Potential Integrals for Uniform and Linear Source Distributions on Polygonal and Polyhedral Domains

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Abstract—**Formulas for the potentials due to uniform and linearly varying source distributions defined on simply shaped domains are systematically developed and presented. Domains considered are infinite planar strips, infinite cylinders of polygonal cross sections, planar surfaces with polygonal boundaries, and volumetric regions with polyhedral boundaries. The expressions obtained are compact in form and their application in the numerical solution of electromagnetics problems by the method of moments is illustrated.**

INTRODUCTION

In the numerical solution of problems in electromagnetics, one is frequently faced with the evaluation of static potential integrals associated with source distributions defined on elementary source regions such as line segments, polygons, and polyhedrons. In two dimensions, for example, potential integrals due to constant or linearly varying source density distributions confined to cylinders whose cross-sections are line segments or polygons are often required. In three dimensions, potentials due to sources confined to planar polygonal or polyhedral regions are required. Potential integrals for such distributions have found use, for example, in the numerical solution of the following problems:

1) static or quasi-static electric and magnetic problems formulated as integral equations [1], [2], [3];
2) the evaluation of "self-term" contributions to the moment matrix in time-harmonic electromagnetic radiation and scattering problems [4], [5];
3) time domain-problems in which time-retardation effects over subdomains are neglected [6].

In this communication, analytical expressions are obtained for all of the most commonly encountered of these potential integrals. In a few instances, the formulas given have appeared previously in the literature. Nevertheless, we include them for completeness together with the observation that the derivations given and the expressions obtained are generally much more concise than those appearing elsewhere in the literature. These features permit the expressions to be easily checked and translated into computer programs. The integrals are also expressed in terms of readily identifiable geometrical quantities, which facilitates programming the expressions and evaluating them in certain limiting cases.

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EVALUATION OF POTENTIAL INTEGRALS

Surface Sources Distributed on an Infinite Strip

Let sources be distributed on an infinite strip such as that shown in Fig. 1(a). The source density is assumed to be invariant along the direction of the strip axis. Projection of the strip onto a plane $\mathcal{P}$ with unit normal $\hat{\mathbf{n}}$ parallel to the strip axis defines a line segment C, a generator of the strip, depicted in greater detail in Fig. 2. The strip potential is evaluated as a superposition of that due to a distribution of elemental line sources. Each line source is assumed to contribute a potential proportional to $\ln P$, where $P = |\boldsymbol{\rho} - \boldsymbol{\rho}'|$ and $\boldsymbol{\rho}$ is the projection of $\mathbf{r}$ onto $\mathcal{P}$ while $\boldsymbol{\rho}'$ is the similar projection of $\mathbf{r}'$. Position vector $\mathbf{r}$ is the vector from the origin to the observation point while $\mathbf{r}'$ is that to a source point on the strip. The perpendicular distance from the point located by $\boldsymbol{\rho}$ to the line segment C or its extension is designated P^0, as shown in Fig. 2. C is parameterized by the arc length variable l' measured from the plane which is perpendicular to the extension of C and which passes through the point located by $\mathbf{r}$. P^0 and l' represent, in effect, a pair of rectangular coordinates in $\mathcal{P}$ locating points on C. In terms of l', the endpoints of C are located at l^+ and l^-.

Distances measured in $\mathcal{P}$ from $\boldsymbol{\rho}$ to given endpoints $\boldsymbol{\rho}^+$ and $\boldsymbol{\rho}^-$ of C are denoted P^+ and P^-, respectively. The quantities P^0, $\hat{\mathbf{P}}^0$, $P^\pm$, and $l^\pm$ of Fig. 2 are readily calculated in terms of $\boldsymbol{\rho}$, $\boldsymbol{\rho}^+$, and $\boldsymbol{\rho}^-$ by the following sequence of computations:

$$\hat{\mathbf{l}} = \frac{\boldsymbol{\rho}^+ - \boldsymbol{\rho}^-}{|\boldsymbol{\rho}^+ - \boldsymbol{\rho}^-|}, \ \hat{\mathbf{u}} = \hat{\mathbf{l}} \times \hat{\mathbf{n}}, \ l^\pm = (\boldsymbol{\rho}^\pm - \boldsymbol{\rho}) \cdot \hat{\mathbf{l}},$$

$$P^0 = |(\boldsymbol{\rho}^\pm - \boldsymbol{\rho}) \cdot \hat{\mathbf{u}}|, \quad P^\pm = |\boldsymbol{\rho}^\pm - \boldsymbol{\rho}| = \sqrt{(P^0)^2 + (l^\pm)^2},$$

$$\hat{\mathbf{P}}^0 = \frac{(\boldsymbol{\rho}^\pm - \boldsymbol{\rho}) - l^\pm \hat{\mathbf{l}}}{P^0}.$$

Note that $\hat{\mathbf{u}} = \pm\hat{\mathbf{P}}^0$, the sign depending on which end of C corresponds to the vector $\boldsymbol{\rho}^+$. Other quantities appearing in Fig. 2 are used in subsequent sections.

An integral proportional to the potential of a uniform source distribution on the strip is now easily evaluated in terms of the quantities defined:

$$\int_C \ln P \, dl' = \int_{l^-}^{l^+} \ln \sqrt{(P^0)^2 + (l')^2} \, dl'$$
$$= l^+ \ln P^+ - l^- \ln P^- + P^0 \left(\tan^{-1} \frac{l^+}{P^0} - \tan^{-1} \frac{l^-}{P^0} \right) - (l^+ - l^-). \tag{1}$$

This and the following integral, though trivial to evaluate, are

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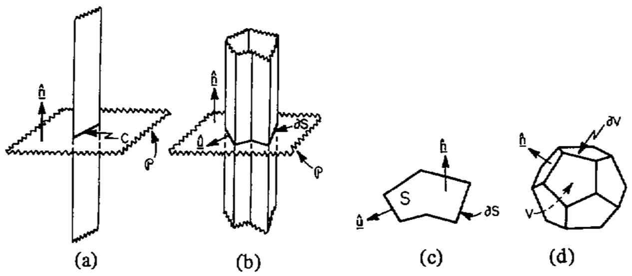

Fig. 1. Four examples. (a) An infinite strip with line-segment cross section C. (b) An infinite cylinder with polygonal cross section S. (c) A planar surface S with polygonal boundary ∂S. (d) A volumetric region V with polyhedral boundary ∂V.

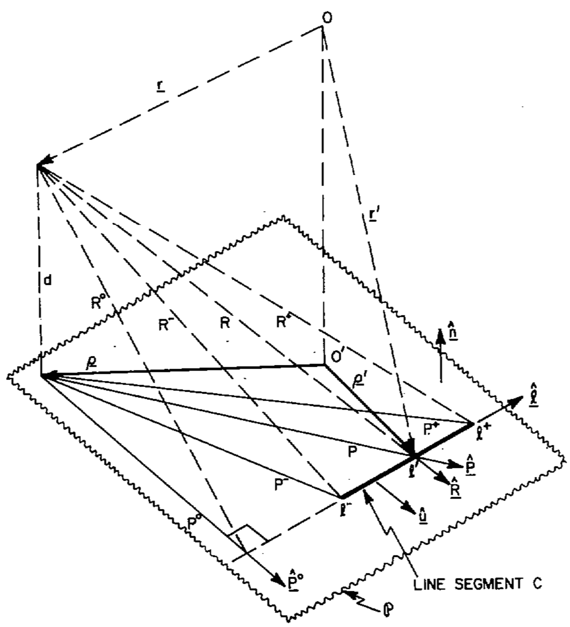

Fig. 2. Geometrical quantities associated with the line segment C lying in the plane $\mathcal{P}$. The observation point for the potential is located by the position vector r with respect to the coordinate origin 0.

included for completeness and to facilitate introduction of the geometrical quantities used in the computation of potential integrals in subsequent sections.

To treat a source distribution which varies linearly on C, we consider the vector-valued integral

$$\int_C \hat{\mathbf{l}} l' \ln P \, dl' = \hat{\mathbf{l}} \int_{l^-}^{l^+} l' \ln \sqrt{(P^0)^2 + (l')^2} \, dl'$$

$$= \frac{\hat{\mathbf{l}}}{2} \{(P^+)^2 \ln P^+ - (P^-)^2 \ln P^- - \tfrac{1}{2}[(l^+)^2 - (l^-)^2]\} \tag{2}$$

where $\hat{\mathbf{l}}$ is the unit vector tangent to C in the direction of increasing l'. The integral in (2) is given a vector character merely to emphasize its correspondence to (4), (6), and (8) which follow. Obviously, the potential of a more general linear variation of the source distribution can easily be synthesized as an appropriate linear combination of (1) and (2).

As an application of (1), consider the numerical evaluation of the integral

$$\int_{l^-}^{l^+} H_0^{(2)}(kP) \, dl'$$

arising when pulse functions are used as basis functions in a solution by the method of moments of two-dimensional time-harmonic electromagnetic scattering problems [7]. $H_0^{(2)}$, the two-dimensional Green's function, is the Hankel function of zero order and second kind; $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber associated with the assumed $\exp(j\omega t)$ time variation. If ρ is on the source strip, the integrand of the above integral is singular at $\rho = \rho'$ as evidenced by the small argument expansion of $H_0^{(2)}$:

$$H_0^{(2)}(kP) \underset{P \to 0}{\longrightarrow} 1 - j\frac{2}{\pi}\left(\ln\frac{kP}{2} + \gamma\right)$$

where $\gamma = 0.5772 \ldots$ is Euler's constant. When P vanishes within the integration domain, the integral is difficult to evaluate numerically due to the presence of the logarithmic singularity. Subtraction of this singular term from the integrand leaves a bounded integrand, however, and hence the integral may be cast into a form which can be numerically evaluated by rewriting it as

$$\int_{l^-}^{l^+} H_0^{(2)}(kP) \, dl' = \int_{l^-}^{l^+} \left[H_0^{(2)}(kP) + j\frac{2}{\pi}\ln P\right] dl' - j\frac{2}{\pi}\int_{l^-}^{l^+} \ln P \, dl'$$

where the logarithmically singular contribution has been subtracted from and added to the original integral. The integrand of the first integral on the right of the equality is nonsingular and can be numerically integrated readily; that of the second integral is singular but the integral is merely (1) and hence is analytically integrable.

Volume Sources Distributed Within a Polygonal Cylinder

We consider next potential integrals associated with volumetric source distributions confined to the interior of an infinite cylinder having a polygonal cross section S (cf., Fig. 1(b)). Source densities are assumed to be invariant along the cylinder axis, which is oriented parallel to $\hat{\mathbf{n}}$, a unit vector normal to S. The boundary of the polygonal cross section S is denoted ∂S and has a right-hand orientation with respect to $\hat{\mathbf{n}}$. It comprises a series of line segments, the ith one of which is denoted by $\partial_i S$.

Evaluation of the potential integral is accomplished by employing an appropriate Gauss integral theorem (divergence theorem, etc.) to first transform an integration over S to one over the boundary ∂S of S. Application of such theorems requires that the integrand be continuously differentiable on the domain of integration S. This is not the case, however, when the endpoint of the vector ρ falls in S or on ∂S. In this case, a region S_ϵ, defined as the intersection of S and a small disk of radius ϵ centered at ρ, is isolated for separate treatment. Integrals over S_ϵ or its boundary, ∂S_ϵ, may be evaluated by expression them in terms of a local polar coordinate system with its origin centered at the point ρ. Such integrals occasionally involve the angular extent $\alpha(\rho)$ of the circular arc portion of ∂S_ϵ lying within S. Thus, if the point located by ρ falls outside S, S_ϵ is empty and $\alpha(\rho) = 0$; if it falls inside S, $\alpha(\rho) = 2\pi$ (Fig. 3(a)); if it falls on ∂S but not at

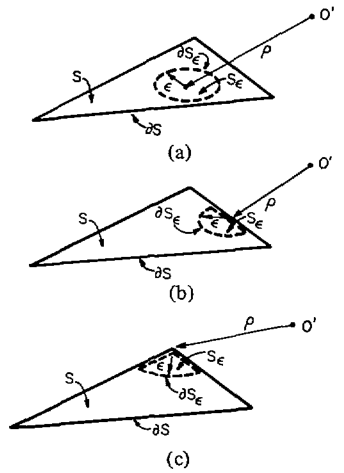

Fig. 3. Region S_ϵ corresponding to a point ρ. (a) Point inside a triangle S. (b) Point on an edge of S. (c) Point at a vertex of S.

a corner, $\alpha(\rho) = \pi$ (Fig. 3(b)); if it falls at a vertex of S, $\alpha(\rho)$ becomes the angle between the two edges of S meeting at the vertex (Fig. 3(c)). In the following it is convenient always to assume that the point located by ρ lies within or on the boundary of S; whenever this is not the case, the correct derivation is recovered by assuming $\epsilon = 0$ and $\alpha(\rho) = 0$.

In the evaluation of each of the integrals of this and the following sections, the strategy is to apply Gauss' theorems a sufficient number of times to reduce the integrals on the boundary edges of the original integration domain. In the present case, these are the edges $\partial_i S$ of the polygon. Consequently, the geometry defined in Fig. 2 is also applicable to the present problem if the line segment C there is taken to be $\partial_i S$, the ith edge of the planar polygon S with normal $\hat{\mathbf{n}}$. In the following, we also append a subscript i to any quantity defined in Fig. 2 when the quantity is to be associated with the ith edge.

With these preliminaries in hand, the potential integral for a uniform source distribution may be evaluated as follows:

$$\int_S \ln P\, dS'$$
$$= \lim_{\epsilon\to 0} \frac{1}{2} \int_{S-S_\epsilon} \nabla_s' \cdot \left[\left(P \ln P - \frac{P}{2}\right)\hat{\mathbf{P}}\right] dS'$$
$$+ \lim_{\epsilon\to 0} \int_{S_\epsilon} \ln P\, dS' = \lim_{\epsilon\to 0} \frac{1}{2} \int_{\partial(S-S_\epsilon)} \left(P \ln P - \frac{P}{2}\right)$$
$$\hat{\mathbf{P}} \cdot \hat{\mathbf{u}}\, dl' + \lim_{\epsilon\to 0} \alpha(\rho) \frac{\epsilon^2}{2} (\ln \epsilon - \tfrac{1}{2})$$
$$= \frac{1}{2} \sum_i \int_{\partial_i S} \left(P \ln P - \frac{P}{2}\right) \hat{\mathbf{P}} \cdot \hat{\mathbf{u}}_i\, dl'$$
$$= \frac{1}{2} \sum_i \mathbf{P}_i^0 \cdot \hat{\mathbf{u}}_i \int_{\partial_i S} (\ln P - \tfrac{1}{2})\, dl'$$
$$= \frac{1}{2} \sum_i \mathbf{P}_i^0 \cdot \hat{\mathbf{u}}_i \Bigg[l_i^+ \ln P_i^+ - l_i^- \ln P_i^-$$
$$+ P_i^0 \left(\tan^{-1} \frac{l_i^+}{P_i^0} - \tan^{-1} \frac{l_i^-}{P_i^0}\right) - \frac{3}{2}(l_i^+ - l_i^-)\Bigg] \quad (3)$$

where $\mathbf{P}_i^0 = P_i^0 \hat{\mathbf{P}}_i^0$, $\hat{\mathbf{P}}$ is the vector $(\rho' - \rho)/P$, and $\hat{\mathbf{u}}$ is the outward normal vector on ∂S lying in the plane of S (cf., Fig. 2). We have also used the fact that $\hat{\mathbf{u}}_i \cdot \hat{\mathbf{P}} P = \mathbf{P}_i^0 \cdot \hat{\mathbf{u}}_i$ is constant on $\partial_i S$. The summation is over all the edges of S. The surface divergence operator, $\nabla_s' \cdot (\cdots)$, involves differentiations with respect to source (primed) coordinates only. Note that the limit of the integral over the circular arc portion of $\partial(S - S_\epsilon)$ vanishes since the integrand remains bounded whereas the domain of integration vanishes as $\epsilon \to 0$. A limiting argument convinces one that the contribution to the sum in (3) from any edge is zero if ρ lies on the edge or on its extension. The steps used to evaluate (3) are common to the remaining integrals of this and subsequent sections and hence are summarized here as follows.

1) The integral over S is partitioned into two integrals over $S - S_\epsilon$ and S_ϵ, respectively. It is convenient to view each integral as a limit with $\epsilon \to 0$, although the value of the sum is, of course, independent of ϵ.

2) The integrand of the integral on $S - S_\epsilon$ is written as the differential (involving the surface del operator ∇_s) of some quantity, and a Gauss integral theorem (in this case, the surface divergence theorem [8]) is used to transform the integral into one over the boundary $\partial(S - S_\epsilon)$ of $S - S_\epsilon$. The orientation of the integration path is assumed to be right-handed with respect to $\hat{\mathbf{n}}$.

3) The limit of the integral over S_ϵ is evaluated; this limit vanishes by inspection when the integrand is bounded and the domain of integration vanishes. When the integrand is unbounded, the integral is evaluated explicitly by the procedure discussed at the beginning of this section and the limit of the integral as $\epsilon \to 0$ is then determined.

4) The limit of the integral over $\partial(S - S_\epsilon)$ is evaluated; occasionally the contribution from the portion of ∂S_ϵ in S vanishes since the integrand remains bounded while the domain of integration vanishes. When this is not the case, the limit of the integral is explicitly evaluated in polar coordinates.

5) The remaining integral over ∂S is decomposed into a sum of line integrals over the subboundaries (edges) $\partial_i S$.

6) The line integral over $\partial_i S$ is evaluated in terms of the geometrical quantities defined in Fig. 2.

For a source density varying linearly over a polygonal cylinder, two independent directions for the linear variation can be considered. Both cases can be treated simultaneously, however, by considering the *vector-valued* integral

$$\int_S (\rho' - \rho) \ln P\, dS'$$
$$= \lim_{\epsilon\to 0} \frac{1}{2} \int_{S-S_\epsilon} \nabla_s' \left(P^2 \ln P - \frac{P^2}{2}\right) dS'$$
$$+ \lim_{\epsilon\to 0} \int_{S_\epsilon} \hat{\mathbf{P}} P \ln P\, dS'$$
$$= \lim_{\epsilon\to 0} \frac{1}{2} \int_{\partial(S-S_\epsilon)} \left(P^2 \ln P - \frac{P^2}{2}\right) \hat{\mathbf{u}}\, dl'$$
$$= \frac{1}{2} \sum_i \hat{\mathbf{u}}_i \int_{\partial_i S} \left(P^2 \ln P - \frac{P^2}{2}\right) dl'$$

$$= \frac{1}{2} \sum_i \hat{\mathbf{u}}_i \Bigg\{ [(P_i^0)^2 + \tfrac{1}{3}(l_i^+)^2]\, l_i^+ \ln P_i^+ - [(P_i^0)^2 + \tfrac{1}{3}(l_i^-)^2]\, l_i^- \ln P_i^- + \tfrac{2}{3}(P_i^0)^3 \left(\tan^{-1} \frac{l_i^+}{P_i^0} - \tan^{-1} \frac{l_i^-}{P_i^0} \right) - \tfrac{5}{18}[(l_i^+)^3 - (l_i^-)^3] - \tfrac{7}{6}(P_i^0)^2 (l_i^+ - l_i^-) \Bigg\} \tag{4}$$

which is also evaluated via steps 1)-6) and with the aid of [9, eq. (623.2)]. In this case, the appropriate Gauss integral theorem transforms the surface integral of the (surface) gradient of a scalar to a vector line integral around the surface boundary [8]. If ρ is on the ith edge or its extension, $P_i^0 = 0$, $P_i^\pm = |l_i^\pm|$, and the corresponding arctangent terms make no contribution to the sum in (4).

If the actual source distribution varies linearly with distance from some arbitrary point ρ_a along the direction of the unit vector $\hat{\mathbf{a}}$ in the plane of S, then the distribution is proportional to $\hat{\mathbf{a}} \cdot (\rho' - \rho_a)$. The potential hence is proportional to

$$\int_S \hat{\mathbf{a}} \cdot (\rho' - \rho_a) \ln P \, dS'$$

$$= \hat{\mathbf{a}} \cdot \int_S (\rho' - \rho) \ln P \, dS' + \hat{\mathbf{a}} \cdot (\rho - \rho_a) \int_S \ln P \, dS'$$

which is seen to be a linear combination of (3) and (4).

Surface Sources Distributed on Polygons

If surface sources are distributed on a planar polygon S, (cf., Fig. 1(c)), the potential of an elemental source on S is no longer of logarithmic form. Instead, the potential observed at a point $\mathbf{r}$ due to an elemental source on S ar $\mathbf{r}'$ is proportional to $1/R = 1/|\mathbf{r} - \mathbf{r}'|$. To facilitate evaluation of the potential integrals, the distances $R_i^0 = \sqrt{(P_i^0)^2 + d^2}$ and $R_i^\pm = \sqrt{(P_i^\pm)^2 + d^2}$, the latter associated with the endpoints of $\partial_i S$ (the ith edge of S), are introduced (Fig. 2). The distance d is the height of the observation point above the plane of S, measured positively in the direction of $\hat{\mathbf{n}}$, and may be calculated by

$$d = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}_i^\pm)$$

where $\mathbf{r}_i^+$ ($\mathbf{r}_i^-$) is a given position vector to the upper (lower) endpoint of $\partial_i S$. The vectors $\hat{\mathbf{l}}_i$ and $\rho_i^\pm$ are now defined in terms of the line segment endpoints as

$$\hat{\mathbf{l}}_i = \frac{\mathbf{r}_i^+ - \mathbf{r}_i^-}{|\mathbf{r}_i^+ - \mathbf{r}_i^-|}$$

and

$$\rho_i^\pm = \mathbf{r}_i^\pm - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r}_i^\pm).$$

The potential due to a uniform source distribution on S is now found to be proportional to

$$\int_S \frac{dS'}{R}$$

$$= \lim_{\epsilon \to 0} \int_{S - S_\epsilon} \nabla_s' \cdot \left(\frac{R}{P} \hat{\mathbf{P}} \right) dS' + \lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{dS'}{R}$$

$$= \lim_{\epsilon \to 0} \int_{\partial(S - S_\epsilon)} \frac{R}{P} \hat{\mathbf{P}} \cdot \hat{\mathbf{u}} \, dl' + \lim_{\epsilon \to 0} \alpha(\rho)(\sqrt{\epsilon^2 + d^2} - |d|)$$

$$= -\alpha(\rho)|d| + \sum_i \int_{\partial_i S} \frac{R}{P} \hat{\mathbf{P}} \cdot \hat{\mathbf{u}}_i \, dl'$$

$$= -\alpha(\rho)|d| + \sum_i \mathbf{P}_i^0 \cdot \hat{\mathbf{u}}_i \int_{\partial_i S} \left(\frac{1}{R} + \frac{d^2}{P^2 R} \right) dl'$$

$$= -\alpha(\rho)|d| + \sum_i \hat{\mathbf{P}}_i^0 \cdot \hat{\mathbf{u}}_i \left[P_i^0 \ln \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} + |d| \left(\tan^{-1} \frac{|d| l_i^+}{P_i^0 R_i^+} - \tan^{-1} \frac{|d| l_i^-}{P_i^0 R_i^-} \right) \right]$$

$$= \sum_i \hat{\mathbf{P}}_i^0 \cdot \hat{\mathbf{u}}_i \left[P_i^0 \ln \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} - |d| \times \left(\tan^{-1} \frac{P_i^0 l_i^+}{(R_i^0)^2 + |d| R_i^+} - \tan^{-1} \frac{P_i^0 l_i^-}{(R_i^0)^2 + |d| R_i^-} \right) \right]. \tag{5}$$

The last integral above may be evaluated with the aid of [9, eqs. (200.01), (387.)] Note the appearance of the "residue" contribution involving the angle $\alpha(\rho)$ of the portion of ∂S_ϵ in S. This term is combined with the sum by expressing it as a sum of the angles between adjacent vertices around the polygon,

$$\alpha(\rho) = \sum_i \hat{\mathbf{P}}_i^0 \cdot \hat{\mathbf{u}}_i \left(\tan^{-1} \frac{l_i^+}{P_i^0} - \tan^{-1} \frac{l_i^-}{P_i^0} \right),$$

and employing the identity

$$\tan^{-1} \frac{l_i^\pm}{P_i^0} - \tan^{-1} \frac{|d| l_i^\pm}{P_i^0 R_i^\pm} = \tan^{-1} \frac{P_i^0 l_i^\pm}{(R_i^0)^2 + |d| R_i^\pm}$$

where each arctangent function is evaluated on its principal branch. If $\mathbf{r}$ is on an edge or its extension, it is easily shown that the contribution to the sum in (5) from that edge vanishes.

Integral (5) has been evaluated for triangular patches by several authors [1], [2], [3], [6], [10], but the formula presented here has advantages in terms of accuracy, conciseness, and convenience for numerical work. By contrast, the formula presented in [1], for example, is found to be inaccurate for moderate-to-large separation distances between observation and source points because it contains many terms which must approximately cancel one another. The present form is also symmetric in the vertex indices, which not only is intuitively satisfying, but also has the practical consequence of rendering it suitable for programming using loop operations when the vertex information is stored in column vector form. This latter feature stands in contrast to the formulas given in [1], [2], [3], [10], which would be difficult to extend to arbitrary polygonal regions.

The corresponding integral for linearly varying source distributions is again evaluated in vector form:

$$\int_S \frac{\rho' - \rho}{R} \, dS'$$

$$= \lim_{\epsilon \to 0} \int_{S - S_\epsilon} \nabla_s' R \, dS' + \lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{\rho' - \rho}{R} \, dS'$$

$$= \lim_{\epsilon \to 0} \int_{\partial(S - S_\epsilon)} R\hat{\mathbf{u}} \, dl' = \lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{P}{R} \mathbf{P} \, dS'$$

$$= \sum_i \hat{\mathbf{u}}_i \int_{\partial_i S} R \, dl'$$

$$= \frac{1}{2} \sum_i \hat{\mathbf{u}}_i \left[(R_i^0)^2 \ln \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} + l_i^+ R_i^+ - l_i^- R_i^- \right] \tag{6}$$

where the last integral is evaluated by [9, eq. (230.01)]. Comments similar to those in the previous paragraph apply in comparing (6) to the formula given in [2] for the potential due to a linear source distribution on a triangle.

In the solution of electromagnetic scattering problems by the method of moments with triangular patch modeling [4], the above integrals are used in the numerical evaluation of moment matrix elements. For example, one may evaluate the time-harmonic scalar potential due to a uniform charge density distributed on a triangular patch T by the same subtraction-and-addition of the singularity approach used in the two-dimensional case. The scalar potential is proportional to

$$\int_T \frac{e^{-jkR}}{R} \, dS' = \int_T \frac{e^{-jkR} - 1}{R} \, dS' + \int_T \frac{dS'}{R}.$$

The first integral on the right has a bounded integrand for every observation point and hence can be integrated numerically; the second integral is a special case of (5) and hence can be analytically evaluated.

An application of (6) is also found in [4]. There vector-valued basis functions are introduced which are proportional to the vector $\rho' - \rho_m$, where ρ_m is the projection onto the plane of triangle T of the position vector $\mathbf{r}$ to the mth vertex of T. Vector potentials due to these basis functions are then proportional to

$$\int_T (\rho' - \rho_m) \frac{e^{-jkR}}{R} \, dS'$$

$$= \int_T (\rho' - \rho) \frac{e^{-jkR}}{R} \, dS' + (\rho - \rho_m) \int_T \frac{e^{-jkR}}{R} \, dS'$$

$$= \int_T (\rho' - \rho) \frac{e^{-jkR} - 1}{R} \, dS' + (\rho - \rho_m) \int_T \frac{e^{-jkR} - 1}{R} \, dS'$$

$$+ \int_T \frac{\rho' - \rho}{R} \, dS' + (\rho - \rho_m) \int_T \frac{dS'}{R}.$$

The last two integrals are merely (5) and (6) with S specialized to T, and the two preceding ones always have bounded integrands which can be numerically integrated. The discussion following (4) with regard to synthesizing the potential of a source distribution with arbitrary linear variation in a polygon S also applies to (6).

It is also noted that the integrals (5) and (6) are used directly in time-domain formulations of electromagnetic scattering problems, as illustrated in [6].

Volume Sources Distributed on a Polyhedron

Consider next a distribution of sources within an arbitrary polyhedral region V (cf., Fig. 1(d)). The boundary of V is denoted by ∂V and has an outward unit normal $\mathbf{n}$. ∂V comprises a number of faces, the jth one of which is designated $\partial_j V$. The jth face, in turn, is a polygon having a boundary $\partial\partial_j V$ which comprises a number of edges, the ith one of which is designated $\partial_i\partial_j V$. The strategy employed to evaluate potentials of source distributions over V is merely an extension of that of the previous sections: Gauss integral theorems are first used to transform integrals over V into integrals over $\partial V = \Sigma_j \partial_j V$. Then integrals over $\partial_j V$ are transformed into integrals over the polyhedron edges, $\partial_i\partial_j V$. When the observation point $\mathbf{r}$ is in V or on ∂V, however, before the appropriate Gauss theorem may be applied it is necessary to exclude for separate treatment a region V_δ, the intersection of V and a sphere of radius δ centered at $\mathbf{r}$ (cf., Fig. 4). The boundary of this region is designated ∂V_δ, and the solid angle of the spherical sector of ∂V_δ contained in V is designated $\Omega(\mathbf{r})$.

With these considerations, the potential of a uniform source distributed in a polyhedron V is found to be proportional to

$$\int_V \frac{dV'}{R}$$

$$= \lim_{\delta \to 0} \frac{1}{2} \int_{V - V_\delta} \nabla' \cdot \hat{\mathbf{R}} \, dV' + \lim_{\delta \to 0} \int_{V_\delta} \frac{dV'}{R}$$

$$= \lim_{\delta \to 0} \frac{1}{2} \int_{\partial(V - V_\delta)} \hat{\mathbf{R}} \cdot \hat{\mathbf{n}} \, dS' + \lim_{\delta \to 0} \frac{\delta^2}{2} \Omega(\mathbf{r})$$

$$= \frac{1}{2} \sum_j \int_{\partial_j V} \hat{\mathbf{R}} \cdot \hat{\mathbf{n}}_j \, dS' = -\frac{1}{2} \sum_j d_j \int_{\partial_j V} \frac{dS'}{R}$$

$$= \frac{1}{2} \sum_j d_j \left\{ \sum_i \hat{\mathbf{P}}_{ij}^0 \cdot \hat{\mathbf{u}}_{ij} \right.$$

$$\times \left[|d_j| \left(\tan^{-1} \frac{P_{ij}^0 l_{ij}^+}{(R_{ij}^0)^2 + |d_j| R_{ij}^+} \right. \right.$$

$$\left. \left. \left. - \tan^{-1} \frac{P_{ij}^0 l_{ij}^-}{(R_{ij}^0)^2 + |d_j| R_{ij}^-} \right) - P_{ij}^0 \ln \frac{R_{ij}^+ + l_{ij}^+}{R_{ij}^- + l_{ij}^-} \right] \right\} \tag{7}$$

where $\hat{\mathbf{R}}$ is the unit vector from $\mathbf{r}$ to $\mathbf{r}'$. The double subscript ij denotes a quantity associated with the ith edge of the jth face of V, while a quantity associated with the face only carries the single subscript j. Note that the last equality in (7) follows from (5). The logarithmic terms associated with the two faces attached to a given edge can be combined in (7) to halve the number of logarithm evaluations.

The case in which V is a rectangular parallelepiped has also been considered by MacMillan [11] and Waldvogel [12]. The above result can easily be reduced to that of MacMillan when $\mathbf{r}$ is exterior to V, but is apparently in disagreement with his result for $\mathbf{r}$ interior to V. It appears that his formula is not intended to apply to the case in which the observation point falls in the source region. Waldvogel has also obtained an expression for the potential when V is an arbitrary polyhedron [13]. Equation (7) is equivalent to his expression, but is obtained in a much simpler fashion. Okon has derived formulas for the potentials of uniformly charged tetrahedra and parallelepipeds [14]. His formulas, however, are much too lengthy to compare with (7). In contrast to the formula given in (7), we point out that the derivations of the formulas in [11]-[14] consume an average of seven pages of text, while the final formulas for the potentials in [11] and [14] average more than five pages of text.

The corresponding integral for a linearly varying source distribution is evaluated in vector form as

$$\begin{aligned}
\int_V \frac{\mathbf{r}'-\mathbf{r}}{R}\, dV'
&= \lim_{\delta\to 0}\int_{V-V_\delta} \nabla' R\, dV' + \lim_{\delta\to 0}\int_{V_\delta} \hat{\mathbf{R}}\, dV' \\
&= \lim_{\delta\to 0}\int_{\partial(V-V_\delta)} R\hat{\mathbf{n}}\, dS' = \sum_j \hat{\mathbf{n}}_j \int_{\partial_j V} R\, dS' \\
&= \sum_j \hat{\mathbf{n}}_j \left[\lim_{\epsilon\to 0}\int_{\partial_j V - S_{\epsilon_j}} \nabla_s' \cdot \left(\frac{R^3}{3P_j}\hat{\mathbf{P}}_j\right) dS' \right.\\
&\qquad \left. + \lim_{\epsilon\to 0}\int_{S_{\epsilon j}} R\, dS'\right] \\
&= \frac{1}{3}\sum_j \hat{\mathbf{n}}_j \left[\lim_{\epsilon\to 0}\int_{\partial(\partial_j V - S_{\epsilon j})} \frac{R^3}{P_j}\hat{\mathbf{P}}_j \cdot \hat{\mathbf{u}}_j\, dl'\right] \\
&= \frac{1}{3}\sum_j \hat{\mathbf{n}}_j \left[-\alpha_j(\mathbf{r})\,|d_j|^3 + \sum_i \int_{\partial_i\partial_j V} \frac{R^3}{P_j}\hat{\mathbf{P}}_j \cdot \hat{\mathbf{u}}_{ij}\, dl'\right] \\
&= \frac{1}{3}\sum_j \hat{\mathbf{n}}_j \left[-\alpha_j(\mathbf{r})\,|d_j|^3 \right.\\
&\qquad \left. + \sum_i \mathbf{P}_{ij}^0\, \hat{\mathbf{u}}_{ij} \int_{\partial_i\partial_j V}\left(R + \frac{d_j^2}{R} + \frac{d_j^4}{P_j^2 R}\right) dl'\right] \\
&= \frac{1}{3}\sum_j \hat{\mathbf{n}}_j \left\{\sum_i \hat{\mathbf{P}}_{ij}^0 \cdot \hat{\mathbf{u}}_{ij}\left[\frac{P_{ij}^0[(R_{ij}^0)^2 + 2d_j^2]}{2}\right.\right.\\
&\qquad \times \ln\frac{R_{ij}^+ + l_{ij}^+}{R_{ij}^- + l_{ij}^-} + \frac{P_{ij}^0}{2}\left(l_{ij}^+R_{ij}^+ - l_{ij}^-R_{ij}^-\right) - |d_j|^3 \\
&\qquad \left.\left. \times\left(\tan^{-1}\frac{P_{ij}^0 l_{ij}^+}{(R_{ij}^0)^2 + |d_j|R_{ij}^+} - \tan^{-1}\frac{P_{ij}^0 l_{ij}^-}{(R_{ij}^0)^2 + |d_j|R_{ij}^-}\right)\right]\right\}.
\end{aligned} \tag{8}$$

Note that the last integral in (8) has terms of the same form as terms of (5) and (6).

As an application, integrals (7) and (8) are used in the evaluation of matrix elements in the moment matrix derived in [5]. There current and charge sources are distributed in tetrahedral volumes and the time-harmonic vector and scalar potentials they produce are required. The procedure for performing the numerical integrations is simply the generalization to a three-dimensional source region of that described following (6).

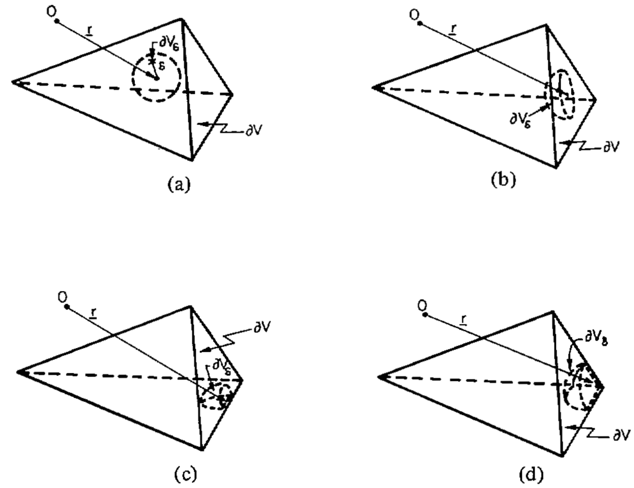

Fig. 4. Region V_δ corresponding to a point $\mathbf{r}$. (a) Point inside a tetrahedron V. (b) Point on a face of V. (c) Point on an edge of V. (d) Point at a vertex of V.

SUMMARY

Formulas for the potentials due to uniform and linearly varying source distributions defined on domains of simple shape are systematically derived and presented. Domains considered are infinite planar strips, infinite cylinders of polygonal cross sections, planar surfaces with polygonal boundaries, and volumetric regions bounded by polyhedrons. Of particular note is the compactness of the formulas obtained. Applications of the formulas to the numerical solution of electromagnetics problems by the method of moments are given.

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