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Electrodynamics of a Layered Electron Gas. I. Single Layer*

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The electrodynamics of a compressible charged layer placed in a uniform rigid neutralizing background is studied with a hydrodynamic description. The resulting plasma oscillations and screening differ considerably from those in bulk media. The induced charge density and scalar potential are evaluated both for a harmonic perturbation and for a uniformly moving charge, which also yields the corresponding power loss of the test charge. Retardation is shown to be significant only at long wavelengths, where other effects (for example, the presence of adjacent layers) are likely to predominate.

1. INTRODUCTION

An electron gas has long provided a simple model for the conduction electrons in metals. For example, the ground-state energy of the electron gas yields a first approximation to that of a metal, and the electrodynamic effects of plasma oscillations [1, 2] and screening [3, 4] illuminate similar phenomena found in the actual physical systems [5–7]. More recently, the suggested existence of effectively two-dimensional electron layers [8–10] has led to a corresponding interest in a two-dimensional electron gas, where the charged particles are confined to a plane and neutralized by an inert uniform rigid positive plane background. For a single layer, the plasma oscillations and screening differ qualitatively from those in a three-dimensional medium, owing to the presence of electromagnetic fields in the surrounding vacuum. A similar difference persists even for a periodic array of such layers. Previous studies of these layered configurations have relied on the powerful but abstract formalism of many-body theory [8, 9], which tends somewhat to obscure the rather direct physical basis for the resulting electrodynamic effects. For this reason we here analyze the same physical system with a hydrodynamic model, in which the charged fluid is characterized by a local density and velocity

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field. This treatment clarifies the role of the altered dimensionality in the physical phenomena. Section 2 reviews the hydrodynamic description of a three-dimensional charged fluid, including the response to a harmonic perturbation of frequency ω and to a uniformly moving test charge. The two-dimensional problem is formulated in Section 3, whose solution again yields the response to a harmonic perturbation (Section 4) and a moving test charge (Section 5). The role of retardation is considered in Section 6.

2. Three-Dimensional Medium

To emphasize the relation between a charged layer and a bulk continuum, we first review the hydrodynamic description of a classical three-dimensional electron fluid embedded in a rigid uniform positive background. Electrical neutrality requires that the equilibrium electron charge density $-eN_0$ precisely cancel that of the background eN_0, where N is a three-dimensional number density. In a dynamical situation, the electron density is altered to $N(\mathbf{R}, t)$, and the equation of continuity requires

$$\partial N/\partial t + \nabla \cdot (N\mathbf{V}) = 0, \tag{1}$$

where $\mathbf{V}(\mathbf{R}, t)$ is the electron velocity and $\mathbf{R}$ and $\mathbf{V}$ denote three-dimensional vectors. Newton's second law provides the equation of motion (Euler's equation) [11, 12]

$$\begin{aligned}\partial \mathbf{V}/\partial t + (\mathbf{V} \cdot \nabla)\mathbf{V} = &- (mN)^{-1} \nabla P \\ &- (e/m)(\mathbf{E} + c^{-1}\mathbf{V} \times \mathbf{B}),\end{aligned} \tag{2}$$

where P is the pressure, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, and m and $-e$ are the electronic mass and charge. These equations must be augmented by Maxwell's equations relating $\mathbf{E}$ and $\mathbf{B}$ to the electric charges and currents. For definiteness, consider an external charge density $\rho_{\text{ex}}(\mathbf{R}, t)$ and current $\mathbf{j}_{\text{ex}}(\mathbf{R}, t)$, which necessarily obey the continuity equation

$$\partial \rho_{\text{ex}}/\partial t + \nabla \cdot \mathbf{j}_{\text{ex}} = 0. \tag{3}$$

The total charge density ρ_{tot} and current $\mathbf{j}_{\text{tot}}$ then differ from ρ_{ex} and $\mathbf{j}_{\text{ex}}$ by the induced contributions

$$\rho_{\text{in}}(\mathbf{R}, t) = -eN(\mathbf{R}, t) + eN_0, \tag{4a}$$

$$\mathbf{j}_{\text{in}}(\mathbf{R}, t) = -eN(\mathbf{R}, t)\, \mathbf{V}(\mathbf{R}, t). \tag{4b}$$

In the present context, Maxwell's equations are most conveniently expressed through the scalar and vector potentials $\varphi(\mathbf{R}, t)$ and $\mathbf{A}(\mathbf{R}, t)$; in the Lorentz gauge [13],

$$\nabla \cdot \mathbf{A} + c^{-1}\, \partial\varphi/\partial t = 0, \tag{5}$$

they satisfy the usual wave equations

$$[\nabla^2 - c^{-2}(\partial/\partial t)^2]\, \varphi(\mathbf{R}, t) = -4\pi\rho_{\text{tot}}(\mathbf{R}, t), \tag{6a}$$

$$[\nabla^2 - c^{-2}(\partial/\partial t)^2]\, \mathbf{A}(\mathbf{R}, t) = -4\pi c^{-1}\mathbf{j}_{\text{tot}}(\mathbf{R}, t). \tag{6b}$$

These quantities yield the fields in (2) by the relations

$$\mathbf{E} = -\nabla\varphi - c^{-1}\, \partial\mathbf{A}/\partial t, \tag{7a}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{7b}$$

This general approach may be simplified by considering the *linear* response of an initially stationary system to an applied perturbation. In this case, the induced density

$$N'(\mathbf{R}, t) \equiv N(\mathbf{R}, t) - N_0\,, \tag{8}$$

the velocity $\mathbf{V}$, and the fields $\mathbf{E}$ and $\mathbf{B}$ are of first order, so that the magnetic terms in (2) become negligible. Consequently, (1) and (2) take the simpler linearized forms

$$\partial N'/\partial t + N_0 \nabla \cdot \mathbf{V} = 0, \tag{9}$$

$$\partial\mathbf{V}/\partial t = -(mN_0)^{-1}\, \nabla P - (e/m)\mathbf{E}. \tag{10}$$

Moreover, we assume an equation of state for the electron fluid

$$P = P(N), \tag{11}$$

so that ∇P may be rewritten as $(\partial P/\partial N)_0\, \nabla N'$. For convenience, this thermodynamic derivative will be abbreviated

$$(\partial P/\partial N)_0 = mS^2, \tag{12}$$

because S would be the thermodynamic speed of sound in a neutral system. Equations (10) and (12) together yield

$$\partial\mathbf{V}/\partial t = -(S^2/N_0)\, \nabla N' - (e/m)\mathbf{E}. \tag{13}$$

The electric field is now determined self-consistently from (6a), (6b), and (7a) with

$$\rho_{\text{tot}} = \rho_{\text{ex}} - eN' \tag{14a}$$

$$\mathbf{j}_{\text{tot}} = \mathbf{j}_{\text{ex}} - eN_0\mathbf{V}. \tag{14b}$$

These equations are easily solved with a Fourier transform in space and time. If ρ_{ex} has the integral representation

$$\rho_{ex}(\mathbf{R}, t) = (2\pi)^{-4} \int d^3K \int d\omega\, e^{i(\mathbf{K}\cdot\mathbf{R}-\omega t)} \rho_{ex}(\mathbf{K}, \omega), \tag{15}$$

with analogous forms for the other quantities, the Fourier transforms of (9), (13), and (6) give

$$\omega N'(\mathbf{K}, \omega) = N_0 \mathbf{K} \cdot \mathbf{V}(\mathbf{K}, \omega), \tag{16}$$

$$\omega \mathbf{V}(\mathbf{K}, \omega) = (S^2/N_0)\, \mathbf{K} N'(\mathbf{K}, \omega) - (e/m)\, \mathbf{K}\varphi(\mathbf{K}, \omega) + (e/mc)\, \omega \mathbf{A}(\mathbf{K}, \omega), \tag{17}$$

$$(K^2 - \omega^2/c^2)\, \varphi(\mathbf{K}, \omega) = 4\pi[\rho_{ex}(\mathbf{K}, \omega) - eN'(\mathbf{K}, \omega)], \tag{18}$$

$$(K^2 - \omega^2/c^2)\, \mathbf{A}(\mathbf{K}, \omega) = 4\pi c^{-1}[\mathbf{j}_{ex}(\mathbf{K}, \omega) - eN_0\mathbf{V}(\mathbf{K}, \omega)]. \tag{19}$$

It is helpful to separate the vector quantities into transverse and longitudinal parts [13], for example

$$\mathbf{A}(\mathbf{K}, \omega) = \mathbf{A}^t(\mathbf{K}, \omega) + \mathbf{A}^l(\mathbf{K}, \omega), \tag{20}$$

where

$$\mathbf{K} \cdot \mathbf{A}^t(\mathbf{K}, \omega) = 0; \qquad \mathbf{K} \times \mathbf{A}^l(\mathbf{K}, \omega) = 0. \tag{21}$$

The transverse components may be solved directly to give

$$\mathbf{V}^t(\mathbf{K}, \omega) = (e/mc)\, \mathbf{A}^t(\mathbf{K}, \omega), \tag{22a}$$

$$(K^2 + \Omega_p{}^2/c^2 - \omega^2/c^2)\, \mathbf{A}^t(\mathbf{K}, \omega) = 4\pi c^{-1} \mathbf{j}^t_{ex}(\mathbf{K}, \omega), \tag{22b}$$

where

$$\Omega_p{}^2 = 4\pi N_0 e^2/m \tag{23}$$

defines the usual plasma frequency. Equation (22b) shows that transverse plane waves necessarily have the dispersion relation

$$\omega_K{}^t = (\Omega_p{}^2 + c^2K^2)^{1/2}. \tag{24}$$

It is not difficult to evaluate $\mathbf{A}^t(\mathbf{R}, t)$ arising from a transverse source $\mathbf{j}^t_{ex}$, and the resulting fields are similar to those for a localized current distribution in vacuum [13, Chapter 9]. The major difference is the absence of radiation for $\omega < \Omega_p$, since the wave number $K = c^{-1}(\omega^2 - \Omega_p{}^2)^{1/2}$ would then be imaginary.

We now turn to the longitudinal components, where Eqs. (5) and (16) give

$$\mathbf{V}^l(\mathbf{K}, \omega) = (\mathbf{K}\omega/K^2N_0)\, N'(\mathbf{K}, \omega), \tag{25a}$$

$$\mathbf{A}^l(\mathbf{K}, \omega) = (\mathbf{K}\omega/K^2c)\, \varphi(\mathbf{K}, \omega). \tag{25b}$$

Direct substitution into the remaining equations yields the solutions

$$N'(\mathbf{K}, \omega) = -\frac{e^{-1}\Omega_p{}^2 \rho_{\text{ex}}(\mathbf{K}, \omega)}{(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K^2}, \tag{26}$$

$$\varphi_{\text{ex}}(\mathbf{K}, \omega) = \frac{4\pi\rho_{\text{ex}}(\mathbf{K}, \omega)}{K^2 - (\omega + i\eta)^2/c^2}, \tag{27a}$$

$$\varphi_{\text{in}}(\mathbf{K}, \omega) = \frac{4\pi\Omega_p{}^2\rho_{\text{ex}}(\mathbf{K}, \omega)}{[(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K^2][K^2 - (\omega + i\eta)^2/c^2]}, \tag{27b}$$

where the total potential φ_{tot} has been separated into the contributions φ_{ex} and φ_{in} arising from the external charge density ρ_{ex} and the induced charge density $-eN'$, respectively. Moreover, the frequency has been given an infinitesimal positive imaginary part $\omega + i\eta$, which ensures analyticity in the upper half ω plane and hence the proper causal behavior [12, Section 62; 14]. Equation (26) evidently allows a homogeneous solution (plasma oscillations) with a dispersion relation

$$\omega_K{}^l = (\Omega_p{}^2 + S^2K^2)^{1/2}. \tag{28}$$

Note furthermore, that (26) is independent of the speed of light c; it is, therefore, unaffected by the inclusion of retardation, in contrast to the induced potential φ_{in}, where the last factor in the denominator would be replaced by K^2 for strictly instantaneous interactions.

To allow a unified description, we derive S^2 from the equation of state for a degenerate spin-1/2 ideal Fermi gas. The usual relations for the Fermi wavenumber

$$K_F{}^3 = 3\pi^2 N_0 \tag{29}$$

and the ground-state energy per particle

$$(3/5)\, E_F = (3/10)\, \hbar^2 K_F{}^2/m \tag{30}$$

immediately give the result

$$m^{-1}\, \partial P/\partial N = S^2 = (1/3)\, V_F{}^2, \tag{31}$$

where $V_F = \hbar K_F/m$ is the Fermi velocity [6, Section 1.3; 7, Section 16]. In fact (31) holds only for low frequencies, where ω is much less than the inverse collision time; in the opposite limit, a compressional wave becomes essentially one-dimensional, and the right side of (31) must be multiplied by a factor 9/5 [13, Section 10.9]. For simplicity, this last correction will be largely ignored in the subsequent work. In the more general case of an ideal Fermi gas in D dimensions, the factor 1/3 in (31) must be replaced by D^{-1}, with an additional high-frequency factor $3D(D + 2)^{-1}$.

The three-dimensional dispersion relation (28) has several interesting features. Since $\omega \geqslant \Omega_p$ for all K, a harmonic perturbation cannot excite such modes unless its frequency exceeds the bulk plasma frequency. Indeed, as shown below, the induced charge density and potential dramatically change character at $\omega = \Omega_p$. This point is particularly evident in the density of states for collective modes

$$\mathscr{N}(\omega) = \sum_{\mathbf{K}} \delta(\omega - \omega_{\mathbf{K}}^{l}). \tag{32}$$

Transforming the sum to an integral over wave vectors, we obtain the expression

$$\mathscr{N}(\omega) = \begin{cases} 0 & \omega < \Omega_p , \\ (\mathscr{V}\omega/2\pi^2 S^3)(\omega^2 - \Omega_p^2)^{1/2} & \omega > \Omega_p , \end{cases} \tag{33}$$

where $\mathscr{V}$ is the volume of the sample. Another notable aspect of three-dimensional plasma oscillations is the existence of a critical velocity V_c, defined as the minimum value of the phase velocity $\omega_K{}^l/K$. Equation (28) shows that this quantity decreases monotonically for increasing K and ultimately approaches the limiting value S, implying

$$V_c = S. \tag{34}$$

If a particle moves slower than S, it cannot excite plasma oscillations, and the medium correspondingly becomes effectively "transparent." A final observation on (28) concerns the group velocity $d\omega_K{}^l/dK$, which is always less than S and approaches zero as $K \to 0$.

It is interesting to consider the induced charge density $\rho_{\text{in}}(\mathbf{R}, \omega) = -eN'(\mathbf{R}, \omega)$ arising from a harmonic perturbation $\rho_{\text{ex}}(\mathbf{R}, \omega)$. The spatial Fourier transform of (26) may be manipulated to give

$$\rho_{\text{in}}(\mathbf{R}, \omega) = \int d^3R'\, Q(\mathbf{R} - \mathbf{R}', \omega)\, \rho_{\text{ex}}(\mathbf{R}', \omega), \tag{35a}$$

where

$$Q(\mathbf{R}, \omega) = -\Omega_p^2 \int \frac{d^3K}{(2\pi)^3} \frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{K^2S^2 + \Omega_p^2 - (\omega + i\eta)^2}. \tag{35b}$$

If $|\omega| < \Omega_p$, the integrand has purely imaginary poles in the complex K plane, but if $|\omega| > \Omega_p$, the infinitesimal sign factor $i\eta$ becomes significant, for the poles now lie just off the real K axis. A straightforward calculation gives the general result

$$Q(\mathbf{R}, \omega) = \begin{cases} -(\Omega_p^2/4\pi RS^2) \exp[-RS^{-1}(\Omega_p^2 - \omega^2)^{1/2}] & |\omega| < \Omega_p , \\ -(\Omega_p^2/4\pi RS^2) \exp[iRS^{-1} \operatorname{sgn} \omega(\omega^2 - \Omega_p^2)^{1/2}] & |\omega| > \Omega_p , \end{cases} \tag{36}$$

where $\operatorname{sgn}\omega = \omega/|\omega|$. In the static limit, this expression reduces to the usual Thomas–Fermi one [6, Section 3.1; 7, Section 14], with the corresponding screening constant

$$K_{TF} = \Omega_p/S = (6\pi N_0 e^2/E_F)^{1/2}, \tag{37}$$

proportional to $N_0^{1/6}$ [see Eqs. (29) and (30)]. Although the dynamic screening becomes less effective as ω increases, the induced charge density remains localized for all $|\omega| < \Omega_p$. For still higher frequencies, however, ρ_{in} becomes an outgoing spherical wave with wave number determined by (28); it represents energy radiated in the form of plasma oscillations. This behavior confirms the qualitative inferences drawn from (33). In an analogous way, Eq. (27) may be used to evaluate the potential produced by a harmonic perturbation. As the results are quite similar to those obtained above for the induced charge density, they will not be given explicitly.

For a final application, consider the response to a test charge Ze moving uniformly with a velocity $\mathbf{V}$, where

$$\begin{aligned} \rho_{\text{ex}}(\mathbf{R}, t) &= Ze\,\delta(\mathbf{R} - \mathbf{V}t), \\ \mathbf{j}_{\text{ex}}(\mathbf{R}, t) &= Ze\mathbf{V}\,\delta(\mathbf{R} - \mathbf{V}t). \end{aligned} \tag{38}$$

The corresponding Fourier transforms

$$\begin{aligned} \rho_{\text{ex}}(\mathbf{K}, \omega) &= 2\pi Ze\,\delta(\omega - \mathbf{K}\cdot\mathbf{V}), \\ \mathbf{j}_{\text{ex}}(\mathbf{K}, \omega) &= 2\pi Ze\mathbf{V}\,\delta(\omega - \mathbf{K}\cdot\mathbf{V}) \end{aligned} \tag{39}$$

may be combined with (26) to give the induced charge density

$$\rho_{\text{in}}(\mathbf{R}, t) = -Ze\Omega_p^2 \int \frac{d^3K}{(2\pi)^3} \frac{e^{i\mathbf{K}\cdot(\mathbf{R}-\mathbf{V}t)}}{K^2S^2 + \Omega_p^2 - (\mathbf{K}\cdot\mathbf{V} + i\eta)^2}\,. \tag{40}$$

The somewhat lengthy integration is treated in Appendix A, where we obtain

$$\rho_{\text{in}}(\mathbf{R}, t) = -\frac{Ze\Omega_p^2}{S(S^2 - V^2)^{1/2}} \frac{\exp(-\Omega_p\bar{R}/S)}{4\pi\bar{R}} \tag{41a}$$

for $V < S$ and

$$\rho_{\text{in}}(\mathbf{R}, t) = \frac{-Ze\Omega_p^2}{S(V^2 - S^2)^{1/2}} \frac{\theta(Vt - R_\parallel)\,\theta(-\bar{R}^2)}{2\pi|\bar{R}|} \cos(\Omega_p|\bar{R}|/S) \tag{41b}$$

for $V > S$. Here $\theta(x)$ denotes the unit step function $\frac{1}{2}[1 + \operatorname{sgn} x]$, and

$$\bar{R}^2 = R_\perp^2 + (R_\parallel - Vt)^2(1 - V^2/S^2)^{-1} \tag{42}$$

with $\mathbf{R}_\perp$ and $R_\parallel$ the projections of $\mathbf{R}$ perpendicular and parallel to $\mathbf{V}$. Note that the induced charge is anisotropic but localized for $V < S$, because $\bar{R}$ is then real. As $V \to S$ from below, ρ_{in} becomes singular, acquiring the characteristics of a shock wave. For $V > S$, Eq. (42) can change sign, and the signal is confined to a region of imaginary $\bar{R}$

$$R_\parallel + (V^2/S^2 - 1)^{1/2} R_\perp < Vt, \tag{43}$$

representing the interior of a conical wave front with a half angle $\arcsin(S/V)$ at its apex. The existence of a transition at $V = S$ illustrates the previous qualitative remarks concerning the critical velocity (34). Although the dispersion in (28) means that the detailed structure of the wave front (41b) differs from that of a sonic shock wave in a gas, the existence of a maximum group velocity S leads to the same geometric conditions as in the more familiar Mach cone [11, Section 79].

It is interesting to evaluate the energy lost by the moving particle. For definiteness, consider first the transverse vector potential, which follows directly from Eqs. (22b) and (38)

$$\mathbf{A}^t(\mathbf{R}, t) = 4\pi Zec \int \frac{d^3K\, d\omega}{(2\pi)^3} \frac{e^{i(\mathbf{K}\cdot\mathbf{R}-\omega t)}\, \delta(\omega - \mathbf{K}\cdot\mathbf{V})[K^2\mathbf{V} - \mathbf{K}(\mathbf{K}\cdot\mathbf{V})]}{[c^2K^2 + \Omega_p{}^2 - (\omega + i\eta)^2]\, K^2}. \tag{44}$$

The resulting electric field $\mathbf{E}^t(\mathbf{R}, t) = -c^{-1}\, \partial\mathbf{A}^t(\mathbf{R}, t)/\partial t$ gives the power lost through excitation of transverse modes

$$\begin{aligned} \mathscr{P}^t &= Ze\mathbf{V}\cdot\mathbf{E}^t(\mathbf{V}t, t) \\ &= 4\pi Z^2e^2 i \int \frac{d^3K\, d\omega}{(2\pi)^3} \frac{\omega\, \delta(\omega - \mathbf{K}\cdot\mathbf{V})[K^2V^2 - (\mathbf{K}\cdot\mathbf{V})^2]}{[c^2K^2 + \Omega_p{}^2 - (\omega + i\eta)^2]\, K^2}. \end{aligned} \tag{45}$$

In cylindrical polar coordinates $\mathbf{K} = (\mathbf{K}_\perp, K_\parallel)$ with $\mathbf{V}$ as polar axis, we use the delta function to write

$$\begin{aligned} \mathscr{P}^t &= \frac{Z^2e^2 i}{2\pi^2\, V} \int d^2K_\perp \int_{-\infty}^{\infty} d\omega \\ &\quad\times \frac{\omega K_\perp{}^2 V^2}{[(cK_\perp)^2 + \Omega_p{}^2 + (c\omega/V)^2 - (\omega + i\eta)^2][K_\perp{}^2 + (\omega/V)^2]}. \end{aligned} \tag{46}$$

Since the denominator can never vanish for any $V < c$, the limit $\eta \to 0$ can now be taken; the resulting frequency integral then vanishes by symmetry. Hence, transverse fields do no work on the test particle, which, conversely, radiates no transverse electromagnetic waves.

The situation is quite different for the longitudinal modes, where the induced charges $-eN'$ and currents $-eN_0\mathbf{V}$ produce a scalar potential (27) and longitudinal vector potential (25b)

$$\varphi_{\text{in}}(\mathbf{R}, t) = \frac{Ze\Omega_p{}^2}{2\pi^2} \int \frac{d^3K\, d\omega\, \delta(\omega - \mathbf{K}\cdot\mathbf{V})\, e^{i\mathbf{K}\cdot(\mathbf{R}-\mathbf{V}t)}}{[(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K^2][K^2 - c^{-2}(\omega + i\eta)^2]}, \tag{47a}$$

$$\mathbf{A}^l_{\text{in}}(\mathbf{R}, t) = \frac{Ze\Omega_p{}^2}{2\pi^2 c} \int \frac{d^3K\, d\omega\, \delta(\omega - \mathbf{K}\cdot\mathbf{V})\, e^{i\mathbf{K}\cdot(\mathbf{R}-\mathbf{V}t)}\mathbf{K}\omega}{[(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K^2][K^2 - c^{-2}(\omega + i\eta)^2]\, K^2}. \tag{47b}$$

The associated electric field follows from (7a), and the resulting force on the particle $Ze\mathbf{E}^l_{\text{in}}(\mathbf{V}t, t)$ again provides the power

$$\begin{aligned}\mathscr{P}^l &= Ze\mathbf{V}\cdot\mathbf{E}^l_{\text{in}}(\mathbf{V}t, t)\\ &= -\frac{i(Ze\Omega_p)^2}{2\pi^2} \int \frac{d^3K\, d\omega\, \delta(\omega - \mathbf{K}\cdot\mathbf{V})\, \omega}{[(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K^2]\, K^2}.\end{aligned} \tag{48}$$

Note that (48) is independent of c, indicating that retardation and similar relativistic effects are irrelevant. Proceeding as in (46), we find

$$\begin{aligned}\mathscr{P}^l &= -i\,\frac{(Ze\Omega_p)^2}{2\pi^2 V} \int d^2K_\perp\\ &\quad\times \int_{-\infty}^{\infty} d\omega\, \frac{\omega}{[(\omega + i\eta)^2 - \Omega_p{}^2 - S^2K_\perp{}^2 - (S\omega/V)^2][K_\perp{}^2 + (\omega/V)^2]}.\end{aligned} \tag{49}$$

If $V < S$, the integrand has no singularities on the real ω axis, and the integral vanishes by symmetry, just as did Eq. (46). On the other hand, if $V > S$, the first factor in the denominator introduces two simple poles just below the real ω axis. It is then convenient to evaluate the frequency integral with a contour closed in the upper-half plane, which gives the usual expression for the energy loss of a fast particle in a degenerate plasma [6, Section 4.4]

$$\begin{aligned}\mathscr{P}^l &= -\frac{(Ze\Omega_p)^2}{2\pi V} \int d^2K_\perp\, \frac{\theta(V^2 - S^2)}{K_\perp{}^2 + (\Omega_p/V)^2}\\ &\approx -\frac{(Ze\Omega_p)^2}{V}\, \theta(V^2 - S^2) \ln\left(\frac{VK_{\max}}{\Omega_p}\right),\end{aligned} \tag{50}$$

where $K_{\max}$ is an upper cutoff.

To account for this energy loss, we first observe that the longitudinal potentials produce no radiation, because $\mathbf{B}$ and the Poynting vector both vanish. As a result, Eq. (50) must arise from mechanical energy associated with the excitation of plasma oscillations. The corresponding instantaneous energy flux across a unit area $d\mathbf{A}$ is given by $P(\mathbf{R}, t)\,\mathbf{V}(\mathbf{R}, t)\cdot d\mathbf{A}$, which may be rewritten to leading order with (12) as

$mS^2N'(\mathbf{R}, t)\,\mathbf{V}(\mathbf{R}, t) \cdot d\mathbf{A}$. This form shows the importance of the longitudinal character of the oscillation, because $N' = 0$ for a transverse mode. An inverse Fourier transform of (25a) and (26) leads to the desired quantities, and the remaining integration over a cylindrical surface coaxial with the trajectory of the moving charge may be performed as in the theory of Cherenkov radiation [15]. A rather lengthy calculation yields the expected result that all the power lost by the moving particle (50) appears as mechanical energy radiated to infinity in the form of plasma oscillations.

3. Plasma Oscillations in a Single Layer

The preceding analysis requires only slight modifications for a single layer. Since the charges are confined to the plane $z = 0$, the three-dimensional number density becomes singular and must be written

$$N(\mathbf{R}, t) = n(\mathbf{r}, t)\,\delta(z), \tag{51}$$

where n denotes the area number density and $\mathbf{r}$ the coordinate in the xy plane. If $\mathbf{v}(\mathbf{r}, t)$ is the corresponding two-dimensional velocity, the two-dimensional analogs of (9) and (13) become

$$\partial n'/\partial t + n_0 \nabla \cdot \mathbf{v} = 0, \tag{52}$$

$$\partial \mathbf{v}/\partial t = -n_0^{-1} s^2\, \nabla n' - (e/m)\, \hat{z} \times (\mathbf{E} \times \hat{z})|_{z=0}\,, \tag{53}$$

where $n'(\mathbf{r}, t) = n(\mathbf{r}, t) - n_0$ is the small perturbation. Here s is a velocity defined by the relation

$$ms^2 = (\partial p/\partial n)_0 \tag{54}$$

and p is a "two-dimensional" pressure, with the dimensions of a force per unit length. For definiteness, we treat the electron fluid as a two-dimensional ideal spin-1/2 Fermi gas, with Fermi wave number

$$k_F = (2\pi n_0)^{1/2}. \tag{55}$$

The energy per particle becomes

$$\epsilon_F/2 = \hbar^2 k_F{}^2/4m, \tag{56}$$

and the pressure p is (minus) the derivative of the total energy with respect to area:

$$p = \hbar^2 \pi n_0{}^2/2m. \tag{57a}$$

Finally, Eqs. (54) and (57a) yield

$$s^2 = v_F{}^2/2, \tag{57b}$$

where $v_F = \hbar k_F/m$. As in the three-dimensional case, (57b) should be multiplied at high frequencies by 3/2, but this correction will be ignored.

Equations (52) and (53) must be combined with Maxwell's equations; for simplicitly, we temporarily neglect the effect of retardation (see, however, Section 6), thus retaining only those terms that persist in the nonrelativistic limit. In particular, the scalar potential now determines the *three-dimensional* electric field by

$$\mathbf{E}(\mathbf{r}, z, t) = -\nabla\varphi(\mathbf{r}, z, t), \tag{58}$$

where

$$\nabla^2\varphi(\mathbf{r}, z, t) = -4\pi\rho_{\text{ex}}(\mathbf{r}, z, t) + 4\pi en'(\mathbf{r}, t)\,\delta(z). \tag{59}$$

To solve (52), (53), and (59), introduce a Fourier transform in time and in the xy plane

$$\rho_{\text{ex}}(\mathbf{r}, z, t) = \int \frac{d^2k\, d\omega}{(2\pi)^3}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\rho_{\text{ex}}(\mathbf{k}, z, \omega), \tag{60}$$

with similar definitions for the other quantities. A simple calculation leads to the equations

$$\omega n'(\mathbf{k}, \omega) = n_0\mathbf{k}\cdot\mathbf{v}(\mathbf{k}, \omega), \tag{61a}$$

$$\omega\mathbf{v}(\mathbf{k}, \omega) = n_0^{-1}s^2\mathbf{k}n'(\mathbf{k}, \omega) - (e/m)\,\mathbf{k}\varphi(\mathbf{k}, z=0, \omega), \tag{61b}$$

$$(\partial^2/\partial z^2 - k^2)\,\varphi(\mathbf{k}, z, \omega) = -4\pi\rho_{\text{ex}}(\mathbf{k}, z, \omega) + 4\pi en'(\mathbf{k}, \omega)\,\delta(z). \tag{62}$$

The first two (algebraic) equations involve $\varphi(\mathbf{k}, z, \omega)$ evaluated at $z = 0$. To derive this last quantity, define a Green's function $g(z, z')$ that obeys the differential equation

$$(\partial^2/\partial z^2 - k^2)\,g(z, z') = -\delta(z - z'). \tag{63}$$

A standard analysis gives

$$g(z, z') = (2k)^{-1}\exp(-k\,|\,z - z'\,|), \tag{64}$$

where $k = |\,k^2\,|^{1/2}$ is positive. The solution of (62) now becomes

$$\varphi(\mathbf{k}, z, \omega) = 2\pi k^{-1}\int_{-\infty}^{\infty} dz'\exp(-k\,|\,z - z'\,|)\,\rho_{\text{ex}}(\mathbf{k}, z', \omega) - 2\pi ek^{-1}n'(\mathbf{k}, \omega)\,e^{-k|z|}. \tag{65}$$

The remaining steps in solving (61) are straightforward and yield

$$n'(\mathbf{k}, \omega) = \frac{-2\pi n_0 ekm^{-1} \int_{-\infty}^{\infty} dz' \, e^{-k|z'|} \rho_{\text{ex}}(\mathbf{k}, z'\omega)}{(\omega + i\eta)^2 - s^2k^2 - 2\pi n_0 e^2 km^{-1}}, \tag{66}$$

$$\varphi_{\text{ex}}(\mathbf{k}, z, \omega) = 2\pi k^{-1} \int_{-\infty}^{\infty} dz' \exp(-k \mid z - z' \mid) \, \rho_{\text{ex}}(\mathbf{k}, z', \omega), \tag{67a}$$

$$\varphi_{\text{in}}(\mathbf{k}, z, \omega) = \frac{2\pi e^{-k|z|}(2\pi n_0 e^2/m) \int_{-\infty}^{\infty} dz' \, e^{-k|z'|} \rho_{\text{ex}}(\mathbf{k}, z', \omega)}{(\omega + i\eta)^2 - s^2k^2 - 2\pi n_0 e^2 km^{-1}}, \tag{67b}$$

where the potential has been separated into contributions of the test charges and the induced charges, respectively.

Our first example will be the free plasma oscillations in the plane, which have the dispersion relation

$$\omega_k = (2\pi n_0 e^2 km^{-1} + s^2k^2)^{1/2}. \tag{68}$$

At short wave lengths, ω_k approaches a nondispersive wave with propagation speed s; at long wave lengths, however, ω_k varies like $k^{1/2}$, in marked contrast to the three-dimensional case (28). This latter behavior seems first to have been suggested by Ferrell [16]; it arises from the electromagnetic fields in the vacuum surrounding the plane, with an associated reduction in the screening. Since ω_k increases monotonically from zero, an external perturbation of arbitrarily low frequency can always excite collective modes; hence, the characteristic three-dimensional absorption edge at Ω_p is here entirely absent. Moreover, the group velocity and the phase velocity both diverge like $k^{-1/2}$ as $k \to 0$ and approach s from above as $k \to \infty$.

The dispersion relation (68) changes character at the wave number

$$k_{TF} = 2\pi n_0 e^2/ms^2, \tag{69a}$$

which turns out to be equivalent to the usual definition of the Thomas–Fermi screening constant [8, 9]. Using (55) and (57b), we see that

$$k_{TF} = 2me^2/\hbar^2 = 2a_0^{-1}, \tag{69b}$$

where a_0 is the Bohr radius (perhaps modified by a factor m^*/m if the effective mass differs from m). Note that k_{TF}^{-1} in two dimensions is comparable with an atomic size and is strictly independent of the equilibrium number density. Although this situation is usually contrasted with that in three-dimensions, it is helpful to recall that K_{TF} there varied only weakly with N_0 [see (37) and the following discussion]. Furthermore, K_{TF}^{-1} had a similar typical numerical value because $N_0^{-1/3}$

is generally of order a_0. The density of states associated with two-dimensional plasmons (68) is

$$\mathcal{N}(\omega) = \sum_{\mathbf{k}} \delta(\omega - \omega_k)$$

$$= \mathcal{O}\!\ell(2\pi)^{-2} \int d^2k\, \delta(\omega - \omega_k)$$

$$= (\mathcal{O}\!\ell\omega/2\pi s^2)\{1 - [1 + (2\omega/k_{TF}s)^2]^{-1/2}\}, \qquad (70)$$

where $\mathcal{O}\!\ell$ is the area of the layer. As mentioned above, $\mathcal{N}(\omega)$ has no sharp discontinuity but instead varies smoothly down to $\omega = 0$.

4. Response to a Harmonic Perturbation

As in the three-dimensional system (Section 2), it is valuable to study the linear response of the charged layer to a harmonically oscillating perturbation. The inverse spatial Fourier transform of (66) gives the induced charge density at the coordinate **r** in the xy plane. With some rearrangement, it may be written in a form analogous to (35a)

$$\rho_{\text{in}}(\mathbf{r}, \omega) = -en'(\mathbf{r}, \omega) = \int d^2r' \int dz'\, Q(\mathbf{r} - \mathbf{r}', z', \omega)\, \rho_{\text{ex}}(\mathbf{r}', z', \omega), \qquad (71\text{a})$$

where

$$Q(\mathbf{r} - \mathbf{r}', z', \omega) = -\int \frac{d^2k}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-k|z'|} k k_{TF}}{k^2 + k k_{TF} - (\omega + i\eta)^2/s^2} \qquad (71\text{b})$$

characterizes the response at the point **r** to a unit perturbation with frequency ω located off the plane at the point $(\mathbf{r}', z')$. The angular integral yields a Bessel function [17]

$$Q(r, z', \omega) = -\frac{k_{TF}}{2\pi} \int_0^\infty \frac{k^2\, dk\, J_0(kr)\, e^{-k|z'|}}{k^2 + k k_{TF} - (\omega + i\eta)^2/s^2}, \qquad (72)$$

but the remaining integral cannot be evaluated in terms of tabulated functions. If we define the integral (see Appendix B)

$$I(a, b, c) \equiv \int_0^\infty \frac{dx\, J_0(ax)\, e^{-bx}}{x + c} \qquad (73)$$

for $b > 0$, then a simple decomposition with partial fractions leads to the form [18]

$$Q(r, z', \omega) = -\frac{k_{TF}}{2\pi(r^2 + z'^2)^{1/2}}$$
$$-\frac{k_{TF}^2}{2\pi(\lambda_1 + \lambda_2)}\left\{\left[\left(\frac{\omega}{k_{TF}s}\right)^2 - \lambda_2\right] I(k_{TF}r, k_{TF} \mid z' \mid, -\lambda_2 - i\eta \operatorname{sgn} \omega)\right.$$
$$\left. - \left[\left(\frac{\omega}{k_{TF}s}\right)^2 + \lambda_1\right] I(k_{TF}r, k_{TF} \mid z' \mid, \lambda_1)\right\}, \tag{74}$$

where λ_1 and λ_2 are positive functions of $\omega/k_{TF}s$, defined by

$$\begin{aligned} \lambda_1 &\equiv [\tfrac{1}{4} + (\omega/k_{TF}s)^2]^{1/2} + \tfrac{1}{2} \\ \lambda_2 &\equiv [\tfrac{1}{4} + (\omega/k_{TF}s)^2]^{1/2} - \tfrac{1}{2}. \end{aligned} \tag{75}$$

Another quantity of interest is the potential $\varphi(\mathbf{r}, z, \omega)$ produced by a harmonic perturbation. With a spatial Fourier transform, (67b) may be reexpressed as

$$\varphi_{\text{in}}(\mathbf{r}, z, \omega) = \int d^2r' \int dz'\, G_{\text{in}}(\mathbf{r} - \mathbf{r}', z, z', \omega)\, \rho_{\text{ex}}(\mathbf{r}', z', \omega), \tag{76}$$

where

$$\begin{aligned} G_{\text{in}}(\mathbf{r} - \mathbf{r}', z, z', \omega) &= -\frac{k_{TF}}{2\pi} \int d^2k\, \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-k|z|} e^{-k|z'|}}{k^2 + kk_{TF} - (\omega + i\eta)^2/s^2} \\ &= -k_{TF} \int_0^\infty \frac{k\, dk\, J_0(k \mid \mathbf{r} - \mathbf{r}' \mid)\, e^{-k|z|} e^{-k|z'|}}{k^2 + kk_{TF} - (\omega + i\eta)^2/s^2} \end{aligned} \tag{77}$$

represents the potential at $(\mathbf{r}, z)$ of the charge induced in the layer by a unit harmonic test change at $(\mathbf{r}', z')$. As expected from the lack of translational symmetry normal to the xy plane, G_{in} depends separately on z and z'. This integral may be expressed in terms of (73)

$$\begin{aligned} G_{\text{in}}(r, z, z', \omega) = {} & -k_{TF}(\lambda_1 + \lambda_2)^{-1}[\lambda_1 I(k_{TF}r, k_{TF} \mid z \mid + k_{TF} \mid z' \mid, \lambda_1) \\ & + \lambda_2 I(k_{TF}r, k_{TF} \mid z \mid + k_{TF} \mid z' \mid, -\lambda_2 - i\eta \operatorname{sgn} \omega)], \end{aligned} \tag{78}$$

where λ_1 and λ_2 are defined in (75). Since the induced charge density in the xy plane (74) and the resulting potential (78) have a very similar structure, we shall concentrate on the more general quantity G_{in}.

The simplest case is the static limit, when $\lambda_1 = 1$ and $\lambda_2 = 0$. Equation (78) then reduces to

$$G_{\text{in}}(r, z, z', \omega = 0) = -k_{TF} I(k_{TF}r, k_{TF} \mid z \mid + k_{TF} \mid z' \mid, 1). \tag{79}$$

Although the integral is still rather complicated, it becomes tractable for $\mathbf{r} = 0$ (source point and observation point on the same line perpendicular to the xy plane) and for $z = z' = 0$ (source point and observation point both in the xy plane). If $\mathbf{r} = 0$, use of (B.4) gives the *total* potential induced at $(0, z)$ by a static charge Ze at $(0, z')$

$$\varphi_{\text{tot}}(0, z) = Ze \mid z - z' \mid^{-1} \\ - Zek_{TF} \exp[k_{TF}(\mid z \mid + \mid z' \mid)] \, E_1[k_{TF}(\mid z \mid + \mid z' \mid)], \qquad (80)$$

where we have added the direct potential of the test charge in vacuum. Here E_1 denotes the exponential integral [17, Section 5.1]. If both the observation point and the source point lie near the plane $(\mid z \mid + \mid z' \mid \ll k_{TF}^{-1})$, (80) reduces to

$$\varphi_{\text{tot}}(0, z) \approx Ze \mid z - z' \mid^{-1} + Zek_{TF} \ln[e^{\gamma} k_{TF}(\mid z \mid + \mid z' \mid)] \qquad (81)$$

with $\gamma \approx 0.577$; hence, the induced charges provide only a small correction to the direct Coulomb potential of the test charge. In the asymptotic region $(\mid z \mid + \mid z' \mid \gg k_{TF}^{-1})$, however, (80) becomes

$$\varphi_{\text{tot}}(0, z) \sim Ze \left[\frac{1}{\mid z - z' \mid} - \frac{1}{\mid z \mid + \mid z' \mid} + \frac{1}{k_{TF}(\mid z \mid + \mid z' \mid)^2} + \cdots \right], \qquad (82)$$

which exhibits a qualitatively different form according as the source and observation points lie on the same or opposite sides of the xy plane. If z and z' have opposite signs, the leading terms of (82) cancel to produce a potential that falls off like $(\mid z \mid + \mid z' \mid)^{-2}$. If z and z' have the same sign, the resulting potential instead has a longer range and varies inversely with the distances. Such behavior arises from the screening of the source by the charged layer. Finally, Eq. (82) again varies as an inverse square if either the source or the observation point is in the xy plane.

The other simple static case occurs when the test charge is located in the xy plane, and the potential is measured in the same plane. Equation (B.2) then gives [19]

$$\varphi_{\text{tot}}(r, 0) = Ze\{r^{-1} - \tfrac{1}{2}\pi k_{TF}[\mathbf{H}_0(k_{TF} r) - Y_0(k_{TF} r)]\}, \qquad (83)$$

where $\mathbf{H}_0$ and Y_0 denote a Struve function and a Neumann function [17, Sections 9.1 and 12.1]. For $k_{TF} r \ll 1$, Eq. (83) becomes

$$\varphi_{\text{tot}}(r, 0) \approx Ze[r^{-1} + k_{TF} \ln(\tfrac{1}{2} e^{\gamma} k_{TF} r)], \qquad (84)$$

which is qualitatively similar to (81). For large r, however, the leading terms of order r^{-1} cancel exactly, giving

$$\varphi_{\text{tot}}(r, 0) \sim Ze/k_{TF}^2 r^3 \qquad (k_{TF} r \gg 1). \tag{85}$$

Thus, a test charge in the xy plane produces an anisotropic potential, varying asymptotically like r^{-3} in the xy plane but like z^{-2} along the z axis. More generally, (B.8) gives the following asymptotic form for the total potential at (r, z) produced by a static test charge at $(0, z')$

$$\varphi_{\text{tot}}(r, z) \sim Ze \left\{ \frac{1}{[r^2 + (z - z')^2]^{1/2}} - \frac{1}{[r^2 + (|z| + |z'|)^2]^{1/2}} + \frac{|z| + |z'| + k_{TF}^{-1}}{[r^2 + (|z| + |z'|)^2]^{3/2}} + \cdots \right\}, \tag{86}$$

which holds for $r^2 + (|z| + |z'|)^2 \gg k_{TF}^{-2}$; it reproduces the various limits discussed previously.

We shall briefly consider the static induced charge density $\rho_{\text{in}}(r)$ produced by a static test charge Ze placed a distance z' along the z axis. Equations (74) and (75) give

$$\rho_{\text{in}}(r) = -\frac{Zek_{TF}}{2\pi(r^2 + z'^2)^{1/2}} + \frac{Zek_{TF}^2}{2\pi} I(k_{TF} r, k_{TF} |z'|, 1). \tag{87}$$

If the charge is off the xy plane ($z' \neq 0$), then ρ_{in} is everywhere bounded, attaining its maximum value at $r = 0$

$$\rho_{\text{in}}(0) = -\frac{Zek_{TF}}{2\pi |z'|} + \frac{Zek_{TF}^2}{2\pi} \exp(k_{TF} |z'|)\, E_1(k_{TF} |z'|). \tag{88}$$

For $z' = 0$, on the other hand, $\rho_{\text{in}}(r)$ diverges for small r like $-Zek_{TF}/2\pi r$. In all cases, $\rho_{\text{in}}(r)$ is integrable, and the total screening charge is precisely $-Ze$, which follows most simply by integrating the static limit of (71b) over the xy plane.

As the final topic in this section, we analyze the dynamic response, when the form of the dispersion relation (68) implies that propagating plasma waves will be excited at any finite frequency. For simplicity, the harmonic test charge will be placed in the xy plane, where the total dynamic potential will also be evaluated. Hence we consider [see (78) and (B.12)]

$$\begin{aligned} \varphi_{\text{tot}}(r, 0, \omega) &= Zer^{-1} + ZeG_{\text{in}}(r, z = 0, z' = 0, \omega) \\ &= \varphi^{(+)}(r, 0, \omega) + \varphi^{(0)}(r, 0, \omega), \end{aligned} \tag{89}$$

where ω is assumed positive and

$$\begin{aligned}\varphi^{(+)}(r, 0, \omega) &= -Ze\pi ik_{TF}\lambda_2(\lambda_1 + \lambda_2)^{-1} H_0^{(1)}(k_{TF}r\lambda_2),\\ \varphi^{(0)}(r, 0, \omega) &= Zer^{-1} - Zek_{TF}(\lambda_1 + \lambda_2)^{-1} [\lambda_1 I(k_{TF}r, 0, \lambda_1) - \lambda_2 I(k_{TF}r, 0, \lambda_2)].\end{aligned} \tag{90}$$

The first term $\varphi^{(+)}$ represents an outward propagating cylindrical plasma wave, because the Hankel function $H_0^{(1)}(x)$ has the asymptotic form $(2/\pi x)^{1/2}\exp[i(x-\pi/4)]$ [17]; the amplitude of $\varphi^{(+)}$ decreases to zero as $\omega \to 0$ and the wave number $k_{TF}\lambda_2$ is determined by the dispersion relation (68). The remaining term $\varphi^{(0)}$ in (89) is purely real and, therefore, corresponds to a stationary (nonpropagating) wave. For any finite frequency, $\varphi^{(+)}$ ultimately decreases like $r^{-1/2}$ for $k_{TF}r\lambda_2 \to \infty$, whereas $\varphi^{(0)}$ falls off more rapidly ($\sim r^{-1}$). For $\omega = 0$, however, λ_2 and $\varphi^{(+)}$ vanish identically, and $\varphi^{(0)}$ reverts to the r^{-3} behavior found previously in (85).

To illustrate these relations, we evaluated the dimensionless functions $(Zek_{TF})^{-1}\,\mathrm{Re}[\varphi^{(+)}e^{-i\omega t}]$ and $(Zek_{TF})^{-1}\,\mathrm{Re}[\varphi^{(0)}\,e^{-i\omega t}]$. Figure 1 shows their spatial forms at $t = 0$ for a frequency $\omega/k_{TF}s = \sqrt{3}/2$, when $\lambda_1 = 3/2$ and $\lambda_2 = 1/2$; it indicates the distinction between the two contributions, for only the first propagates outward. For comparison, Fig. 1 also contains the static limit, when $\varphi^{(+)}$ vanishes, and $\varphi^{(0)}$ is much smaller than its value at finite ω. This shows that a dynamic perturbation induces a stronger and longer-range response than does a static one.

In the general case of a harmonic perturbation a distance z' off the xy plane, the stationary-wave contribution to $\varphi_{\text{tot}}(r, z, \omega)$ is rather complicated, but the propagating-wave component is merely multiplied by a factor $\exp[-k_{TF}\lambda_2(|\,z\,| + |\,z'\,|)]$. Since the qualitative behavior is similar to that in Fig. 1, this question will not be considered further.

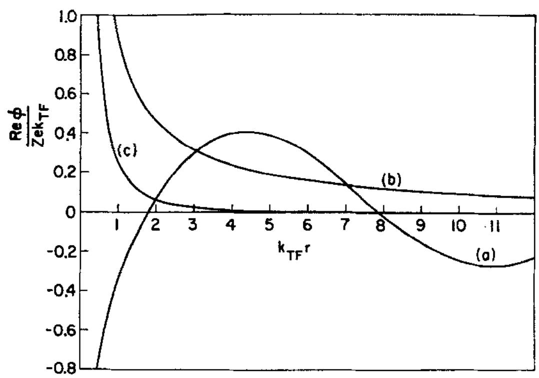

FIG. 1. The dynamic potentials (a) $(Zek_{TF})^{-1}\,\mathrm{Re}\,\varphi^{(+)}$ and (b) $(Zek_{TF})^{-1}\,\mathrm{Re}\,\varphi^{(0)}$ in the plane $z = 0$, produced by an oscillating test charge Ze located in the plane $z = 0$ [see Eq. (90)]. The curves are drawn for the value $\omega/k_{TF}s = \frac{1}{2}\sqrt{3}$. Curve (c) shows the corresponding static potential (83).

5. Response to a Moving Charge

The final application of this formalism will be the electrodynamic response of the charged layer to a test charge Ze moving uniformly with velocity $\mathbf{V} = \mathbf{u} + v\hat{\mathbf{z}}$, where $\mathbf{u}$ lies in the xy plane and v is taken as positive. The external charge density is

$$\rho_{\text{ex}}(\mathbf{r}, z, t) = Ze\delta(\mathbf{r} - \mathbf{u}t)\,\delta(z - z_0 - vt), \tag{91}$$

where z_0 denotes the separation of the charge and the plane if $v = 0$ and otherwise may be taken as zero. Since the limiting process $v \to 0$ is rather delicate, we first consider the case $v \neq 0$ and, hence, set $z_0 = 0$. The corresponding Fourier transform in $\mathbf{r}$ and t becomes

$$\rho_{\text{ex}}(\mathbf{k}, z, \omega) = Zev^{-1} \exp[iv^{-1}(\omega - \mathbf{k}\cdot\mathbf{u})z] \tag{92}$$

and a combination with (67b) yields the induced potential

$$\begin{aligned}\varphi_{\text{in}}(\mathbf{k}, z, \omega) &= \frac{2\pi e^{-k|z|} s^2 k_{TF} Zev^{-1} \int_{-\infty}^{\infty} dz' \exp[-k\,|\,z'\,|\, + iv^{-1}(\omega - \mathbf{k}\cdot\mathbf{u})\,z']}{(\omega + i\eta)^2 - s^2k^2 - s^2kk_{TF}} \\ &= \frac{4\pi Zes^2k_{TF}e^{-k|z|}}{(\omega + i\eta)^2 - s^2k^2 - s^2kk_{TF}}\,\frac{kv}{k^2v^2 + (\omega - \mathbf{k}\cdot\mathbf{u})^2}\,. \end{aligned}\tag{93}$$

Equation (58) then provides the induced electric field; as in Section 2, the power added to the particle follows directly

$$\begin{aligned}\mathscr{P}(t) &= Ze\mathbf{V}\cdot\mathbf{E}_{\text{in}}(\mathbf{u}t, vt, t) \\ &= \frac{Z^2e^2s^2k_{TF}}{2\pi^2}\int \frac{d^2k\,d\omega\, e^{i(\mathbf{k}\cdot\mathbf{u}-\omega)t}e^{-kv|t|}(-i\mathbf{k}\cdot\mathbf{u} + kv\,\operatorname{sgn} t)\,kv}{[(\omega + i\eta)^2 - s^2k^2 - s^2kk_{TF}][k^2v^2 + (\omega - \mathbf{k}\cdot\mathbf{u})^2]}\,. \end{aligned}\tag{94}$$

Here the planar geometry implies that $\mathscr{P}(t)$ is time dependent, owing to the lack of translational invariance normal to the plane.

Unfortunately, (94) is still complicated, and we, therefore, consider only the case of a particle moving along the z axis ($\mathbf{u} = 0$, $v > 0$), when (94) reduces to

$$\mathscr{P}(t) = \frac{(Zes)^2\,k_{TF}}{2\pi^2}\int \frac{d^2k\,d\omega\, e^{-i\omega t}v^2k^2\,\operatorname{sgn} t\, e^{-kv|t|}}{[(\omega + i\eta)^2 - s^2k^2 - s^2kk_{TF}](k^2v^2 + \omega^2)}\,. \tag{95}$$

The frequency integral may be evaluated by deforming the contour into the upper or lower half plane according as t is negative or positive; the residue theorem then gives

$$\begin{aligned}\mathscr{P}(t) = &-(Zes)^2\,k_{TF}v\,\operatorname{sgn} t \int_0^\infty \frac{dk\,ke^{-2kv|t|}}{(s^2 + v^2)\,k + s^2k_{TF}} \\ &- 2(Zesv)^2\,\theta(t)\,k_{TF}\int_0^\infty \frac{dk\,k^2e^{-kvt}}{(s^2 + v^2)\,k + s^2k_{TF}}\,\frac{\sin\omega_k t}{\omega_k}\,, \end{aligned}\tag{96}$$

where ω_k is fixed by the two-dimensional plasma dispersion relation (68). The first term of (96) is odd in t and does not lead to a net loss of energy; it represents an instantaneous electrostatic force between the test charge and the induced charges in the plane. The second term contributes only after the particle has crossed the charged plane; it represents energy lost through the excitation of plasma oscillations. The total energy loss arises solely from the last term of (96)

$$\begin{aligned}\Delta E &= \int_{-\infty}^{\infty} dt\, \mathscr{P}(t) \\ &= -2(Zesv)^2\, k_{TF} \int_0^{\infty} \frac{dk\, k^2}{(s^2 + v^2)\, k + s^2 k_{TF}} \int_0^{\infty} dt\, e^{-kvt} \frac{\sin \omega_k t}{\omega_k} \\ &= -2(Zesv)^2\, k_{TF} \int_0^{\infty} \frac{dk\, k}{[(s^2 + v^2)\, k + s^2 k_{TF}]^2}\,. \end{aligned} \tag{97}$$

This integral diverges logarithmically and must be cut off at some upper limit $k_{\max}$. To account for such behavior, it is easy to verify that the plasmon contribution to $\mathscr{P}(t)$ varies like t^{-1} as $t \to 0$. Thus the present model predicts a singular loss of energy on crossing the plane, which may be ascribed to the large group velocity of long-wave length plasmons in (68). A more realistic calculation of the dispersion relation (such as in Section 6) would likely remedy this defect.

It is also interesting to consider the energy loss for a test charge moving parallel to the plane ($z = z_0$, $v = 0$), when the external charge density becomes

$$\rho_{\text{ex}}(\mathbf{r}, z, t) = Ze\delta(\mathbf{r} - \mathbf{u}t)\, \delta(z - z_0), \tag{98a}$$

$$\rho_{\text{ex}}(\mathbf{k}, z, \omega) = 2\pi Ze\delta(z - z_0)\, \delta(\omega - \mathbf{k} \cdot \mathbf{u}). \tag{98b}$$

The corresponding induced potential

$$\varphi_{\text{in}}(\mathbf{k}, z, \omega) = \frac{4\pi^2 Zes^2 k_{TF} \exp[-k(|z| + |z_0|)]\, \delta(\omega - \mathbf{k} \cdot \mathbf{u})}{(\omega + i\eta)^2 - s^2k^2 - s^2 k k_{TF}} \tag{99}$$

readily gives the power added to the test charge

$$\mathscr{P} = -\frac{i(Zes)^2\, k_{TF}}{2\pi} \int \frac{d^2k \exp(-2k\, |z_0|)\, \mathbf{k} \cdot \mathbf{u}}{(\mathbf{k} \cdot \mathbf{u} + i\eta)^2 - s^2k^2 - s^2 k k_{TF}}\,, \tag{100}$$

which is time independent because the configuration is steady. Equation (100) may be rewritten with partial fractions

$$\begin{aligned}\mathscr{P} = &-\frac{i(Zes)^2\, k_{TF}}{4\pi} \int d^2k \exp(-2k\, |z_0|) \\ &\times \left[\frac{1}{\omega_k - \mathbf{k} \cdot \mathbf{u} + i\eta} - \frac{1}{\omega_k - \mathbf{k} \cdot \mathbf{u} - i\eta}\right],\end{aligned} \tag{101}$$

where (68) determines ω_k and the first term has been obtained with the change of variables ($\mathbf{k} \to -\mathbf{k}$). In the limit $\eta \to 0$, the quantity in brackets is precisely $-2\pi i\delta(\omega_k - \mathbf{k} \cdot \mathbf{u})$, yielding

$$\begin{aligned}\mathscr{P} &= -\tfrac{1}{2}(Zes)^2 k_{TF} \int_0^\infty k\,dk \exp(-2k \mid z_0 \mid) \int_0^{2\pi} d\chi\, \delta(\omega_k - ku\cos\chi) \\ &= -(Zes)^2 k_{TF} \int_0^\infty \frac{k\,dk \exp(-2k \mid z_0 \mid)}{(k^2u^2 - \omega_k{}^2)^{1/2}}\, \theta(k^2u^2 - \omega_k{}^2).\end{aligned} \tag{102}$$

This integral is simply related to a standard representation of certain Bessel functions [18, p. 172], and we eventually find

$$\mathscr{P} = -\frac{(Zek_{TF}s^2)^2}{2(u^2 - s^2)^{3/2}}\, \theta(u^2 - s^2)\, e^{-\alpha}[K_0(\alpha) + K_1(\alpha)], \tag{103a}$$

where

$$\alpha \equiv \mid z_0 \mid s^2 k_{TF}(u^2 - s^2)^{-1}. \tag{103b}$$

The rate of energy loss is exponentially small for $\alpha \gg 1$, and it increases like $\mid z_0 \mid^{-1}$ as $z_0 \to 0$. It is striking that (103) exhibits a critical velocity $u_c = s$, whereas no such effect occurs in (94).

6. Retardation

The previous treatment of plasma oscillations in a charged layer has neglected retardation, thereby producing certain unphysical features such as a divergent group velocity for plasmons as $k \to 0$. We here improve the theory by returning to the full set of Maxwell's equations. For simplicity, we consider only the self-excited collective modes, which permits us to deal with homogeneous equations. In the linearized theory (52) and (53) are still valid, but the electric field is now the full expression (7a) obtained from (6a) and (6b) with $\rho_{\text{ex}} = \mathbf{j}_{\text{ex}} = 0$. A Fourier transform in the xy plane and with respect to t yields the set of equations

$$\omega n'(\mathbf{k}, \omega) = n_0 \mathbf{k} \cdot \mathbf{v}(\mathbf{k}, \omega), \tag{104}$$

$$\begin{aligned}\omega \mathbf{v}(\mathbf{k}, \omega) = \mathbf{k} n_0^{-1} s^2 n'(\mathbf{k}, \omega) + (e/m) \\ \times \{-\mathbf{k}\varphi(\mathbf{k}, z = 0, \omega) + (\omega/c)\, \hat{z} \times [\mathbf{A}(\mathbf{k}, z = 0, \omega) \times \hat{z}]\},\end{aligned} \tag{105}$$

$$(\partial^2/\partial z^2 - k^2 + \omega^2/c^2)\, \varphi(\mathbf{k}, z, \omega) = 4\pi e n'(\mathbf{k}, \omega)\, \delta(z), \tag{106}$$

$$(\partial^2/\partial z^2 - k^2 + \omega^2/c^2)\, \mathbf{A}(\mathbf{k}, z, \omega) = 4\pi e n_0 c^{-1} \mathbf{v}(\mathbf{k}, \omega)\, \delta(z). \tag{107}$$

The last two equations may be solved exactly as in (65), and we obtain

$$\varphi(\mathbf{k}, z, \omega) = \frac{-2\pi e n'(\mathbf{k}, \omega)}{(k^2 - \omega^2/c^2)^{1/2}} \exp[-(k^2 - \omega^2/c^2)^{1/2} \mid z \mid], \tag{108a}$$

$$\mathbf{A}(\mathbf{k}, z, \omega) = \frac{-2\pi e n_0 \mathbf{v}(\mathbf{k}, \omega)}{c(k^2 - \omega^2/c^2)^{1/2}} \exp[-(k^2 - \omega^2/c^2)^{1/2} \mid z \mid], \tag{108b}$$

with $\mathbf{A}$ automatically lying in the xy plane.

It is first convenient to study the component of (105) and (108b) perpendicular to $\mathbf{k}$; direct elimination of $\mathbf{A}_\perp$ yields the relation

$$\omega \left[1 + \frac{2\pi e^2 n_0}{mc^2(k^2 - \omega^2/c^2)^{1/2}}\right] \mathbf{v}_\perp(\mathbf{k}, \omega) = 0. \tag{109}$$

Since neither the first nor the second factor can vanish at finite frequency, we see that $\mathbf{v}_\perp$ (and, hence, $\mathbf{A}_\perp$) $= 0$. A similar analysis for (104), (106), and the parallel components $\mathbf{v}_\parallel$ and $\mathbf{A}_\parallel$ gives the desired dispersion relation

$$\omega^2 = s^2 k^2 + s^2 k_{TF}(k^2 - \omega^2/c^2)^{1/2}, \tag{110}$$

which generalizes (68) to incorporate retardation. Note that the two-dimensional plasma dispersion relation now depends explicitly on the speed of light. This situation should be contrasted with the three-dimensional case (28), where the longitudinal dispersion relation was unaffected by the use of a retarded potential [see the discussion following (28)].

Although (110) can be solved exactly for ω, the result is somewhat unwieldy, and we instead concentrate on its approximate form in various limits. If $k \gg k_{TF}$, then the first term on the right side of (110) dominates

$$\omega_k \approx sk \qquad (k \gg k_{TF}) \tag{111}$$

because $\omega_k/c \approx sk/c$ is then much smaller than k. On the other hand, if $k \ll k_{TF}$, then the second term dominates, and the term s^2k^2 is negligible. As k decreases below k_{TF}, the dispersion relation is given approximately by

$$\omega_k \approx s(kk_{TF})^{1/2} \qquad (k_{TF}s^2/c^2 \ll k \ll k_{TF}) \tag{112}$$

until ω_k/c becomes comparable with k. Equations (111) and (112) are just those obtained from (68) in Section 3. For still smaller wave numbers, however, the dispersion relation is qualitatively altered and approaches that of an electromagnetic wave

$$\omega_k \approx ck \qquad (k \ll k_{TF}s^2/c^2). \tag{113}$$

The important point is that the group velocity for $k \ll k_{TF}$ rises continuously like $k^{-1/2}$ until it approaches the speed of light (near $k_{TF}s^2/c^2$). For smaller k, retardation becomes significant, and the resulting group velocity remains bounded by c. Since k_{TF}^{-1} is typically an atomic dimension, we see that retardation can play a role only at very long wave lengths.

The present model is still very idealized because it neglects the presence of other charged layers. A subsequent paper will analyze the electrodynamics of a layered array in detail, and it is here sufficient to note that the major alteration in (68) occurs for wave lengths long compared to the spacing d between layers. If retardation is again neglected, (68) spreads out into a band associated with various wave numbers q for propagation perpendicular to the surface of the layers. In particular, we find the generalized expression

$$\omega_k^2 = s^2k^2 + \frac{s^2k_{TF}k \sinh kd}{\cosh kd - \cos qd}, \tag{114}$$

which reduces to (68) as $d \to \infty$. For fixed k, the allowed frequencies lie in the range

$$s^2k^2 + s^2k_{TF}k \tanh(\tfrac{1}{2}kd) \leqslant \omega_k^2 \leqslant s^2k^2 + s^2k_{TF}k \coth(\tfrac{1}{2}kd) \tag{115}$$

associated with the values $|qd| = \pi$ and $qd = 0$, respectively. Moreover, (114) yields a finite group velocity for all k. In principle, it might be important to generalize (114) further to include retardation; the corresponding dispersion relation is obtained with the replacement $k \to (k^2 - \omega^2/c^2)^{1/2}$ in the second term on the right side. For most cases, however, the effect of neighboring layers (which is important for $k \lesssim d^{-1}$) dominates the effect of retardation, and it is usually permissible to use the instantaneous potential, as in Sections 3–5.

Appendix A

We here evaluate the integral in (40) that determines the charge induced by a uniformly moving test charge

$$\rho_{\text{in}}(\mathbf{R}, t) = -Ze\Omega_p^2 \int \frac{d^3K}{(2\pi)^3} \frac{e^{i\mathbf{K}\cdot(\mathbf{R}-\mathbf{V}t)}}{K^2S^2 + \Omega_p^2 - (\mathbf{K}\cdot\mathbf{V} + i\eta)^2}. \tag{A.1}$$

If $V < S$, the denominator is always positive, and the sign factor $i\eta$ is irrelevant. In this case, a change of variables

$$\begin{aligned} \mathbf{K}_\perp' &= \mathbf{K}_\perp, \\ \mathbf{K}_\parallel' &= \mathbf{K}_\parallel(1 - V^2/S^2)^{1/2} \end{aligned} \tag{A.2}$$

immediately reduces (A.1) to

$$\rho_{\text{in}}(\mathbf{R}, t) = -\frac{Ze\Omega_p^2}{S(S^2 - V^2)^{1/2}} \int \frac{d^3K'}{(2\pi)^3} \frac{e^{i\mathbf{K}'\cdot\bar{\mathbf{R}}}}{K'^2 + \Omega_p^2/S^2}, \tag{A.3}$$

where the vector $\bar{R}$ is defined by

$$\begin{aligned} \bar{\mathbf{R}}_\perp &= \mathbf{R}_\perp, \\ \bar{R}_\parallel &= (R_\parallel - Vt)(1 - V^2/S^2)^{-1/2}, \end{aligned} \tag{A.4}$$

and the subscripts $\parallel$ and $\perp$ denote the projections parallel and perpendicular to $\mathbf{V}$. Equation (A.3) now has spherical symmetry, and a standard analysis gives

$$\rho_{\text{in}}(\mathbf{R}, t) = -\frac{Ze\Omega_p^2}{S(S^2 - V^2)^{1/2}} \frac{\exp(-\bar{R}\Omega_p/S)}{4\pi\bar{R}}, \tag{A.5}$$

where $\bar{R}$ is the length of the vector $\bar{\mathbf{R}}$.

The calculation for $V > S$ is more complicated, as the preceding change of variables no longer produces a simple scalar product in the denominator. Instead, we note that the integrand of (A.1) as a function of $K_\parallel$ has simple poles at the points

$$\pm i[(K_\perp^2 S^2 + \Omega_p^2)/(S^2 - V^2)]^{1/2} - i\eta V(V^2 - S^2)^{-1}. \tag{A.6}$$

For $V < S$, the second term is superfluous, and the poles are located symmetrically on the imaginary axis. For $V > S$, however, the first term of (A.6) is real, so that the second term becomes essential. Both poles then lie in the lower-half $K_\parallel$ plane, with the result that (A.1) vanishes identically for $R_\parallel > Vt$. In the opposite case ($R_\parallel < Vt$), a contour integral yields [17]

$$\begin{aligned} \rho_{\text{in}}(\mathbf{R}, t) &= \frac{-Ze\Omega_p^2}{V^2 - S^2} \theta(Vt - R_\parallel) \int \frac{d^2K_\perp}{(2\pi)^2} \exp(i\mathbf{K}_\perp \cdot \mathbf{R}_\perp) \frac{\sin K_1(Vt - R_\parallel)}{K_1} \\ &= \frac{-Ze\Omega_p^2}{V^2 - S^2} \theta(Vt - R_\parallel) \int_0^\infty \frac{K_\perp\, dK_\perp}{2\pi} J_0(K_\perp R_\perp) \frac{\sin K_1(Vt - R_\parallel)}{K_1}, \end{aligned} \tag{A.7}$$

where

$$K_1^2 = (S^2 K_\perp^2 + \Omega_p^2)(V^2 - S^2)^{-1}. \tag{A.8}$$

If we change variables from $K_\perp$ to K_1, the integral assumes a standard form [20], and we obtain

$$\rho_{\text{in}}(\mathbf{R}, t) = \frac{-Ze\Omega_p^2 \theta(Vt - R_\parallel)\, \theta(-\bar{R}^2)}{2\pi S(V^2 - S^2)^{1/2}} \frac{\cos(\Omega_p \mid \bar{R} \mid /S)}{\mid \bar{R} \mid}, \tag{A.9}$$

where

$$|\bar{R}| = [(Vt - R_\parallel)^2 (V^2/S^2 - 1)^{-1} - R_\perp^2]^{1/2}, \tag{A.10}$$

in accordance with the notation (A.4) used previously for $V < S$.

Appendix B

This appendix studies the integral

$$I(a, b, c) = \int_0^\infty dx \, \frac{J_0(ax)\, e^{-bx}}{x + c} \tag{B.1}$$

used in Section 4, where b and c are temporarily assumed positive. This quantity becomes simple in two separate limiting cases, $b = 0$ and $a = 0$. For definiteness, consider first [18, p. 436; 20, p. 685]

$$\begin{aligned} I(a, 0, c) &= \int_0^\infty dx \, \frac{J_0(ax)}{x + c} \\ &= (1/2)\pi[\mathbf{H}_0(ac) - Y_0(ac)], \end{aligned} \tag{B.2}$$

with the approximate behavior [17, Sections 9.1 and 12.1]

$$I(a, 0, c) \approx \ln(2e^{-\gamma}/ac) \qquad (ac \ll 1), \tag{B.3a}$$

$$I(a, 0, c) \sim (ac)^{-1}[1 - (ac)^{-2} + \cdots] \qquad (ac \gg 1). \tag{B.3b}$$

Here $\mathbf{H}_0$ and Y_0 denote Struve and Neumann functions, and $\gamma \approx 0.577$. Alternatively [17, p. 230],

$$\begin{aligned} I(0, b, c) &= \int_0^\infty dx \, \frac{e^{-bx}}{x + c} \\ &= e^{bc} E_1(bc), \end{aligned} \tag{B.4}$$

with the approximate behavior

$$I(0, b, c) \approx \ln(e^{-\gamma}/bc) \qquad (bc \ll 1), \tag{B.5a}$$

$$I(0, b, c) \sim (bc)^{-1}[1 - (bc)^{-1} + \cdots] \qquad (bc \gg 1). \tag{B.5b}$$

The general case cannot be evaluated in closed form, but a more convenient expression [19] follows from the integral representation

$$(x + c)^{-1} = \int_0^\infty d\sigma \exp[-\sigma(x + c)]. \tag{B.6}$$

Substitution into (B.1) gives [18]

$$
\begin{aligned}
I(a, b, c) &= \int_0^\infty d\sigma\, e^{-\sigma c} \int_0^\infty dx\, J_0(ax)\, e^{-(b+\sigma)x} \\
&= \int_0^\infty d\sigma \frac{e^{-\sigma c}}{[a^2 + (b+\sigma)^2]^{1/2}} .
\end{aligned}
\tag{B.7}
$$

The denominator of (B.7) may be expanded for large $a^2 + b^2$

$$
\begin{aligned}
I(a, b, c) &\sim \frac{1}{(a^2 + b^2)^{1/2}} \int_0^\infty d\sigma\, e^{-\sigma c} \left[1 - \frac{b\sigma + \sigma^2/2}{a^2 + b^2} + \cdots\right] \\
&= \frac{1}{(a^2 + b^2)^{1/2}} \left[\frac{1}{c} - \frac{b + c^{-1}}{c^2(a^2 + b^2)} + \cdots\right]
\end{aligned}
\tag{B.8}
$$

which holds for $c(a^2 + b^2) \gg 1$.

The final modification required here may be obtained by writing $J_0(ax)$ as the sum of two Hankel functions $(1/2)[H_0^{(1)}(ax) + H_0^{(2)}(ax)]$, analytic respectively in the upper half and lower half x planes

$$
I(a, b, c) = \tfrac{1}{2} \int_0^\infty dx\, [H_0^{(1)}(ax) + H_0^{(2)}(ax)] \frac{e^{-bx}}{x + c} .
\tag{B.9}
$$

For real positive c, the contour may be deformed to the upper or lower half imaginary axis with no added contributions [17, Sections 9.1 and 9.6]:

$$
\begin{aligned}
I(a, b, c) &= \tfrac{1}{2} \int_0^\infty i\, dy \left[H_0^{(1)}(iay) \frac{e^{-iby}}{iy + c} - H_0^{(2)}(-iay) \frac{e^{iby}}{-iy + c}\right] \\
&= (2/\pi) \int_0^\infty dy\, K_0(ay) \operatorname{Re} \left(\frac{e^{-iby}}{iy + c}\right).
\end{aligned}
\tag{B.10}
$$

If, however, c is analytically continued to the value $-d - i\eta$, where d is real and positive and $\eta \to 0^+$, then the first term acquires an additional part from the pole at $x = d + i\eta$:

$$
I(a, b, -d - i\eta) = \pi i H_0^{(1)}(ad)\, e^{-bd} + (2/\pi) \int_0^\infty dy\, K_0(ay) \operatorname{Re} \left(\frac{e^{-iby}}{iy - d}\right).
\tag{B.11}
$$

In general, (B.10) and (B.11) are not directly related, but they assume a simple form for $b = 0$

$$
\begin{aligned}
I(a, 0, -d - i\eta) &= \pi i H_0^{(1)}(ad) - \frac{2d}{\pi} \int_0^\infty dy \frac{K_0(ay)}{y^2 + d^2} \\
&= \pi i H_0^{(1)}(ad) - I(a, 0, d).
\end{aligned}
\tag{B.12}
$$

When combined with (B.2), this last relation was used in Section 4.

Notes added in proof.

1. For an external charge density $\rho_{ex}(\mathbf{k}, z, \omega) = \rho_{ex}(\mathbf{k}, \omega)\delta(z)$ located in the xy plane, it is convenient to characterize the response of the charged layer with a dielectric function $\epsilon(\mathbf{k}, \omega)$. If the left side of (53) is augmented with a phenomenological relaxation term $\mathbf{v}/\tau$, the ratio of Eqs. (67b) to (67a) gives

$$\epsilon(\mathbf{k}, \omega) = \left(\frac{\varphi_{in}(\mathbf{k}, z, \omega)}{\varphi_{ex}(\mathbf{k}, z, \omega)} + 1\right)^{-1} = 1 - \frac{2\pi n_0 e^2 k/m}{\omega(\omega + i/\tau) - s^2k^2}.$$

Correspondingly, the conductivity (defined as the ratio of the induced surface current to the tangential component of the total electric field) is

$$\sigma(\mathbf{k}, \omega) = \frac{i\omega n_0 e^2/m}{\omega(\omega + i/\tau) - s^2k^2}.$$

Each of these quantities is analytic in the upper half ω plane and satisfies the usual Kramers–Kronig relation [12, Sec. 62].

2. If s in (54) and (57b) is replaced by the adiabatic speed of sound $(2p/mn_0)^{1/2}$ in an ideal two-dimensional Fermi gas with pressure p, temperature T, and number density n_0, the resulting theory describes the electrodynamics of a charged layer for any T and n_0 whenever the screening length (69a) $ms^2/2\pi n_0 e^2 = p/\pi n_0^2 e^2$ much exceeds the interparticle spacing $n_0^{-1/2}$. In the classical limit, such a model may describe an electron layer on the surface of liquid He [T. R. Brown and C. C. Grimes, *Phys. Rev. Lett.* **29** (1972), 1233] and Ne. I am grateful to T. R. Brown for suggesting this possibility.

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