Gauss-Legendre Quadrature for the Evaluation of Integrals Involving the Hankel Function

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Abstract

The boundary integral method for the two dimensional Helmholtz equation requires the approximate evaluation of the integral $\int_{-1}^{1} g(x) H_0^{(1)} \left(\lambda\sqrt{(x-a)^2+b^2}\right) dx$, where g is a polynomial. In particular, Gauss-Legendre quadrature is considered when the source point is close to the interval of integration; that is $-1 \leq a \leq 1$ and $0 < b \ll 1$ so that the integral is nearly weakly singular. It is shown that the real and imaginary parts of the integral must be considered separately. The sinh transformation can be used to improve the truncation error of the imaginary part, but must not be used for the real part. An asymptotic error analysis is given.

Key words: Numerical Integration, Boundary Element Method, Helmholtz' Equation, Nearly Singular Integrals, Nonlinear Coordinate Transformation, Sinh Function, Hankel Function.

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1 Introduction

In applying the boundary integral method to problems arising from the two–dimensional Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0, \tag{1.1}$$

where $\lambda > 0$, we need to evaluate integrals Ig say, where

$$Ig := \int_{-1}^{1} g(x) H_0^{(1)}\left(\lambda\sqrt{(x-a)^2+b^2}\right)\, dx. \tag{1.2}$$

Throughout we shall assume that g is a real polynomial of low degree. The function $H_0^{(1)}$ is the Hankel function of the first kind of order zero which is defined in terms of the zero order Bessel functions of the first and second kind as

$$H_0^{(1)}(z) = J_0(z) + iY_0(z). \tag{1.3}$$

(See, for example, Abramowitz and Stegun [1, §9.1.3]). Of particular interest, in the context of the boundary integral method, is the evaluation of Ig when the so–called "source point" z_0, where

$$z_0 := a + ib, \tag{1.4}$$

is "close" to the interval of integration $(-1, 1)$ when viewed in the complex z–plane, where $z = x + iy$. In particular, we shall assume that $-1 \leq a \leq 1$ and that $0 < b \ll 1$.

The Helmholtz equation arises naturally in the study of many modelling problems with a wave-like behaviour. In particular, these include acoustic wave scattering problems [2], analysis of integrated optical wave guides [3], acoustic and aerospace research [4], natural convection flows [5] and failure analysis of sheet metals [6]. As with most boundary element method implementations, the accurate evaluation of weakly singular and nearly weakly singular integrals is significant in obtaining accurate solutions to the governing equations. Weakly singular integrals can be handled using analytical integration techniques [7–9] or numerically [10]. As mentioned above, we will consider here the numerical evaluation of nearly weakly singular integrals where the source point is close to the interval of integration.

Throughout this paper we shall determine appropriate values of Ig by using n–point Gauss-Legendre quadrature for various $n \in \mathbb{N}$. Our principal aim is to obtain asymptotic estimates of the truncation error under the assumption that n is large although, as we shall see, these estimates are often good for modest values of n. However, the real and imaginary parts of Ig have totally different characteristics. Let us consider these in reverse order; we have

$$\Im Ig = \int_{-1}^{1} g(x) Y_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\, dx. \tag{1.5}$$

Now $Y_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)$ has branch points at $z_0, \overline{z_0}$ where the singularity is logarithmic (see [1, §9.1.11]). Consequently, if z_0 is close to $(-1,1)$, the integrand will have a "peak" in the neighbourhood of the point a on $(-1,1)$ and, for modest values of n, n–point Gauss-Legendre quadrature will not be very good. In order to diminish the truncation error for a given value of n we shall make a transformation of the variable of integration in (1.5). We shall write

$$x = a + b\sinh(\mu u - \eta), \tag{1.6}$$

where μ and η are chosen so that the interval $-1 \leq x \leq 1$ corresponds to $-1 \leq u \leq 1$. We find that

$$\mu = \frac{1}{2}\left(\operatorname{arcsinh}\left(\frac{1+a}{b}\right) + \operatorname{arcsinh}\left(\frac{1-a}{b}\right)\right), \tag{1.7}$$

$$\eta = \frac{1}{2}\left(\operatorname{arcsinh}\left(\frac{1+a}{b}\right) - \operatorname{arcsinh}\left(\frac{1-a}{b}\right)\right). \tag{1.8}$$

As we shall show, in §§3 and 4, this transformation gives rise to a dramatic decrease in the truncation error when we again apply n–point Gauss–Legendre quadrature to the transformed integral. We shall give asymptotic estimates of the truncation errors for both the untransformed and transformed integrals. These estimates will be compared with the actual values of the truncation errors in a few examples. Actual errors for Gauss-Legendre quadrature in Tables 1–6 were obtained using MATLAB; the asymptotic errors were computed using Mathematica.

Consider now $\Re Ig$ which, from (1.2) and (1.3), is given by

$$\Re Ig = \int_{-1}^{1} g(x) J_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\,dx. \tag{1.9}$$

Unlike the previous integral we see, from [1, §9.1.10], that $J_0\left(\lambda\sqrt{(z-a)^2+b^2}\right)$ is an entire function in $\mathbb{C}$. The fact that the points z_0 and $\overline{z_0}$ are "close" to the interval $(-1,1)$ now makes no difference to the evaluation of $\Re Ig$. In §2 we shall obtain an estimate of the truncation error when n–point Gauss-Legendre quadrature is used to evaluate (1.9). We shall compare the analytic estimate with the actual truncation error in a few examples. However, we shall also show with these examples how *disastrous* the truncation errors are when n–point Gauss-Legendre quadrature is applied to the integral $\Re Ig$ after using the transformation given by (1.6) — (1.8). This, of course, is plausible since there are now no singular points close to the interval of integration.

Before leaving this introduction, let us outline the analysis of Donaldson and Elliott [11] which gives the truncation error in terms of a contour integral. This contour integral representation will be the starting point for all the subsequent error analysis.

Suppose we use n–point Gauss–Legendre quadrature to evaluate the integral $\int_{-1}^{1} f(x)\ dx$. Let us further suppose that the definition of f can be continued from the interval $(-1, 1)$ into the complex z-plane $\mathbb{C}$, where $z = x + iy$. For $\rho > 1$, let $\mathcal{E}_\rho$ denote one of the family of confocal ellipses, with foci at ($\pm$1,0), defined by $|z + \sqrt{z^2 - 1}| = \rho$. The ellipse $\mathcal{E}_\rho$ has semi–major and semi–minor axes given respectively by $(\rho + 1/\rho)/2$ and $(\rho - 1/\rho)/2$. (For further discussion see Davis [12, pp 19–20].) If $\mathcal{E}_\rho$ is described in the positive (i.e. anti–clockwise) direction, and if f is analytic on and within $\mathcal{E}_\rho$, then Donaldson and Elliott [11] write

$$\int_{-1}^{1} f(x)\ dx = \sum_{k=1}^{n} \lambda_{k,n} f(x_{k,n}) + \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} k_n(z) f(z)\ dz. \tag{1.10}$$

Here $x_{k,n}$, for k=1(1)n, are the n simple zeros of the Legendre polynomial P_n and $\lambda_{k,n}$ are the appropriate weights, see [1, §25.4.29]. The function k_n is analytic in $\mathbb{C} \setminus [-1, 1]$ and depends only on the fact that we are using n–point Gauss–Legendre quadrature. For $n \gg 1$, k_n can be approximated in terms of elementary functions by

$$k_n(z) \sim \frac{c_n}{\left(z + \sqrt{z^2 - 1}\right)^{2n+1}}, \qquad z \in \mathbb{C} \setminus [-1, 1], \tag{1.11}$$

where

$$c_n := \frac{2\pi\left(\Gamma(n+1)\right)^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}. \tag{1.12}$$

In (1.11), $\sqrt{z^2 - 1}$ is defined so that $|z + \sqrt{z^2 - 1}| > 1$ for all $z \in \mathbb{C} \setminus [-1, 1]$.

2 Truncation errors for $\int_{-1}^{1} x^k J_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\, dx$

Let us consider the truncation error when n–point Gauss-Legendre quadrature is used to evaluate the integral I_k where

$$I_k := \int_{-1}^{1} x^k J_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\, dx, \tag{2.1}$$

k being a non-negative integer. From (1.10) the truncation error, let us denote it by $E_n I_k$, is given by

$$E_n I_k = \frac{1}{2\pi i}\int_{\mathcal{E}_\rho} k_n(z) z^k J_0\left(\lambda\sqrt{(z-a)^2+b^2}\right)\, dz, \tag{2.2}$$

for any $\rho > 1$. In order to estimate $E_n I_k$ we shall assume that $n \gg 1$ and replace k_n by (1.11) and (1.12) to give

$$E_n I_k \sim \frac{c_n}{2\pi i}\int_{\mathcal{E}_\rho} \frac{z^k J_0\left(\lambda\sqrt{(z-a)^2+b^2}\right)\, dz}{(z+\sqrt{z^2-1})^{2n+1}}. \tag{2.3}$$

Since the integrand is analytic in $\mathbb{C}\setminus[-1,1]$, let us replace $\mathcal{E}_\rho$ by C_R say, a circle with centre at the origin and radius R where we assume that $R \gg 1$. If on C_R we replace $\sqrt{(z-a)^2+b^2}$ by $(z-a)$ and $z+\sqrt{z^2-1}$ by $2z$ we find

$$E_n I_k \simeq \frac{c_n}{2^{2n+1}}\frac{1}{2\pi i}\int_{C_R} \frac{J_0(\lambda(z-a))}{z^{2n+1-k}}\, dz. \tag{2.4}$$

From Abramowitz and Stegun [1, §9.1.75] we have

$$J_0(\lambda(z-a)) = 2{\sum_{l=0}^{\infty}}' J_l(\lambda a) J_l(\lambda z), \tag{2.5}$$

where $\sum'$ denotes a sum whose first term is halved. Substituting (2.5) into (2.4) gives

$$E_n I_k \simeq \frac{c_n}{2^{2n}}{\sum_{l=0}^{\infty}}' J_l(\lambda a)\frac{1}{2\pi i}\int_{C_R} \frac{J_l(\lambda z)}{z^{2n+1-k}}\, dz. \tag{2.6}$$

Since, from Abramowitz and Stegun [1, §9.1.10],

$$J_l(\lambda z) = \left(\frac{\lambda z}{2}\right)^l \sum_{s=0}^{\infty} \frac{(-1)^s (\lambda^2/4)^s z^{2s}}{s!(l+s)!} \tag{2.7}$$

it follows that

$$\frac{1}{2\pi i}\int_{C_R} \frac{J_l(\lambda z)\,dz}{z^{2n+1-k}} = \sum_{s=0}^{\infty} \frac{(-1)^s \lambda^{l+2s}}{2^{l+2s} s!(l+s)!} \frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^{2n+1-k-2s-l}}. \tag{2.8}$$

By Cauchy's residue theorem we have that

$$\frac{1}{2\pi i}\int_{C_R} \frac{dz}{z^{2n+1-k-2s-l}} = \begin{cases} 1, \text{ when } 2n-k-2s-l=0, \\ \\ 0, \text{ otherwise.} \end{cases} \tag{2.9}$$

Thus the integral takes the value 1 when

$$s = n - (k+l)/2, \tag{2.10}$$

and let us recall that s, n, k and l are non–negative integers. Consequently, if k is even we must choose l to be even and, when k is odd, l must be odd.

Suppose first that $k = 2p$, $p = 0, 1, 2, \ldots$ and $l = 2m$ for $m = 0, 1, 2, \ldots$ also. From (2.6)–(2.10) we find, after some algebra, that

$$E_n I_{2p} \simeq \frac{c_n}{2^{2n}}(-1)^{n-p}\left(\frac{\lambda}{2}\right)^{2n-2p} \sideset{}{'}\sum_{m=0}^{n-p} \frac{(-1)^m J_{2m}(\lambda a)}{(n-p-m)!(n-p+m)!}, \tag{2.11}$$

for $p = 0, 1, 2, \ldots$.

On the other hand if k is odd, suppose $k = 2p+1$ for $p = 0, 1, 2, \ldots$ then if we write $l = 2m+1$

for $m = 0, 1, 2\ldots$ we find from (2.6)–(2.10), again after some algebra, that

$$E_n I_{2p+1} \simeq \frac{c_n}{2^{2n}}(-1)^{n-p+1}\left(\frac{\lambda}{2}\right)^{2n-2p-1}\sum_{m=0}^{n-p-1}{}' \frac{(-1)^m J_{2m+1}(\lambda a)}{(n-p-1-m)!(n-p+m)!}, \tag{2.12}$$

for $p = 0, 1, 2, \ldots$.

In Tables 1–3 we have compared the actual truncation errors with the asymptotic estimates given by equations (2.11) and (2.12) when $k = 0, 1$ and 2. In all cases we see that even with n as small as 6, the asymptotic estimates agree well with the actual values of the truncation errors. Even though (2.11) and (2.12) are independent of b we have predicted the errors correctly to one significant digit. In the last column of each Table we have recorded the actual truncation errors when the same n-point Gauss-Legendre quadrature rule is applied to the transformed integrals. Almost without exception the truncation error is dramatically *worse* so that the transformation should not be applied to these integrals. However, for the integrals involving the Bessel functions of the second kind exactly the opposite is true, as we shall see in the next two sections.

3 The Untransformed Integral $\int_{-1}^{1} x^k Y_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\,dx$

We shall firstly consider the truncation error when n–point Gauss-Legendre quadrature is applied to the integral L_k say, where

$$L_k := \int_{-1}^{1} x^k Y_0\left(\lambda\sqrt{(x-a)^2+b^2}\right)\,dx, \tag{3.1}$$

k being a non-negative integer. In the next section, where we consider the effects of the sinh transformation, we shall discuss the integrals M_k say, where

$$M_k := \int_{-1}^{1} (x-a)^k Y_0 \left(\lambda \sqrt{(x-a)^2 + b^2} \right) dx. \tag{3.2}$$

There is, of course, a simple relationship between the L_k and the M_k. We have

$$M_k = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} a^{k-l} L_l, \quad L_k = \sum_{l=0}^{k} \binom{k}{l} a^{k-l} M_l. \tag{3.3}$$

Consider the integrals L_k. We may continue the definition of $Y_0 \left(\lambda \sqrt{(x-a)^2 + b^2} \right)$ into the complex z–plane, where $z = x + iy$, to give the function $Y_0 \left(\lambda \sqrt{(z-z_0)(z-\overline{z_0})} \right)$ where, recall (1.4), $z_0 = a + ib$. From (1.10) we have that the truncation error, $E_n L_k$ say, is given by

$$E_n L_k = \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} k_n(z) z^k Y_0 \left(\lambda \sqrt{(z-z_0)(z-\overline{z_0})} \right) dz, \tag{3.4}$$

where $\mathcal{E}_\rho$ is one of the confocal ellipses on and within which the function $Y_0 \left(\lambda \sqrt{(z-z_0)(z-\overline{z_0})} \right)$ is analytic. If we assume that $n \gg 1$ and replace k_n by (1.11) and (1.12) we shall take as our starting point

$$E_n L_k \sim \frac{c_n}{2\pi i} \int_{\mathcal{E}_\rho} \frac{z^k Y_0 \left(\lambda \sqrt{(z-z_0)(z-\overline{z_0})} \right) dz}{\left(z + \sqrt{z^2 - 1} \right)^{2n+1}}. \tag{3.5}$$

Now the Bessel function $Y_0 \left(\lambda \sqrt{(z-z_0)(z-\overline{z_0})} \right)$ has logarithmic branch points at z_0 and $\overline{z_0}$. In order to evaluate $E_n L_k$ we increase ρ so that the contour becomes as shown in Figure 1. Let us first define

$$\xi_0 := z_0 + \sqrt{z_0^2 - 1}, \tag{3.6}$$

where $-\pi < \arg(z \pm 1) < \pi$, so that $|\xi_0| > 1$. Following Hough [13], a branch cut B_{z_0}, from z_0

to the point at infinity, is defined as

$$B_{z_0} = \{z \in \mathbb{C} : z(s) = (\xi_0 s + 1/(\xi_0 s))/2,\ 1 \leq s < \infty\}. \tag{3.7}$$

It might be noted that Hough writes $s = e^t$ with $0 \leq t < \infty$, but this simpler algebraic parameterisation is sufficient in this context. Although we shall not prove it here, it can be shown that B_{z_0} is the arc of a hyperbola from ξ_0 to ∞, the hyperbola having foci at the points $(\pm 1, 0)$. No matter how we choose $z_0 \in \mathbb{C} \setminus [-1, 1]$, the cut B_{z_0} never crosses the interval $-1 \leq \Re z \leq 1$. Along this cut it is readily shown that

$$z(s) + \sqrt{z^2(s) - 1} = \xi_0 s, \quad 1 \leq s < \infty. \tag{3.8}$$

Returning to equation (3.5), let $E_n L_k(z_0)$ denote the contribution to the truncation error $E_n L_k$ from the neighbourhood of the branch point at z_0. That is, see Figure 1,

$$E_n L_k(z_0) \sim \frac{c_n}{2\pi i} \int_{\mathrm{AB} \cup \mathrm{CD}} \frac{z^k Y_0\left(\lambda\sqrt{(z - z_0)(z - \overline{z_0})}\right)\, dz}{\left(z + \sqrt{z^2 - 1}\right)^{2n+1}}. \tag{3.9}$$

Since, see Abramowitz and Stegun [1, §9.1.13], $Y_0(z) = (2/\pi)\log(z/2) + \mathrm{O}(1)$ near $z = 0$ then

$$Y_0\left(\lambda\sqrt{(z - z_0)(z - \overline{z_0})}\right) = \frac{1}{\pi}\log(z - z_0) + \mathrm{O}(1), \tag{3.10}$$

near $z = z_0$. On approximating Y_0 in (3.9) by (3.10) we have

$$E_n L_k(z_0) \simeq \frac{c_n}{\pi} \frac{1}{2\pi i} \int_{\mathrm{AB} \cup \mathrm{CD}} \frac{z^k \log(z - z_0)\, dz}{\left(z + \sqrt{z^2 - 1}\right)^{2n+1}}. \tag{3.11}$$

Let $(z - z_0)|_{\mathrm{AB}}$ denote the value of $(z - z_0)$ along AB and $(z - z_0)|_{\mathrm{CD}}$ denote its value along CD. Then, from Figure 1, we see that

$$(z - z_0)|_{\mathrm{CD}} = (z - z_0)|_{\mathrm{AB}} e^{-2\pi i}. \tag{3.12}$$

Again, from (3.7), as we describe AB, s goes from ∞ to 1 whereas, along CD, s goes from 1 to ∞. Recalling (3.8) we have from (3.11) that

$$E_n L_k(z_0) \simeq -\frac{c_n}{\pi}\frac{1}{\xi_0^{2n+1}}\int_1^{\infty} s^{-(2n+1)} z^k(s) z'(s)\, ds. \tag{3.13}$$

From (3.7) we have firstly that

$$z'(s) = \frac{\xi_0}{2}\left(1 - \frac{1}{\xi_0^2 s^2}\right) \tag{3.14}$$

and also

$$z^k(s) = \frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{l}\xi_0^{2l-k} s^{2l-k}. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13) gives, after some algebra,

$$E_n L_k(z_0) \sim -\frac{c_n}{\pi 2^{k+1}}\sum_{l=0}^{k}\binom{k}{l}\frac{1}{(2n+k-2l)\xi_0^{2n+k-2l}}\left(1 - \frac{(2n+k-2l)}{(2n+k+2-2l)}\frac{1}{\xi_0^2}\right). \tag{3.16}$$

Now z_0 is any point from $\mathbb{C} \setminus [-1,1]$ so if $E_n L_k(\overline{z_0})$ denotes the contribution to $E_n L_k$ from the neighbourhood of $\overline{z_0}$ then we find

$$E_n L_k(\overline{z_0}) = \overline{E_n L_k(z_0)}. \tag{3.17}$$

To sum up: from the neighbourhoods of the branch points at z_0 and $\overline{z_0}$ we have that

$$\begin{aligned} E_n L_k(z_0 \cup \overline{z_0}) &= 2\Re\left\{E_n L_k(z_0)\right\} \\ &\sim -\frac{c_n}{\pi 2^k}\sum_{l=0}^{k}\binom{k}{l}\frac{1}{(2n+k-2l)}\times \\ &\qquad \Re\left\{\frac{1}{\xi_0^{2n+k-2l}}\left(1 - \frac{(2n+k-2l)}{(2n+k+2-2l)}\frac{1}{\xi_0^2}\right)\right\}. \end{aligned} \tag{3.18}$$

In addition to the neighbourhoods of the branch points we must also consider the contributions to $E_n L_k$ from the remainder of the contour. We have used the method of steepest descents in order to do this and have found that in all cases the contribution from the remainder of the contour is considerably smaller than that from (3.18). The details of this analysis are omitted. In Table 4 we give a few comparisons of the actual truncation error with that predicted by (3.18) in the case when $k = 0$ and $n = 30$. Since we see that (3.18) is independent of λ we have included in Table 4 the actual errors for $\lambda = 1$ and $\lambda = 5$.

We note from Table 4 that the truncation error varies little between $\lambda = 1$ and $\lambda = 5$. The asymptotic estimates given by (3.18) agree, both in sign and order of magnitude, with the actual errors and there seems to be, at worst, a factor of three difference between the actual error and its estimates. Let us now consider a comparable analysis for the transformed integral.

4 The Transformed Integral $\int_{-1}^{1} (x-a)^k Y_0(\lambda\sqrt{(x-a)^2+b^2})\, dx$

We consider now the integral M_k, see (3.2). After the transformation of the variable of integration as given by (1.6)–(1.8) we have

$$M_k = \mu b^{k+1} \int_{-1}^{1} \sinh^k(\mu u - \eta)\cosh(\mu u - \eta) Y_0(\lambda b \cosh(\mu u - \eta))\, du. \tag{4.1}$$

If we introduce the complex w-plane, where $w = u + iv$ then, by (1.10), we have that the truncation error $E_n M_k$ say, for n-point Gauss–Legendre quadrature, is given by

$$E_n M_k = \mu b^{k+1} \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} k_n(w) \sinh^k(\mu w - \eta)\cosh(\mu w - \eta) Y_0(\lambda b \cosh(\mu w - \eta))\, dw. \tag{4.2}$$

Here $\mathcal{E}_\rho$ is initially taken to be an ellipse, with foci at $(\pm 1, 0)$, on and within which $Y_0(\lambda b \cosh(\mu w - \eta))$ is analytic. Now $Y_0(\lambda b \cosh(\mu w - \eta))$ will have logarithmic branch points where $\cosh(\mu w - \eta) = 0$. Let w_0 and $\overline{w_0}$ be the branch points closest to the interval

$[-1, 1]$; then

$$w_0 = \frac{\eta}{\mu} + i\frac{\pi/2}{\mu}, \tag{4.3}$$

where μ and η are defined in (1.7) and (1.8) respectively. In order to estimate E_nM_k we shall again assume that n is large and replace k_n by (1.11) and (1.12) so that

$$E_nM_k \sim \mu c_n b^{k+1} \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} \frac{\sinh^k(\mu w - \eta)\cosh(\mu w - \eta) Y_0(\lambda b \cosh(\mu w - \eta))}{(w + \sqrt{w^2 - 1})^{2n+1}}\, dw. \tag{4.4}$$

In order to evaluate the contour integral we let ρ increase so that the contour is as shown in Figure 2. The integrand does not tend to zero as $\rho \to \infty$ so we shall assume that the error E_nM_k can be written as

$$E_nM_k = E_nM_k(w_0) + E_nM_k(\overline{w_0}) + E_nM_k(rem). \tag{4.5}$$

The first two terms represent the contributions to E_nM_k from the neighbourhoods of the branch points at w_0 and $\overline{w_0}$ respectively. The term $E_nM_k(rem)$ represents the contribution to E_nM_k from the "remainder" of the contour.

Let us first consider the evaluation of $E_nM_k(rem)$. To do this we shall use the method of steepest descents. This can be summarised as follows, see Donaldson and Elliott [11]. Given an integral I where

$$I = \int_C A(z) \exp \chi(z)\, dz \tag{4.6}$$

over some contour C, the saddle points z_j, $j = 1(1)J$ are defined to be such that $\chi'(z_j) = 0$. The function A is assumed to be "slowly varying" in the neighbourhoods of each z_j. If complex numbers α_j, $j = 1(1)J$, are defined by

$$|\alpha_j| = 1 \quad \text{and} \quad \arg \alpha_j = \frac{\pi}{2} - \frac{1}{2} \arg \chi''(z_j) \tag{4.7}$$

then, by the method of steepest descents, I is given by

$$I \simeq \sum_{j=1}^{J} \frac{\sqrt{2\pi}\alpha_j A(z_j) \exp\chi(z_j)}{|\chi''(z_j)|^{1/2}}. \tag{4.8}$$

Recalling equation (4.4), let us write

$$\exp\chi(w) = \frac{\mu b^{k+1} c_n}{2\pi i} \frac{\sinh^k(\mu w - \eta)\cosh(\mu w - \eta) Y_0(\lambda b \cosh(\mu w - \eta))}{(w + \sqrt{w^2-1})^{2n+1}}, \tag{4.9}$$

so that $A(w) := 1$. Then

$$\begin{aligned}\chi'(w) = \frac{k\mu}{\tanh(\mu w - \eta)} &+ \mu\tanh(\mu w - \eta) - \frac{2n+1}{\sqrt{w^2-1}} \\ &+ \lambda b\mu \sinh(\mu w - \eta)\frac{Y_0'(\lambda b\cosh(\mu w - \eta))}{Y_0(\lambda b \cosh(\mu w - \eta))},\end{aligned} \tag{4.10}$$

and we want to find those values of w such that $\chi'(w) = 0$. It turns out that there are four saddle points which are given by w_1, $\overline{w_1}$, w_2 and $\overline{w_2}$ in Figure 2. Unfortunately, we have been unable to give analytic expressions for these points but have evaluated them numerically for given values of λ, k, a, b and n. As we shall see in Tables 5 and 6 these numerical approximations yield good results; but more of that later. Let us return to equation (4.10) and assume that $\chi'(w) = 0$ occurs when $|\lambda b \cosh(\mu w - \eta)|$ is large. Then, from Abramowitz and Stegun [1, §9.1.28 and §9.2.2], we find that

$$\begin{aligned}\chi'(w) \simeq \frac{k\mu}{\tanh(\mu w - \eta)} &+ \mu\tanh(\mu w - \eta) - \frac{2n+1}{\sqrt{w^2-1}} \\ &+ \lambda b\mu \sinh(\mu w - \eta)\frac{\tan(\lambda b\cosh(\mu w - \eta)) + 1}{\tan(\lambda b \cosh(\mu w - \eta)) - 1}.\end{aligned} \tag{4.11}$$

Recalling that w_0 is defined by equation (4.3), we can write

$$\cosh(\mu w - \eta) = i\sinh(\mu(w - w_0)), \quad \sinh(\mu w - \eta) = i\cosh(\mu(w - w_0)) \tag{4.12}$$

so that we may rewrite (4.11) as

$$\begin{aligned}\chi'(w) \simeq & k\mu \tanh(\mu(w-w_0)) + \frac{\mu}{\tanh(\mu(w-w_0))} - \frac{2n+1}{\sqrt{w^2-1}} \\ & + i\lambda b\mu \cosh(\mu(w-w_0)) \frac{\tanh(\lambda b \sinh(\mu(w-w_0))) - i}{\tanh(\lambda b \cosh(\mu(w-w_0))) + i}.\end{aligned} \tag{4.13}$$

Let us further assume that where $\chi'(w) = 0$ we will have $\mu(w - w_0) = X + iY$, where X and Y are real numbers with X sgn $X \gg 1$ and $-\pi/2 < Y < \pi/2$. Under this assumption, since λ and b are positive it is not difficult to show that

$$\tanh(\mu(w-w_0)) = \text{sgn } X \tag{4.14}$$

and

$$i\left(\frac{\tanh(\lambda b \sinh(\mu(w-w_0))) - i}{\tanh(\lambda b \cosh(\mu(w-w_0))) + i}\right) \simeq \text{sgn } X. \tag{4.15}$$

We now have

$$\chi'(w) \simeq (k+1)\mu \,\text{sgn } X + \lambda b\mu \cosh(\mu(w-w_0))\,\text{sgn } X - \frac{2n+1}{\sqrt{w^2-1}}. \tag{4.16}$$

If we first assume that $X \gg 1$, then we want saddle points w satisfying

$$\chi'(w) \simeq (k+1)\mu + \lambda b\mu \cosh(\mu(w-w_0)) - \frac{2n+1}{\sqrt{w^2-1}} = 0. \tag{4.17}$$

In spite of these simplifications we have been unable to find a neat analytic expression for w_1 such that $\chi'(w_1) = 0$. However, we have used Mathematica to determine w_1 numerically. As an example, for $\lambda = 1$, $k = 0$, $a = 0$, $b = 0.001$ and $n = 25$ we have found that $w_1 = 1.25864 + 0.15992i$. As a check, we find that $X = \Re\{\mu(w_1 - w_0)\} = 9.5667$ and $\tanh X = 0.9999999902$ to 10 decimal places. To proceed with the saddle point method, from (4.17) we compute $\chi''(w_1)$ from

$$\chi''(w) \simeq \lambda b\mu^2 \sinh(\mu(w-w_0)) + \frac{(2n+1)w}{(w^2-1)^{3/2}}. \tag{4.18}$$

Finally, recall (4.7), we have

$$\alpha_1 = \exp(i(\pi - \arg\chi''(w_1))/2). \tag{4.19}$$

With these values and evaluating $\exp\chi(w_1)$ from equation (4.9), we obtain $E_nM_k(w_1)$ say, the contribution to $E_nM_k(rem)$ from the saddle point at w_1. By repeating the above arguments we find that there is also a saddle point at $\overline{w_1}$ and, in an obvious notation, $E_nM_k(\overline{w_1}) = \overline{E_nM_k(w_1)}$ so that

$$E_nM_k(w_1 \cup \overline{w_1}) = 2\Re\{E_nM_k(w_1)\}. \tag{4.20}$$

This has been obtained under the assumption that $X \gg 1$. Suppose now that $X \ll -1$. Then, in place of (4.17), we have

$$\chi'(w) \simeq -(k+1)\mu - \lambda b\mu\cosh(\mu(w-w_0)) - \frac{2n+1}{\sqrt{w^2-1}}. \tag{4.21}$$

Suppose w_2 is such that $\chi'(w_2) = 0$. Again we do not have an analytic expression for w_2 so that we need to resort to numerical methods. In particular, for $\lambda = 5$, $k = 2$, $a = 0.5$, $b = 0.0001$ and $n = 20$, we find that $w_2 = -1.01759 + 0.0695593i$ from which it follows that $X = \Re\mu(w_2 - w_0) = -10.4806$ and $\tanh X = -0.9999999984$, to 10 decimal places. From (4.21) we have

$$\chi''(w_2) \simeq -\lambda b\mu^2\sinh(\mu(w_2-w_0)) + \frac{(2n+1)w_2}{(w_2^2-1)^{3/2}} \tag{4.22}$$

and in this case, see Figure 2, we have chosen

$$\alpha_2 = \exp(-i(\pi + \arg\chi''(w_2))/2). \tag{4.23}$$

Again, on evaluating $\exp\chi(w_2)$ from (4.9) we obtain from (4.8) $E_nM_k(w_2)$ say, the contribution to $E_nM_k(rem)$ from the saddle point at w_2. Equally well, by a similar argument there is also a saddle point at $\overline{w_2}$ and it can be shown that $E_nM_k(\overline{w_2}) = \overline{E_nM_k(w_2)}$ so that

$$E_nM_k(w_2 \cup \overline{w_2}) = 2\Re\{E_nM_k(w_2)\}. \tag{4.24}$$

From (4.20) and (4.24) we obtain a numerical estimate for $E_nM_k(rem)$ given by

$$E_nM_k(rem) = 2\Re\{E_nM_k(w_1) + E_nM_k(w_2)\}. \tag{4.25}$$

Let us now consider $E_nM_k(w_0)$, the contribution to the truncation error E_nM_k from the neighbourhood of the logarithmic singularity at w_0. We shall later consider $E_nM_k(\overline{w_0})$. Referring to Figure 2, we shall introduce the branch cut B_{w_0} at the point w_0. If we define

$$\zeta_0 := w_0 + \sqrt{w_0^2 - 1}, \tag{4.26}$$

then B_{w_0} is defined by

$$B_{w_0} = \{w \in \mathbb{C} : w(s) = (\zeta_0 s + 1/(\zeta_0 s))/2, 1 \leq s < \infty\}, \tag{4.27}$$

cf. (3.7). We define, see (4.4), $E_nM_k(w_0)$ by

$$\begin{aligned} E_nM_k(w_0) \sim \frac{\mu c_n b^{k+1}}{2\pi i} \times \\ \int_{AB\cup CD} \frac{\sinh^k(\mu w - \eta)\cosh(\mu w - \eta) Y_0(\lambda b \cosh(\mu w - \eta))}{(w + \sqrt{w^2-1})^{2n+1}}\, dw. \end{aligned} \tag{4.28}$$

Since $\cosh(\mu w_0 - \eta) = 0$ we have, for w in the neighbourhood of w_0, that

$$\begin{aligned} \sinh(\mu w - \eta) &\simeq \sinh(\mu w_0 - \eta), \quad \text{and} \\ \cosh(\mu w - \eta) &\simeq \mu(w - w_0)\sinh(\mu w_0 - \eta). \end{aligned} \tag{4.29}$$

From Abramowitz and Stegun [1, §9.1.13] we have for w near w_0 that

$$\begin{aligned} Y_0(\lambda b \cosh(\mu w - \eta)) &= Y_0(\lambda b \mu(w - w_0)\sinh(\mu w_0 - \eta)) \\ &= \frac{2}{\pi}\log(w - w_0) + O(1). \end{aligned} \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.28) gives

$$E_nM_k(w_0) \simeq \frac{2c_nb^{k+1}\mu^2}{\pi}\sinh^k(\mu w_0-\eta)\frac{1}{2\pi i}\int_{AB\cup CD}\frac{(w-w_0)\log(w-w_0)dw}{(w+\sqrt{w^2-1})^{2n+1}}. \tag{4.31}$$

But we have, in §3, already evaluated integrals of this form. From equation (3.11) we may write

$$E_nM_k(w_0) = 2b^{k+1}\mu^2\sinh^k(\mu w_0-\eta)\{E_nL_1(w_0)-w_0E_nL_0(w_0)\}. \tag{4.32}$$

From the results of equation (3.16) with $k=0$ and $k=1$ we find, after some algebra, that

$$E_nM_k(w_0) \sim -\frac{c_nb^{k+1}\mu^2\sinh^{k+1}(\mu w_0-\eta)}{2\pi(2n-1)(2n)\zeta_0^{2n-1}}\left[1-\frac{(2n-1)}{(n+1)}\frac{1}{\zeta_0^2}+\frac{n(2n-1)}{(n+1)(2n+3)}\frac{1}{\zeta_0^4}\right]. \tag{4.33}$$

Since it follows that $E_nM_k(\overline{w_0}) = \overline{E_nM_k(w_0)}$ we have that

$$\begin{aligned} E_nM_k(w_0\cup\overline{w_0}) \sim -&\frac{c_nb^{k+1}\mu^2}{\pi(2n-1)(2n)}\times \\ &\Re\left\{\frac{\exp(i\pi(k+1)/2)}{\zeta_0^{2n-1}}\left(1-\frac{(2n-1)}{(n+1)}\frac{1}{\zeta_0^2}+\frac{n(2n-1)}{(n+1)(2n+3)}\frac{1}{\zeta_0^4}\right)\right\}. \end{aligned} \tag{4.34}$$

To sum up: the estimate of the truncation error E_nM_k is given by the sum of equations (4.25) and (4.34). It now remains to see how this estimate compares with the actual error in a few cases.

In Table 5 we have considered the case where $\lambda = 1$, $k = 2$, $n = 20$ and various values of a and b. From the Table we observe that the error estimate agrees to one significant figure with the actual truncation error. We also observe from the last two columns that the contribution to the error from the "remainder" of the contour is far in excess of that from the branch points at w_0 and $\overline{w_0}$.

Consider now Table 6 where we have chosen $\lambda = 2$, $k = 0$ and $n = 25$ for various values of a and b. From the Table we note that once again the error estimates agree with actual truncation errors to one significant digit. However, in this case, and in contrast to Table 5, most of the truncation error comes from the neighbourhood of the branch points except in the particular case when $a = 1$ and $b = 0.0001$.

Finally, we might compare the order of magnitudes of the actual truncation errors in Tables 4 and 6. In Table 4 we have chosen $\lambda = 1$ and 5 with $k = 0$ and $n = 30$ and we see that the truncation errors are of the order of 10^{-2}. However, in Table 6 again with $k = 0$ although we have chosen $\lambda = 2$ and $n = 25$ we find errors of order 10^{-9} say. Even though we have chosen a smaller value of n this is a dramatic improvement and justifies the use of the sinh transformation in this case.

References

[1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover Publications Inc., New York, 1972.

[2] W. S. Hall, W. H. Robertson, Boundary element methods for acoustic wave scattering, in: C. A. Brebbia (Ed.), Boundary Elements X, Vol. 4, Springer–Verlag, Berlin, 1988, pp. 301–315.

[3] T. Lu, D. Yevick, A vectorial boundary element method analysis of integrated optical waveguides, Journal of Lightwave Technology 21 (8) (2003) 1793–1801.

[4] H. Schippers, Boundary–integral–equation methods for screen problems in acoustic and electromagnetic aerospace research, Journal of Engineering Mathematics 34 (1–2) (1998) 143–162.

[5] P. S. Ramesh, M. H. Lean, A boundary integral formulation for natural–convection flows,

Communications in Applied Numerical Methods 8 (6) (1992) 407–415.

[6] A. Brunet, F. Morestin, H. Walter-Leberre, Failure analysis of anisotropic sheet-metals using a non–local plastic damage model, Journal of Materials Processing Technology 170 (1–2) (2005) 457–470.

[7] C. Mei, C. S. Pates, Analysis of random structure–acoustic interaction problems using coupled boundary element and finite element methods, Technical Report 147, Old Dominion University Research Foundation, Norfolk, VA (1994).

[8] H. L. Pina, Analytical integration in the 2D boundary element method, Communications in Numerical Methods in Engineering 13 (9) (1997) 715–725.

[9] K. M. Singh, M. Tanaka, Analytical integration of weakly singular integrals in boundary element analysis of Helmholtz and advection–diffusion equations, Computer Methods in Applied Mechanics and Engineering 189 (2000) 625–640.

[10] P. S. Ramesh, M. H. Lean, Accurate integration of singular kernels in boundary integral formulations for Helmoltz equation, International Journal for Numerical Methods in Engineering 31 (1991) 1055–1068.

[11] J. D. Donaldson, D. Elliott, A unified approach to quadrature rules with asymptotic estimates of their remainders, SIAM Journal of Numerical Analysis 9 (1972) 573–602.

[12] P. J. Davis, Interpolation and Approximation, Dover Publications Inc., New York, 1975.

[13] D. M. Hough, Asymptotic estimation of Gaussian quadrature error for a nonsingular integral in potential theory, Mathematics of Computation 71 (2001) 717–727.

Captions

Table 1: $k = 0$, $\lambda = 3$, $n = 6$.

Table 2: $k = 1$, $\lambda = 2$, $n = 6$.

Table 3: $k = 2$, $\lambda = 1$, $n = 6$.

Table 4: Truncation errors and estimates for $k = 0$, $n = 30$.

Table 5: Truncation errors and estimates for $\lambda = 1$, $k = 2$, $n = 20$.

Table 6: Truncation errors and estimates for $\lambda = 2$, $k = 0$, $n = 25$

Figure 1: z–plane

Figure 2: w–plane

a	b	Actual Truncation Error	Equation (2.11) with $p = 0$	Truncation Error Transformed Integral
0.0	0.1	$+1.59 \times 10^{-7}$	$+1.85 \times 10^{-7}$	$+5.03 \times 10^{-3}$
0.25	0.01	$+1.19 \times 10^{-7}$	$+1.39 \times 10^{-7}$	$+3.91 \times 10^{-2}$
0.5	0.001	$+2.03 \times 10^{-8}$	$+2.34 \times 10^{-8}$	$+1.15 \times 10^{-1}$
0.75	0.0001	-8.84×10^{-8}	-1.03×10^{-7}	$+2.37 \times 10^{-1}$
1.00	0.0001	-1.52×10^{-7}	-1.77×10^{-7}	$+1.32 \times 10^{-1}$

Table 1

		Actual	Equation (2.12)	Truncation Error
a	b	Truncation Error	with $p = 0$	Transformed Integral
0.0	0.1	-1.10×10^{-20}	0	-1.10×10^{-20}
0.25	0.01	-4.10×10^{-9}	-4.11×10^{-9}	$+1.95 \times 10^{-2}$
0.5	0.001	-7.12×10^{-9}	-7.28×10^{-9}	$+1.32 \times 10^{-1}$
0.75	0.0001	-8.28×10^{-9}	-8.80×10^{-9}	$+3.33 \times 10^{-1}$
1.00	0.0001	-7.25×10^{-9}	-8.32×10^{-9}	-6.87×10^{-2}

Table 2

a	b	Actual Truncation Error	Equation (2.12) with $p = 1$	Truncation Error Transformed Integral
0.0	0.1	-4.89×10^{-11}	-5.01×10^{-11}	-1.53×10^{-3}
0.25	0.01	-4.70×10^{-11}	-4.86×10^{-11}	-9.64×10^{-4}
0.5	0.001	-4.14×10^{-11}	-4.44×10^{-11}	$+1.64 \times 10^{-2}$
0.75	0.0001	-3.25×10^{-11}	-3.77×10^{-11}	$+5.37 \times 10^{-2}$
1.00	0.0001	-2.10×10^{-11}	-2.88×10^{-11}	-1.49×10^{-1}

Table 3

		Actual error	Actual error	Equation
a	b	$\lambda = 1$	$\lambda = 5$	(3.18)
0.0	0.01	-2.85×10^{-2}	-2.86×10^{-2}	-3.54×10^{-2}
0.25	0.001	$+7.87 \times 10^{-2}$	$+7.89 \times 10^{-2}$	$+5.65 \times 10^{-2}$
0.5	0.0001	-3.74×10^{-2}	-3.75×10^{-2}	-4.87×10^{-2}
0.75	0.001	-1.58×10^{-2}	-1.58×10^{-2}	-4.89×10^{-3}
1.00	0.01	-1.28×10^{-5}	-1.28×10^{-5}	-1.38×10^{-5}

Table 4

		Actual	Estimated Error	Equation	Equation
a	b	Error	(4.34) and (4.25)	(4.34)	(4.25)
0.0	0.01	3.05×10^{-11}	3.13×10^{-11}	-9.17×10^{-13}	3.23×10^{-11}
0.25	0.001	1.19×10^{-8}	1.18×10^{-8}	-1.40×10^{-14}	1.18×10^{-8}
0.5	0.0001	4.19×10^{-7}	3.80×10^{-7}	$+4.09 \times 10^{-16}$	3.80×10^{-7}
0.75	0.001	7.90×10^{-8}	8.12×10^{-8}	-1.98×10^{-14}	8.12×10^{-8}
1.00	0.01	4.64×10^{-14}	4.65×10^{-14}	$+1.99 \times 10^{-21}$	4.65×10^{-14}

Table 5

a	b	Actual Error	Estimated Error (4.34) and (4.25)	Equation (4.34)	Equation (4.25)
0.0	0.0001	-1.025×10^{-8}	-1.06×10^{-8}	-9.82×10^{-9}	-7.51×10^{-10}
0.25	0.001	$+5.37 \times 10^{-10}$	$+5.42 \times 10^{-10}$	$+4.95 \times 10^{-10}$	$+4.65 \times 10^{-11}$
0.5	0.01	-7.70×10^{-11}	-7.71×10^{-11}	-7.68×10^{-11}	-3.04×10^{-13}
0.75	0.001	-1.99×10^{-9}	-2.00×10^{-9}	-2.16×10^{-9}	$+1.51 \times 10^{-10}$
1.00	0.0001	-9.83×10^{-12}	-10.1×10^{-12}	-5.83×10^{-19}	-10.1×10^{-12}

Table 6

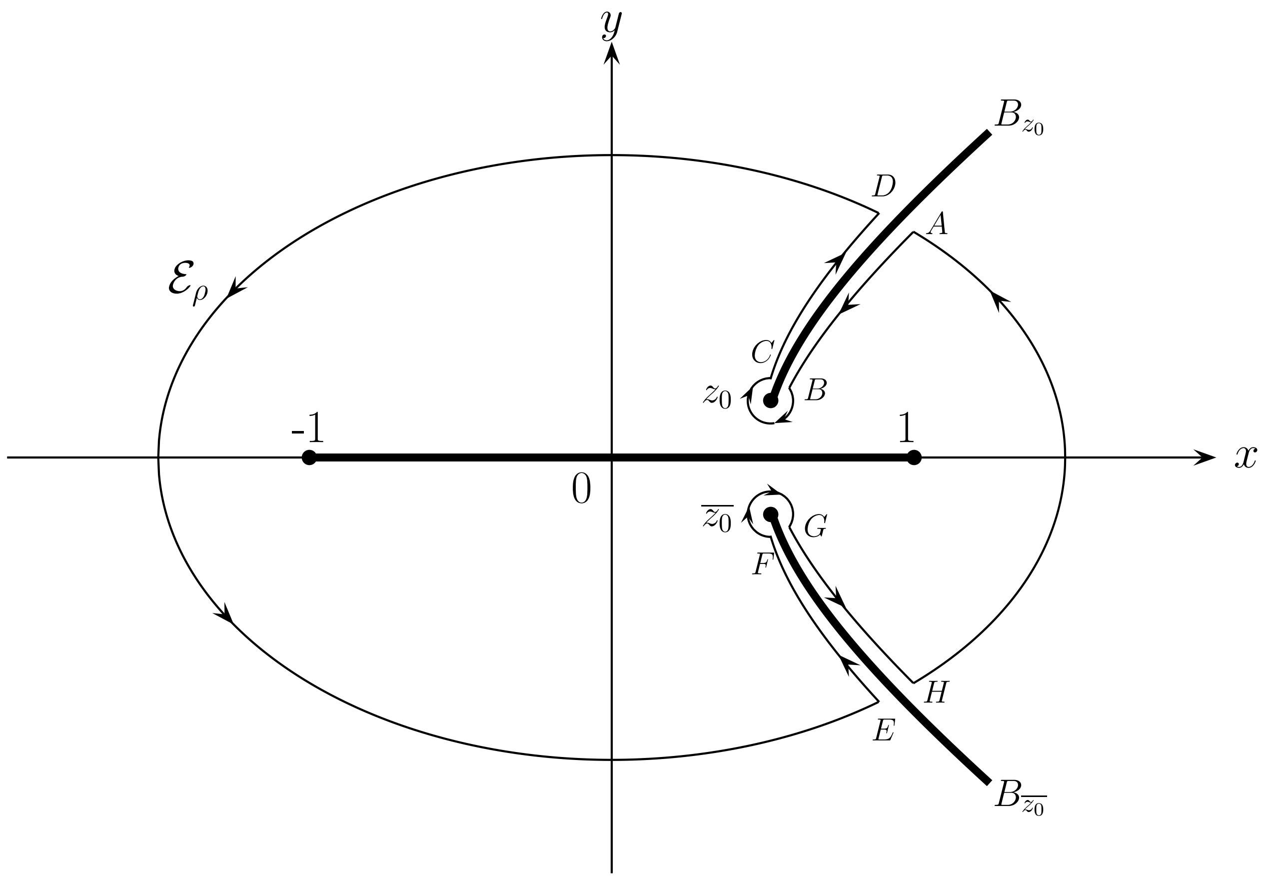
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Fig. 1.

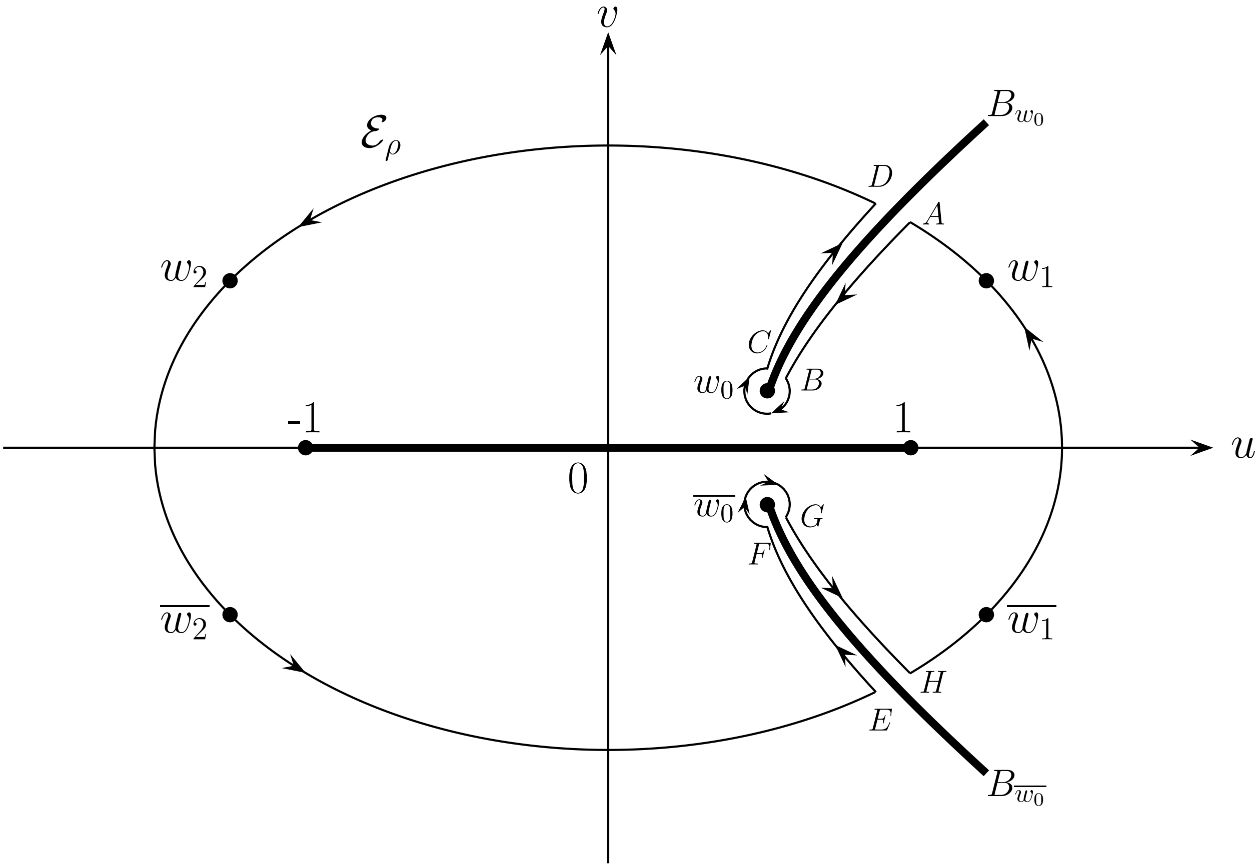
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Fig. 2.