Effective Boundary Conditions for Reflection and Transmission by an Absorbing Shell of Arbitrary Shape

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Abstract—**The reflection and transmission properties of an absorbing shell of arbitrary shape can be described by two algebraic boundary conditions relating the four tangential fields E_+, H_+, E_-, H_- at corresponding points on the outer surface S_+ and inner surface S_-, respectively. The boundary conditions contain first-order curvature correction terms, but it is still necessary that the shell curvature be reasonably small compared to the attenuation per unit distance in the shell material. Especially simple conditions are obtained for shells thin compared to the attenuation length. The shell boundary conditions can also be used to determine an effective surface impedance for a conducting body coated by multiple layers of absorbing material.**

I. Introduction

Many practical scattering problems involve determination of the transmission and reflection properties of shells of finite conductivity. One example is the shielding problem in which we must find the transmission through one or more such shells. Another is the determination of the radar cross section of a body which has been coated by one or more layers of absorbing material.

An exact treatment requires the solution of a differential or integral equation for propagation in the shell material, a difficult task for all but the simplest geometries. To circumvent this difficulty, accurate simplified treatments have been developed for some special cases. Thus, in dealing with a shell of very high conductivity but thin compared to the attenuation length, its effect is represented by a jump in tangential H proportional to tangential E; the two-dimensional form of the jump boundary condition is given by Oshiro and Cross [1] and is also implicit in the work of Schmitt [2]. In another important case, Weston and Hemenger [3] and Bowman and Weston [4] have shown that, given a conducting sphere or cylinder coated with multiple layers of absorbing material (with no intervening air spaces), we can describe its reflection properties in terms of an impedance boundary condition at the outermost surface.

Here we shall develop a more general result, a pair of boundary conditions which describe reflection and transmission by an absorbing shell of arbitrary shape. By going to appropriate limits, we then obtain the jump boundary conditions for thin shells of high conductivity and also dual boundary conditions for thin magnetic shells. By stacking shells and eliminating intermediate boundary conditions, we derive an equivalent surface impedance for a multilayer absorber-coated body of arbitrary shape. The previously known approximate boundary conditions [1]–[4] thus emerge as specializations of the general shell boundary conditions.

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II. Synopsis of Results

General Boundary Conditions

The shell geometry is shown in Fig. 1. The thickness is Δ, the outer surface is S_+, the inner S_-; S_0 is the middle surface [7], and n is the outward-directed unit normal to S_0. The shell material is characterized by the complex constitutive parameters

$$\begin{aligned} \epsilon^* &= \epsilon - \frac{\sigma}{i\omega} \\ \mu^* &= \mu - \frac{\tau}{i\omega}, \end{aligned} \tag{1}$$

or alternatively by the wave impedance

$$Z = (\mu^*/\epsilon^*)^{1/2} \tag{2}$$

and the propagation constant

$$k = \omega(\mu^*\epsilon^*)^{1/2}. \tag{3}$$

The attenuation per unit distance is

$$k_I = Im\,(k) \tag{4}$$

and the complex "attenuation" in passing through the shell is given by

$$\alpha = -ik\Delta. \tag{5}$$

Instead of working directly in terms of the tangent fields, it is convenient to define effective electric and magnetic surface currents on S_+ and S_- by

$$\begin{aligned} K_{e+} &= n \times H_+, & K_{m+} &= -n \times E_+ \\ K_{e-} &= -n \times H_-, & K_{m-} &= n \times E_-, \end{aligned} \tag{6}$$

where we are neglecting any difference between n and the normals at corresponding points of S_+ and S_- (see Appendix).

We can now write the two boundary conditions for shells of small curvature in the following three equivalent forms:

$$K_{m+} + Zn \times K_{e+} + e^{-\alpha}K_{m-} + Ze^{-\alpha}n \times K_{e-} = 0 \tag{7a}$$

$$e^{-\alpha}K_{m+} - Ze^{-\alpha}n \times K_{e+} + K_{m-} - Zn \times K_{e-} = 0 \tag{7b}$$

$$\left.\begin{aligned} K_{m+} &= -K_{m-}\cosh\alpha + Zn\times K_{e-}\sinh\alpha && (8a)\\ K_{e+} &= -K_{e-}\cosh\alpha - (1/Z)n\times K_{m-}\sinh\alpha && (8b)\end{aligned}\right\}$$

$$\left.\begin{aligned} K_{m-} &= -K_{m+}\cosh\alpha - Zn\times K_{e+}\sinh\alpha && (9a)\\ K_{e-} &= -K_{e+}\cosh\alpha + (1/Z)n\times K_{m+}\sinh\alpha && (9b)\end{aligned}\right\}.$$

The first form exhibits the symmetry of the two conditions; the second is especially appropriate for reflection problems, and the third is especially useful for shielding problems.

When curvature is significant, we must define a few additional quantities. First we establish on S_0 a right-handed orthogonal coordinate system (e_u, e_v, n) in which e_u and e_v point along the directions of principal curvature. The principal curvatures are κ_u and κ_v, defined as positive for the convex shell illustrated. The maximum curvature (at a point) is

$$\kappa = \max\{|\kappa_u|, |\kappa_v|\}, \tag{10}$$

and the minimum radius of curvature is then $(1/\kappa)$. We further define the unit dyadic I, the curvature correction terms

$$p = \tfrac{1}{2}i(\kappa_v - \kappa_u)/k \tag{11}$$

$$P = p(e_u e_u - e_v e_v) \tag{12}$$

$$q = \tfrac{1}{2}\Delta(\kappa_v + \kappa_u) \tag{13}$$

and the modified surface currents

$$\hat{K}_\pm = (1 \pm \tfrac{1}{2}q)K_\pm. \tag{14}$$

Quantities of $O(p^2)$ and $O(q^2)$ will be considered negligibly small, and we note that the dyadic P vanishes on a sphere.

We can now state the curvature-corrected boundary conditions in the following three alternative forms:

$$\left.\begin{aligned} &(I+\tfrac{1}{2}P)\cdot\hat{K}_{m+}+Z(I-\tfrac{1}{2}P)\cdot(n\times\hat{K}_{e+})\\ &\quad+e^{-\alpha}(I+\tfrac{1}{2}P)\cdot\hat{K}_{m-}+Ze^{-\alpha}(I-\tfrac{1}{2}P)\cdot(n\times\hat{K}_{e-}) = 0 && (15a)\\ &e^{-\alpha}(I-\tfrac{1}{2}P)\cdot\hat{K}_{m+}-Ze^{-\alpha}(I+\tfrac{1}{2}P)\cdot(n\times\hat{K}_{e+})\\ &\quad+(I-\tfrac{1}{2}P)\cdot\hat{K}_{m-}-Z(I+\tfrac{1}{2}P)\cdot(n\times\hat{K}_{e-}) = 0 && (15b)\end{aligned}\right\}$$

$$\left.\begin{aligned} \hat{K}_{m+} &= -(I\cosh\alpha - P\sinh\alpha)\cdot\hat{K}_{m-}\\ &\quad + Zn\times\hat{K}_{e-}\sinh\alpha && (16a)\\ \hat{K}_{e+} &= -(I\cosh\alpha - P\sinh\alpha)\cdot\hat{K}_{e-}\\ &\quad - (1/Z)n\times\hat{K}_{m-}\sinh\alpha && (16b)\end{aligned}\right\}$$

$$\left.\begin{aligned} \hat{K}_{m-} &= -(I\cosh\alpha + P\sinh\alpha)\cdot\hat{K}_{m+}\\ &\quad - Zn\times\hat{K}_{e+}\sinh\alpha && (17a)\\ \hat{K}_{e-} &= -(I\cosh\alpha + P\sinh\alpha)\cdot\hat{K}_{e+}\\ &\quad + (1/Z)n\times\hat{K}_{m+}\sinh\alpha && (17b)\end{aligned}\right\}.$$

These three forms are not exactly equivalent because the $O(p^2)$ terms have been neglected.

In Section III, we discuss in detail the conditions under which the above equations are valid. At this point, we simply note that the most important restriction is

$$(\kappa/k_I)^2 \ll 1, \tag{18}$$

which means that waves traveling laterally in the shell must damp out in a distance which is small compared to the minimum radius of curvature.

Electromagnetically Thin Shells

We define an electromagnetically thin shell as one that is thin compared to the complex attenuation length, so that the approximation

$$e^{-\alpha} = 1 - \alpha \tag{19}$$

is satisfactory.

There are two important types of thin shells. A thin electric shell is characterized by the electric layer admittance

$$Y_e = -i\omega\epsilon^*\Delta = \alpha/Z \tag{20}$$

and can be described by the boundary conditions

$$K_{m+} = -K_{m-} = -\left(\frac{1}{Y_e}\right)n\times K_{e\Delta} \tag{21}$$

with

$$K_{e\Delta} = K_{e+} + K_{e-}. \tag{22}$$

A thin magnetic shell is characterized by the magnetic layer impedance

$$Z_m = -i\omega\mu^*\Delta = \alpha Z \tag{23}$$

and can be described by the boundary conditions

$$K_{e+} = -K_{e-} = \left(\frac{1}{Z_m}\right)n\times K_{m\Delta} \tag{24}$$

with

$$K_{m\Delta} = K_{m+} + K_{m-}. \tag{25}$$

The physical significance of (21) and (22) is that tangential E is the same on both surfaces of the thin electric shell, and is proportional to the effective current $K_{e\Delta}$ flowing in the shell. Similarly, for the thin magnetic shell, tangential H is proportional to an effective magnetic current $K_{m\Delta}$ in the shell. There are no curvature correction terms, since it is implicit in the approximation (19) that such corrections are negligible.

Surface Impedance of an Absorber-Coated Body

A perfect conductor coated by multiple layers of absorbing material can be described by an impedance boundary condition of form

$$K_m + \bar{Z}\cdot(n\times K_e) = 0 \tag{26}$$

imposed at the outer surface of the outermost layer. To determine the dyadic surface impedance $\bar{Z}$ for any number of layers, we begin with the formula for a single layer coated onto a perfect conductor:

$$\bar{Z} = Z(I + P\tanh\alpha)^{-1}\tanh\alpha. \tag{27}$$

Then we add layers, one at a time, calculating the $\bar{Z}$ for each new configuration by means of the formula for a single layer coated onto a body with impedance $\bar{Z}_-$, which is

$$\begin{aligned}\bar{Z} = {} & [I + P\tanh\alpha + \bar{Z}_-(1/Z)\tanh\alpha]^{-1}\\ & \cdot[ZI\tanh\alpha + \bar{Z}_-\cdot(I - P\tanh\alpha)].\end{aligned} \tag{28}$$

There are no correction terms in q. The dyadics are all diagonal and thus exceptionally easy to invert.

When curvature is negligible, we can work with a scalar surface impedance $\bar{Z}_s$, and (27) and (28) become

$$\bar{Z}_s = Z \tanh \alpha \tag{27a}$$

$$\bar{Z}_s = (Z \tanh \alpha + \bar{Z}_{s-})/[1 + (\bar{Z}_{s-}/Z) \tanh \alpha]. \tag{28a}$$

III. Derivation of the General Boundary Conditions

The derivation of the various sets of boundary conditions in (7) through (9) and (15) through (17) can readily be reduced to the derivation of (15a). For, once we have this, we can obtain (15b) by interchanging the roles of S_+ and S_-. Then we can find (16) and (17) by straightforward algebra, and finally we get (7) through (9) by omitting terms in p and q.

In deriving (15a), we follow the same approach we have used elsewhere to derive the impedance boundary condition for a solid body [5]. In brief, this method involves writing down an integral equation for K_{m+} and then approximating the integrals analytically.

The desired integral equation for the magnetic current at point r on S_+ is

$$\tfrac{1}{2}K_{m+} + L_+\cdot K_{m+} - ZM_+\cdot K_{e+} + L_-\cdot K_{m-} - ZM_-\cdot K_{e-} = 0. \tag{29}$$

Here

$$L_\pm\cdot K_\pm = -\int_{S_\pm} dS'[n_+ \times (\nabla' G \times I)]\cdot K_\pm' \tag{30}$$

$$\begin{aligned} M_+\cdot K_+ = &-(1/ik)\int_{S_+} dS' n_+ \times [k^2 G K_+' \\ &+ (\kappa_u' + \kappa_v')\nabla' G(n_+' - n_+)\cdot K_+ \\ &+ (K_+' - K_+)\cdot \nabla_S'\nabla' G] \end{aligned} \tag{31a}$$

$$\begin{aligned} M_-\cdot K_- = &-(1/ik)\int_{S_-} dS'[n_+ \times (k^2 G I \\ &+ \nabla'\nabla' G)]\cdot K_-', \end{aligned} \tag{31b}$$

where G is Green's function

$$G = e^{ikR}/4\pi R, \qquad R = |r - r'|, \tag{32}$$

primes indicate functions of the integration point r', and ∇_S' is the tangential gradient with respect to r', as defined in [6]. The integrals over S_+ are to be interpreted as principal value integrals over the punctured surface $\bar{S}_+$, as defined in [5]. Equation (29) follows from (1.9) of [5] upon separating the integral over S into two parts, one over S_+ and the other over S_-, and using (2.5) of [5] to simplify the latter. It is important to note that n_+ is the negative of the n in [5].

The operators L_+ and M_+ are of the same form as the operators L_c and M_c in [5] and can be estimated in the same way. Equation (29) can thus be approximated by

$$\tfrac{1}{2}(I+P)\cdot K_{m+} + \tfrac{1}{2}Zn\times K_{e+} + L_-\cdot K_{m-} - ZM_-\cdot K_{e-} = 0. \tag{33}$$

The estimation of L_- and M_- follows a pattern similar to that used in [5], but requires a considerable amount of background material which we have compiled in the Appendix. In the case of L_-, we start with the appropriate form of (30) and argue as in the Appendix that the integration can be limited to the region S_c and then transformed into an integral over the disc D_-. Using the notation of the Appendix, we can write this latter integral as

$$\begin{aligned} L_-\cdot K_- \approx \int_{D_-} dP_-' \frac{dG}{dR}\frac{\rho}{R}[s^*(n - n') - (s^*\cdot n)I] \\ \cdot[K_- + (K_-' - K_-)], \end{aligned} \tag{34}$$

where the quantities in the integrand are still evaluated on S_c.

The integral in (34) can be broken up into two parts, one involving K_-, the other $(K_-' - K_-)$. The second part we show to be negligibly small by the same technique used in [5] and based on (3.14), (A.6), (A.7), and (A.9) of that reference. In the first part, K_- can be taken outside the integral sign and the remaining integral is then an approximation to L_-. It can be written as

$$\begin{aligned} L_- \approx -\int_{D_-} dP_-' \frac{1}{R}\frac{dG}{dR}\{\Delta I + \rho^2[\tfrac{1}{2}\bar{\kappa} I \\ - (\kappa_u \cos^2\theta e_x e_x + \kappa_v \sin^2\theta e_y e_y)]\}, \end{aligned} \tag{35}$$

where we have applied (61) and (64) and eliminated high-order terms.

Next we set (72) and (73) into (35), and then we argue that only a small error is introduced by extending the range of integration to the entire P_- plane. We thus get

$$\begin{aligned} L_- \approx -\int_\Delta^\infty dl \int_{-\pi}^{\pi} d\theta \Big\{(1 - \Delta\bar{\kappa})\Delta \frac{d}{dl} G_l I \\ - 2lG_l[\tfrac{1}{2}\bar{\kappa} I - (\kappa_u \cos^2\theta e_x e_x + \kappa_v \sin^2\theta e_y e_y)]\Big\}, \end{aligned} \tag{36}$$

where we have excluded high-order terms and terms arising from (72) and (73) which integrate to zero. Evaluating (36), we obtain

$$L_- \approx \tfrac{1}{2}e^{-\alpha}[(1 - q)I + P]. \tag{37}$$

Turning to M_-, we rewrite (31b) as

$$\begin{aligned} M_-\cdot K_- = &\left[ik\int_{S_-} dS' G n_+ \times I\right]\cdot K_- \\ &- (1/ik)\int_{S_-} dS'(K_-\cdot\nabla_S')(n_+ \times \nabla' G) \\ &+ ik\int_{S_-} dS' G(n_+ \times I)\cdot(K_-' - K_-) \\ &- (1/ik)\int_{S_-} dS' n_+ \times (K_-' - K_-)\cdot\nabla_S'\nabla' G. \end{aligned} \tag{38}$$

The second integral can be put in a more tractable form using identity (2.5) of [5] and can then be shown to be negligible. The third integral can be shown to be negligible by the same type of arguments we used in treating the second part of

(34). The fourth integral can be disposed of in the same way, with (A.13) of [5] also being used. Thus M_- can be estimated from the first term alone. Evaluating this with the aid of (71), we find

$$M_- \approx -\tfrac{1}{2}e^{-\alpha}(1-q)\boldsymbol{n} \times I. \tag{39}$$

Setting (37) and (39) into (33), multiplying the result by

$$2(I + \tfrac{1}{2}q)\cdot(I - \tfrac{1}{2}P),$$

and eliminating the second-order terms in p and q, we finally obtain (15a), which was to be proven.

Let us now state the conditions under which the derivation is valid, and then discuss their significance. These conditions are as follows.

Condition 1: The shell curvature and its rate of change are small enough compared to the attenuation per unit distance so that

$$(|k|/k_I)(\kappa^2 + |\nabla_S \kappa_u| + |\nabla_S k_v|)/\kappa_I^2 \ll 1. \tag{40}$$

Condition 2: The attenuation length is small enough compared to the distance h to the nearest significant source so that

$$(k_I h)^2 \gg 1. \tag{41}$$

Condition 3: In each of the adjacent volumes V_+ and V_-, either the wave number—k_+ in V_+, k_- in V_-—is small enough compared to the shell attenuation rate so that

$$(k_\pm/k_I)^2 \ll 1 \tag{42}$$

or, alternatively, the medium is a good enough absorber so that a condition of form (40) holds in it.

Condition 4: The shell is thin enough compared to its minimum radius of curvature so that

$$(|k|/k_I)^2(\kappa\Delta)^2 \ll 1. \tag{43}$$

Condition 5: The normal and principal curvature directions and the principal curvatures can be assumed the same at corresponding points of S_0, S_+, and S_-.

Condition 6: The region S_c described in the Appendix and a similarly defined region S_{c+} on S_+ can be chosen large enough so that the L and M can be approximated by integrals over these regions, but still small enough so that each region is connected and approximately paraboloidal.

Note that these are local conditions. Wherever they hold, the effective boundary conditions will hold, regardless of what may happen elsewhere on the shell.

The first three conditions are essentially those under which the impedance boundary condition for a solid body can be derived [5]. The second form of Condition 3 was not, however, given in the solid body study [5] because the case of two absorbers in contact was not considered there.

We can see from the Appendix how Condition 1, which is the same as (67), enters into the approximation process. In most cases of interest, the real and imaginary parts of k are of the same order. Furthermore, the rate of change of curvature is ordinarily not a problem except perhaps near sharp bends. Thus Condition 1 can usually be replaced by the simpler form (18). In the important class of problems involving plane wave scattering, Condition 2 is of no interest.

The last three conditions pertain specifically to the shell problem. They arise from the approximations made in the Appendix.

Condition 4, which is the same as (68), can usually be replaced by the simpler form

$$(\kappa\Delta)^2 \ll 1, \tag{44}$$

which is only restrictive near sharp edges and bends.

For shells of constant thickness, satisfaction of Condition 4 implies satisfaction of Condition 5. Condition 5 will still not be violated when Δ is a slowly changing function of position, and the boundary conditions will be unchanged in form. Even when Condition 5 is violated, it is possible to derive effective boundary conditions, but the derivation is more complicated.

Condition 6 will be restrictive near bends and edges where the curvature changes rapidly or suddenly. It will also be restrictive when the shell doubles back on itself to form a narrow slot, thus limiting the dimension τ of S_c or S_{c+} to the slot width.

It is clear from this discussion that Conditions 1 and 3 are the most important conditions. In practice, it turns out that Condition 1 is more important—we are more likely to encounter excessively high curvature than excessively high frequencies.

We have been concerned here only with homogeneous shells. It is worth noting, however, that boundary conditions can be derived which take into account slow spatial variations of Z and k.

IV. Electromagnetically Thin Shells

A shell is electromagnetically thin if approximation (19) holds or, equivalently, if we can neglect $O(\alpha^2)$ terms. An immediate consequence of this condition is that

$$q = O(\kappa\Delta) = O(\alpha\kappa/k_I) = O(\alpha^2) + O[(\kappa/k_I)^2] \tag{45}$$

and

$$P \sinh \alpha = O(\alpha p) \tag{46}$$

are negligible.

Let us for the moment characterize the shell material by both a layer admittance Y_e as defined in (20) and a layer impedance Z_m as defined in (23). This is completely equivalent to characterizing the material by an α and a Z.

We can now simplify (16a) and (17a) to get

$$\boldsymbol{K}_{m\Delta} = -\tfrac{1}{2}Z_m\boldsymbol{n} \times (\boldsymbol{K}_{e+} - \boldsymbol{K}_{e-}) \tag{47}$$

with $\boldsymbol{K}_{m\Delta}$ defined by (25). This says that in crossing from S_- to S_+, tangential $\boldsymbol{E}$ jumps by an amount proportional to the average value of tangential $\boldsymbol{H}$. Similarly, (16b) and (17b) yield

$$\boldsymbol{K}_{e\Delta} = \tfrac{1}{2}Y_e\boldsymbol{n} \times (\boldsymbol{K}_{m+} - \boldsymbol{K}_{m-}) \tag{48}$$

with $\boldsymbol{K}_{e\Delta}$ defined by (22).

At this point, it is necessary to develop some way of evaluating the relative significance of $K_{e\Delta}$ and $K_{m\Delta}$. To this end, let us assume the shell to be immersed in a medium with wave impedance Z_0. If the sources are far enough away from the shell, the incident field will have

$$\begin{aligned} E &= O(Z_0 H) \\ H &= O(Y_0 E), \end{aligned} \tag{49}$$

where

$$Y_0 = 1/Z_0. \tag{50}$$

It follows that for a jump in E to be significant, it must be significant compared to Z_0H, and similarly for H and Y_0E. We thus rewrite (47) and (48) as

$$K_{m\Delta} = -\tfrac{1}{2}Z_0(Z_m/Z_0)n \times (K_{e+} - K_{e-}) \tag{51}$$

$$K_{e\Delta} = \tfrac{1}{2}Y_0(Y_e/Y_0)n \times (K_{m+} - K_{m-}), \tag{52}$$

with (Z_m/Z_0) and (Y_e/Y_0) measuring the size and significance of the jumps.

Now we note that

$$(Y_e/Y_0)(Z_m/Z_0) = \alpha^2. \tag{53}$$

Thus, either (Y_e/Y_0) and (Z_m/Z_0) are both of order α, o- else one is significant and the other is not. The latter situation is the one of most practical interest. If it is (Y_e/Y_0) that is significant, then $K_{m\Delta}$ is effectively zero and (47) and (48) simplify to (21); we call the shell a thin electric shell because it is characterized by an effective electric current. Similarly, if (Z_m/Z_0) is large, then $K_{e\Delta}$ is effectively zero and we obtain (24); we call the shell a thin magnetic shell because it is characterized by an effective magnetic current. The value of Z_m for a thin electric shell and the value of Y_e for a thin magnetic shell are of no interest. The results for a thin electric shell agree with (30) of Oshiro and Cross [1].

We usually deal with shells in free space. Thin electric shells will then be composed of high-conductivity material, and thin magnetic shells will be composed of ferrite. It is interesting, however, to note that the same shell can be electric, magnetic, or intermediate, depending on the Z_0 of the medium in which it is immersed.

V. Absorber-Coated Bodies

The key formula in the treatment of absorber-coated bodies is (28). This can be derived by eliminating $\hat{K}_{m-}$ and $\hat{K}_{e-}$ from (16a) and (16b) with the aid of the boundary condition

$$\hat{K}_{m-} + \overline{Z}_{-}\cdot(n \times \hat{K}_{e-}) = 0. \tag{54}$$

Equation (27) for a single layer coated onto a perfect conductor is obtained simply by setting

$$\overline{Z}_{-} = 0 \tag{55}$$

in (28).

For two absorbing layers coated onto a perfect conductor of small curvature, the calculational technique of Section II yields

$$\overline{Z}_s = Z_1 \frac{Z_1 \tanh \alpha_1 + Z_2 \tanh \alpha_2}{Z_1 + Z_2 \tanh \alpha_1 \tanh \alpha_2}, \tag{56}$$

where subscript 1 refers to the outer layer and subscript 2 to the inner layer. This is equivalent to (7) of Bowman and Weston [4]; similarly, (27), when specialized for a singly coated cylinder, is equivalent to their (12). Thus, the results here are consistent with those of Bowman and Weston.

VI. Concluding Remarks

We have shown here that the scattering properties of a sufficiently lossy shell can be described completely by two boundary conditions and that these can be derived directly from the integral equations for propagation in the shell. Under the assumptions stated in Section III, the boundary conditions can be expressed in the symmetric form (15). Alternatively, they can be expressed in the form (16), in which the K_+ are given as explicit functions of the K_-, or in the form (17), in which the K_- are given as explicit functions of the K_+. When curvature is negligible, the three alternative sets of boundary conditions reduce to the simpler forms (7), (8), and (9), respectively.

By specializing the boundary conditions to electromagnetically thin shells, we have shown that these fall into two main classifications: a) thin electric shells, characterized by the layer admittance Y_e and supporting an effective electric shell current; and b) thin magnetic shells, characterized by the layer impedance Z_m and supporting an effective magnetic shell current. There is also an intermediate case in which the shell supports weak electric and magnetic currents. Interestingly, the classification of a thin shell depends on the wave impedance of the surrounding medium.

We have also used the general boundary conditions to derive formulas for the effective surface impedance of a multiply coated body. These formulas are extremely useful in radar cross-section reduction work.

The shell boundary conditions greatly simplify the numerical solution of scattering problems. This is true even when the boundary conditions are not valid everywhere, as is the case for shells with sharp edges or apertures. In such problems, a more accurate description of propagation in the shell can be used wherever necessary, and there is no difficulty in combining it with the boundary condition approximation.

Appendix

Background Material for Estimation of L_- and M_-

Referring to the geometry of Fig. 1, we say that r_0 on S_0, r_+ on S_+, and r_- on S_- are corresponding points if r_+ and r_- are the closest points of S_+ and S_- along the normal n to S_0 at r_0. We shall assume here that the normals n_+ and n_- at these points do not differ significantly from n, and that the difference in the principal curvatures and in the directions of principal curvature is also negligible.

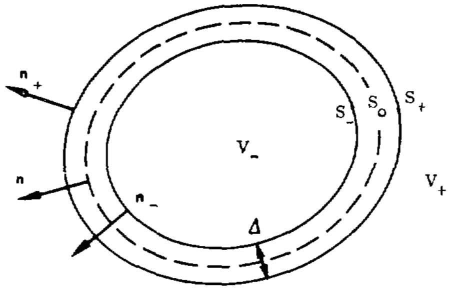

Fig. 1. Shell geometry.

We introduce a tangent–normal Cartesian coordinate system ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) which is aligned with the principal curvature coordinate system ($\mathbf{e}_u$, $\mathbf{e}_v$, $\boldsymbol{n}$), and has its origin at $\boldsymbol{r}_-$; we also introduce an associated cylindrical coordinate system ($\mathbf{e}_\rho$, $\mathbf{e}_\theta$, $\mathbf{e}_z$). We then define S_c as the region of S_- contained within the cylinder

$$-\tau \leq z \leq \tau, \qquad \rho \leq \tau, \tag{57}$$

where

$$\tau^2 = 2c\Delta/k_I + c^2/k_I^2, \tag{58}$$

with c a number of order unity. We assume that c is small enough so that S_c is connected and can be approximated by the osculating paraboloid

$$z = f(\rho, \theta) = -\tfrac{1}{2}\rho^2\bar{\kappa}, \tag{59}$$

with

$$\bar{\kappa} = \kappa_u \cos^2\theta + \kappa_v \sin^2\theta \tag{60}$$

and κ_u and κ_v evaluated at $\boldsymbol{r}_0$.

Let us now designate an arbitrary point on S_c by $\boldsymbol{r}'$. Then we readily find from (59) and (60) that the normal $\boldsymbol{n}'$ at $\boldsymbol{r}'$ satisfies

$$\boldsymbol{n} - \boldsymbol{n}' \approx -\rho(\kappa_u \cos\theta \mathbf{e}_x + \kappa_v \sin\theta \mathbf{e}_y) \tag{61}$$

and

$$\boldsymbol{n}\cdot\boldsymbol{n}' \approx 1. \tag{62}$$

Furthermore, the vector from $\boldsymbol{r}'$ to $\boldsymbol{r}_+$ is

$$\boldsymbol{r}_+ - \boldsymbol{r}' = \rho \mathbf{s}^* \tag{63}$$

with

$$\mathbf{s}^* = -(\cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y) + [(\Delta/\rho) + \tfrac{1}{2}\rho\bar{\kappa}]\mathbf{e}_z. \tag{64}$$

The magnitude of this vector is

$$R = |\boldsymbol{r}_+ - \boldsymbol{r}'| = l[1 + (\rho\Delta/l^2)\rho\bar{\kappa} + \tfrac{1}{4}(\rho^2/l^2)(\rho\bar{\kappa})^2]^{1/2} \tag{65}$$

with

$$l = (\rho^2 + \Delta^2)^{1/2}. \tag{66}$$

At this point we must impose the inequalities

$$(|k|/k_I)(\kappa^2 + |\nabla_S \kappa_u| + |\nabla_S \kappa_v|)/k_I^2 \ll 1 \tag{67}$$

$$(|k|/k_I)^2(\kappa\Delta)^2 \ll 1. \tag{68}$$

Under these restrictions and with the maximum value of ρ restricted by (58), we see that (65) can be approximated on S_c by

$$R = l[1 + \tfrac{1}{2}(\Delta\rho/l^2)\rho\bar{\kappa}] \tag{69}$$

and the exponential in Green's function can be approximated by

$$e^{ikR} = e^{ikl}[1 + \tfrac{1}{2}ik\Delta\rho^2\bar{\kappa}/l]. \tag{70}$$

We can now justify the further approximations

$$\begin{aligned} lG &= lG_l + \frac{1}{2}\left(\Delta\bar{\kappa}\rho^2 \frac{d}{dl} G_l\right) \\ &= (1 - \Delta\bar{\kappa})lG_l + \frac{1}{2}\Delta\bar{\kappa}\frac{d}{dl}(\rho^2 G_l) \end{aligned} \tag{71}$$

$$\begin{aligned} \frac{l}{R}\frac{d}{dR}G &= \frac{d}{dl}G_l + \frac{1}{2}\Delta\bar{\kappa}\rho^2\frac{d}{dl}\left(\frac{1}{l}\frac{d}{dl}G_l\right) \\ &= (1 - \Delta\bar{\kappa})\frac{d}{dl}G_l \\ &\quad + \frac{1}{2}\Delta\bar{\kappa}\frac{d}{dl}\left(\frac{\rho^2}{l}\frac{d}{dl}G_l\right), \end{aligned} \tag{72}$$

$$\begin{aligned} \rho^2\frac{l}{R}\frac{d}{dR}G &= -2lG_l + 4\Delta\bar{\kappa}lG_l \\ &\quad + (1 - 2\Delta\bar{\kappa})\frac{d}{dl}(\rho^2 G_l) \\ &\quad + \frac{1}{2}\Delta\bar{\kappa}\frac{d}{dl}\left(\frac{\rho^4}{l}\frac{d}{dl}G_l\right). \end{aligned} \tag{73}$$

Here

$$G_l = e^{ikl}/4\pi l, \tag{74}$$

and we have used

$$d\rho/dl = l/\rho. \tag{75}$$

The maximum value of R on S_c will be roughly equal to the maximum value of l. This can be obtained using (58) and is

$$l_{\max} = \Delta + c/k_I \tag{76}$$

at $\rho = \tau$. Thus the magnitude of e^{ikR} at $\rho = \tau$ is down by a factor of about e^{-c} from its value at $\boldsymbol{r}$. It follows that, provided c is sufficiently large, the integrals over S_- in the definitions of L_- and M_- can be approximated by integrals over S_c.

Going one step further, we define P_- as the plane normal to $\boldsymbol{n}$ through $\boldsymbol{r}_-$ and define D_- as the disk in P_- with center $\boldsymbol{r}_-$ and radius τ. Then D_- is the projection of S_c on P_- and we can transform an integral over S_c into an integral over D_-. This is especially simple since, by (62), the surface elements are essentially equal, so that

$$dS_c = dP_- = \rho d\rho d\theta = l dl d\theta. \tag{77}$$

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Bounds on the Electric Field Outside a Radiating System

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Abstract—**Upper and lower bounds on the (spatial) maximum electric field outside a radiating system have been obtained and studied. The results should be useful in the design of systems where microwave breakdown is an important consideration.**

I. Introduction

RECENTLY designers of high-power radiating systems (for ECM and other applications) have found that the power that may be transmitted can be severely limited by the breakdown of the air around the antenna. In fact, in the design of systems for UHF and VHF applications to reentry vehicles, breakdown is the paramount consideration. Since the breakdown generally occurs in the near field of the antenna (and in particular in the spatial region outside the antenna where the near *electric* field is a *maximum*), the antenna near fields have taken on a new importance and it is not sufficient merely to design an antenna with given *far*-field requirements.

The antenna Q has been generally used as a qualitative measure of the near electric-field strength. However, Q is a measure of the stored near-field energy and does not necessarily give any of the detailed structure of the near electric field. For example, two antennas with the same Q could have different near electric-field distributions and, hence, different breakdown characteristics. In fact, as we shall see later, the antenna with the lowest Q is not necessarily optimum from the standpoint of having the lowest[1] maximum electric fields outside it.

In the past, Chu [1] and others [2], [3], have been able to relate the Q to the size of an antenna. In particular, small antennas must have a high Q, while large antennas can be made to have a small Q. It would be desirable to do the same thing for the near electric fields, but, unfortunately, this is not possible, since E depends on current distribution and antenna shape in addition to the antenna size. However, it is possible to determine bounds on the near E-field, which are functions of only the antenna size. It is the purpose of this paper to obtain these bounds. In particular, since breakdown occurs in the spatial region of maximum field strength, it is desirable to obtain a lower bound on the maximum field, while keeping the radiated power P_R fixed. That is, for a given antenna size and radiated power, what is the smallest possible value of the *maximum* E-field outside the antenna? Knowing this value the designer could then attempt to design his system to achieve it. However, if the smallest possible maximum E-field (for a given size antenna) exceeds the breakdown field strength, then no amount of design can prevent breakdown from occurring and the antenna size must be increased.

It is also desirable to obtain an upper bound on the maximum E-field outside the antenna. That is, for a given antenna size, what is the largest possible E-field that can occur outside the antenna if we choose the poorest design possible? Obviously, if this value is below the breakdown field strength, the antenna design is not very critical.

II. Derivation of a Lower Bound on the Maximum E-Field

In order to derive results that are applicable to general antennas, it is useful to enclose the radiating system in an imaginary sphere S_1, which is just large enough to enclose the antenna. Our object in this section, then, will be to derive

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[1] Since breakdown occurs in the region of maximum field strength, we, obviously, would like to make this maximum as small as possible, while still having the required radiated power.