THE ELECTROMAGNETIC FIELDS OF A HORIZONTAL DIPOLE IN THE PRESENCE OF A CONDUCTING HALF-SPACE[1]

JAMES R. WAIT

ABSTRACT

The problem considered is a horizontal electric dipole which is located above or below the plane surface of a conducting half-space. Expressions for the fields are obtained using three different approaches. The formulas developed are quite simple and, taken together, the whole range of distances from the far-zone to the near-zone is adequately covered.

INTRODUCTION

It is well known that a horizontal wire antenna will radiate horizontally polarized electromagnetic waves along the surface of the earth. It is not quite so well known that such an antenna will also radiate vertically polarized radiation. In fact the statement is sometimes made that the field of a horizontal electric dipole in the direction of maximum field is proportional to the field of a vertical *magnetic* dipole. Actually, at low radio frequencies, the field radiated from a low horizontal antenna is predominantly vertically polarized and is proportional to the field of a vertical *electric* dipole.

At very low frequencies (less than 30 kc/s) it would be expected that the radiated field of a horizontal antenna is almost purely polarized in the vertical direction. However, if the range is comparable to the wavelength it is not immediately obvious that this situation prevails. It is the purpose of this paper to investigate this general problem. Expressions are derived for the electromagnetic fields of a horizontal electric dipole which is over, on, or below the surface of a homogeneous conducting half-space.

To facilitate discussion of the past work in this field three overlapping distance ranges are considered. These are:

(i) $\rho \gg \lambda_0$,
(ii) $\rho \sim \lambda_0$,
(iii) $\rho \ll \lambda_0$,

where ρ is the horizontal separation between transmitter and receiver and λ_0 is the free-space wavelength.

Case (i), the radiation field, has been treated in detail by Sommerfeld (1926) and Norton (1937) for the case where the dipole and the observer are in the upper half-space (i.e., the air). The extension to a spherical earth has also been developed (Wait 1956). The radiation field of the horizontal dipole immersed in the half-space has been treated by Moore (1951) in a doctoral thesis and by Banös and Wesley (1953). The problem was also discussed by Wait (1953*a*) in a published note.

Case (ii), the near field, has been considered by Norton (1937), who states that his formulas are valid for distances down to a wavelength. His results are

[1]Manuscript received March 23, 1961.
Contribution from the Central Radio Propagation Laboratory, National Bureau of Standards, Boulder, Colo.

Can. J. Phys. Vol. 39 (1961)

restricted to the upper half-space. This intermediate distance region has also been treated by Banös and Wesley (1953) for the dipole in the half-space.

Case (iii), the quasi-static case, has been treated by Foster (1931) and Fock (1933) for the horizontal dipole on the surface and by Lien (1953) and Wait (1959) for the dipole inside the conducting half-space.

In the present paper simple field expressions are obtained for the three distance ranges mentioned above. No attempt was made to adapt specialized results developed by others; rather, self-contained derivations are given for each case. However, some of the methods are very similar to those employed by the author in earlier papers treating the fields of a magnetic dipole in and over a conducting half-space (Wait 1951, 1952, 1953*b*, 1954, 1959; and Wait and Campbell 1953).

FORMAL SOLUTION

The earth is assumed to be a homogeneous medium with conductivity σ_1, dielectric constant ϵ_1. With reference to a Cartesian co-ordinate system (x,y,z) it occupies the half-space $z < 0$. The air with dielectric constant ϵ_0 occupies the space $z > 0$. The magnetic permeability of the whole space is assumed to be constant and equal to μ. The electric dipole, of infinitesimal length ds, is situated at $(0,0,h)$ and is oriented in the x direction.

The formal solution of this problem was given by Sommerfeld (1926). The method he used is now considered to be straightforward. Expressions for the components of the Hertz vector are derived by matching tangential fields at the interface $z = 0$. He showed that for $z > 0$ and $h \geqq 0$,

$$\text{(1)} \qquad \frac{\Pi_x}{C_0} = \frac{e^{-\gamma_0 R_0}}{R_0} - \frac{e^{-\gamma_0 R_1}}{R_1} + 2\int_0^\infty \frac{e^{-u_0(z+h)}}{u_0+u_1} J_0(\lambda\rho)\lambda\, d\lambda,$$

$$\Pi_y = 0,$$

$$\text{(2)} \qquad \frac{\Pi_z}{C_0} = -2\frac{\partial}{\partial x}\int_0^\infty \frac{(u_0-u_1)e^{-u_0(z+h)}}{\gamma_0^2 u_1+\gamma_1^2 u_0} J_0(\lambda\rho)\lambda\, d\lambda,$$

where

$$R_0 = [(z-h)^2+\rho^2]^{\frac{1}{2}}, \qquad R_1 = [(z+h)^2+\rho^2]^{\frac{1}{2}}, \qquad \rho = (x^2+y^2)^{\frac{1}{2}},$$

$$\gamma_0^2 = -\epsilon_0\mu\omega^2, \qquad \gamma_1^2 = i\sigma_1\mu\omega - \epsilon_1\mu\omega^2,$$

$$u_0^2 = \lambda^2+\gamma_0^2, \qquad u_1^2 = \lambda^2+\gamma_1^2, \qquad \text{and } C_0 = Ids/(4\pi i\epsilon_0\omega).$$

Time enters according to the factor $\exp(i\omega t)$.

The fields are then to be obtained from:

$$\text{(3)} \qquad \mathbf{E} = -\gamma_0^2\mathbf{\Pi} + \text{grad div}\,\mathbf{\Pi},$$

$$\text{(4)} \qquad \mathbf{H} = i\epsilon_0\,\omega\,\text{curl}\,\mathbf{\Pi}.$$

When $z < 0$ and $h \geqq 0$ it is found that

(5) $$\frac{\Pi_x}{C_1} = 2\int_0^\infty \frac{e^{uz}e^{-u_0h}}{u_0+u_1} J_0(\lambda\rho)\lambda\, d\lambda,$$

$$\Pi_y = 0,$$

(6) $$\frac{\Pi_z}{C_1} = -2\frac{\partial}{\partial x}\int_0^\infty \frac{(u_0-u_1)e^{-u_0h}e^{uz}}{\gamma_0^2u_1+\gamma_1^2u_0} J_0(\lambda\rho)\lambda\, d\lambda,$$

where $C_1 = Ids/[4\pi(\sigma_1+i\epsilon_1\omega)]$.

FAR-ZONE FIELDS

In the present work it is assumed consistently that $|\gamma_1| \gg |\gamma_0|$. This situation is quite typical for low radio frequencies. Exceptions may occur when the ground is frozen. Because of the largeness of γ_1^2 it is permissible to replace u_1 by γ_1. Thus

(7) $$\frac{\Pi_x}{C_0} \cong \frac{e^{-\gamma_0R_0}}{R_0} - \frac{e^{-\gamma_0R_1}}{R_1} + 2\int_0^\infty \frac{e^{-u_0(z+h)}}{u_0+\gamma_1} J_0(\lambda\rho)\lambda\, d\lambda$$

and

(8) $$\frac{\Pi_z}{C_0} \cong \frac{2}{\gamma_1}\frac{\partial}{\partial x}\int_0^\infty\left(\frac{1}{u_0+\gamma_0^2/\gamma_1} - \frac{1}{u_0+\gamma_1}\right) e^{-u_0(z+h)}\lambda J_0(\lambda\rho)\lambda\, d\lambda.$$

These integrals are of a type which has already been evaluated by Sommerfeld (1926). They may also be treated by a modified saddle-point method (Wait 1957). Therefore, if $k\rho \gg 1$

(9) $$\frac{\Pi_x}{C_0} \cong \frac{e^{-\gamma_0R_0}}{R_0} - \frac{e^{-\gamma_0R_1}}{R_1} + 2[1-(p'/w')^{\frac{1}{2}}(1-F(w'))]\frac{e^{-\gamma_0R_1}}{R_1}$$

where

$$p' = -(ikR_1/2)(\gamma_1/\gamma_0)^2,$$

$$w' = p'\left(1+\frac{z+h}{R_1}\frac{\gamma_0}{\gamma_1}\right)^2,$$

and

$$F(w') \cong -\frac{1}{2w'}.$$

Similarly,

(10) $$\frac{\Pi_z}{C_0} \cong \frac{2}{\gamma_1}\frac{\partial}{\partial x}[1-i(\pi p)^{\frac{1}{2}}e^{-w}\operatorname{erfc}(iw^{\frac{1}{2}})]\frac{e^{-\gamma_0R_1}}{R_1}$$

where

$$p = -(ikR_1/2)(\gamma_0/\gamma_1)^2,$$

and

$$w = p\left(1+\frac{z+h}{R_1}\cdot\frac{\gamma_1}{\gamma_0}\right)^2.$$

It is immediately evident that the fields derived from Π_x are very similar to those of a vertical magnetic dipole whereas the fields derived from Π_z are related to those of a vertical electric dipole. This is particularly evident when these expressions for Π_x and Π_z are written in the form

$$\frac{\Pi_x}{C_0} \cong 2F(p')(1+\gamma_1 z)(1+\gamma_1 h)e^{-\gamma_0\rho}/\rho \tag{11}$$

$$\cong [2\,\gamma_0/(\gamma_1\rho)^2](1+\gamma_1 z)(1+\gamma_1 h)e^{-\gamma_0\rho},$$

and

$$\frac{\Pi_z}{C_0} \cong -[2\,\gamma_0/(\gamma_1\rho)]F(p)(\cos\phi)(1+\gamma_0^2 z/\gamma_1)(1+\gamma_0^2 h/\gamma_1)e^{-\gamma_0\rho}, \tag{12}$$

which are valid for small heights (i.e., z^2 and $h^2 \ll \rho^2$). It is seen that the factors which depend on height have the form $(1+ik\Delta z)$ in which $\Delta = \gamma_1/\gamma_0$ or $\Delta = \gamma_0/\gamma_1$.

Using equations (11) and (12) it is a relatively straightforward matter to obtain the following expressions for the fields:

$$E_\rho \cong -\frac{i\mu\omega}{\gamma_1}H_\phi \cong \frac{i\mu\omega Ids}{2\pi\rho}\frac{\gamma_0^2}{\gamma_1^2}F(p)e^{-\gamma_0\rho}(\cos\phi), \qquad \text{for } z = h = 0, \tag{13}$$

$$E_\phi \cong \frac{i\mu\omega}{\gamma_1}H_\rho \cong \frac{i\mu\omega Ids}{\pi\rho^2}\frac{\gamma_0}{\gamma_1^2}e^{-\gamma_0\rho}(\sin\phi), \qquad \text{for } z = h = 0, \tag{14}$$

$$E_z \cong \frac{i\mu\omega Ids}{2\pi\rho}\frac{\gamma_0}{\gamma_1}F(p)e^{-\gamma_0\rho}(\cos\phi)G_V, \qquad \text{for } z \text{ and } h \geqq 0, \tag{15}$$

$$H_z \cong \frac{Ids}{2\pi\rho^2}\frac{\gamma_0^2}{\gamma_1^2}e^{-\gamma_0\rho}(\sin\phi)G_H, \qquad \text{for } z \text{ and } h \geqq 0, \tag{16}$$

where

$$F(p) = 1-i(\pi p)^{\frac{1}{2}}e^{-p}\operatorname{erfc}(ip^{\frac{1}{2}}),$$

$$G_V = (1+\gamma_0^2 z/\gamma_1)(1+\gamma_0^2 h/\gamma_1),$$

and

$$G_H = (1+\gamma_1 z)(1+\gamma_1 h).$$

Equations (13) to (16) are valid subject to the following approximations:

(i) $|\gamma_0\rho| = k\rho \gg 1$ (i.e., far-field),
(ii) h^2 and $z^2 \ll \rho^2$ (i.e., small heights),
(iii) $|\gamma_1| \gg |\gamma_0|$ (i.e., highly conducting ground).

NEAR-ZONE FIELDS

The use of the saddle-point method to reduce the exact integral forms for Π_x and Π_z to the forms given by equations (9) and (10) requires that $k\rho \gg 1$. When $k\rho$ becomes comparable with unity it is necessary to use a different tack.

In the case $h = z = 0$ it is shown in the Appendix that

$$\frac{\Pi_x}{C_0} = \frac{2}{(\gamma_1^2-\gamma_0^2)\rho^3}[(1+\gamma_0\rho)e^{-\gamma_0\rho}-(1+\gamma_1\rho)e^{-\gamma_1\rho}]. \tag{17}$$

This is an exact result. Now under the restriction that $|\gamma_1\rho| \gg 1$, this simplifies to

$$\frac{\Pi_x}{C_0} \cong \frac{2}{\gamma_1^2\rho^3}(1+\gamma_0\rho)e^{-\gamma_0\rho}. \tag{18}$$

For $|\gamma_0\rho| \gg 1$, this agrees with equation (11). Furthermore, since $|\gamma_1^2| \gg |\gamma_0^2|$ it is reasonable to assume that the factor $\gamma_1^2 u_0 + \gamma_0^2 u_1$ in the exact integral form for Π_z can be replaced by $\gamma_1^2 u_0$ for the near zone. Thus

$$\begin{aligned}\frac{\Pi_z}{C_0} &\cong \frac{2}{\gamma_1}\frac{\partial}{\partial x}\int_0^\infty u_0^{-1}e^{-u_0(z+h)}J_0(\lambda\rho)d\lambda \\ &\cong \frac{2}{\gamma_1}\frac{\partial}{\partial x}(e^{-\gamma_0 R_1}/R_1).\end{aligned} \tag{19}$$

For $z = h = 0$, this becomes

$$\frac{\Pi_z}{C_0} \cong -(2/\gamma_1\rho^2)(1+\gamma_0\rho)e^{-\gamma_0\rho}\cos\phi. \tag{20}$$

When $|\gamma_0\rho| \gg 1$ and *provided* $p \ll 1$, this agrees with equation (12).

The corresponding forms for Π_x and Π_z when z and $h > 0$ are

$$\frac{\Pi_x}{C_0} \cong (2/\gamma_1^2\rho^3)(1+\gamma_0\rho)e^{-\gamma_0\rho}G_H, \tag{21}$$

$$\frac{\Pi_z}{C_0} \cong -(2/\gamma_1\rho^2)(1+\gamma_0\rho)e^{-\gamma_0\rho}(\cos\phi)G_V. \tag{22}$$

The resulting field expressions are then given by

$$E_\rho \cong -\frac{i\mu\omega}{\gamma_1}H_\phi \cong \frac{i\mu\omega I ds}{2\pi\rho^3\gamma_1^2}(1+\gamma_0\rho+\gamma_0^2\rho^2)e^{-\gamma_0\rho}(\cos\phi) \qquad \text{for } z = h = 0, \tag{23}$$

$$E_\phi \cong \frac{i\mu\omega}{\gamma_1}H_\rho \cong \frac{i\mu\omega I ds}{\pi\rho^3\gamma_1^2}(1+\gamma_0\rho)e^{-\gamma_0\rho}(\sin\phi), \qquad \text{for } z = h = 0, \tag{24}$$

$$E_z \cong \frac{i\mu\omega I ds}{2\pi\rho^2\gamma_1}(1+\gamma_0\rho)e^{-\gamma_0\rho}(\cos\phi)G_V, \qquad \text{for } z \text{ and } h \geqq 0, \tag{25}$$

$$H_z \cong \frac{I ds}{2\pi\rho^4\gamma_1^2}(3+3\gamma_0\rho+\gamma_0^2\rho^2)e^{-\gamma_0\rho}(\sin\phi)G_H, \qquad \text{for } z \text{ and } h \geqq 0. \tag{26}$$

These expressions are valid subject to the following approximations:

(i) $|p| \ll 1$, (i.e., small 'numerical' distances)
(ii) h^2 and $z^2 \ll \rho^2$,
(iii) $|\gamma_1| \gg |\gamma_0|$.

It is seen that the range of validity of equations (13) to (16) and of equations (23) to (26) overlap when, simultaneously, $|p| \ll 1$ and $|\gamma_0\rho| \gg 1$.

The preceding discussion has been restricted to the case where the observer is in the upper half-space. The analysis for the situation where the observer is in the lower half-space (i.e., $z < 0$) is very similar. The exact integral expressions for Π_x and Π_z then contain a factor exp $(u_1 z)$ in the integrand. If $|\gamma_1| \gg |\gamma_0|$ and $|\gamma_1\rho| \gg 1$, this factor may be approximated by exp $(\gamma_1 z)$ and thus taken outside the integral. The resulting expressions for the fields are identical with equations (23) to (26) for the field components E_ρ, H_ϕ, E_ϕ, H_ρ, and H_z but now G_V and G_H are defined by

$$G_V \cong (1+\gamma_0^2 h/\gamma_1)\exp(\gamma_1 z), \tag{27}$$

$$G_H \cong (1+\gamma_1 h)\exp(\gamma_1 z), \tag{28}$$

and (23) and (24) have a factor $\exp(\gamma_1 z)$ on the right-hand sides. The E_z field component is not continuous at the interface. In fact,

$$E_z(z \to 0-) \cong (\gamma_0^2/\gamma_1^2)E_z(z \to 0+),$$

which indicates that equations (15) and (25) for E_z are applicable to $z < 0$ if G_V is replaced by G_V^1 where

$$G_V^1 = (1+\gamma_0^2 h/\gamma_1)(\gamma_0^2/\gamma_1^2)\exp(\gamma_1 z). \tag{29}$$

The other obvious extension is to the case where the horizontal dipole is lowered into the conducting half-space (i.e., $h < 0$). A factor exp $(u_1 h)$ is now contained within the integrand of the exact integral expressions. This can be approximated by exp $(\gamma_1 z)$ subject again to the limitations that $|\gamma_1\rho| \gg 1$ and $|\gamma_1^2| \gg |\gamma_0^2|$. For $z > 0$, the resulting expressions for the fields are again given by equations (13) to (16) and (23) to (26) if

$$G_V \cong \exp(\gamma_1 h)(1+\gamma_0^2 z/\gamma_1), \tag{30}$$

$$G_H \cong \exp(\gamma_1 h)(1+\gamma_1 z), \tag{31}$$

and (23) and (24) for $z = 0$ are now multiplied by exp $(\gamma_1 h)$.

For $z < 0$

$$G_V \cong \exp(\gamma_1 h)\exp(\gamma_1 z) \cong G_H \tag{32}$$

except for the E_z component which requires that

$$G_V \cong (\gamma_0^2/\gamma_1^2)\exp(\gamma_1 h)\exp(\gamma_1 z). \tag{33}$$

Furthermore, (23) and (24) are still valid provided the right-hand sides are multiplied by exp $(\gamma_1 h)$ exp $(\gamma_1 z)$.

QUASI-STATIC APPROACH

At very short distances where the condition $|\gamma_1\rho| \gg 1$ becomes violated it is necessary to use a quasi-static treatment. If $|\gamma_0\rho| \ll 1$ it is permissible to regard the fields in the upper insulating half-space as solutions of Laplace's equation. This is equivalent to setting γ_0 equal to zero in the exact integral expressions for Π_x and Π_z. For $z < 0$ and $h \leqq 0$ these may be written

$$\Pi_x = \frac{Ids}{4\pi(\sigma+i\omega\epsilon)}\left[\frac{e^{-\gamma R_0}}{R_0}-\frac{e^{-\gamma R_1}}{R_1}+2\int_0^\infty \frac{e^{u(z+h)}}{u+\lambda}J_0(\lambda\rho)\lambda\,d\lambda\right] \tag{34}$$

and

$$\Pi_z = \frac{Ids}{2\pi(\sigma+i\omega\epsilon)}\frac{\partial}{\partial x}\int_0^\infty \frac{e^{u(z+h)}}{u+\lambda}J_0(\lambda\rho)\,d\lambda \tag{35}$$

where $\gamma^2 = \gamma_1^2 = i\mu\omega(\sigma+i\epsilon\omega)$ and $u = u_1 = (\lambda^2+\gamma^2)^{\frac{1}{2}}$. (The subscript 1 is dropped for the sake of brevity in the remainder of this section.)

As z and h approach zero (from negative values) it easily follows, from equation (17), that

$$\Pi_x = \frac{Ids}{2\pi(\sigma+i\omega\epsilon)}\frac{1}{\gamma^2\rho^3}[1-(1+\gamma\rho)e^{-\gamma\rho}]. \tag{36}$$

The integral involved in Π_z is not quite so simple. However, it may be reduced as follows

$$\int_0^\infty \frac{J_0(\lambda\rho)}{u+\lambda}d\lambda = \frac{1}{\gamma^2}\int_0^\infty (u-\lambda)J_0(\lambda\rho)\,d\lambda. \tag{37}$$

Since

$$\int_0^\infty \lambda J_0(\lambda\rho)d\lambda = 0$$

and using

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\lambda^2\right)J_0(\lambda\rho) = 0,$$

it is seen that

$$\gamma^2\int_0^\infty \frac{J_0(\lambda\rho)}{u+\lambda}d\lambda = \left(\gamma^2-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\int_0^\infty \frac{J_0(\lambda\rho)}{u}d\lambda. \tag{38}$$

According to Watson (1944) the integral on the right-hand side is equal to $I_0(\gamma\rho/2)K_0(\gamma\rho/2)$ where I_0 and K_0 are modified Bessel functions of the first and second kind, both with argument $\gamma\rho/2$. Thus, for $z = h = 0-$

$$\Pi_z = \frac{Ids\cos\phi}{2\pi(\sigma+i\epsilon\omega)\gamma^2}\frac{\partial}{\partial\rho}\left(\gamma^2-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)[I_0(\gamma\rho/2)K_0(\gamma\rho/2)]. \tag{39}$$

In a similar fashion it is found that, for $z = h = 0-$,

$$\frac{\partial\Pi_x}{\partial z} = -\frac{Ids}{2\pi(\sigma+i\epsilon\omega)\gamma^2}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\gamma^2\right)[I_0(\gamma\rho/2)K_0(\gamma\rho/2)] \tag{40}$$

and

$$\operatorname{div}\mathbf{\Pi} = -\frac{Ids\cos\phi}{2\pi(\sigma+i\epsilon\omega)\rho^2}. \tag{41}$$

The field components in the interface are then obtained from

$$(42a)\qquad E_\rho = -\gamma^2\Pi_x \cos\phi + \frac{\partial}{\partial\rho}\operatorname{div}\mathbf{\Pi},$$

$$(42b)\qquad E_\phi = \gamma^2\Pi_x \sin\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}\operatorname{div}\mathbf{\Pi},$$

$$(42c)\qquad H_\rho = (\sigma+i\epsilon\omega)\left[\frac{\partial\Pi_z}{\rho\partial\phi}+\frac{\partial\Pi_x}{\partial z}\sin\phi\right],$$

$$(43a)\qquad H_\phi = (\sigma+i\epsilon\omega)\left[\frac{\partial\Pi_x}{\partial z}\cos\phi-\frac{\partial\Pi_z}{\partial\rho}\right],$$

$$(43b)\qquad H_z = -(\sigma+i\epsilon\omega)\frac{\partial\Pi_x}{\partial\rho}\sin\phi.$$

Within the limits of quasi-static theory E_z vanishes inside the conducting medium for $h = 0$ and z tending to zero from negative values (i.e., $z \to 0-$). On the other hand, if $h = 0$ and z tends to zero from the positive values (i.e., $z \to 0+$) it is found that

$$(44)\qquad E_z = -\frac{i\mu\omega Ids}{2\pi}\frac{\partial}{\partial x}\int_0^\infty \frac{J_0(\lambda\rho)}{u+\lambda}\,d\lambda$$

$$(45)\qquad = -\frac{i\mu\omega Ids}{2\pi\gamma^2}\cos\phi\,\frac{\partial}{\partial\rho}\left(\gamma^2-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\left[I_0\left(\frac{\gamma\rho}{2}\right)K_0\left(\frac{\gamma\rho}{2}\right)\right].$$

In the limiting case where $|\gamma\rho| \gg 1$ it is seen that

$$(46)\qquad E_z \cong \frac{i\mu\omega Ids}{2\pi\gamma\rho^2}\cos\phi.$$

The field components E_ρ, E_ϕ, and H_ϕ in this quasi-static approximation may then be written concisely in the form

$$(47)\qquad E_\rho = \frac{Ids}{2\pi(\sigma+i\epsilon\omega)\rho^3}\left[1+(1+\gamma\rho)e^{-\gamma\rho}\right]\cos\phi,$$

$$(48)\qquad E_\phi = \frac{Ids}{2\pi(\sigma+i\epsilon\omega)\rho^3}\left[2-(1+\gamma\rho)e^{-\gamma\rho}\right]\sin\phi,$$

$$(49)\qquad H_z = \frac{Ids}{2\pi\gamma^2\rho^4}\left[3-(3+3\gamma\rho+\gamma^2\rho^2)e^{-\gamma\rho}\right]\sin\phi.$$

These results are valid for $z = h = 0$ and $|\gamma_0\rho| \ll 1$. The value of $\gamma\rho$ is unrestricted. In particular, if $|\gamma\rho|$ approaches zero the above reduce to

$$(50)\qquad E_\rho = \frac{Ids}{\pi(\sigma+i\epsilon\omega)\rho^3}\cos\phi,$$

$$(51)\qquad E_\phi = \frac{Ids}{2\pi(\sigma+i\epsilon\omega)\rho^3}\sin\phi,$$

$$(52)\qquad H_z = \frac{Ids}{4\pi\rho^2}\sin\phi.$$

These elementary forms are consistent with potential theory.

Another interesting check is to note that if, simultaneously, $|\gamma_0\rho| \ll 1$ and $|\gamma\rho| \gg 1$, equations (47), (48), and (49) agree with (23), (24), and (26) for $z = h = 0$.

Apparently the field components H_ρ and H_ϕ for $z = h = 0$ cannot be written in a concise form. The higher derivatives of I_0K_0 can be reduced to a linear combination of the functions I_0K_0, I_0K_1, I_1K_0, and I_1K_1 where I_1 and K_1 are modified Bessel functions of order unity. These forms will not be written out. However, it is of interest to note that if $|\gamma\rho| \gg 1$ each of these products can be replaced by $1/(\gamma\rho)$ where the error is of order $1/(\gamma\rho)^2$. To this approximation

$$H_\rho \cong \frac{Ids \sin \phi}{\pi\gamma\rho^3} \cong \frac{\gamma}{i\mu\omega} E_\phi, \tag{53}$$

$$H_\phi \cong -\frac{Ids \cos \phi}{2\pi\gamma\rho^3} \cong -\frac{\gamma}{i\mu\omega} E_\rho. \tag{54}$$

The previous discussion of the quasi-static case was restricted to the case $z = h = 0$. The manner in which the fields vary with z or h in this range (i.e., $|\gamma_0\rho| \ll 1$) is quite complicated. For example, the fields do not generally vary in a simple exponential manner with depth into the conductor. Nevertheless, for the case z and $h < 0$, it is possible to express the fields in terms of modified Bessel functions. Setting $\gamma_0 = 0$ it is found that

$$\Pi_x = \frac{Ids}{4\pi(\sigma+i\epsilon\omega)}\left[\frac{e^{-\gamma R_0}}{R_0} - P + \frac{2}{\gamma^2}\left(\frac{\partial^2 P}{\partial z^2} - \frac{\partial^3 N}{\partial z^3} + \gamma^2\frac{\partial N}{\partial z}\right)\right] \tag{55}$$

and

$$\Pi_z = \frac{Ids}{2\pi(\sigma+i\epsilon\omega)}\left(\frac{\partial^3 N}{\partial x\partial z^2} - \frac{\partial^2 P}{\partial x \partial z}\right) \tag{56}$$

where

$$N = \int_0^\infty \frac{e^{u(z+h)}}{u} J_0(\lambda\rho)d\lambda, \tag{57}$$

$$P = \int_0^\infty \frac{e^{u(z+h)}}{u} J_0(\lambda\rho)\lambda\, d\lambda = \frac{e^{-\gamma R_1}}{R_1}. \tag{58}$$

The form of equations (55) and (56) may be readily verified by carrying out the indicating differentiations and comparing the results with equations (34) and (35).

The integral N is known. From Foster (1931) or Magnus and Oberhettinger (1954)

$$N = I_0[(\gamma/2)[R_1+(z+h)]]K_0[(\gamma/2)[R_1-(z+h)]]. \tag{59}$$

This result can be checked by noting that equation (59) reduces to the integral in equation (38) when $z+h = 0$, and by seeing that N satisfies the wave equation $(\nabla^2-\gamma^2)N = 0$ when z and $h < 0$.

The electric field in the conducting half-space may now be found from

$$\mathbf{E} = -\gamma^2\mathbf{\Pi}+\text{grad div }\mathbf{\Pi}.$$

Expressing these in Cartesian co-ordinates it is found, after some simplification, that

$$(60)\quad E_x = \frac{Ids}{4\pi(\sigma+i\epsilon\omega)}\left[-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{e^{-\gamma R_0}}{R_0}-P\right)-2\left(\frac{\partial^3 N}{\partial y^2\partial z}+\frac{\partial^2 P}{\partial z^2}\right)\right],$$

$$(61)\quad E_y = \frac{Ids}{4\pi(\sigma+i\epsilon\omega)}\left[\frac{\partial^2}{\partial x\partial y}\left(\frac{e^{-\gamma R_0}}{R_0}-P\right)+2\frac{\partial^3 N}{\partial x\partial y\partial z}\right],$$

$$(62)\quad E_z = \frac{Ids}{4\pi(\sigma+i\epsilon\omega)}\left[-\frac{\partial^2}{\partial x\partial z}\left(\frac{e^{-\gamma R_0}}{R_0}+P\right)\right].$$

As a check on these results it may be noted that as $|\gamma R_0|$ and $|\gamma R_1|\to 0$, the electric field components can be expressed as gradients of a scalar potential. Thus

$$(63)\qquad \mathbf{E} = \frac{Ids}{4\pi(\sigma+i\epsilon\omega)}\,\text{grad}\left[\frac{\partial}{\partial x}\left(\frac{1}{R_0}+\frac{1}{R_1}\right)\right].$$

This result is consistent with potential theory.

Another interesting limiting case is when $|\gamma\rho|\gg 1$ and $z+h\ll\rho$. Under these conditions the Bessel functions I_0 and K_0 may be replaced by the first terms in their respective asymptotic expansions. This leads to the asymptotic approximations

$$(64)\qquad N\cong(1/\gamma\rho)\exp(\gamma(z+h))$$

and

$$\partial N/\partial z\cong(1/\rho)\exp(\gamma(z+h)).$$

The field components are then given by

$$(65)\qquad E_x\cong-\frac{Ids}{2\pi(\sigma+i\epsilon\omega)}e^{\gamma(z+h)}\frac{\partial^2}{\partial y^2}\left(\frac{1}{\rho}\right),$$

$$(66)\qquad E_y\cong\frac{Ids}{2\pi(\sigma+i\epsilon\omega)}e^{\gamma(z+h)}\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{\rho}\right).$$

Expressing these in cylindrical co-ordinates

$$(67)\qquad E_\rho\cong\frac{Ids\cos\phi}{2\pi(\sigma+i\epsilon\omega)\rho^3}e^{\gamma(z+h)},$$

$$(68)\qquad E_\phi\cong\frac{Ids\sin\phi}{\pi(\sigma+i\epsilon\omega)\rho^3}e^{\gamma(z+h)}.$$

These are in agreement with equations (23) and (24) under the conditions that $|\gamma_0\rho|\ll 1$, $|\gamma\rho|\gg 1$, and z and $h\leq 0$.

The manner in which the exponential factor $\exp(\gamma(z+h))$ occurs is rather interesting. It is only in the integral N that this factor emerges. The integral P, within the approximation that $|\gamma\rho|\gg 1$, gives a negligible contribution.

This exponential factor is usually obtained by somewhat heuristic arguments. In the quasi-static treatment it is established on a sound basis and the limits of its validity are given.

ACKNOWLEDGMENTS

I would like to thank Mrs. Eileen Brackett for her assistance in preparing the manuscript and Mr. Kenneth Spies for his careful reading of the paper. I also wish to thank F. J. Zucker for suggesting the problem. The research was sponsored by the U.S. Air Force Cambridge Research Laboratories under contract A58-40.

REFERENCES

BANÖS, A. and WESLEY, J. P. 1953. Report from Marine Physical Laboratory, University of California.
FOCK, V. A. 1933. Ann. phys. **17**, 401.
FOSTER, R. M. 1931. Bell System Tech. J. **10**, 408.
LIEN, R. H. 1953. J. Appl. Phys. **24**, 1.
MAGNUS, W. and OBERHETTINGER, F. 1954. Functions of mathematical physics (Chelsea Publ. Co. New York), p. 133.
MOORE, R. K. 1951. Ph.D. Thesis, Cornell University.
NORTON, K. A. 1937. Proc. I.R.E. **25**, 1203.
SOMMERFELD, A. N. 1926. Ann. Physik, **81**, 1135.
WAIT, J. R. 1951. Can. J. Phys. **29**, 577.
——— 1952. J. Appl. Phys. **23**, 497.
——— 1953*a*. J. Appl. Phys. **24**, 958 (in equation (6) V should be replaced by U).
——— 1953*b*. Trans. Am. Geophys. Union, **34**, 185.
——— 1954. Can. J. Technol. **32**, 1.
——— 1956. Can. J. Phys. **34**, 586.
——— 1957. J. Research Natl. Bur. Standards, **59**, 365.
——— 1959. Appl. Sci. Research, **B8**, 213.
WAIT, J. R. and CAMPBELL, L. L. 1953. J. Geophys. Research, **58**, 167.
WATSON, G. N. 1944. Bessel functions, 2nd ed. (Cambridge University Press).

APPENDIX

It is of interest to observe that the vertical component of the magnetic field may be expressed in closed form. The first step is to note that Π_z evaluated at $h = 0$ and $z = 0+$ is given by

$$\frac{\Pi_z}{C_0} = 2\int_0^\infty \frac{J_0(\lambda\rho)}{u_0+u_1}\lambda\, d\lambda. \tag{69}$$

Multiplying numerator and denominator by u_1-u_0 it follows that

$$\frac{\Pi_z}{C_0} = \frac{2}{\gamma_1^2-\gamma_0^2}\left[\int_0^\infty \lambda u_1 J_0(\lambda\rho)d\lambda - \int_0^\infty \lambda u_0 J_0(\lambda\rho)d\lambda\right]. \tag{70}$$

From Watson (1944)

$$\int_0^\infty u_1^{-1}\lambda J_0(\lambda\rho)\, d\lambda = \rho^{-1}e^{-\gamma_1\rho} \tag{71}$$

and using

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\lambda^2\right) J_0(\lambda\rho) = 0 \tag{72}$$

it follows that

$$\frac{\Pi_x}{C_0} = \frac{2}{\gamma_1^2-\gamma_0^2}\left[\left(\gamma_1^2-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\frac{e^{-\gamma_1\rho}}{\rho}-\left(\gamma_0^2-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\frac{e^{-\gamma_0\rho}}{\rho}\right] \tag{73}$$

$$= \frac{-2}{(\gamma_1^2-\gamma_0^2)\rho^3}\left[(1+\gamma_1\rho)e^{-\gamma_1\rho}-(1+\gamma_0\rho)e^{-\gamma_0\rho}\right].$$

Then

$$H_z = -(i\epsilon_0\omega)\frac{\partial\Pi_x}{\partial\rho}\sin\phi \tag{74}$$

$$= \frac{Ids\sin\phi}{2\pi(\gamma_1^2-\gamma_0^2)\rho^4}\left[(3+3\gamma_0\rho+\gamma_0^2\rho^2)e^{-\gamma_0\rho}-(3+3\gamma_1\rho+\gamma_1^2\rho^2)e^{-\gamma_1\rho}\right].$$

This is an exact result valid when $z = h = 0$. In the limiting case for $|\gamma_1\rho| \gg 1$, it agrees with equation (26), and if $|\gamma_0\rho| \ll 1$, it agrees with equation (49). It should be noted that strictly the integrals in equations (70) and (71) are divergent. The operations performed, using these relations, can be made rigorous by simply replacing

$$\int_0^\infty [\ldots]\,d\lambda \text{ by } \lim_{\alpha\to 0}\int_0^\infty e^{-\lambda\alpha}[\ldots]\,d\lambda.$$