$$D\{\sigma\} \equiv \left\langle \left[\frac{\sigma - \langle \sigma \rangle}{\langle \sigma \rangle} \right]^2 \right\rangle$$

$$= \left\langle \left[\frac{|E|^2 - \langle |E|^2 \rangle}{\langle |E|^2 \rangle} \right]^2 \right\rangle. \qquad (20)$$

The notation $\delta|E| = |E| - \langle |E| \rangle$ allows the right side of (20) to be expanded and written as

$$\left\langle \left[\frac{|E|^2 - \langle |E|^2 \rangle}{\langle |E|^2 \rangle} \right]^2 \right\rangle$$

$$= 4 \frac{\langle \delta |E|^2 \rangle}{\langle |E| \rangle^2} \left[1 + 0 \left\{ \frac{\delta |E|}{|E|} \right\} \right] \qquad (21)$$

so that, since both $\delta|E| \leq |\delta E|$ and, by assumption, $|\delta E| \ll |E|$, the factor in square brackets on the right of (21) is essentially unity and

$$D\{\sigma\} \approx 4 \frac{\langle \delta |E|^2 \rangle}{\langle |E| \rangle^2} = 4 \left\langle \left[\frac{|E| - \langle |E| \rangle}{\langle |E| \rangle} \right]^2 \right\rangle. \qquad (22)$$

By combining (19) with (22), the required expression (8) is obtained.

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TE-Wave Scattering by a Dielectric Cylinder of Arbitrary Cross-Section Shape

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Abstract—The theory and equations are developed for the scattering pattern of a dielectric cylinder of infinite length and arbitrary cross-section shape. The harmonic incident wave is assumed to have its electric vector perpendicular to the axis of the cylinder, and the fields are assumed to have no variations along this axis.

Although some investigators have approximated the field within the dielectric body by the incident field, a more accurate solution is obtained here by treating the field as an unknown function which is determined by solving a system of linear equations.

Scattering patterns obtained by this method are presented for dielectric shells of circular and semicircular cross section, and for a thin plane dielectric slab of finite width. The results for the circular shell agree accurately with the exact classical solution. The effects of surface-wave excitation and mutual interaction among the various portions of the shell are included automatically in this solution.

I. Introduction

Although rigorous solutions are available for scattering by homogeneous dielectric cylinders of circular or elliptical cross section, only approximate solutions exist for cylinders of other shapes. If the cross-section dimensions of the cylinder are small in comparison with the wavelength, accurate results may be obtained with the variational formulation or the quasi-static solutions, while the ray optics approach is often successful for large cylinders.

The technique employed here is an extension of the one developed by Richmond[1] for the case where the incident electric vector is parallel with the cylinder axis. When the electric vector is perpendicular to the axis, the problem is more complicated but the solution proceeds in a similar manner. The cross section of the dielectric cylinder is divided into cells which are small enough so that the electric field intensity is nearly uniform in each cell. The electric field intensity within each cell is initially considered to be an unknown quantity, and a system of linear equations is generated by enforcing at the center of each cell the condition that the total field must equal the sum of the incident and scattered fields. This system of equations is solved with the aid of a digital computer to evaluate the electric field intensity in each cell. It is then straightforward to calculate the distant scattering pattern of the cylinder.

The solution approaches the exact solution if a sufficiently large number of cells is employed. Calcula-

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[1] J. H. Richmond, "Scattering by a dielectric cylinder of arbitrary cross-section shape," *IEEE Trans. on Antennas and Propagation*, vol. AP-13, pp. 334–341, May 1965.

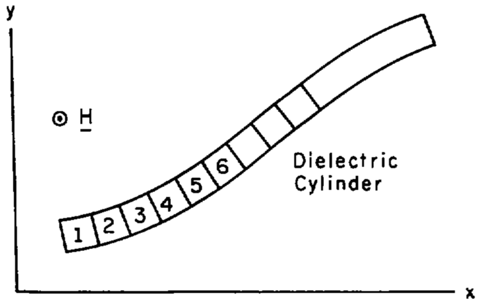

Fig. 1. Cross section of a dielectric cylinder, showing the coordinate system and the division into small cells.

tions can be obtained for a dielectric shell of arbitrary shape as quickly and systematically as for a circular shell. With a large digital computer it is now possible to obtain accurate solutions for dielectric cylinders having cross-section areas up to one square wavelength, and several techniques exist for extending the maximum size. Dielectric cylinders which are dissipative or inhomogeneous offer no complication.

II. The Basic Theory

Consider a harmonic wave incident in free space on a dielectric cylinder of arbitrary cross-section shape as indicated in Fig. 1. The time factor $e^{j\omega t}$ is understood, and it is assumed that the incident electric field intensity $\boldsymbol{E}^i$ has no z component, where the z axis is taken to be parallel with the axis of the cylinder. Thus,

$$\boldsymbol{E}^i = \hat{x}E_x{}^i(x, y) + \hat{y}E_y{}^i(x, y) \tag{1}$$

where $\hat{x}$ and $\hat{y}$ represent unit vectors parallel with the x and y axes. The dielectric cylinder is assumed to have the same permeability as free space ($\mu = \mu_0$). The dielectric material is assumed to be linear and isotropic, but it may be inhomogeneous with respect to the transverse coordinates:

$$\epsilon = \epsilon(x, y) \tag{2}$$

where ϵ represents the complex permittivity.

Let $\boldsymbol{E}$ represent the total electric field intensity; that is, the field generated by the source in the presence of the dielectric cylinder. The "scattered field" $\boldsymbol{E}^s$ is defined to be the difference between the total and the incident fields. Thus,

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s. \tag{3}$$

The scattered field may be generated by an equivalent electric current radiating in unbounded free space, where the current density is

$$J = j\omega(\epsilon - \epsilon_0)\boldsymbol{E} \tag{4}$$

with ω representing the angular frequency $2\pi f$.

For convenience, let us divide the cross-section area of the dielectric cylinder into cells which are sufficiently small that the dielectric constant and the electric field intensity are essentially uniform over each cell. The division into cells is indicated in Fig. 1. If the dielectric shell is not thin in comparison with the wavelength, two or more rows of cells must be used instead of the single row shown in Fig. 1.

It is possible to use numerical integration to calculate the scattered field from each rectangular cell shown in Fig. 1, but a substantial reduction in computer time is obtained by replacing each rectangular cell with an equivalent circular cell of the same cross-section area. Comparatively simple expressions are given below for the scattered fields of a circular cell.

If harmonic electric current densities J_x and J_y are distributed uniformly in a circular cylinder of radius a, these currents generate in free space a field given by

$$E_x{}^s(x, y) = K\{[k\rho y^2 H_0(k\rho) + (x^2 - y^2)H_1(k\rho)]J_x + xy[2H_1(k\rho) - k\rho H_0(k\rho)]J_y\} \tag{5}$$

$$E_y{}^s(x, y) = K\{xy[2H_1(k\rho) - k\rho H_0(k\rho)]J_x + [k\rho x^2 H_0(k\rho) + (y^2 - x^2)H_1(k\rho)]J_y\} \tag{6}$$

where ρ represents the distance from the center of the cylinder to the observation point, $k = 2\pi/\lambda$, λ is the wavelength in free space, $H_0(k\rho)$ and $H_1(k\rho)$ represent the Hankel functions with the superscript (2) understood,

$$K = -\frac{\pi a J_1(ka)}{2\omega\epsilon_0\rho^3} \tag{7}$$

and $J_1(ka)$ is the Bessel function of order one. Equations (5) and (6) represent the scattered field from one circular cell centered at the origin, with the observation point located outside the cell. These equations can be derived with the aid of the vector potential. If the observation point is at the center of the circular cell, the scattered field is given by

$$E_x{}^s(0, 0) = -[\pi ka H_1(ka) - 4j]J_x/(4\omega\epsilon_0) \tag{8}$$

$$E_y{}^s(0, 0) = -[\pi ka H_1(ka) - 4j]J_y/(4\omega\epsilon_0). \tag{9}$$

Equations (5)–(9) give the field scattered from a single cell, say cell n. The scattered field of the entire cylinder is, according to the superposition theorem, given by a summation of such terms with x and y replaced by $x - x_n$ and $y - y_n$ where x_n and y_n denote the coordinates of the center of cell n.

If (3) is enforced at the center of cell m, we obtain two scalar equations

$$E_{xm} - E_{xm}^s = E_{xm}^i \tag{10}$$

$$E_{ym} - E_{ym}^s = E_{ym}^i \tag{11}$$

where E_{xm}, E_{xm}^s, and E_{xm}^i represent the x components of the total, scattered, and incident electric field intensities in cell m. From (4), the equivalent electic current density in cell n is related to the total field in that cell as follows:

$$J_{xn} = j\omega\epsilon_0(\epsilon_n - 1)E_{xn} \tag{12}$$

$$J_{yn} = j\omega\epsilon_0(\epsilon_n - 1)E_{yn} \tag{13}$$

where ϵ_n is the complex relative dielectric constant in cell n. Equations (5)–(13) can be combined to obtain two linear relations among the total fields in the various cells:

$$\sum_{n=1}^{N} (A_{mn}E_{xn} + B_{mn}E_{yn}) = E_{xm}^i \tag{14}$$

$$\sum_{n=1}^{N} (C_{mn}E_{xn} + D_{mn}E_{yn}) = E_{ym}^i. \tag{15}$$

N represents the total number of cells, and the coefficients are found from (5)–(15) to be

$$A_{mn} = K'[k\rho(y_m - y_n)^2H_0(k\rho) + ((x_m - x_n)^2 - (y_m - y_n)^2)H_1(k\rho)] \tag{16}$$

$$B_{mn} = C_{mn} = K'(x_m - x_n)(y_m - y_n)[2H_1(k\rho) - k\rho H_0(k\rho)] \tag{17}$$

$$D_{mn} = K'[k\rho(x_m - x_n)^2H_0(k\rho) + ((y_m - y_n)^2 - (x_m - x_n)^2)H_1(k\rho)] \tag{18}$$

where

$$K' = \frac{j\pi a_nJ_1(ka_n)(\epsilon_{n-1})}{2\rho^3} \tag{19}$$

$$\rho = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \tag{20}$$

and a_n represents the radius of the circular cell having the same area as cell n. From (8) and (9), the diagonal elements are found to be

$$A_{mm} = D_{mm} = 1 + (\epsilon_m - 1)[0.25j\pi ka_mH_1(ka_m) + 1] \tag{21}$$

$$B_{mm} = C_{mm} = 0. \tag{22}$$

Equations (14) and (15) represent a system of $2N$ linear equations if we let $m=1, 2, 3, \cdots, N$, thus enforcing (10) and (11) at the center of each cell. This system of equations is solved with the aid of a digital computer to determine the field distribution in the dielectric cylinder. Thereafter it is not difficult to compute the distant scattering pattern of the cylinder.

If the cells are rectangular with a thickness differing greatly from the width, it is not reasonable to replace them with equivalent circular cells. Thus, it is assumed here that the cross section is divided into cells having essentially square cross sections.

Sections III–VII describe the calculation of the distant scattering pattern and present the numerical results obtained by this method for dielectric cylinders of various cross-section shapes.

III. Formulating the Scattered Field

Once the system of linear equations [(14) and (15)] has been solved, the scattered field can be calculated at any point in space. If the observation point is outside the dielectric region, (5), (6), (12), and (13) can be used.

Let ρ_n denote the distance from the center of cell n to the observation point (x, y). The distant scattering pattern of the dielectric cylinder is obtained by employing the asymptotic form for the Hankel function of large argument and taking

$$\rho_n = \rho_0 - x_n \cos\phi - y_n \sin\phi \tag{23}$$

where ρ_0 and ϕ are the cylindrical coordinates of the distant observation point. The distant scattered field has only a ϕ component which is given by

$$E^s = j\sqrt{j\pi k/2\rho_0}\, e^{-jk\rho_0} \cdot \sum_{n=1}^{N} (\epsilon_n - 1)a_nJ_1(ka_n)[E_{xn}\sin\phi - E_{yn}\cos\phi] \cdot e^{jk(x_n\cos\phi + y_n\sin\phi)}. \tag{24}$$

If the incident field is a plane wave, the bistatic echo width of the cylinder is given by

$$W/\lambda = (\pi/2)\left|\frac{k}{E^i}\sum_{n=1}^{N} (\epsilon_n - 1)a_nJ_1(ka_n) \cdot [E_{xn}\sin\phi - E_{yn}\cos\phi]e^{jk(x_n\cos\phi + y_n\sin\phi)}\right|^2. \tag{25}$$

IV. Scattering by a Slender Dielectric Rod

It is possible to test the solution developed above by applying it to the problem of scattering by a slender dielectric rod. Consider a harmonic plane wave, traveling in the x direction, to be incident on a dielectric rod whose axis coincides with the z axis. Let the incident wave be polarized in the y direction. If the radius a of the rod is small in comparison with the wavelength, the electric field will be nearly uniform throughout the rod so that no division into cells is necessary. That is, the total number of cells will be $N=1$. In this case (14) and (15) yield only two linear equations

$$A_{11}E_{x1} + B_{11}E_{y1} = 0 \tag{26}$$

$$C_{11}E_{x1} + D_{11}E_{y1} = 1. \tag{27}$$

The solution is readily obtained with the aid of (21) and (22):

$$E_{x1} = 0 \tag{28}$$

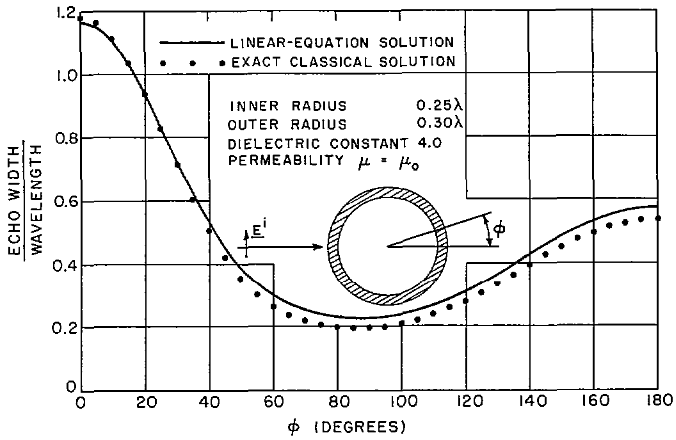

Fig. 2. Calculated scattering pattern of dielectric cylindrical shell of circular cross-section shape (lossless case).

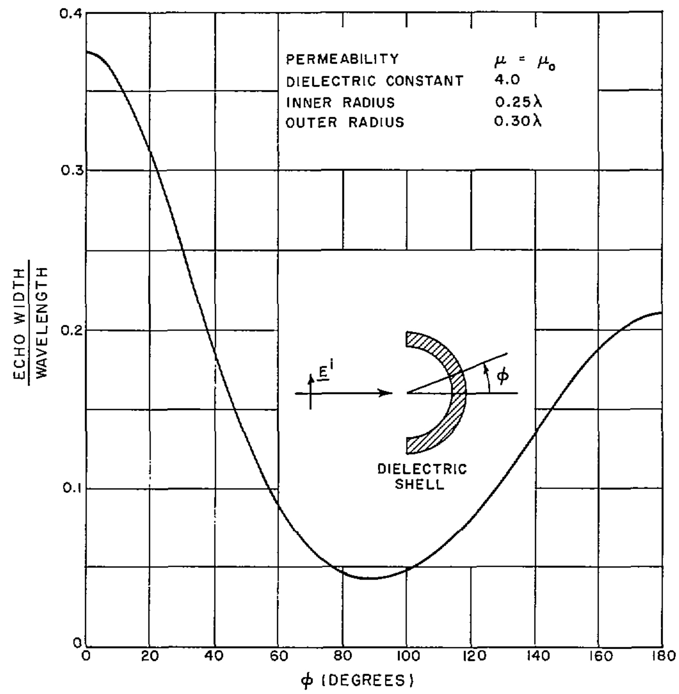

Fig. 3. Calculated scattering pattern of dielectric cylindrical shell of semicircular cross-section shape (lossless case).

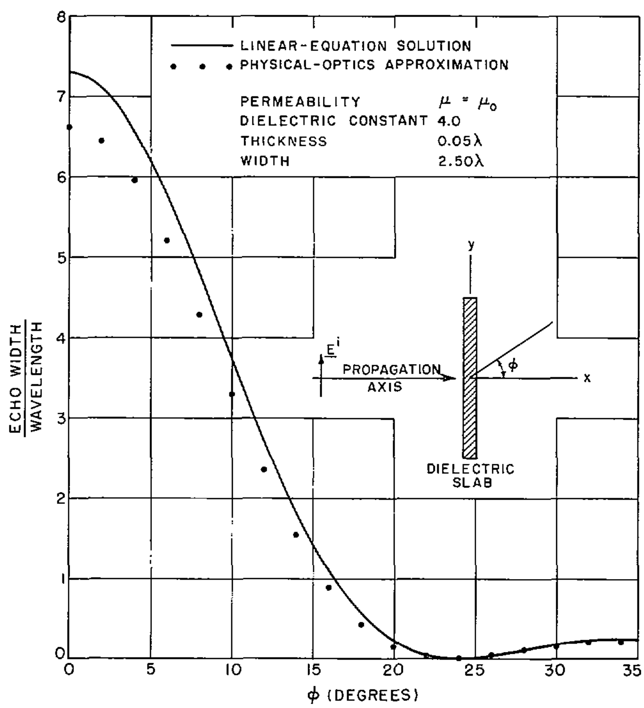

Fig. 4. Calculated scattering patterns of a homogeneous, lossless, plane dielectric slab with a plane wave having normal incidence.

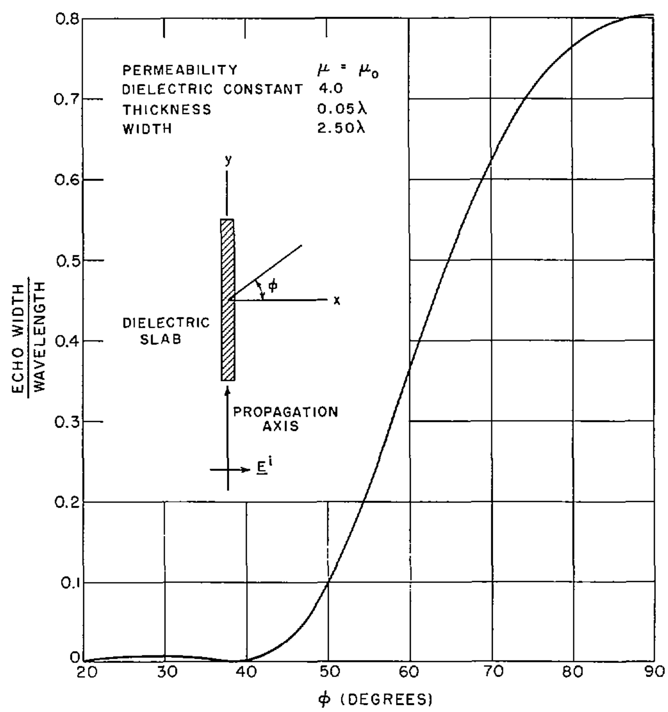

Fig. 5. Linear equation solution for the scattering pattern of a homogeneous, lossless, plane dielectric slab with a plane wave having grazing incidence.

$$E_{y1} = \frac{1}{1 + (\epsilon_1 - 1)[0.25j\pi kaH_1(ka) + 1]}. \qquad (29)$$

From (25) the echo width of the slender dielectric rod is given by

$$W/\lambda = (\pi/2) \mid ka(\epsilon_1 - 1)J_1(ka)E_{y1} \cos \phi \mid^2. \qquad (30)$$

Comparison with the exact classical solution for the circular dielectric cylinder[2] shows that (29) and (30) are accurate when the radius is small in comparison with the wavelength in the dielectric material. In particular, the linear equation solution is accurate when

$$a/\lambda < \frac{0.06}{\sqrt{\epsilon_r}} \qquad (31)$$

where λ is the wavelength in free space and ϵ_r is the relative permittivity.

V. Numerical Results for the Circular Cylindrical Shell

Figure 2 shows the scattering pattern of a cylindrical shell of circular cross section, calculated with the aid of (14)–(22) and (25). The result show excellent agreement with the exact classical solution which is also shown.

In this example, the dielectric shell was divided into 34 cells with each cell having a thickness of 0.05λ. Ordinarily there would be 68 complex linear equations, but these were reduced to 34 equations by taking advantage of the even symmetry of E_y and the odd symmetry of E_x about the x axis. The entire calculation was accomplished in 0.8 minute with an IBM 7094 computer, including the solution for the field distribution in the dielectric shell and the distant scattering pattern.

VI. The Semicircular Cylindrical Shell

Although no exact solution is available for the semicircular cylindrical shell, the linear-equation solution proceeds in the same manner as for the complete circular shell or any other cross-section shape.

Figure 3 shows the scattering pattern for a dielectric cylindrical shell of semicircular cross-section shape, calculated with the linear-equation technique. The computer program for this problem provides for an arbitrary angle of incidence and an arbitrary portion of a complete circular shell. An approximate solution is possible with ray-optical methods but the accuracy would be questionable for semicircular cylinders of small radii. Furthermore, such a solution becomes complicated when one includes multiple reflected rays and the effects of refraction.

[2] J. R. Wait, *Electromagnetic Radiation from Cylindrical Structures*. New York: Pergamon, 1959, pp. 142–145.

VII. The Plane Dielectric Slab of Finite Width and Thickness

Figures 4 and 5 show the calculated scattering patterns of a plane dielectric slab with a plane wave having normal and grazing incidence, respectively. For comparison, the physical-optics solution is also shown for normal incidence. The physical-optics solution is based on the approximation that the electric field intensity within the slab is the same as that in an infinitely wide slab. The two solutions show reasonably good agreement in Fig. 4.

In the linear-equation solutions, the dielectric slab was divided into 50 square cells in a single row.

VIII. Conclusions

A technique is described for calculating accurately the scattered fields of a dielectric cylinder of infinite length and arbitrary cross-section shape. The harmonic incident field is assumed to have its electric vector perpendicular to the axis of the cylinder.

The cross section of the cylinder is divided into small cells and a system of linear equations is obtained for the electric field intensities in these cells. These equations are solved with the aid of a digital computer to determine the field distribution in the dielectric cylinder. The scattered field at any point in space can then be calculated by integrating the known fields over the cross-section area of the cylinder.

Examples are included to show the scattering patterns of dielectric rods, circular and semicircular cylindrical shells, and plane slabs of finite width and thickness. Any other shape can be treated simply by inserting into the equations the coordinates which define the cross-section shape, the complex dielectric constant at the center of each cell, and the incident electric field intensity in each cell.