IMPEDANCE BOUNDARY CONDITIONS FOR IMPERFECTLY CONDUCTING SURFACES

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Summary

It is shown how the exact electromagnetic boundary conditions at the surface of a material of large refractive index can be approximated to yield the usual impedance or Leontovich boundary conditions. These conditions relate the tangential components of the electric and magnetic fields (or the normal components and their normal derivatives) via a surface impedance which is a function only of the electromagnetic properties of the material. They are valid for surfaces whose radii of curvature are large compared with the penetration depth, and also for materials which are not homogeneous but whose properties vary slowly from point to point. As the refractive index (or conductivity) increases to infinity, the conditions go over uniformly to the conditions for perfect conductivity.

§ 1. *Introduction.* In its most straightforward form an impedance boundary condition is one which relates the tangential components of the electric and magnetic fields via an impedance factor which is a function of the properties of the surface and, possibly, of the field which is incident upon it. The concept of a surface impedance is, of course, not new, and has long been used in a variety of engineering calculations. On the other hand, the idea of incorporating this impedance into the initial formulation of a boundary value problem appears to date only from the beginning of the last war.

During the early 1940's a considerable number of Russian papers were published dealing with various aspects of propagation over the earth, and in these an attempt was made to take into account the properties of actual ground materials by specifying an impedance boundary condition at the surface. This represented a departure from the (then) accepted practice of studying in complete detail certain problems of a very idealized nature, and paved the way for a discussion of propagation over an inhomogeneous, as well as a rough, earth. It was shown that the impedance boundary condition is a valid approximation to the exact condition when the refractive index of the ground is large compared with unity, and the surface impedance can be expressed directly in terms of the electromagnetic properties of the material. These boundary conditions are usually attributed to Leontovich (see, for example, Fock [1])) and were described by Leontovich [2]) himself in 1948. They were first applied to a physical problem by Alpert [3]) in 1940, and were used extensively in Russian work throughout the war. A short summary of their application to propagation problems has been given by Feinberg [4]).

Unfortunately, the proofs associated with these conditions are not readily accessible. Although the conditions are frequently employed in modern electromagnetic theory, it would often appear that either their degree of generality or the restrictions which they require are not fully appreciated. It is the purpose of the present paper to collect in one place some of the proofs associated with these conditions as they apply to the surface of a material of large but finite refractive index. This also serves to provide the necessary background for a subsequent paper in which impedance boundary conditions are developed for a surface which is perfectly conducting but geometrically rough.

In § 2 the exact electromagnetic boundary conditions are briefly discussed. The approximate conditions for a flat interface between a homogeneous isotropic medium and free space are derived in § 3, and the flat interface is generalized to a surface of large radius of curvature in § 4. The necessary modifications when the properties of the medium vary from point to point are given in § 5.

§ 2. *Exact boundary conditions.* At the interface between two homogeneous isotropic media neither of which is perfectly conducting, an electromagnetic field satisfies the boundary conditions

$$[\hat{\boldsymbol{n}} \times \boldsymbol{E}] = 0, \tag{1}$$

$$[\hat{\boldsymbol{n}} \cdot \boldsymbol{D}] = 0, \tag{2}$$

$$[\hat{\boldsymbol{n}} \times \boldsymbol{H}] = 0, \tag{3}$$

$$[\hat{\boldsymbol{n}} \cdot \boldsymbol{B}] = 0, \tag{4}$$

where $\boldsymbol{n}$ is a unit vector normal and the square brackets denote the discontinuities in the corresponding field components on crossing the boundary. In these equations $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field vectors in terms of which $\boldsymbol{B} = \mu \boldsymbol{H}$, where μ is the permeability, and $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, where ε is the complex permittivity *)

$$\varepsilon = \varepsilon' + i \frac{\sigma}{\omega}.$$

A consequence of using a complex (rather than a real) permitivity is that no surface charge distribution appears on the right-hand side of (2).

Equations (1) through (4) are not all independent and therefore constitute a set of boundary conditions at the interface which are more than sufficient. If, for example, the first two are selected, the use of Maxwell's equations shows that (3) and (4) are satisfied automatically. Similarly if the conditions (3) and (4) upon the magnetic field are selected; and indeed, a specification of all the tangential components ($\boldsymbol{E}$ and $\boldsymbol{H}$), or both normal components will suffice. On the other hand, (1) and (4) or (2) and (3) do not constitute sufficient sets since, for example, (1) is not independent of (4).

It should be emphasized that in spite of the so-called "proofs" presented in many textbooks the boundary conditions (1) through (4) cannot be verified by experiments carried out in a homogeneous medium, nor is the author aware of any method by which they can be deduced from Maxwell's equations. In consequence, it appears necessary to regard them as an essential postulate of electromagnetic theory, and the consequent agreement between theory and experiment then provides the evidence in favour of their validity.

It will be observed that the boundary conditions relate the field in the first medium (which we shall henceforth regard as free space) to that in the second medium, and in practice are not always easy to apply in the solution of problems. When the second medium is perfectly conducting, however, the fields therein are identically zero and the only fields to be considered are those in free space. In this case (1) and (4) reduce to

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0, \tag{5}$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0, \tag{6}$$

*) A time variation $e^{i\omega t}$ is assumed.

but (2) and (3) are replaced by

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{D} = \delta, \tag{7}$$

$$\hat{\boldsymbol{n}} \times \boldsymbol{H} = \boldsymbol{K}, \tag{8}$$

where δ and $\boldsymbol{K}$ are surface distributions of charge and current respectively. Since these are known only when the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ have been determined, (7) and (8) do not represent boundary conditions in the usual sense, and we are therefore left with (5) and (6) from which to determine the fields in free space. On the other hand, a further degeneracy now appears and whereas two conditions were required when the medium was not perfectly conducting, a single equation now suffices. Thus, for example, (5) alone *) specifies the fields at all points, and (6), (7) and (8) can all be deduced therefrom.

When the refractive index N of the second medium relative to free space is large compared with unity, boundary conditions can be derived which are analogous to (5) and (6) in that the only fields which appear are those in free space (medium 1). This permits a considerable simplification in the analysis of any scattering or diffraction problem involving bodies which are not perfectly conducting, since it avoids the need to calculate the fields within the body. These new conditions are an approximation to (1) through (4), and their derivation is based on the neglect of terms $O(1/N^2)$ in comparison with unity. We shall first obtain the conditions for an infinite flat interface and later generalize the results so as to apply to a more practical set of circumstances.

§ 3. *Approximate boundary conditions for a flat interface.* Consider a homogeneous isotropic medium whose permittivity, permeability and conductivity are ε', μ and σ respectively. It is assumed that this medium occupies the region $z < 0$ of a Cartesian coordinate system (x, y, z). The halfspace $z > 0$ is free space, the permittivity and permeability of which are ε_0 and μ_0.

Relative to free space the complex refractive index of the medium is

$$N = \sqrt{\frac{\mu}{\mu_0}\left(\frac{\varepsilon'}{\varepsilon_0} + i\,\frac{\sigma}{\omega\varepsilon_0}\right)},$$

*) Although a radiation condition (or its equivalent) must also be imposed if the region is infinite in extent, and an edge condition if this is appropriate.

and boundary conditions at the interface $z = 0$ will now be derived under the assumption that $|N|$ is large compared with unity. It will be appreciated that this requirement is satisfied by a material whose dielectric constant $\varepsilon'/\varepsilon_0$ is large, as well as by a material of high conductivity. For the purposes of the analysis it is convenient to introduce the parameter η defined by $\eta = \mu/(\mu_0 N)$; thus

$$\eta = \frac{1}{\sqrt{\frac{\mu_0}{\mu}\left(\frac{\varepsilon'}{\varepsilon_0} + i\,\frac{\sigma}{\omega\varepsilon_0}\right)}} \tag{9}$$

and is zero for perfect conductivity.

Let us denote by $(\boldsymbol{E}, \boldsymbol{H})$ the electromagnetic field in $z > 0$, and by $(\boldsymbol{E}', \boldsymbol{H}')$ the field in $z < 0$. From the divergence condition we have

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{10}$$

and similarly

$$\frac{\partial E_x'}{\partial x} + \frac{\partial E_y'}{\partial y} + \frac{\partial E_z'}{\partial z} = 0. \tag{11}$$

At the interface $z = 0$ the tangential components of the electric field are continuous, so that

$$E_x = E_x', \quad E_y = E_y'$$

and hence, by tangential differentiation,

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x'}{\partial x}, \quad \frac{\partial E_y}{\partial y} = \frac{\partial E_y'}{\partial y}.$$

Eq. (10) and (11) then give

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z'}{\partial z}. \tag{12}$$

In the medium, however,

$$\frac{\partial^2 E_z'}{\partial x^2} + \frac{\partial^2 E_z'}{\partial y^2} + \frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' = 0 \tag{13}$$

where k is the propagation constant in free space. If $|N| \gg 1$, the field is rapidly varying in the z direction, leading to a large value of

$\partial^2 E_z'/\partial z^2$, and by comparison with this the x and y derivatives are small. This fact is, perhaps, most clearly seen by considering a plane wave incident on the boundary from the direction of free space. Because of the large value of $|N|$, application of Snell's law shows that the transmitted field is deflected toward the normal. For a fixed direction of incidence, the angle between the direction of the transmitted field and the normal is $O(1/|N|)$, which implies that $\partial^2 E_z'/\partial x^2$ and $\partial^2 E_z'/\partial y^2$ are smaller than $\partial^2 E_z'/\partial z^2$ by a factor of order $|N|^2$. Accordingly, in (13) the first two derivatives can be neglected in comparisons with the third, and the equation then becomes

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2 N^2 E_z' = 0. \tag{14}$$

The solution of this is

$$E_z' = A e^{ikNz} + B e^{-ikNz}, \tag{15}$$

where A and B are constants as yet undetermined. If N is defined to have positive imaginary part, the fact that the medium is infinite in extent implies that A must be zero, since the field E_z' must correspond to propagation in the negative z direction. Hence

$$E_z' = B e^{-ikNz} \tag{16}$$

from which we obtain

$$\frac{\partial E_z'}{\partial z} = -ikNE_z'. \tag{17}$$

But from (2)

$$E_z' = \frac{\varepsilon_0}{\varepsilon} E_z \tag{18}$$

at the interface, and this can be combined with equation (17) to give

$$\frac{\partial E_z'}{\partial z} - = ikN \frac{\varepsilon_0}{\varepsilon} E_z = -ik\eta E_z \tag{19}$$

at $z = 0$. Using (12) we now have

$$\frac{\partial E_z}{\partial z} = -ik\eta E_z, \tag{20}$$

and this is one of the required boundary conditions at the interface. Eq. (20) is accurate to the first order in η.

A similar analysis can be developed for the normal component of the magnetic field. From the divergence condition we obtain

$$\frac{\partial H_z}{\partial z} = \frac{\partial H_z'}{\partial z},$$

(cf (12)) and since we also have

$$\frac{\partial H_z'}{\partial z} = -ikNH_z',$$

(cf (17)) it follows that

$$\frac{\partial H_z}{\partial z} = -ikNH_z'.$$

But at the boundary $z = 0$

$$H_z' = \frac{\mu_0}{\mu} H_z$$

and hence

$$\frac{\partial H_z}{\partial z} = -\frac{ik}{\eta} H_z. \tag{21}$$

This is the second of two boundary conditions at the interface, and is accurate to order η.

It will be observed that (21) differs from (20) in having η replaced by $1/\eta$, and this is in accordance with the interpretation of η as an impedance associated with the surface. The point will be elaborated upon in a moment, but for the time being it is sufficient to note that (20) and (21) specify the behaviour of the normal components of both $\boldsymbol{E}$ and $\boldsymbol{H}$ at the interface, and therefore represent a sufficient set of boundary conditions.

For some applications an alternative (but entirely equivalent) representation of these boundary conditions proves more convenient. Taking first (20), since

$$\boldsymbol{E} = -\frac{Z}{ik} \nabla \times \boldsymbol{H}$$

where $Z = 1/Y = \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of free space, and since $\nabla \cdot \boldsymbol{E} = 0$, the boundary condition can be written as

$$\frac{\partial}{\partial x}(E_x + \eta Z H_y) = -\frac{\partial}{\partial y}(E_y - \eta Z H_x). \tag{22}$$

Similarly, the boundary condition (21) gives

$$\frac{\partial}{\partial y}(E_x + \eta Z H_y) = \frac{\partial}{\partial x}(E_y - \eta Z H_x) \tag{23}$$

and by eliminating $E_x + \eta Z H_y$ and $E_y - \eta Z H_x$ successively between these equations, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{24}$$

where $\Phi = E_x + \eta Z H_y$ or $E_y - \eta Z H_x$. This equation can be solved by assuming a separable form for Φ. If $\Phi = \Phi_1(x)\Phi_2(y)$, then

$$\frac{\partial^2 \Phi_1}{\partial x^2} + a^2 \Phi_1 = 0, \quad \frac{\partial^2 \Phi_2}{\partial y^2} - a^2 \Phi_2 = 0,$$

where a^2 is some separation constant, and the solutions are

$$\Phi_1 = A_1 e^{iax} + B_1 e^{-iax},$$

$$\Phi_2 = A_2 e^{ay} + B_2 e^{-ay}$$

where A_1, B_1, A_2 and B_2 are constants as yet undefined. If a is not purely real, both A_1 and B_1 must be identically zero since otherwise Φ_1 would become exponentially large for large x (either positive or negative). In this case Φ_1, and hence Φ, is zero. If a is purely real, the same argument applied to the variable y shows that Φ_2 is zero, leading to the same conclusion as regards Φ. Since Φ is therefore zero,

$$E_x = -\eta Z H_y, \tag{25}$$

$$E_y = \eta Z H_x, \tag{26}$$

and this is the alternative statement of the boundary conditions at the interface $z = 0$. In this form the conditions simply state that ηZ is the effective impedance of the surface as seen by a field in free space. For comparison with this, the impedance of a perfectly conducting surface is zero.

§ 4. *Extension to a curved interface.* In order to generalize these conditions for application to surfaces which are not flat, it is first necessary to express (20) and (21), (25) and (26) in forms which do not explicitly involve the coordinate system. If E_n and H_n are the field components normal to the boundary, and if n is a coordinate whose positive direction is outwards as regards the medium, (20) and (21) can be written as

$$\frac{\partial E_n}{\partial n} = -ik\eta E_n, \tag{27}$$

$$\frac{\partial H_n}{\partial n} = -\frac{ik}{\eta} H_n. \tag{28}$$

For the second pair of conditions a vector form is more convenient, and following Leontovich [2]), (25) and (26) are combined to give

$$\boldsymbol{E} - (\hat{\boldsymbol{n}} \cdot \boldsymbol{E})\hat{\boldsymbol{n}} = \eta Z \hat{\boldsymbol{n}} \times \boldsymbol{H}. \tag{29}$$

Of the three scalar equations contained herein, only two are independent.

We now turn to a consideration of the boundary conditions at a curved interface between the medium and free space. As in the case of the flat interface the object is to determine approximate boundary conditions in which only the fields in free space appear. It is clear, however, that unless restrictions are placed upon the shape of the boundary, these conditions will involve the geometrical properties of the surface as well as the electrical parameters of the medium and in consequence may vary from point to point on the surface. Such conditions would be of little practical value. On the other hand, by restricting the type of surface to be allowed, the curvature effects can be made negligible, and the boundary conditions then reduce to those obtained for an infinite flat surface.

A rigorous derivation of the restrictions which must be placed on the type of surface in order that (27) through (29) be valid is beyond the scope of this paper, and for details of the analysis reference is made to Rytov [5]) and Leontovich [2]). The actual limitations, however, can be arrived at by a semi-intuitive argument.

It will be recalled that in the analysis of the flat boundary the assumption was made that

$$|N| \gg 1, \tag{30}$$

and this is sufficient to ensure that within the medium the field is slowly varying along the surface and behaves essentially as a plane wave propagating in the direction of the inward normal. Let us now seek to apply (29) or (27) and (28) to each point on a curved surface.

In order that the field shall vary little within a wavelength along the surface, a restriction must be placed upon the radii of curvature, and a trivial analysis shows that the requirement is

$$|N|\, k\rho \gg 1, \tag{31}$$

where ρ is the smallest radius of curvature at the point in question. If (31) is satisfied, any correction to the boundary condition (29) consequent upon the curvature is negligible (see Leontovich [2])).

For a surface which is open (implying that the medium is infinite in extent) and which has no inward normal intersecting the surface in a second point, the restrictions (30) and (31) are sufficient to justify the application of the flat surface conditions. For a closed surface, however, a difficulty arises when the conduction current in the medium is negligible compared with the displacement current. The inward travelling field then suffers little or no attenuation, and accordingly may appear as an outward travelling field on the farther side of the surface. This is contrary to the assumption made in the derivation of the flat surface condition. For this reason it is necessary for the field within the medium to be attenuated at a rate such that the penetration depth δ is small compared with ρ, giving rise to the additional restriction

$$\delta \ll \rho. \tag{32}$$

If $\sigma \gg \omega\varepsilon'$, (32) can be written as

$$\sqrt{\frac{\mu}{\mu_0}\,\frac{\sigma}{2\omega\varepsilon_0}}\; k\rho \gg 1,$$

which in turn reduces to the inequality (31) if the conduction current dominates. On the other hand, if the displacement current dominates, the inequality (32) represents an additional restriction which is stronger than (31).

The difficulty which arises with a dielectric medium has been noted by Leontovich [2]), who also points out that for a body made of this material the boundary condition (29) can be justified only under very restricted circumstances. For a body of general shape the boundary conditions are only applicable if the medium is conducting and satisfies the inequality (32). The importance of this restriction, rather than (31), can be seen from a study of the few exact solutions which are known for bodies which are not perfectly conducting. For example, if a plane wave is incident on a sphere of radius ρ, the exact solution can be found as a sum of vector wave functions whose coefficients are functions of N. If it is now assumed that $|N|k\rho \gg 1$, these coefficients reduce to the forms which would have been obtained by using the condition (29) apart from additional terms involving $\tan Nk\rho$. Such terms only disappear if $\tan Nk\rho$ can be replaced by $-i$ to the leading order in N, i.e. if $|\mathrm{Im}\, N|k\rho \gg 1$. Similarly, if a field is incident upon an infinite slab of (uniform) thickness d, the exact solution contains an exponential factor e^{2iNkd} corresponding to internal reflection from the lower surface, and the approximate boundary conditions would then be valid only if the terms containing this factor can be taken zero. This in turn requires an attenuation of the inward travelling field subject to a restriction of the form (32) with ρ replaced by d. It is of interest to note that (32) is here required even though the surface is flat.

In summary, we now have that for a homogeneous isotropic body whose refractive index N and smallest radius of curvature or dimension ρ are such that

$$|N| \gg 1, \tag{30}$$

$$|\mathrm{Im}\, N|k\rho \gg 1, \tag{33}$$

the boundary conditions at its surface can be written as

$$\frac{\partial E_n}{\partial n} = -ik\eta E_n, \tag{27}$$

$$\frac{\partial H_n}{\partial n} = -\frac{ik}{\eta} H_n, \tag{28}$$

where $\eta = \mu/(\mu_0 N)$. These are equivalent to the single vector condition

$$\boldsymbol{E} - (\hat{\boldsymbol{n}} \cdot \boldsymbol{E})\hat{\boldsymbol{n}} = \eta Z \hat{\boldsymbol{n}} \times \boldsymbol{H}. \tag{29}$$

In some circumstances it may be possible to replace (33) by the weaker restriction

$$|N|\, k\rho \gg 1, \tag{31}$$

but such cases must be regarded as exceptional. In this connection it is of interest to note that Fock [1]) in his discussion of these boundary conditions ignores the distinction between dielectric and conducting media, and gives only the restrictions (30) and (31).

Eq. (27) through (29) are approximations to the exact boundary conditions correct to the first order in η, and accordingly in any solution obtained using these conditions there is no (physical) justification for retaining terms which are of a higher order in η. A consequence of this is that if the fields are capable of expansion in series of ascending (positive) powers of η, the perfectly conducting approximation (corresponding to $\eta = 0$) can be inserted into the right-hand sides of (27) and (29) and into the left-hand side of (28). In general, such expansions will be valid, though a problem in which this is not true is the incidence of an H-polarized plane wave on an imperfectly conducting half-plane (Senior [6])[7]). In this case, however, the failure may well be due to the additional assumption of a "thin" body *) implicit in the problem.

§ 5. *An inhomogeneous medium.* Let us now go on to consider the problem in which the medium is not homogeneous, so that the refractive index varies from point to point. This variation will be attributed to ε alone and μ will be regarded as spatially invariant. We shall again begin by assuming an infinite flat interface between the medium and free space.

At any point (x, y, z) within the medium

$$\nabla \cdot (\varepsilon \boldsymbol{E}') = 0, \tag{34}$$

and since

$$\nabla \cdot (\varepsilon \boldsymbol{E}') = \varepsilon \nabla \cdot \boldsymbol{E}' + \boldsymbol{E}' \cdot \nabla \varepsilon,$$

(34) can be written as

$$\frac{\partial E_x'}{\partial x} + \frac{\partial E_y'}{\partial y} = -\frac{\partial E_z'}{\partial z} - \frac{1}{\varepsilon} \boldsymbol{E}' \cdot \nabla \varepsilon. \tag{35}$$

In free space, however,

$$\nabla \cdot \boldsymbol{E} = 0, \tag{36}$$

and using the continuity of the tangential components across the interface, we now have

$$\frac{\partial E_z}{\partial z} = -\frac{\partial E_x'}{\partial x} - \frac{\partial E_y'}{\partial y} = \frac{\partial E_z'}{\partial z} + \frac{1}{\varepsilon} \boldsymbol{E}' \cdot \nabla \varepsilon \tag{37}$$

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*) The mathematical requirement here is $d \ll \lambda$, where d is the thickness of the half-plane, and by assuming that the half-plane is tipped with a semi-circular cylinder it can be shown that the boundary conditions are applicable if $\delta \ll d \ll \lambda$.

at $z = 0$. In terms of η, however, $\varepsilon = (\mu/\mu_0)(\varepsilon_0/\eta^2)$, so that $(1/\varepsilon)\nabla\varepsilon = (-2/\eta)\nabla\eta$, and this can be inserted into (37) to give

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z'}{\partial z} - \frac{2}{\eta}\left(E_x' \frac{\partial\eta}{\partial x} + E_y' \frac{\partial\eta}{\partial y} + E_z' \frac{\partial\eta}{\partial z}\right) \tag{38}$$

at $z = 0$. But at the interface

$$E_x' = E_x, \quad E_y' = E_y, \quad E_z' = (\mu_0/\mu)\eta^2 E_z,$$

and using these relations (38) becomes

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z'}{\partial z} - \frac{2}{\eta}\left(E_x \frac{\partial\eta}{\partial x} + E_y \frac{\partial\eta}{\partial y} + \frac{\mu_0}{\mu}\eta^2 E_z \frac{\partial\eta}{\partial z}\right). \tag{39}$$

In arriving at this equation no approximations have been made, and the second term on the right-hand side can be interpreted as a correction to the boundary condition resulting from the variation of η throughout the medium. If

$$\left|\frac{1}{k\eta}\nabla\eta\right| \ll 1, \tag{40}$$

which implies that the relative variation is small, E_x, E_y and E_z will not differ substantially from the values appropriate to a homogeneous medium. For such a medium it was shown in § 3 that

$$E_x, E_y = O(\eta), \quad E_z = O(1),$$

and in (39) it is now seen that the lateral variation of η is more important than the normal variation. Indeed, if $\partial\eta/\partial x$, $\partial\eta/\partial y$ and $\partial\eta/\partial z$ are all comparable with one another, the effect produced by the z variation of η is smaller by an order of magnitude. The z variation can therefore be neglected and henceforth η will be assumed to be a function of x and y only.

The next step is to obtain an expression for $\partial E_z'/\partial z$ in terms of the free space field. From the field equations

$$\boldsymbol{E}' = -\frac{\sqrt{\mu_0\varepsilon_0}}{ik\varepsilon}\nabla \times \boldsymbol{H}', \tag{41}$$

$$\boldsymbol{H}' = \frac{\sqrt{\mu_0\varepsilon_0}}{ik\mu}\nabla \times \boldsymbol{E}' \tag{42}$$

we have

$$\boldsymbol{E}' = \frac{1}{k^2N^2}\nabla \times \nabla \times \boldsymbol{E}', \tag{43}$$

and since

$$\nabla \times \nabla \times \boldsymbol{E}' = \nabla(\nabla \cdot \boldsymbol{E}') - \nabla^2 \boldsymbol{E}'$$
$$= -\nabla\left(\frac{1}{\varepsilon}\boldsymbol{E}' \cdot \nabla\varepsilon\right) - \nabla^2\boldsymbol{E}',$$

the equation for the field within the medium can be written

$$\nabla^2\boldsymbol{E}' + k^2N^2\boldsymbol{E}' + \nabla\left(\frac{1}{\varepsilon}\boldsymbol{E}' \cdot \nabla\varepsilon\right) = 0. \tag{44}$$

If ε is now expressed in terms of the refractive index N using

$$\varepsilon = \frac{\mu_0}{\mu}\varepsilon_0 N^2,$$

(44) becomes

$$\nabla^2\boldsymbol{E}' + k^2N^2\boldsymbol{E}' + 2\nabla\left(\frac{1}{N}\boldsymbol{E}' \cdot \nabla N\right) = 0, \tag{45}$$

and since the tangential derivatives of $\boldsymbol{E}'$ are again negligible in comparison with the normal derivative, (45) reduces to

$$\frac{\partial^2\boldsymbol{E}'}{\partial z^2} + k^2N^2\boldsymbol{E}' + 2\nabla\left(\frac{1}{N}\boldsymbol{E}' \cdot \nabla N\right) = 0. \tag{46}$$

In particular,

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2N^2E_z' + 2\frac{\partial}{\partial z}\left[\frac{1}{N}\left(E_x'\frac{\partial N}{\partial x} + E_y'\frac{\partial N}{\partial y}\right)\right] = 0. \tag{47}$$

In order to determine E_z' from (47) it is necessary to know the variation of E_x' and E_y' in the z direction, and for this purpose the x and y components of (46) are employed. To the first order the variation of N can be neglected, and we then have

$$\frac{\partial^2 E_x'}{\partial z^2} + k^2N^2E_x' = 0,$$

the solution of which is

$$E_x' = (E_x')_{z=0}e^{-ikNz}, \tag{48}$$

since the field in the medium must behave as a wave travelling in the negative z-direction. Moreover, at $z = 0$, $E_x' = E_x$ and hence

$$E_x' = E_x e^{-ikNz}. \tag{49}$$

Similarly,

$$E_y' = E_y e^{-ikNz}, \tag{50}$$

and (47) can now be written as

$$\frac{\partial^2 E_z'}{\partial z^2} + k^2N^2E_z' + \alpha e^{-ikNz} = 0, \tag{51}$$

where

$$\alpha = -2ik\left(E_x \frac{\partial N}{\partial x} + E_y \frac{\partial N}{\partial y}\right). \tag{52}$$

α is, of course, independent of z.

The complete solution of (51) is obtained by adding a particular integral to the general solution Ae^{-ikNz}, where A is some constant. The former can be taken as

$$\frac{\alpha z}{2ikN} e^{-ikNz},$$

giving

$$E_z' = e^{-ikNz}\left(A + \frac{\alpha z}{2ikN}\right). \tag{53}$$

Hence

$$\frac{\partial E_z'}{\partial z} = -ikNE_z' + \frac{\alpha}{2ikN} e^{-ikNz},$$

and at $z = 0$ this reduces to

$$\begin{aligned}\frac{\partial E_z'}{\partial z} &= -ikNE_z' + \frac{\alpha}{2ikN} = \\ &= -ik\eta E_z + \frac{1}{\eta}\left(E_x \frac{\partial \eta}{\partial x} + E_y \frac{\partial \eta}{\partial y}\right)\end{aligned} \tag{54}$$

by using the expression for α. If this is now inserted into (34) bearing in mind that $\partial\eta/\partial z = 0$, a boundary condition is obtained in the form

$$\frac{\partial E_z}{\partial z} = -ik\eta E_z - \frac{1}{\eta}\left(E_x \frac{\partial \eta}{\partial x} + E_y \frac{\partial \eta}{\partial y}\right) \tag{55}$$

at the interface $z = 0$. Apart from the presence of the tangen ial components E_x and E_y consequent upon the variation of ε through the medium this equation is the same as (20).

A boundary condition for the normal component of the magnetic field can be obtained by an analysis similar to the above. Since $\nabla \cdot \boldsymbol{H} = \nabla \cdot \boldsymbol{H}' = 0$, the continuity of the tangential components of $\boldsymbol{H}$ across the interface leads to the equation

$$\frac{\partial H_z}{\partial z} = \frac{\partial H_z'}{\partial z} \tag{56}$$

(cf (37))at $z = 0$. Inside the medium the field equations give

$$\begin{aligned}\boldsymbol{H}' &= \frac{\mu_0 \varepsilon_0}{\mu k^2} \nabla \times \left(\frac{1}{\varepsilon} \nabla \times \boldsymbol{H}'\right) \\ &= \frac{\mu_0 \varepsilon_0}{\mu k^2} \left[\frac{1}{\varepsilon} \nabla \times \nabla \times \boldsymbol{H}' + \nabla \frac{1}{\varepsilon} \times (\nabla \times \boldsymbol{H}')\right] \\ &= -\frac{1}{k^2 N^2} \left(\nabla^2 \boldsymbol{H}' + 2ikN \frac{\mu_0}{\mu} Y \boldsymbol{E}' \times \nabla N\right)\end{aligned}$$

and hence

$$\nabla^2 \boldsymbol{H}' + k^2 N^2 \boldsymbol{H}' + 2ikN \frac{\mu_0}{\mu} Y \boldsymbol{E}' \times \nabla N = 0. \tag{57}$$

In particular, the z-component of (57) is

$$\frac{\partial^2 H_z'}{\partial z^2} + k^2 N^2 H_z' + 2ikN \frac{\mu_0}{\mu} Y \left(E_x' \frac{\partial N}{\partial y} - E_y' \frac{\partial N}{\partial x}\right) = 0,$$

where the x and y derivatives have been neglected in comparison with the z, and by using the expressions for E_x' and E_y' given by (49) and (50) respectively we arrive at the equation

$$\frac{\partial^2 H_z'}{\partial z^2} + k^2 N^2 H_z' + \beta \mathrm{e}^{-ikNz} = 0 \tag{58}$$

(cf (51)), where

$$\beta = 2ikN \frac{\mu_0}{\mu} Y \left(E_x \frac{\partial N}{\partial y} - E_y \frac{\partial N}{\partial x}\right) \tag{59}$$

(cf (52)). The solution of (58) is

$$H_z' = \mathrm{e}^{-ikNz}\left(B + \frac{\beta z}{2ikN}\right) \tag{60}$$

(cf (53)), where B is some constant, and hence at $z = 0$

$$\begin{aligned}\frac{\partial H_z'}{\partial z} &= -\,ikNH_z' + \frac{\beta}{2ikN} = \\ &= -\,ikN\,\frac{\mu_0}{\mu}\,H_z + \frac{\mu_0}{\mu}\,Y\left(E_x\,\frac{\partial N}{\partial y} - E_y\,\frac{\partial N}{\partial x}\right) \qquad (61)\\ &= -\,\frac{ik}{\eta}\,H_z + \frac{Y}{\eta}\left(E_y\,\frac{\partial \eta}{\partial x} - E_x\,\frac{\partial \eta}{\partial y}\right).\end{aligned}$$

If this is substituted into (56), the boundary condition on the normal component of $\boldsymbol{H}$ at the interface is

$$\frac{\partial H_z}{\partial z} = -\,\frac{ik}{\eta}\,H_z + \frac{Y}{\eta}\left(E_y\,\frac{\partial \eta}{\partial x} - E_x\,\frac{\partial \eta}{\partial y}\right), \qquad (62)$$

which is analogous to the condition (21). As with the condition (55), the variation of η has introduced the tangential components E_x and E_y into a boundary condition which is otherwise the same as for a homogeneous medium.

From (55) and (62) boundary conditions can be derived involving only the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$. Using the equation $\nabla \cdot \boldsymbol{E} = 0$ and the expression for E_z in terms of H_x and H_y, (55) becomes

$$\frac{\partial E_x}{\partial x} - \frac{1}{\eta}\,E_x\,\frac{\partial \eta}{\partial x} + \eta Z\,\frac{\partial H_y}{\partial x} = -\,\frac{\partial E_y}{\partial y} - \frac{1}{\eta}\,E_y\,\frac{\partial \eta}{\partial y} - \eta Z\,\frac{\partial H_x}{\partial y}$$

which reduces to

$$\frac{\partial}{\partial x}\left(\frac{E_x}{\eta} + ZH_y\right) = -\,\frac{\partial}{\partial y}\left(\frac{E_y}{\eta} - ZH_x\right).$$

Similarly (62) can be written as

$$\frac{\partial}{\partial y}\left(\frac{E_x}{\eta} + ZH_y\right) = \frac{\partial}{\partial x}\left(\frac{E_y}{\eta} - ZH_x\right)$$

and by means of the analysis given in § 3 (eq. (24) et seq.) it now follows that $(E_x/\eta + ZH_y)$ and $(E_y/\eta - ZH_x)$ are both identically zero at the interface. Hence,

$$E_x = -\,\eta Z H_y, \qquad (63)$$

$$E_y = \eta Z H_x, \qquad (64)$$

which are of precisely the same form as the conditions (25) and (26) for a homogeneous medium. In particular, the tangential derivatives of η do not enter into these equations in spite of the fact that they appear in (55) and (62). Thus, the conditions (63) and (64) are relatively insensitive to changes in the medium, and any correction terms arising from the inhomogeneity must be of higher order than those considered here. Indeed, if η is regarded as a function of z as well as x and y, it can be shown that

$$E_x = -\eta Z H_y \left[1 + O\left(\frac{1}{k}\frac{\partial \eta}{\partial z}\right)\right]$$

(see Rytov [5]), and by virtue of (40) $k^{-1}\,\partial\eta/\partial z \ll \eta$.

In spite of the simplicity of equations (63) and (64), these boundary conditions are of little practical value as they stand. Although the coordinates x and y do not occur explicitly in these equations, the material parameter η is itself a function of x and y, and accordingly the boundary conditions vary with position on the interface. This is a source of difficulty in any attempt to employ these conditions in the solution of an actual problem.

On the other hand, if it is assumed that the variations of η are random but uniform in some statistical sense, the difficulty can be overcome in a manner which is satisfactory for many practical applications. Such an assumption is, of course, additional to the restriction (40) and implies that if a large sample of the surface is chosen, the values of η within this sample are substantially the same independently of the portion of the surface from which the sample is taken. Under these circumstances it is to be expected that the field will (in general) be a function of the statistical properties of the surface, rather than of individual features, and this leads us to consider an average field satisfying an averaged boundary condition. Such an average is obtained either by moving the transmitter and receiver whilst maintaining their positions relative to the plane $z = 0$ (so that different samples of surface appear beneath them), or by replacing the given surface by others of a family whose statistical properties are the same. The boundary conditions satisfied by the average field ($\boldsymbol{E}$, $\boldsymbol{H}$) can be found by the simple process of averaging equations (63) and (64). Bearing in mind that to the first order in η, H_x and H_y can be replaced by the components $H_x{}^0$ and $H_y{}^0$ for a perfectly conducting surface, (63) and (64) give

$$\bar{E}_x = -\bar{\eta} Z H_y{}^0, \quad E_y = \bar{\eta} Z H_x{}^0,$$

which can be replaced by

$$\bar{E}_x = -\bar{\eta} Z \bar{H}_y, \tag{65}$$

$$\bar{E}_y = \bar{\eta} Z \bar{H}_x \tag{66}$$

to the first order in η. Similarly, if the correction terms in (55) and (62) are neglected, the averaged versions are

$$\frac{\partial \bar{E}_z}{\partial z} = -ik\bar{\eta}\bar{E}_z, \tag{67}$$

$$\frac{\partial \bar{H}_z}{\partial z} = -\frac{ik}{\bar{\eta}}\bar{H}_z. \tag{68}$$

The above results are valid for statistically uniform surfaces whose refractive index $N = \mu/(\mu_0\eta)$ satisfies the restrictions (30) and (40). It will be observed that the average fields are determined by the average value of η, and not by the average values of ε or σ. This is in accordance with the conclusion reached by Feinberg [8]) under the same restrictions but by a somewhat circuitous analysis.

These boundary conditions can be generalized so as to apply to a curved surface in the manner described in § 4. The restrictions under which this is valid are the same as in § 4, and will not be repeated here.

Acknowledgement. This research was sponsored by the Rome Air Development Center of the U.S. Air Force under Contract AF-30-(602)-2099.

Received 26th July, 1960.

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