{pmatrix} \quad (3.14)$$

So we get that in two dimensional TE polarization, we have

$$(1+b_1\partial_x^2)E_x = -(a_0+a_1\partial_x^2)H_y$$

and in TM polarization, we have

$$(1+b_2\partial_x^2)E_y = (a_0+a_2\partial_x^2)H_x.$$

So, for a plane wave the first order IBC (3.14) can be written as

$$\begin{pmatrix} 1-b_1k_x^2 & 0 \\ 0 & 1-b_2k_x^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a_0-a_1k_x^2 & 0 \\ 0 & a_0-a_2k_x^2 \end{pmatrix}\begin{pmatrix} -H_y \\ H_x \end{pmatrix} \quad (3.15)$$

According to (C.9), we get first order approximation of impedance in two dimensional cases for each polarization

$$\boxed{Z_{2Dj}: \quad (1+b_j\partial_x^2)\mathbf{E}_{tg} = (a_0+a_j\partial_x^2)\mathbf{n}\times\mathbf{H}} \quad (3.16)$$

and the impedance Z_{2Dj} is the following rational function of k_x^2

$$Z_{2Dj} = \frac{a_0 - a_jk_x^2}{1-b_jk_x^2}, \quad j=1,2 \quad (3.17)$$

The coefficients indicated by $j = 1, 2$ correspond to polarizations TE and TM respectively. These coefficients (a_0, a_j, and b_j) are determined by equating this first order impedance Z_{2Dj} and the exact impedance. From (C.11) and (C.12) we can express exact impedance for TE and TM polarization:

$$Z_{TE}^{ex} = \sqrt{\frac{\mu}{\varepsilon}}\frac{k_z}{k}\tan(k_z d) = z_0\sqrt{\mu_r\varepsilon_r - \left(\frac{k_x}{k_0}\right)^2}\tan\left(\sqrt{\mu_r\varepsilon_r - \left(\frac{k_x}{k_0}\right)^2}k_0 d\right)/\varepsilon_r \tag{3.18}$$

$$Z_{TM}^{ex} = \sqrt{\frac{\mu}{\varepsilon}}\frac{k}{k_z}\tan(k_z d) = \frac{z_0\mu_r\tan\left(\sqrt{\mu_r\varepsilon_r - \left(\frac{k_x}{k_0}\right)^2}k_0 d\right)}{\sqrt{\mu_r\varepsilon_r - \left(\frac{k_x}{k_0}\right)^2}} \tag{3.19}$$

If $\theta = 0$, we get that $a_0 = Z(0) = Z^{ex}(0)$, for a normally incident wave, which is known as the Leontovich boundary condition and we get

$$a_0 = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}\tan\left(\omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}d\right)$$

We calculate other coefficients a_j and b_j, using two arbitrary angles θ_1 and θ_2 by:

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{bmatrix} -k_x^2(\theta_1) & k_x^2 Z_j^{ex}(\theta_1) \\ -k_x^2(\theta_2) & k_x^2 Z_j^{ex}(\theta_2) \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\theta_1) - a_0 \\ Z_j^{ex}(\theta_2) - a_0 \end{pmatrix}$$

The indices correspond to TE and TM polarizations, as in (3.17). The arbitrary angles θ_1 and θ_2 should be in the angle range $]0, \pi/2[$. Here we take $k_x^2 = k_0^2\sin^2(\theta)$ as [BSb].

Note that in two dimensional case we suppose that $\partial_y = 0$. The three dimensional impedance Z_{3D} should correspond to Z_{2Dj} in two dimensional case. Coefficients a_1, b_1 and a_2, b_2 should correspond to those in two dimension, TE and TM coefficients, respectively.

3.2.3 Second order form in 2D case

The equation (3.16) can be extended to second order polynimials in ∂_x^2:

$$\mathbf{E}_{tg} + b_j\partial_x^2\mathbf{E}_{tg} + b_j'\partial_x^4\mathbf{E}_{tg} = a_0(\mathbf{n}\times\mathbf{H}) + a_j\partial_x^2(\mathbf{n}\times\mathbf{H}) + a_j'\partial_x^4(\mathbf{n}\times\mathbf{H}) \tag{3.20}$$

or it can be reduced to constant:

$$\mathbf{E}_{tg} = a_0(\mathbf{n}\times\mathbf{H}). \tag{3.21}$$

We will call the equation (3.20) second order IBC (IBC2), the equation (3.21) zeroth order IBC (IBC0), which is also known as Leontovich IBC. And we will call the eqution (3.16) as first order IBC (IBC1). Note that (3.20) with $a_j' = b_j' = 0$ derives to (3.16). As well as with $a_j = b_j = 0$, the equation (3.16) derives to (3.21).

3.3 Sufficient uniqueness conditions

We have defined HOIBC. Now we want to establish sufficient uniqueness conditions (SUC) for the solutions of Maxwell's equations associated with these IBCs, with help of the uniqueness theorem mentioned earlier in chapter 1.

Problem 3.3.1 *Find* $(\boldsymbol{E}, \boldsymbol{H})$ *such that*

$$\begin{cases} \mathrm{rot}\boldsymbol{E} + ik_0\mu\boldsymbol{H} = 0 \quad in\ \Omega^+ \\ \mathrm{rot}\boldsymbol{H} - ik_0\varepsilon\boldsymbol{E} = 0 \quad in\ \Omega^+ \\ \lim_{r\to\infty} r(\boldsymbol{E}\times\boldsymbol{n}_r + \boldsymbol{H}) = 0 \end{cases} \tag{3.22}$$

with a boundary conditions

$$\boldsymbol{E}_{tg} = Z(\boldsymbol{n}\times\boldsymbol{H}) \quad on\ \Gamma$$

where $Z = Z_1\boldsymbol{\tau} + Z_2\boldsymbol{\nu}$ *is a matrix with constant values.*

The following theorem gives us sufficient uniqueness conditions for this problem.

Theorem 3.3.1 *The* ***problem 3.3.1*** *admits a unique solution if the following relations are verified*

$$\begin{cases} \Im(\mu) \leq 0, \\ \Im(\varepsilon) \leq 0, \\ \Re(Z_1) \geq 0, \\ \Re(Z_2) \geq 0. \end{cases} \tag{3.23}$$

Proof. According to the theorem 1.3.1 from chapter 1, the problem admits a unique solution if

$$\Re\left(k_0\int_\Gamma(\mathbf{n}\times\mathbf{H})\cdot\mathbf{E}^*\right) \geq 0.$$

The boundary condition of the problem 3.3.1 says that $\mathbf{E}_{tg} = Z(\mathbf{n}\times\mathbf{H}) = -Z_1H_\nu\boldsymbol{\tau} + Z_2H_\tau\boldsymbol{\nu}$, where Z_1 and Z_2 are complex constants. So we need to show that

$$0 \leq \Re\int_\Gamma -H_\nu(-Z_1^*H_\nu^*) + H_\tau(Z_2^*H_\tau^*) = \Re\int_\Gamma Z_1^*|H_\nu|^2 + Z_2^*|H_\tau|^2$$

And with theorem hypothesis:

$$\Re(Z_1^*) = \Re(Z_1) \geq 0 \ \ and \ \ \Re(Z_2^*) = \Re(Z_2) \geq 0,$$

the hypothesis (1.9) are verified. So the problem with IBC0 admits a unique solution. Q.E.D.

[BSc]. □

The condition $\Re(Z) \geq 0$ is invariant to incident angle, so applying it for SIBC we get that coefficient a_0 in HOIBC also should verify condition

$$\Re(a_0) \geq 0.$$

We are going to study a more complicated boundary condition, that is called first order IBC. For the problem 3.22 with a boundary condition IBC1

$$\mathbf{E}_{tg} + b_j L(\mathbf{E}_{tg}) = a_0(\mathbf{n} \times \mathbf{H}) + a_j L(\mathbf{n} \times \mathbf{H}) \quad on\ \Gamma, \tag{3.24}$$

where L is a complex differential operator, also known as **Hodge** operator. We assume that

$$\begin{cases} (\mathbf{n} \times \mathbf{H}) \cdot \tau = (\mathbf{n} \times \mathbf{H}) \cdot \nu = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open, \\ \mathbf{E}_{tg} \cdot \tau = \mathbf{E}_{tg} \cdot \nu = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open. \end{cases} \tag{3.25}$$

Theorem 3.3.2 *The Maxwell problem (3.22) with the boundary conditions (3.24) and (3.25) has a unique solution if the following relations are verified*

$$\begin{cases} \Im(\mu) \leq 0, \\ \Im(\varepsilon) \leq 0, \\ a_j - b_j^* a_0 \neq 0, \\ \Re(a_j - b_j^* a_0) = 0, \\ \Im(a_0^* a_j)\Im(a_j - b_j^* a_0) \geq 0, \\ \Im(b_j)\Im(a_j - b_j^* a_0) \geq 0. \end{cases} \tag{3.26}$$

Proof. see [BSc]. □

We study yet another more complicated HOIBC, which is called second-order IBC that increases power of Hodge operator. For the SUC of the problem (3.22) with IBC2, we assume that

$$\mathbf{E}_{tg} + b_j L(\mathbf{E}_{tg}) + b'_j L^2(\mathbf{E}_{tg}) = a_0(\mathbf{n} \times \mathbf{H}) + a_j L(\mathbf{n} \times \mathbf{H}) + a'_j L^2(\mathbf{n} \times \mathbf{H}) \quad on\ \Gamma, \tag{3.27}$$

where $L^2(\cdot) = L \circ L(\cdot)$. And all coefficients a_j, a'_j, b_j and b'_j are defined locally and depend on incident angle. With a next conditions on a bound

$$\begin{cases} (\mathbf{n} \times \mathbf{H}) \cdot \tau = (\mathbf{n} \times \mathbf{H}) \cdot \nu = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open, \\ \mathbf{E}_{tg} \cdot \tau = \mathbf{E}_{tg} \cdot \nu = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open \\ X_1 \cdot \nu = X_2 \cdot \nu = Y_1 \cdot \nu = Y_2 \cdot \nu = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open \\ X_1 \cdot \tau = X_2 \cdot \tau = Y_1 \cdot \tau = Y_2 \cdot \tau = 0 \quad on\ \partial\Gamma \quad if \quad \Gamma \quad is \quad open. \end{cases} \tag{3.28}$$

where

$$\begin{cases} X_1 = \nabla_\Gamma \mathrm{div}_\Gamma \mathbf{E}_{tg} = L_D \mathbf{E}_{tg} \\ X_2 = \nabla_\Gamma \mathrm{div}_\Gamma (\mathbf{n} \times \mathbf{H}) = L_D(\mathbf{n} \times \mathbf{H}) \\ Y_1 = \mathbf{rot}(\mathbf{n}(\mathrm{rot}_\Gamma \mathbf{E}_{tg})) = L_R \mathbf{E}_{tg} \\ Y_2 = \mathbf{rot}(\mathbf{n}(\mathrm{rot}_\Gamma(\mathbf{n} \times \mathbf{H}))) = L_R(\mathbf{n} \times \mathbf{H}) \end{cases} \tag{3.29}$$

So for the IBC2 we have following uniqueness theorem

Theorem 3.3.3 *The Maxwell problem (3.22) with a boundary condition (3.27), (3.28) and (3.29) admits a unique solution if next relations are verified*

$$\begin{cases} \Im(\mu) \leq 0, \\ \Im(\varepsilon) \leq 0, \\ \Delta \neq 0, \\ \Re[\Delta^*(a_j b_j'^* - a_j' b_j^*)] = 0, \\ \Re[\Delta^*(a_j' - a_0 b_j'^*)] = 0, \\ \alpha \Im b_j' + \beta \Im(b_j b_j'^*) \leq 0, \\ \alpha \Im(a_j' a_0^*) - \beta \Im(a_j' a_j^*) \leq 0, \\ -\alpha \Im b_j + \beta \Im b_j' \leq 0, \\ \alpha \Im(a_j a_0^*) - \beta \Im(a_j' a_0^*) \geq 0 \end{cases} \tag{3.30}$$

with

$$\Delta = (a_0 b_j^* - a_j)(a_j b_j'^* - a_j' b_j^*) - (a_j' - a_0 b_j'^*)^2$$

$$\alpha = \Im[\Delta^*(a_j b_j'^* - a_j' b_j^*)]$$

$$\beta = \Im[\Delta^*(a_0 b_j'^* - a_j')]$$

Proof. see [BSc]. □

All these three theorems are based on the theorem 1.3.1 and its condition $\Re(k_0 \int_\Gamma \mathbf{E}^* \cdot (\mathbf{n} \times \mathbf{H}) ds) \geq 0$. So, to verify this condition, coefficients of HOIBC should verify conditions mentioned in these theorems.

3.4 Conclusion

We assume that imedance operator Z is equal to Z^{ex} on infinitesimally thin tangent plane, where tangent plane and surface of the object have the same properties (ε, μ, d). We introduced the approximation of impedance boundary conditions by equation (3.13) for three dimentional case and equation (3.16) for two dimentional case. We proposed zero order IBC that is known as Leontovich IBC, and second order HOIBC with different coefficients a_0, a_1, a_2, b_1 and b_2. We calculate coefficients as a solution of a system of linear equations. We defined coefficients for 2D case and we use them for 3D case. Further in next chapter we will propose different ways to determine the coefficients. The sufficient uniqueness conditions are represented as restrictions on coefficients.

CHAPTER 4

Higher order impedance boundary condition's coefficients

Contents

4.1 Introduction

In this chapter, we propose some different methods to calculate coefficients for HOIBC [BS, R-Sb]. The impedance operator for normal incident wave is equal to constant a_0 that is known as Leontovich IBC (or standard IBC). As was mentioned earlier, the impedance operator depends on layer thickness, on the dielectric characteristics of medium of this layer. Also it depends on the incident angle of the incident electromagnetic plane wave. So another two coefficients depend on the incedent angle in 2D first order IBC. Numerical results demonstrates the relative advantages of calculations using IBC0, IBC1 and IBC2 with respect to the angle of incidence.

4.2 Approximation of coefficients

Wave formulas of electromagnetic fields give the quantity of wave number x-component, that was noted in the previous chapter, at calculation coefficients of approximate polynomials. Where from we denote $\xi = -(\frac{k_x}{k_0})^2 = -\sin^2(\theta)$ and input it in the exact impedance values (3.18), (3.19), we get:

$$Z_{TE}^{ex}(\xi) = z_0\sqrt{\mu_r\varepsilon_r + \xi}\tan\left(\sqrt{\mu_r\varepsilon_r + \xi}k_0 d\right)/\varepsilon_r$$

$$Z_{TM}^{ex}(\xi) = \frac{z_0\mu_r\tan\left(\sqrt{\mu_r\varepsilon_r + \xi}k_0 d\right)}{\sqrt{\mu_r\varepsilon_r + \xi}}.$$

Here we approximate impedance as ratio of polynomials of ξ.

The simplest IBC is Leontovich IBC, as were already mentioned several times $Z = const$. Usually, it is taken for incident wave perpendicular to plane

$$Z_1 = Z_2 = Z_{1,2}^{ex}(\theta = 0)$$

$$a_0 = z_0\sqrt{\frac{\mu_r}{\varepsilon_r}}\tan\left(\sqrt{\mu_r\varepsilon_r}k_0 d\right) \quad (LIBC)$$

In fact, we can take an arbitrary angle value in permitted range $[0, \pi/2]$. If the angle is not zero, then impedances are different to each other

$$Z_1 = Z_1^{ex}(\theta) \; and \; Z_2 = Z_2^{ex}(\theta)$$

in different polarizations.

4.2.1 First order IBC

First order polynomial

The accuracy of the boundary condition may be improved by including the coefficient a_1, thus approximation derive first order polynomial form [R-Sb]. Let coefficient $a_0 = Z_1^{ex}(0)$ and a_j is defined for arbitrary $\theta_1 \in]0, \pi/2[$

$$a_j = \frac{Z_j^{ex}(\theta_1) - a_0}{\xi(\theta_1)}$$

$$b_j = 0.$$

We can approximate impedance Z^{ex} as the first order Taylor polynomial for ξ near to zero [Stupfel], which yields

$$a_1 = z_0\frac{k_0 d}{2\varepsilon_r} + z_0\frac{\tan(\sqrt{\mu_r\varepsilon_r}k_0 d)}{2\sqrt{\mu_r\varepsilon_r}\varepsilon_r}$$

$$+z_0\frac{k_0 d\tan^2(\sqrt{\mu_r\varepsilon_r}k_0 d)}{2\varepsilon_r} \quad (Taylor)$$

$$b_1 = 0.$$

Ratio of polynomials

Now we suppose that b_1 is different to zero and we approximate the impedance as ratio of polynomials. Stupfel mentioned Padé approximation, which considers second order Taylor approximation as follows

$$Z_1^{ex}(\xi) = c_0 + c_1\xi + c_2\xi^2 + O(\xi^2) \approx \frac{a_0 + a_1\xi}{1 + b_1\xi}, \tag{4.1}$$

where coefficients c_0, c_1 and c_2 are Taylor coefficients:

$$c_0 = z_0\sqrt{\frac{\mu_r}{\varepsilon_r}}\tan\left(\sqrt{\mu_r\varepsilon_r}k_0 d\right)$$

$$c_1 = z_0\frac{k_0 d}{2\varepsilon_r} + z_0\frac{\tan(\sqrt{\mu_r\varepsilon_r}k_0 d)}{2\sqrt{\mu_r\varepsilon_r}\varepsilon_r}$$
$$+z_0\frac{k_0 d\tan^2(\sqrt{\mu_r\varepsilon_r}k_0 d)}{2\varepsilon_r}$$
$$c_2 = \frac{z_0 k_0 d}{8\varepsilon_r^2\mu_r} + \left(\frac{z_0 k_0^2 d^2}{4\varepsilon_r\sqrt{\varepsilon_r\mu_r}} - \frac{z_0}{8\varepsilon_r(\varepsilon_r\mu_r)^{3/2}}\right)\tan(\sqrt{\varepsilon_r\mu_r}k_0 d)$$
$$+\frac{z_0 k_0 d}{8\varepsilon_r^2\mu_r}\tan^2(\sqrt{\varepsilon_r\mu_r}k_0 d) + \frac{z_0 k_0^2 d^2}{2\varepsilon_r\sqrt{\varepsilon_r\mu_r}}\tan^3(\sqrt{\varepsilon_r\mu_r}k_0 d).$$

By multiplying this Taylor polynomial by denominator polynomial and equating the product to the numerator polynomial, we derive equations, where coefficients of Pade approximation may be determined through coefficients of Taylor approximation. So, from (4.1) we have

$$a_0 + a_1\xi \approx c_0 + (c_1 + c_0 b_1)\xi + (c_2 + c_1 b_1)\xi^2 + O(\xi^2),$$

that gives following equations

$$a_0 = c_0,$$
$$a_1 = c_1 + c_0 b_1$$
$$0 = c_2 + c_1 b_1.$$

Finally, we get the coefficients of Pade approximation

$$b_1 = -\frac{c_2}{c_1},$$
$$a_1 = c_1 - c_0\frac{c_2}{c_1} \quad (Pad).$$

Another method to calculate the coefficients was used in previous chapter. For different θ_1, θ_2 from $]0,\pi/2[$, we get different values of $\xi_i = -k_0^2\sin^2(\theta_i)$ and two linear equations [R-Sb]

$$Z_1^{ex}(\xi_1) - b_1\xi_1 Z_1^{ex}(\xi_1) - a_0 + a_1\xi_1 = 0$$
$$Z_1^{ex}(\xi_2) - b_1\xi_2 Z_1^{ex}(\xi_2) - a_0 + a_1\xi_2 = 0.$$

That can be solved in matrix form as follows

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{bmatrix} \xi_1 & -\xi_1 Z_j^{ex}(\xi_1) \\ \xi_2 & -\xi_2 Z_j^{ex}(\xi_2) \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\xi_1) - a_0 \\ Z_j^{ex}(\xi_2) - a_0 \end{pmatrix}$$

and we get the coefficients of so-called Collocation approximation [BS]

$$a_1 = \frac{-\xi_2 Z_j^{ex}(\xi_2)(Z_j^{ex}(\xi_2) - a_0) + \xi_1 Z_j^{ex}(\xi_1)(Z_j^{ex}(\xi_2) - a_0)}{\xi_1\xi_2(Z_j^{ex}(\xi_1) - Z_j^{ex}(\xi_2))}$$
$$b_1 = \frac{-\xi_2(Z_j^{ex}(\xi_1) - a_0) + \xi_1(Z_j^{ex}(\xi_2) - a_0)}{\xi_1\xi_2(Z_j^{ex}(\xi_1) - Z_j^{ex}(\xi_2))}.$$

4.2.2 Second order IBC

We can apply all methods that were proposed earlier for IBC2, too. We get Taylor approximation, if we admit that b_j and b'_j are zero. If they are not zero, we get ratio of polynomials approximation. Padé approximation uses fourth order Taylor expansion. Whence we get five equation for five unknown coefficients. And in the last method, the coefficients are calculated by solving system of linear equations for different $\theta_k \in]0, \pi/2[,\ k = 1, 2, 3, 4$

$$\begin{pmatrix} a_j \\ a'_j \\ b_j \\ b'_j \end{pmatrix} = \begin{bmatrix} \xi_1 & \xi_1^2 & -\xi_1 Z_j^{ex}(\xi_1) & -\xi_1^2 Z_j^{ex}(\xi_1)^2 \\ ... & ... & ... & ... \\ \xi_4 & \xi_4^2 & -\xi_4 Z_j^{ex}(\xi_4) & -\xi_4^2 Z_j^{ex}(\xi_4)^2 \end{bmatrix}^{-1} \begin{pmatrix} Z_j^{ex}(\xi_1) - a_0 \\ ... \\ ... \\ Z_j^{ex}(\xi_4) - a_0 \end{pmatrix}$$

Here, we suppose that in the last two methods we get invertible matrices.

4.3 Numerical results

In order to illustrate the relative accuracy of approximated boundary conditions compared to the exact IBC, we present here some examples. And we will see that HOIBC consider the incident angle parameter. We will see the difference between IBC0 and exact IBC.

Let us consider a mono-layer dielectric coating with characteristics $\varepsilon_r = 4.0$, $\mu_r = 1.0$ and $d = 0.005\lambda_0$. Figure 4.3 shows values exact IBC, Leontovich IBC, first order and second order impedance boundary conditions, in TE polarization. Where the angle of incidence of the plane wave ϕ has angle range $]0, \pi[$. The IBC0 was taken as an impedance of a perpendicular incidence wave. To calculate first-order IBC approximation we used $\phi = 0\ , \pi/6\ , \pi/3$. To calculate second-order IBC we used $\phi = 0\ , \pi/8\ , \pi/6\ , \pi/4\ , \pi/3$.

On the figure 4.3, we can easily see that the difference between IBC0 and exact IBC increases. While the difference between exact IBC and IBC1 is very small, as the difference between exact IBC and IBC2.

But we can see the error of IBC1 and IBC2 approximations on the figure 4.4. As the angle of incidence increases the error of first-order IBC approximation reaches 0.39Ω.

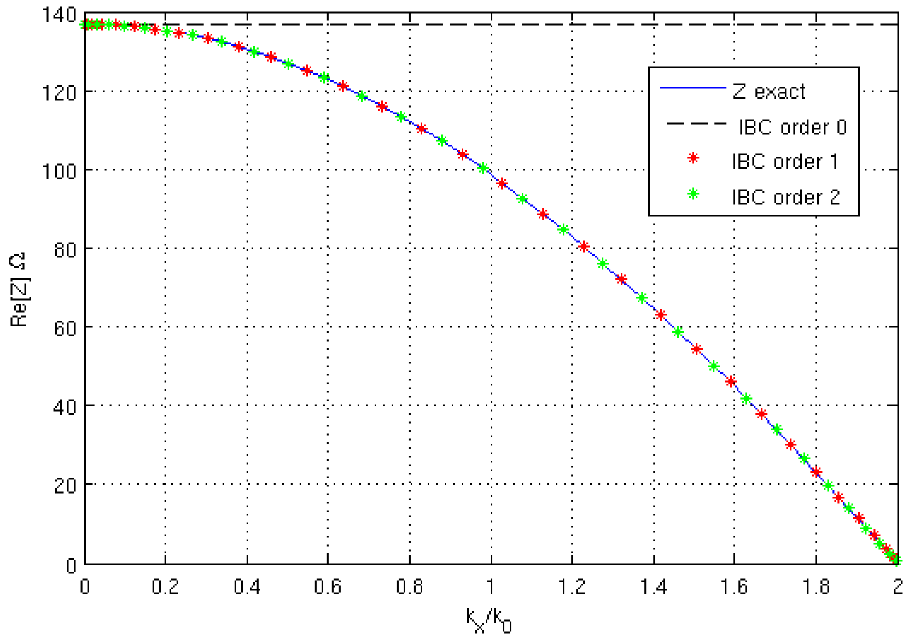

Figure 4.1: Comparison of the exact impedance, Leontovich impedance (IBC0), first-order (IBC1) and second-order (IBC2) IBC in TE polarisation.

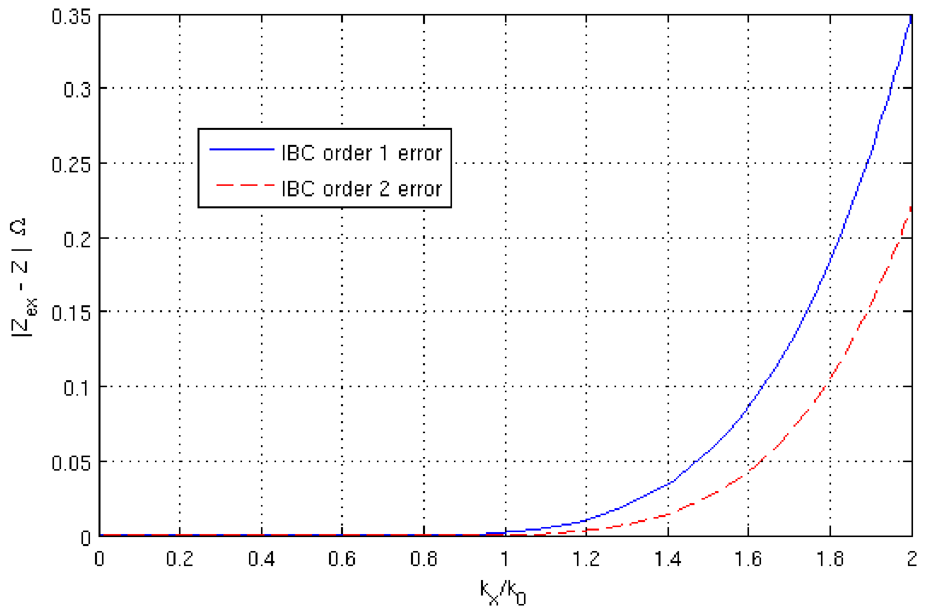

Figure 4.2: Errors of first-order (IBC1) and second-order (IBC2) IBC in TE polarisation.

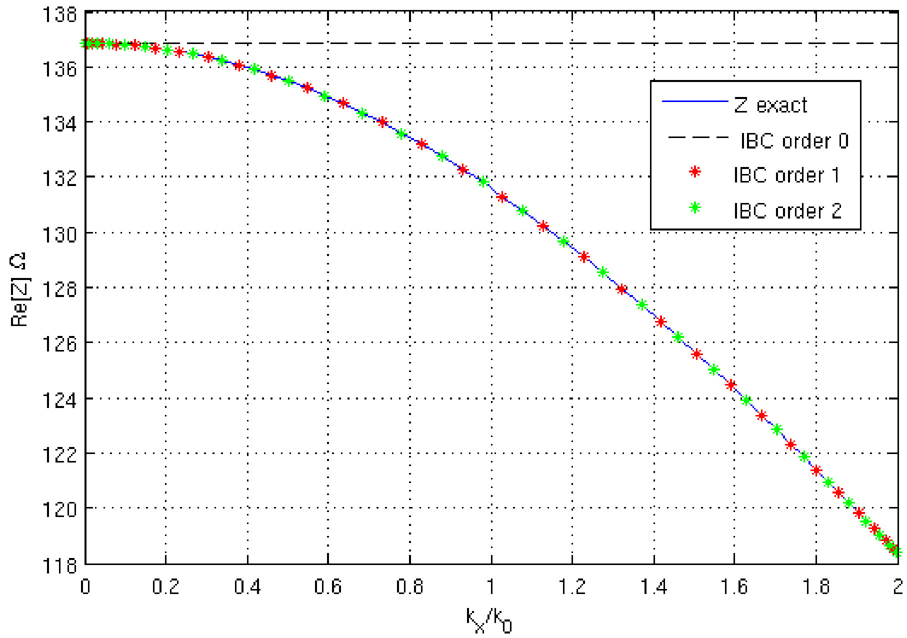

Figure 4.3: Comparison of the exact impedance, Leontovich impedance (IBC0), first-order (IBC1) and second-order (IBC2) IBC in TM polarisation.

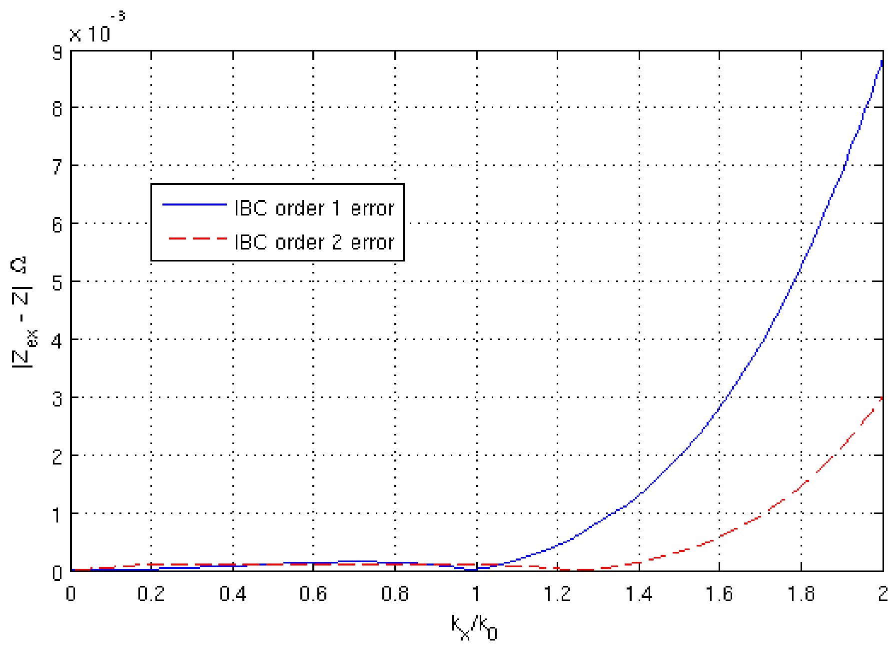

Figure 4.4: Errors of first-order (IBC1) and second-order (IBC2) IBC in TM polarisation.

Chapter 5

Two dimensional variational problem

Contents

5.1 Introduction

We recall integral form of Maxwell's equations with help of the operators $(B-S)$ and Q

$$< Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > + < (P+Q)\mathbf{M}, \mathbf{\Psi}_J > = < IE^{inc}, \mathbf{\Psi}_J > \tag{5.1}$$

$$- < (P+Q)\mathbf{J}, \mathbf{\Psi}_M > + < \frac{1}{Z_0}(B-S)\mathbf{M}, \mathbf{\Psi}_M > = < IH^{inc}, \mathbf{\Psi}_M > . \tag{5.2}$$

The variational form of scattering problem of electromagnetic waves with a constant or Leontovich impedance boundary condition is well-known [TL]. And the existence and uniqueness theorems were proved.

Here we apply the first and second order HOIBC for two dimensional problem that were defined in chapter 3. The problems for TE and TM polarizations will be presented separately. The existence and uniqueness theorems based on alternative of Fredholm or theorem 2.3.3. The continuity and coercivity conditions impose their own restrictions on HOIBC.

5.2 Variational problem with impedance boundary condition

Two dimensional case system is invariant in one direction, so object surface Γ becomes a curved contour, that we will call C. We have the curvilinear abscissa l along

C and normal to the contour unit vector $\mathbf{n}$. We set the local frame $(\boldsymbol{\tau},\ \boldsymbol{\nu},\ \mathbf{n})$, where $\boldsymbol{\tau}$ is a unit vector tangent to the contour C in l direction, and $\boldsymbol{\nu}$ can be defined as $\boldsymbol{\nu} = \mathbf{n} \times \boldsymbol{\tau}$. We suppose that our two dimensional system does not depend on ν parameter, however variable ν component is depend on l.

We said that impedance boundary conditions are described by the following

$$\mathbf{E}_{tg} = Z(\mathbf{n} \times \mathbf{H}).$$

According to the definition of electromagnetic current densities, we have

$$\mathbf{E}_{tg} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \mathbf{n} \times \mathbf{M} \quad on\ \Gamma;$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{J} \quad on\ \Gamma.$$

So we rewrite impedance boundary condition as follows

$$\mathbf{n} \times \mathbf{M} = Z\mathbf{J}. \tag{5.3}$$

And we approximate the operator Z, as a ratio of polynomials of differential operators, as indicated in chapter 3. So, we recall first order IBC

$$(1 + b_j d_l^2)(\mathbf{n} \times \mathbf{M}) = (a_0 + a_j d_l^2)\mathbf{J} \tag{5.4}$$

and the second order IBC

$$(1 + b_j d_l^2 + b_j' d_l^4)(\mathbf{n} \times \mathbf{M}) = (a_0 + a_j d_l^2 + a_j' d_l^4)\mathbf{J} \tag{5.5}$$

where $j = 1, 2$ correspond to TE and TM polarizations, respectively. The invariance in one direction for two dimensional model allow us to simplify the Hodge operator as the second partial derivative on the contour. Where the electromagnetic current densities $\mathbf{n} \times \mathbf{M}$ and $\mathbf{J}$ have $\boldsymbol{\tau}$ direction for TE polarization, and $\boldsymbol{\nu}$ direction for TM polarization.

- TE: $\partial_x^2 \mathbf{J} = \boldsymbol{\tau}\partial_x^2 J_\tau = \boldsymbol{\tau} d_l^2 J_\tau$ and $\partial_x^2(\mathbf{n} \times \mathbf{M}) = -\boldsymbol{\tau}\partial_x^2 M_\nu = -\boldsymbol{\tau} d_l^2 M_\nu$;
- TM: $\partial_x^2 \mathbf{J} = \boldsymbol{\nu}\partial_x^2 J_\nu = \boldsymbol{\nu} d_l^2 J_\nu$ and $\partial_x^2(\mathbf{n} \times \mathbf{M}) = \boldsymbol{\nu}\partial_x^2 M_\tau = \boldsymbol{\nu} d_l^2 M_\tau$.

Next, we consider the integral form of the boundary conditions and we present variational formulation of the problem with these IBC1 and IBC2 conditions.

5.2.1 Problem with first order IBC

We weakly write the equation (5.4), we multiply it by a test function and integrate along the contour C. In EFIE test function is noted as Ψ_j, so we use it to insert our boundary condition in this equation.

$$\int_C (1 + b_j d_l^2)(\mathbf{n} \times \mathbf{M}) \cdot \boldsymbol{\Psi}_J dl = \int_C (a_0 + a_j d_l^2)\mathbf{J} \cdot \boldsymbol{\Psi}_J dl.$$

That gives us

$$\int_C (\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl = \int_C (a_0 + a_j d_l^2)\mathbf{J}\cdot\mathbf{\Psi}_J dl - \int_C b_j d_l^2(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl.$$

We put it in the operator P and obtain:

$$< P\mathbf{M},\mathbf{\Psi}_J >= \frac{1}{2}\int_C (\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl = \frac{a_0}{2}\int_C \mathbf{J}\cdot\mathbf{\Psi}_J dl$$

$$+\frac{a_j}{2}\int_C d_l^2\mathbf{J}\cdot\mathbf{\Psi}_J dl - \frac{b_j}{2}\int_C d_l^2(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_J dl \tag{5.6}$$

We use $(\mathbf{n}\times\mathbf{\Psi}_M)$ as a test function, to insert IBC1 in MFIE

$$\int_C (1+b_j d_l^2)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl = \int_C (a_0 + a_j d_l^2)\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl.$$

We take the first part of right side

$$\int_C \mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl = \frac{1}{a_0}\int_C (1+b_j d_l^2)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl - \frac{1}{a_0}\int_C a_j d_l^2\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl.$$

And using the formula of vector analysis

$$\mathbf{\Psi}_M\cdot(\mathbf{n}\times\mathbf{J}) = -\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M),$$

we put it in P operator with weakly form of IBC1

$$< P\mathbf{J},\mathbf{\Psi}_M >= \frac{1}{2}\int_C (\mathbf{n}\times\mathbf{J})\cdot\mathbf{\Psi}_M dl = -\frac{1}{2}\int_C \mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl =$$

$$= -\frac{1}{2a_0}\int_C (\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl - \frac{b_j}{2a_0}\int_C d_l^2(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl$$

$$+\frac{a_j}{2a_0}\int_C d_l^2\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)dl \tag{5.7}$$

First, we observe TE polarization, where P operator becomes:

$$\int_C PM_\nu\ \Psi_{J\tau}dl = \frac{a_0}{2}\int_C J_\tau\ \Psi_{J\tau}dl + \frac{a_1}{2}\int_C d_l^2 J_\tau\ \Psi_{J\tau}dl + \frac{b_1}{2}\int_C d_l^2 M_\nu\ \Psi_{J\tau}dl$$

and

$$\int_C PJ_\tau\ \Psi_{M\nu}dl = -\frac{1}{2a_0}\int_C M_\nu\ \Psi_{M\nu}dl - \frac{b_1}{2a_0}\int_C d_l^2 M_\nu\ \Psi_{M\nu}dl - \frac{a_1}{2a_0}\int_C d_l^2 J_\tau\ \Psi_{M\nu}dl$$

for EFIE and MFIE, respectively.

We put them in the variational equations (1.16) and (1.17) and get:

$$iZ_0\iint_C kG(l,l')\ J_\tau(l')\ \Psi_{J\tau}\ [\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] - \frac{1}{k}G(l,l')\ d_l' J_\tau(l')\ d_l\Psi_{J\tau}(l)\ dl'dl$$

$$
\begin{aligned}
+ \iint_C \Psi_{J\tau}(l)\ M_\nu(l')\ [\boldsymbol{\tau}(l) \times \boldsymbol{\nu}(l')] \cdot \nabla_\Gamma G(l,l')\ dl'dl \\
+\frac{a_0}{2} \int_C J_\tau\ \Psi_{J\tau} dl + \frac{a_1}{2} \int_C d_l^2 J_\tau\ \Psi_{J\tau} dl \\
+ \frac{b_1}{2} \int_C d_l^2 M_\nu\ \Psi_{J\tau} dl = \int_C E_\tau^{inc}\ \Psi_{J\tau} dl
\end{aligned} \tag{5.8}
$$

and

$$
\begin{aligned}
- \iint_C \Psi_{M\nu}(l)\ J_\tau(l')\ [\boldsymbol{\nu(l)} \times \boldsymbol{\tau(l')}] \cdot \nabla_\Gamma G(l,l') dl'dl \\
+\frac{i}{Z_0} \iint_C kG(l,l')\ M_\nu(l')\ \Psi_{M\nu}(l)[\boldsymbol{\nu(l')} \cdot \boldsymbol{\nu(l)}] dl'dl \\
+\frac{1}{2a_0} \int_C M_\nu\ \Psi_{M\nu} dl + \frac{b_j}{2a_0} \int_C d_l^2 M_\nu\ \Psi_{M\nu} dl \\
+ \frac{a_j}{2a_0} \int_C d_l^2 J_\tau\ \Psi_{M\nu} dl = \int_C H_\nu^{inc}\ \Psi_{M\nu} dl
\end{aligned} \tag{5.9}
$$

for EFIE and MFIE, respectively.

In the equations (5.8) and (5.9) we have scalar products $[\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] = 1$ and $[\boldsymbol{\nu}(l)\cdot \boldsymbol{\nu}(l')] = 1$, and vector products $[\boldsymbol{\tau}(l) \times \boldsymbol{\nu}(l')] = \mathbf{n}(l)$ and $[\boldsymbol{\nu}(l) \times \boldsymbol{\tau}(l')] = -\mathbf{n}(l')$. The operator S contains surface divergence operator that becomes differential operator

$$\mathrm{div}_\Gamma \mathbf{J} = \mathrm{div}_\Gamma(\boldsymbol{\tau} J_\tau) = d_l J_\tau;$$

$$\mathrm{div}_\Gamma \mathbf{M} = \mathrm{div}_\Gamma(\boldsymbol{\nu} M_\nu) = d_\nu M_\nu \equiv 0,$$

because the model is invariance in ν parameter.

By doing integration by parts, we have

$$\frac{b_1}{2} \int_C d_l^2 M_\nu(l) \Psi_{J\tau}(l) dl = -\frac{b_1}{2} \int_C d_l M_\nu(l)\ d_l \Psi_{J\tau}(l) dl. \tag{5.10}$$

Finally we combine two equations (5.8)-(5.9) to present next variational problem:

Problem 5.2.1 *Find $U = (J_\tau, M_\nu) \in [H^1(C)]^2$ such that:*

$$A(U,\Psi) = \int_C E_\tau^{inc} \Psi_{J\tau} dl + \int_C H_\nu^{inc} \Psi_{M\nu} dl \tag{5.11}$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}) \in [H^1(C)]^2$, where the bilinear form A is defined as:

$$A(U,\Psi) = iZ_0 \iint_C kG(l,l') J_\tau(l') \Psi_{J\tau}\ [\tau(l) \cdot \tau(l')] - \frac{1}{k} G(l,l') d_l' J_\tau(l') d_l \Psi_{J\tau}(l) dl'dl$$

$$+ \iint_C \Psi_{J\tau}(l) M_\nu(l')\ \mathbf{n}(l) \cdot \nabla_\Gamma G(l,l') dl'dl + \iint_C \Psi_{M\nu} J_\tau\ \mathbf{n}(l') \cdot \nabla_\Gamma G(l,l') dl'dl$$

$$+\frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl + \frac{a_0}{2}\int_C J_\tau \Psi_{J\tau} dl + \frac{1}{2a_0}\int_C M_\nu \Psi_{M\nu} dl$$
$$-\frac{a_1}{2}\int_C d_l J\, d_l \Psi_{J\tau} dl - \frac{b_1}{2}\int_C d_l M\, d_l \Psi_{J\tau} dl - \frac{b_1}{2a_0}\int_C d_l M\, d_l \Psi_{M\nu} dl - \frac{a_1}{2a_0}\int_C d_l J\, d_l \Psi_{M\nu} dl \tag{5.12}$$

We present similar variational problem for TM polarization:

Problem 5.2.2 *Find $U = (J_\nu, M_\tau) \in [H^1(C)]^2$ such that:*

$$A(U,\Psi) = \int_C E_\nu^{inc} \Psi_{J\nu} dl + \int_C H_\tau^{inc} \Psi_{M\tau} dl \tag{5.13}$$

for all $\Psi = (\Psi_{J\nu}, \Psi_{M\tau}) \in [H^1(C)]^2$, where the bilinear form A is defined as:

$$A(U,\Psi) = iZ_0 \iint_C kG(l,l')J_\nu(l')\Psi_{J\nu} dl'dl - \iint_C \Psi_{J\nu}(l)M_\tau(l')\ \mathbf{n}(l')\cdot\nabla_\Gamma G(l,l')dl'dl$$
$$-\iint_C \Psi_{M\tau} J_\nu\ \mathbf{n}(l)\cdot\nabla_\Gamma G(l,l')dl'dl + \frac{i}{Z_0}\iint_C kG(l,l')M_\tau(l')\Psi_{M\tau}(l)[\tau(l)\cdot\tau(l')]$$
$$-\frac{1}{k}G(l,l')d_l' M_\tau(l') d_l \Psi_{M\tau}(l) dl'dl + \frac{a_0}{2}\int_C J_\nu \Psi_{J\nu} dl - \frac{1}{2a_0}\int_C M_\tau \Psi_{M\tau} dl$$
$$-\frac{a_2}{2}\int_C d_l J\, d_l \Psi_{J\nu} dl + \frac{b_2}{2}\int_C d_l M\, d_l \Psi_{J\nu} dl + \frac{b_2}{2a_0}\int_C d_l M\, d_l \Psi_{M\tau} dl - \frac{a_2}{2a_0}\int_C d_l J\, d_l \Psi_{M\tau} dl \tag{5.14}$$

5.2.2 Problem with second order IBC

The equation (5.5) passes the same way as IBC1 to become weak. The weak formulations replace operator P in EFIE and MFIE equations. Finally, we assemble them to define the bilinear form:

$$A(U,\Psi) = iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\boldsymbol{\tau}(l)\cdot\boldsymbol{\tau}(l')] - \frac{1}{k}G(l,l')d_l' J_\tau(l') d_l \Psi_{J\tau}(l) dl'dl$$
$$+\iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_\Gamma G(l,l')dl'dl + \iint_C \Psi_{M\nu} J_\tau\ \mathbf{n}(l')\cdot\nabla_\Gamma G(l,l')dl'dl$$
$$+\frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl + \frac{a_0}{2}\int_C J_\tau \Psi_{J\tau} dl + \frac{1}{2a_0}\int_C M_\nu \Psi_{M\nu} dl$$
$$+\frac{a_1}{2}\int_C d_l^2 J_\tau \Psi_{J\tau} dl + \frac{b_1}{2}\int_C d_l^2 M_\nu \Psi_{J\tau} dl + \frac{b_1}{2a_0}\int_C d_l^2 M_\nu \Psi_{M\nu} dl + \frac{a_1}{2a_0}\int_C d_l^2 J_\tau \Psi_{M\nu} dl$$
$$+\frac{a_1'}{2}\int_C d_l^4 M_\nu\ \Psi_{J\tau} dl + \frac{b_1'}{2}\int_C d_l^4 M_\nu\ \Psi_{J\tau} + \frac{b_1'}{2a_0}\int_C d_l^4 M_\nu\ \Psi_{M\nu} dl + \frac{a_1'}{2a_0}\int_C d_l^4 J_\tau\ \Psi_{M\nu} dl$$

for TE polarization. And with integration by parts, we get

$$A(U,\Psi) = iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\boldsymbol{\tau}(l)\cdot\boldsymbol{\tau}(l')] - \frac{1}{k}G(l,l')d'_l J_\tau(l')d_l\Psi_{J\tau}(l)dl'dl$$

$$+ \iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_\Gamma G(l,l')dl'dl + \iint_C \Psi_{M\nu}J_\tau\ \mathbf{n}(l')\cdot\nabla_\Gamma G(l,l')dl'dl$$

$$+\frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl + \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl$$

$$-\frac{a_1}{2}\int_C d_lJ_\tau\ d_l\Psi_{J\tau}dl - \frac{b_1}{2}\int_C d_lM_\nu\ d_l\Psi_{J\tau}dl - \frac{b_1}{2a_0}\int_C d_lM_\nu\ d_l\Psi_{M\nu}dl - \frac{a_1}{2a_0}\int_C d_lJ_\tau\ d_l\Psi_{M\nu}dl$$

$$+\frac{a'_1}{2}\int_C d_l^2M_\nu\ d_l^2\Psi_{J\tau}dl + \frac{b'_1}{2}\int_C d_l^2M_\nu\ d_l^2\Psi_{J\tau} + \frac{b'_1}{2a_0}\int_C d_l^2M_\nu\ d_l^2\Psi_{M\nu}dl + \frac{a'_1}{2a_0}\int_C d_l^2J_\tau\ d_l^2\Psi_{M\nu}dl$$

Problem 5.2.3 *Find $U = (J_\tau, M_\nu) \in [H^1(C)]^2$ such that*

$$A(U,\Psi) = \int_C E_\tau^{inc}\Psi_{J\tau}dl + \int_C H_\nu^{inc}\Psi_{M\nu}dl$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}) \in [H^1(C)]^2$.

5.3 Existence and uniqueness theorem

We are going to show that our variational problem in TE has a unique solution using theorem 2.3.3 from chapter 2. For TM problem it will be analogous. It is necessary to determine the continuity and the coercivity of the bilinear form $A(U,\Psi)$.

For the sake of simplicity we consider the operator $A(U,\Psi)$ as a sum of three bilinear operator

$$A_1(U,\Psi) = \iint_C Z_0(B-S)J_\tau\Psi_{J\tau}dl'dl + \iint_C \frac{1}{Z_0}(B-S)M_\nu\Psi_{M\nu}dl'dl + \iint_C QM_\nu\Psi_{J\tau}dl'dl$$

$$+\iint_C QJ_\tau\Psi_{M\nu}dl'dl + \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl$$

$$A_2(U,\Psi) = -\frac{a_1}{2}\int_C d_lJ\ d_l\Psi_{J\tau}dl - \frac{b_1}{2a_0}\int_C d_lM\ d_l\Psi_{M\nu}dl$$

and

$$A_3(U,\Psi) = -\frac{b_1}{2}\int_C d_lM\ d_l\Psi_{J\tau}dl - \frac{a_1}{2a_0}\int_C d_lJ\ d_l\Psi_{M\nu}dl$$

where

$$A = A_1 + A_2 + A_3$$

5.3.1 Continuity

Lemma 5.3.1 *The bilinear form $A(U,\Psi)$ (6.1) is continuous on $[H^1(C)]^2$.*

Proof. : To prove the continuity of the bilinear form $A(U,\Psi)$, we have to show that there exists $\beta>0$ such that

$$|A(U,\Psi)| \le \beta\|U\|_{H^1(C)}\|\Psi\|_{H^1(C)} \tag{5.15}$$

for all $U,\Psi \in H^1(C)$.

From the works in the past [TL], we have that for operator $A_1(U,\Psi)$ there exists constant $\beta_1>0$ such that

$$|A_1(U,\Psi)| \le \beta_1\|U\|_{H^1(C)}\|\Psi\|_{H^1(C)}$$

It remains to prove continuity for last integrals in bilinear form. Using Cauchy-Schwarz inequality we get:

$$|A_2(U,\Psi)+A_3(U,\Psi)| \le$$

$$\left|\frac{a_1}{2}\int_C d_lJ\ d_l\Psi_{J\tau}dl\right| + \left|\frac{b_1}{2}\int_C d_lM\ d_l\Psi_{J\tau}dl\right| + \left|\frac{b_1}{2a_0}\int_C d_lM\ d_l\Psi_{M\nu}dl\right| + \left|\frac{a_1}{2a_0}\int_C d_lJ\ d_l\Psi_{M\nu}dl\right| \le$$

$$\left|\frac{a_1}{2}\right|\|d_lJ\|_{L^2}\|\Psi_{J\tau}\|_{L^2} + \left|\frac{b_1}{2}\right|\|d_lM\|_{L^2}\|\Psi_{J\tau}\|_{L^2} + \left|\frac{b_1}{2a_0}\right|\|d_lM\|_{L^2}\|\Psi_{M\nu}\|_{L^2} + \left|\frac{a_1}{2a_0}\right|\|d_lJ\|_{L^2}\|\Psi_{M\nu}\|_{L^2}$$

$$\le \beta_2\|U\|_{H^1(C)}\|\Psi\|_{H^1(C)} \quad , \ \textit{where } \beta_2 \ge 0.$$

Finally, we take $\beta=\beta_1+\beta_2\ge 0$ for which (5.15) is true. □

5.3.2 Coercivity

Lemma 5.3.2 *The bilinear form $A(U,\Psi)$ is coercive on $H^1(C)$; i.e., there exists $\gamma>0$ and γ' such that*

$$\Re[A(U,U^*)] \ge \gamma\|U\|^2_{H^1(C)} - \gamma'\|U\|^2_{L^2(C)}, \quad \forall U \in [H^1(C)]^4.$$

Proof. First of all, we take $\Psi=U^*$ and get

$$\begin{aligned} A(U,U^*) = iZ_0 \iint_C kGJ_\tau J_\tau^*[\boldsymbol{\tau}(l')\cdot\boldsymbol{\tau}(l)] - \frac{1}{k}d_l'J_\tau(l')d_lJ_\tau^*(l)dl'dl \\ + \iint_C J_\tau^*M_\nu\mathbf{n}(l)\cdot\nabla_\Gamma G dl'dl + \iint_C M_\nu^*J_\tau\mathbf{n}(l')\cdot\nabla_\Gamma G dl'dl \\ + \frac{i}{Z_0}\iint_C kGM_\nu M_\nu^*[\boldsymbol{\nu}(l')\cdot\boldsymbol{\nu}(l)]dl'dl + \frac{a_0}{2}\int_C J_\tau J_\tau^* dl + \frac{1}{2a_0}\int_C M_\nu M_\nu^* dl \\ -\frac{a_1}{2}\int_C d_lJ_\tau\ d_lJ_\tau^*dl - \frac{b_1}{2}\int_C d_lM_\nu\ d_lJ_\tau^*dl - \frac{b_1}{2a_0}\int_C d_lM_\nu\ d_lM_\nu^*dl - \frac{a_1}{2a_0}\int_C d_lJ_\tau\ d_lM_\nu^*dl \end{aligned} \tag{5.16}$$

As in continuity, from works in the past [TL], we have that there exists $\gamma_1 > 0$ for the operator $A_1(U, \Psi)$, such that $\forall U \in V$

$$\Re[A_1(U, U^*)] \geq \frac{\Re(a_0)}{2}\|J_\tau\|^2_{L^2(C)} + \frac{\Re(a_0)}{2|a_0|^2}\|M_\nu\|^2_{L^2(C)} + \gamma_1\left(\|J_\tau\|^2_{H^1(C)} + \|M_\nu\|^2_{H^1(C)}\right)$$

Next we take operator A_2

$$A_2 = -\frac{a_1}{2}\int_C d_l J_\tau \; d_l J_\tau^* dl - \frac{b_1}{2a_0}\int_C d_l M_\nu \; d_l M_\nu^* dl$$

where real part

$$\Re(A_2) = -\frac{\Re(a_1)}{2}\|d_l J_\tau\|^2_{L^2(C)} - \Re(\frac{b_1}{2a_0})\|d_l M_\nu\|^2_{L^2(C)}$$

And it remains

$$A_3 = -\frac{b_1}{2}\int_C d_l M_\nu \; d_l J_\tau^* dl - \frac{a_1}{2a_0}\int_C d_l J_\tau \; d_l M_\nu^* dl$$

where real part

$$\Re(A_3) = \Re\left(-\frac{b_1}{2}\int_C d_l M_\nu \; d_l J_\tau^* dl - \frac{a_1}{2a_0}\int_C d_l J_l \; d_l M_\nu^* dl\right) =$$

$$= -\Re\left[\left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2}\right)\int_C d_l M_\nu \; d_l J_\tau^* dl\right] =$$

$$= -\Re\left[\int_C \frac{1}{|a_0|^{1/2}}\left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2}\right)^{1/2} d_l M_\nu \cdot |a_0|^{1/2}\left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2}\right)^{1/2} d_l J_l^* dl\right]$$

We note $q = b_1|a_0| + a_1^* a_0/|a_0|$, so

$$\Re(A_3) \geq -\frac{|q|}{4}\|d_l J_\tau\|^2_{L^2(C)} - \frac{|q|}{4|a_0|^2}\|d_l M_\nu\|^2_{L^2(C)}.$$

Sufficient uniqueness conditions (3.26) says that $\Re(a_1 - b_1^* a_0) = 0$ or $\Re(a_1) = \Re(b_1 a_0^*)$. So the sum of operators A_2 and A_3 get

$$\Re(A_2) + \Re(A_3) \geq$$

$$-\frac{1}{2}\left(\Re(a_1) + \frac{|q|}{2}\right)\|d_l J_\tau\|^2_{L^2(C)} - \frac{1}{2|a_0|^2}\left(\Re(a_1) + \frac{|q|}{2}\right)\|d_l M_\nu\|^2_{L^2(C)}$$

So, if $\Re(a_1) + \frac{|q|}{2} = 0$, we get that

$$\Re(A) = \Re(A_1) + \Re(A_2) + \Re(A_3) \geq$$

$$\geq \gamma_1\|U\|^2_{H^1(C)} - c\|U\|^2_{L^2(C)}$$

That gives us coercivity of bilinear form $A(U, \Psi)$. □

Theorem 5.3.1 *The problem (5.11) admits a unique solution $U \in [H^1(C)]^2$ for any $\Psi \in [H^1(C)]^2$, if coefficients satisfy*

$$\Re(a_1) + \frac{|a_0||b_1 + a_1^*/a_0^*|}{2} = 0. \tag{5.17}$$

Proof. Lemmas 5.3.1-5.3.2 give us that the bilinear form $A(U, \Psi)$ verifies hypothesis of the theorem 2.3.3. □

5.4 Conclusion

We include 2D HOIBC in integral equations by replacing operator P and combined $EFIE$ and $MFIE$ to represent variational form of 2D scattering problem. The problem presentation helps us to determine the functional spaces of the problem. The integrals are taken over curvilinear contour. The existence and uniqueness theorem is important contribution to work with HOIBC. The coercivity of an operator A requires restriction (5.17) on the coefficients. Further we propose discretization of the two dimensional problem and numerical results.

Chapter 6

Two dimensional discretization

Contents

6.1 Introduction

We again recognize that the two dimensional case is invariant in one direction and a two dimensional object has a unidimensional boundary. So we will use finite element method on a curvilinear contour. We use auxiliary variables X, Y that were mentioned in [BSb], to avoid integration by parts. So we take formulation of bilinear form $A(U, \Psi)$ presented in the previous chapter before carrying out integration by parts

$$\begin{aligned} A(U,\Psi) = iZ_0 \iint_C kG(l,l')J_\tau(l')\Psi_{J\tau}\ [\tau(l)\cdot\tau(l')] - \frac{1}{k}G(l,l')d_l' J_\tau(l')d_l\Psi_{J\tau}(l)dl'dl \\ + \iint_C \Psi_{J\tau}(l)M_\nu(l')\ \mathbf{n}(l)\cdot\nabla_\Gamma G(l,l')dl'dl + \iint_C \Psi_{M\nu}J_\tau\ \mathbf{n}(l')\cdot\nabla_\Gamma G(l,l')dl'dl \\ + \frac{i}{Z_0}\iint_C kG(l,l')M_\nu(l')\Psi_{M\nu}(l)dl'dl + \frac{a_0}{2}\int_C J_\tau\Psi_{J\tau}dl + \frac{1}{2a_0}\int_C M_\nu\Psi_{M\nu}dl \\ + \frac{a_1}{2}\int_C d_l^2 J\ \Psi_{J\tau}dl + \frac{b_1}{2}\int_C d_l^2 M\Psi_{J\tau}dl + \frac{b_1}{2a_0}\int_C d_l M\Psi_{M\nu}dl + \frac{a_1}{2a_0}\int_C d_l^2 J\Psi_{M\nu}dl \end{aligned} \tag{6.1}$$

where we replace $d_l J_\tau$ and $d_l M_\nu$ by auxiliary variables X and Y, respectively. And we introduce two weak equations

$$\int_C (X - d_l J_\tau)\ X' dl = 0;$$

$$\int_C (Y - d_l M_\nu)\; Y' dl = 0.$$

And finally we get bilinear form for $U = (J_\tau, M_\nu, X, Y) \in [H^1(C)]^4$

$$A(U, \Psi) = iZ_0 \iint_C kG(l,l')\; J_\tau(l')\; \Psi_{J\tau}\; [\boldsymbol{\tau}(l') \cdot \boldsymbol{\tau}(l)] - \frac{1}{k} G(l,l')\; d'_l J_\tau(l')\; d_l \Psi_{J\tau}(l) dl' dl$$

$$+ \iint_C \Psi_{J\tau}(l)\; M_\nu(l')\; \mathbf{n}(l) \cdot \nabla_\Gamma G(l,l') dl' dl + \iint_C \Psi_{M\nu}\; J_\tau\; \mathbf{n}(l') \cdot \nabla_\Gamma G(l,l') dl' dl$$

$$+ \frac{i}{Z_0} \iint_C kG(l,l')\; M_\nu(l')\; \Psi_{M\nu}(l) dl' dl + \frac{a_0}{2} \int_C J_\tau\; \Psi_{J\tau} dl + \frac{1}{2a_0} \int_C M_\nu\; \Psi_{M\nu} dl$$

$$+ \frac{a_1}{2} \int_C d_l X\; \Psi_{J\tau} dl + \frac{b_1}{2} \int_C d_l Y\; \Psi_{J\tau} dl + \frac{b_1}{2a_0} \int_C d_l Y\; \Psi_{M\nu} dl + \frac{a_1}{2a_0} \int_C d_l X\; \Psi_{M\nu} dl$$

$$+ c_1 \int_C X\; X' dl - c_1 \int_C d_l J_\tau\; X' dl + d_1 \int_C Y\; Y' dl - d_1 \int_C d_l M_\nu\; Y' dl \tag{6.2}$$

for all $\Psi = (\Psi_{J\tau}, \Psi_{M\nu}, X', Y') \in [H^1(C)]^4$.

Here, we study only TE polarization case. We assume that TM polarization has analogous results.

6.2 Discretization

We approximate the curve C by means of N straight *line segments* C_i, satisfying the general overlapping conditions for a finite element method. The line segments are also called *elements* and provide a contour piecewise linear approximation $C^h = \cup_{i=1}^N C_i$, where h is a positive parameter such that $\lim_{h \to 0} N(h) = +\infty$. Curvilinear structures include geometry modeling errors in this approach. These errors can only be reduced by decreasing the segments lengths; i.e. by increasing the number of segments N. We denote nodes from 1 to N. We consider V_h a finite dimensional subspace of $H^1(C^h)$

$$V_h = \Big\{ v_h : C^h \to \mathbb{R},\; v_h \in H^1(C^h),\; v_h|_{C_i} \in P_1,\; \forall i \in 1, ..., N \Big\} \subset H^1(C^h)$$

where P_1 is the space of first degree polynomials, and

$$W_h = \Big\{ w_h : C^h \to \mathbb{R},\; w_h \in H^1(C^h),\; w_h|_{C_i} \in P_0,\; \forall i \in 1, ..., N \Big\} \subset L^2(C^h)$$

where P_0 is the space of constant functions.

We observe discretization of the unknowns by basis functions

$$J_\tau \approx J_\tau^h(l) = \sum_{i=1}^N J_{\tau i} \phi_i(l) \;\; \in V_h \tag{6.3}$$

$$M_\nu \approx M_\nu^h(l) = \sum_{i=1}^N M_{\nu i} \psi_i(l) \;\; \in W_h \tag{6.4}$$

$$X \approx X^h(l) = \sum_{i=1}^{N} X_i \psi_i(l) \ \in W_h \tag{6.5}$$

and

$$Y \approx Y^h(l) = \sum_{i=1}^{N} Y_i \psi_i(l) \ \in W_h. \tag{6.6}$$

where $\phi_i \in V_h$ and $\psi_i \in W_h$

So the bilinear form $A(U, \Psi)$ in (6.2) can be written as

$$A(U^h, \Psi^h) = iZ_0 \sum_{i,j=1}^{N} \left(\iint_{C^h} kG \ \phi_j \ \phi_i \ [\vec{\tau}_j(l') \cdot \vec{\tau}_i(l)] - \frac{1}{k} G \ d_l' \phi_j \ d_l \phi_i \ dl' dl \right) J^h_{\tau j}$$

$$+ \sum_{i,j=1}^{N} \left(\iint_{C^h} \psi_j \ \phi_i \ \mathbf{n}_i \cdot \nabla_\Gamma G dl' dl \right) M^h_{\nu j}$$

$$+ \sum_{i,j=1}^{N} \left(\iint_{C^h} \phi_j \ \psi_i \ \mathbf{n}_j \cdot \nabla_\Gamma G dl' dl \right) J^h_{\tau j} + \frac{i}{Z_0} \sum_{i,j=1}^{N} \left(\iint_{C^h} kG \ \psi_j \ \psi_i \ dl' dl \right) M^h_{\nu j}$$

$$+ \frac{a_0}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} \phi_j \ \phi_i dl \right) J^h_{\tau j} + \frac{1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) M^h_{\nu j}$$

$$+ \frac{a_1}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \phi_i dl \right) X^h_j + \frac{b_1}{2} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \phi_i dl \right) Y^h_j$$

$$+ \frac{b_1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) Y^h_j + \frac{a_1}{2a_0} \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) X^h_j$$

$$+ \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) X^h_j - \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) J^h_{\tau j}$$

$$+ \sum_{i,j=1}^{N} \left(\int_{C^h} \psi_j \ \psi_i dl \right) Y^h_j - \sum_{i,j=1}^{N} \left(\int_{C^h} d_l \psi_j \ \psi_i dl \right) M^h_{\nu j}$$

In order to simplify the equations in matrix form we define the following matrices

$$(B - S)_{ij} = i \iint_{C^h} kG(l, l') \ \phi_j(l') \ \phi_i \ [\boldsymbol{\tau}_j \cdot \boldsymbol{\tau}_i] - \frac{1}{k} G(l, l') \ d_l' \phi_j(l') \ d_l \phi_i(l) \ dl' dl$$

$$Q_{ij} = \iint_{C^h} \phi_i(l) \ \psi_j(l') \ \mathbf{n}_i \cdot \nabla_\Gamma G(l, l') \ dl' dl$$

$$B_{ij} = i \iint_{C^h} kG(l, l') \ \psi_j(l') \ \psi_i(l) \ dl' dl$$

$$I1_{ij} = \int_{C^h} \phi_i(l) \ \phi_j(l) \ dl$$

$$I2_{ij} = \int_{C^h} \psi_i(l)\ \psi_j(l)\ dl$$

$$D1_{ij} = \int_{C^h} \phi_i(l)\ d_l\psi_j(l)\ dl$$

$$D3_{ij} = \int_{C^h} \psi_i(l)\ d_l\psi_j(l)\ dl$$

$$D5_{ij} = \int_{C^h} \psi_i(l)\ d_l\phi_j(l)\ dl$$

Note: $[I2]$ is the nonsingular diagonal matrix, therefore it is invertible.

So we can write our system in a matrix form:

$$\begin{bmatrix} Z_0[B-S] + \frac{a_0}{2}[I1] & [Q] & \frac{a_1}{2}[D1] & \frac{b_1}{2}[D1] \\ [Q]^T & \frac{1}{Z_0}[B] + \frac{1}{2a_0}[I2] & \frac{a_1}{2a_0}[D3] & \frac{b_1}{2a_0}[D3] \\ -[D5] & 0 & [I2] & 0 \\ 0 & -[D3] & 0 & [I2] \end{bmatrix} \begin{pmatrix} \overline{J}^h \\ \overline{M}^h \\ \overline{X}^h \\ \overline{Y}^h \end{pmatrix} = \begin{pmatrix} \overline{E}^h \\ \overline{H}^h \\ 0 \\ 0 \end{pmatrix} \tag{6.7}$$

Where right-side vectors $\overline{E}^h$, $\overline{H}^h$ are defined as follows:

$$E_i^h = \int_{C^h} \mathbf{E}^{inc} \cdot \boldsymbol{\phi}_i dl;$$

$$H_i^h = \int_{C^h} \mathbf{H}^{inc} \cdot \boldsymbol{\psi}_i dl.$$

And vectors $\overline{J}^h$ and $\overline{M}^h$ are unknowns.

From the last two lines in (E.1), we get

$$-[D5]\ \overline{J}^h + [I2]\ \overline{X}^h = 0\ \rightarrow\ \overline{X}^h = [I2]^{-1}\ [D5]\ \overline{J}^h;$$

$$-[D3]\ \overline{M}^h + [I2]\ \overline{Y}^h = 0\ \rightarrow\ \overline{Y}^h = [I2]^{-1}\ [D3]\ \overline{M}^h.$$

We put X, Y in first two equations.

$$\begin{bmatrix} [A1] & [A2] \\ [A3] & [A4] \end{bmatrix} \begin{pmatrix} \overline{J}^h \\ \overline{M}^h \end{pmatrix} = \begin{pmatrix} \overline{E}^h \\ \overline{H}^h \end{pmatrix} \tag{6.8}$$

where matrices are defined as

$$[A1] = Z_0[B-S] + \frac{a_0}{2}[I1] + \frac{a_1}{2}[D1]\ [I2]^{-1}\ [D5]$$

$$[A2] = [Q] + \frac{b_1}{2}[D1]\ [I2]^{-1}\ [D3]$$

$$[A3] = [Q]^T + \frac{a_1}{2a_0}[D3]\ [I2]^{-1}\ [D5]$$

$$[A4] = \frac{1}{Z_0}[B] + \frac{1}{2a_0}[I2] + \frac{b_1}{2a_0}[D3]\ [I2]^{-1}\ [D3].$$

Finally we get the matrix equation to be solved. The matrices $[I1]$ and $[I2]$ are simple. In $[D1]$, $[D3]$ and $[D5]$ we use differentials that was defined in (6.11)-(6.12). Matrices $[B-S]$ and $[Q]$ are more complicated, they involve double-integral with the Green function.

6.3 Assembly of matrices

Basis and test functions for values tangent to boundary in τ direction depend on l, denoted by ϕ_i, which belongs to class P_1 - partially continuous first degree polynomials. Basis and test functions for values tangent to boundary in ν direction depend on l, denoted by ψ_j, which belongs to class P_0 - partially constant (see Fig.6.1 a-b). Hence, we define functions ϕ_i and ψ_j as

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{6.9}$$

$$\psi_j(x) = \begin{cases} \frac{1}{x_{j+1}-x_j} & x \in [x_j, x_{j+1}] \\ 0 & x \notin [x_j, x_{j+1}]. \end{cases} \tag{6.10}$$

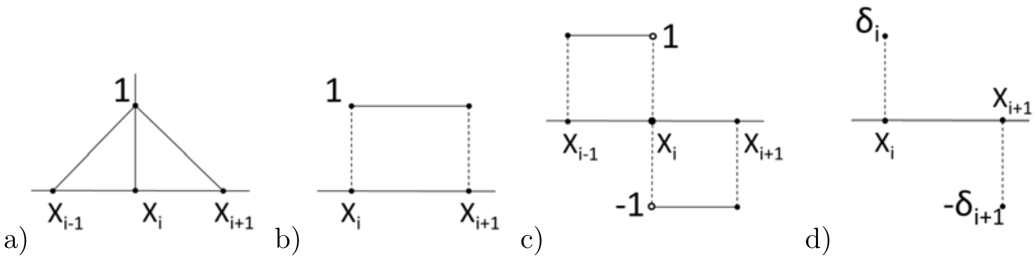

Figure 6.1: Basis functions a)P_1 type and b)P_0 type. Derivative of basis functions c)P_1 type and d)P_0 type

We observe that the derivative of a P_1 function is a function of class P_0. Thus we can express $d_l\phi_i$ with the basis functions ψ_j. Whereas the derivative of functions ψ_j, we express as difference of Dirac functions in breaking points (see Fig.6.1 c-d):

$$d_l\phi_i(l) = \psi_{i-1}(l) - \psi_i(l); \tag{6.11}$$

$$d_l\psi_j(l) = \delta_j - \delta_{j+1}. \tag{6.12}$$

We use Gaussian quadrature to calculate the integral

$$\int_{C_j} f(x)dx \approx \sum_{g=1}^{n_g} f(x_g)p_g.$$

6.3.1 Calculation of the matrix (B-S)

The element of the matrix $(B-S)$ are calculated on segments associated to functions ϕ_i and ϕ'_j.

$$(B-S)_{ij} = i \iint_C G(l,l')(k\phi'_j\phi_i[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{k}d'_l\phi'_j d_l\phi_i)dl'dl =$$

$$= i\int_{C_i+C_{i-1}} dl \int_{C'_j+C'_{j-1}} G(l,l')(k\phi'_j\phi_i[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{k}(\psi'_{j-1} - \psi'_j)(\psi_{i-1} - \psi_i))dl'$$

For the sake of simplification, we want to show calculation of the simple part

$$Int_{ij} = i\iint_{C_iC'_j} G(l,l')(k\phi'_j\phi_i[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{k}\psi'_j\psi_i)dl'dl$$

where $\psi_i = \frac{1}{x_{i+1}-x_i}$ is a constant on the element C_i and $\psi'_j = \frac{1}{x_{j+1}-x_j}$ on C'_j.

According to features of Green's function $G(l,l')$, we separate calculation into two cases. First case when arguments l and l' are apart from each other.

Apart elements

If the elements have enough big distance from each other, we can be sure in convergence of integral and we use Gauss points approach:

$$Int_{ij} \approx i\sum_{g=1}^{n_g}\sum_{g'=1}^{n'_g} p_g p'_g \frac{\pi}{i} H_0^{(2)}(k\rho_{gg'}) \left[k\phi'_{jg'}\phi_{ig}[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{kh'_j h_i}\right]$$

Close elements

On the other hand if the elements are close to each other, we should expand Green function:

$$G = \frac{\pi}{i}H_0^{(2)}(k\rho) = \underbrace{\frac{\pi}{i}H_0^{(2)}(k\rho) + 2\ln(\rho)}_{\to G|_1} - \underbrace{2\ln(\rho)}_{\to G|_2} \tag{6.13}$$

When $\rho \to 0$, we have

$$G|_1 = \frac{\pi}{i}H_0^{(2)}(k\rho) - 2\ln(\rho) \to \frac{\pi}{i} - 2(\gamma + \ln(\frac{k}{2})) \tag{6.14}$$

So for the calculation of double integral $Int_{ij}|_1$ we can use Gauss points approach:

$$Int_{ij}|_1 \approx i\left\{\frac{\pi}{i} - 2(\gamma + \ln(\frac{k}{2}))\right\}\sum_{g=1}^{n_g}\sum_{g'=1}^{n'_g} p_g p'_g \left[k\phi'_{jg'}\phi_{ig}[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{kh'_j h_i}\right]$$

and to calculate the remaining part, we integrate over Γ with help of Gauss points and over Γ' we use analytical integral

$$Int_{ij}|_2 \approx -2i\sum_{g=1}^{n_g} p_g \int_{C_j} ln(\rho(l_g,l')) \left[k\phi'_j\phi_{ig}[\vec{\tau}'_j \cdot \vec{\tau}_i] - \frac{1}{kh'_j h_i}\right] dl'.$$

6.3.2 Calculation of the matrix Q

The elements of matrix Q are calculated on segments associated to functions ϕ_i and ψ'_j.

$$Q_{ij} = -i \iint_C \phi_i(l)\psi_j(l')\mathbf{n}_i \cdot \nabla_\Gamma G(l,l')dl'dl$$

where function ψ'_j is defined only on a segment C_j and gradient of Green function ∇G is expressed

$$\nabla G(l,l') = -\frac{\pi k}{i\rho} H_1^{(2)}(k\rho)\vec{\rho}.$$

So we can write

$$Q_{ij} = i \int_{C_i+C_{i-1}} \int_{C_j} \phi_i(l)\psi_j(l')\frac{\pi k}{\rho} H_1^{(2)}(k\rho)\mathbf{n}_i \cdot \vec{\rho}.$$

As in $(B-S)$ matrix, for apart elements we use Gauss points approach.

Close elements

According to the property of $H_1^{(2)}(k\rho)$ for $\rho \to 0$

$$\frac{k}{\rho}\left[H_1^{(2)}(k\rho) - \frac{2i}{\pi k\rho} + \frac{i}{\pi}k\rho\ln(\rho)\right] \to -\frac{i}{\pi}k^2\ln(k/2) + k^2\left(\frac{1}{2} + \frac{i}{2\pi}(1-2\gamma)\right)$$

$$\nabla G(l,l') = \underbrace{-\frac{\pi k_j}{i\rho}\left[H_1^{(2)}(k\rho) - \frac{2i}{\pi k\rho} + \frac{i}{\pi}k\rho\ln(\rho)\right]\vec{\rho}}_{\to GG|_1} - \underbrace{\left[\frac{2}{\rho^2} - k^2\ln(\rho)\right]\vec{\rho}}_{\to GG|_2}$$

For ρ small enough

$$GG|_1 \approx -i\left[k\ln(k/2) - \frac{\pi}{i}k^2\left(\frac{1}{2} + \frac{i}{2\pi}(1-2\gamma)\right)\right]\sum_{g=1}^{n_g}\sum_{g'=1}^{n_{g'}} p_g p'_g \phi_{ig}\psi'_{jg'}\mathbf{n}_i \cdot \vec{\rho}_{gg'}$$

and

$$GG|_2 \approx i\sum_{g=1}^{n_g}\sum_{g'=1}^{n_{g'}} p_g p'_g \phi_{ig}\psi'_{jg'}\left[\frac{2}{\rho_{gg'}} - k^2\ln(\rho_{gg'})\right]\mathbf{n}_i \cdot \vec{\rho}_{gg'}.$$

6.4 Numerical results

6.4.1 Radar cross section

Radiation theory teaches us that the energy is intercepted by an object can be reflected, absorbed or transmitted through the target. We can assume that most of the energy is reflected. The spatial distribution of this energy depends on the size, shape and composition of the target, and on the frequency and nature of the incident wave. This distribution of energy is called scattering, and the target itself is often referred to as a scatterer. The radar cross section (RCS) of the body is a measure of the energy scattered in a particular direction for a given illumination [R-Sb].

Bistatic scattering is the name given to the situation when the scattering direction is not back toward the source of the radiation. If $\mathbf{E}$ and $\mathbf{H}$ represent fields scattered by an object illuminated by incident plane wave $\mathbf{E}^{inc}$ traveling in the direction of the unit vector $\mathbf{k}$, the *bistatic radar cross section* in the observation direction $\mathbf{r}$ is

$$\sigma(\mathbf{r},\mathbf{k}) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is defined as the area through which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The *monostatic radar cross section* is defined as the radar cross section observed in the back scattering direction, $\sigma(-\mathbf{k},\mathbf{k})$.

In two dimensions, the bistatic radar cross section for scattering by a cylindrical object illuminated by an incident plane wave $\mathbf{E}^{inc}$ traveling in the direction of the unit vector $\mathbf{k}$ normal to the cylinder axis is

$$\sigma(\rho,\mathbf{k}) = \lim_{\rho\to\infty} 2\pi\rho \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is the equivalent width across which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The monostatic radar cross section is $\sigma(-\mathbf{k},\mathbf{k})$. That is defined for cylinders as the ratio of the total scattered power per unit length to the power density of the incident wave.

The units for RCS are square meters. As RCS can span a wide range of values, a logarithmic decibel scale is also used with a typical reference value σ_{ref} equal to $1m^2$:

$$\sigma_{dBm^2} = 10\ \log_{10}(\frac{\sigma}{\sigma_{ref}}) \tag{6.15}$$

6.4.2 Numerical tests

Let us consider conducting circular cylinder depicted in figure 6.2 coated with thin dielectric layer. The radius of the inner conductor is $r = 50mm$ and the thickness of the coating is d. It is assumed that the incident field is propagating normal to the axis of the cylinder. And we consider both TE and TM polarizations. In order to illustrate several key points the case of a simple dielectric coating will be considered.

An exact solution of the scattering problem depicted in figure 6.2 is obtained by expanding the incident field, the scattered field outside the cylinder, and the total field inside the cylinder coating in terms of a series of cylindrical wave functions and applying the appropriate boundary conditions at each interface.

Since the coefficients appearing in the HOIBC were derived by considering the planar canonical problem it is expected that the solution should be most accurate for cylinders with large radius of curvature and thin coating, where the geometrical approximation is a good one.

In order to illustrate these points scattering by three typical coated cylinders will be considered next. Figures 6.4-6.5 show the monostatic RCS for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 0.1\lambda_0$, and coating parameters $\varepsilon_r = 4.0 - 0.5i$ and $\mu_r = 1.0$. The exact series solution is presented along with the HOIBC and SIBC solutions. We computed monostatic RCS for different frequencies to see how do results depend on frequency. In TE-polarization we can see that results of SIBC jumps in range between $6GHz$ and $8GHz$ (see fig. 6.4). Much bigger difference, we can see in TM-polarization between $7GHz$ and $9GHz$ (see fig. 6.5).

Next we consider bistatic RCS for different scattering angles. Figures 6.6-6.7 show the bistatic radar cross section for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 1.5mm$, and coating parameters $\varepsilon_r = 10 - 5i$ and $\mu_r = 1$, for fixed frequency $f = 6.8GHz$ in TE and TM polarizations. The exact series solution is presented along with the SIBC and HOIBC order 1 and order 2 solutions.

After we increase thickness of a boundary and decrease frequency, so we considered bistatic RCS for different scattering angles. Figures 6.8-6.9 shows the bistatic radar cross section for a coated conducting cylinder with inner radius λ_0, coating thickness $d = 3mm$ and frequency $f = 3.4GHz$, coating parameters $\varepsilon_r = 10 - 5i$ and $\mu_r = 1.0$, in TE and TM polarizations. The exact series solution is presented along with the SIBC and HOIBC order 1 and order 2 solutions.

Here we comput bistatic RCS for coated circular cylinder with parameters, $d = 0.1\lambda_0, \varepsilon_r = 4 - 0.5i$ and $\mu_r = 1$. And we compare to Rahmat-Samii results for same test. The backscatter direction is $\phi = 180°$. Results for exact formulation, SIBC or Leontovich IBC formulation and the formulation based on the planar higher order IBC are presented in the figure 6.10. As can be seen in the figure, the results using the planar HOIBC are in excellent agreement with the exact solution over most of the angular range, while SIBC solutions give only the average behavior of the scattered field.

Next we consider conducting plate with open boundary thin dielectric layer (see fig. 6.3). Figures 6.11-6.12 show the bistatic RCS for layer thickness $d = 4mm$ and frequency $f = 6.8GHz$. This example is interesting because it shows that method works even for open boundaries. And we can see that it solves problem much better than with Leontovich IBC. But it is difficult to see difference between first order and second order IBCs.

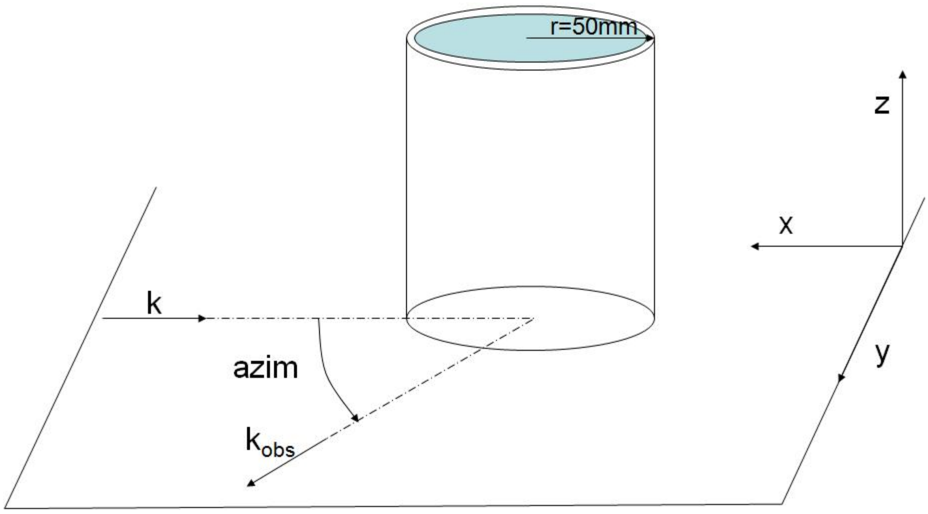

Figure 6.2: Cylinder

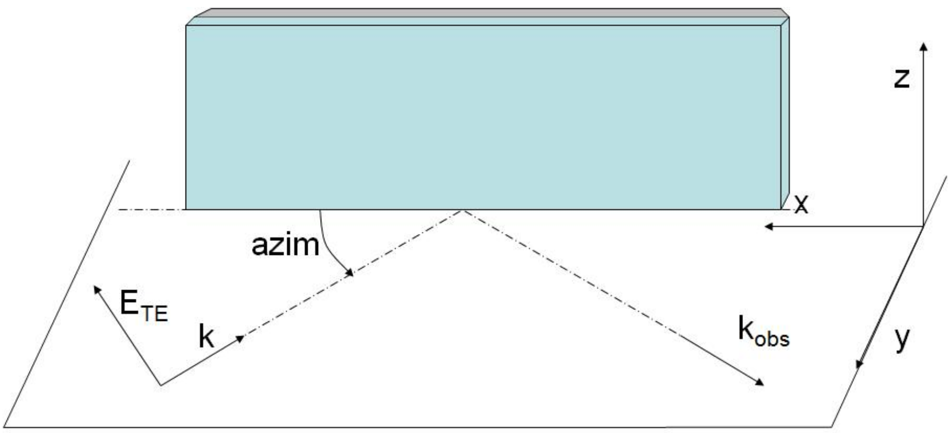

Figure 6.3: Plate with thin layer

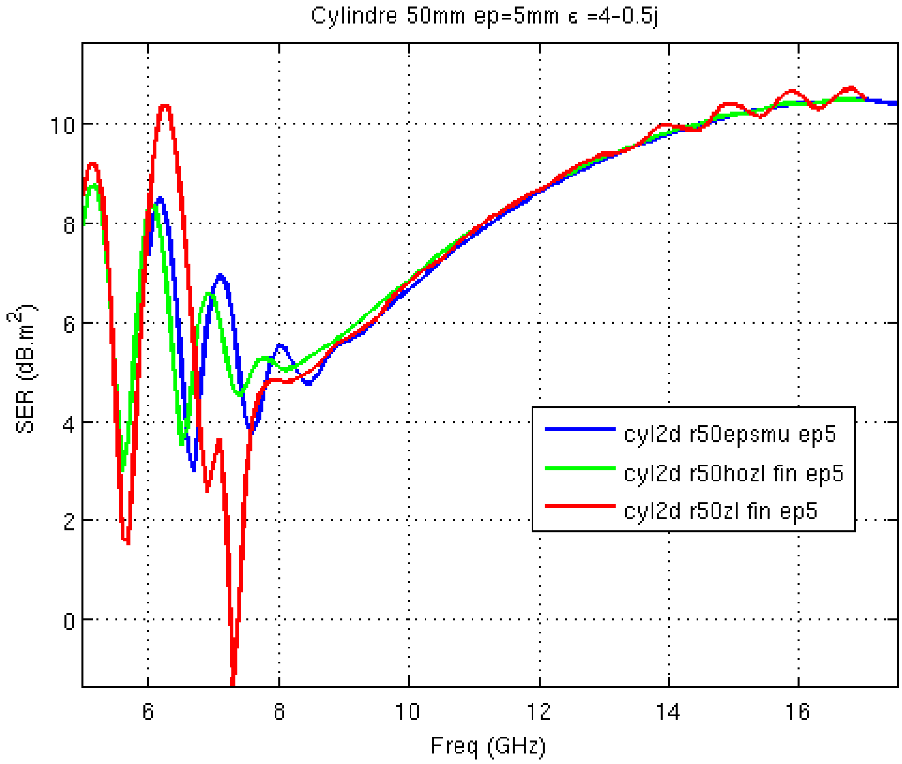

Figure 6.4: Monostatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\varepsilon_r = 4.0 - j0.5$ and $\mu_r = 1.0$, with TE polarization

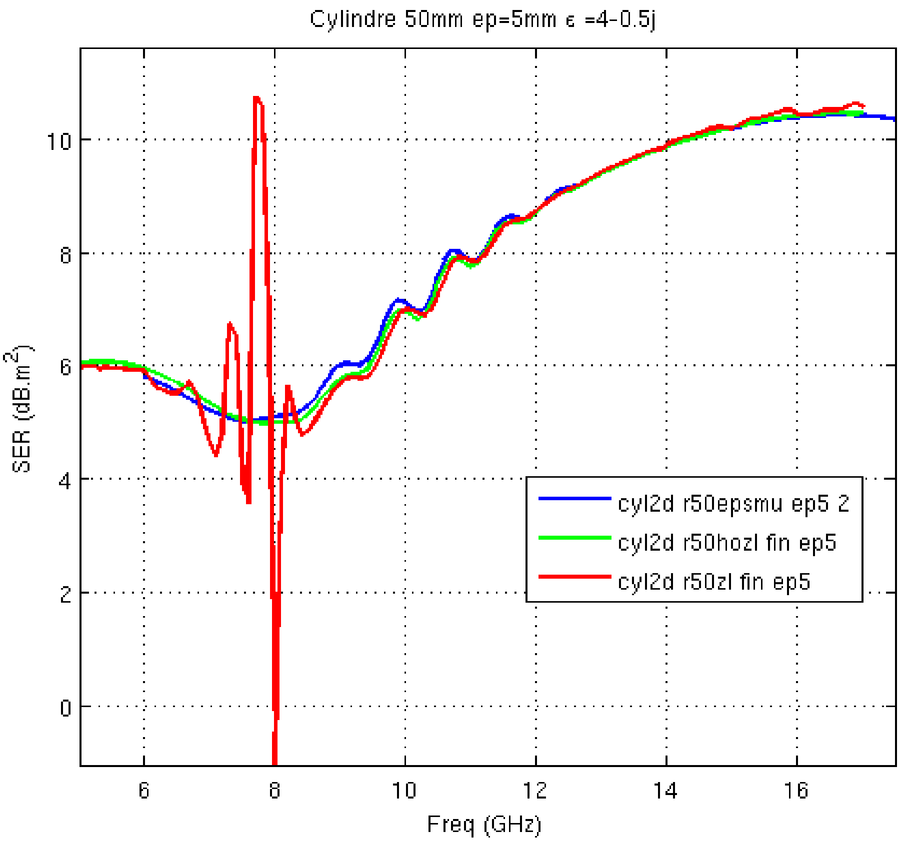

Figure 6.5: Monostatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\varepsilon_r = 4.0 - j0.5$ and $\mu_r = 1.0$, with TM polarization

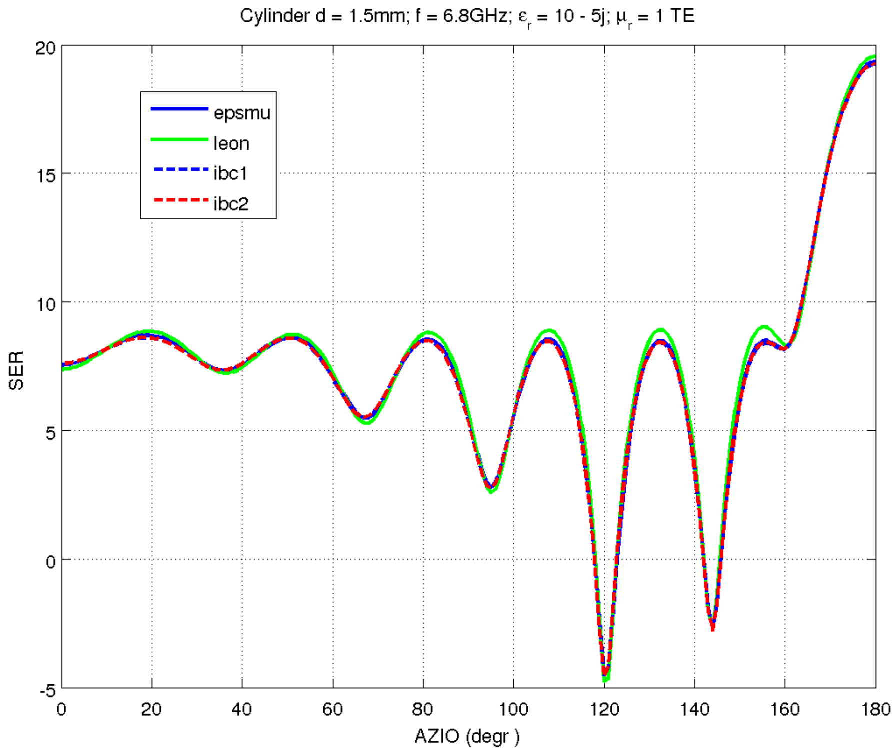

Figure 6.6: Bistatic RCS for a coated circular cylinder, when $d = 1.5mm$, $\varepsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TE polarization

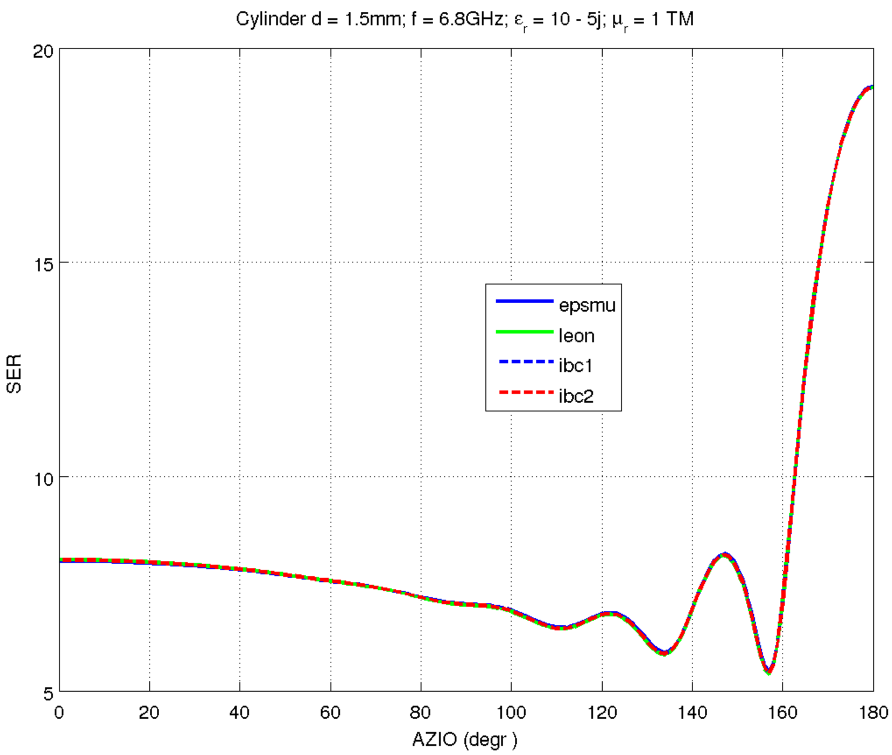

Figure 6.7: Bistatic RCS for a coated circular cylinder, when $d = 1.5mm$, $\varepsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TM polarization

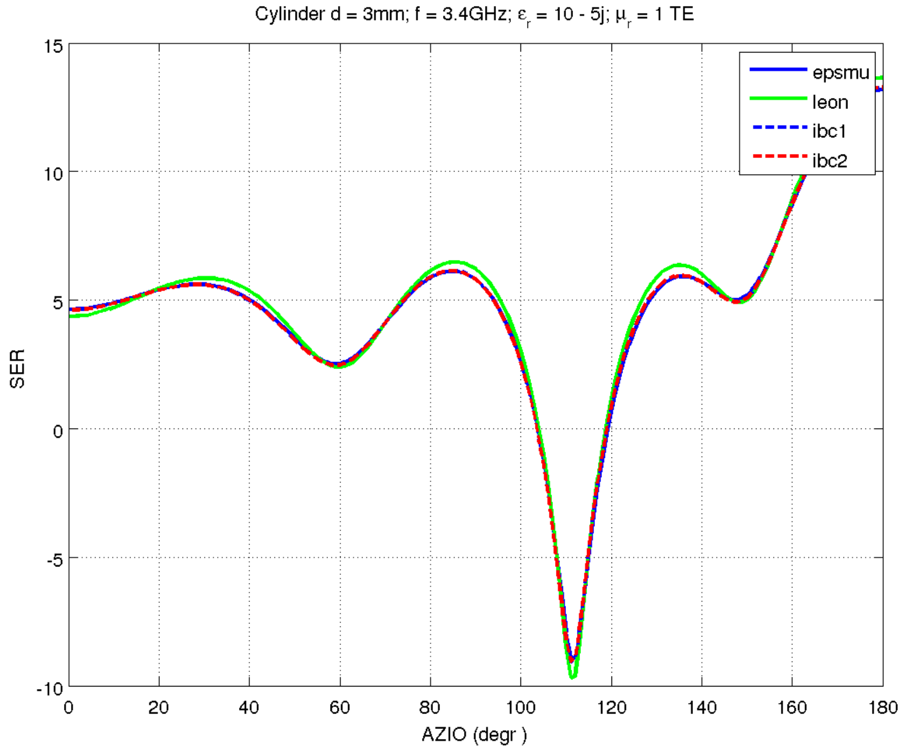

Figure 6.8: Bistatic RCS for a coated circular cylinder, when $d = 3mm$, $\varepsilon_r = 10-5j$, $\mu_r = 1.0$, and $f = 3.4GHz$ with TE polarization

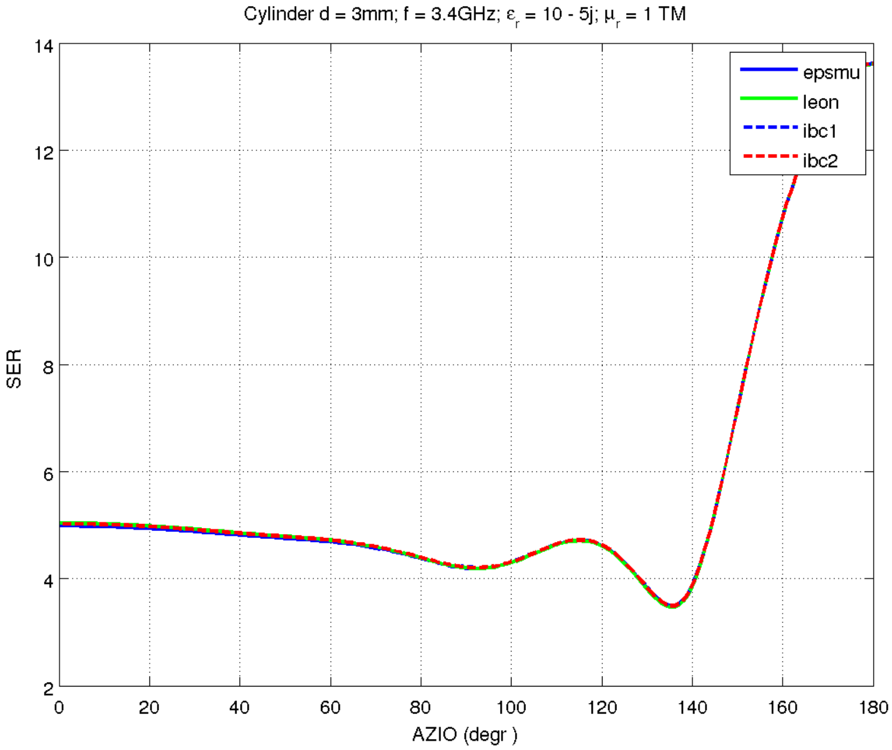

Figure 6.9: Bistatic RCS for a coated circular cylinder, when $d = 3mm$, $\varepsilon_r = 10-5j$, $\mu_r = 1.0$, and $f = 3.4GHz$ with TM polarization

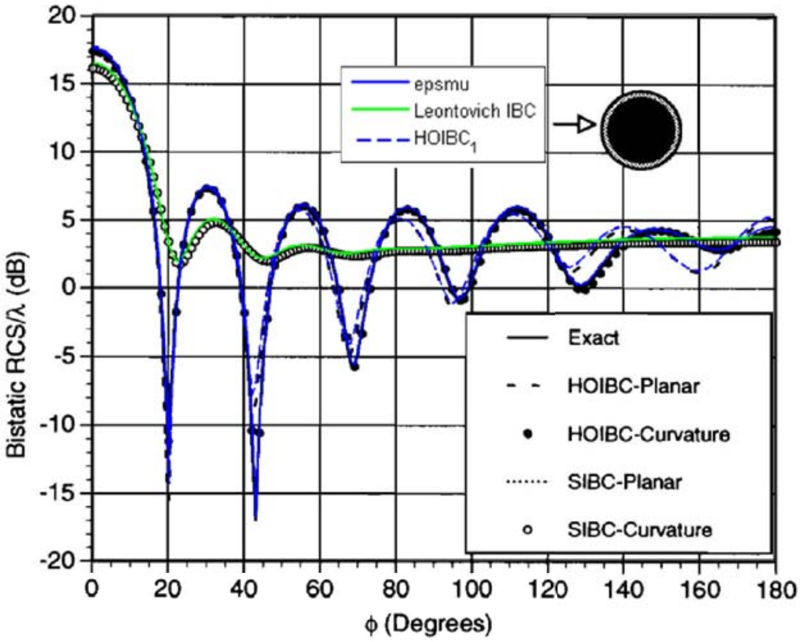

Figure 6.10: Bistatic RCS for a coated circular cylinder, when $d = 0.1\lambda_0$, $\varepsilon_r = 4 - 0.5j$, $\mu_r = 1$ with TE polarization

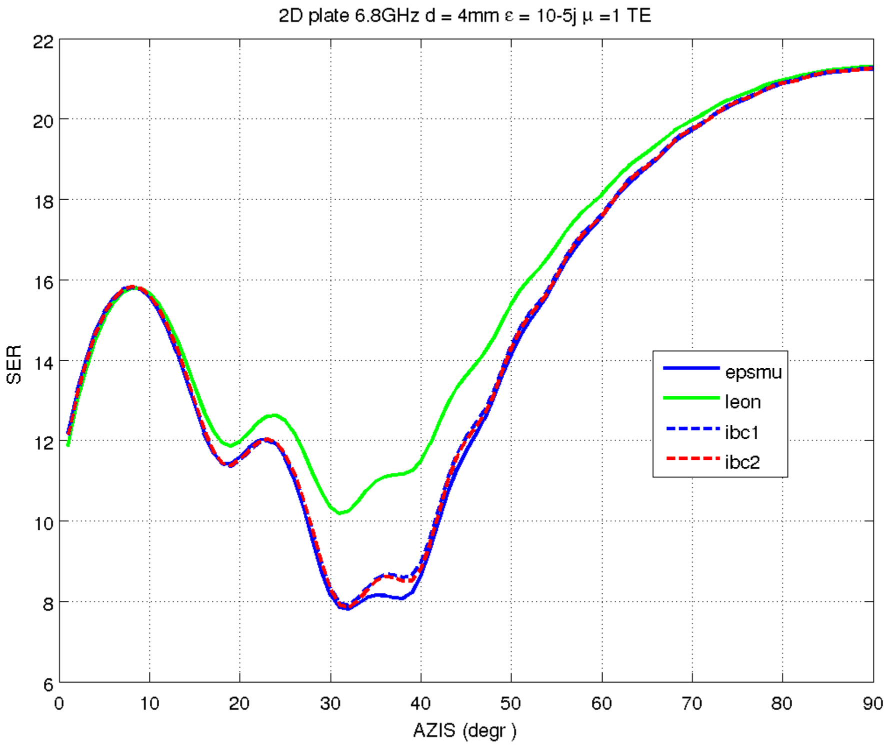

Figure 6.11: Bistatic RCS for a coated 2D plate, when $d = 4mm$, $\varepsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TE polarization

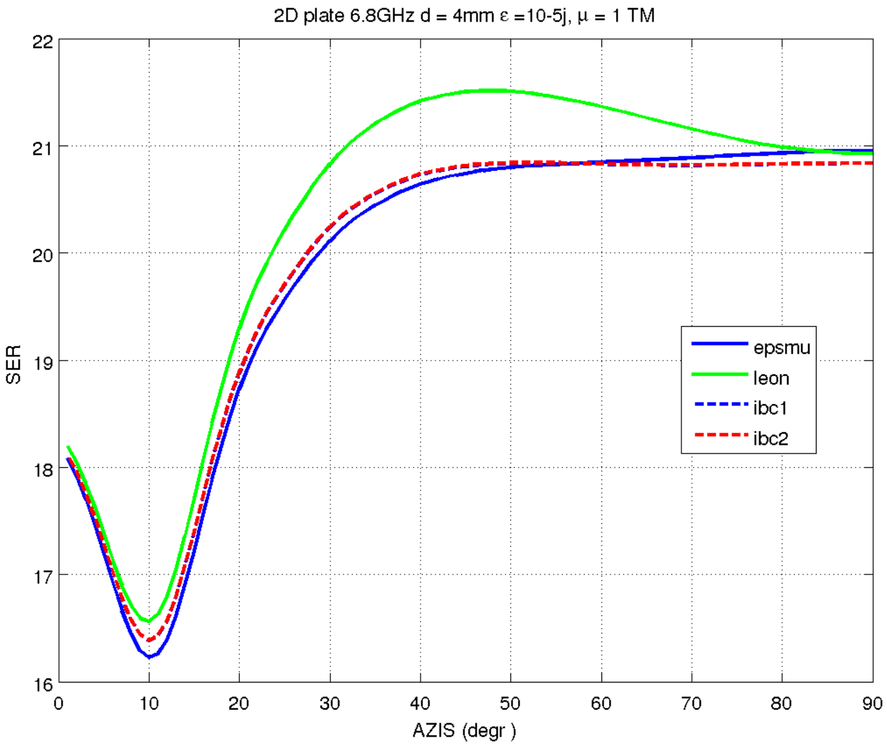

Figure 6.12: Bistatic RCS for a coated 2D plate, when $d = 4mm$, $\varepsilon_r = 10 - 5j$, $\mu_r = 1.0$, and $f = 6.8GHz$ with TM polarization

Chapter 7

Three dimensional variational problem

Contents

7.1 Introduction

The difficulties of three dimensional scattering problem is that we cannot use the polarization decomposition as in the two dimensional case, because the incident wave that perform in one point as TE polarization could perform as TM polarization in another point. Hence, we cannot combine them under the banner of one polarization. So in three dimensional case we combine TE and TM polarizations.

Recently, in different studies different ways to solve this problem were found. For the scattering problem with Leontovich boundary condition, system doesn't depend on incident angle, cause impedance assumed as constant [TL]. Rahmat-Samii used higher order boundary condition for body of revolution. They break three dimensional body into two dimensional strips and apply well known two dimensional methods.

We try to stay in three dimensional geometry.

7.2 Problem with HOIBC

Remind that we want to use integral methods to solve Maxwell's equations. Here we also apply system of integral equations

$$< Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > + < (P+Q)\mathbf{M}, \mathbf{\Psi}_J > = < I\mathbf{E}^{inc}, \mathbf{\Psi}_J > \qquad (7.1)$$

$$-<(P+Q)\mathbf{J},\mathbf{\Psi}_M>+<\frac{1}{Z_0}(B-S)\mathbf{M},\mathbf{\Psi}_M>=<I\mathbf{H}^{inc},\mathbf{\Psi}_M> \tag{7.2}$$

where the operators $(B-S)$ and $(P+Q)$ are defined in chapter 1. And we recall impedance boundary condition (5.3) that relates current densities $\mathbf{J}$ and $\mathbf{M}$

$$\mathbf{n}\times\mathbf{M}=Z\mathbf{J}. \tag{7.3}$$

where Z is the impedance operator. We approximate impedance operator Z as differential form that was mentioned in (3.13) chapter 3 and we call it three dimensional HOIBC:

$$\boxed{(I+b_1L_D-b_2L_R)(\mathbf{n}\times\mathbf{M})=(a_0I+a_1L_D-a_2L_R)\mathbf{J}}; \tag{7.4}$$

where differential operators L_D and L_R are acting on a vector field $\mathbf{V}$ tangent to the surface Γ

$$L_D(\mathbf{V})=\nabla_\Gamma(\mathrm{div}_\Gamma\mathbf{V})$$
$$L_R(\mathbf{V})=\mathbf{rot}_\Gamma(\mathrm{rot}_\Gamma\mathbf{V}).$$

We multiply three dimensional HOIBC (7.4) by test functions and integrate on the surface Γ, that gives us a weak form of the boundary condition. So, we take $\mathbf{\Psi}_J$ and $\mathbf{n}\times\mathbf{\Psi}_M$ as test functions to get the weak forms for EFIE and MFIE, respectively

$$\int_\Gamma(I+b_1L_D-b_2L_R)(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_Jds=\int_\Gamma(a_0I+a_1L_D-a_2L_R)\mathbf{J}\cdot\mathbf{\Psi}_Jds, \tag{7.5}$$

$$\int_\Gamma(I+b_1L_D-b_2L_R)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds=\int_\Gamma(a_0I+a_1L_D-a_2L_R)\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds \tag{7.6}$$

where from

$$\int_\Gamma(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_Jds=\int_\Gamma(a_0I+a_1L_D-a_2L_R)\mathbf{J}\cdot\mathbf{\Psi}_J-(b_1L_D-b_2L_R)(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_Jds \tag{7.7}$$

and

$$\int_\Gamma\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds=\frac{1}{a_0}\int_\Gamma(I+b_1L_D-b_2L_R)(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds$$
$$-\frac{1}{a_0}\int_\Gamma(a_1L_D-a_2L_R)\mathbf{J}\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds \tag{7.8}$$

We insert the HOIBC into EFIE through (7.7) replacing the operator P that was defined in (1.14):

$$<P\mathbf{M},\mathbf{\Psi}_J>=\frac{1}{2}\int_\Gamma\mathbf{\Psi}_J\cdot(\mathbf{n}\times\mathbf{M})ds$$

$$=\frac{a_0}{2}\int_\Gamma\mathbf{J}\cdot\mathbf{\Psi}_Jds+\frac{a_1}{2}\int_\Gamma L_D\mathbf{J}\cdot\mathbf{\Psi}_Jds-\frac{a_2}{2}\int_\Gamma L_R\mathbf{J}\cdot\mathbf{\Psi}_Jds$$
$$-\frac{b_1}{2}\int_\Gamma L_D(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_Jds+\frac{b_2}{2}\int_\Gamma L_R(\mathbf{n}\times\mathbf{M})\cdot\mathbf{\Psi}_Jds.$$

And using the formula of vector analysis

$$\mathbf{\Psi}_M \cdot (\mathbf{n} \times \mathbf{J}) = -\mathbf{J} \cdot (\mathbf{n} \times \mathbf{\Psi}_M),$$

we put weak form of HOIBC (7.8) into MFIE by replacing $< P\mathbf{J}, \mathbf{\Psi}_M >$:

$$< P\mathbf{J}, \mathbf{\Psi}_M >= \frac{1}{2}\int_\Gamma \mathbf{\Psi}_M \cdot (\mathbf{n} \times \mathbf{J}) ds = -\frac{1}{2}\int_\Gamma (\mathbf{n} \times \mathbf{\Psi}_M) \cdot \mathbf{J} ds$$

$$= \frac{-1}{2a_0}\int_\Gamma (\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds - \frac{b_1}{2a_0}\int_\Gamma L_D(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds + \frac{b_2}{2a_0}\int_\Gamma L_R(\mathbf{n}\times\mathbf{M})\cdot(\mathbf{n}\times\mathbf{\Psi}_M)ds$$

$$+\frac{a_1}{2a_0}\int_\Gamma L_D\mathbf{J} \cdot (\mathbf{n} \times \mathbf{\Psi}_M)ds - \frac{a_2}{2a_0}\int_\Gamma L_R\mathbf{J} \cdot (\mathbf{n} \times \mathbf{\Psi}_M)ds.$$

According to properties of vector analysis [VB], we have

$$< L_D\mathbf{A}, \mathbf{\Psi}_\Gamma >= \int_\Gamma \nabla_\Gamma(\mathrm{div}_\Gamma\mathbf{A})\cdot\mathbf{\Psi}_\Gamma ds = -\int_\Gamma (\mathrm{div}_\Gamma\mathbf{A})(\mathrm{div}_\Gamma\mathbf{\Psi}_\Gamma)ds + \int_{\partial\Gamma}(\mathrm{div}_\Gamma\mathbf{A})(\mathbf{\Psi}_\Gamma\cdot\boldsymbol{\nu})dl$$

$$< L_R\mathbf{A}, \mathbf{\Psi}_\Gamma >= \int_\Gamma \mathbf{rot}_\Gamma(\mathrm{rot}_\Gamma\mathbf{A})\cdot\mathbf{\Psi}_\Gamma ds = \int_\Gamma (\mathrm{rot}_\Gamma\mathbf{A})(\mathrm{rot}_\Gamma\mathbf{\Psi}_\Gamma)ds + \int_{\partial\Gamma}(\mathrm{rot}_\Gamma\mathbf{A})\mathbf{n}\cdot(\mathbf{\Psi}_\Gamma\times\boldsymbol{\nu})dl =$$

$$= \int_\Gamma \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{A})\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{\Psi}_\Gamma)ds - \int_{\partial\Gamma} \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{A})(\mathbf{\Psi}_\Gamma \cdot \boldsymbol{\tau})dl$$

So we get for EFIE

$$< P\mathbf{M}, \mathbf{\Psi}_J >= \frac{a_0}{2}\int_\Gamma \mathbf{J} \cdot \mathbf{\Psi}_J ds \tag{7.9}$$

$$-\frac{a_1}{2}\int_\Gamma \mathrm{div}_\Gamma\mathbf{J}\ \mathrm{div}_\Gamma\mathbf{\Psi}_J ds - \frac{a_2}{2}\int_\Gamma \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{J})\ \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{\Psi}_J)ds$$

$$+\frac{b_1}{2}\int_\Gamma \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{M})\ \mathrm{div}_\Gamma\mathbf{\Psi}_J ds + \frac{b_2}{2}\int_\Gamma \mathrm{div}_\Gamma\mathbf{M}\ \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{\Psi}_J)ds$$

$$+\frac{a_1}{2}\int_{\partial\Gamma} \mathrm{div}_\Gamma\mathbf{J}\ (\mathbf{\Psi}_J \cdot \boldsymbol{\nu})dl + \frac{a_2}{2}\int_{\partial\Gamma} \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{J})\ (\mathbf{\Psi}_J \cdot \boldsymbol{\tau})dl$$

$$-\frac{b_1}{2}\int_{\partial\Gamma} \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{M})\ (\mathbf{\Psi}_J \cdot \boldsymbol{\nu})dl + \frac{b_2}{2}\int_{\partial\Gamma} \mathrm{div}_\Gamma\mathbf{M}\ (\mathbf{\Psi}_J \cdot \boldsymbol{\tau})dl;$$

and for MFIE

$$< P\mathbf{J}, \mathbf{\Psi}_M >= \frac{-1}{2a_0}\int_\Gamma \mathbf{M} \cdot \mathbf{\Psi}_M ds \tag{7.10}$$

$$+\frac{b_1}{2a_0}\int_\Gamma \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{M})\ \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{\Psi}_M)ds + \frac{b_2}{2a_0}\int_\Gamma \mathrm{div}_\Gamma\mathbf{M}\ \mathrm{div}_\Gamma\mathbf{\Psi}_M ds$$

$$-\frac{a_1}{2a_0}\int_\Gamma \mathrm{div}_\Gamma\mathbf{J}\ \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{\Psi}_M)ds + \frac{a_2}{2a_0}\int_\Gamma \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{J})\ \mathrm{div}_\Gamma\mathbf{\Psi}_M ds$$

$$-\frac{b_1}{2a_0}\int_{\partial\Gamma} \mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{M})\ (\mathbf{\Psi}_M \cdot \boldsymbol{\tau})dl - \frac{b_2}{2a_0}\int_{\partial\Gamma} \mathrm{div}_\Gamma\mathbf{M}\ (\mathbf{\Psi}_M \cdot \boldsymbol{\nu})dl$$

$$+\frac{a_1}{2a_0}\int_{\partial\Gamma}\mathrm{div}_\Gamma\mathbf{J}\ (\mathbf{\Psi}_M\cdot\boldsymbol{\tau})dl-\frac{a_2}{2a_0}\int_{\partial\Gamma}\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{J})\ (\mathbf{\Psi}_M\cdot\boldsymbol{\nu})dl$$

We suppose that boundary Γ is closed, so all integrals on $\partial\Gamma$ are zero. We put (7.9) and (7.10) in the variational problem (7.1) and (7.2) respectively. Now we define bilinear operator

$$A(U,\Psi)=<Z_0(B-S)\mathbf{J},\mathbf{\Psi}_J>+\frac{1}{Z_0}<(B-S)\mathbf{M},\mathbf{\Psi}_M>$$

$$+<Q\mathbf{M},\mathbf{\Psi}_J>-<Q\mathbf{J},\mathbf{\Psi}_M>+\frac{a_0}{2}<\mathbf{J},\mathbf{\Psi}_J>+\frac{1}{2a_0}<\mathbf{M},\mathbf{\Psi}_M>$$

$$-\frac{a_1}{2}<\mathrm{div}_\Gamma\mathbf{J},\mathrm{div}_\Gamma\mathbf{\Psi}_J>-\frac{a_2}{2}<\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{J}),\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_J)>$$

$$+\frac{b_1}{2}<\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{M}),\mathrm{div}_\Gamma\mathbf{\Psi}_J>-\frac{b_2}{2}<\mathrm{div}_\Gamma\mathbf{M},\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_J)>$$

$$-\frac{b_1}{2a_0}<\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{M}),\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_M)>-\frac{b_2}{2a_0}<\mathrm{div}_\Gamma\mathbf{M},\mathrm{div}_\Gamma\mathbf{\Psi}_M>$$

$$+\frac{a_1}{2a_0}<\mathrm{div}_\Gamma\mathbf{J},\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_M)>-\frac{a_2}{2a_0}<\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{J}),\mathrm{div}_\Gamma\mathbf{\Psi}_M>$$

that combines integral equations (7.1)-(7.2), where operator P parts are replaced.

Finally, we introduce the following problem:

Problem 7.2.1 *Find* $U=(\mathbf{J},\mathbf{M})$ *such that*

$$A(U,\Psi)=F(\Psi)\tag{7.11}$$

for all $\Psi=(\mathbf{\Psi}_J,\mathbf{\Psi}_M)$. *Where*

$$F(\Psi)=\int_\Gamma\mathbf{E}^{inc}\cdot\mathbf{\Psi}_Jds+\int_\Gamma\mathbf{H}^{inc}\cdot\mathbf{\Psi}_Mds.$$

Since $\mathrm{rot}_\Gamma\mathbf{A}$ and $\mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{A})$ do not make sense for $\mathbf{A}\in H^{-1/2}(\mathrm{div},\Gamma)$, we follow the method that we see in [BA]. We introduce auxiliary unknowns $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{M}}$ from $H^{-1/2}(\mathrm{div},\Gamma)$ with test functions $\tilde{\mathbf{\Psi}}_J$ and $\tilde{\mathbf{\Psi}}_M$ such that

$$\int_\Gamma\boldsymbol{\lambda}_J\cdot(\tilde{\mathbf{\Psi}}_J-\mathbf{n}\times\mathbf{\Psi}_J)ds=0$$

$$\int_\Gamma\boldsymbol{\lambda}_M\cdot(\tilde{\mathbf{\Psi}}_M-\mathbf{n}\times\mathbf{\Psi}_M)ds=0$$

$$\int_\Gamma\boldsymbol{\lambda}'_J\cdot(\tilde{\mathbf{J}}-\mathbf{n}\times\mathbf{J})ds=0$$

$$\int_\Gamma\boldsymbol{\lambda}'_M\cdot(\tilde{\mathbf{M}}-\mathbf{n}\times\mathbf{M})ds=0$$

where $\boldsymbol{\lambda}_J,\boldsymbol{\lambda}_M,\boldsymbol{\lambda}'_J,\boldsymbol{\lambda}'_M$ are known as Lagrange multipliers. Under these assumptions, we will replace $(\mathbf{n}\times\mathbf{\Psi}_J,\mathbf{n}\times\mathbf{\Psi}_M,\mathbf{n}\times\mathbf{J},\mathbf{n}\times\mathbf{M})$ by $(\tilde{\mathbf{\Psi}}_J,\tilde{\mathbf{\Psi}}_M,\tilde{\mathbf{J}},\tilde{\mathbf{M}})$, respectively.

And we will write the problem as

Problem 7.2.2 *Find* $U = (\mathbf{J}, \mathbf{M}, \tilde{\mathbf{J}}, \tilde{\mathbf{M}}) \in V = [H^{-1/2}(div, \Gamma) \cap L^2(\Gamma)]^4$ *and* $\lambda = (\boldsymbol{\lambda}_J, \boldsymbol{\lambda}_M) \in [H^{-1/2}(\Gamma)]^2$ *such that*

$$\begin{cases} A(U, \Psi) + B^T(\lambda, \Psi) = F(\Psi) \\ B(U, \lambda') = 0 \end{cases} \tag{7.12}$$

for all $\Psi = (\boldsymbol{\Psi}_J, \boldsymbol{\Psi}_M, \tilde{\boldsymbol{\Psi}}_J, \tilde{\boldsymbol{\Psi}}_M) \in V = [H^{-1/2}(div, \Gamma) \cap L^2(\Gamma)]^4$ *and* $\lambda' = (\boldsymbol{\lambda}'_J, \boldsymbol{\lambda}'_M) \in W = [H^{-1/2}(\Gamma)]^2$.

Where right hand side operator is defined as next

$$F(\Psi) = \int_\Gamma \mathbf{E}^{inc} \cdot \boldsymbol{\Psi}_J ds + \int_\Gamma \mathbf{H}^{inc} \cdot \boldsymbol{\Psi}_M ds.$$

Bilinear operator with Lagrange multiplier λ is

$$B(U, \lambda') = \int_\Gamma \boldsymbol{\lambda}'_J \cdot (\tilde{\mathbf{J}} - \mathbf{n} \times \mathbf{J}) ds + \int_\Gamma \boldsymbol{\lambda}'_M \cdot (\tilde{\mathbf{M}} - \mathbf{n} \times \mathbf{M}) ds$$

and

$$A(U, \Psi) = iZ_0 \iint_\Gamma kG\ (\mathbf{J} \cdot \boldsymbol{\Psi}_J) - \frac{1}{k} G \ \mathrm{div} \boldsymbol{\Psi}_J \ \mathrm{div} \mathbf{J} ds ds' \tag{7.13}$$

$$+ \frac{i}{Z_0} \iint_\Gamma kG\ (\boldsymbol{\Psi}_M \cdot \mathbf{M}) - \frac{1}{k} G \ \mathrm{div} \boldsymbol{\Psi}_M \ \mathrm{div} \mathbf{M} ds ds'$$

$$+ \iint_\Gamma \nabla' G \cdot (\boldsymbol{\Psi}_J \times \mathbf{M}) ds ds' - i \iint_\Gamma \nabla' G \cdot (\boldsymbol{\Psi}_M \times \mathbf{J}) ds ds'$$

$$+ \frac{a_0}{2} \int_\Gamma \mathbf{J} \cdot \boldsymbol{\Psi}_J ds + \frac{1}{2a_0} \int_\Gamma \mathbf{M} \cdot \boldsymbol{\Psi}_M ds$$

$$- \frac{a_1}{2} \int_\Gamma \mathrm{div}_\Gamma \mathbf{J} \ \mathrm{div}_\Gamma \boldsymbol{\Psi}_J ds - \frac{a_2}{2} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{J}} \ \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_J ds$$

$$+ \frac{b_1}{2} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{M}} \ \mathrm{div}_\Gamma \boldsymbol{\Psi}_J ds - \frac{b_2}{2} \int_\Gamma \mathrm{div}_\Gamma \mathbf{M} \ \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_J ds$$

$$- \frac{b_1}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{M}} \ \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_M ds - \frac{b_2}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \mathbf{M} \ \mathrm{div}_\Gamma \boldsymbol{\Psi}_M ds$$

$$+ \frac{a_1}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \mathbf{J} \ \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_M ds - \frac{a_2}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{J}} \ \mathrm{div}_\Gamma \boldsymbol{\Psi}_M ds$$

is the bilinear operator on $V \times V$.

7.3 Existence and uniqueness theorem

According to theorem 2.3.4 from chapter 2, there exists a unique solution of the problem 7.2.2, if the bilinear operator $A(U,\Psi)$ verifies the continuity and coercivity conditions and the operator $B(U,\lambda')$ satisfies the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, also known as inf-sup condition.

For the sake of simplicity we consider the operator $A(U,\Psi)$ as a sum of three bilinear operator

$$A_1(U,\Psi) = < Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{Z_0} < (B-S)\mathbf{M}, \mathbf{\Psi}_M >$$

$$+ < Q\mathbf{M}, \mathbf{\Psi}_J > - < Q\mathbf{J}, \mathbf{\Psi}_M > + \frac{a_0}{2} < \mathbf{J}, \mathbf{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M}, \mathbf{\Psi}_M >$$

$$A_2(U,\Psi) = -\frac{a_1}{2} < \mathrm{div}_\Gamma \mathbf{J}, \mathrm{div}_\Gamma \mathbf{\Psi}_J > - \frac{a_2}{2} < \mathrm{div}_\Gamma \tilde{\mathbf{J}}, \mathrm{div}_\Gamma \tilde{\mathbf{\Psi}}_J >$$

$$-\frac{b_1}{2a_0} < \mathrm{div}_\Gamma \tilde{\mathbf{M}}, \mathrm{div}_\Gamma \tilde{\mathbf{\Psi}}_M > - \frac{b_2}{2a_0} < \mathrm{div}_\Gamma \mathbf{M}, \mathrm{div}_\Gamma \mathbf{\Psi}_M >$$

and

$$A_3(U,\Psi) = \frac{b_1}{2} < \mathrm{div}_\Gamma \tilde{\mathbf{M}}, \mathrm{div}_\Gamma \mathbf{\Psi}_J > - \frac{b_2}{2} < \mathrm{div}_\Gamma \mathbf{M}, \mathrm{div}_\Gamma \tilde{\mathbf{\Psi}}_J >$$

$$+\frac{a_1}{2a_0} < \mathrm{div}_\Gamma \mathbf{J}, \mathrm{div}_\Gamma \tilde{\mathbf{\Psi}}_M > - \frac{a_2}{2a_0} < \mathrm{div}_\Gamma \tilde{\mathbf{J}}, \mathrm{div}_\Gamma \mathbf{\Psi}_M >$$

where

$$A = A_1 + A_2 + A_3$$

7.3.1 Continuity of operator A

Lemma 7.3.1 *The bilinear operator $A(U,\Psi)$ (7.13) is continuous on V for all $\Psi \in V$.*

Proof. : We have to show that there exists $C > 0$ such that for all $\Psi \in V$

$$|A(U,\Psi)| \leq C\|U\|_V \|\Psi\|_V$$

According to theorems 2.3.1 - 2.3.2 and triangle inequality (property of a norm) we have

$$|A_1(U,\Psi)| \leq | < Z_0(B-S)\mathbf{J}, \mathbf{\Psi}_J > | + |Z_0^{-1}| \, | < (B-S)\mathbf{M}, \mathbf{\Psi}_M > | + | < Q\mathbf{M}, \mathbf{\Psi}_J > |$$

$$+| < Q\mathbf{J}, \mathbf{\Psi}_M > | + \frac{|a_0|}{2} | < \mathbf{J}, \mathbf{\Psi}_J > | + \frac{1}{2|a_0|} | < \mathbf{M}, \mathbf{\Psi}_M > | \leq$$

$$\leq \|Z_0(B-S)\mathbf{J}\|_{-1/2,\mathrm{rot}_\Gamma} \|\mathbf{\Psi}_J\|_{-1/2,\mathrm{div}_\Gamma}$$

$$+|Z_0^{-1}| \|(B-S)\mathbf{M}\|_{-1/2,\mathrm{rot}_\Gamma} \|\mathbf{\Psi}_M\|_{-1/2,\mathrm{div}_\Gamma} + \|Q\mathbf{M}\|_{-1/2,\mathrm{rot}_\Gamma} \|\mathbf{\Psi}_J\|_{-1/2,\mathrm{div}_\Gamma}$$

$$+\|Q\mathbf{J}\|_{-1/2,\mathrm{rot}_\Gamma} \|\mathbf{\Psi}_M\|_{-1/2,\mathrm{div}_\Gamma} + \frac{a_0}{2} \|\mathbf{J}\|_{L^2(\Gamma)} \|\mathbf{\Psi}_J\|_{L^2(\Gamma)} + \frac{1}{2a_0} \|\mathbf{M}\|_{L^2(\Gamma)} \|\mathbf{\Psi}_M\|_{L^2(\Gamma)}$$

$$\leq C_1\|U\|_V\|\Psi\|_V$$

We consider the rest part as a sum of $A_2(U,\Psi)+A_3(U,\Psi)$:

$$|A_2(U,\Psi)+A_3(U,\Psi)| \leq \frac{|a_1|}{2}|<\mathrm{div}_\Gamma\mathbf{J},\mathrm{div}_\Gamma\mathbf{\Psi}_J>|+\frac{|a_2|}{2}|<\mathrm{div}_\Gamma\tilde{\mathbf{J}},\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_J>|$$

$$+\frac{|b_1|}{2}|<\mathrm{div}_\Gamma\tilde{\mathbf{M}},\mathrm{div}_\Gamma\mathbf{\Psi}_J>|+\frac{|b_2|}{2}|<\mathrm{div}_\Gamma\mathbf{M},\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_J>|$$

$$+\frac{|b_1|}{2|a_0|}|<\mathrm{div}_\Gamma\tilde{\mathbf{M}},\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_M>|+\frac{|b_2|}{2|a_0|}|<\mathrm{div}_\Gamma\mathbf{M},\mathrm{div}_\Gamma\mathbf{\Psi}_M>|$$

$$+\frac{|a_1|}{2|a_0|}|<\mathrm{div}_\Gamma\mathbf{J},\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_M>|+\frac{|a_2|}{2|a_0|}|<\mathrm{div}_\Gamma\tilde{\mathbf{J}},\mathrm{div}_\Gamma\mathbf{\Psi}_M>|$$

that gives us the following inequality

$$|A_2(U,\Psi)+A_3(U,\Psi)| \leq \frac{|a_1|}{2}\|\mathrm{div}_\Gamma\mathbf{J}\|_{-1/2,H(\Gamma)}\|\mathrm{div}_\Gamma\mathbf{\Psi}_J\|_{-1/2,\Gamma}+\frac{|a_2|}{2}\|\mathrm{div}_\Gamma\tilde{\mathbf{J}}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_J\|_{-1/2,\Gamma}$$

$$+\frac{|b_1|}{2}\|\mathrm{div}_\Gamma\tilde{\mathbf{M}}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\mathbf{\Psi}_J\|_{-1/2,\Gamma}+\frac{|b_2|}{2}\|\mathrm{div}_\Gamma\mathbf{M}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_J\|_{-1/2,\Gamma}$$

$$+\frac{|b_1|}{2|a_0|}\|\mathrm{div}_\Gamma\tilde{\mathbf{M}}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_M\|_{-1/2,\Gamma}+\frac{|b_2|}{2|a_0|}\|\mathrm{div}_\Gamma\mathbf{M}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\mathbf{\Psi}_M\|_{-1/2,\Gamma}$$

$$+\frac{|a_1|}{2|a_0|}\|\mathrm{div}_\Gamma\mathbf{J}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\tilde{\mathbf{\Psi}}_M\|_{-1/2,\Gamma}+\|\frac{|a_2|}{2|a_0|}\|\mathrm{div}_\Gamma\tilde{\mathbf{J}}\|_{-1/2,\Gamma}\|\mathrm{div}_\Gamma\mathbf{\Psi}_M\|_{-1/2,\Gamma}$$

$$\leq C_2\|U\|_V\|\Psi\|_V$$

So, the sum of these two parts gives us $|A(U,\Psi)|=|A_1+A_2+A_3|\leq|A_1(U,\Psi)|+|A_2(U,\Psi)+A_3(U,\Psi)|\leq C\|U\|_V\|\Psi\|_V$ where $C=C_1+C_2$. □

7.3.2 Coercivity of operator A

Lemma 7.3.2 *Bilinear form $A(U,\Psi)$ verifies coercivity inequality for all $U\in V=[H^{-1/2}(div,\Gamma)\cap L^2(\Gamma)]^4$.*

Proof. : We have to show that there exist $\alpha>0$ such that

$$\Re[A(U,U^*)]\geq\alpha\|U\|_V^2-C\|U\|_H^2,\ \forall U\in V.$$

From [TL], we know that there exists α_1 such that

$$\Re(A_1)=\Re(<Z_0(B-S)\mathbf{J},\mathbf{J}^*>)+\Re(<Z_0^{-1}(B-S)\mathbf{M},\mathbf{M}^*>)+\Re(<Q\mathbf{M},\mathbf{J}^*>)$$

$$-\Re(<Q\mathbf{J},\mathbf{M}^*>)+\Re(\frac{a_0}{2}\int_\Gamma\mathbf{J}\cdot\mathbf{J}^*ds)+\Re(\frac{1}{2a_0}\int_\Gamma\mathbf{M}\cdot\mathbf{M}^*ds)\geq$$

$$\geq \alpha \left(\|\mathbf{J}\|^2_{-1/2,\mathrm{div}_\Gamma} + \|\mathbf{M}\|^2_{-1/2,\mathrm{div}_\Gamma} \right) + \frac{\Re(a_0)}{2} \|\mathbf{J}\|^2_{L^2(\Gamma)} + \frac{\Re(a_0)}{2|a_0|^2} \|\mathbf{M}\|^2_{L^2(\Gamma)}$$

We can easily show that

$$\Re(A_2) = -\Re(\frac{a_1}{2} \int_\Gamma \mathrm{div}_\Gamma \mathbf{J}\ \mathrm{div}_\Gamma \mathbf{J}^* ds) - \Re(\frac{a_2}{2} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{J}}\ \mathrm{div}_\Gamma \tilde{\mathbf{J}}^* ds)$$

$$-\Re(\frac{b_1}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{M}}\ \mathrm{div}_\Gamma \tilde{\mathbf{M}}^* ds) - \Re(\frac{b_2}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \mathbf{M}\ \mathrm{div}_\Gamma \mathbf{M}^* ds) =$$

$$-\frac{\Re(a_1)}{2} \|\mathrm{div}_\Gamma \mathbf{J}\|^2_{L^2(\Gamma)} - \frac{\Re(a_2)}{2} \|\mathrm{div}_\Gamma \tilde{\mathbf{J}}\|^2_{L^2(\Gamma)}$$

$$-\frac{\Re(b_1 a_0^*)}{2|a_0|^2} \|\mathrm{div}_\Gamma \tilde{\mathbf{M}}\|^2_{L^2(\Gamma)} - \frac{\Re(b_2 a_0^*)}{2|a_0|^2} \|\mathrm{div}_\Gamma \mathbf{M}\|^2_{L^2(\Gamma)}$$

For the rest, we do as follows:

$$\Re(A_3) = \Re(\frac{b_1}{2} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{M}}\ \mathrm{div}_\Gamma \mathbf{J}^* ds) - \Re(\frac{b_2}{2} \int_\Gamma \mathrm{div}_\Gamma \mathbf{M}\ \mathrm{div}_\Gamma \tilde{\mathbf{J}}^* ds)$$

$$+\Re(\frac{a_1}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \mathbf{J}\ \mathrm{div}_\Gamma \tilde{\mathbf{M}}^* ds) - \Re(\frac{a_2}{2a_0} \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{J}}\ \mathrm{div}_\Gamma \mathbf{M}^* ds) =$$

$$= \Re \left\{ \left(\frac{b_1}{2} + \frac{a_1^*}{2a_0^*} \right) \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{M}}\ \mathrm{div}_\Gamma \mathbf{J}^* ds \right\}$$

$$-\Re \left\{ \left(\frac{b_2}{2} + \frac{a_2^*}{2a_0^*} \right) \int_\Gamma \mathrm{div}_\Gamma \tilde{\mathbf{J}}^*\ \mathrm{div}_\Gamma \mathbf{M} ds \right\} =$$

$$= \Re \left\{ \int_\Gamma \frac{1}{|a_0|^{1/2}} \left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2} \right)^{1/2} \mathrm{div}_\Gamma \tilde{\mathbf{M}}\ \cdot |a_0|^{1/2} \left(\frac{b_1}{2} + \frac{a_1^* a_0}{2|a_0|^2} \right)^{1/2} \mathrm{div}_\Gamma \mathbf{J}^* ds \right\}$$

$$-\Re \left\{ \int_\Gamma |a_0|^{1/2} \left(\frac{b_2}{2} + \frac{a_2^* a_0}{2|a_0|^2} \right)^{1/2} \mathrm{div}_\Gamma \tilde{\mathbf{J}}^* \cdot \frac{1}{|a_0|^{1/2}} \left(\frac{b_2}{2} + \frac{a_2^* a_0}{2|a_0|^2} \right)^{1/2} \mathrm{div}_\Gamma \mathbf{M} ds \right\}$$

We note $q_1 = b_1|a_0| + a_1^* a_0/|a_0|$ and $q_2 = b_2|a_0| + a_2^* a_0/|a_0|$, so

$$\Re(A_3) \geq -\frac{|q_1|}{4} \|\mathrm{div}_\Gamma \mathbf{J}\|^2_{L^2(\Gamma)} - \frac{|q_1|}{4|a_0|^2} \|\mathrm{div}_\Gamma \tilde{\mathbf{M}}\|^2_{L^2(\Gamma)}$$

$$-\frac{|q_2|}{4} \|\mathrm{div}_\Gamma \tilde{\mathbf{J}}\|^2_{L^2(\Gamma)} - \frac{|q_2|}{4|a_0|^2} \|\mathrm{div}_\Gamma \mathbf{M}\|^2_{L^2(\Gamma)}$$

We have conditions on coefficients from two dimensional case, which say that $\Re(a_j) + \frac{|q_j|}{2} = 0$, where $j = 1, 2$ (see equation (5.17) in chapter 5); and from the sufficient uniqueness conditions we have that $\Re(a_j) = \Re(b_j^* a_0)$. So for the A_2 and A_3, we have

$$\Re(A_2) + \Re(A_3) \geq 0.$$

Finally we have for the entire operator A that

$$\Re(A) = \Re(A_1) + \Re(A_2) + \Re(A_3) \geq$$

$$\geq \alpha \left(\|\mathbf{J}\|^2_{-1/2,\mathrm{div}_\Gamma} + \|\mathbf{M}\|^2_{-1/2,\mathrm{div}_\Gamma} \right) + \frac{\Re(a_0)}{2} \|\mathbf{J}\|^2_{L^2(\Gamma)} + \frac{\Re(a_0)}{2|a_0|^2} \|\mathbf{M}\|^2_{L^2(\Gamma)}$$

□

7.3.3 Ladyzhenskaya-Babuška-Brezzi condition

Lemma 7.3.3 *Bilinear form $B(U,\lambda)$ verifies next inequality*

$$\sup_{\|U\|_V=1} |B(U,\lambda)| \geq \beta\|\lambda\|_W, \ \forall\lambda \in W = [H^{-1/2}(\Gamma)]^3 \times [H^{-1/2}(\Gamma)]^3$$

where $U \in V = [H^{-1/2}(div,\Gamma) \cap L^2(\Gamma)]^4$ and $\beta > 0$.

Proof. : Here we have to show that there exists $\beta > 0$ such that

$$\sup_{\|U\|_V=1} \left| \int_\Gamma \boldsymbol{\lambda}_J \cdot (\tilde{\mathbf{J}} - n \times \mathbf{J}) + \boldsymbol{\lambda}_M \cdot (\tilde{\mathbf{M}} - n \times \mathbf{M}) ds \right| \geq \beta\|\lambda\|_W$$

First we take

$$\boldsymbol{J} = 0; \ \boldsymbol{M} = 0; \ \tilde{\boldsymbol{J}} = \frac{\hat{\mathbf{J}}}{\|\hat{\mathbf{J}}\|_V} \ \ \textit{and} \ \ \tilde{\boldsymbol{J}} = \frac{\hat{\mathbf{M}}}{\|\hat{\mathbf{M}}\|_V}$$

where

$$\hat{\mathbf{J}} = \int_{\Gamma\backslash x} \frac{\boldsymbol{\lambda}_J}{|x-y|} ds_y \quad \textit{and} \quad \hat{\mathbf{M}} = \int_{\Gamma\backslash x} \frac{\boldsymbol{\lambda}_M}{|x-y|} ds_y$$

so we get following inequality

$$\sup_{\|U\|_V=1} \left| \int_\Gamma \boldsymbol{\lambda}_J \cdot (\tilde{\mathbf{J}} - n \times \mathbf{J}) + \boldsymbol{\lambda}_M \cdot (\tilde{\mathbf{M}} - n \times \mathbf{M}) ds \right| \geq$$

$$\geq \frac{1}{\|\hat{\mathbf{J}}\|_{-1/2,\mathrm{div}_\Gamma}} \iint_{\Gamma\Gamma} \frac{\boldsymbol{\lambda}_J(x)\boldsymbol{\lambda}_J(y)}{|x-y|} ds_y ds_x + \frac{1}{\|\hat{\mathbf{M}}\|_{-1/2,\mathrm{div}_\Gamma}} \iint_{\Gamma\Gamma} \frac{\boldsymbol{\lambda}_M(x)\boldsymbol{\lambda}_M(y)}{|x-y|} ds_y ds_x \tag{7.14}$$

Since each of double-integrals verifies the Planchard-Nédélec inequality, [CD, NP]

$$\iint_{\Gamma\Gamma} \frac{\boldsymbol{\lambda}(x)\boldsymbol{\lambda}(y)}{|x-y|} ds_y ds_x \geq \beta\|\boldsymbol{\lambda}\|^2_{-1/2,\Gamma} \tag{7.15}$$

So we get

$$(7.14) \geq \frac{1}{\|\hat{\mathbf{J}}\|_{-1/2,\mathrm{div}_\Gamma}} \beta_J \|\boldsymbol{\lambda}_J\|^2_{-1/2,\Gamma} + \frac{1}{\|\hat{\mathbf{M}}\|_{-1/2,\mathrm{div}_\Gamma}} \beta_M \|\boldsymbol{\lambda}_M\|^2_{-1/2,\Gamma} \tag{7.16}$$

Also, we have that there exist $C_J > 0$ and $C_M > 0$ such that, [CD, NP]

$$\|\hat{\mathbf{J}}\|_{-1/2,\mathrm{div}_\Gamma} \leq C_J \|\boldsymbol{\lambda}_J\|_{-1/2,\Gamma} \ \ \textit{and} \ \ \|\hat{\mathbf{M}}\|_{-1/2,\mathrm{div}_\Gamma} \leq C_M \|\boldsymbol{\lambda}_M\|_{-1/2,\Gamma} \tag{7.17}$$

Finally

$$(7.16) \geq \frac{\beta_J}{C_J} \|\boldsymbol{\lambda}_J\|_{-1/2,\Gamma} + \frac{\beta_M}{C_M} \|\boldsymbol{\lambda}_M\|_{-1/2,\Gamma} \geq \beta\|\lambda\|_W \tag{7.18}$$

where $\beta = \min(\beta_J/C_J; \beta_M/C_M)$.

□

Theorem 7.3.1 *The problem (7.2.2) admits a unique solution* $U \in V = [H^{-1/2}(div, \Gamma) \cap L^2(\Gamma)]^4$ *and* $\lambda \in [H^{-1/2}(\Gamma)]^2$*, if coefficients satisfy*

$$\Re(a_j) + \frac{|a_0||b_j + a_j^*/a_0^*|}{2} = 0 \quad for \quad j = 1, 2. \tag{7.19}$$

Proof. Lemmas 7.3.1-7.3.3 give us that the bilinear forms $A(U, \Psi)$ and $B(U, \lambda)$ verify hypothesis of the theorem 2.3.4. □

7.4 Conclusion

In this chapter, we set 3D HOIBC (7.4) into EFIE and MFIE (7.1 - 7.2) and we introduce the problem 7.2.1. We introduce the problem 7.2.2 by introducing auxiliary variables $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{M}}$ and Lagrange multipliers λ_J and λ_M. We proved the coercivity and continuity of the operator $A(U, \Psi)$ and that the operator $B(U, \lambda)$ satisfies the Inf-Sup condition. And according to Fredholm alternative from chapter 2, it follows that there exists the unique solution of the problem. We should note that the restrictions on the coefficients in 2D case were used for 3D case. Further we propose the discretization of the problem 7.2.2 in chapter 8 and the problem 7.2.1 in chapter 9.

Chapter 8

3D discretization

Contents

8.1 Introduction

In this chapter, we will see discretization of variational problem 7.2.2 based on Rao-Wilton-Glisson basis functions and auxiliary basis functions introduced by Bendali [BA] for Lagrange multipliers. We recall bilinear form that was formulated in the previous chapter. So we solve the problem

$$\begin{cases} A(U,\Psi) + B^T(\lambda,\Psi) =< \mathbf{E}^{inc}, \boldsymbol{\Psi}_J > + < \mathbf{H}^{inc}, \boldsymbol{\Psi}_M > \\ B(U,\lambda') = 0 \end{cases} \tag{8.1}$$

where

$$B(U,\lambda') =< \boldsymbol{\lambda}'_J, \tilde{\mathbf{J}} > - < \boldsymbol{\lambda}'_J, \mathbf{n}\times\mathbf{J} > + < \boldsymbol{\lambda}'_M, \tilde{\mathbf{M}} > - < \boldsymbol{\lambda}'_M, \mathbf{n}\times\mathbf{M} >$$

and

$$\begin{aligned} A(U,\Psi) =&< Z_0(B-S)\mathbf{J}, \boldsymbol{\Psi}_J > + Z_0^{-1} < (B-S)\mathbf{M}, \boldsymbol{\Psi}_M > \\ &+ < Q\mathbf{M}, \boldsymbol{\Psi}_J > - < Q\mathbf{J}, \boldsymbol{\Psi}_M > + \frac{a_0}{2} < \mathbf{J}, \boldsymbol{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M}, \boldsymbol{\Psi}_M > \\ &- \frac{a_1}{2} < \mathrm{div}_\Gamma \mathbf{J}, \mathrm{div}_\Gamma \boldsymbol{\Psi}_J > - \frac{a_2}{2} < \mathrm{div}_\Gamma \tilde{\mathbf{J}}, \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_J > \\ &+ \frac{b_1}{2} < \mathrm{div}_\Gamma \tilde{\mathbf{M}}, \mathrm{div}_\Gamma \boldsymbol{\Psi}_J > - \frac{b_2}{2} < \mathrm{div}_\Gamma \mathbf{M}, \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_J > \\ &- \frac{b_1}{2a_0} < \mathrm{div}_\Gamma \tilde{\mathbf{M}}, \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_M > - \frac{b_2}{2a_0} < \mathrm{div}_\Gamma \mathbf{M}, \mathrm{div}_\Gamma \boldsymbol{\Psi}_M > \\ &+ \frac{a_1}{2a_0} < \mathrm{div}_\Gamma \mathbf{J}, \mathrm{div}_\Gamma \tilde{\boldsymbol{\Psi}}_M > - \frac{a_2}{2a_0} < \mathrm{div}_\Gamma \tilde{\mathbf{J}}, \mathrm{div}_\Gamma \boldsymbol{\Psi}_M > \end{aligned}$$

The discretization of unknowns $\mathbf{J}$ and $\mathbf{M}$ should verify a condition – flow conservation of these currents. One way to ensure this is to use the basis functions of Rao-Wilton-Glisson.

In [BA] were introduced basis function for Lagrange multipliers, so that they can express $\mathbf{J}$ in terms of $\mathbf{M}$ according to boundary condition equation given below

$$\int_\Gamma \mathbf{L} \cdot (\mathbf{n} \times \mathbf{M} - ikZ\mathbf{J}) ds = 0.$$

We use these basis functions to eliminate auxiliary unknowns.

8.2 Discretization

The first step is to approach the surface of the obstacle by a surface Γ_h composed of finite number of two dimensional elements. These elements are triangular facets denoted by T_i for $i = 1$ to N_T:

$$\Gamma_h = \bigcup_{i=1}^{N_T} T_i.$$

We will call it an initial mesh (or original mesh). We denote by N_e the total number of edges of the mesh component Γ_h. Let $\{\mathbf{f}_i\}_{i=1,N_e}$ be a base of Rao-Wilton-Glisson functions, where each function correspond to one edge. We decompose the electric and magnetic currents:

$$\mathbf{J}(y) = \sum_{l=1}^{N_e} J_l \mathbf{f}_l(y), \quad \mathbf{M}(y) = \sum_{l=1}^{N_e} M_l \mathbf{f}_l(y),$$

and auxiliary unknowns

$$\tilde{\mathbf{J}}(x) = \sum_{l=1}^{N_e} \tilde{J}_l \mathbf{f}_l(x), \quad \tilde{\mathbf{M}}(x) = \sum_{l=1}^{N_e} \tilde{M}_l \mathbf{f}_l(x).$$

And Lagrange multipliers are represented as

$$\boldsymbol{\lambda}_J = \sum_{k=1}^{N_e} \lambda_{Jk} \mathbf{g}_k, \quad \boldsymbol{\lambda}_M = \sum_{k=1}^{N_e} \lambda_{Mk} \mathbf{g}_k,$$

where $\mathbf{g_k}$ are basis functions that we will define later with help of functions introduced by Bendali.

Considering to these assumptions we are looking for approximated solution of the problem (8.1). We introduce the following system

$$\begin{cases} A^h(U_h, \Psi_h) + B^h(\lambda_h, \Psi_h) = \sum_{i=1}^{N_e} < \mathbf{E}^{inc}, \mathbf{f}_i > + \sum_{i=1}^{N_e} < \mathbf{H}^{inc}, \mathbf{f}_i > \\ B^h(U_h, \lambda_h) = 0 \end{cases} \tag{8.2}$$

where

$$A^h(U_h,\Psi_h) = \sum_{i,j=1}^{N_e} < Z_0(B-S)\mathbf{f}_j,\mathbf{f}_i > J_j + Z_0^{-1}\sum_{i,j=1}^{N_e} < (B-S)\mathbf{f}_j,\mathbf{f}_i > M_j$$

$$+\sum_{i,j=1}^{N_e} < Q\mathbf{f}_j,\mathbf{f}_i > M_j - \sum_{i,j=1}^{N_e} < Q\mathbf{f}_j,\mathbf{f}_i > J_j + \frac{a_0}{2}\sum_{i,j=1}^{N_e} < \mathbf{f}_j,\mathbf{f}_i > J_j + \frac{1}{2a_0}\sum_{i,j=1}^{N_e} < \mathbf{f}_j,\mathbf{f}_i > M_j$$

$$-\frac{a_1}{2}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > J_j - \frac{a_2}{2}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > \tilde{J}_j$$

$$+\frac{b_1}{2}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > \tilde{M}_j - \frac{b_2}{2}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > M_j$$

$$-\frac{b_1}{2a_0}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > \tilde{M}_j - \frac{b_2}{2a_0}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > M_j$$

$$+\frac{a_1}{2a_0}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > J_j - \frac{a_2}{2a_0}\sum_{i,j=1}^{N_e} < \mathrm{div}_\Gamma\mathbf{f}_j,\mathrm{div}_\Gamma\mathbf{f}_i > \tilde{J}_j;$$

and

$$B^h(U_h,\lambda_h') = \sum_{i,k=1}^{N_e} < \mathbf{g}_k,\mathbf{f}_i > \tilde{J}_i - \sum_{i,k=1}^{N_e} < \mathbf{g}_k,\mathbf{n}\times\mathbf{f}_i > J_i$$

$$+\sum_{i,k=1}^{N_e} < \mathbf{g}_k,\mathbf{f}_i > \tilde{M}_i - \sum_{i,k=1}^{N_e} < \mathbf{g}_k,\mathbf{n}\times\mathbf{f}_i > M_i$$

In order to express our problem in matrix form we define the following matrices

$$(B-S)_{i,j} = i\iint_{\Gamma_h} kG(s,s')\mathbf{f}_j(s')\cdot\mathbf{f}_i(s) - \frac{1}{k}G(s,s')(\mathrm{div}_\Gamma\mathbf{f}_i)(\mathrm{div}_\Gamma'\mathbf{f}_j)dsds'$$

$$Q_{i,j} = -i\iint_{\Gamma_h}[\mathbf{f}_i(s)\times\mathbf{f}_j(s')]\cdot\nabla_\Gamma' G(s,s')dsds'$$

$$I_{i,j} = \int_{\Gamma_h}\mathbf{f}_i\cdot\mathbf{f}_j ds$$

$$D_{i,j} = \int_{\Gamma_h}(\mathrm{div}_\Gamma\mathbf{f}_j)(\mathrm{div}_\Gamma\mathbf{f}_i)ds$$

$$C_{Hi,j} = \int_{\Gamma_h}\mathbf{g}_i\cdot\mathbf{f}_j ds$$

$$C_{Ki,j} = \int_{\Gamma_h}\mathbf{g}_i\cdot(\mathbf{n}\times\mathbf{f}_j)ds$$

Note: We will define $\mathbf{g}_i$ basis function such that $[C_H]$ is a nonsingular diagonal matrix, therefore it is invertible.

For the sake of simplicity we define

$$[A1] = [(B-S)] + \frac{a_0}{2}[I] - \frac{a_1}{2}[D], \quad [A_2] = [(B-S)] + \frac{1}{2a_0}[I] - \frac{b_2}{2a_0}[D]$$

According to the equation system (8.2), we get the following matrix form equation

$$\begin{pmatrix} [A1] & [Q] & 0 & \frac{b_1}{2}[D] & [C_K]^T & 0 \\ [Q]^T & [A2] & -\frac{a_2}{2a_0}[D] & 0 & 0 & [C_K]^T \\ 0 & -\frac{b_2}{2}[D] & -\frac{a_2}{2}[D] & 0 & [C_H]^T & 0 \\ \frac{a_1}{2a_0}[D] & 0 & 0 & -\frac{b_1}{2a_0}[D] & 0 & [C_H]^T \\ [C_K] & 0 & [C_H] & 0 & 0 & 0 \\ 0 & [C_K] & 0 & [C_H] & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{J} \\ \overline{M} \\ \overline{\tilde{J}} \\ \overline{\tilde{M}} \\ \overline{\lambda_J} \\ \overline{\lambda_M} \end{pmatrix} = \begin{pmatrix} \overline{E} \\ \overline{H} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where right-side vectors $\overline{E}$, $\overline{H}$ are defined as follows:

$$E_i = \int_{\Gamma_h} \mathbf{E}^{inc} \cdot \mathbf{f}_i ds;$$

$$H_i = \int_{\Gamma_h} \mathbf{H}^{inc} \cdot \mathbf{f}_i ds.$$

The last two lines of the system give us

$$\begin{aligned} [C_K]\overline{J} + [C_H]\overline{\tilde{J}} = 0 \to \overline{\tilde{J}} = -[C_H]^{-1}[C_K]\overline{J} \\ [C_K]\overline{M} + [C_H]\overline{\tilde{M}} = 0 \to \overline{\tilde{M}} = -[C_H]^{-1}[C_K]\overline{M} \end{aligned} \tag{8.3}$$

The third and fourth lines with (8.3) give us expression of Lagrange multipliers in terms of $\overline{J}$ and $\overline{M}$

$$\begin{aligned} -\frac{b_2}{2}[D]\overline{M} - \frac{a_2}{2}[D]\overline{\tilde{J}} + [C_H]^T\overline{\lambda_J} = 0 \to \\ \overline{\lambda_J} = \frac{b_2}{2}[C_H]^{-T}[D]\overline{M} + \frac{a_2}{2}[C_H]^{-T}[D]\overline{\tilde{J}} = \\ = \frac{b_2}{2}[C_H]^{-T}[D]\overline{M} - \frac{a_2}{2}[C_H]^{-T}[D][C_H]^{-1}[C_K]\overline{J} \end{aligned} \tag{8.4}$$

$$\begin{aligned} \frac{a_2}{2a_0}[D]\overline{J} - \frac{b_1}{2a_0}[D]\overline{\tilde{M}} + [C_H]^T\overline{\lambda_M} = 0 \to \\ \overline{\lambda_M} = -\frac{a_2}{2a_0}[C_H]^{-T}[D]\overline{J} + \frac{b_1}{2a_0}[C_H]^{-T}[D]\overline{\tilde{M}} = \\ = -\frac{a_2}{2a_0}[C_H]^{-T}[D]\overline{J} - \frac{b_1}{2a_0}[C_H]^{-T}[D][C_H]^{-1}[C_K]\overline{M} \end{aligned} \tag{8.5}$$

So we replace the unknowns (8.3), (8.4) and (8.5) in first two lines and if we define $[C_{KH}] = [C_H]^{-1}[C_K]$ and $[C_{KH}]^T = [C_K]^T[C_H]^{-T}$, we get

$$\begin{bmatrix} [A1] - \frac{a_2}{2}[C_{KH}]^T[D][C_{KH}] & [Q] + \frac{b_1}{2}[D][C_{KH}] + \frac{b_2}{2}[C_{KH}]^T[D] \\ [Q]^T + \frac{a_2}{2a_0}[D][C_{KH}] - \frac{a_1}{2a_0}[C_{KH}]^T[D] & [A2] - \frac{b_1}{2a_0}[C_{KH}]^T[D][C_{KH}] \end{bmatrix} \begin{pmatrix} \overline{J} \\ \overline{M} \end{pmatrix} = \begin{pmatrix} \overline{E} \\ \overline{H} \end{pmatrix}$$

8.3 Assembly of matrices

We introduce local numbering of a triangle T. The vertices $(a_j^T)_{j=1,3}$ are arranged in clockwise order. Triangle edges are numbered so that the edge T_j' connects vertices a_j^T and a_{j+1}^T.

Moreover, we give an orientation $\boldsymbol{\nu}_n$ to each edge n. Consider the two triangles sharing this edge. We note T_n^+ the triangle so that the direction of the edge n coincides with the forward direction (locally defined) of this triangle. For the other triangle, which we will denote T_n^-, the direction of the edge coincides with the indirect sense.

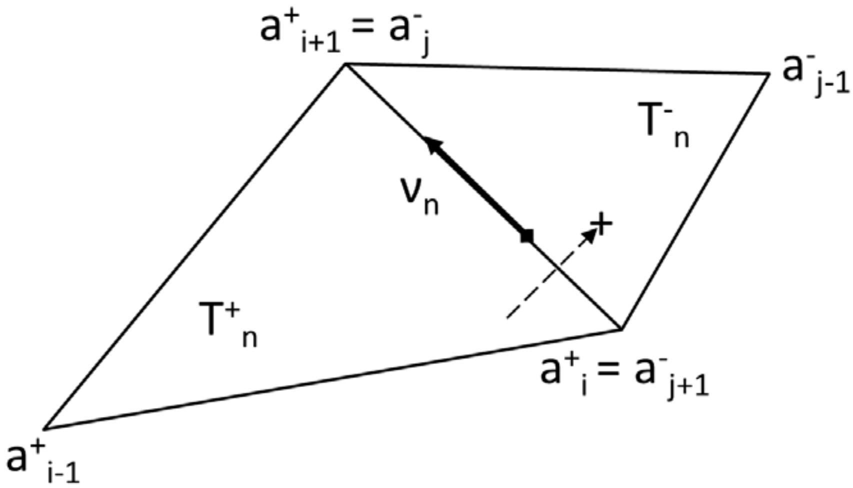

Figure 8.1: Triangles T_n^+ and T_n^- adjacent to edge n

8.3.1 RWG basis functions

Each basis function is associated with an edge and ensures the conservation of flux through this edge. If we denote $|T|$ the area of a triangle T, the n^{th} basis function is defined as follows:

Definition 4 *If n is the i^{th} local edge of triangle T_n^+ and the j^{th} of triangle T_n^- then:*

$$\mathbf{f}_n(x) = \begin{cases} \frac{l_n}{2|T_n^+|}(x - a_{i-1}^+) \quad if\ x \in T_n^+ \\ \frac{l_n}{2|T_n^-|}(a_{j-1}^- - x) \quad if\ x \in T_n^- \\ 0 \quad if\ x \notin T_n^+ \cup T_n^- \end{cases} \tag{8.6}$$

The density is proportional to $\mathrm{div}_\Gamma \mathbf{f}_n$, *where*

$$\mathrm{div}_\Gamma \mathbf{f}_n(x) = \begin{cases} +\frac{l_n}{|T_n^+|} & on\ T_n^+ \\ -\frac{l_n}{|T_n^-|} & on\ T_n^- \\ 0 & elsewhere. \end{cases} \tag{8.7}$$

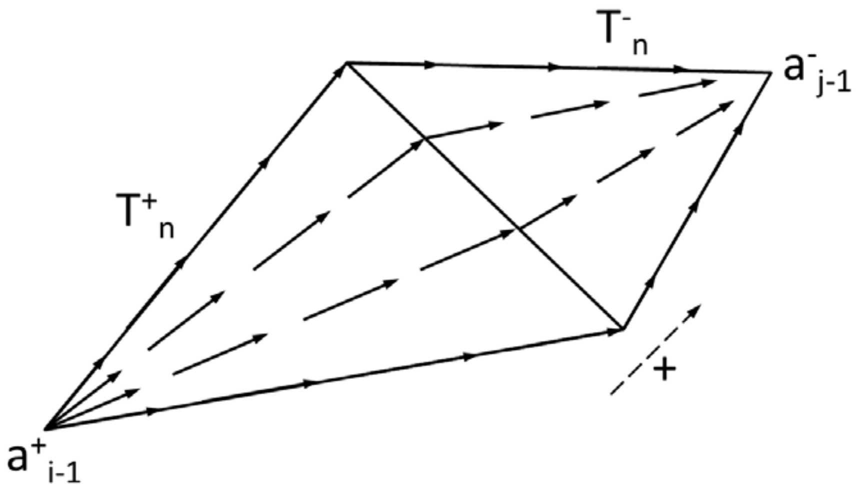

Figure 8.2: Rao-Wilton-Glisson basis function on triangle elements associated to the edge n

On each triangle, the current is written as a linear combination of three basis functions associated to three edges of a triangle. Since the basis function is defined on two triangles, namely adjacent to edge i, T_i^+ and T_i^-, we define $[(B-S)]$ matrix elements

$$(B-S)_{i,j} = iZ_0 \sum_{t1,t2=1}^{N_{tr}} J_j \int_{T_{t1}} \int_{T_{t2}} kG(s,s')\mathbf{f}_j(s')\cdot\mathbf{f}_i(s) - \frac{1}{k}G(s,s')(\mathrm{div}_\Gamma \mathbf{f}_i)(\mathrm{div}'_\Gamma \mathbf{f}_j)ds'd =$$

$$\int_{T_i^+\cup T_i^-} \int_{T_j^+\cup T_j^-} kG(s,s')\mathbf{f}_j(s')\cdot\mathbf{f}_i(s) - \frac{1}{k}G(s,s')(\mathrm{div}_\Gamma \mathbf{f}_i)(\mathrm{div}'_\Gamma \mathbf{f}_j)ds'ds.$$

Analogically

$$Q_{i,j} = -i\int_{T_i^+\cup T_i^-} \int_{T_j^+\cup T_j^-} [\mathbf{f}_i(s)\times\mathbf{f}_j(s')]\cdot\nabla'_\Gamma G(s,s')ds'ds.$$

For lonely integrals we have

$$I_{i,j} = \frac{a_0}{2}\sum_{t=1}^{N_{tr}} J_j \int_{T_t} \mathbf{f}_j\cdot\mathbf{f}_i ds = \int_{T_i^+\cup T_i^-} \mathbf{f}_j\cdot\mathbf{f}_i ds.$$

We should note that the integral is not zero only if j is a number of edge of the triangles T_i^+ and T_i^-. Analogically, we define

$$D_{i,j} = \int_{T_i^+ \cup T_i^-} (\mathrm{div}_\Gamma \mathbf{f}_j)(\mathrm{div}_\Gamma \mathbf{f}_i) ds.$$

8.3.2 Basis functions proposed by Bendali

Now we discretize the Lagrange multipliers $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$. There was no a-priori to satisfy conservation of flow, therefore no reason to prefer a decomposition of the same elements as the currents. We will therefore choose a discretization that simplifies the elimination of both $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{J}}$ according to equation $\int_\Gamma \tilde{\boldsymbol{\lambda}}_J \cdot (\tilde{\mathbf{J}} - \mathbf{n} \times \mathbf{J}) ds = 0$.

For simplicity, we will take the basis functions of degree 1 for the decomposition of multipliers. Thus the integrated term is at most of degree 2 and the integration of triangle can therefore be carried out accurately by a sum of the midpoints of the edges.

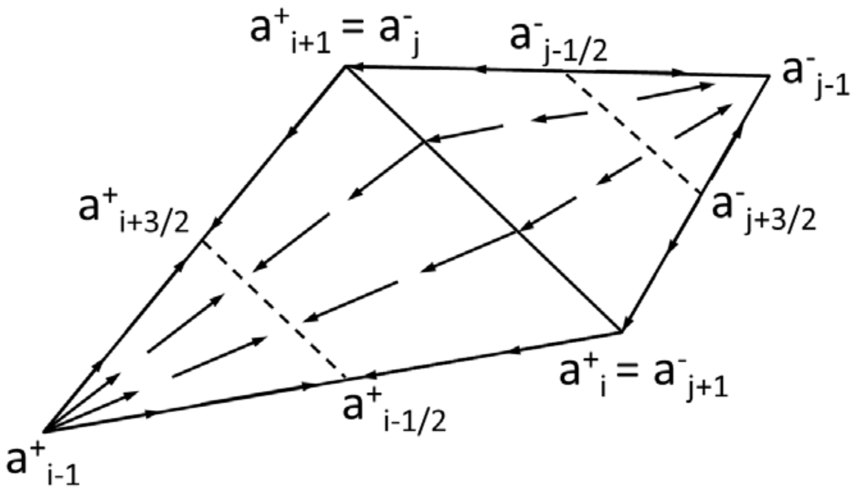

Figure 8.3: Basis function proposed by Bendali on triangle elements

$$\int_T \tilde{\boldsymbol{\lambda}}_J \cdot \tilde{\mathbf{J}} ds = \sum_{i=1}^{3} \frac{|T|}{3} \tilde{\boldsymbol{\lambda}}_J(a^T_{i+1/2}) \cdot \tilde{\mathbf{J}}(a^T_{i+1/2}),$$

where $a^T_{i+1/2}$ denotes the middle point of the local i^{th} edge of triangle T.

It is then interesting to choose a decomposition of $\boldsymbol{\lambda}_J$ of the basis functions vanishing on two of these three points and having the direction of the edge on the third. It is therefore natural that we come to the basis functions proposed by Bendali, Fares and Gay [BA]. Function associated to the edge n which is common to the triangles T_n^+ and T_n^- (notation according to the orientation of the edge) is defined by:

$$\mathbf{g}_n(x) = \begin{cases} (1 - 2\omega^+_{i+2}(x))(\boldsymbol{\nu}_n \times \mathbf{n}^+), & \forall x \in T_n^+ \\ (1 - 2\omega^-_{j+2}(x))(\boldsymbol{\nu}_n \times \mathbf{n}^-), & \forall x \in T_n^- \end{cases} \tag{8.8}$$

where $\boldsymbol{\nu}_n$ is a direction vector of the edge n and $\{\omega_i\}_{i=1,3}$ are barycentric coordinates. So

$$\int_T \tilde{\boldsymbol{\lambda}}_J \cdot \tilde{\mathbf{J}} ds = \frac{1}{2} \sum_{n=1}^{N_e} \left(\int_{T_n^+} \tilde{\boldsymbol{\lambda}} \cdot \tilde{\mathbf{J}} ds + \int_{T_n^-} \tilde{\boldsymbol{\lambda}} \cdot \tilde{\mathbf{J}} ds \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_e} \left(\frac{|T_n^+|}{3} \sum_{i=1}^{3} \tilde{\boldsymbol{\lambda}}_J(a^+_{i+1/2}) \cdot \tilde{\mathbf{J}}(a^+_{i+1/2}) + \frac{|T_n^-|}{3} \sum_{j=1}^{3} \tilde{\boldsymbol{\lambda}}_J(a^-_{j+1/2}) \cdot \tilde{\mathbf{J}}(a^-_{j+1/2}) \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N_e} \left(\frac{|T_n^+|}{3} \sum_{i=1}^{3} \tilde{\lambda}_{(i,T_n^+)} J_{(i,T_n^+)} + \frac{|T_n^-|}{3} \sum_{j=1}^{3} \tilde{\lambda}_{(j,T_n^-)} J_{(j,T_n^-)} \right),$$

where we tentatively adopt the local numbering: (i, T_n^+) denotes the i^{th} edge of the triangle T_n^+.

Elimination of auxiliary currents

Suppose that $\tilde{\boldsymbol{\lambda}}_n$ has all its coefficients zero except that associated to the n^{th} edge that is 1. The integration on the surface is reduced to integrals on the triangles T_n^+ and T_n^-

$$\int_\Gamma \mathbf{g}_n(s) \cdot \tilde{\mathbf{J}} ds = \frac{|T_n^+| + |T_n^-|}{3} \tilde{J}_n.$$

On the other side we have

$$\int_\Gamma \mathbf{g}_n(s) \cdot (\mathbf{n} \times \mathbf{J}) ds = \int_{T_n^+ \cup T_n^-} \mathbf{g}_n(s) \cdot (\mathbf{n} \times \mathbf{J}) ds,$$

that is calculated with help of Gaussian quadrature. And we get next equation

$$\frac{|T_n^+| + |T_n^-|}{3} \tilde{J}_n = \int_{T_n^+ \cup K_n^-} \mathbf{g}_n(s) \cdot (\mathbf{n} \times \mathbf{J}) ds \tag{8.9}$$

and

$$\tilde{J}_n = \frac{3}{|T_n^+| + |T_n^-|} \int_{T_n^+ \cup T_n^-} \mathbf{g}_n(s) \cdot (\mathbf{n} \times \mathbf{J}) ds \tag{8.10}$$

on each edge of the mesh.

Matrix form of system

The integrals with Lagrange multipliers have their own special place. And approximation of these integrals were discussed while defining of basis functions for Lagrange multipliers. So we define

$$C_{Hi,j} = \begin{cases} \frac{|T_i^+|+|T_i^-|}{3} & for\ i = j \\ 0 & else, \end{cases}$$

so C_H is the diagonal and invertible matrix. And C_K is

$$C_{Ki,j} = \int_{T_i^+ \cup T_i^-} \mathbf{g}_i \cdot (\mathbf{n}(s) \times \mathbf{f}_j(s)) ds.$$

Finally, if we combine all edges together, equation (8.9) can be written in matrix form as $[C_H]\overline{\tilde{J}} + [C_K]\overline{J} = 0$ and equation (8.10) as $\overline{\tilde{J}} = -[C_H]^{-1}[C_K]\overline{J}$.

8.4 Conclusion

We discretized the surface of a three dimensional object by triangulation. We approximated the unknowns in terms of RWG basis functions and the Lagrange multipliers in terms of the basis functions that introduced by Bendali. These specially constructed basis functions allow us to get diagonal matrices. So we eliminated the Lagrange multipliers and the auxiliary unknowns. And we propose the matrix form of the discrete problem for unknowns $(\mathbf{J}, \mathbf{M})$.

Chapter 9

3D discretization using the Buffa-Christiansen functions

Contents

9.1 Introduction

We are going to define another set of basis functions, which was introduced by Christiansen. We call them Buffa-Christiansen (BC) functions [FAa, FAb]. We are not going to discuss the advantages or disadvantages of this type of discretization or discuss its properties. We propose the way to use it in solving 3D scattering problem.

We recall bilinear operator that we consider in three dimensional case

$$
\begin{aligned}
A(U,\Psi) =&< Z_0(B-S)\mathbf{J},\mathbf{\Psi}_J > + \frac{1}{Z_0} < (B-S)\mathbf{M},\mathbf{\Psi}_M > \\
&+ < Q\mathbf{M},\mathbf{\Psi}_J > - < Q\mathbf{J},\mathbf{\Psi}_M > + \frac{a_0}{2} < \mathbf{J},\mathbf{\Psi}_J > + \frac{1}{2a_0} < \mathbf{M},\mathbf{\Psi}_M > \\
&- \frac{a_1}{2} < \mathrm{div}_\Gamma\mathbf{J}, \mathrm{div}_\Gamma\mathbf{\Psi}_J > - \frac{a_2}{2} < \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{J}), \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_J) > \\
&+ \frac{b_1}{2} < \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{M}), \mathrm{div}_\Gamma\mathbf{\Psi}_J > + \frac{b_2}{2} < \mathrm{div}_\Gamma\mathbf{M}, \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_J) > \\
&- \frac{b_1}{2a_0} < \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{M}), \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_M) > - \frac{b_2}{2a_0} < \mathrm{div}_\Gamma\mathbf{M}, \mathrm{div}_\Gamma\mathbf{\Psi}_M > \\
&+ \frac{a_1}{2a_0} < \mathrm{div}_\Gamma\mathbf{J}, \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{\Psi}_M) > - \frac{a_2}{2a_0} < \mathrm{div}_\Gamma(\mathbf{n}\times\mathbf{J}), \mathrm{div}_\Gamma\mathbf{\Psi}_M >
\end{aligned} \tag{9.1}
$$

and the problem that was introduced in chapter 7.

Problem 9.1.1 *Find $U = (\mathbf{J}, \mathbf{M})$ such that*

$$A(U, \Psi) = F(\Psi) \tag{9.2}$$

for all $\Psi = (\mathbf{\Psi}_J, \mathbf{\Psi}_M)$. Where

$$F(\Psi) = \int_\Gamma \mathbf{E}^{inc} \cdot \mathbf{\Psi}_J ds + \int_\Gamma \mathbf{H}^{inc} \cdot \mathbf{\Psi}_M ds.$$

9.2 Barycentric refinement. The Buffa-Christiansen functions

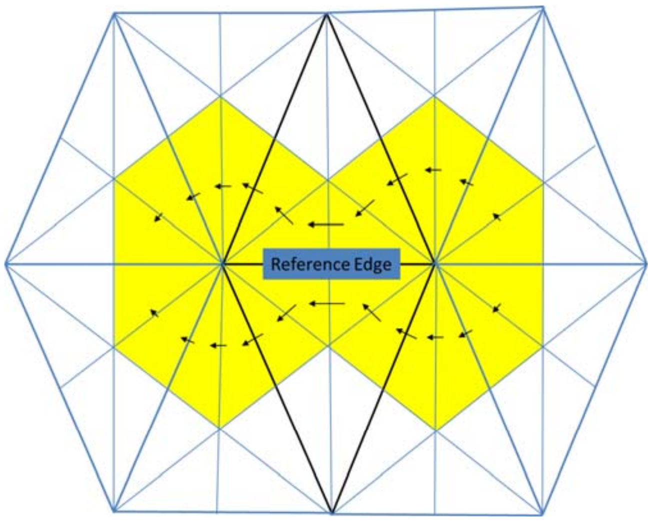

Figure 9.1: Buffa-Christiensen function

The barycentric refinement of Γ_h is defined by dividing each triangle $T \in \Gamma_h$, into six triangles by drawing the six edges joining the barycenter of T with vertexes as well as the midpoints of edges. The barycentric refinement of Γ_h is denoted Γ_h^b and we will call it barycentric mesh.

Generally speaking, basis and testing functions can be categorized into two different kinds: the divergence-conforming and curl-conforming functions. A typical divergence-conforming function is the curvilinear Rao-Wilton-Glisson (RWG) function. In this chapter, we will denote RWG functions as $\mathbf{f}_R$. By rotating it with respect to the normal vector, a commonly used curl-conforming function $\mathbf{n} \times \mathbf{f}_R$ can be obtained. The Buffa-Christiansen (BC) basis functions are defined on triangles of a barycentric mesh that have common vertex with reference edge (see Fig. 9.1), denoted as f_B. These functions are div-conforming on this barycentric mesh. The Buffa-Christiansen basis functions are also quasi-curl-conforming on the original mesh. [YJN]

- RWG function $\mathbf{f}_R$ is a divergence-conforming
- rotation of RWG function $\mathbf{n} \times \mathbf{f}_R$ is a curl-conforming
- BC function $\mathbf{f}_B$ is a quasi-curl-conforming on original mesh, and div-conforming on a barycentric mesh
- rotation of BC function $\mathbf{n} \times \mathbf{f}_B$ is a quasi-divergence-conforming on original mesh, and curl-conforming on a barycentric mesh.

The figure 9.2 shows the definition domains of this four functions. We denote the space combined by the Rao-Wilton-Glisson functions, as X_{RWG}. X^b_{RWG} is the space of functions combined by the RWG functions $\mathbf{f}^b$ defined on a berycentric mesh. The Buffa-Christiansen functions are linear combinations of $\mathbf{f}^b \in X^b_{RWG}$ functions. And we denote the space of combinations of BC functions as X_{BC}.

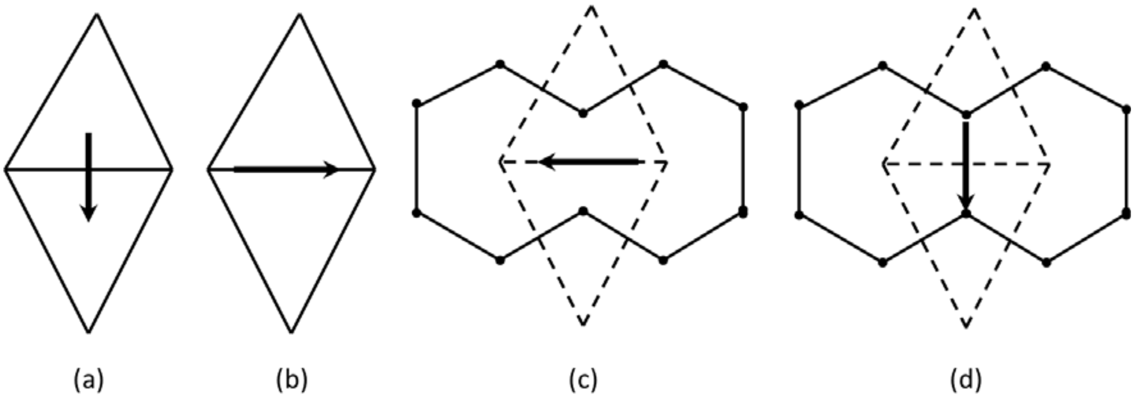

Figure 9.2: The outline of the definition domains of the basis functions and its' rotations: (a)-(b) Rao-Wilton-Glisson function f and $n \times f$, resp.; (c)-(d) Buffa-Christiansen functions f_{BC} and $n \times f_{BC}$, resp.

We introduce two operators K and R from [FAa] (K was denoted as P in [FAa]), such that

$$K : X^b_{RWG} \to X_{BC} \qquad R : X^b_{RWG} \to X_{RWG}$$

In the formulation (9.1), we have terms with divergence of rotation of unknowns. These terms have no sense if we take these unknown as divergence-conforming functions. For this reason we want to represent rotation of unknowns in terms of RWG functions on barycentric mesh.

9.3 Matrices form

We introduce Rao-Wilton-Glisson $\{\mathbf{f}_{Ri}\}_{i=1}^N$ and Buffa-Christiansen $\{\mathbf{f}_{Bj}\}_{j=1}^N$ the basis functions on initial mesh. And $\{\mathbf{f}^b_i\}_{i=1}^{N^b}$ Rao-Wilton-Glisson basis function on barycentric mesh. We can represent RWG functions $\mathbf{f}_{Rj}$ on initial mesh as a linear combination of functions $\mathbf{f}_i^b$ on barycentric mesh

$$\mathbf{f}_{Rj} = \sum_{l=1}^{N^b} c_{jl}\mathbf{f}_l^b = (c_{j1}, .., c_{jN^b}) \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix}$$

And we will define BC functions $\mathbf{f}_{Bj}$ as a linear combination of RWG functions $\mathbf{f}_i^b$ on barycentric mesh

$$\mathbf{f}_{Bi} = \sum_{k=1}^{N^b} d_{ik}\mathbf{f}_k^b = (d_{i1}, .., d_{iN^b}) \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix}$$

The operators K and R are transition matrices $[K]$ and $[R]$ that express functions in X_{BC} and X_{RWG} as linear combinations of functions in X_{RWG}^b, respectively.

$$\begin{pmatrix} \mathbf{f}_{R1} \\ \vdots \\ \mathbf{f}_{RN_e} \end{pmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1N^b} \\ \vdots & c_{jl} & \vdots \\ c_{N_e1} & \cdots & c_{N_eN^b} \end{bmatrix} \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix} = [R] \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_{B1} \\ \vdots \\ \mathbf{f}_{BN_e} \end{pmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1N^b} \\ \vdots & d_{ik} & \vdots \\ d_{N_e1} & \cdots & d_{N_eN^b} \end{bmatrix} \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix} = [K] \begin{pmatrix} \mathbf{f}_1^b \\ \vdots \\ \mathbf{f}_{N^b}^b \end{pmatrix}$$

We develop the unknown function $\mathbf{J}$ on RWG basis function

$$\mathbf{J} = \sum_{i=1}^{N_e} J_i\mathbf{f}_{Ri} = (J_1, .., J_N) \begin{pmatrix} \mathbf{f}_{R1} \\ \vdots \\ \mathbf{f}_{RN} \end{pmatrix} = \overline{J}^T \begin{pmatrix} \mathbf{f}_{R1} \\ \vdots \\ \mathbf{f}_{RN} \end{pmatrix},$$

as well as test function

$$\mathbf{\Psi}_J = \sum_{i=1}^{N_e} \mathbf{f}_{Ri} = (1_1, .., 1_{N_e}) \begin{pmatrix} \mathbf{f}_{R1} \\ \vdots \\ \mathbf{f}_{RN_e} \end{pmatrix} = \overline{I}^T \begin{pmatrix} \mathbf{f}_{R1} \\ \vdots \\ \mathbf{f}_{RN_e} \end{pmatrix}.$$

Now we develop rotation of unknown function $\mathbf{J}$

$$\mathbf{n} \times \mathbf{J} = \sum_{i=1}^{N_e} J_i\mathbf{n} \times \mathbf{f}_{Ri} = \overline{J}^T \begin{pmatrix} \mathbf{n} \times \mathbf{f}_{R1} \\ \vdots \\ \mathbf{n} \times \mathbf{f}_{RN_e} \end{pmatrix}.$$

We apply Gram matrix $[G]$ between $\mathbf{n}\times\mathbf{f}_R$ and $\mathbf{f}_B$ that links the two basis [FAa], such that

$$\begin{pmatrix} \mathbf{n} \times \mathbf{f}_{R1} \\ \vdots \\ \mathbf{n} \times \mathbf{f}_{RN_e} \end{pmatrix} = [G] \begin{pmatrix} \mathbf{f}_{B1} \\ \vdots \\ \mathbf{f}_{BN_e} \end{pmatrix}.$$

So we get

$$\mathbf{n} \times \mathbf{J} = \overline{J}^T [G] \begin{pmatrix} \mathbf{f}_{B1} \\ \vdots \\ \mathbf{f}_{BN_e} \end{pmatrix}$$

Analogically

$$\mathbf{n} \times \mathbf{\Psi}_J = \overline{I}^T [G] \begin{pmatrix} \mathbf{f}_{B1} \\ \vdots \\ \mathbf{f}_{BN_e} \end{pmatrix}. \tag{9.3}$$

In the formulation (9.1) we use the following integrals that we express in matrices form

$$< \text{div}_\Gamma \mathbf{J}, \text{div}_\Gamma \mathbf{\Psi}_J >= \sum_{i,j}^{N_e} J_j \int_{\Gamma^h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma \mathbf{f}_{Rj} ds =$$

$$= \overline{I}^T \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma \mathbf{f}_{Rj} ds \right] \overline{J} = \overline{I}^T [L_D] \overline{J}$$

and

$$< \text{div}_\Gamma (\mathbf{n} \times \mathbf{J}), \text{div}_\Gamma \mathbf{\Psi}_M >= \sum_{i,j}^{N_e} J_j \int_{\Gamma^h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma (\mathbf{n} \times \mathbf{f}_{Rj}) ds =$$

$$= \overline{I}^T \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma \mathbf{f}_{Bj} ds \right] [G]^T \overline{J}.$$

Anallogically

$$< \text{div}_\Gamma \mathbf{J}, \text{div}_\Gamma (\mathbf{n} \times \mathbf{\Psi}_M) >= \overline{I}^T [G] \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Bi} \ \text{div}_\Gamma \mathbf{f}_{Rj} ds \right] \overline{J}$$

$$< \text{div}_\Gamma (\mathbf{n} \times \mathbf{J}), \text{div}_\Gamma (\mathbf{n} \times \mathbf{\Psi}_J) >= \overline{I}^T [G] \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Bi} \ \text{div}_\Gamma \mathbf{f}_{Bj} ds \right] [G]^T \overline{J}$$

Here, we define the following matrices

$$[L_D] = \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma \mathbf{f}_{Rj} ds \right]$$

$$[L_{DC}] = \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Ri} \ \text{div}_\Gamma \mathbf{f}_{Bj} ds \right] [G]^T$$

$$[L_{DB}] = [G] \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Bi} \ \text{div}_\Gamma \mathbf{f}_{Rj} ds \right]$$

and

$$[L_{DBC}] = [G] \left[\int_{\Gamma_h} \text{div}_\Gamma \mathbf{f}_{Bi} \ \text{div}_\Gamma \mathbf{f}_{Bj} ds \right] [G]^T.$$

We will call them "div-div" matrices.

And by defining the following matrices, we can propose matrices form of the problem.

$$[A1] = Z_0[(B-S)] + \frac{a_0}{2}[I] - \frac{a_1}{2}[L_D] - \frac{a_2}{2}[L_{DBC}]$$

$$[A2] = [Q] + \frac{b_1}{2}[L_{DC}] + \frac{b_2}{2}[L]_{DB}$$

$$[A3] = [Q]^T - \frac{a_2}{2a_0}[L_{DC}] + \frac{a_1}{2a_0}[L_{DB}]$$

$$[A4] = \frac{1}{Z_0}[(B-S)] + \frac{1}{2a_0}[I] - \frac{b_2}{2a_0}[L_D] - \frac{b_1}{2a_0}[L_{DBC}]$$

In the bilinear form $A(U,\Psi)$ we will write as the following matrix equation:

$$\begin{bmatrix} A1 & A2 \\ A3 & A4 \end{bmatrix} \begin{pmatrix} \overline{J} \\ \overline{M} \end{pmatrix} = \begin{pmatrix} \overline{E} \\ \overline{H} \end{pmatrix}$$

9.4 Assembly of elementary matrices

9.4.1 div-div matrices

In order to calculate the matrix $[L_{DB}]$, we have to know the following matrix

$$\left[\int_{\Gamma_h} \mathrm{div}_\Gamma \mathbf{f}_{Bi}\ \mathrm{div}_\Gamma \mathbf{f}_{Rj} ds\right] =$$

where we use operators K and R to express $\mathbf{f}_{Bi}$ and $\mathbf{f}_{Rj}$ in terms of RWG function on barycentric mesh

$$= \left[\sum_{k,l=1}^{N^b} d_{ik} c_{jl} \int_{\Gamma_h^b} \mathrm{div}_\Gamma \mathbf{f}_k^b \mathrm{div}_\Gamma \mathbf{f}_l^b ds\right]$$

$$= \begin{bmatrix} d_{11} & \cdots & d_{1N^b} \\ \vdots & d_{ik} & \vdots \\ d_{N_e1} & \cdots & d_{N_eN^b} \end{bmatrix} \left[\int_{\Gamma_h^b} \mathrm{div}_\Gamma \mathbf{f}_k^b \mathrm{div}_\Gamma \mathbf{f}_l^b ds\right] \begin{bmatrix} c_{11} & \cdots & c_{1N^b} \\ \vdots & c_{jl} & \vdots \\ c_{N_e1} & \cdots & c_{N_eN^b} \end{bmatrix}^T$$

$$= [K] \left[\int_{\Gamma_h^b} \mathrm{div}_\Gamma \mathbf{f}_k^b \mathrm{div}_\Gamma \mathbf{f}_l^b ds\right] [R]^T$$

We define matrix of div-/div-conforming functions on barycentric mesh

$$[L_D^b]_{kl} = \int_{\Gamma_h^b} \mathrm{div}_\Gamma \mathbf{f}_k^b\ \mathrm{div}_\Gamma \mathbf{f}_l^b ds.$$

Finally, we get

$$[L_{DB}]\overline{J} = [G] \left[\int_{\Gamma_h} \mathrm{div}_\Gamma \mathbf{f}_{Bi}\ \mathrm{div}_\Gamma \mathbf{f}_{Rj} ds\right] \overline{J} = [G][K][L_D^b][R]^T \overline{J}$$

And we obtain three matrices

- $[L_{DC}] = [R][L_D^b][K]^T[G]^T$
- $[L_{DB}] = [G][K][L_D^b][R]^T$
- $[L_{DBC}] = [G][K][L_D^b][K]^T[G]^T$

9.4.2 Gram matrix $[G]$

We take $[G]$ as the invers Gram matrix between $\mathbf{n} \times \mathbf{f}$ and $\mathbf{f}_{BC}$ that links the two discretizations in [FAa]. Where the author Andriulli defines it as

$$[G] = ([R]^T[G^b][K])^{-1}$$

where $[G^b] \in \mathbb{R}^{N^b \times N^b}$ is the Gram matrix linking div- and rot-conforming RWG functions defined on barycentric mesh

$$G^b_{i,j} = \int_{\Gamma_h^b} \mathbf{n} \times \mathbf{f}_i^b \cdot \mathbf{f}_j^b ds.$$

9.4.3 $[R]$ and $[K]$ matrices

This method was described in [FAa]. The $[R]$ matrix means the expansion of the Rao-Wilton-Glisson basis functions on the original mesh in terms of the Rao-Wilton-Glisson basis function on the barycentric mesh.

$$[R]: \quad \mathbf{f} = \sum_{i=1}^{14} c_i \mathbf{f}_i^b,$$

$$\mathbf{f}(x) = -\frac{l}{l_2}\mathbf{f}_2(x) + \frac{l}{6l_3}\mathbf{f}_3(x) - \frac{l}{3l_4}\mathbf{f}_4(x) + \frac{l}{3l_6}\mathbf{f}_6(x) + \mathbf{f}_7(x) + \tag{9.4}$$
$$+\mathbf{f}_8(x) - \frac{l}{3l_9}\mathbf{f}_9(x) + \frac{l}{3l_{11}}\mathbf{f}_{11}(x) - \frac{l}{6l_{12}}\mathbf{f}_{12}(x) + \frac{l}{l_{14}}\mathbf{f}_{14}(x) \quad \forall x \in \Gamma$$

(see (2.42) in [FAa]), where l_i denotes edge length in the barycentric mesh and l is length of the reference edge, see figure 9.3.

And, the $[K]$ matrix means expansion of the Buffa-Christiansen basis functions on the original mesh in terms of the Rao-Wilton-Glisson basis function on the barycentric mesh.

$$[K]: \quad \mathbf{f}_{BC} = \sum c_i \mathbf{f}_i^b,$$

(see (2.45) in [FAa]), where the coefficients are

$$c_0 = \frac{1}{2l_0}, c_1 = \frac{4}{10l_1}, c_2 = \frac{3}{10l_2}, c_3 = \frac{2}{10l_3}, c_4 = \frac{1}{10l_4}, c_5 = 0$$
$$c_6 = -\frac{1}{10l_6}, c_7 = -\frac{2}{10l_7}, c_8 = -\frac{3}{10l_8}, c_9 = -\frac{4}{10l_9}, c_{0'} = -\frac{1}{2l_{0'}}, c_{1'} = -\frac{3}{8l_{1'}}$$
$$c_{2'} = -\frac{2}{8l_{2'}}, c_{3'} = -\frac{1}{8l_{3'}}, c_{4'} = 0, c_{5'} = \frac{1}{8l_{5'}}, c_{6'} = -\frac{2}{8l_{6'}}, c_{7'} = \frac{3}{8l_{7'}}. \tag{9.5}$$

see figure 9.4. These coefficients were found and defined in [FAa].

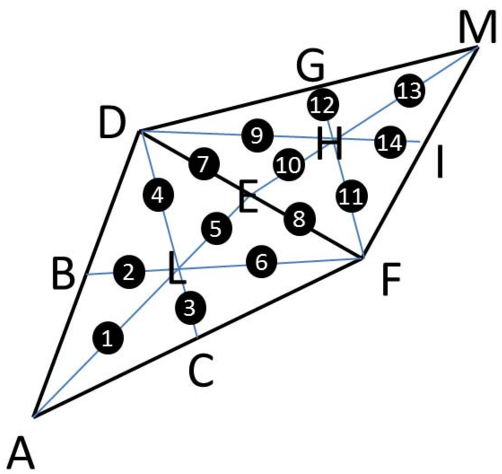

Figure 9.3:

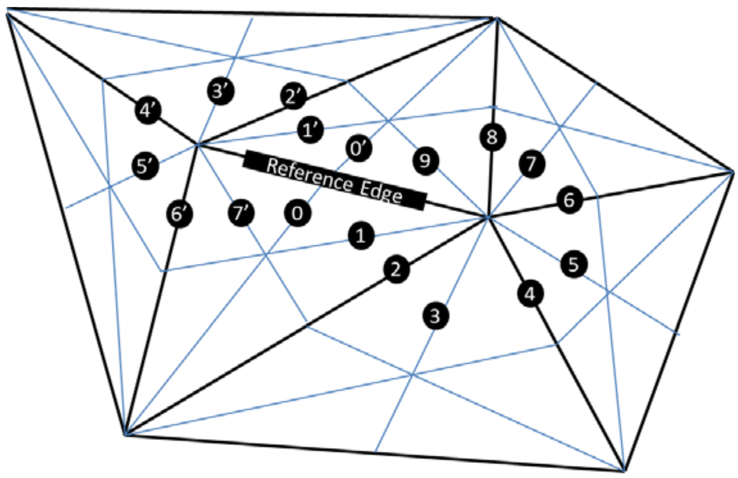

Figure 9.4:

9.5 Conclusion

In this chapter, we discretized the problem 9.1.1 with help of Buffa-Christiansen basis functions. We proposed decompositions of RWG and BC basis functions on initial mesh in terms of RWG basis functions on barycentric mesh. That permitted us to calculate $\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{J})$, $\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{M})$. The Gram matrix links rotated RWG basis functions on initial mesh with BC basis functions. Finally, we proposed the matrix form of discretization of the problem 9.1.1. The matrices $[R]$, $[K]$ and $[G]$ are studied more precisely in [FAa].

Chapter 10

3D Numerical results

Contents

In previous chapters we proposed the variational formulation and discretizations for tri-dimesional scattering problem. Here we present our numerical results.

10.1 Structure of the code

The scheme of the code can be seen in the figure below (figure 10.1).The code begins in the main file named $leonto.f$. The main file calls $matleos.f$ to assemble the matrix $MAT(ndl, ndl)$ of the problem; $ssmbg.f$ determines the second member $SMB(ndl)$ of the system. To solve this system of linear equations we use Gauss method that we apply in $gaussc.f$, then we obtain the solution $SOL(ndl)$. The last subroutine $ser.f$ calculates bistatic RCS in angular range $[0°, 180°]$.

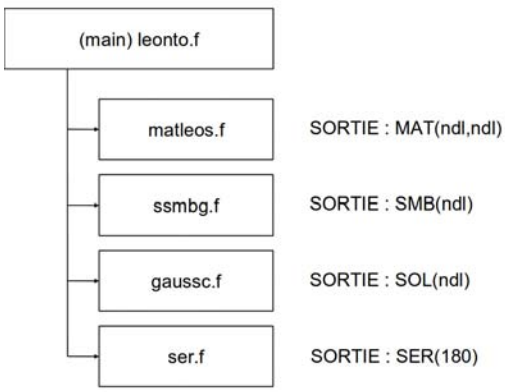

Figure 10.1: The scheme of the ucp code

The subroutine *matleos.f* applies to next scheme (see figure 10.2). The subroutines *ctsb*1.*f*, *ctsb*2.*f* and *ctsb*3.*f* calculate three double-integrals. The impedance boundary conditions are calculated in *choibc.f*, that we use to calculate both leontovich and higher order IBCs.

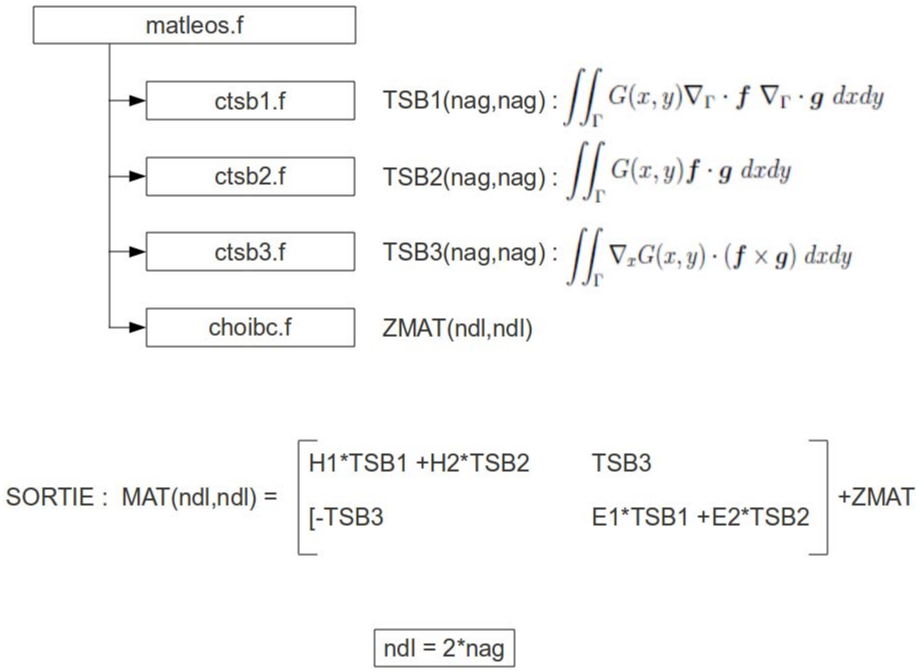

Figure 10.2: The scheme of the subroutine *matleos.f*.

10.2 Singular integrals

The main difficulty in calculating double-ingerals is the singularity. In $ctsb1.f$ and $ctsb2.f$, we encounter singularity because of Green function. To avoid this problem, we use the analytical method proposed by Salles [NS]. And in case there is no singularity threat we use Gaussian quadrature with 9 and 6 gauss points (see figure 10.3).

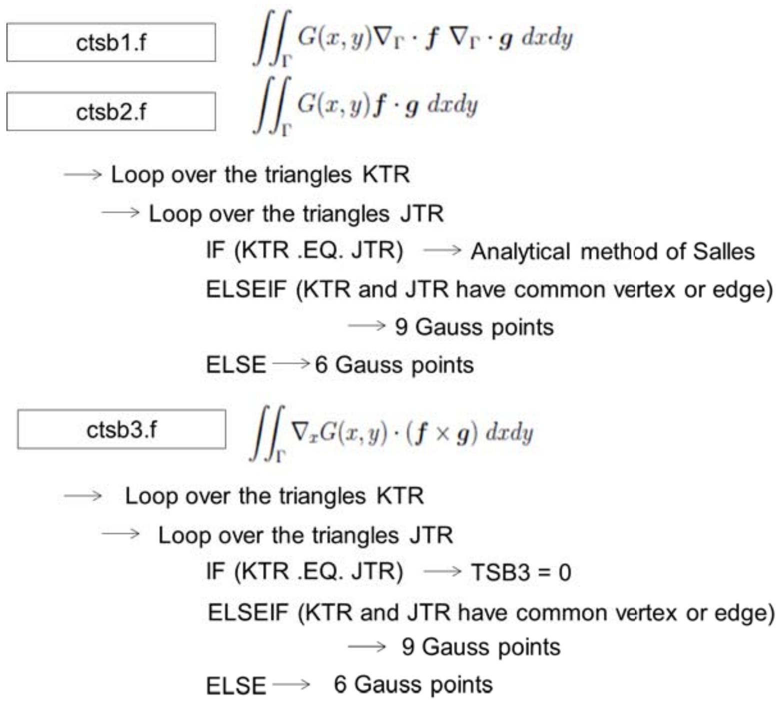

Figure 10.3: The subroutines $ctsb1.f$, $ctsb2.f$ and $ctsb1.3$.

10.3 Tests on a perfect electric sphere

First to validate the code, we test it on a perfect electric conducting (PEC) sphere with different incident fields' frquencies and different sphere-meshes and we compare it with Mie series. In figure 10.4, we take the sphere's radius is 1m; incident frequency $\omega = 1.2GHz$. We plot RCS in TM polarization calculated by Mie Series (continuous red line); blue dash-dot line is the ucp code calculated on the mesh $\lambda/3$; and green dash-dot line is the ucp code calculated on the mesh $\lambda/4$.

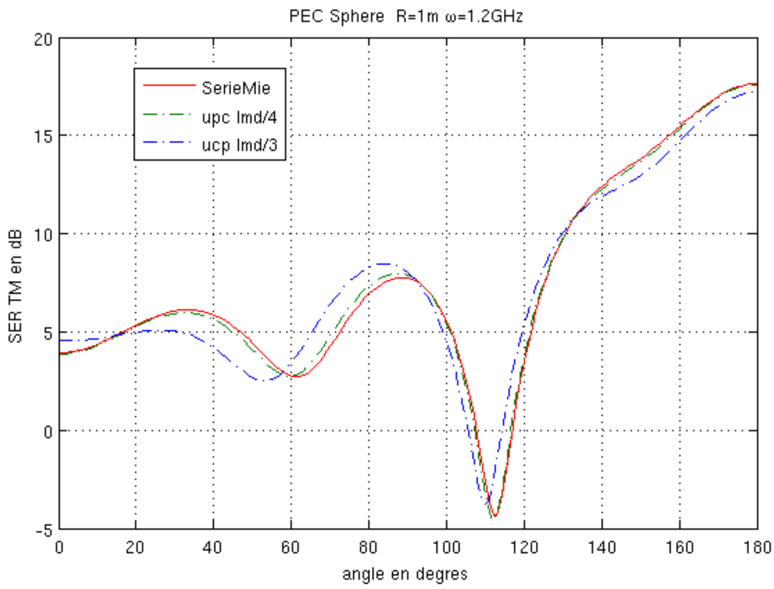

Figure 10.4: PEC Sphere $\omega = 1.2GHz$; Mie series, ucp code for meshes $\lambda/3$ and $\lambda/4$

In figure 10.5, we take the sphere's radius is 1m; incident frequency $\omega = 2.82GHz$. As in previous figure we plot RCS in TM polarization calculated by Mie Series (continuous red line); blue dash-dot line is the ucp code calculated on the mesh $\lambda/1$; and green dash-dot line is the ucp code calculated on the mesh $\lambda/2$.

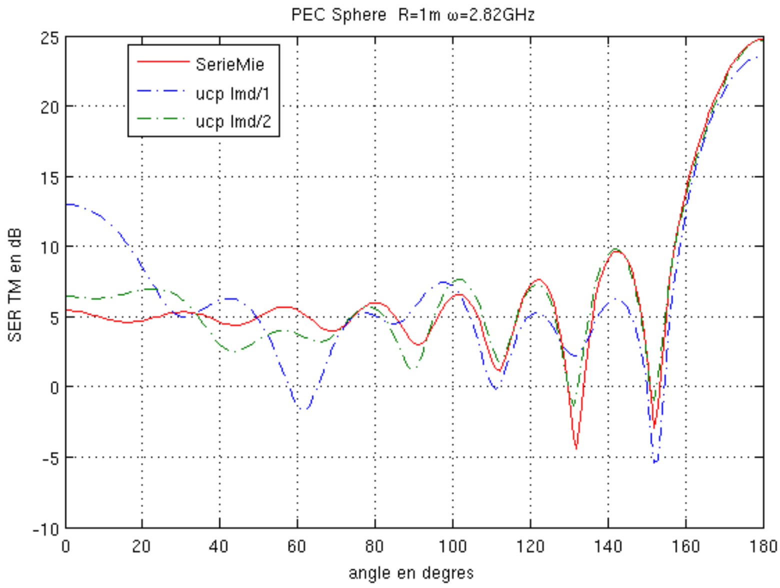

Figure 10.5: PEC Sphere $\omega = 2.82GHz$; Mie series, ucp code for meshes $\lambda/1$ and $\lambda/2$

We can see that it improves with the refined mesh.

10.4 Some results with Higher Order Impedance Boundary Conditions

Here we propose some numerical results on PEC spheres with a thin dielectric layer. We compare Mie series with leontovich and higher order impedance boundary conditions calculted by ucp code.

First we take the sphere with radius 1m, frequency of incident fields $\omega = 1.2GHz$, layer thickness $d = 0.05m$, layer caracteristics are $\varepsilon_r = 3$ and $\mu_r = 1$, and the sphere is meshed as $\lambda/3$. We plot RCS in TM polarization calculated by Mie Series (continues red line); blue dash-dot line is the ucp code calculated for leontovich IBC; and green dash-dot line is the ucp code calculated with higher-order IBC (see figure 10.6).

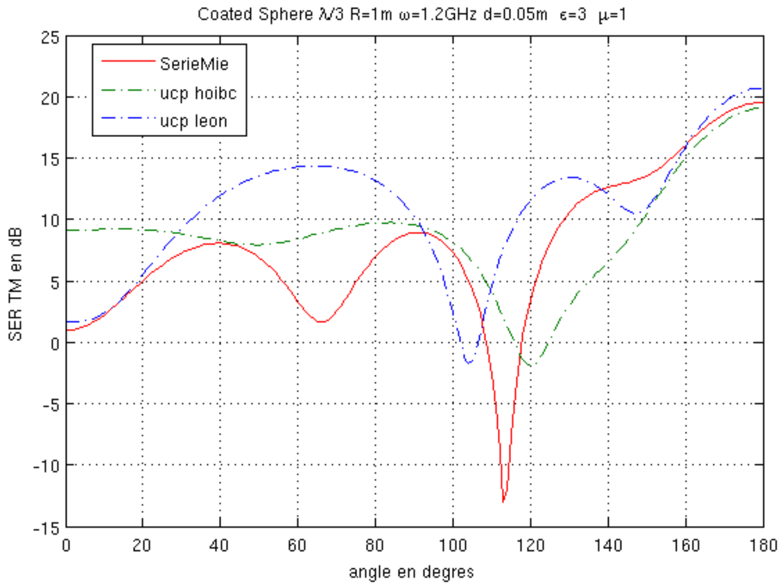

Figure 10.6: Coeted Sphere $\lambda/3$, frequency $\omega = 1.2GHz$, layer thickness $d = 0.05m$, $\varepsilon_r = 3$ and $\mu_r = 1$.

In the second test, we take sphere with the same charateristics as in the previous test, but the sphere is meshed as $\lambda/4$. We plot RCS in TM polarization calculated by Mie Series (continuous red line); blue dash-dot line is the ucp code calculated for leontovich IBC; and green dash-dot line is the ucp code calculated with higher-order IBC (see figure 10.7) .

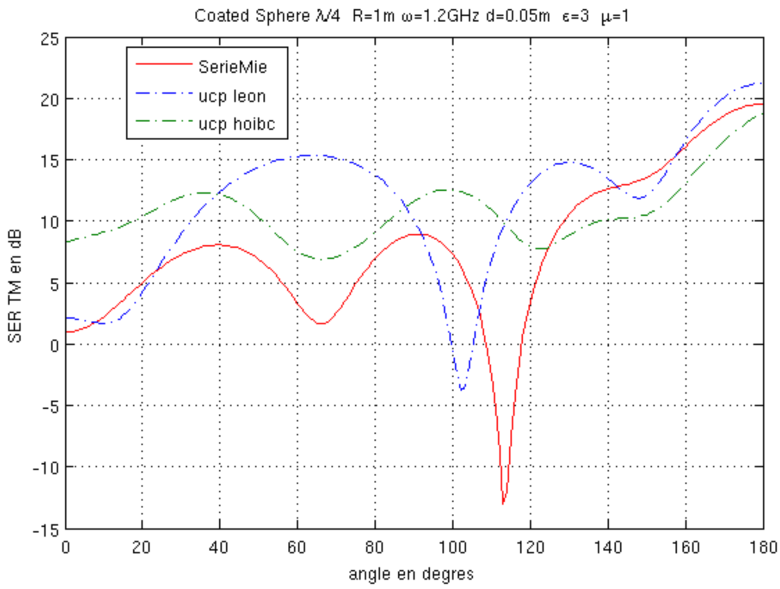

Figure 10.7: Coeted Sphere $\lambda/4$, frequency $\omega = 1.2GHz$, layer thickness $d = 0.05m$, $\varepsilon_r = 3$ and $\mu_r = 1$.

The last case, we test on a sphere with radius 1m, frequency of incident fields $\omega = 2.82GHz$, layer thickness $d = 0.05m$, layer characteristics are $\varepsilon_r = 3$ and $\mu_r = 1$, and the sphere is meshed as $\lambda/1$. We plot RCS in TM polarization calculated by Mie Series (continuous red line); blue dash-dot line is the ucp code calculated for leontovich IBC; and green dash-dot line is the ucp code calculated with higher-order IBC (see figure 10.8).

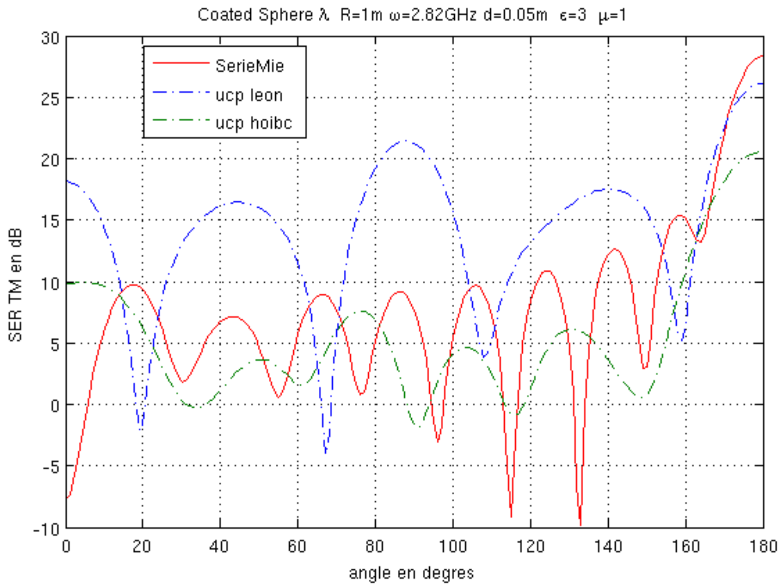

Figure 10.8: Coeted Sphere $\lambda/1$, frequency $\omega = 2.82GHz$, layer thickness $d = 0.05m$, $\varepsilon_r = 3$ and $\mu_r = 1$.

10.5 Conclusion

In this chapter we tested our code on the spheres meshes that are not very small. But even in these tests we see that higher order impedance boundary conditions method approaches to the exact solution more than the leontovich IBC method. So we expect that we can get the more precise results if we test it with a mesh $\lambda/10$. In a refined mesh, triangles that are close to each other can issue singularity problem. So to avoid this problem we should use the same analitycal method of Salles for triangles that have common vertex or edge.

Conclusion

The main subject of this thesis is to propose a new formulation of electromagnetic scattering problem with higher order impedance boundary condition. We proposed both two dimensional and three dimensional formulation. We saw that, if though we do not know the incident angle, the HOIBC takes this angle into account. Therefore, the solution of our formulation is more accurate than that with Leontovich impedance boundary condition. We saw that the definition of HOIBC in [R-Sb] and in [BSb] is the same in the two dimensional case. Even if [R-Sb] used partial differentials and [BSb] used Hodge differential operator. We proposed to define 3D HOIBC, which was introduced in [R-Sb], in terms of decomposed Hodge operators (L_D and L_R). TE and TM polarizations help us to decompose problem on two simple ones in two dimentional case, where we can easily determine coefficients for 2D HOIBC and utilize them for 3D HOIBC too. We have proved the existence and uniqueness theorems, hence we set some restrictions on the coefficients. We proposed new 3D problem formulation. We found that 3D formulation has difficulties with operators. And we proposed two different way to avoid this difficulties in discrete problem. Finally we presented some 3D numerical results for scattering problem.

In [R-Sb] it was discussed HOIBC that includes curvature of a coating. In [BS] it was studied multi-layer coating cases. In practice the surfaces are inhomogeneous. We might have interest to pursue further research in these topics.

Appendix A

Notations and Physical constants

A.1 Notations

- Bold $\mathbf{v}$ denotes vector in $\mathbb{R}^3$ on (x, y, z) basis;
- overline $\overline{A}$ is a vector of discretization values, so unknown $\mathbf{X} \approx \sum_{i=1}^{N_e} X_i \mathbf{f}_i$ or $\mathbf{X} \approx \overline{X} \cdot \overline{\mathbf{f}}$;
- we denote matrices in square brackets [B];
- complex number $z \in \mathbb{C}$ decompose as real part $z_r = \Re z$ and imaginary part $z_i = \Im z$;
- power aster means complex conjugated value $z^* = z_r - iz_i$;
- sometimes partial derivatives $\frac{\partial}{\partial x}$ are denoted by ∂_x.

A.2 Physical constants

- $\pi = 3,14159265358979323 84$;
- $c = 299,792458$ speed of light;
- $\mu_0 = 4\pi \cdot 10^{-7}$ free space permeability in (Henry/m);
- $\varepsilon_0 = 10^{-12}/(\mu_0 c^2)$ free space permittivity in (Farad/m);
- $Z0 = \sqrt{\mu_0/\varepsilon_0} = \mu_0 c \cdot 10^6$ impedance in (Ohm);
- $\omega = 2\pi f$;
- $\lambda = \dfrac{c}{f}$ wave length;
- $k_0 = \omega\sqrt{\varepsilon_0 \mu_0} = \dfrac{\omega}{c}$ wave number.

Appendix B

Integral formulations

The integral method places their unknowns on the boundaries of the object and it takes into account the boundary conditions. It consists in expressing the electromagnetic field as a function of potentials defined on Γ. The Stratton-Chu formulation helps us to characterize the electromagnetic fields in terms of surface currents and charges [TL]. These currents and charges are solutions of integral equations on Γ.

B.1 Integral formulation of the problem

We are looking for a presentation of the fields $(\mathbf{E}, \mathbf{H})$ outside the object in terms of electromagnetic currents $(\mathbf{J}, \mathbf{M})$ on the surface Γ [JCN]. We have Maxwell's equations on the surface Γ :

$$\begin{cases} \mathbf{rot}\mathbf{E} - ikZ_0\mathbf{H} = \mathbf{M}\delta_\Gamma \\ \mathbf{rot}\mathbf{H} + ikZ_0^{-1}\mathbf{E} = \mathbf{J}\delta_\Gamma \end{cases} \tag{B.1}$$

Superposition theorem lets us consider two cases. In the first case we suppose that $\mathbf{M} = 0$, in the second $\mathbf{J} = 0$. In both cases we want to get $\mathbf{E}$ and $\mathbf{M}$ expressions in terms of $\mathbf{J}$ and $\mathbf{M}$. After we combine these cases to get general form of fields $\mathbf{E}$ and $\mathbf{M}$.

In case $\mathbf{M} = 0$, we are looking for $\mathbf{E}$ in the form

$$\mathbf{E} = \mathbf{A} + \nabla V \tag{B.2}$$

with Lorentz gauge condition $\mathrm{div}\mathbf{A} = k^2 V$. The divergence of the second equation of (B.1) with support of gauge condition, shows that V satisfies scalar Helmholtz equation

$$ikZ_0^{-1}\mathrm{div}\mathbf{E} = \mathrm{div}\mathbf{J} \tag{B.3}$$

$$\mathrm{div}\mathbf{A} + \mathrm{div}\nabla V = \frac{1}{ikZ_0^{-1}}\mathrm{div}\mathbf{J} \tag{B.4}$$

$$\Delta V + k^2 V = -ik^{-1}Z_0\mathrm{div}\mathbf{J} \tag{B.5}$$

It can be expressed as a potential simple layer density of $-ik^{-1}Z_0\mathrm{div}\mathbf{J}$

$$V(x) = ik^{-1}Z_0\int_\Gamma G(x,y)\mathrm{div}_\Gamma\mathbf{J}(y)dy \tag{B.6}$$

In this expression G is the Green kernel, fundamental solution of Helmholtz equation, i.e. verifies Maxwell's equations and radiation condition.

We have that $\mathbf{A}$ verifies the next vector Helmholtz equation

$$\Delta\mathbf{A} + k^2\mathbf{A} = \nabla\mathrm{div}\mathbf{A} - \mathbf{rot}(\mathbf{rot}\mathbf{A}) + k^2\mathbf{A} \tag{B.7}$$

According to earlier condition right side of first term equals to $k^2\nabla V$. When (B.2) allows us to express (B.7) as

$$= k^2\nabla V - \mathbf{rot}(\mathbf{rot}\mathbf{A}) + k^2\mathbf{E} - k^2\nabla V = k^2\mathbf{E} - \mathbf{rot}(\mathbf{rot}\mathbf{E}) = -ikZ_0\mathbf{J}$$

The last equivalence derived from Maxwell's equations (B.1).

Then $\mathbf{A}$ is a potential of simple layer density $-ikZ_0\mathbf{J}$. So we can write

$$\mathbf{A}(x) = ikZ_0\int_\Gamma G(x,y)\mathbf{J}(y)dy \tag{B.8}$$

$$\mathbf{E}(x) = ikZ_0\left(\int_\Gamma G(x,y)\mathbf{J}(y)dy + \frac{1}{k^2}\nabla_x\int_\Gamma G(x,y)\mathrm{div}_\Gamma\cdot\mathbf{J}(y)dy\right) \tag{B.9}$$

$$\mathbf{H}(x) = \frac{1}{ikZ_0}\mathbf{rot}\mathbf{E} = \mathbf{rot}\int_\Gamma G(x,y)\mathbf{J}(y)dy \tag{B.10}$$

We write them via the operators

$$\begin{cases}\mathbf{E}(x) = ikZ_0(B-S)\mathbf{J}(x)\\ \mathbf{H}(x) = -Q\mathbf{J}(x)\end{cases} \tag{B.11}$$

where $(B-S)$ and Q are the potentials defined in chapter 1:

$$(B-S)\mathbf{J}(x) = \int_\Gamma G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma\mathbf{J}(y)dy \tag{B.12}$$

$$Q\mathbf{J}(x) = -\mathbf{rot}_x\int_\Gamma G(x,y)\mathbf{J}(y)dy = \int_\Gamma \nabla_x G(x,y)\times\mathbf{J}(y)dy \tag{B.13}$$

By an argument of symmetry, the second case $\mathbf{J} = 0$ derives the same way

$$\mathbf{E}(x) = \mathbf{rot}_x\int_\Gamma G(x,y)\mathbf{M}(y)dy \tag{B.14}$$

$$\mathbf{H}(x) = -ikZ_0^{-1}\left(\int_\Gamma G(x,y)\mathbf{M}(y)dy + \frac{1}{k^2}\nabla_y\int_\Gamma G(x,y)\mathrm{div}_\Gamma\mathbf{M}(y)dy\right) \tag{B.15}$$

Potentials express

$$\begin{cases}\mathbf{E}(x) = Q\mathbf{M}(x)\\ \mathbf{H}(x) = -ikZ_0^{-1}(B-S)\mathbf{M}(x)\end{cases} \tag{B.16}$$

Finally we combine $(B.11)$ and $(B.16)$ to get Stratton-Chu formula:

$$\begin{cases}\mathbf{E}(x) = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M}(x)\\ \mathbf{H}(x) = -Q\mathbf{J}(x) + ikZ_0^{-1}(B-S)\mathbf{M}(x)\end{cases} \tag{B.17}$$

These expressions are valable in external domain Ω^+. The impedance condition associates fields values on a boundary Γ. We have to determine external limit values on Γ. Using the classical jump relations, we can express the respective limiting boundary tangential values of $\mathbf{E}$ and $\mathbf{H}$ by

$$\begin{cases} \mathbf{E}_{tg}^{inc} = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M} + \frac{1}{2}\mathbf{E}_{tg} \\ \mathbf{H}_{tg}^{inc} = -Q\mathbf{J} + ikZ_0^{-1}(B-S)\mathbf{M}(x) + \frac{1}{2}\mathbf{H}_{tg} \end{cases} \tag{B.18}$$

We recall that on a boundary, we have

$$\mathbf{E}_{tg} = \mathbf{n} \times \mathbf{M}$$

$$\mathbf{H}_{tg} = -\mathbf{n} \times \mathbf{J}$$

where the subscript t designates the tangential component $\mathbf{E}_{tg} := \mathbf{n} \times (\mathbf{E} \times \mathbf{n})$ of the respective vector field on Γ.

Thus the electromagnetic field can be expressed in Ω^+ in terms of the equivalent currents and charges by the familiar Stratton-Chu formula

$$\begin{cases} \mathbf{E}_{tg}^{inc} = ikZ_0(B-S)\mathbf{J}(x) - Q\mathbf{M} + \frac{1}{2}\mathbf{n} \times \mathbf{M} \\ \mathbf{H}_{tg}^{inc} = -Q\mathbf{J} + ikZ_0^{-1}(B-S)\mathbf{M}(x) - \frac{1}{2}\mathbf{n} \times \mathbf{J} \end{cases} \tag{B.19}$$

Or in integral form

$$\begin{cases} \mathbf{E}_{tg}^{inc} = & ikZ_0 \int_\Gamma \left(G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{J} \right) dy \\ & + \frac{1}{2}\mathbf{n} \times \mathbf{M} - \int_\Gamma \nabla_x G(x,y) \times \mathbf{M} dy \\ \mathbf{H}_{tg}^{inc} = & -\frac{1}{2}\mathbf{n} \times \mathbf{J} - \int_\Gamma \nabla_x G(x,y) \times \mathbf{J} dy \\ & + ikZ_0^{-1} \int_\Gamma \left(G(x,y)\mathbf{M}(y) + \frac{1}{k^2}\nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{M} \right) dy \end{cases} \tag{B.20}$$

Later, to write variational formulation, we will multiply it by a test function and integrate over the surface Γ.

B.2 Double-integral formulation

We are looking for a variational formulation, for the sake of which we multiply the integral equations (B.20) by test functions and integrate it over the boundary Γ. Note that the integral over Γ must be comprise in the sense of an integral over $\Gamma \backslash \{y\}$. We will keep this notation thereafter. We will use a divergence-conforming functions $\boldsymbol{\psi}$ such that $div(\boldsymbol{\psi})$ is defined on the surface and $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ on Γ. For the sake of simplicity, we index the gradients that are related to the index values. We deduce:

$$\int_\Gamma \mathbf{E}_{tg}^{inc} \cdot \boldsymbol{\psi} dx = \int_\Gamma \left[\frac{1}{2}\mathbf{n} \times \mathbf{M} + ikZ_0 \int_\Gamma G(x,y)\mathbf{J} dy \right.$$
$$\left. + \frac{iZ_0}{k} \int_\Gamma \nabla_x G(x,y)\mathrm{div}_\Gamma \mathbf{J} dy - \int_\Gamma \nabla_x G(x,y) \times \mathbf{M} dy \right] \cdot \boldsymbol{\psi} dx \tag{B.21}$$

We may change this formula thanks to the relations between the operators [VB]

$$\nabla_x G(x,y)\cdot\boldsymbol{\psi}(x) = \mathrm{div}_\Gamma(G\boldsymbol{\psi}) - G\mathrm{div}_\Gamma\boldsymbol{\psi}$$

so that:

$$\iint_{\Gamma\Gamma}\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\nabla_x G(x,y)\cdot\boldsymbol{\psi}(x)dydx \tag{B.22}$$

$$= \iint_{\Gamma\Gamma}\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\mathrm{div}_\Gamma[G(x,y)\boldsymbol{\psi}(x)]dydx - \iint_{\Gamma\Gamma}\frac{Z_0}{k}G(x,y)\mathrm{div}_\Gamma\mathbf{J}(y)\mathrm{div}_\Gamma\boldsymbol{\psi}(x)dydx$$

We are interested in the first term of this equality and replace the integral on Γ by an integral over the volume $\Omega_\varepsilon = \varepsilon\times\Gamma$ based on this surface and height ε. The surface integral is the limit of this integral when $\varepsilon\to 0$:

$$\iint_{\Gamma\Gamma}\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\mathrm{div}_\Gamma[G(x,y)\boldsymbol{\psi}(x)]dydx = \lim_{\varepsilon\to 0}\iint_{\Omega_\varepsilon\Gamma}\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\mathrm{div}_\Gamma[G(v,y)\boldsymbol{\psi}(v)]dydv$$

$$= \int_\Gamma\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\left\{\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\mathrm{div}_\Gamma[G(v,y)\boldsymbol{\psi}(v)]dv\right\}dy \tag{B.23}$$

And using the definition of the divergence, the boundary condition and the invariance of $\boldsymbol{\psi}$ along z and along the depth of the volume Ω_ε:

$$\int_{\Omega_\varepsilon}\mathrm{div}_\Gamma[G(v,y)\boldsymbol{\psi}(v)]dv = -\int_\Gamma G(x,y)\boldsymbol{\psi}(x)\cdot\mathbf{n}dx$$

$$= -\int_{side\ surface}G(x,y)\boldsymbol{\psi}(x)\cdot\mathbf{n}dx = \varepsilon C \tag{B.24}$$

where C is a constant. This integral tends to 0 when ε goes to 0, so the first term on the right in (B.22) is zero:

$$\iint_{\Gamma\Gamma}\frac{Z_0}{k}\mathrm{div}_\Gamma\mathbf{J}(y)\nabla_x G(x,y)dy\cdot\boldsymbol{\psi}(x)dx = -\iint_{\Gamma\Gamma}\frac{Z_0}{k}G(x,y)\mathrm{div}_\Gamma\mathbf{J}(y)\mathrm{div}_\Gamma\boldsymbol{\psi}(x)dydx \tag{B.25}$$

So we replace these integrals in (B.21). Another term combination product will be written as:

$$[\nabla_y G(x,y)\times\mathbf{M}(y)]\cdot\boldsymbol{\psi}(x) = -[\boldsymbol{\psi}(x)\times\mathbf{M}(y)]\cdot\nabla_y G(x,y) \tag{B.26}$$

By injection (B.25) and (B.26) in (B.21), we arrive at:

$$\int_\Gamma\mathbf{E}^{inc}_{tg}\cdot\boldsymbol{\psi}dx = -i\iint_{\Gamma\Gamma}\left[kZ_0G(x,y)\mathbf{J}\cdot\boldsymbol{\psi} - \frac{Z_0}{k}G(x,y)\mathrm{div}_\Gamma\mathbf{J}\mathrm{div}_\Gamma\boldsymbol{\psi}\right]dydx$$

$$+\frac{1}{2}\int_\Gamma\boldsymbol{\psi}\cdot(\mathbf{n}\times\mathbf{M})dx + \iint_{\Gamma\Gamma}(\boldsymbol{\psi}\times\mathbf{M})\cdot\nabla_y G(x,y)dydx \tag{B.27}$$

We develop the second equation of (B.20) in the same way. Finally, we get double integral formulation (1.16)-(1.17):

$$< Z_0(B-S)\mathbf{J},\boldsymbol{\Psi}_J > + < (P+Q)\mathbf{M},\boldsymbol{\Psi}_J > = < IE^{inc},\boldsymbol{\Psi}_J > \tag{B.28}$$

$$- < (P+Q)\mathbf{J},\boldsymbol{\Psi}_M > + < \frac{1}{Z_0}(B-S)\mathbf{M},\boldsymbol{\Psi}_M > = < IH^{inc},\boldsymbol{\Psi}_M > \tag{B.29}$$

with the operators $(B-S)$ and $(P+Q)$ defined in (1.13)-(1.14).

Appendix C

Exact impedance boundary conditions

Here, we derive an exact tensor IBC for one layer of isotropic material, set on a perfect electric conductor (PEC) medium, that is mentioned in [R-Sb](equations (3.19)-(3.20)), by a technique that presented in [BS]. We consider a plane isotropic medium infinite in the x and y directions, the z axis is directed normal to the coating medium and $\mathbf{n}$ is a normal vector. In the thin layer, we represent $E_x(z)$ by its Taylor series expansion around z_0, that is fixed value of z in thin layer. Separating odd and even power terms

$$E_x(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^{2n}}{(2n)!} \partial_z^{2n} E_x(z_0) + \sum_{n=0}^{\infty} \frac{(z - z_0)^{2n+1}}{(2n + 1)!} \partial_z^{2n+1} E_x(z_0). \tag{C.1}$$

The Cartesian components of $\mathbf{E}$ satisfy Helmholtz equation in thin layer. Hence, $\partial_z^2 E_x(z) = -(k^2 + \nabla_{tg}^2)E_x(z)$, and $\partial_z^{2n} E_x(z) = (-1)^n(k^2 + \nabla_{tg}^2)^n E_x(z)$. Consequently

$$\sum_{n=0}^{\infty} \frac{(z - z_0)^{2n}}{(2n)!} \partial_z^{2n} E_x(z_0) = \cos[(z_0 - z)\sqrt{k^2 + \nabla_{tg}^2}]E_x(z_0). \tag{C.2}$$

Also, we have $\partial_z^{2n+1} E_x(z) = (-1)^n(k^2 + \nabla_{tg}^2)^n \partial_z E_x(z)$ and, from Maxwell's equations, $\partial_z E_x(z) = -i(\partial_{yz}^2 H_z(z) - \partial_z^2 H_y(z))/(\omega\varepsilon)$. Because $H_y(z)$ satisfies Helmholtz equation and $\nabla \cdot \mathbf{H} = 0$, we get $\partial E_x(z) = -i(k^2 H_y(z) - \partial_{xy}^2 H_x(z) + \partial_x^2 H_y(z))/(\omega\varepsilon)$. Hence

$$\sum_{n=0}^{\infty} \frac{(z - z_0)^{2n+1}}{(2n + 1)!} \partial_z^{2n+1} E_x(z_0) = \frac{i}{\omega\varepsilon} \frac{\sin[d\sqrt{k^2 + \nabla_{tg}^2}]}{\sqrt{k^2 + \nabla_{tg}^2}} \times \left[k^2 H_y - \partial_{xy}^2 H_x + \partial_x^2 H_y\right](z_0). \tag{C.3}$$

Finally, inserting (C.2) and (C.3) in (C.1), we obtain

$$E_x(z) = \cos[(z_0 - z)\sqrt{k^2 + \nabla_{tg}^2}]E_x(z_0) + \frac{i}{\omega\varepsilon} \frac{\sin[d\sqrt{k^2 + \nabla_{tg}^2}]}{\sqrt{k^2 + \nabla_{tg}^2}} \times \left[k^2 H_y - \partial_{xy}^2 H_x + \partial_x^2 H_y\right](z_0) \tag{C.4}$$

Similarly, we obtain for $E_y(z)$

$$E_y(z) = \cos[(z_0 - z)\sqrt{k^2 + \nabla_{tg}^2}]E_y(z_0) - \frac{i}{\omega\varepsilon} \frac{\sin[d\sqrt{k^2 + \nabla_{tg}^2}]}{\sqrt{k^2 + \nabla_{tg}^2}} \times \left[k^2 H_x - \partial_{xy}^2 H_y + \partial_y^2 H_x\right](z_0). \tag{C.5}$$

That we can write in vector form as

$$\mathbf{E}_{tg}(z) = \cos[(z_0 - z)\sqrt{k^2 + \nabla_{tg}^2}]\mathbf{E}_{tg}(z_0)$$
$$+\frac{i}{\omega\varepsilon}\frac{\sin[d\sqrt{k^2+\nabla_{tg}^2}]}{\sqrt{k^2+\nabla_{tg}^2}}\begin{bmatrix} k^2+\partial_x^2 & \partial_{xy}^2 \\ \partial_{xy}^2 & k^2+\partial_y^2 \end{bmatrix}[\mathbf{n}\times\mathbf{H}](z_0) \tag{C.6}$$

Now, we take $z = -d$ on the interface between thin layer and PEC, and $z_0 = 0$ is on the interface between freespace and dielectric layer. So $\mathbf{E}_{tg}(z) = 0$ and $(z_0 - z) = d$, and we obtain

$$\mathbf{E}_{tg}(0) = -\frac{i}{\omega\varepsilon}\frac{\tan[d\sqrt{k^2+\nabla_{tg}^2}]}{\sqrt{k^2+\nabla_{tg}^2}}\begin{bmatrix} k^2+\partial_x^2 & \partial_{xy}^2 \\ \partial_{xy}^2 & k^2+\partial_y^2 \end{bmatrix}[\mathbf{n}\times\mathbf{H}](0) \tag{C.7}$$

We assume that the plane-wave fields are written in the following forms:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}_1 E_0 e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$$
$$\mathbf{H}(\mathbf{r},t) = \mathbf{e}_2 H_0 e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}$$

where $\mathbf{e}_1,\ \mathbf{e}_2$ are two constant real unit vectors, and $E_0,\ H_0$ are complex amplitudes which are constant in space and time.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-i(k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}})\cdot\mathbf{r}}$$
$$\partial_x\mathbf{E}(\mathbf{r}) = -ik_x\mathbf{E}(\mathbf{r})$$
$$\partial_x^2\mathbf{E}(\mathbf{r}) = -k_x^2\mathbf{E}(\mathbf{r})$$

So we can replace partial derivatives by k_x and k_y components

$$\partial_x = -ik_x,\ and\ \partial_y = -ik_y \tag{C.8}$$

or

$$\partial_x^2 = -k_x^2,\ \partial_{xy}^2 = -k_xk_y\ and\ \partial_y^2 = -k_y^2. \tag{C.9}$$

We obtain

$$\mathbf{E}_{tg}(0) = -i\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{\tan[k_zd]}{kk_z}\right)\begin{bmatrix} k^2-k_x^2 & -k_xk_y \\ -k_xk_y & k^2-k_y^2 \end{bmatrix}[\mathbf{n}\times\mathbf{H}](0) \tag{C.10}$$

where $\nabla_{tg}^2 = \partial_x^2 + \partial_y^2$, $k = \omega\sqrt{\mu\varepsilon}$ and $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$. We should make some calculations

$$\frac{(k_x^2-k^2)}{kk_z} = \frac{(k_x^2-k^2)(k_x^2+k_y^2)}{kk_z(k_x^2+k_y^2)} = \frac{k_x^2(k_x^2+k_y^2-k^2)-k^2k_y^2}{kk_z(k_x^2+k_y^2)} = -\frac{k_x^2k_z^2+k^2k_y^2}{kk_z(k_x^2+k_y^2)}$$

So the following exact impedances are obtained for the dielectric coated conductor

$$Z_{xy}(k_x,k_y) = Z_{yx}(k_x,k_y) = i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_xk_y}{kk_z}\tan[k_zd] \tag{C.11}$$

$$Z_{xx}(k_x, k_y) = -i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_x^2 k_z^2 + k_y^2 k^2}{k k_z(k_x^2 + k_y^2)}\tan[k_z d] \tag{C.12}$$

and

$$Z_{yy}(k_x, k_y) = -i\sqrt{\frac{\mu}{\varepsilon}}\frac{k_y^2 k_z^2 + k_x^2 k^2}{k k_z(k_y^2 + k_x^2)}\tan[k_z d]. \tag{C.13}$$

Z is the impedance tensor of wave numbers k_x, k_y, wave frequency and coating at each point of a surface.

Appendix D

Coefficients calculation MATLAB script

```
function [a0 a1 a2 b1 b2] = coef2()
%---------------------------------------------------------------
%    a0 = ik0 Z0 ethan(0) Leontovich IBC
%
%    TE polarization
%    ik0 Z0 ethan(theta) = (a0 + a1*ksi)/(1 + b1*ksi)
%    ou ksi = -( k0 * sin(theta) )^2
%
%    TM polarization
%    ik0 Z0 ethan(theta) = (a0 + a2*ksi)/(1 + b2*ksi)
%---------------------------------------------------------------

ci = sqrt(-1); %complex number
c = 299792458; %Speed of light m/s
mu0 = 4*pi*1e-7; %Free space permeability
eps0 = 1e-12/(mu0 * c^2); %Free space permittivity
z0=376.73; %Free space impedance

Theta = [pi/6; pi/3]; %angular range
eps = 5; %Relative permittivity
mu = 1; %Relative permeability
f = 0.45; % frequency GHz
d = 0.05; %Layer thickness

k0 = 20.96*f; %Free space wave number
k = sqrt(eps*mu)*k0; %Wave number

%Leontovich impedance constant
a0 = ci*k0*z0*sqrt(mu/eps)*tan(k*d);

%---------------------------------------------------------------
pol = 1; %  TE polarization
Zexct = Zexx2(Theta, eps, mu, d, pol, f);
```

```
ksi = -k0^2*(sin(Theta).^2);

a1b1 = [ksi -ksi.*Zexct]^(-1) *(Zexct - a0);

a1 = a1b1(1);
b1 = a1b1(2);

%--------------------------------------------------------------
pol = 2; %   TM polarization
Zexct = Zexx2(Theta, eps, mu, d, pol, f);

ksi = -k0^2*(sin(Theta).^2);

a1b1 = [ksi -ksi.*Zexct]^(-1) *(Zexct - a0);

a2 = a1b1(1);
b2 = a1b1(2);

%--------------------------------------------------------------
function Z = Zexx2(Theta, eps,mu, d, pol, f)
% Calculation of Z_exact(theta)
%a0 = ci*k0*z0*sqrt(mu/eps)*tan(k*d);

ci =sqrt(-1); %complex number
c = 299792458;
mu0 = 4*pi*1e-7;
eps0 = 1e-12/(mu0 * c^2);
z0 = 376.73;
k0 = 20.96*f;

tankd = tan(k0*sqrt(eps*mu - sin(Theta).^2)*d);

if(pol==1)
    Z = ci*k0*z0*sqrt(eps*mu - sin(Theta).^2).*tankd/eps;

elseif(pol==2)
    Z = ci*z0*mu*tankd./sqrt(eps*mu - sin(Theta).^2)/k0;

else
    Z = 0;
    sprintf('should be pol=1 for TE and pol=2 for TM')
end
```

Appendix E

2D discretization matrices

We introduce basis matrices,

$$E_{ij} = \left\{ \begin{array}{cc} 1 & i = j-1 \\ -1 & i = j \end{array} \right\}$$

$$M_{ij} =< \psi_j, \psi_i >= \{\frac{1}{h_i} \quad i = j\}$$

where $[M]$ is an invertible diagonal matrix.

$$S_{ij} =< d_l\psi_j, \phi_i >= \left\{ \begin{array}{cc} \frac{1}{h_j} & i = j \\ -\frac{1}{h_j} & i = j+1 \end{array} \right\}$$

$$P_{ij} =< d_l\psi_j, \psi_i >= \frac{1}{2} \left\{ \begin{array}{cc} \frac{1}{h_i h_j} & i = j-1 \\ -\frac{1}{h_i h_j} & i = j+1 \end{array} \right\}$$

where basis functions ϕ and ψ are defined earlier in (6.9)-(6.12) in chapter 6.

Note: Here matrices $[M]$, $[S]$ and $[P]$ correspond to matrices $[I2]$, $[D1]$ and $[D3]$ from the chapter 6, respectively.

Now we define matrix $[T]$ that corresponds to matrix $[D5]$ in chapter 6:

$$T_{ij} =< d_l\phi_j, \psi_i >=< \psi_{j-1} - \psi_j, \psi_i >= (ME)_{ij}.$$

And matrices:

$$M^{-1}_{ij} = \{h_i \quad i = j\}$$

$$(M^{-1}P)_{ij} = \frac{1}{2} \left\{ \begin{array}{cc} \frac{1}{h_j} & i = j-1 \\ -\frac{1}{h_j} & i = j+1 \end{array} \right\}$$

We need to find next matrices from (6.8):

$$[D1][I2]^{-1}[D5] \equiv [S][M]^{-1}[M][E] \equiv [S][E]$$

$$\boxed{(SE)_{ij} = \left\{ \begin{array}{cc} \frac{1}{h_i} & i = j-1 \\ -(\frac{1}{h_i} + \frac{1}{h_{i-1}}) & i = j \\ \frac{1}{h_{i-1}} & i = j+1 \end{array} \right\}}$$

$$[D1][I2]^{-1}[D3] \equiv [S][M]^{-1}[P]$$

$$\boxed{(SM^{-1}P)_{ij} = \frac{1}{2}\left\{\begin{array}{cc} \frac{1}{h_i h_j} & i = j-1 \\ -\frac{1}{h_{i-1} h_j} & i = j \\ -\frac{1}{h_i h_j} & i = j+1 \\ \frac{1}{h_{i-1} h_j} & i = j+2 \end{array}\right\}}$$

$$[D3][I2]^{-1}[D5] \equiv [P][M]^{-1}[M][E] \equiv [P][E]$$

$$\boxed{(PE)_{ij} = \frac{1}{2}\left\{\begin{array}{cc} \frac{1}{h_i h_{j-1}} & i = j-2 \\ -\frac{1}{h_i h_j} & i = j-1 \\ -\frac{1}{h_i h_{j-1}} & i = j \\ \frac{1}{h_i h_j} & i = j+1 \end{array}\right\}}$$

$$[D3][I2]^{-1}[D3] \equiv [P][M]^{-1}[P]$$

$$\boxed{(PM^{-1}P)_{ij} = \frac{1}{4}\left\{\begin{array}{cc} \frac{1}{h_i h_j h_{j+1}} & i = j-2 \\ -\frac{1}{h_i h_j}\left(\frac{1}{h_{j-1}} + \frac{1}{h_{j+1}}\right) & i = j \\ \frac{1}{h_i h_{i-1} h_j} & i = j+2 \end{array}\right\}}$$

For IBC2 we have next matrix of a problem:

$$\left[\begin{array}{cccccccc} Z_0[B-S] + \frac{a_0}{2}[I1] & [Q] & \frac{a_1}{2}[D1] & \frac{b_1}{2}[D1] & 0 & 0 & \frac{a_1'}{2}[D1] & \frac{b_1'}{2}[D1] \\ [Q]^T & \frac{1}{Z_0}[B] + \frac{1}{2a_0}[I2] & \frac{a_1}{2a_0}[D3] & \frac{b_1}{2a_0}[D3] & 0 & 0 & \frac{a_1'}{2a_0}[D3] & \frac{b_1'}{2a_0}[D3] \\ -[D5] & 0 & [I2] & 0 & 0 & 0 & 0 & 0 \\ 0 & -[D3] & 0 & [I2] & 0 & 0 & 0 & 0 \\ 0 & 0 & -[D3] & 0 & [I2] & 0 & 0 & 0 \\ 0 & 0 & 0 & -[D3] & 0 & [I2] & 0 & 0 \\ 0 & 0 & 0 & 0 & -[D3] & 0 & [I2] & 0 \\ 0 & 0 & & 0 & 0 & 0-[D3] & 0 & [I2] \end{array}\right] \tag{E.1}$$

with auxiliary unkowns X_2, Y_2, X_3, Y_3, such that

$$< d_l^4 J, \psi_j > = < d_l^3 X_1, \psi_j > = < d_l^2 X_2, \psi_j > = < d_l X_3, \psi_j >$$

We have next equations form

$$< X_3, \psi_j > = < d_l X_2, \psi_j > \Rightarrow \ [I2]\overline{X_3} = [D3]\overline{X_2} \ \Rightarrow \ \overline{X_3} = [I2]^{-1}[D3]\overline{X_2}$$

$$< X_2, \psi_j > = < d_l X_1, \psi_j > \Rightarrow \ [I2]\overline{X_2} = [D3]\overline{X_1} \ \Rightarrow \ \overline{X_2} = [I2]^{-1}[D3]\overline{X_1}$$

$$< X_1, \psi_j > = < d_l J, \psi_j > \Rightarrow \ [I2]\overline{X_1} = [D3]\overline{J} \ \Rightarrow \ \overline{X_1} = [I2]^{-1}[D3]\overline{J}$$

The same equations for Y_3, Y_2, Y_1 and M. Finally, we need to find next matrices

$$[D1]\left([I2]^{-1}[D3]\right)^2 [I2]^{-1}[D5] \equiv [S]\left([M]^{-1}[P]\right)^2 [M]^{-1}[M][E] \equiv [S]\left([M]^{-1}[P]\right)^2 [E]$$

$$(S(M^{-1}P)^2E)_{ij} = \frac{1}{4}\left\{\begin{array}{cc} \frac{1}{h_ih_{i+1}h_{j-1}} & i=j-3 \\ -\frac{1}{h_ih_{i+1}}(\frac{1}{h_j}+\frac{1}{h_{i-1}}) & i=j-2 \\ -\frac{1}{h_i}(\frac{1}{h_ih_{i-1}}+\frac{1}{h_ih_{i+1}}-\frac{1}{h_{i-1}h_{i+1}}) & i=j-1 \\ \frac{1}{h_{i-1}h_{i-1}}(\frac{1}{h_{i-2}}+\frac{1}{h_i})+\frac{1}{h_ih_i}(\frac{1}{h_{i-1}}+\frac{1}{h_{i+1}}) & i=j \\ \frac{1}{h_{i-1}}(\frac{1}{h_{i-1}h_{i-2}}+\frac{1}{h_ih_{i-1}}-\frac{1}{h_ih_{i-2}}) & i=j+1 \\ -\frac{1}{h_{i-1}h_{i-2}}(\frac{1}{h_i}+\frac{1}{h_{j-1}}) & i=j+2 \\ \frac{1}{h_{i-1}h_jh_{j+1}} & i=j+3 \end{array}\right\}$$

$$[D1]\ ([I2]^{-1}[D3])^2\ [I2]^{-1}[D3] \equiv [S]\ ([M]^{-1}[P])^2\ [M]^{-1}[P] \equiv [S]\ ([M]^{-1}[P])^3$$

$$(S(M^{-1}P)^3)_{ij} = \frac{1}{8}\left\{\begin{array}{cc} \frac{1}{h_ih_{i+1}h_jh_{j-1}} & i=j-3 \\ -\frac{1}{h_ih_{i-1}h_jh_{j-1}} & i=j-2 \\ -\frac{1}{h_ih_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j-1}h_{j-2}}) & i=j-1 \\ \frac{1}{h_{i-1}h_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j-1}h_{j-2}}) & i=j \\ \frac{1}{h_ih_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j+1}h_{j+2}}) & i=j+1 \\ -\frac{1}{h_{i-1}h_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j+1}h_{j+2}}) & i=j+2 \\ -\frac{1}{h_ih_jh_{j+1}h_{j+2}} & i=j+3 \\ \frac{1}{h_{i-1}h_jh_{j+1}h_{j+2}} & i=j+4 \end{array}\right\}$$

$$[D3]\ ([I2]^{-1}[D3])^2\ [I2]^{-1}[D5] \equiv [P]\ ([M]^{-1}[P])^2\ [M]^{-1}[M][E] \equiv [P]\ ([M]^{-1}[P])^2\ [E]$$

$$(P(M^{-1}P)^2E)_{ij} = \frac{1}{8}\left\{\begin{array}{cc} \frac{1}{h_ih_{i+1}h_{i+2}h_{j-1}} & i=j-4 \\ -\frac{1}{h_ih_{i+1}h_jh_{j-1}} & i=j-3 \\ -\frac{1}{h_ih_{i+1}}(\frac{1}{h_ih_{i+1}}+\frac{1}{h_{i+1}h_j}+\frac{1}{h_ih_{i-1}}) & i=j-2 \\ \frac{1}{h_ih_{i+1}}(\frac{1}{h_ih_{i+1}}+\frac{1}{h_{i+1}h_{j+1}}+\frac{1}{h_ih_{i-1}}) & i=j-1 \\ \frac{1}{h_ih_{i-1}}(\frac{1}{h_ih_{i+1}}+\frac{1}{h_{i-1}h_{i-2}}+\frac{1}{h_ih_{i-1}}) & i=j \\ -\frac{1}{h_ih_{i-1}}(\frac{1}{h_ih_{i+1}}+\frac{1}{h_{i-1}h_{j-1}}+\frac{1}{h_ih_{i-1}}) & i=j+1 \\ -\frac{1}{h_ih_{i-1}h_jh_{j-1}} & i=j+2 \\ \frac{1}{h_ih_{i-1}h_jh_{j+1}} & i=j+3 \end{array}\right\}$$

$$[D3]\ ([I2]^{-1}[D3])^2\ [I2]^{-1}[D3] \equiv [P]\ ([M]^{-1}[P])^2\ [M]^{-1}[P] \equiv [P]\ ([M]^{-1}[P])^3$$

$$(P(M^{-1}P)^3)_{ij} = \frac{1}{16}\left\{\begin{array}{cc} \frac{1}{h_ih_{i+1}h_jh_{j-1}h_{j-2}} & i=j-4 \\ -\frac{1}{h_ih_{i+1}h_jh_j}(\frac{1}{h_{j-1}}+\frac{1}{h_{j+1}})-\frac{1}{h_ih_jh_{j-1}h_{j-2}}(\frac{1}{h_{i-1}}+\frac{1}{h_{i+1}}) & i=j-2 \\ \frac{1}{h_ih_{i-1}h_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j-1}h_{j-2}}) & \\ +\frac{1}{h_ih_{i+1}h_j}(\frac{1}{h_jh_{j-1}}+\frac{1}{h_jh_{j+1}}+\frac{1}{h_{j+1}h_{j+2}}) & i=j \\ -\frac{1}{h_ih_{i-1}h_jh_j}(\frac{1}{h_{j-1}}+\frac{1}{h_{j+1}})-\frac{1}{h_ih_jh_{j+1}h_{j+2}}(\frac{1}{h_{i-1}}+\frac{1}{h_{i+1}}) & i=j+2 \\ \frac{1}{h_ih_{i-1}h_jh_{j+1}h_{j+2}} & i=j+4 \end{array}\right\}$$

Appendix F
Mie Scattering

The Mie solution to Maxwell's equations (also known as the Lorenzo-Mie solution, the Lorenzo-Mieo-Debye solution or Mie scattering) describes the scattering of electromagnetic radiation by a sphere. The solution takes the form of an analytical infinite series. It is named after Gustav Mie.

F.1 Mie coefficients for coated spheres

Mie coefficients a_n and b_n of coated spheres can to compute cross sections and scattering diagrams. The coated sphere has an inner radius R with size parameter $x = kR$ (k is the wave number in the ambient medium) amd m_1 is the inner-medium refractive index relative to the ambient medium (which is suppose to be PEC), a coating of outer radius $R + d$ (d is the thickness of coated layer) with relative refractive index m_2, and size parameter $y = k(R + d)$.

One form used to compute the Mie coefficients of coated spheres is th following:

$$a_n = \frac{(\tilde{D}_n/m_2 + n/y)\psi_n(y) - \psi_{n-1}(y)}{(\tilde{D}_n/m_2 + n/y)\xi_n(y) - \xi_{n-1}(y)}; \quad b_n = \frac{(m_2\tilde{G}_n + n/y)\psi_n(y) - \psi_{n-1}(y)}{(/m_2\tilde{G}_n + n/y)\xi_n(y) - \xi_{n-1}(y)}$$

$$\tilde{D}_n = \frac{D_n(m_2 y) - A_n\chi'_n(m_2 y)/\psi_n(m_2 y)}{1 - A_n\chi_n(m_2 y)}; \quad \tilde{G}_n = \frac{D_n(m_2 y) - B_n\chi'_n(m_2 y)/\psi_n(m_2 y)}{1 - B_n\chi_n(m_2 y)}$$

$$A_n = \psi_n(m_2 x)\frac{mD_n(m_1 x) - D_n(m_2 x)}{mD_n(m_1 x)\chi_n(m_2 x) - \chi'_n(m_2 x)};$$

$$B_n = \psi_n(m_2 x)\frac{D_n(m_1 x)/m - D_n(m_2 x)}{D_n(m_1 x)\chi_n(m_2 x)/m - \chi'_n(m_2 x)}; \quad m = \frac{m_2}{m_1}$$

F.2 The scattered far field

If the detailed shape of the angular scattering pattern is required, e.g. to get the phase matrix or phase function for radiative-transfer calculations (Chandrasekhar, 1960), the scattering functions S_1 and S_2 are required. These functions describe the scattered field $\mathbf{E}_s$. The scattered far field in spherical coordinates $(E_{s\theta}, E_{s\phi})$ for a unit-amplitude incident field (where the time variation $exp(-i\omega t)$ has been omitted) is given by

$$\begin{aligned} E_{s\theta} &= \frac{e^{ikr}}{-ikr}\cos\phi \cdot S_2(\cos\theta) \\ E_{s\phi} &= \frac{e^{ikr}}{ikr}\sin\phi \cdot S_1(\cos\theta) \end{aligned} \tag{F.1}$$

with the scattering amplitudes S_1 and S_2

$$\begin{aligned} S_1(\cos\theta) &= \textstyle\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}(a_n\pi_n + b_n\tau_n); \\ S_2(\cos\theta) &= \textstyle\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}(a_n\tau_n + b_n\pi_n) \end{aligned} \tag{F.2}$$

$E_{s\theta}$ is the scattered far-field component in the scattering plane, defined by the incident and scattered directions, and $E_{s\phi}$ is the orthogonal component. The angle ϕ is the angle between the incident electric field and the scattering plane. The functions $\pi(\cos\theta)$ and $\tau(\cos\theta)$ describe the angular scattering patterns of the spherical harmonics used to describe S_1 and S_2 and follow from the recurrence relations

$$\pi_n = \frac{2n-1}{n-1}\cos\theta \cdot \pi_{n-1} - \frac{n}{n-1}\pi_{n-2}; \;\; \tau_n = n\cos\theta \cdot \pi_n - (n+1)\pi_{n-1} \tag{F.3}$$

starting with (Deirmendjian, 1969)

$$\pi_0 = 0; \;\; \pi_1 = 1; \;\; \pi_2 = 3\cos\theta; \;\; \tau_0 = 0; \;\; \tau_1 = \cos\theta; \;\; \tau_2 = 3\cos(2\theta)$$

F.3 The exact RCS

The bistatic radar cross section, defined at $\phi = 0$ as (for TE polarization)

$$RCS(\theta) = 10 \;\log\left(\frac{4\pi}{k_0^2}|S_2(\cos\theta)|^2\right)$$

and at $\phi = \pi/2$ as (for TM polarization)

$$RCS(\theta) = 10 \;\log\left(\frac{4\pi}{k_0^2}|S_1(\cos\theta)|^2\right)$$

Bibliography

[BA] A. Bendali, M'B. Fares and Jean Gay. *A Boundary-Element Solution of the Leontovich Problem.* IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 47, no. 10, pages 1597–1605, October 1999. (Cited on pages 66, 73, 74 and 79.)

[BS] Bruno Stupfel and Olivier Marceaux. *High-Order Impedance Boundary Conditions for Multilayer Coated 3-D Objects.* IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 48, no. 3, pages 429–436, MARCH 2000. (Cited on pages 1, 4, 19, 20, 27, 29, 101 and 109.)

[BSa] Bruno Stupfel. *Impedance Boundary Conditions for Finite Planar or Curved Frequency Selective Surfaces Embedded in Dielectric Layers.* IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 53, no. 11, pages 3654–3662, November 2005. (Cited on pages 1, 4 and 7.)

[BSb] Bruno Stupfel and Yannick Pion. *Impedance Boundary Conditions for Finite Planar or Curved Frequency Selective Surfaces.* IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 53, no. 4, pages 1415–1424, April 2005. (Cited on pages 1, 4, 18, 19, 21, 23, 43 and 101.)

[BSc] Bruno Stupfel, Dorcas Poget and Jean Claude Nédélec. *Sufficient uniqueness conditions for the solution of the time harmonic Maxwell's equations associated with surface impedance boundary conditions.* Journal of Computational Physics, vol. 230, no. 12, pages 4571–4587, June 2011. (Cited on pages 1, 4, 24, 25 and 26.)

[CD] Christian Daveau and Jacques Laminie. *Mixed and Hybrid Formulations For The Three-Dimensional Magnetic Problem.* Numerical Methods for Partial Differential Equations, vol. 18, no. 1, pages 85–104, January 2002. (Cited on page 71.)

[FAa] Francesco Andriulli. *Well-Posed Boundary Element Formulations in Electromagnetics.* PhD thesis, Electrical Engineering; The University of Michigan, 2008. (Cited on pages 83, 85, 86, 89 and 90.)

[FAb] Francesco P. Andriulli and Annalisa Buffa and Snorre Christiansen and Kristof Cools and Hakan Bağci gci and Femke Olyslager and Eric Michielssen. *A Multiplicative Calderon Preconditioner for the Electric Field Integral Equation.* IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 56, no. 8, pages 2398–2412, August 2008. (Cited on page 83.)

[FL] Frédérique LE LOUER. *Optimisation de Forme dâAntennes Lentilles Intégrées aux Ondes Millimétriques.* PhD thesis, Mention Mathematiques et

Applications, L'UNIVERSITE DE RENNES 1, September 2009. (Cited on page 7.)

[JCN] Jean Claude Nédélec. Acoustic and electromagnetic equations: Integral representations for harmonic problems, volume 144. Springer édition, 2000. (Cited on pages 16, 17 and 105.)

[JDJ] John David Jackson. Classical electrodynamics. Wiley, first edition édition, 1962. (Cited on page 20.)

[MC] Michel Cessenat. *Mathematical Methods in Electromagnetism: Linear Theory and Applications*. World Scientific, 1996. (Cited on page 12.)

[NP] Jean Claude Nédélec and J. Planchard. *Une méthode variationnelle d'éléments finis pour la résolution numérique d'un problème extérieur dans* $\mathbb{R}^3$. RAIRO, pages 105–129, December 1973. (Cited on page 71.)

[NS] Nicolas Salles. *Calcul des singularités dans les méthodes dâéquations intégrales variationnelles*. PhD thesis, Mathématiques appliquées, Université Paris-Sud 11, September 2013. (Cited on page 93.)

[R-Sa] Yahya Rahmat-Samii and Daniel J.Hoppe. *Scattering by Superquadric Dielectric-Coated Cylinders Using Higher Order Impedance Boundary Conditions*. IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 40, no. 12, pages 1513–1522, December 1992. (Cited on pages 1 and 3.)

[R-Sb] Yahya Rahmat-Samii and Daniel J.Hoppe. Impedance boundary conditions in electromagnetics. Taylor & francis édition, 1995. (Cited on pages 1, 3, 6, 19, 20, 27, 28, 29, 50, 101 and 109.)

[SV] T.B.A. Senior and J.L. Volakis. Approximate boundary conditions in electromagnetics. IEE ELECTROMAGNETIC WAVES SERIES 41. 1995. (Cited on page 6.)

[TE] Isabelle Terrasse. *Résolution mathémathique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés*. PhD thesis, L'école Polytechnique, 1993. (Cited on page 15.)

[TL] Virginie Lange. *Equations intégrales espace-temps pour les équations de Maxwell : calcul du champ diffracté par un obstacle dissipatif*. PhD thesis, Mathématiques appliquées, Bordeaux 1, 1995. (Cited on pages 6, 16, 33, 39, 40, 63, 69 and 105.)

[VB] Jean G. Van Bladel. Electromagnetic fields. Wiley, second édition, 2007. (Cited on pages 65 and 108.)

[YJN] Su Yan, Jian-Ming Jin and Zaiping Nie. *Improving the Accuracy of the Second-Kind Fredholm Integral Equations by Using the Buffa-Christiansen*

Functions. IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, vol. 59, no. 4, pages 1299–1310, April 2011. (Cited on page 84.)

Electromagnetic Scattering Problem with Higher Order Impedance Boundary Conditions and Integral Methods.

Abstract: The main subject of this thesis is to propose a new variational formulation of electromagnetic scattering problem with approximate impedance boundary conditions. We consider a perfect conductor coated with a thin dielectric layer. The impedance operator is approximated as a ratio of polynomials of differential operators, so that the boundary conditions are presented as an equation of these polynomials. We call this condition as higher order IBC (HOIBC). We propose the formulation of the problem, the discretization and the numerical results in two dimensional case. Also we elaborate the formulation and some different methods of discretization for three dimensional case. Finally we presented some numerical results of three dimensional case.

Keywords: Higher-order impedance boundary conditions, scattering problem, integral method, three dimensional method.

Résumé Dans cette thèse, on a étudié le problème de diffraction d'une onde électromagnétique en régime harmonique. On se place dans le cas d'un objet parfaitement conducteur recouvert d'une couche mince de diélectrique. On présente en 2D et 3D une nouvelle formulation intégrale couplée avec des conditions d'impédance d'ordre élevée. On fait l'étude théorique pour l'existence et l'unicité pour chaque formulation. Des résultats numériques sont donnés et valident les méthodes. En 3D une autre approche est proposée en utilisant les fonctions de base de Buffa-Christiansen.

Mots clefs: Conditions aux limites d'impédance d'ordre élevée, problème de diffraction, méthode intégrale, méthode tridimensionnel.