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Optical Systems with Resolving Powers Exceeding the Classical Limit*

W. LUKOSZ†

Institut A für Physik, Technische Hochschule, 33 Braunschweig, Germany

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A new theorem on the ultimate limit of performance of optical systems is established: Not the bandwidth of the transferred spatial frequencies but only the number of degrees of freedom of the optical message transmitted by a given optical system is invariant. It is therefore possible (a) to extend the bandwidth by reducing the object area, (b) to extend the bandwidth in the x direction while proportionally reducing it in the y direction, so that the two-dimensional bandwidth is constant, and (c) to double the bandwidth when transmitting information about one state of polarization only.

To achieve this, the optical systems are modified by inserting two suitable masks (generally gratings) into optically conjugate planes of object and image space. The transfer and spread function of the modified systems are calculated for the case of coherent illumination.

INDEX HEADINGS: Optical systems; Diffraction; Polarization; Resolving power; Coherence.

1. INTRODUCTION

THE limit of resolution of optical systems set by diffraction was, for the case of coherent illumination, first stated by Abbe. The modern and more exact formulation of this classical resolution limit is that optical systems transfer a limited band of spatial frequencies, the bandwidth depending on the angular aperture of the system and the wavelength of the light used.

In this paper the following new theorem is established: For a given optical system not the bandwidth of transferred spatial frequencies but only the number N of degrees of freedom of the optical message transmitted by the system is constant. According to von Laue[1] this number N is given by the product of object area times optical bandwidth, times a factor 2 which is due to the existence of two independent states of polarization, times the number of temporal degrees of freedom.

According to our theorem of the invariance of the number N of degrees of freedom of the message transmitted it must be possible:

(a) to extend the optical bandwidth above the classical value by reducing the object area. These systems are described in Sec. 4.

(b) to extend the bandwidth in the x direction above the classical value while proportionally reducing it in the y direction, so that the two-dimensional bandwidth is constant. In Sec. 5 the necessary modifications of the optical system are described.

(c) to double the classical bandwidth when transmitting information about one state of polarization only. Methods achieving this have been reported by Lohmann *et al.*[2] and are discussed from our point of view in Sec. 6.

(d) to extend the bandwidth of transferred spatial frequencies above its classical value by reducing the bandwidth of transferred temporal frequencies. A system of this type has already been realized by Lukosz and Marchand,[3] though described then in a different way.

All these systems for transferring spatial frequencies with a bandwidth exceeding the classical value or in other words having a resolving power exceeding the classical limit (they may be called "superresolving"

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† Present address: Lehrstuhl B für Theoretische Physik, Technische Hochschule, Braunschweig.

[1] M. von Laue, Ann. Physik 44, 1197 (1914).

[2] A. Lohmann, Opt. Acta 3, 97 (1956); W. Gärtner and A. Lohmann, Z. Physik 174, 18 (1963); A. W. Lohmann and D. P. Paris, Appl. Opt. 3, 1037 (1965).

[3] W. Lukosz, Z. Naturforsch. 18a, 436 (1963); W. Lukosz and M. Marchand, Opt. Acta. 10, 241 (1963).

systems) are shown to have linear and approximately space-invariant imaging properties. They can therefore be described by their spread or transfer functions which are calculated. First, however, we restate in Sec. 2 the results of the spatial-filter theory of optical systems used later and review the concept of the degrees of freedom of an optical wave field or message. A feature common to all the "superresolving" systems is—as we show in Sec. 3—the insertion of suitable masks (generally gratings) into conjugate planes of object and image space.

Although, in this paper, consideration is confined to coherent illumination, the "superresolving" systems described in this paper also work when the object is illuminated by partially coherent or incoherent light.

2. FILTER THEORY OF COHERENT OPTICAL IMAGING

We review the filter theory of optical imaging using coherent light, which essentially is the modern version of Abbe's theory.[4] The theory is based on the assumption that the system has linear and space invariant imaging properties. The condition of linearity is met as a consequence of the linearity of Maxwell's equations of electrodynamics and of the derived scalar wave equation for a field periodic with the frequency ω

$$\Delta U+k^2U=0, \qquad (2.1)$$

where $k^2=\omega^2/c^2$, which serves as the basis of the theory of optical imaging. The condition of space invariance, which is only approximately fulfilled for most optical systems, demands that all points in the object field should be equivalent. This means that the amplitude distribution in the image of a point source (spread function) should not change as the source explores the object field.

The amplitude distribution $A'(x,y)$ in the image produced by a linear and space-invariant system is the convolution of the amplitude distribution $A(x,y)$ of the object and of the spread function $F_A(x,y)$

$$A'(x,y)=\int\int_{-\infty}^{+\infty} A(\bar{x},\bar{y})F_A(x-\bar{x},\, y-\bar{y})d\bar{x}d\bar{y}. \qquad (2.2)$$

To simplify the notation we assume that the magnification is unity, so (x,y) denotes a point in the object plane and its optically conjugate point in the image plane.

The object amplitude $A(x,y)$ is decomposed into its spatial-frequency components

$$A(x,y)=\frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty} a(k_x,k_y) \times\exp[i(k_xx+k_yy)]dk_xdk_y. \qquad (2.3)$$

Here (k_x,k_y) denotes the two-dimensional spatial frequency. The spatial-frequency spectrum $a(k_x,k_y)$ can be calculated if $A(x,y)$ is known, by the use of the inversion formula

$$a(k_x,k_y)=\int\int_{-\infty}^{+\infty} A(x,y)\exp[-i(k_xx+k_yy)]dxdy. \qquad (2.4)$$

According to Eqs. (2.3) and (2.4), $A(x,y)$ and $a(k_x,k_y)$ are Fourier transforms of each other. If $A'(x,y)-a'(k_x,k_y)$ and $F_A(x,y)-f_a(k_x,k_y)$ constitute Fourier-transform pairs defined analogously to $A(x,y)-a(k_x,k_y)$, the Fourier transform of Eq. (2.2) is

$$a'(k_x,k_y)=a(k_x,k_y)f_a(k_x,k_y). \qquad (2.5)$$

Equation (2.5) shows that the linear and space-invariant optical system acts as a filter for spatial frequencies: Each spatial frequency is transferred from the object to the image plane independently of all other frequencies present; its amplitude is multiplied by the transfer function $f_a(k_x,k_y)$ of the system. All the spatial frequencies for which $f_a=0$ do not appear in the image.

Next we give a more physical description of the optical system as a spatial-frequency filter. We start by remembering that the Fourier transform is the mathematical expression for a Fraunhofer diffraction process. The wave equation (2.1) has the solutions

$$U_{\mathbf{k}}(x,y,z)=\text{const}\exp[i(k_xx+k_yy+k_zz)] \qquad (2.6)$$

with $k_x^2+k_y^2+k_z^2=k^2$, which form a complete set of functions. Therefore the wave field $U(x,y,z)$ behind any coherently illuminated object in the plane $z=0$—assumed to be illuminated by a source in the $z<0$ half-space—can be represented by a superposition of solutions (2.6)[5–7]

$$U(x,y,z)=\frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty} u(k_x,k_y) \times\exp[i(k_xx+k_yy+k_zz)]dk_xdk_y \qquad (2.7)$$

with $k_z=+(k^2-k_x^2-k_y^2)^{\frac{1}{2}}$. For $k_x^2+k_y^2\leq k^2$ and therefore real k_z the solutions (2.6) represent plane waves with wave vector $\mathbf{k}(k_x,k_y,k_z)$. Alternatively a plane wave can be described by the angles $(\pi/2-\alpha_x,\, \pi/2-\alpha_y)$

[4] *Die Lehre von der Bildentstehung im Mikroskop von E. Abbe*, bearbeitet und herausgegeben von O. Lummer und F. Reiche (Vieweg, Braunschweig, 1910).

[5] D. Gabor, *Progress in Optics, Vol. I*, E. Wolf, Ed. (North-Holland Publishing Co., Amsterdam, 1961), p. 109.

[6] A. Lohmann and H. Wegener, Z. Physik 143, 431 (1955).

[7] G. Toraldo di Francia, Rev. Opt. 28, 597 (1949).

between $\mathbf{k}$ and the x and y axes

$$k_x = k \sin\alpha_x; \quad k_y = k \sin\alpha_y. \tag{2.8}$$

For $k_x^2 + k_y^2 > k^2$ and therefore imaginary k_z the expansion (2.7) contains "evanescent" waves. Their amplitudes decrease exponentially in the $+z$ direction, so they are practically damped out in a few wavelengths distance behind the object. In the back focal plane of a lens (Fraunhofer plane) that part of the spatial-frequency spectrum $u(k_x,k_y)$ corresponding to plane waves is observed.

Fraunhofer diffraction can also be approximately realized by illuminating the object with a spherical wave, the spatial-frequency spectrum of the object being observed in the plane containing the light source (cf. Born and Wolf[8]), or more precisely on the "Fraunhofer sphere" described about a point in the middle of the object field as center and passing through the point light source.

Now we are ready to discuss the optical systems shown in Figs. 1 and 2 used with plane and spherical waves working as spatial-frequency filters according to Eqs. (2.2) and (2.5). In the telescopic system of Fig. 1 a diaphragm in the Fraunhofer plane having an amplitude transmittance $f(k_x,k_y)$ is the actual filter for the spatial frequencies. The object with transmittance $A(x,y)$ is illuminated by a plane wave with wave vector $\mathbf{k}^s(k_x^s,k_y^s,k_z^s)$. As the position of the object's spatial-frequency spectrum $a(k_x,k_y)$ on the diaphragm depends on the obliquity of illumination, so does the system's transfer function

$$f_a(k_x,k_y) = f(k_x + k_x^s,\, k_y + k_y^s). \tag{2.9}$$

The filtered spectrum $a'(k_x,k_y)$ is transformed by the second lens of the system into a set of plane waves, their interference producing the amplitude distribution in the image plane. This process may be regarded—by an observer behind the image plane—as the virtual

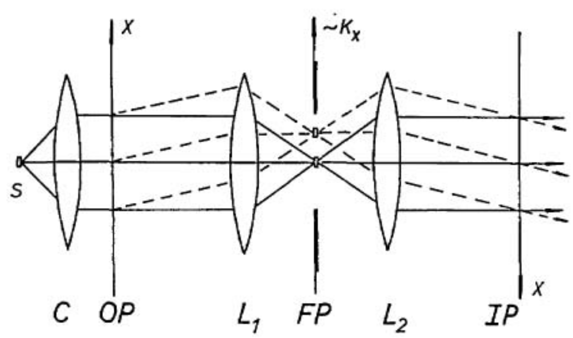

FIG. 1. Telescopic system (of unit magnification) used with plane waves as a spatial-frequency filter. S light source; C condenser; OP, IP object and image plane; L_1, L_2 lenses of equal focal length; FP Fraunhofer plane with filtering diaphragm or aperture stop.

[8] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York and London, 1959), pp. 381, 480.

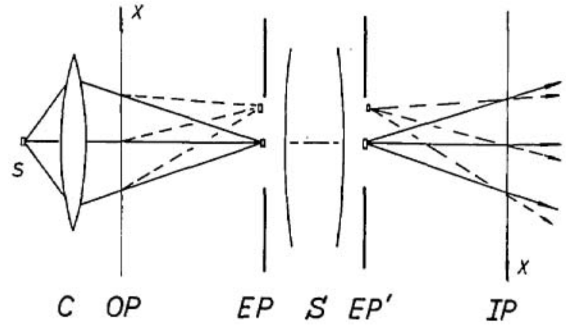

FIG. 2. Arbitrary optical system used with spherical waves as a spatial-frequency filter. S light source; C condenser; OP, IP object and image plane; EP, EP' entrance and exit pupil.

diffraction of the plane wave coming via condenser and optical system from the original light source, through a fictitious film having the amplitude transmittance $A'(x,y)$. So we understand $A'(x,y)$ in Eq. (2.2) to be the transmittance of this fictitious film and not directly the amplitude in the image plane. This has the advantage that the photometric constants and the phase factor depending on the angle of incidence of the illuminating wave are eliminated. Taking the Fourier transform of the transfer function (2.9) we obtain the spread function

$$F_A(x,y) = F(x,y)\exp[i(k_x^s x + k_y^s y)], \tag{2.10}$$

where F is the Fourier transform of f.

In the case of an arbitrary optical system—cf. Fig. 2—the object with transmittance $A(x,y)$ is illuminated by a converging spherical wave so that its spatial-frequency spectrum $a(k_x,k_y)$ is formed on the Fraunhofer sphere in the entrance pupil of the system. Only the part of the spatial-frequency spectrum falling into the entrance pupil is imaged by the system into the exit pupil, all spatial frequencies lying outside the pupil are excluded. The essential assumption, which is closely connected with the condition of space-invariant imaging properties, is: The amplitude at a given point on the Fraunhofer sphere in the exit pupil (described about a point in the middle of the image plane as center, and passing through the center of the exit pupil) is equal to the amplitude at the corresponding point on the Fraunhofer sphere in the entrance pupil multiplied by the pupil function $f(k_x,k_y)$, which outside the pupil is $f=0$ (cf. Ref. 8). From each point on the Fraunhofer sphere in the exit pupil, secondary spherical waves are emitted; their interference produces the amplitude distribution in the image plane. We may regard this process as the virtual diffraction of the diverging spherical wave, coming via condenser and system from the original light source, by a fictitious film having the transmittance $A'(x,y)$. The dependence of the transfer function on the obliquity of illumination is again given by Eq. (2.9), where f is now the pupil function of the system.

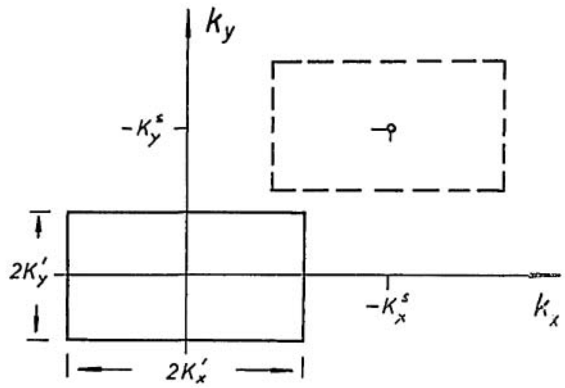

FIG. 3. Spatial frequencies transferred by an optical system with rectangular aperture for (a) central and (b) oblique illumination.

The "ideal" system would have a transfer function $f(k_x,k_y)=1$ for all spatial frequencies. All real optical systems transfer only a limited band of spatial frequencies. For illustration, we consider an aberration-free system with a rectangular aperture; the angular extents of the aperture in the x and y direction are $2\alpha_x'$ and $2\alpha_y'$, respectively. The pupil function of the system is

$$f(k_x,k_y)=\begin{cases}1 & \text{for } |k_x|\leq k_x',\ |k_y|\leq k_y' \\ 0 & \text{elsewhere}\end{cases} \tag{2.11}$$

with $k_x'=k\sin\alpha_x'$; $k_y'=k\sin\alpha_y'$. According to Eq. (2.9) the angle of incidence of the illuminating wave determines which spatial frequencies are passed by the system (cf. Fig. 3)

$$\begin{aligned}-k_x'-k_x^s\leq k_x\leq k_x'-k_x^s;\\ -k_y'-k_y^s\leq k_y\leq k_y'-k_y^s.\end{aligned} \tag{2.12}$$

But the optical bandwidth W—defined as the two-dimensional area of the transferred spatial frequencies—is independent of the obliquity of illumination and in the example considered is equal to $W=k_x'k_y'/\pi^2$.

Next we review the concept of the number of degrees of freedom of a coherent optical wave field, first introduced by von Laue.[1] It is defined as the number of independent real parameters necessary to describe a (classical) wave field completely. We now calculate the number N of degrees of freedom of the wave field in the image space of the system with the rectangular aperture described above, assuming the image area to be rectangular with widths L_x and L_y. To simplify the derivation we use the following conceptual device: Because obviously no information is added we may assume the amplitude distribution to be repeated doubly periodically with L_x and L_y in the x and y directions, respectively. By this procedure, the spatial-frequency spectrum of the amplitude distribution is made discrete; the allowed spatial frequencies are

$$k_x=2\pi n_x/L_x,\quad k_y=2\pi n_y/L_y;\quad n_x,n_y=0,\pm1,\cdots. \tag{2.13}$$

The number of spectra lying in the free aperture of the system is $(1+L_xk_x'/\pi)(1+L_yk_y'/\pi)$. For each of these spectra there are two independent states of polarization—for instance the two perpendicular linear polarization states—and each is a function of time. To obtain the number of degrees of freedom N of the wave field we have to multiply the number of spectra by a factor of 2 due to the two independent states of polarization and by the number N_t of temporal degrees of freedom, where $N_t=2(1+\Delta\nu T)$ if $\Delta\nu$ denotes the bandwidth of temporal frequencies and T the observation time. For a strictly monochromatic wave field $N_t=2$ because both amplitude *and* phase of the wave have to be specified

$$N=2\cdot N_t\cdot(1+L_xk_x'/\pi)(1+L_yk_y'/\pi). \tag{2.14}$$

For $L_xk_x'\gg1$ and $L_yk_x'\gg1$, N becomes approximately

$$N\simeq2\cdot N_t\cdot L_xL_y\cdot k_x'k_y'/\pi^2=2\cdot N_t\cdot S\cdot W, \tag{2.14$'$}$$

where W is the optical bandwidth and $S=L_xL_y$ the image area. More recently Gabor,[5,9] Gamo,[10] MacKay,[11] Toraldo di Francia,[12] Miyamoto[13] (cf. also Brillouin[14]) have shown that the amplitude distribution in a monochromatic image can be specified completely by giving the values of the complex amplitude at certain sampling points. The necessary number of sampling points is of course half the number N of degrees of freedom of the wave field, because it is irrelevant which parameters are chosen to describe the field.

As the wave field in the image space of the system is the carrier of the optical information, we may call N also the number of degrees of freedom of the optical message transmitted by the system. von Laue's theorem says essentially: The number N of degrees of freedom of the message transmitted by a given system is constant *because* the bandwidth of spatial frequencies transferred by the system is constant. The "new theorem" presented in this paper says: Only the number N of degrees of freedom of the optical message transmitted by a given system is invariant; the bandwidth is not constant and can be extended above the classical value by reducing one of the other factors in the formula (2.14) for N.

3. PROBLEM OF EXCEEDING THE CLASSICAL RESOLUTION LIMIT

The problem to be discussed is how to exceed the classical resolution limit, more precisely how to extend the bandwidth of transferred spatial frequencies while maintaining the space-invariant imaging properties of

[9] D. Gabor, in *Astronomical Optics*, Zdenek Kopal, Ed. (North-Holland Publishing Co., Amsterdam, 1956), p. 17.
[10] H. Gamo, in *Progress in Optics*, *Vol. III*, E. Wolf, Ed. North-Holland Publishing Co., Amsterdam, 1964), p. 202.
[11] D. M. MacKay, Inform. Control **1**, 148 (1958).
[12] G. Toraldo di Francia, J. Opt. Soc. Am. **45**, 497 (1955).
[13] K. Miyamoto, J. Opt. Soc. Am. **50**, 856 (1960; **51**, 910 (1961).
[14] L. Brillouin, *Science and Information Theory* (Academic Press Inc., New York, 1956), Ch. 8.

the system, when the aperture of the system and the wavelength of light are given. This can be possible only if an unnecessary supposition is contained in the derivation of the classical value of the optical bandwidth in Sec. 2. The only essential supposition made in addition to the assumed validity of the scalar wave theory is that the object field is illuminated uniformly. This supposition seems to be indispensable for obtaining space-invariant imaging properties, but we see that actually it is not.

When the object is illuminated by one plane (or, in the other case a spherical) wave, the band of transmitted spatial frequencies depends—according to Sec. 2—on the obliquity of the illumination. Therefore, it is tempting to illuminate the object with several coherent waves under different angles of incidence with the hope that we might be able by some means to combine the different bands of transmitted spatial frequencies into one band with a width exceeding the classical limit. Because of the interference of these coherent waves, the illumination of the object plane is nonuniform. This replacement of the uniform illumination of the object by an arbitrary (nonuniform) but also coherent illumination does not alter the number N of degrees of freedom of the wave field in the image space of the system or, in other words, of the optical message transmitted. The invariance of the number of degrees of freedom shows that it is not possible to extend the bandwidth of the system without sacrificing something (usable object field, etc.) for it. The different possibilities were listed in Sec. 1 and are expounded in detail in the following sections. The idea common to all these systems is to insert two masks[15] M and M' into optically conjugate planes of object and image space (cf. Fig. 4). The mask M in object space (a grating, a birefringent prism, etc.) splits an incident wave into several coherent waves illuminating the object under different angles of incidence. (If M is behind the object plane the coherent waves illuminating the object are virtual.) A spatial-frequency component (k_x,k_y) in the object diffracts each of these waves into a new direction. (In Fig. 4 the spatial frequency $k_x=k_y=0$ has been chosen which produces no change of direction of the illuminating waves.) So, in the image space of the system there are several waves

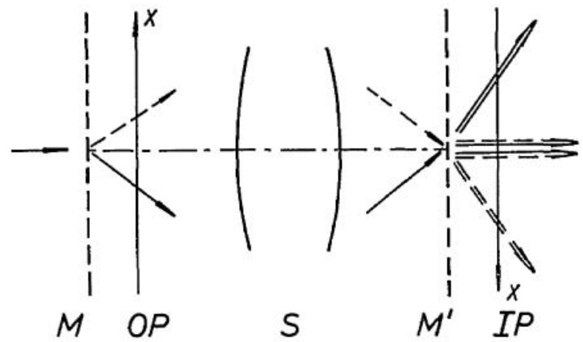

FIG. 4. The essential feature common to all systems with a resolving power exceeding the classical limit is the insertion of two masks M and M' into optically conjugate planes of object and image space. Each of the above masks is assumed to be a grating producing only two diffraction orders. OP, IP object and image plane.

corresponding to the same spatial-frequency component of the object. But in a system with space-invariant imaging properties a spatial-frequency component in the object gives rise to only the same spatial frequency in the image and to no other spatial frequencies. In physical terms this means that there should be only one wave corresponding to one spatial frequency of the object. It is the objective of the mask M' to impart to all the waves in image space corresponding to the same spatial frequency the same direction also. Figure 4 shows that a part of the amplitude of all waves is diffracted into the same direction if M' is a grating with a grating constant optically conjugate to that of M; unfortunately, but necessarily also waves with other "unwanted" directions are produced. To fulfill its objective, the mask M should be able to distinguish between waves corresponding to the same spatial frequency but produced by the different waves illuminating the object. We may say that the problem is how to "earmark" the coherent waves illuminating the object.

4. INCREASE OF RESOLUTION BY REDUCTION OF THE OBJECT FIELD

The principle of the optical arrangement giving increased resolution for a reduced object field is sketched in Fig. 5. Both masks M and M' are gratings; M is inserted into a plane somewhere between object and the optical system, M' into the conjugate plane of image space. The lateral magnification between these conjugate planes must equal the ratio of the grating constants of M and M' (in other words the grating constants of M and M' must be optically conjugate, also). The rulings of the two gratings must also be parallel to each other.

Figure 5 depicts—"anschaulich," but not quite rigorously—how the gain in resolution comes about. The masks are assumed to be line gratings that produce only two diffraction orders. A spherical wave radiated by a luminous point P in the object plane is split by the

[15] If only *one* mask is inserted into object space a moiré pattern produced by object and mask appears in the image plane. The experiments of Blanc-Lapierre *et al.* [A. Blanc-Lapierre, M. Perot, and G. Peri, Compt. Rend. **236**, 1540 (1953)] and Wolter [H. Wolter, Physica **24**, 457 (1958); **26**, 75 (1960); Opt. Acta **7**, 53 (1960)] {cf. also Herzberger [M. Herzberger, Optik **22**, 645 (1965)]} and the similar electron-microscopic investigations of crystal lattices and their defects [J. W. Menter, Advan. Phys. **7**, 299 (1958)] show that it is possible—if some *a priori* knowledge about the object exists—to obtain information about structures "unresolvable" by the same system used with uniform illumination. The reason is, that the moiré pattern contains the spatial difference and sum frequencies of the spatial frequencies of the object and the mask. But this appearance of spatial frequencies not existing in the object shows that the imaging is not space invariant. This makes the evaluation rather complicated [A. W. Lohmann and D. P. Paris, J. Opt. Soc. Am. **55**, 1007 (1965)].

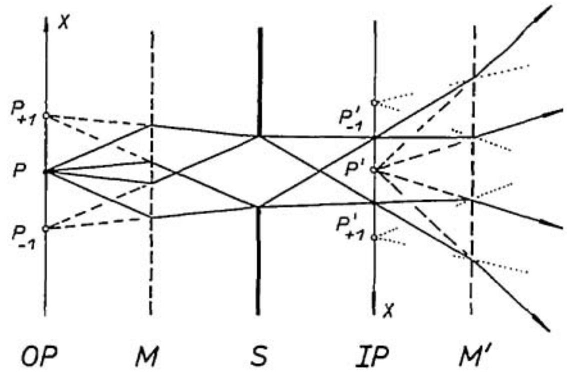

FIG. 5. Principle of optical arrangement increasing the resolution for a reduced field of view. *OP* object plane; *S* optical system with aperture stop; *IP* image plane. The masks *M* and *M'*—in optically conjugate planes of object and image space—are gratings, which in this figure are assumed to produce only two diffraction orders.

mask M into two (approximately) spherical waves virtually originating from the points P_{+1} and P_{-1}, which are laterally shifted from P by $\pm\frac{1}{2}\Delta x$, where

$$\Delta x = 2z_0\lambda/d. \tag{4.1}$$

Here z_0 denotes the distance between the object plane and M, and d the grating constant of M. The points P_{+1} and P_{-1} are imaged by the system into P_{+1}' and P_{-1}', respectively, in the image plane. The spherical waves proceeding from P_{+1}' and P_{-1}' are each split by the diffraction process at M' into two spherical waves. Two of these waves—one actually originating from P_{+1}', the other from P_{-1}'—are virtually issued by the same point P'. The modified optical system images P into P' with an apparent angular aperture in the x direction twice as great as the aperture of the same optical system used in the conventional way without masks. As the bandwidth is proportional to the angular aperture, this means that the bandwidth of the transmitted spatial frequencies is increased by the factor 2 in the k_x direction. The angular aperture in the y direction and consequently the bandwidth in the k_y direction are not changed in the present example. Besides P' there are two additional images of P, P_{+2}' and P_{-2}'. In the case of extended objects, the condition that the additional images do not overlap the central image is that an object amplitude $\neq 0$ exists only in strips of width Δx parallel to the y axis separated by dark strips of equal width. An illuminated object has to be covered by a screen consisting of alternating transparent and opaque strips of width Δx aligned parallel to the lines of the gratings M and M', where Δx is given by Eq. (4.1). This means that the usable object area is reduced to $\frac{1}{2}$ of its original value, in accordance with the theorem of the invariance of the number of degrees of freedom of the optical message transmitted.

How the system performs with linear gratings producing more than two diffraction orders and with crossed gratings follows from the more rigorous theory presented next. To simplify the notation, we treat the case of unit-magnification between object and image space. We want to calculate the pupil function $\tilde{f}(k_x,k_y)$ of the modified system consisting of the optical system and the masks M and M', the pupil function $f(k_x,k_y)$ of the optical system used in the conventional way being given. The spatial-frequency transfer function is derived from the corresponding pupil function according to Eq. (2.9). The masks M and M' are two-dimensional gratings with amplitude transmittances

$$M(x,y) = \sum_{j,l=0,\pm1,\cdots} m_{j,l} \exp[2\pi i(jx/d_x + ly/d_y)], \tag{4.2a}$$

$$M'(x,y) = \sum_{j,l=0,\pm1,\cdots} m_{j,l}' \times \exp[2\pi i(jx/d_x + ly/d_y)]. \tag{4.2b}$$

Both M and M' are doubly periodic with d_x, d_y in the x and y directions, respectively; but the grating form factors of M and M' are allowed to be different. The first step is the evaluation of the effect of the mask M in object space. Let the actual amplitude distribution in the object plane $z=0$ be $U(x,y,z=0)$. Then an observer somewhere behind the mask M in the plane $z=z_0$ sees the apparent amplitude distribution $U_M(x,y,z=0)$ in the object plane. The wave field $U_M(x,y,z)$ for $z=z_0$ is identical with the actually observed wave field, which is the original field $U(x,y,z)$ modified by the presence of the mask. The amplitude directly behind the mask—assumed to be a strictly two-dimensional transparency—is the product of the amplitude in front of the mask times its transmittance; therefore:

$$U_M(x,y,z=z_0) = U(x,y,z=z_0)M(x,y). \tag{4.3}$$

Using plane-wave expansions for the wave fields U and U_M [cf. Eq. (2.7)]

$$U_{(M)}(x,y,z) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty} u_{(M)}(k_x,k_y) \times \exp i(k_x x + k_y y + k_z z)dk_x dk_y,$$

where

$$k_z = +[k^2 - k_x^2 - k_y^2]^{\frac{1}{2}}, \tag{4.4}$$

we obtain from Eq. (4.3) a relation between the spatial-frequency spectra of the apparent and the actual amplitude distribution in the object plane

$$u_M(k_x,k_y) = \sum_{j,l=0,\pm1,\cdots} m_{j,l} u(k_x - 2\pi j/d_x,\ k_y - 2\pi l/d_y) \times \exp iz_0\{[k^2 - (k_x - 2\pi j/d_x)^2 - (k_y - 2\pi l/d_y)^2]^{\frac{1}{2}} - (k^2 - k_x^2 - k_y^2)^{\frac{1}{2}}\}. \tag{4.5}$$

The physical interpretation of Eq. (4.5) is: The plane waves incident on the mask M are diffracted into different orders. Therefore the plane wave with wave vector (k_x,k_y,k_z) observed behind the mask M is a superposition of different plane waves originally having the wave vectors $(k_x',k_y'k_z')$, where $k_x'=k_x-2\pi j/d_x$, $k_y'=k_y-2\pi l/d_y$, $k_z'=[k^2-k_x'^2-k_y'^2]^{\frac{1}{2}}$. The apparent amplitude distribution in the object plane $U_M(x,y,z=0)$ is calculated by taking the Fourier transform of Eq. (4.5). If the angles between the wave vectors and the z axis are small, we may expand the square root in the exponential factor of Eq. (4.5) up to quadratic terms and obtain

$$U_M(x,y,z=0)=\sum_{j,l=0,\pm1,\cdots} m_{j,l}U(x-\Delta x_j,y-\Delta y_l)$$
$$\times\exp i[(x-\Delta x_j/2)(k_x^s)_j+(y-\Delta y_l/2)(k_y^s)_l] \quad (4.6)$$

with $\Delta x_j=-jz_0\lambda/d_x$, $\Delta y_l=-lz_0\lambda/d_y$ and $(k_x^s)_j=2\pi j/d_x=k\sin(\alpha_x^s)_j$, $(k_y^s)_l=2\pi l/d_y=k\sin(\alpha_y^s)_l$. So the apparent object is a superposition of the original object shifted laterally by Δx_j, Δy_l, and illuminated obliquely under the angles of incidence $(\alpha_x^s)_j$, $(\alpha_y^s)_l$, with j, $l=0$, ±1, ±2, $\cdots$. This result justifies the description of the system at the beginning of Sec. 4; moreover it establishes the connection with the considerations of Sec. 3, where it has been stressed that in order to extend the optical bandwidth it is necessary to illuminate the object by several distinguishable waves under different angles of incidence. Here the different illuminating waves produced by the mask M are virtual and may be distinguished by the part of the field from which they seem to be issued, provided that the laterally shifted versions of the original object do not overlap.

Next we consider the imaging by the optical system with the pupil function $f(k_x,k_y)$. The spatial-frequency spectrum of the amplitude distribution in the image plane $z=0$ is

$$u_M'(k_x,k_y)=u_M(k_x,k_y)f(k_x,k_y). \quad (4.7)$$

This image is viewed through the mask M' at the distance z_0' behind the image plane. The effect of M' in the image space is analogous to that of M in object space. The apparent amplitude distribution in the image plane $z=0$ is denoted by $\hat{U}(x,y,z=0)$; its spatial frequency spectrum is denoted by $\hat{u}(k_x,k_y)$ which in analogy to Eq. (4.5) is given by

$$\hat{u}(k_x,k_y)=\sum_{j,l=0,\pm1,\cdots} m_{j,l}'u_M'(k_x-2\pi j/d_x,\, k_y-2\pi l/d_y)$$
$$\times\exp iz_0'\{[k^2-(k_x-2\pi j/d_x)^2-(k_y-2\pi l/d_y)^2]^{\frac{1}{2}}$$
$$-(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\}. \quad (4.8)$$

Combining Eq. (4.7) and (4.8) and assuming that M and M' are in optically conjugate planes of object and image space, we finally obtain

$$\hat{u}(k_x,k_y)=\sum_{p,q=0,\pm1,\cdots} f_{p,q}(k_x,k_y)$$
$$\times u(k_x-2\pi p/d_x,\, k_y-2\pi q/d_y)$$
$$\cdot\exp iz_0\{[k^2-(k_x-2\pi p/d_x)^2-(k_y-2\pi q/d_y)^2]^{\frac{1}{2}}$$
$$-(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\}, \quad (4.9)$$

where

$$f_{p,q}(k_x,k_y)=\sum_{j,l=0,\pm1,\cdots} m_{j,l}'m_{p-j,q-l}$$
$$\times f(k_x-2\pi j/d_x,\, k_y-2\pi l/d_y). \quad (4.10)$$

The term $p=q=0$ corresponds to the central image of the object. The spatial-frequency spectrum of this image is given by

$$\hat{u}(k_x,k_y)=u(k_x,k_y)\hat{f}(k_x,k_y), \quad (4.11)$$

showing that the imaging by the modified optical system is space invariant; the effective pupil function of the modified system is

$$\hat{f}(k_x,k_y)\equiv f_{p=0,q=0}(k_x,k_y)=\sum_{j,l=0,\pm1,\cdots}\hat{m}_{j,l}$$
$$\times f(k_x-2\pi j/d_x,\, k_y-2\pi l/d_y), \quad (4.12)$$

with

$$\hat{m}_{j,l}=m_{j,l}'m_{-j,-l}.$$

If we take the Fourier transforms of both sides of Eq. (4.12), we find the amplitude spread function $\hat{F}(x,y)$ of the modified system to be the product of the spread function $F(x,y)$ of the system used in the conventional way without masks and the cross-correlation function $\hat{M}(x,y)$ of the amplitude transmittances of M and M'

$$\hat{F}(x,y)=F(x,y)\hat{M}(x,y), \quad (4.13)$$

with

$$\hat{M}(x,y)=\frac{1}{d_xd_y}\int_0^{d_y}\int_0^{d_x} M(\bar{x},\bar{y})M'(\bar{x}+x,\,\bar{y}+y)dxdy$$
$$=\sum_{j,l=0,\pm1,\cdots}\hat{m}_{j,l}\exp 2\pi i(jx/d_x+ly/d_y). \quad (4.14)$$

The terms with p and/or $q\neq0$ correspond to the additional laterally displaced images of the object. For the additive contribution of the term p, q to the amplitude distribution in the image plane we obtain, if the square roots in the exponential factor of Eq. (4.9) may be approximated by an expansion up to quadratic terms,

$$\hat{U}_{p,q}(x,y)=\int\int_{-\infty}^{+\infty} U(\bar{x}-\Delta x_p,\,\bar{y}-\Delta y_q)F_{p,q}(x-\bar{x},\,y-\bar{y})$$
$$\times\exp 2\pi i[(\bar{x}-\Delta x_p/2)p/d_x+(\bar{y}-\Delta y_q/2)q/d_y]d\bar{x}d\bar{y}, \quad (4.15)$$

where

$$F_{p,q}(x,y)=F(x,y)M_{p,q}(x,y), \quad (4.16)$$

and

$$M_{p,q}(x,y)=\frac{1}{d_xd_y}\int_0^{d_y}\int_0^{d_x} M(\bar{x},\bar{y})M'(\bar{x}+x,\bar{y}+y) \times\exp[-2\pi i(p\bar{x}/d_x+q\bar{y}/d_y)]d\bar{x}d\bar{y}. \quad (4.17)$$

The essential result is that the lateral shift of the image is Δx_p, Δy_q, given by Eq. (4.6). The spread function $F_{p,q}(x,y)$ describing the imaging of the object into the laterally shifted images depends on p and q.

As an example we again consider the case in which M and M' are line gratings, with grating constant d, that produce only two diffraction orders. We allow for a lateral shift Δ between M and M' in the x direction, perpendicular to the lines of the gratings.

$$M(x,y)=\cos 2\pi x/d;\quad M'(x,y)=\cos 2\pi(x-\Delta)/d. \quad (4.18)$$

The cross-correlation function of M and M' is

$$\hat{M}(x,y)=\tfrac{1}{2}\cos[2\pi(x-\Delta)/d]. \quad (4.19)$$

The effective pupil function of the modified system becomes

$$\hat{f}(k_x,k_y)=\tfrac{1}{4}\exp(-2\pi i\Delta/d)f(k_x-2\pi/d,\, k_y) +\tfrac{1}{4}\exp(2\pi i\Delta/d)f(k_x+2\pi/d,\, k_y) \quad (4.20)$$

and in the special case $\Delta=0$

$$\hat{f}(k_x,k_y)=\tfrac{1}{4}[f(k_x-2\pi/d,\, k_y)+f(k_x+2\pi/d,\, k_y)]. \quad (4.21)$$

This result shows that the bandwidth in the k_x direction can be doubled. That only half the original object area can be used has been explained above. From Eq. (4.12) we infer that with line gratings producing more than two diffraction orders, the bandwidth in the k_x direction can be extended by more than a factor of two, and that with crossed gratings the bandwidth can be extended in both the k_x and the k_y directions. From Eqs. (4.12) and (4.15) it can be proved that an extension of the optical bandwidth by a factor n necessitates the reduction of the usable object field to $1/n$ of its original value.

5. INCREASE OF RESOLUTION IN x DIRECTION BY REDUCTION OF RESOLUTION IN y DIRECTION

We consider objects having a spatial-frequency spectrum whose extension, though wide in the k_x direction, is narrow in the k_y direction so that the bandwidth of the imaging optical system in the k_y direction is not fully exploited. (The extreme cases of such objects are one-dimensional objects.) The principle of the optical arrangement giving increased resolution in the k_x direction while sacrificing the unneeded resolution in the k_y direction is shown in Fig. 6. Both masks M and M' are line gratings; M is inserted into a plane somewhere between the object and the optical system and M' into

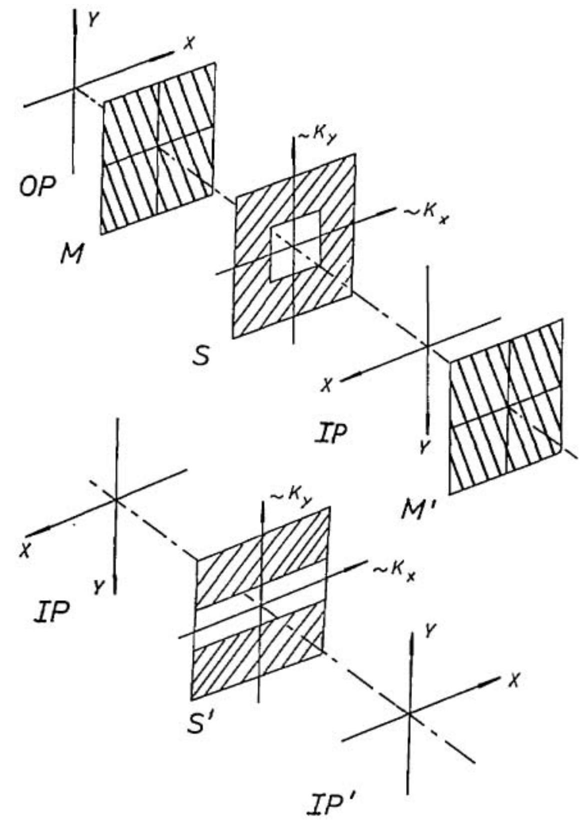

FIG. 6. Principle of an optical arrangement increasing resolution in the k_x direction by sacrificing resolution in the k_y direction. M, M' masks (gratings); OP, IP object and image plane, L (band limited) lens. Below, second imaging stage $IP\rightarrow IP'$, in which all spatial frequencies $|k_y|>\bar{k}_y$ are filtered out by the system S'.

the conjugate plane of image space. The grating constants of M and M' must be optically conjugate, that is, the ratio of the grating constants of M and M' must equal the lateral magnification between the conjugate planes. The lines of M and M' have to be adjusted parallel to each other.

In the following analysis we allow for different grating form factors of M and M' and for a lateral shift Δ in the $\tilde{x}$ direction (perpendicular to the lines of the gratings) between M and M'. If α denotes the angle between the lines of the gratings and the y axis

$$\tilde{x}=x\cos\alpha+y\sin\alpha. \quad (5.1)$$

The amplitude transmittances of M and M' are given by

$$M(x,y)\equiv M(\tilde{x})=\sum_{j=0,\pm1,\cdots} m_j\exp(2\pi ij\tilde{x}/d) \quad (5.2a)$$

$$M'(x,y)\equiv M'(\tilde{x})=\sum_{j=0,\pm1,\cdots} m_j'\exp(2\pi ij\tilde{x}/d). \quad (5.2b)$$

The suitable orientation (angle α) and grating constant d of the masks are specified below, during the development of the theory of the system. There the objective of the subsequent spatial-filtering process—effective in the k_y direction only—also becomes evident. A special case of this optical arrangement, with M and M' directly in the object and image planes, has been used by Grimm and Lohmann[16] for imaging one-dimensional objects with increased resolution.

We want to calculate the pupil function $\hat{f}(k_x,k_y)$ of the modified system consisting of the optical system itself and the masks M and M', the pupil function $f(k_x,k_y)$ of the system used in the conventional way being given. To simplify the notation, we assume unit magnification between object and image space. We denote the actual distribution in the object plane $z=0$ by $U(x,y,z=0)$. Its spatial-frequency spectrum $u(k_x,k_y)$ is assumed to have a width $2\bar{k}_y$ smaller than the bandwidth of the system in the k_y direction

$$u(k_x,k_y)=0 \quad \text{for} \quad |k_y|>\bar{k}_y. \tag{5.3}$$

In particular, we consider optical systems with rectangular apertures of widths $2k_x'$, $2k_y'$ in the k_x and k_y directions, respectively, and we assume $\bar{k}_y=k_y'/N$ where $N=1, 2, \cdots$.

The first step is again the evaluation of the effect of the mask M in object space. The procedure is analogous to that of Sec. 4. The apparent amplitude distribution in the object plane $z=0$ seen by an observer somewhere behind the mask M in the plane $z=z_0$ is denoted by $U_M(x,y,z=0)$. From

$$U_M(x,y,z=z_0)=U(x,y,z=z_0)M(x,y) \tag{5.4}$$

we obtain for the spatial-frequency spectrum of the apparent amplitude distribution

$$\begin{aligned} u_M(k_x,k_y) &= \sum_{j=0,\pm1,\cdots} m_j u(k_x-2\pi_j \cos\alpha/d,\ k_y-2\pi j \sin\alpha/d) \\ &\times \exp iz_0\{[k^2-(k_x-2\pi j\cos\alpha/d)^2-(k_y-2\pi j\sin\alpha/d)^2]^{\frac{1}{2}} \\ &\qquad -(k^2-k_x^2-k_y^2)^{\frac{1}{2}}\}. \end{aligned} \tag{5.5}$$

Equation (5.5) can be interpreted as follows: The mask M produces several coherent waves illuminating the object under different angles of incidence. So the object spectrum appears several times in the aperture plane laterally shifted by $k_x^s=j2\pi\cos\alpha/d$; $k_y^s=j2\pi\sin\alpha/d$; $j=0, \pm1, \ldots$. We choose the grating constant d and the angle α of the mask in such a way that the different spectra do not overlap. Then the spectra produced by the different illuminating waves are distinguishable, and we may say in connection with Sec. 3 that the different illuminating waves are "earmarked" by their wave-vector components k_y. If $\bar{k}_y=k_y'/N$, only $1/N$ of the band width of the system in the k_y direction is exploited. In this case we choose a mask producing (at least) N different coherent illuminating waves. Consequently the object spectrum appears N times in the aperture plane. If the lateral shift between adjacent spectra in the k_y direction is $2\bar{k}_y=2k_y'/N$, N object spectra fall into the free aperture of the system. The bands of spatial frequencies falling into the free aperture are different if the shift between adjacent spectra in the k_x direction is chosen to be $2k_x'$. In this case the angle α obeys the relation

$$\tan\alpha=k_y'/Nk_x'. \tag{5.6}$$

Then the bandwidth of the modified system is increased by a factor N in the k_x direction and reduced by a factor $1/N$ in the k_y direction, so that the two-dimensional bandwidth remains invariant.

The next step in the theory is the imaging by the optical system with the pupil function $f(k_x,k_y)$. The spatial-frequency spectrum of the amplitude distribution $U_M'(x,y,z=0)$ in the image plane $z=0$ is

$$u_M'(k_x,k_y)=u_M(k_x,k_y)f(k_x,k_y). \tag{5.7}$$

This image is viewed through the mask M' in the plane $z=z_0$. The effect of M' is analogous to that of M in object space. The apparent amplitude distribution in the image plane is denoted by $\bar{U}(x,y,z=0)$. From

$$\bar{U}(x,y,z=z_0)=U_M'(x,y,z=z_0)M'(x,y) \tag{5.8}$$

and Eqs. (5.5) and (5.7) we obtain for the spatial-frequency spectrum of the apparent amplitude distribution in the image plane

$$\bar{u}(k_x,k_y)=\sum_{p=0,\pm1,\cdots} f_p(k_x,k_y) \times u\left(k_x-\frac{2\pi p\cos\alpha}{d}, k_y-\frac{2\pi p\sin\alpha}{d}\right), \tag{5.9}$$

where

$$f_p(k_x,k_q)=\sum_{j=0,\pm1,\cdots} m_j' m_{p-j} \times f\left(k_x-\frac{2\pi j\cos\alpha}{d}, k_y-\frac{2\pi j\sin\alpha}{d}\right). \tag{5.10}$$

By a spatial-filtering process, realized for instance by a second imaging system with the pupil function

$$g(k_y)=\begin{cases}1 & \text{for} \quad |k_y| \leq k_y \\ 0 & \phantom{\text{for}} \quad |k_y| > k_y\end{cases}, \tag{5.11}$$

all spatial frequencies $|k_y|>\bar{k}_y$ are eliminated. The effect of this, provided that the different object spectra do not overlap in the aperture plane, is that the terms $p\neq0$ in Eq. (5.9) do not contribute to the spatial-frequency spectrum $\hat{u}(k_x,k_y)$ of the final image:

$$\hat{u}(k_x,k_y)=\hat{u}(k_x,k_y)g(k_y)=u(k_x,k_y)\hat{f}(k_x,k_y), \tag{5.12}$$

[16] M. A. Grimm and A. W. Lohmann, J. Opt. Soc. Am. **55**, 600A (1965); **56**, 1151 (1966).

where

$$\hat{f}(k_x,k_y)=g(k_y)\sum_{j=0,\pm1,\cdots}\hat{m}_j \times f\left(k_x-\frac{2\pi j\cos\alpha}{d},\, k_y-\frac{2\pi j\sin\alpha}{d}\right),$$
$$\text{with}\quad \hat{m}_j\text{-}m_j' m_{-j}. \tag{5.13}$$

This means that the optical system has space-invariant imaging properties; $\hat{f}(k_x,k_y)$ is the effective pupil function. The spread function $\hat{F}(x,y)$ of the modified system is obtained by Fourier transforming Eq. (5.13)

$$\hat{F}(x,y)=\int_{-\infty}^{+\infty}[F(x,\bar{y})\hat{M}(x,\bar{y})]G(y-\bar{y})d\bar{y}, \tag{5.14}$$

where

$$\hat{M}(x,y)\equiv\hat{M}(\bar{x})=\frac{1}{d}\int_0^d M(\bar{x})M'(\bar{x}+\bar{x})d\bar{x}$$
$$=\sum_{j=0,\pm1,\cdots}\hat{m}_j\exp(2\pi ij\bar{x}/d) \tag{5.15}$$

is the cross-correlation function of M and M', and

$$G(y)=(\bar{k}_y/\pi)\sin\bar{k}_y y/\bar{k}_y y \tag{5.16}$$

is the spread function of the spatial-filtering process. We observe that the distance z_0 between the object plane and the mask M does not appear in Eqs. (5.13) and (5.14) for the pupil and spread function of the modified system. This means that the gain in bandwidth (in the k_x direction) is achieved for all object planes before (in the sense of the direction of propagation of light) the mask M. The same result holds true for the systems described in Secs. 4 and 6. The modified systems, image like conventional systems with increased aperture.

6. INCREASE OF RESOLUTION FOR ONE STATE OF POLARIZATION

In the number of degrees of freedom of the message transmitted by an optical system, a factor 2 appeared because of the existence of two independent states of polarization of the light—cf. Eq. (2.14). According to our theorem of the invariance of N it must be possible to double the two-dimensional bandwidth if information about only one state of polarization is transmitted. An optical arrangement for achieving this has been described by Lohmann *et al.*[2] The mask M used consists of a polarizer followed by—in the direction of the propagation of the light—a birefringent prism (for instance a Wollaston prism). This polarizer selects the state of linear polarization for which information is transmitted. No information about the perpendicular direction of polarization is passed by the system. All information is obtained with one setting of the polarizer, if the object is nonbirefringent, that means if it influences both states of polarization in the same way. So the method is especially suited for objects of this type.

We stress that it is not necessary, as described in Ref. 2, to insert the mask into the object plane; the mask can be inserted anywhere between the object and the optical system. Into the optically conjugate plane of image space the mask M', consisting of a similar birefringent prism followed by an analyzer, has to be inserted. The birefringent prisms correspond to line gratings with sinusoidal transmittances, which produce only two diffraction orders. The grating constant d depends on the parameters of the prisms—cf. Ref. 2. (From the point of view of Sec. 3 the first mask M produces virtually two illuminating waves "earmarked" by their orthogonal linear states of polarization.) Therefore the following formulas for the pupil function $\hat{f}(k_x,k_y)$ and the spread function $\hat{F}(x,y)$ of the modified system are identical with Eqs. (4.19) and (4.20)

$$\hat{f}(k_x,k_y)=\tfrac{1}{4}\exp(-2\pi i\Delta/d)f(k_x-2\pi/d,\,k_y)+\tfrac{1}{4}\exp(2\pi i\Delta/d)f(k_x+2\pi/d,\,k_y) \tag{6.1}$$

$$\hat{F}(x,y)=F(x,y)\hat{M}(x,y), \tag{6.2}$$

where $\hat{M}(x,y)=\frac{1}{2}\cos[2\pi(x-\Delta)/d]$. Here $f(k_x,k_y)$ and $F(x,y)$ are again the pupil function and spread function, respectively, of the optical system used in the conventional way. It has been assumed that the orientation of the prism wedges is parallel to the y axis; Δ is the lateral shift between M and M' in the x direction.

Alfred Kastler to Receive Nobel Prize

The news has just been received that Professor Kastler will be awarded the Nobel Prize in Physics for his work on optical pumping which was cited when he was awarded the first C. E. K. Mees International Medal by the Optical Society in 1962. Professor Kastler is Professor of Physics of the Faculty of Sciences of Paris. See the August 1963 issue of the Journal, pages 900–910 for a portrait of Professor Kastler, the citation for the Mees Medal, an outline of his researches, and his lecture at the Rochester meeting of the Society, entitled "Displacement of Energy Levels of Atoms by Light."