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The electromagnetic field of a horizontal electric dipole in the presence of a three-layered region

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The electromagnetic field generated by a horizontal electric dipole in the air over the surface of a two-layered region is determined for continuous-wave excitation. The region of interest consists of a conductor coated with an electrically thin layer of dielectric under a half-space of air. Simple explicit formulas are derived for the field at all points in all three regions, including the surface wave. Typical applications are to microstrip circuits and antennas and to remote sensing from the arctic ice.

I. INTRODUCTION

An interesting and practically important three-layered region consists of a half-space of air (Region 0, $z\leqslant 0$) over a dielectric layer with the uniform thickness l (Region 1, $0\leqslant z\leqslant l$) that coats a conducting or dielectric medium (Region 2, $l\leqslant z$), as illustrated in Fig. 1. The three regions are characterized, respectively, by the following wave numbers:

$$k_0=\omega/c=\omega(\mu_0\epsilon_0)^{1/2};$$

$$k_1=\epsilon_{1r}^{1/2}k_0;\ \ k_2=k_0(\epsilon_{2r}+i\sigma_2/\omega\epsilon_0)^{1/2}, \tag{1}$$

where ϵ_r is the relative permittivity and use is made of the time dependence $e^{-i\omega t}$. In the applications of primary interest, Region 2 is a good conductor, so that

$$\omega\epsilon_0\epsilon_{2r}\ll\sigma_2;\ \ k_2\sim(i\omega\mu_0\sigma_2)^{1/2}, \tag{2}$$

but in the derivations and final formulas k_2 is general as defined in Eq. (1). The properties of the three regions at the operating frequency are assumed to satisfy the following inequalities:

$$k_0^2\ll k_1^2\ll|k_2^2|, \tag{3a}$$

$$k_1^2l^2\ll 1. \tag{3b}$$

Examples of layered regions that satisfy these conditions at widely different frequencies are: (1) the dielectric-coated conductors used in microstrip in the range $1\leqslant f\leqslant 100$ GHz, and (2) ice-coated sea water in the very low-frequency range $10\leqslant f\leqslant 100$ Hz. These are discussed in turn below.

(1) A horizontal electric dipole located on the dielectric substrate of the microstrip is the basic differential element of a microstrip transmission line or a microstrip patch antenna. Of primary interest is the field when the dipole is in the air on the surface of the dielectric layer. In the case of transmission lines, the field of interest is everywhere along the surface of the dielectric where interacting circuit elements are located. For microstrip patch antennas, the far field at all points in the air is of primary interest, but the field on the dielectric surface is also needed to determine the coupling among elements in an array.

(2) Remote sensing with a horizontal dipole on the ice surface of a frozen lake or ocean is an application of special interest. It provides a possible means for locating submarines under the arctic ice. In the practical arrangement a horizontal transmitting dipole is located on the surface of the ice and a receiver is moved over the surface to detect the electric or magnetic field scattered by the submarine. Details of the technique when there is no ice layer and both the transmitting and receiving antennas are in the ocean have been presented elsewhere.[1]

It is the purpose of this paper to outline the rigorous analytical foundations and the derivation of simple formulas for the six components of the electromagnetic field at all points in the three regions when the horizontal dipole is at any point in the air, including especially on the surface of the dielectric.

II. GENERAL INTEGRALS FOR THE FIELD IN THE AIR

The field generated by a horizontal electric dipole with unit electric moment ($Ih_e = 1$ Am) located at a height d' over the surface of the dielectric layer is complicated. It is convenient to determine first the field in the air. To simplify the notation, let $z' = -z$ be directed upward from the dielectric into the air. The general integrals are like those for a two-layered region given in King and Smith[2] and King[3] but with generalized reflection coefficients. In the cylindrical coordinates ρ, $\phi' = -\phi$, $z' = -z$, the components of the field in the desired arrangement of the integrals with a negative image are

$$E_{0\rho}(\rho,\phi',z') = -\frac{\omega\mu_0}{4\pi k_0^2}\cos\phi'[F_{\rho 0}(\rho,z'-d') - F_{\rho 0}(\rho,z'+d') + F_{\rho 1}(\rho,z'+d')], \tag{4}$$

$$E_{0\phi'}(\rho,\phi',z') = \frac{\omega\mu_0}{4\pi k_0^2}\sin\phi'[F_{\phi' 0}(\rho,z'-d') - F_{\phi' 0}(\rho,z'+d') + F_{\phi' 1}(\rho,z'+d')], \tag{5}$$

where

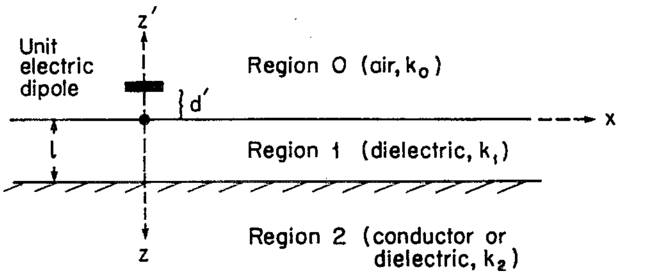

FIG. 1. Unit electric dipole at height d' over plane boundary ($z=0$) between air and a sheet of dielectric with thickness l over a conducting or dielectric half-space.

$$\left.\begin{array}{l} F_{\rho 0}(\rho,z'-d') \\ F_{\phi' 0}(\rho,z'-d') \end{array}\right\} = \int_0^\infty \left\{ \frac{\gamma_0}{2}[J_0(\lambda\rho) \mp J_2(\lambda\rho)] + \frac{k_0^2}{2\gamma_0} \times [J_0(\lambda\rho) \pm J_2(\lambda\rho)] \right\} e^{i\gamma_0|z'-d'|}\lambda d\lambda,$$

$$\left.\begin{array}{l} F_{\rho 0}(\rho,z'+d') \\ F_{\phi' 0}(\rho,z'+d') \end{array}\right\} = \int_0^\infty \left\{ \frac{\gamma_0}{2}[J_0(\lambda\rho) \mp J_2(\lambda\rho)] + \frac{k_0^2}{2\gamma_0} \times [J_0(\lambda\rho) \pm J_2(\lambda\rho)] \right\} e^{i\gamma_0|(z'+d')|}\lambda d\lambda, \tag{7}$$

$$F_{\rho 1}(\rho,z'+d') = F_{\rho 2}(\rho,z'+d') + F_{\rho 3}(\rho,z'+d'), \tag{8}$$

$$F_{\phi' 1}(\rho,z'+d') = F_{\phi' 2}(\rho,z'+d') + F_{\phi' 3}(\rho,z'+d'), \tag{9}$$

with

$$\left.\begin{array}{l} F_{\rho 2}(\rho,z'+d') \\ F_{\phi' 2}(\rho,z'+d') \end{array}\right\} = \frac{1}{2}\int_0^\infty \gamma_0(Q_3+1)[J_0(\lambda\rho) \mp J_2(\lambda\rho)] \times e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{10}$$

$$\left.\begin{array}{l} F_{\rho 3}(\rho,z'+d') \\ F_{\phi' 3}(\rho,z'+d') \end{array}\right\} = -\frac{k_0^2}{2}\int_0^\infty \gamma_0^{-1}(P_3-1) \times [J_0(\lambda\rho) \pm J_2(\lambda\rho)] e^{i\gamma_0(z'+d')}\lambda d\lambda; \tag{11}$$

$$E_{0z'}(\rho,\phi',z') = \frac{i\omega\mu_0}{4\pi k_0^2}\cos\phi'[F_{z'0}(\rho,z'-d') - F_{z'0}(\rho,z'+d') + F_{z'1}(\rho,z'+d')], \tag{12}$$

where

$$F_{z'0}(\rho,z'-d') = \pm\int_0^\infty J_1(\lambda\rho)e^{i\gamma_0|z'-d'|}\lambda^2 d\lambda; \quad \begin{cases} z'>d' \\ 0\leqslant z'\leqslant d' \end{cases}, \tag{13}$$

$$F_{z'0}(\rho,z'+d') = \int_0^\infty J_1(\lambda\rho)e^{i\gamma_0(z'+d')}\lambda^2 d\lambda, \tag{14}$$

$$F_{z'1}(\rho,z'+d') = \int_0^\infty (Q_3+1)J_1(\lambda\rho)e^{i\gamma_0(z'+d')}\lambda^2 d\lambda; \tag{15}$$

$$B_{0\rho}(\rho,\phi',z') = -\frac{\mu_0}{4\pi}\sin\phi'[G_{\rho 0}(\rho,z'-d') - G_{\rho 0}(\rho,z'+d') + G_{\rho 1}(\rho,z'+d')], \tag{16}$$

$$B_{0\phi'}(\rho,\phi',z') = -\frac{\mu_0}{4\pi}\cos\phi'[G_{\phi' 0}(\rho,z'-d') - G_{\phi' 0}(\rho,z'+d') + G_{\phi' 1}(\rho,z'+d')], \tag{17}$$

where

$$G_{\rho 0}(\rho,z'-d') = G_{\phi' 0}(\rho,z'-d') = \pm\int_0^\infty J_0(\lambda\rho)e^{i\gamma_0|z'-d'|}\lambda d\lambda; \quad \begin{cases} z'>d' \\ 0\leqslant z'\leqslant d' \end{cases}, \tag{18}$$

$$G_{\rho 0}(\rho,z'+d') = G_{\phi' 0}(\rho,z'+d') = \int_0^\infty J_0(\lambda\rho)e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{19}$$

$$G_{\rho 1}(\rho,z'+d') = G_{\rho 2}(\rho,z'+d') + G_{\rho 3}(\rho,z'+d'), \tag{20}$$

$$G_{\phi' 1}(\rho,z'+d') = G_{\phi' 2}(\rho,z'+d') + G_{\phi' 3}(\rho,z'+d'), \tag{21}$$

with

$$\left.\begin{array}{l} G_{\rho 2}(\rho,z'+d') \\ G_{\phi' 2}(\rho,z'+d') \end{array}\right\} = \frac{1}{2}\int_0^\infty (Q_3+1)[J_0(\lambda\rho) \pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{22}$$

$$\left.\begin{array}{l} G_{\rho 3}(\rho,z'+d') \\ G_{\phi' 3}(\rho,z'+d') \end{array}\right\} = -\frac{1}{2}\int_0^\infty (P_3-1)[J_0(\lambda\rho) \mp J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda; \tag{23}$$

$$B_{0z'}(\rho,\phi',z') = \frac{i\mu_0}{4\pi}\sin\phi'[G_{z'0}(\rho,z'-d') - G_{z'0}(\rho,z'+d') + G_{z'1}(\rho,z'+d')], \tag{24}$$

where

$$G_{z'0}(\rho,z'\mp d') = \int_0^\infty \gamma_0^{-1}J_1(\lambda\rho)e^{i\gamma_0|z'\mp d'|}\lambda^2 d\lambda, \tag{25}$$

$$G_{z'1}(\rho,z'+d') = -\int_0^\infty (P_3-1)\gamma_0^{-1}J_1(\lambda\rho) \times e^{i\gamma_0(z'+d')}\lambda^2 d\lambda, \tag{26}$$

III. THE GENERALIZED COEFFICIENTS OF REFLECTION

The quantities Q_3 and P_3 in the integrands of several of the integrals are the negative of the plane-wave reflection coefficients, respectively, of electric and magnetic type. The reflection coefficients for a plane wave incident from the air on the dielectric boundary are readily determined from the boundary conditions. The general formula is

$$f_r=\frac{x_0-x_2-i[(x_0x_2/x_1)-x_1]\tan\gamma_1 l}{x_0+x_2-i[(x_0x_2/x_1)+x_1]\tan\gamma_1 l} \tag{27}$$

The coefficients of electric and magnetic types are

$$-Q_3=f_{er}=f_r \text{ with } x_j=\frac{\gamma_j}{k_j^2};\ \gamma_j=(k_j^2-\lambda^2)^{1/2},\ j=0,1,2, \tag{28a}$$

$$-P_3=f_{mr}=f_r \text{ with } x_j=\gamma_j. \tag{28b}$$

Note that, when $l=0$, $f_r=(x_0-x_2)/(x_0+x_2)$ and, when $l\to\infty$, $f_r=(x_0-x_1)/(x_0+x_1)$—in agreement with Eq. (41) in Ref. 3 for the two-layer region. With (28a) and (28b),

$$\frac{\gamma_0}{2}(Q_3+1)=\frac{k_0^2\gamma_0}{k_1^2}\left(\frac{k_1^2\gamma_2}{k_2^2}-i\gamma_1\tan\gamma_1 l\right)\times\left[\gamma_0-\frac{ik_0^2\gamma_1}{k_1^2}\tan\gamma_1 l+\frac{k_0^2\gamma_2}{k_2^2}-\frac{ik_1^2\gamma_0\gamma_2}{k_2^2\gamma_1}\tan\gamma_1 l\right]^{-1}, \tag{29}$$

$$\frac{k_0^2}{2\gamma_0}(P_3-1)=-k_0^2\left(\frac{\gamma_1}{\gamma_2}-i\tan\gamma_1 l\right)\left[\gamma_1-i\gamma_0\tan\gamma_1 l+\frac{\gamma_1}{\gamma_2}(\gamma_0-i\gamma_1\tan\gamma_1 l)\right]^{-1}. \tag{30}$$

IV. EVALUATION OF INTEGRALS

The integrals in the functions F and G with subscript 0 apply to the isolated dipole and its perfect image. They are well-known and can readily be evaluated without approximation with the help of standard formulas. The exact results are

$$F_{\rho 0}(\rho,z'\pm d')=-e^{ik_0r}\left[\frac{2k_0}{r^2}+\frac{2i}{r^3}+\left(\frac{z'\pm d'}{r}\right)^2\times\left(\frac{ik_0^2}{r}-\frac{3k_0}{r^2}-\frac{3i}{r^3}\right)\right], \tag{31}$$

$$F_{\phi' 0}(\rho,z'\pm d')=-e^{ik_0r}\left(\frac{ik_0^2}{r}-\frac{k_0}{r^2}-\frac{i}{r^3}\right), \tag{32}$$

$$F_{z'0}(\rho,z'\pm d')=-e^{ik_0r}\left(\frac{\rho}{r}\right)\left(\frac{z'\pm d'}{r}\right)\left(\frac{k_0^2}{r}+\frac{3ik_0}{r^2}-\frac{3}{r^3}\right), \tag{33}$$

$$G_{\rho 0}(\rho,z'\pm d')=G_{\phi' 0}(\rho,z'\pm d')=-e^{ik_0r}\left(\frac{z'\pm d'}{r}\right)\left(\frac{ik_0}{r}-\frac{1}{r^2}\right), \tag{34}$$

$$G_{z'0}(\rho,z'\pm d')=-e^{ik_0r}\left(\frac{\rho}{r}\right)\left(\frac{k_0}{r}+\frac{i}{r^2}\right). \tag{35}$$

In these formulas, $r=r_2=[\rho^2+(z'+d')^2]^{1/2}$ for the upper signs, $r=r_1=[\rho^2+(z'-d')^2]^{1/2}$ for the lower signs.

The integrals that contain (Q_3+1) in the integrand can be simplified with the help of a procedure (illustrated in Appendix A) which depends on the inequalities in Eq. (3a). The results from the procedure show that the only radially propagating terms are obtained from the integrals when the variable of integration has the order of magnitude $\lambda\sim k_0$. For larger values of λ in the ranges $\lambda\sim k_1$ and $\lambda\sim k_2$, the contributions are highly attenuated, radially nonpropagating terms that can be omitted from the useful solution. With the approximation (3b), the integrals to be evaluated are

$$\left.\begin{matrix}F_{\rho 2}(\rho,z'+d')\\ F_{\phi' 2}(\rho,z'+d')\end{matrix}\right\}\sim\epsilon'\int_0^\infty\frac{\gamma_0}{\gamma_0+\epsilon'}[J_0(\lambda\rho)\mp J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{36}$$

$$F_{z'1}(\rho,z'+d')\sim 2\epsilon'\int_0^\infty\frac{1}{\gamma_0+\epsilon'}J_1(\lambda\rho)e^{i\gamma_0(z'+d')}\lambda^2 d\lambda, \tag{37}$$

$$\left.\begin{matrix}G_{\rho 2}(\rho,z'+d')\\ G_{\phi' 2}(\rho,z'+d')\end{matrix}\right\}\sim\epsilon'\int_0^\infty\frac{1}{\gamma_0+\epsilon'}[J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{38}$$

where

$$\epsilon'=k_0\epsilon;\ \epsilon=\frac{k_0(k_2^{-1}-il)}{1-ik_1^2l/k_2}. \tag{39}$$

ϵ is a small quantity since $k_0^2\ll k_1^2\ll|k_2|^2, k_0^2l^2\ll k_1^2l^2\ll$.

Similarly, the integrals that include (P_3-1) in the integrand reduce to

$$\left.\begin{matrix}F_{\rho 3}(\rho,z'+d')\\ F_{\phi' 3}(\rho,z'+d')\end{matrix}\right\}\sim\epsilon'\int_0^\infty\frac{1}{1+\gamma_0(\epsilon'/k_0^2)}[J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda$$
$$\sim\epsilon'\int_0^\infty\left(1-\frac{\gamma_0\epsilon'}{k_0^2}+\frac{\gamma_0^2\epsilon'^2}{k_0^4}\cdots\right)\times[J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{40}$$

$$\left.\begin{matrix}G_{\rho 3}(\rho,z'+d')\\ G_{\phi' 3}(\rho,z'+d')\end{matrix}\right\}\sim\frac{\epsilon'}{k_0^2}\int_0^\infty\frac{\gamma_0}{1+\gamma_0(\epsilon'/k_0^2)}[J_0(\lambda\rho)\mp J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda$$
$$\sim\frac{\epsilon'}{k_0^2}\int_0^\infty\gamma_0\left(1-\frac{\gamma_0\epsilon'}{k_0^2}+\frac{\gamma_0^2\epsilon'^2}{k_0^4}\cdots\right)\times[J_0(\lambda\rho)\mp J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda, \tag{41}$$

$$G_{z'1}(\rho,z'+d')\sim\frac{2\epsilon'}{k_0^2}\int_0^\infty\frac{1}{1+\gamma_0(\epsilon'/k_0^2)}J_1(\lambda\rho)\times e^{i\gamma_0(z'+d')}\lambda^2 d\lambda$$
$$\sim\frac{2\epsilon'}{k_0^2}\int_0^\infty\left(1-\frac{\gamma_0\epsilon'}{k_0^2}+\frac{\gamma_0^2\epsilon'^2}{k_0^4}\cdots\right)\times J_1(\lambda\rho)e^{i\gamma_0(z'+d')}\lambda^2 d\lambda. \qquad (42)$$

In the second forms for Eqs. (40)–(42), the expansion

$$\left(1+\frac{\gamma_0\epsilon'}{k_0^2}\right)^{-1}\sim 1-\frac{\gamma_0\epsilon'}{k_0^2}+\frac{\gamma_0^2\epsilon'^2}{k_0^4}\cdots \qquad (43)$$

is made. Note that

$$\frac{\gamma_0\epsilon'}{k_0^2}=\gamma_0(k_2^{-1}-il)\left(1-\frac{ik_1^2 l}{k_2}\right)^{-1}, \qquad (44)$$

so that the assumption underlying the expansion is that both γ_0/k_2 and $\gamma_0 l$ are small. Since $k_0^2 \ll k_1^2 \ll |k_2^2|$ and $k_0^2 l^2 \ll k_1^2 l^2 \ll 1$, this is certainly true not only when $\lambda\sim k_0$ but also when $\lambda\sim k_1$. Since contributions to the integral come largely when $\lambda\sim k_0$, the series of integrated terms obtained from the second forms in Eqs. (40)–(42) should converge rapidly. This turns out to be true.

The integrals in Eqs. (36)–(38) are evaluated in Appendix B; the integrals in Eqs. (40)–(42) in Appendix C.

V. INTEGRATED FORMULAS FOR THE COMPONENTS OF THE FIELD

When the integrated formulas for $F_0(\rho,z'-d')$, $F_0(\rho,z'+d')$, $F_2(\rho,z'+d')$, and $F_3(\rho,z'+d')$ are combined in Eqs. (4), (5), and (12), the final formulas for the components of the electric field are

$$E_{0\rho}(\rho,\phi',z')=\frac{\omega\mu_0}{4\pi k_0}\cos\phi'\Bigg(e^{ik_0r_1}\left[\frac{2}{r_1^2}+\frac{2i}{k_0r_1^3}+\left(\frac{z'-d'}{r_1}\right)^2\left(\frac{ik_0}{r_1}-\frac{3}{r_1^2}-\frac{3i}{k_0r_1^3}\right)\right]-e^{ik_0r_2}\left[\frac{2}{r_2^2}+\frac{2i}{k_0r_2^3}+\left(\frac{z'+d'}{r_2}\right)^2\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0r_2^3}\right)\right]+2e^{ik_0r_2}\left\{\epsilon\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)-\epsilon^2\left[\frac{ik_0}{r_2}-\frac{1}{r_2^2}-\frac{i}{k_0r_2^3}-k_0^2\epsilon\left(\frac{r_2}{\rho}\right)\left(\frac{\pi}{k_0r_2}\right)^{1/2}e^{-iP_2}\mathscr{F}(P_2)\right]\right\}\Bigg), \qquad (45)$$

$$E_{0\phi'}(\rho,\phi',z')=-\frac{\omega\mu_0}{4\pi k_0}\sin\phi'\Bigg(e^{ik_0r_1}\left(\frac{ik_0}{r_1}-\frac{1}{r_1^2}-\frac{i}{k_0r_1^3}\right)-e^{ik_0r_2}\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}-\frac{i}{k_0r_2^3}\right)+2e^{ik_0r_2}\left\{\epsilon\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)-\epsilon^2\left[\frac{2}{r_2^2}+\frac{2i}{k_0r_2^3}+\left(\frac{z'+d'}{r_2}\right)^2\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0r_2^3}\right)+ik_0\epsilon\left(\frac{r_2^2}{\rho^3}\right)\left(\frac{\pi}{k_0r_2}\right)^{1/2}e^{-iP_2}\mathscr{F}(P_2)\right]\right\}\Bigg), \qquad (46)$$

$$E_{0z'}(\rho,\phi',z')=-\frac{\omega\mu_0}{4\pi k_0}\cos\phi'\Bigg\{e^{ik_0r_1}\left(\frac{\rho}{r_1}\right)\left(\frac{z'-d'}{r_1}\right)\left(\frac{ik_0}{r_1}-\frac{3}{r_1^2}-\frac{3i}{k_0r_1^3}\right)-e^{ik_0r_2}\left(\frac{\rho}{r_2}\right)\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0r_2^3}\right)+2\epsilon e^{ik_0r_2}\left[\left(\frac{\rho}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)-k_0^2\epsilon\left(\frac{\pi}{k_0r_2}\right)^{1/2}e^{-iP_2}\mathscr{F}(P_2)\right]\Bigg\}. \qquad (47)$$

Similarly, the substitution of the integrated formulas for $G_0(\rho,z'-d')$, $G_0(\rho,z'+d')$, $G_2(\rho,z'+d')$, and $G_3(\rho,z'+d')$ into Eqs. (16), (17), and (24) gives the following final formulas for the components of the magnetic field:

$$B_{0\rho}(\rho,\phi',z')=\frac{\mu_0}{4\pi}\sin\phi'\Bigg\{e^{ik_0r_1}\left(\frac{z'-d'}{r_1}\right)\left(\frac{ik_0}{r_1}-\frac{1}{r_1^2}\right)-e^{ik_0r_2}\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)+2\epsilon e^{ik_0r_2}\left[\frac{2}{r_2^2}+\frac{2i}{k_0r_2^3}+\left(\frac{z'+d'}{r_2}\right)^2\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0r_2^3}\right)+ik_0\epsilon\left(\frac{r_2^2}{\rho^3}\right)\left(\frac{\pi}{k_0r_2}\right)^{1/2}e^{-iP_2}\mathscr{F}(P_2)\right]\Bigg\}, \qquad (48)$$

$$B_{0\phi'}(\rho,\phi',z')=\frac{\mu_0}{4\pi}\cos\phi'\Bigg\{e^{ik_0r_1}\left(\frac{z'-d'}{r_1}\right)\left(\frac{ik_0}{r_1}-\frac{1}{r_1^2}\right)-e^{ik_0r_2}\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)+2\epsilon e^{ik_0r_2}\left[\frac{ik_0}{r_2}-\frac{1}{r_2^2}-\frac{i}{k_0r_2^3}-k_0^2\epsilon\left(\frac{r_2}{\rho}\right)\left(\frac{\pi}{k_0r_2}\right)^{1/2}e^{-iP_2}\mathscr{F}(P_2)\right]\Bigg\}, \qquad (49)$$

$$B_{0z'}(\rho,\phi',z')=-\frac{\mu_0}{4\pi}\sin\phi'\Bigg(e^{ik_0r_1}\left(\frac{\rho}{r_1}\right)\left(\frac{ik_0}{r_1}-\frac{1}{r_1^2}\right)-e^{ik_0r_2}\left(\frac{\rho}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}\right)+2e^{ik_0r_2}\Bigg\{\epsilon\left(\frac{\rho}{r_2}\right)\left(\frac{z'+d'}{r_2}\right)\times\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0r_2^3}\right)-\epsilon^2\left(\frac{\rho}{r_2}\right)\left[\frac{1}{r_2^2}+\frac{3i}{k_0r_2^3}-\frac{3}{k_0^2r_2^4}+\left(\frac{z'+d'}{r_2}\right)^2\left(\frac{ik_0}{r_2}-\frac{6}{r_2^2}-\frac{15i}{k_0r_2^3}\right)\right]\Bigg\}\Bigg). \qquad (50)$$

In these formulas the small quantity ϵ is defined as follows:

$$\epsilon=\frac{(k_0/k_2)-ik_0l}{1-ik_1^2l/k_2}, \tag{51}$$

$$P_2=\frac{k_0r_2}{2}\left(\frac{\epsilon r_2+z'+d'}{\rho}\right)^2, \tag{52}$$

$$\mathscr{F}(P_2)=\frac{1}{2}(1+i)-\int_0^{P_2}\frac{e^{it}}{(2\pi t)^{1/2}}dt. \tag{53}$$

The integral in $\mathscr{F}(P_2)$ is the well-known and tabulated Fresnel integral.

Of particular interest are the much simpler formulas when both the dipole and the point of observation are on the surface of the dielectric. With $z'=d'=0$, the formulas are

$$E_{0\rho}(\rho,\phi',0)=-\frac{\omega\mu_0\epsilon^2}{2\pi k_0}\cos\phi' e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-\frac{i}{k_0\rho^3}-k_0^2\epsilon\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{54}$$

$$E_{0\phi'}(\rho,\phi',0)=\frac{\omega\mu_0\epsilon^2}{2\pi k_0}\sin\phi' e^{ik_0\rho}\left[\frac{2}{\rho^2}+\frac{2i}{k_0\rho^3}+\frac{ik_0\epsilon}{\rho}\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{55}$$

$$E_{0z'}(\rho,\phi',0)=-\frac{\omega\mu_0\epsilon}{2\pi k_0}\cos\phi' e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-k_0^2\epsilon\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{56}$$

$$B_{0\rho}(\rho,\phi',0)=\frac{\mu_0\epsilon}{2\pi}\sin\phi' e^{ik_0\rho}\left[\frac{2}{\rho^2}+\frac{2i}{k_0\rho^3}+\frac{ik_0\epsilon}{\rho}\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{57}$$

$$B_{0\phi'}(\rho,\phi',0)=\frac{\mu_0\epsilon}{2\pi}\cos\phi' e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-\frac{i}{k_0\rho^3}-k_0^2\epsilon\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{58}$$

$$B_{0z'}(\rho,\phi',0)=\frac{\mu_0\epsilon^2}{2\pi}\sin\phi' e^{ik_0\rho}\left(\frac{1}{\rho^2}+\frac{3i}{k_0\rho^3}-\frac{3}{k_0^2\rho^4}\right), \tag{59}$$

where ϵ is defined in Eq. (51) and

$$p_2=k_0\rho\epsilon^2/2. \tag{60}$$

VI. APPLICATION TO MICROSTRIP

When the three-layered region is microstrip, Region 2 is so highly conducting and k_2 so large that, at all relevant frequencies,

$$|k_0/k_2|\ll k_0l. \tag{61}$$

Hence,

$$\epsilon\sim -ik_0l, \tag{62}$$

and the formulas for the field are the same as when $k_2\to\infty$. The detailed application of the formulas (45)–(50) to problems in the theory of microstrip—including especially microstrip patch antennas—is in preparation.[4]

VII. APPLICATION TO REMOTE SENSING FROM THE ARCTIC ICE

At 25 Hz, the wave number of sea water is $k_2=0.02(1+i)\mathrm{m}^{-1}$. With the ice thickness $l\sim 2.5$ m,

$$k_2^{-1}-il=25(1-i)-2.5i=25-27.5i\ \mathrm{m}. \tag{63}$$

Clearly, the amplitude of the surface wave associated with the layer of ice is much smaller than the amplitude of the surface wave associated with the currents in the sea. In effect, the presence of the ice layer can be ignored without significant error.

VIII. THE ELECTROMAGNETIC FIELD IN REGION 2

Because the dielectric layer is assumed to be electrically thin—$k_1^2l^2\ll 1$—the boundary conditions specify the field throughout Region 1 and on the boundary in Region 2. That is,

$$E_{2\rho}(\rho,\phi,l)=E_{1\rho}(\rho,\phi,0\leqslant z\leqslant l)=E_{0\rho}(\rho,\phi,0), \tag{64}$$

$$E_{2\phi}(\rho,\phi,l)=E_{1\phi}(\rho,\phi,0\leqslant z\leqslant l)=E_{0\phi}(\rho,\phi,0), \tag{65}$$

$$k_2^2E_{2z}(\rho,\phi,l)=k_1^2E_{1z}(\rho,\phi,0\leqslant z\leqslant l)=k_0^2E_{0z}(\rho,\phi,0), \tag{66}$$

$$\mathbf{B}_2(\rho,\phi,l)=\mathbf{B}_1(\rho,\phi,0\leqslant z\leqslant l)=\mathbf{B}_0(\rho,\phi,0). \tag{67}$$

Once the field on the boundary of Region 2 is known and because $|k_2|^2\gg k_1^2$, the field in its interior is obtained from the field on the boundary multiplied by e^{ik_2z}. The electromagnetic waves travel into Region 2 just as if the electrically thin Region 1 were absent. With $\epsilon=k_0/k_2$, the components of the field are

$$E_{2\rho}(\rho,\phi,z)=-\frac{\omega\mu_0k_0}{2\pi k_2^2}\cos\phi e^{ik_2z}e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-\frac{i}{k_0\rho^3}-\frac{k_0^3}{k_2}\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{68}$$

$$E_{2\phi}(\rho,\phi,z)=\frac{\omega\mu_0k_0}{2\pi k_2^2}\sin\phi e^{ik_2z}e^{ik_0\rho}\left[\frac{2}{\rho^2}+\frac{2i}{k_0\rho^3}+\frac{ik_0^2}{k_2\rho}\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{69}$$

$$E_{2z}(\rho,\phi,z)=\frac{\omega\mu_0k_0^2}{2\pi k_2^3}\cos\phi e^{ik_2z}e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-\frac{k_0^3}{k_2}\left(\frac{\pi}{k_0\rho}\right)^{1/2}e^{-ip_2}\mathscr{F}(p_2)\right], \tag{70}$$

$$B_{2\rho}(\rho,\phi,z) = -\frac{\mu_0 k_0}{2\pi k_2}\sin\phi e^{ik_2z}e^{ik_0\rho}\left[\frac{2}{\rho^2}+\frac{2i}{k_0\rho^3}+\frac{ik_0^2}{k_2\rho}\left(\frac{\pi}{k_0\rho}\right)^{1/2} e^{-ip_2}\mathscr{F}(p_2)\right], \quad (71)$$

$$B_{2\phi}(\rho,\phi,z) = -\frac{\mu_0 k_0}{2\pi k_2}\cos\phi e^{ik_2z}e^{ik_0\rho}\left[\frac{ik_0}{\rho}-\frac{1}{\rho^2}-\frac{i}{k_0\rho^3}-\frac{k_0^3}{k_2}\left(\frac{\pi}{k_0\rho}\right)^{1/2} e^{-ip_2}\mathscr{F}(p_2)\right], \quad (72)$$

$$B_{2z}(\rho,\phi,z) = \frac{\mu_0 k_0^2}{2\pi k_2^2}\sin\phi e^{ik_2z}e^{ik_0\rho}\left(\frac{1}{\rho^2}+\frac{3i}{k_0\rho^3}-\frac{3}{k_0^2\rho^4}\right). \quad (73)$$

These formulas are the same as those used in earlier work[5,6] on the "Scattering of lateral waves by buried or submerged objects" [Ref. 5, Eqs. (1a)–(1d)] with the dipoles lying on the surface of the earth and the more recent work[1] on the use of "Lateral electromagnetic waves from a horizontal antenna for remote sensing in the ocean" in which the antenna is at a depth d in the ocean instead of on an electrically thin layer of ice with $k_1^2 l^2 \ll 1$. The factor $e^{ik_1 l} \sim 1$, so that it does not appear in the above formulas. The presence of the ice can be ignored, and the data supplied in the earlier paper can be used with only minor changes. These including setting $d=0$ and making use of vertical bare terminations lowered into the sea through holes in the ice.

IX. CONCLUSION

General expressions have been derived in simple integrated form for the complete electromagnetic field generated by a horizontal electric dipole in the air above a two-layered region consisting of an electrically thin sheet of dielectric on a conducting or dielectric half-space. This includes the field at all points in all three regions. No conditions are imposed other than the basic inequalities $k_0^2 \ll k_1^2 \ll |k_2|^2$ and $k_1^2 l^2 \ll 1$. The effect on the field in the air of the dielectric layer is contained in the important small parameter $\epsilon = [(k_0/k_2) - ik_0 l]/(1 - ik_1^2 l/k_2)$.

Applications of the new formulas are to elements of current in microstrip circuits and antennas and to remote sensing on the arctic ice.

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APPENDIX A: A PROCEDURE OF APPROXIMATION

In order to simplify the complicated integrals that occur in the functions $F_2(\rho,z'+d')$ and $G_2(\rho,z'+d')$, use is made of the following procedure which is carried out explicitly for $F_{\rho 2}(\rho,z'+d')$ as given in Eq. (10) with Eq. (29).

Let

$$f(\lambda;k_0,k_1,k_2) = \frac{\gamma_0}{2}(Q_3+1)e^{i\gamma_0(z'+d')}. \quad (A1)$$

Then,

$$f(\lambda;k_0,k_1,k_2) \sim f_1(\lambda;k_0,k_1,k_2) + f_2(\lambda;k_0,k_1,k_2) + f_3(\lambda;k_0,k_1,k_2) - f_4(\lambda;k_0,k_1,k_2) - f_5(\lambda;k_0,k_1,k_2), \quad (A2)$$

where

$$f_1(\lambda;k_0,k_1,k_2) = f(\lambda\sim k_0;k_0,k_1,k_2),$$

$$f_2(\lambda;k_0,k_1,k_2) = f(\lambda\sim k_1;k_0,k_1,k_2),$$

$$f_3(\lambda;k_0,k_1,k_2) = f(\lambda\sim k_2;k_0,k_1,k_2),$$

$$f_4(\lambda;k_0,k_1,k_2) = f_1(\lambda\sim k_1;k_0,k_1,k_2),$$

$$f_5(\lambda;k_0,k_1,k_2) = f_2(\lambda\sim k_2;k_0,k_1,k_2). \quad (A3)$$

Specifically,

$$f_1 = \frac{k_0^2\gamma_0}{k_1^2}\frac{[(k_1^2/k_2) - ik_1\tan k_1 l]e^{i\gamma_0(z'+d')}}{\gamma_0 - i(k_0^2/k_1)\tan k_1 l + k_0^2/k_2 - i(k_1\gamma_0/k_2)\tan k_1 l}$$

$$\sim \frac{\gamma_0[(k_0^2/k_2) - ik_0^2 l]e^{i\gamma_0(z'+d')}}{\gamma_0(1 - ik_1^2 l/k_2) + k_0^2/k_2 - ik_0^2 l} = \frac{\epsilon'\gamma_0 e^{i\gamma_0(z'+d')}}{\gamma_0+\epsilon'}, \quad (A4)$$

where ϵ' is defined in Eq. (39), and

$$f_2 = \frac{i\lambda k_0^2}{k_1^2}\frac{[(k_1^2/k_2) - i\gamma_1\tan\gamma_1 l]e^{-k_1(z'+d')}}{i\lambda - i\gamma_1(k_0^2/k_1^2)\tan\gamma_1 l + k_0^2/k_2 - i(i\lambda k_1^2/\gamma_1 k_2)\tan\gamma_1 l}, \quad (A5)$$

$$f_3 = \frac{i\lambda k_0^2}{k_1^2}\frac{[\gamma_2(k_1^2/k_2^2) + \lambda\tan(i\lambda l)]e^{-k_2(z'+d')}}{i\lambda + \lambda(k_0^2/k_1^2)\tan(i\lambda l) + \gamma_2(k_0^2/k_2^2) - i(\gamma_2 k_1^2/k_2^2)\tan(i\lambda l)}, \quad (A6)$$

$$f_4=\frac{\epsilon' i\lambda e^{-k_1(z'+d')}}{i\lambda+\epsilon'}, \tag{A7}$$

$$f_5=\frac{i\lambda k_0^2}{k_1^2}\frac{[(k_1^2/k_2)+\lambda\tan(i\lambda l)]e^{-k_2(z'+d')}}{i\lambda+\lambda(k_0^2/k_1^2)\tan(i\lambda l)+k_0^2/k_2-i(k_1^2/k_2)\tan(i\lambda l)}. \tag{A8}$$

The exponential factors $e^{-k_1(z'+d')}$ and $e^{-k_2(z'+d')}$ which occur in f_2, f_3, f_4, and f_5 indicate that these are all nonpropagating terms that can contribute nothing to the outward propagating field from the source. They may, therefore, be omitted so that $F_{\rho 2}(\rho,z'+d')$ is given by Eq. (36). The same procedure applied to the other integrals gives Eqs. (37), (38), and (40)–(42).

APPENDIX B: EVALUATION OF THE INTEGRALS IN EQS. (36)–(38)

In order to evaluate the integrals in Eqs. (36)–(38), let

$$K(\rho,z_d')=\int_0^\infty \frac{\gamma_0}{\gamma_0+\epsilon'}C(\lambda\rho)e^{i\gamma_0 z'_d}\lambda d\lambda, \tag{B1}$$

and

$$J(\rho,z_d')=\int_0^\infty \frac{1}{\gamma_0+\epsilon'}C(\lambda\rho)e^{i\gamma_0 z'_d}\lambda d\lambda, \tag{B2}$$

where $z_d'=z'+d'$ and $C(\lambda\rho)$ stands for $[J_0(\lambda\rho)\mp J_2(\lambda\rho)]$, $\lambda J_1(\lambda\rho)$, or $[J_0(\lambda\rho)\pm J_2(\lambda\rho)]$. Then

$$\frac{dJ(\rho,z_d')}{dz_d'}=iK(\rho,z_d'). \tag{B3}$$

Also,

$$\frac{\gamma_0}{\gamma_0+\epsilon'}-1=-\frac{\epsilon'}{\gamma_0+\epsilon'}, \tag{B4}$$

so that

$$K(\rho,z_d')=K_0(\rho,z_d')-\epsilon' J(\rho,z_d'), \tag{B5}$$

where

$$K_0(\rho,z_d')=\int_0^\infty C(\lambda\rho)e^{i\gamma_0 z'_d}\lambda d\lambda. \tag{B6}$$

Hence,

$$\frac{dJ(\rho,z_d')}{dz_d'}+i\epsilon' J(\rho,z_d')=iK_0(\rho,z_d'). \tag{B7}$$

This is an ordinary differential equation with the solution

$$J(\rho,z_d')=-i\int_{z_d'}^\infty e^{i\epsilon'(z-z'_d)}K_0(\rho,z)dz. \tag{B8}$$

Let

$$K_1(\rho,z_d')=\int_0^\infty \frac{1}{\gamma_0}C(\lambda\rho)e^{i\gamma_0 z'_d}\lambda d\lambda. \tag{B9}$$

Then

$$\frac{dK_1(\rho,z_d')}{dz_d'}=iK_0(\rho,z_d'), \tag{B10}$$

and

$$J(\rho,z_d')=-\int_{z_d'}^\infty e^{i\epsilon'(z-z'_d)}\frac{dK_1(\rho,z)}{dz}dz. \tag{B11}$$

This is readily integrated by parts to give

$$J(\rho,z_d')=K_1(\rho,z_d')-\epsilon' K_2(\rho,z_d'), \tag{B12}$$

where

$$K_2(\rho,z_d')=-i\int_{z_d'}^\infty K_1(\rho,z)e^{i\epsilon'(z-z'_d)}dz. \tag{B13}$$

With (B5) and (B12),

$$K(\rho,z_d')=K_0(\rho,z_d')-\epsilon' K_1(\rho,z_d')+\epsilon'^2K_2(\rho,z_d'). \tag{B14}$$

All of the integrals $K_0(\rho,z_d')$ and $K_1(\rho,z_d')$ as defined in (B6) and (B9) are readily evaluated. Thus with

$$F_2(\rho,z'+d')=\epsilon' K_0(\rho,z'+d')-\epsilon'^2K_1(\rho,z'+d')+\epsilon'^3K_2(\rho,z'+d'), \tag{B15}$$

it follows that

$$F_{\rho 2}(\rho,z'+d')=-2\epsilon'\left[\frac{e^{ik_0(z'+d')}}{\rho^2}+\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}-\frac{1}{\rho^2}\right)e^{ik_0r_2}\right]+\frac{2\epsilon'^2}{k_0}\left[\frac{e^{ik_0(z'+d')}}{\rho^2}+\left(\frac{ik_0}{r_2}-\frac{1}{\rho^2}\right)e^{ik_0r_2}\right]+\epsilon'^3K_{2\rho}(\rho,z'+d'), \tag{B16}$$

$$F_{\phi' 2}(\rho,z'+d')=2\epsilon'\left(\frac{e^{ik_0(z'+d')}}{\rho^2}-\frac{z'+d'}{r_2}\frac{e^{ik_0r_2}}{\rho^2}\right)-\frac{2\epsilon'^2}{k_0}\left(\frac{e^{ik_0(z'+d')}}{\rho^2}-\frac{e^{ik_0r_2}}{\rho^2}\right)+\epsilon'^3K_{2\phi'}(\rho,z'+d'). \tag{B17}$$

Similarly,

$$F_{z'1}(\rho,z'+d')=2\epsilon' K_{1z'}(\rho,z'+d')-2\epsilon'^2K_{2z'}(\rho,z'+d')=-2\epsilon'\left(\frac{\rho}{r_2}\right)\left(\frac{k_0}{r_2}+\frac{i}{r_2^2}\right)e^{ik_0r_2}-2\epsilon'^2K_{2z'}(\rho,z'+d'). \tag{B18}$$

With

$$G_2(\rho,z'+d')=\epsilon' K_{1m}(\rho,z'+d')-\epsilon'^2 K_{2m}(\rho,z'+d'), \tag{B19}$$

$$G_{\rho 2}(\rho,z'+d')=\frac{2\epsilon'}{k_0}\left(\frac{e^{ik_0(z'+d')}}{\rho^2}-\frac{e^{ik_0 r_2}}{\rho^2}\right) -\epsilon'^2 K_{2m\rho}(\rho,z'+d'), \tag{B20}$$

$$G_{\phi' 2}(\rho,z'+d')=-\frac{2\epsilon'}{k_0}\left[\frac{e^{ik_0(z'+d')}}{\rho^2}+\left(\frac{ik_0}{r_2}-\frac{1}{\rho^2}\right)e^{ik_0 r_2}\right]-\epsilon'^2 K_{2m\phi'}(\rho,z'+d'). \tag{B21}$$

The several integrals $K_2(\rho,z'+d')$ can all be put into the form of Eq. (A21) in Ref. 7 with different factors in front of the sign of integration. The resulting integrated formulas are given by Eq. (A28) in Ref. 7 with changes appropriate to the different factors in front of the integral. The results are

$$K_{2\rho}(\rho,z'+d')=-2\left(\frac{r_2}{\rho}\right)\left(\frac{\pi}{k_0 r_2}\right)^{1/2} e^{ik_0 r_2}e^{-iP_2}\mathscr{F}(P_2), \tag{B22}$$

$$K_{2\phi'}(\rho,z'+d')=\frac{2i}{k_0}\left(\frac{r_2^2}{\rho^3}\right)\left(\frac{\pi}{k_0 r_2}\right)^{1/2} e^{ik_0 r_2}e^{-iP_2}\mathscr{F}(P_2), \tag{B23}$$

$$K_{2z'}(\rho,z'+d')=ik_0\left(\frac{\pi}{k_0 r_2}\right)^{1/2} e^{ik_0 r_2}e^{-iP_2}\mathscr{F}(P_2), \tag{B24}$$

$$K_{2m\rho}(\rho,z'+d')=\frac{2i}{k_0}\left(\frac{r_2^2}{\rho^3}\right)\left(\frac{\pi}{k_0 r_2}\right)^{1/2} e^{ik_0 r_2}e^{-iP_2}\mathscr{F}(P_2), \tag{B25}$$

$$K_{2m\phi'}(\rho,z'+d')=-2\left(\frac{r_2}{\rho}\right)\left(\frac{\pi}{k_0 r_2}\right)^{1/2} e^{ik_0 r_2}e^{-iP_2}\mathscr{F}(P_2), \tag{B26}$$

where

$$P_2=\frac{k_0 r_2}{2}\left(\frac{\epsilon' r_2+k_0(z'+d')}{k_0\rho}\right)^2, \tag{B27}$$

and

$$\mathscr{F}(P_2)=\frac{1}{2}(1+i)-\int_0^{P_2}\frac{e^{it}}{(2\pi t)^{1/2}}\,dt. \tag{B28}$$

APPENDIX C: EVALUATION OF INTEGRALS IN EQS. (40)–(42)

With Eq. (43) the integrals to be evaluated in Eqs. (40)–(42) are

$$\left.\begin{matrix} F_{\rho 3}(\rho,z'+d') \\ F_{\phi' 3}(\rho,z'+d') \end{matrix}\right\}=\epsilon'\int_0^\infty [J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda-\frac{\epsilon'^2}{k_0^2}\int_0^\infty \gamma_0[J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda$$
$$+\frac{\epsilon'^3}{k_0^2}\int_0^\infty [J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda d\lambda-\frac{\epsilon'^3}{k_0^4}\int_0^\infty [J_0(\lambda\rho)\pm J_2(\lambda\rho)]e^{i\gamma_0(z'+d')}\lambda^3 d\lambda\cdots. \tag{C1}$$

These are readily integrated using standard formulas. The results are

$$F_{\rho 3}(\rho,z'+d')=2\epsilon'\left(1+\frac{\epsilon'^2}{k_0^2}\right)\left[\frac{e^{ik_0(z'+d')}}{\rho^2}-\frac{z'+d'}{r_2}\frac{e^{ik_0 r_2}}{\rho^2}\right]-\frac{2\epsilon'^2}{k_0}\left[\frac{e^{ik_0(z'+d')}}{\rho^2}+\frac{ie^{ik_0 r_2}}{k_0 r_2^3}-\left(\frac{z'+d'}{r_2}\right)^2\frac{e^{ik_0 r_2}}{\rho^2}\right]$$
$$+\frac{2\epsilon'^3}{k_0^3}\left(\frac{z'+d'}{r_2}\right)\left(\frac{k_0}{r_2^2}+\frac{3i}{r_2^3}-\frac{3}{k_0 r_2^4}\right)e^{ik_0 r_2}. \tag{C2}$$

Here the factor $\epsilon'^2/k_0^2=\epsilon^2 \ll 1$, $\epsilon'^2/k_0=k_0\epsilon^2$, and $\epsilon'^3/k_0^3=\epsilon^3$. Since $\epsilon=[(k_0/k_2)-ik_0 l]/(1-ik_1^2 l/k_2)\ll 1$, the leading terms are

$$F_{\rho 3}(\rho,z'+d')\sim 2\epsilon'\left\{\frac{e^{ik_0(z'+d')}}{\rho^2}-\frac{z'+d'}{r_2}\frac{e^{ik_0 r_2}}{\rho^2}-\frac{\epsilon'}{k_0}\left[\frac{e^{ik_0(z'+d')}}{\rho^2}+\frac{ie^{ik_0 r_2}}{k_0 r_2^3}-\left(\frac{z'+d'}{r_2}\right)^2\frac{e^{ik_0 r_2}}{\rho^2}\right]\right\}. \tag{C3}$$

Similarly,

$$F_{\phi' 3}(\rho,z'+d')\sim -2\epsilon'\left(\frac{e^{ik_0(z'+d')}}{\rho^2}+\left(\frac{z'+d'}{r_2}\right)\left(\frac{ik_0}{r_2}-\frac{1}{r_2^2}-\frac{1}{\rho^2}\right)e^{ik_0 r_2}\right.$$
$$\left.-\frac{\epsilon'}{k_0}\left\{\frac{e^{ik_0(z'+d')}}{\rho^2}+\left[\frac{1}{r_2^2}+\frac{2i}{k_0 r_2^3}+\left(\frac{z'+d'}{r_2}\right)^2\left(\frac{ik_0}{r_2}-\frac{3}{r_2^2}-\frac{3i}{k_0 r_2^3}-\frac{1}{\rho^2}\right)\right]e^{ik_0 r_2}\right\}\right), \tag{C4}$$

$$G_{\rho 3}(\rho,z'+d') = -\frac{2\epsilon'}{k_0^2}\Bigg(k_0 \left\{ \frac{e^{ik_0(z'+d')}}{\rho^2} + \left[\frac{1}{r_2^2} + \frac{2i}{k_0 r_2^3} + \left(\frac{z'+d'}{r_2} \right)^2 \left(\frac{ik_0}{r_2} - \frac{3}{r_2^2} - \frac{3i}{k_0 r_2^3} - \frac{1}{\rho^2} \right) \right] e^{ik_0 r_2} \right\}$$

$$-\epsilon' \left[\frac{e^{ik_0(z'+d')}}{\rho^2} + \left(\frac{z'+d'}{r_2} \right) \left(\frac{ik_0}{r_2} - \frac{1}{r_2^2} - \frac{1}{\rho^2} \right) e^{ik_0 r_2} \right] - \epsilon' \left(\frac{z'+d'}{r_2} \right) \left(-\frac{ik_0}{r_2} + \frac{5}{r_2^2} \right.$$

$$\left. + \frac{12i}{k_0 r_2^3} \right) e^{ik_0 r_2} \Bigg) \quad , \tag{C5}$$

$$G_{\phi' 3}(\rho,z'+d') = \frac{2\epsilon'}{k_0^2} \left\{ k_0 \left[\frac{e^{ik_0(z'+d')}}{\rho^2} + \frac{ie^{ik_0 r_2}}{k_0 r_2^3} - \left(\frac{z'+d'}{r_2} \right)^2 \frac{e^{ik_0 r_2}}{\rho^2} \right] - \epsilon' \left[\frac{e^{ik_0(z'+d')}}{\rho^2} - \frac{z'+d'}{r_2} \frac{e^{ik_0 r_2}}{\rho^2} \right] \right.$$

$$\left. -\epsilon' \left(\frac{z'+d'}{r_2} \right) \left(\frac{1}{r_2^2} + \frac{3i}{k_0 r_2^3} - \frac{3}{k_0^2 r_2^4} \right) e^{ik_0 r_2} \right\}, \tag{C6}$$

$$G_{z' 1}(\rho,z'+d') = \frac{2\epsilon'}{k_0} \left\{ -\left(\frac{\rho}{r_2} \right) \left(\frac{z'+d'}{r_2} \right) \left(\frac{k_0}{r_2} + \frac{3i}{r_2^2} - \frac{3}{k_0 r_2^3} \right) e^{ik_0 r_2} - \frac{\epsilon'}{k_0} \left(\frac{\rho}{r_2} \right) \left[\frac{i}{r_2^2} - \frac{3}{k_0 r_2^3} - \frac{3i}{k_0^2 r_2^4} - \left(\frac{z'+d'}{r_2} \right)^2 \right. \right.$$

$$\left. \left. \times \left(\frac{k_0}{r_2} + \frac{6i}{r_2^2} - \frac{15}{k_0 r_2^3} \right) \right] e^{ik_0 r_2} \right\}. \tag{C7}$$

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