Chapter 5

The Argument Principle

5.1 Counting Zeros and Poles

5.1.1 Local Geometric Behavior of a Holomorphic Function

In this chapter, we shall be concerned with questions that have a geometric, qualitative nature rather than an analytical, quantitative one. These questions center around the issue of the local geometric behavior of a holomorphic function.

5.1.2 Locating the Zeros of a Holomorphic Function

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on a connected, open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P, r) \subseteq U$. We know from the Cauchy integral formula that the values of f on $D(P, r)$ are completely determined by the values of f on $\partial D(P, r)$. In particular, the number and even the location of the zeros of f in $D(P, r)$ are determined in principle by f on $\partial D(P, r)$. But it is nonetheless a pleasant surprise that there is a *simple formula* for the number of zeros of f in $D(P, r)$ in terms of f (and f') on $\partial D(P, r)$. In order to obtain a precise formula, we shall have to agree to count zeros according to multiplicity (see §§3.1.4). We now explain the precise idea.

Let $f : U \to \mathbb{C}$ be holomorphic as before, and assume that f has *some* zeros in U but that f is not identically zero. Fix $z_0 \in U$ such that $f(z_0) = 0$. Since the zeros of f are isolated, there is an $r > 0$ such that $\overline{D}(z_0, r) \subseteq U$ and such that f does not vanish on $\overline{D}(z_0, r) \setminus \{z_0\}$.

Now the power series expansion of f about z_0 has a first non-zero term determined by the least positive integer n such that $f^{(n)}(z_0) \neq 0$. (Note that $n \geq 1$ since $f(z_0) = 0$ by hypothesis.) Thus the power series expansion

S. G. Krantz, *Handbook of Complex Variables*

of f about z_0 *begins* with the n^{th} term:

$$f(z) = \sum_{j=n}^{\infty} \frac{1}{j!}\frac{\partial^j f}{\partial z^j}(z_0)(z-z_0)^j. \tag{5.1.2.1}$$

Under these circumstances we say that f has a zero of *order* n (or *multiplicity* n) at z_0. When $n=1$, then we also say that z_0 is a *simple* zero of f.

5.1.3 Zero of Order n

The concept of zero of "order n," or "multiplicity n," for a function f is so important that a variety of terminology has grown up around it (see also §§3.1.4). It has already been noted that when the multiplicity $n=1$, then the zero is sometimes called *simple*. For arbitrary n, we sometimes say that "n is the order of z_0 as a zero of f." More generally if $f(z_0)=\beta$ so that, for some $n \geq 1$, the function $f(\cdot)-\beta$ has a zero of order n at z_0, then we say either that "f assumes the value β at z_0 to order n" or that "the order of the value β at z_0 is n." When $n>1$, then we call z_0 a *multiple point* of the function f.

The next result provides a method for computing the multiplicity n of the zero at z_0 from the values of f, f' on the boundary of a disc centered at z_0.

5.1.4 Counting the Zeros of a Holomorphic Function

If f is holomorphic on a neighborhood of a disc $\overline{D}(P,r)$ and has a zero of order n at P and no other zeros in the closed disc, then

$$\frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = n. \tag{5.1.4.1}$$

More generally, we consider the case that f has several zeros—with different locations and different multiplicities—inside a disc: Suppose that $f: U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P,r) \subseteq U$. Suppose that f is non-vanishing on $\partial D(P,r)$ and that $z_1, z_2, \ldots, z_k$ are the zeros of f in the interior of the disc. Let n_ℓ be the order of the zero of f at z_ℓ, $\ell = 1, \ldots, k$. Then

$$\frac{1}{2\pi i}\oint_{|\zeta-P|=r} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{\ell=1}^{k} n_\ell. \tag{5.1.4.2}$$

Refer to Figure 5.1 for illustrations of both these situations.

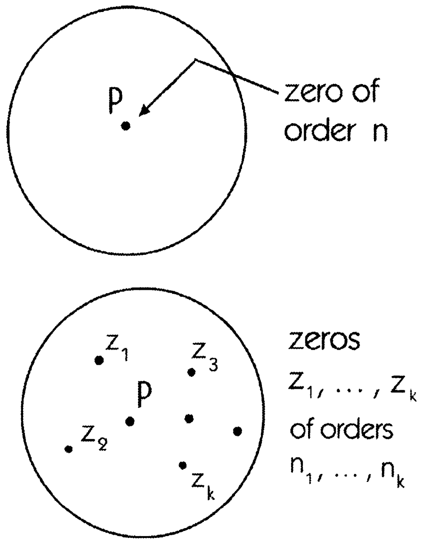

FIGURE 5.1
Locating the zeros of a holomorphic function.

5.1.5 The Argument Principle

This last formula, which is often called the *argument principle*, is both useful and important. For one thing, there is no obvious reason why the integral in the formula should be an integer, much less the crucial integer that it is. Since it is an integer, it is a counting function; and we need to learn more about it.

The integral

$$\frac{1}{2\pi i}\oint_{|\zeta-P|=r}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta \tag{5.1.5.1}$$

can be reinterpreted as follows: Consider the C^1 closed curve

$$\gamma(t)=f(P+re^{it}),\quad t\in[0,2\pi]. \tag{5.1.5.2}$$

Then

$$\frac{1}{2\pi i}\oint_{|\zeta-P|=r}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)}dt, \tag{5.1.5.3}$$

as you can check by direct calculation. The expression on the right is just the index of the curve γ with respect to 0 (with the notion of index that we defined earlier—§§4.4.4). See Figure 5.2. Thus the number of zeros of f (counting multiplicity) inside the circle $\{\zeta : |\zeta - P| = r\}$ is equal to the index of γ with respect to the origin. This, intuitively speaking, is equal to the number of times that the f-image of the boundary circle winds around 0 in $\mathbb{C}$.

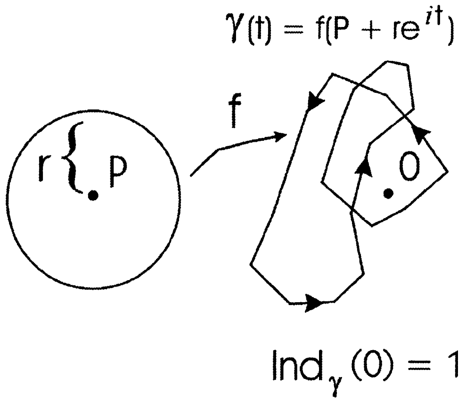

FIGURE 5.2
The argument principle: counting the zeros.

The argument principle can be extended to yield information about meromorphic functions, too. We can see that there is hope for this notion by investigating the analog of the argument principle for a pole.

5.1.6 Location of Poles

If $f : U \setminus \{Q\} \to \mathbb{C}$ is a nowhere-zero holomorphic function on $U \setminus \{Q\}$ with a pole of order n at Q and if $\overline{D}(Q, r) \subseteq U$, then

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = -n. \tag{5.1.6.1}$$

5.1.7 The Argument Principle for Meromorphic Functions

Just as with the argument principle for holomorphic functions, this new argument principle gives a counting principle for zeros and poles of meromorphic functions:

Suppose that f is a meromorphic function on an open set $U \subseteq \mathbb{C}$, that $\overline{D}(P, r) \subseteq U$, and that f has neither poles nor zeros on $\partial D(P, r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k, \tag{5.1.7.1}$$

where $n_1, n_2, \ldots, n_p$ are the multiplicities of the zeros $z_1, z_2, \ldots, z_p$ of f in $D(P, r)$ and $m_1, m_2, \ldots, m_q$ are the orders of the poles $w_1, w_2, \ldots, w_q$ of f in $D(P, r)$.

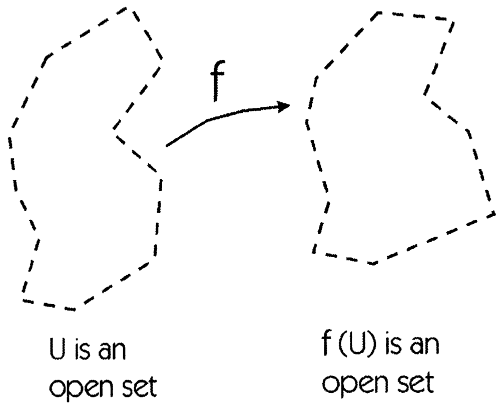

FIGURE 5.3
The open mapping principle.

5.2 The Local Geometry of Holomorphic Functions

5.2.1 The Open Mapping Theorem

The argument principle for holomorphic functions has a consequence that is one of the most important facts about holomorphic functions considered as geometric mappings:

> If $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U, then $f(U)$ is an open set in $\mathbb{C}$.

(5.2.1.1)

See Figure 5.3. The result says, in particular, that if $U \subseteq \mathbb{C}$ is connected and open and if $f : U \to \mathbb{C}$ is holomorphic, then either $f(U)$ is a connected open set (the non-constant case) or $f(U)$ is a single point.

In the subject of topology, a function f is defined to be continuous if the inverse image of any open set under f is also open. In contexts where the $\epsilon - \delta$ definition makes sense, the $\epsilon - \delta$ definition (§§2.2.1, 2.2.2) is equivalent to the inverse-image-of-open-sets definition. By contrast, functions for which the direct image of any open set is open are called "open mappings."

Here is a quantitative, or counting, statement that comes from the proof of the open mapping principle: Suppose that $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order k. Then there are numbers $\delta, \epsilon > 0$ such that each

$q \in D(Q, \epsilon) \setminus \{Q\}$ has exactly k distinct pre-images in $D(P, \delta)$ and each pre-image is a simple point of f.

The considerations that establish the open mapping principle can also be used to establish the fact that if $f : U \to V$ is a one-to-one and onto holomorphic function, then $f^{-1} : V \to U$ is also holomorphic.

5.3 Further Results on the Zeros of Holomorphic Functions

5.3.1 Rouché's Theorem

Now we consider global aspects of the argument principle.

Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subseteq \mathbb{C}$. Suppose also that $\overline{D}(P, r) \subseteq U$ and that, for each $\zeta \in \partial D(P, r)$,

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|. \tag{5.3.1.1}$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)} \, d\zeta. \tag{5.3.1.2}$$

That is, the number of zeros of f in $D(P, r)$ counting multiplicities equals the number of zeros of g in $D(P, r)$ counting multiplicities.

Remark: Rouché's theorem is often stated with the stronger hypothesis that

$$|f(\zeta) - g(\zeta)| < |g(\zeta)| \tag{5.3.1.3}$$

for $\zeta \in \partial D(P, r)$. Rewriting this hypothesis as

$$\left| \frac{f(\zeta)}{g(\zeta)} - 1 \right| < 1, \tag{5.3.1.4}$$

we see that it says that the image γ under f/g of the circle $\partial D(P, r)$ lies in the disc $D(1, 1)$. See Figure 5.4. Our weaker hypothesis that $|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|$ has the geometric interpretation that $f(\zeta)/g(\zeta)$ lies in the set $\mathbb{C} \setminus \{x + i0 : x \leq 0\}$. Either hypothesis implies that the image of the circle $\partial D(P, r)$ under f has the same "winding number" around 0 as does the image under g of that circle.

5.3.2 Typical Application of Rouché's Theorem

EXAMPLE 5.3.2.1 Let us determine the number of roots of the polynomial $f(z) = z^7 + 5z^3 - z - 2$ in the unit disc. We do so by comparing

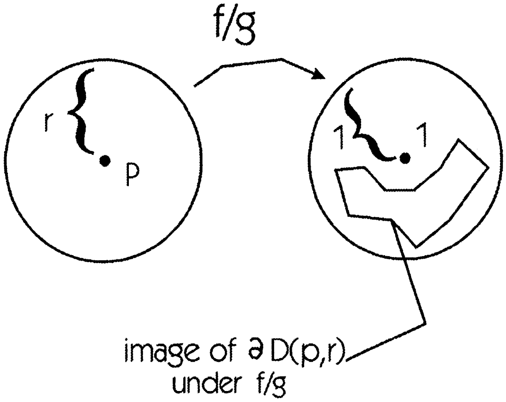

FIGURE 5.4
Rouché's theorem.

the function f to the holomorphic function $g(z) = 5z^3$ on the unit circle. For $|z| = 1$ we have

$$|f(z) - g(z)| = |z^7 - z - 2| \leq 4 < |g(\zeta)| \leq |f(\zeta)| + |g(\zeta)|.$$

By Rouché's theorem, f and g have the same number of zeros, counting multiplicity, in the unit disc. Since g has three zeros, so does f. □

5.3.3 Rouché's Theorem and the Fundamental Theorem of Algebra

Rouché's theorem provides a useful way to locate approximately the zeros of a holomorphic function that is too complicated for the zeros to be obtained explicitly. As an illustration, we analyze the zeros of a non-constant polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z + a_0. \qquad (5.3.3.1)$$

If R is sufficiently large (say $R > \max\{1, n \cdot \max_{0 \leq j \leq n-1} |a_j|\}$) and $|z| = R$, then

$$\frac{|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0|}{|z^n|} < 1. \qquad (5.3.3.2)$$

Thus Rouché's theorem applies on $\overline{D}(0, R)$ with $f(z) = z^n$ and $g(z) = P(z)$. We conclude that the number of zeros of $P(z)$ inside $D(0, R)$, counting multiplicities, is the same as the number of zeros of z^n inside $D(0, R)$, counting multiplicities—namely n. Thus we recover the Fundamental Theorem of Algebra. Incidentally, this example underlines the importance of counting zeros with multiplicities: the function z^n has only one root in the naïve sense of counting the number of points where it is zero; but it has n roots when they are counted with multiplicity.

5.3.4 Hurwitz's Theorem

A second useful consequence of the argument principle is the following result about the limit of a sequence of zero-free holomorphic functions:

> **Hurwitz's theorem** Suppose that $U \subseteq \mathbb{C}$ is a connected open set and that $\{f_j\}$ is a sequence of nowhere-vanishing holomorphic functions on U. If the sequence $\{f_j\}$ converges uniformly on compact subsets of U to a (necessarily holomorphic) limit function f_0, then either f_0 is nowhere-vanishing or $f_0 \equiv 0$.

5.4 The Maximum Principle

5.4.1 The Maximum Modulus Principle

A *domain* in $\mathbb{C}$ is a connected open set (§§1.3.1). A *bounded domain* is a connected open set U such that there is an $R > 0$ with $|z| < R$ for all $z \in U$—or $U \subseteq D(0, R)$.

The Maximum Modulus Principle

> Let $U \subseteq \mathbb{C}$ be a domain. Let f be a holomorphic function on U. If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is constant.

Here is a sharper variant of the theorem:

> Let $U \subseteq \mathbb{C}$ be a domain and let f be a holomorphic function on U. If there is a point $P \in U$ at which $|f|$ has a *local maximum*, then f is constant.

5.4.2 Boundary Maximum Modulus Theorem

The following version of the maximum principle is intuitively appealing, and is frequently useful.

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must in fact occur on ∂U.

In other words,

$$\max_{\overline{U}} |f| = \max_{\partial U} |f|.$$

5.4.3 The Minimum Principle

Holomorphic functions (or, more precisely, their moduli) *can* have interior minima. The function $f(z) = z^2$ on $D(0,1)$ has the property that $z = 0$ is a global minimum for $|f|$. However, it is not accidental that this minimum value is 0:

> Let f be holomorphic on a domain $U \subseteq \mathbb{C}$. Assume that f never vanishes. If there is a point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant. This result is proved by applying the maximum principle to the function $1/f$.

There is also a boundary minimum principle:

> Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Assume that f never vanishes on $\overline{U}$. Then the minimum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded—see [RUD1], [KRA2]) must occur on ∂U.
>
> In other words,
>
> $$\min_{\overline{U}} |f| = \min_{\partial U} |f|.$$

5.4.4 The Maximum Principle on an Unbounded Domain

It should be noted that the maximum modulus theorem is not always true on an unbounded domain. The standard example is the function $f(z) = \exp(\exp(z))$ on the domain $U = \{z = x + iy : -\pi/2 < y < \pi/2\}$. Check for yourself that $|f| = 1$ on the boundary of U. But the restriction of f to the real number line is unbounded at infinity. The theorem does, however, remain true with some additional restrictions. The result known as the Phragmen-Lindelöf theorem is one method of treating maximum modulus theorems on unbounded domains (see [RUD2]).

5.5 The Schwarz Lemma

This section treats certain estimates that bounded holomorphic functions on the unit disc necessarily satisfy. We present the classical, analytic viewpoint in the subject (instead of the geometric viewpoint—see [KRA1]).

5.5.1 Schwarz's Lemma

Let f be holomorphic on the unit disc. Assume that

(5.5.1.1) $|f(z)| \leq 1$ for all z.

(5.5.1.2) $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

If either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation: $f(z) \equiv \alpha z$ for some complex constant α of unit modulus.

Schwarz's lemma enables one to classify the invertible holomorphic self-maps of the unit disc (see [GK]). (Here a *self-map* of a domain U is a mapping $F : U \to U$ of the domain to itself.) These are commonly referred to as the "conformal self-maps" of the disc. The classification is as follows: If $0 \leq \theta < 2\pi$, then define the *rotation through angle* θ to be the function $\rho_\theta(z) = e^{i\theta} z$; if a is a complex number of modulus less than one, then define the associated *Möbius transformation* to be $\varphi_a(z) = [z - a]/[1 - \bar{a}z]$. Any conformal self-map of the disc is the composition of some rotation ρ_θ with some Möbius transformation φ_a. This topic is treated in detail in §6.2.

We conclude this section by presenting a generalization of the Schwarz lemma, in which we consider holomorphic mappings $f : D \to D$, but we discard the hypothesis that $f(0) = 0$. This result is known as the Schwarz-Pick lemma.

5.5.2 The Schwarz-Pick Lemma

Let f be holomorphic on the unit disc. Assume that

(5.5.2.1) $|f(z)| \leq 1$ for all z.

(5.5.2.2) $f(a) = b$ for some $a, b \in D(0,1)$.

Then

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}. \tag{5.5.2.3}$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|. \tag{5.5.2.4}$$

There is a "uniqueness" result in the Schwarz-Pick Lemma. If either

$$|f'(a)| = \frac{1 - |b|^2}{1 - |a|^2} \qquad \text{or} \qquad \left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| = \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right| \quad \text{with } a_1 \neq a_2\,, \tag{5.5.2.5}$$

then the function f is a conformal self-mapping (one-to-one, onto holomorphic function) of $D(0,1)$ to itself.