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Real axis integration of Sommerfeld integrals with applications to printed circuit antennas[a)]

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Printed circuit antennas are becoming an integral part of imaging arrays in microwave, millimeter, and submillimeter wave frequencies. The electrical characteristics of such antennas can be analyzed by solving integral equations of the Fredholm first kind. The kernel involves Sommerfeld integrals which are particularly difficult to solve when source and field points lie on an electrical discontinuity, as it occurs in the determination of the characteristics of printed circuit antennas. An analytic-numeric real axis integration technique has been developed for such integrals and it is combined with piece-wise sinusoidal expansions to solve the Fredholm integral equation for the unknown current density.

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I. INTRODUCTION

When sources of electromagnetic radiation are in the proximity of or on the interface of electrically dissimilar materials, the resulting electromagnetic fields involve Sommerfeld type integrals.[1–8] If the source is near the interface, these integrals include exponentially decaying terms in the integrand which facilitate convergence.[5–8] On the other hand whenever the source and observation points are on the interface these integrals converge very slowly.

The recent evolution of printed circuit antennas[2–4, 9–14] with applications in microwave, millimeter and submillimeter wave frequencies, has necessitated the development of special techniques to solve the resulting Sommerfeld type integrals. The printed circuit antenna is an extended source typically printed on a grounded dielectric substrate. In order to evaluate the electrical properties of such antennas, the current distribution must be obtained by solving Pocklington's integral equation, a Fredholm integral equation of the first kind.[15] In the process of solving for the antenna current distribution, the field and source points lie on the interface between vacuum and the substrate. This renders the resulting Sommerfeld-type integrals very slowly convergent.

In this paper special techniques are developed to address this case. This is done in the effort to obtain the current distribution of dipoles printed on a substrate, as compared to previous approaches where the antenna current was assumed to be uniform and the antennas were chosen to be off the interface.[5–8] The integral equation for the unknown current distribution is reduced into a system of linear equations by applying the method of moments.[16] The solution of the resulting matrix equation leads to the evaluation of integrals which are of the form

$$I = \int_0^\infty \mathscr{L}\{J_0(\lambda\rho)\} f(\lambda)\, d\lambda ,$$

where $\mathscr{L}$ is a self-adjoint operator which involves integration with respect to ρ, $J_0(\lambda\rho)$ is the zeroth order Bessel function of the first kind and $f(\lambda)$ is given by $f(\lambda) = Q(\lambda)/P(\lambda)$ where $Q(\lambda)$ and $P(\lambda)$ are expressions involving transcendental functions. The semi-infinite path of integration along the real axis is divided in four subintervals, with the integrand being treated differently in each one. The fourth subinterval extends from a constant $A > 1$ to infinity, wherein the integrand is a fast varying function and converges very slowly.

The difficulties which are encountered within the fourth subinterval are resolved by approximating the integrand with controlled error and by applying a combination of analytical and numerical evaluation of the resulting integral. This procedure introduces an error which is of the order 10^{-4} of the correct value of the integral.

II. REDUCTION OF INTEGRAL EQUATION TO A SYSTEM OF LINEAR EQUATIONS

The integral equation resulting from the solution of the boundary value problem (see Fig. 1) gives an expression for the radiated electric field in the form

$$\vec{E}(\vec{r}) = \int_L [k^2\bar{\bar{I}} + \vec{\nabla}\vec{\nabla}]\cdot\bar{\bar{G}}(\vec{r},\vec{r}')\cdot\vec{J}(\vec{r}')dr' , \tag{2.1}$$

where $\bar{\bar{G}}(\vec{r},\vec{r}')$ is the Green's function of the problem given by

$$\bar{\bar{G}}(\vec{r},\vec{r}') = \int_0^\infty J_0(\lambda|\vec{r}-\vec{r}'|)\bar{\bar{F}}(|\vec{r}-\vec{r}'|,\lambda)\, d\lambda \tag{2.2}$$

with

$$\bar{\bar{F}}(|\vec{r}-\vec{r}'|,\lambda) = f_x(|\vec{r}-\vec{r}'|\lambda)\hat{x}\hat{x} + f_y(|\vec{r}-r'|\lambda)\hat{z}\hat{x} , \tag{2.3}$$

while $\vec{J}(\vec{r}')$ is the unknown current distribution on the an-

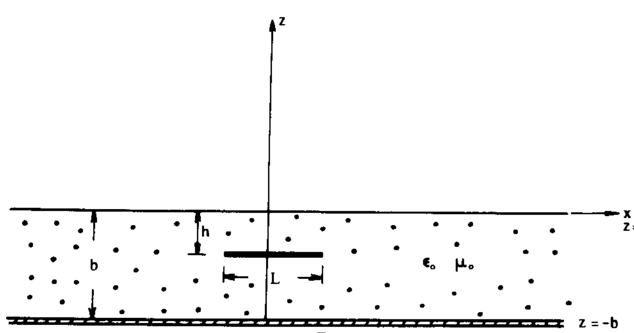

FIG. 1. Horizontal electric dipole embedded in a grounded dielectric slab.

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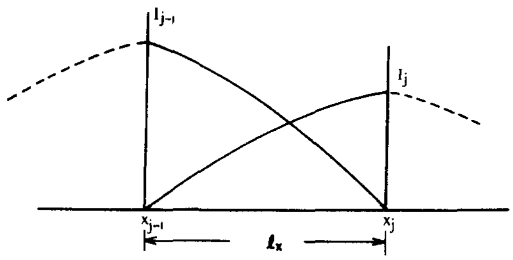

FIG. 2. Piecewise-sinusoidal currents on a wire segment.

tenna. The reduction of the integral equation to a matrix equation requires the expansion of the current distribution in a finite series of basis functions $\phi_i(|\vec{r}'|,x_i)$ as follows

$$\vec{J}(\vec{r}') = \sum_{i=1}^{N} I_i \phi_i(|\vec{r}'|,x_i), \tag{2.4}$$

where I_i are unknown constants to be determined. For the problem of interest the most appropriate set of basis functions has been found to be in the form (see Fig. 2)

$$\phi_i(|\vec{r}'|,x_i) = \left\{ P_{i-1} \frac{\sin[k(x' - x_{i-1})]}{\sin kl_x} + P_i \frac{\sin[k(x_{i+1} - x')]}{\sin kl_x} \right\}, \tag{2.5}$$

where $\vec{r}' = x'\hat{x}$ and

$$P_{i-1} = \begin{Bmatrix} 1, & x_{i-1} \leqslant x' \leqslant x_i \\ 0 & \text{elsewhere} \end{Bmatrix}. \tag{2.6}$$

By combining Eqs. (2.1), (2.2) and (2.4) the radiated field can be written in the form

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} I_i \int_0^\infty \bar{\mathscr{L}}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} \cdot \hat{x} d\lambda, \tag{2.7}$$

where

$$\bar{\mathscr{L}}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} = \int_L [k^2 \bar{I} + \vec{\nabla}\vec{\nabla}] \cdot \bar{F} J_0(\lambda|\vec{r} - \vec{r}'|) dr'. \tag{2.8}$$

The integral equation given by (2.7) is solved by Galerkin's method with a set of weighting functions $\vec{w}_m$ being identical to the basis function, i.e.,

$$\{\vec{w}_m\} = \phi_m(|\vec{r}|,x_m), \tag{2.9}$$

subsequently, an inner product is formulated as follows

$$\langle \vec{w}_m, \vec{a} \rangle = \int_L dr(\vec{w}_m \cdot \vec{a}) \text{ with } \vec{r} = x\hat{x}$$

$$\sum_{i=1}^{N} I_i \int_0^\infty \langle \vec{w}_k, \bar{\mathscr{L}}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} \cdot \hat{x} \rangle d\lambda = \langle \vec{w}_k, \vec{E}(\vec{r}) \rangle \tag{2.10}$$

for the unknown coefficients I_i. By noting that

$$\begin{aligned} &\langle \vec{w}_k, \bar{\mathscr{L}}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} \cdot \hat{x} \rangle \\ &= \langle \phi_k, \hat{x} \cdot \bar{\mathscr{L}}_i \cdot \hat{x} \rangle \\ &= \langle \phi_k, \mathscr{L}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} f(\lambda) \rangle \end{aligned} \tag{2.11}$$

with

$$\mathscr{L}_i \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} = \int_L \phi_i(\vec{r}',x_i) J_0(\lambda|\vec{r} - \vec{r}'|) dr', \tag{2.12}$$

Eq. (2.10) can be written as

$$[I_i] \cdot [Z_{ik}] = [V_k], \tag{2.13}$$

where V_k is the voltage excitation vector given by

$$V_k = \langle \vec{w}_k, \vec{E}(\vec{r}) \rangle = \begin{cases} 1 & \text{feedpoint at } x_k, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.14}$$

and

$$Z_{ik} = \int_0^\infty \mathscr{L}_{ik} \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} f(\lambda) d\lambda. \tag{2.15}$$

Here $\mathscr{L}_{ik}\{\ \}$ is the finite double-integral operator

$$\mathscr{L}_{ik}\{\ \} = \int_L dx \int_L dx' \, \Phi_k(x,x_k) \Phi_i(x',x_i) \{\ \}, \tag{2.16}$$

where

$$\begin{aligned} \Phi_i(x',x_i) = [&\phi_i(x',x_i) + S_{i-1}\delta(x' - x_{i-1}) \\ &+ S_i\delta(x' - x_i) + S_{i+1}\delta(x' - x_{i+1})] \end{aligned} \tag{2.17}$$

and S_{i-1}, S_i, S_{i+1} are independent of x'.

In this manner the integral equation is reduced to a system of linear equations which can be solved for the unknown coefficients I_i. The matrix inversion requires a small fraction of the total computation time, while a considerable part of the computational effort is spent on the evaluation of the elements of the matrix Z.

III. EVALUATION OF THE INTEGRAL IN THE GENERALIZED IMPEDANCE MATRIX

A. Singular points and related surface waves

It can be shown analytically that the matrix elements in the matrix equation are given by

$$Z_{ik} = \int_0^\infty \mathscr{L}_{ik} \{ J_0(\lambda|\vec{r} - \vec{r}'|) \} \left[\frac{A(\lambda,b)}{f_1(\lambda,b) f_2(\lambda,b)} \right] d\lambda, \tag{3.1}$$

where

$$f_1(\lambda,b) = u_0 \sinh(ub) + u \cosh(ub), \tag{3.2}$$

$$f_2(\lambda,b) = \epsilon_r u_0 \cosh(ub) + u \sinh(ub), \tag{3.3}$$

$$\begin{aligned} A(\lambda,b) = &[C_1 \sinh[u(b-h)] + C_2 \sinh(ub)] \\ &\times [D_1 u_0 \cosh(ub) + D_2 u \sinh(ub)] \\ &\times \left[E_1 f_2(\lambda,b)(\cosh(uh) - \frac{u_0}{u} \sinh(uh)) + E_2 \right] \end{aligned} \tag{3.4}$$

with C_i, D_i, E_i $(i = 1,2)$ being constants. Since the technique for the evaluation of the elements Z_{ik} is the same for any finite values of the above constants, for simplicity E_1 is considered equal to zero throughout this paper.

In Eq. (3.1) the integrand is a function of the parameter λ through the radicals

$$u = [\lambda^2 - k^2]^{1/2}, \tag{3.5}$$

$$u_0 = [\lambda^2 - k_0^2]^{1/2}, \tag{3.6}$$

where k_0,k are the wavenumbers in vacuum and substrate, respectively. The sign of the radical u does not affect the single-value of the integrals, as the terms involving the radical are even functions of u. For this reason, only the branch cut contribution by the radical u_0 is considered and its direction is determined by convergence of the integrals, the re-

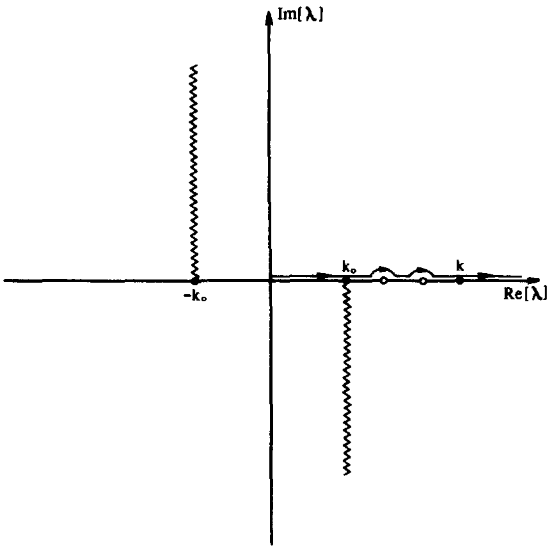

FIG. 3. Path of integration.

quirement that the radiated field is a wave receding from the source and the integrals must be single valued. As a result the restrictions on λ are

$$\operatorname{Re}[\lambda] > 0, \tag{3.7}$$

$$\operatorname{Im}[\lambda] > 0, \tag{3.8}$$

which in turn impose the following behavior for u and u_0:

$$\operatorname{Re}(u_0) > 0, \quad \operatorname{Im}(u_0) > 0, \tag{3.9}$$

$$\operatorname{Re}(u) > 0, \quad \operatorname{Im}(u) > 0. \tag{3.10}$$

A possible position of the branch cuts governed by these inequalities is shown in Fig. 3.

The integrand in Eq. (3.1), with $E_i = 0$, has poles whenever either one of the functions $f_1(\lambda,b)$, $f_2(\lambda,b)$ becomes zero. The zeros of these two functions correspond to surface-wave modes. Particularly the zeros of $f_1(\lambda,b)$ correspond to TE surface waves while the zeros of $f_2(\lambda,b)$ to TM surface waves. In the case of lossless dielectric these TE and TM poles are the roots of the equations $u_0 = -u \coth(ub)$ and $\epsilon_r u_0 = -u \tanh(ub)$. Furthermore, these poles lie within the range $k_0 < \operatorname{Re}(\lambda) < k$.

The infinite integration in Eq. (3.1) is performed along the real axis and it is completed in two steps:

(i) Numerical integration over the interval $[0,A]$ where A satisfies the relationship $\coth((A^2 - k^2)^{1/2} b) \doteq 1$;

(ii) Combination of numerical and analytical integration for the evaluation of the tail contribution which is actually the integration over the path $[A,\infty)$.

Subsequently it is concluded that the (i,k)th element of the impedance matrix can be split into two parts as it is shown below

$$Z_{ik} = Z_{ik}^{(1)} + Z_{ik}^{(2)}, \tag{3.11}$$

where

$$Z_{ik}^{(1)} = \int_0^A \mathscr{L}_{ik}\{ J_0(\lambda |\vec{r} - \vec{r}'|)\} \left[\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} \right] d\lambda \tag{3.12}$$

and

$$Z_{ik}^{(2)} = \int_A^\infty \mathscr{L}_{ik}\{ J_0(\lambda |\vec{r} - \vec{r}'|)\} \left[\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} \right] d\lambda . \tag{3.13}$$

B. Numerical integration $\lambda \in [0,A]$

The first part of each element of the generalized impedance matrix given by

$$Z_{ik}^{(1)} = \int_0^A \mathscr{L}_{ik}\{ J_0(\lambda |\vec{r} - \vec{r}'|)\} \left[\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} \right] d\lambda \tag{3.14}$$

is evaluated numerically. Here $\mathscr{L}_{ik}\{\ \}$ is a double-integral operator with the integrals being finite with respect to space coordinates. In this integral it has been found that the integral operator can make the integrand a slow varying function of λ and therefore it minimizes the error of integration. However, the opposite happens if $\mathscr{L}_{ik}$ operates on the infinite integral as shown below

$$Z_{ik}^{(1)} = \mathscr{L}_{ik} \left\{ \int_0^A J_0(\lambda |\vec{r} - \vec{r}'|) \left[\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} \right] d\lambda \right\} . \tag{3.15}$$

Therefore at first, for the evaluation of $Z_{ik}^{(1)}$, the computation of the space integrals in the operator is performed numerically using a Gaussian-Quadrature method with fixed points. For the integration with respect to λ, because of the existence of poles in the strip $k_0 < \operatorname{Re}(\lambda) < k$, a further division of the integration interval into subintervals takes place as it is shown below:

(i) $0 < \lambda < k_0$. The integration with respect to λ is performed numerically using a modified Romberg–Quadrature method.

(ii) $k_0 < \lambda < k$. For the integration along this interval a singularity extraction technique is used (see Appendix A) which transforms the integral into a finite series plus an integral of a slowly varying function.[2,4] This finite series gives the contribution of the surface wave modes with the number of its terms dependent on the thickness of the dielectric as well as the dielectric constant.

(iii) $k < \lambda < A$. Numerical integration is invoked here in exactly the same manner as it is performed in the first subinterval.

C. Combination of numerical and analytic integration

In this case the integration with respect to λ is extended along the interval $[A,\infty]$ and the integrals which have to be computed are given by

$$Z_{ik}^{(2)} = \int_A^\infty \mathscr{L}_{ik}\{ J_0(\lambda |\vec{r} - \vec{r}'|)\} \left[\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} \right] d\lambda . \tag{3.16}$$

When the source point and the observation point coincide, then $|r - r'| = 0$ and the integral above takes the form

$$Z_{ik}^{(2)} = \int_A^\infty \frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} d\lambda . \tag{3.17}$$

For the integrand in (3.17) it can be proved that when $A \ll \lambda$

$$\frac{A(\lambda,b)}{f_1(\lambda,b) \cdot f_2(\lambda,b)} = O(\lambda^{-a}) \quad (a<1), \tag{3.18}$$

and therefore $Z_{ik}^{(2)}$ becomes infinite. In order to avoid this difficulty which arises in the computation of the matrix element, the dipole is assumed to have a very small but finite radius with the source current being along the axis of the dipole and the observation points being considered on its surface. As a result the distance between source and observation points can never be zero but the minimum value it can take is equal to the radius of the dipole. Since A has already been chosen in such a way that

$$\coth[(A^2 - k^2)^{1/2} b] \doteq 1. \tag{3.19}$$

Equation (3.16) can be approximated as (see Appendix B)

$$\begin{aligned} Z_{ik}^{(2)} = \Bigg[& \frac{2E_2}{\epsilon_r - 1}(D_1 - D_2)[1 - e_1(A)] \\ & + E_2 \frac{\epsilon_r + 1}{\epsilon_r - 1}(D_2\epsilon_r - D_1)\left[1 - \frac{\epsilon_r}{2(\epsilon_r - 1)} e_2(A)\right] \\ & \times \mathscr{L}_{ik}\left\{C_1 \int_A^\infty J_0(\lambda|\vec{r} - \vec{r}'|) e^{-uh} \frac{\lambda}{u} d\lambda \right. \\ & \left. + C_2[1 - e_3(A)] \int_A^\infty J_0(\lambda|\vec{r} - \vec{r}'|) d\lambda \right\} \\ & + \text{Error}, \end{aligned} \tag{3.20}$$

where

$$e_1(A) = \frac{k^2 - k_0^2}{4A^2 - (3k^2 + k_0^2)}, \tag{3.21}$$

$$e_2(A) = \frac{k^2 - k_0^2}{A^2 - k^2 + (\epsilon_r/2(\epsilon_r + 1))(k^2 - k_0^2)}, \tag{3.22}$$

$$e_3(A) = \frac{1}{2}\frac{k^2}{A^2}. \tag{3.23}$$

The error made by the approximation in Eq. (3.20) depends on A. It has been found that if

$$A = O(10^{-2}), \tag{3.24}$$

then

$$\text{Error} = O(10^{-4} Z_{ik}^{(2)}) \tag{3.25}$$

and therefore the approximation considered here is a very good one since the overall error made in the computation of the input impedance is of the order of 0.1%. In Eq. (3.20) the integrals can be written as

$$\begin{aligned} I_1 & \int_A^\infty J_0(\lambda|\vec{r} - \vec{r}'|) e^{-uh} \frac{\lambda}{u} d\lambda \\ & = \frac{1}{(h^2 + |\vec{r} - \vec{r}'|^2 (1 + e_3(A))^2)^{1/2}} \\ & \quad - \int_0^{(A^2 - k^2)^{1/2}} J_0[t\rho(1 + e_3(A))] e^{-th} dt + \mathscr{R}_3, \end{aligned} \tag{3.26}$$

with

$$\mathscr{R}_3 = O(10^{-4} I_1) \tag{3.27}$$

and

$$\begin{aligned} \int_A^\infty & J_0(\lambda|\vec{r} - \vec{r}'|) d\lambda \\ & = \frac{1}{|\vec{r} - \vec{r}'|} - \Big\{ A J_0(A|\vec{r} - \vec{r}'|) \\ & \quad + \frac{\pi A}{2}\Big[J_1(A|\vec{r} - \vec{r}'|)\mathscr{H}_0(A|\vec{r} - \vec{r}'|) \\ & \quad - J_0(A|\vec{r} - \vec{r}'|)\mathscr{H}_1(A|\vec{r} - \vec{r}'|)\Big]\Big\}. \end{aligned} \tag{3.28}$$

By substituting equations above into (3.20) and by ignoring the error terms, $Z_{ik}^{(2)}$ becomes

$$\begin{aligned} Z_{ik}^{(2)} = \Bigg[& \frac{2E_2}{\epsilon_r - 1}(D_1 - D_2)[1 - e_1(A)] \\ & + E_2 \frac{\epsilon_r + 1}{\epsilon_r - 1}(D_2\epsilon_r - D_1)\left[1 - \frac{\epsilon_r}{2(\epsilon_r - 1)} e_2(A)\right]\Bigg] \\ & \mathscr{L}_{ik}\{F(|r - r'|) - \phi(|r - r'|)\}, \end{aligned} \tag{3.29}$$

where $F(|\vec{r} - \vec{r}'|)$ is a fast varying function of $|\vec{r} - \vec{r}'|$ given by

$$\begin{aligned} & F(|\vec{r} - \vec{r}'|) \\ & = \frac{C_1}{1 + e_3(A)} \frac{1}{(|\vec{r} - \vec{r}'|^2 + h^2/(1 + e_3(A))^2)^{1/2}} \\ & \quad + C_2[1 - e_3(A)] \frac{1}{|\vec{r} - \vec{r}'|}, \end{aligned} \tag{3.30}$$

while $\phi(|\vec{r} - \vec{r}'|)$ is a slow varying function of $|\vec{r} - \vec{r}'|$ and is given by

$$\begin{aligned} \phi(|\vec{r} - \vec{r}'|) = C_1 & \int_0^{(A^2 - k^2)^{1/2}} J_0[t\rho(1 + e_3(A))] e^{-th} dt \\ & + C_2[1 - e_3(A)]\Big\{ A J_0(A|\vec{r} - \vec{r}'|) \\ & + \frac{\pi A}{2}\Big[J_1(A|\vec{r} - \vec{r}'|)\mathscr{H}_0(A|\vec{r} - \vec{r}'|) \\ & - J_0(A|\vec{r} - \vec{r}'|)\mathscr{H}_1(A|\vec{r} - \vec{r}'|)\Big]\Big\}. \end{aligned} \tag{3.31}$$

Considering the result of the operation of $\mathscr{L}_{ik}$ on the function $F(|\vec{r} - \vec{r}'|)$, $Z_{ik}^{(2)}$ can be written in a final form as

$$\begin{aligned} Z_{ik}^{(2)} = \Bigg[& \frac{2E_2}{\epsilon_{r-1}}(D_1 - D_2)[1 - e_1(A)] \\ & + E_2 \frac{\epsilon_{r+1}}{\epsilon_{r-1}}(D_2\epsilon_r - D_1)\left[1 - \frac{\epsilon_r}{2(\epsilon_{r+1})} e_2(A)\right]\Bigg] \\ & \times \Bigg[\sum_{n=-1,0} \sum_{m=-1,0,1} A_{nm}\{\tau \ln((\sigma^2 + \tau^2)^{1/2} - \tau) \\ & + (\sigma^2 + \tau^2)^{1/2}\}\Big|_{\tau=\tau_0}^{\tau=\tau_1} \\ & + \mathscr{L}_{ik}\{\mathscr{R}(|\vec{r} - \vec{r}'|) - \phi(|\vec{r} - \vec{r}'|)\}\Bigg], \end{aligned} \tag{3.32}$$

where

$\tau_1 = x_i - x_k + (n - m)l_x$,

$\tau_0 = x_i - x_k + (n - m - 1)l_x$,

l_x = the length of the subinterval considered on the dipole,

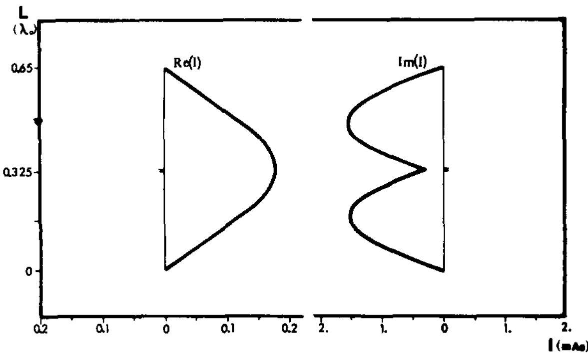

FIG. 4. Current distribution for a printed dipole with $\epsilon_r = 2.35$, $b = 0.1016\lambda_0, L$ '$0.65\lambda_0$.

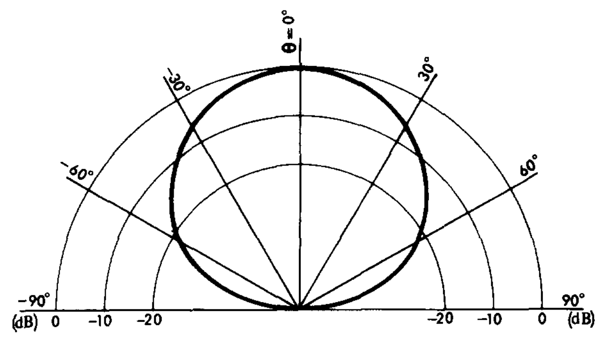

FIG. 6. E-plane normalized power pattern for a printed dipole with $\epsilon_r = 2.35$, $b = 0.1016\lambda_0$ and $L = 0.65\lambda_0$.

and

$\mathscr{R}(|\vec{r} - \vec{r}'|) - \phi(|\vec{r} - \vec{r}'|)$

is a slow-varying function of $|\vec{r} - \vec{r}'|$.

In Eq. (3.32) the term $\mathscr{L}_{ik}\{\ \}$ is evaluated numerically by using the Gaussian-Quadrature method of integration.

IV. AN APPLICATION AND CONCLUSIONS

In the attempt to obtain the current distribution of extended sources of electromagnetic radiation with dipole moments on the interface between two electrically dissimilar materials, the kernel of the resulting Fredholm integral equation of the first kind involves Sommerfeld-type integrals. These integrals include all the physical phenomena of the problem, i.e., radiation into space, evanescent waves as well as surface modes in the substrate. In addition, because of the fact that source and field points must be selected on the substrate surface to solve the integral equation for the current distribution, the Sommerfeld-type integrals converge very slowly and special techniques are necessary for their proper evaluation. This is the crucial contribution of this paper, i.e., a systematic methodology is provided to solve these integrals for sources on dielectric interfaces and subsequently to obtain the current distribution of such extended sources. Previously,[5–8] sources of electromagnetic radiation with a presumed current distribution were selected off the interface, introducing in this manner an exponential decay factor in the integrand of the resulting Sommerfeld-type integrals.

In the process of solving the Fredholm integral equation, a basis set of piece-wise continuous functions has been adopted in the expansion of the unknown current distribution. This device is dictated by two factors: (a) it simplifies the kernel after a series of closed form integrations by parts and (b) it satisfies the boundary conditions of the current distribution on the antenna.[17,18] Finally in order to illustrate the utility of solutions of Eq. (1) for printed circuit antennas, the case of lineal printed dipoles is considered. The current distribution is shown in Fig. 4 for a center-fed, by a unit voltage source, dipole of radius $10^{-4}\lambda_0$ and length $L = 0.65\lambda_0$. The substrate properties are $\epsilon_r = 2.35$ and thickness $b = 0.1016\lambda_0$. The input impedance for this printed circuit dipole is also shown in Fig. 5 and its radiation pattern in Fig. 6.

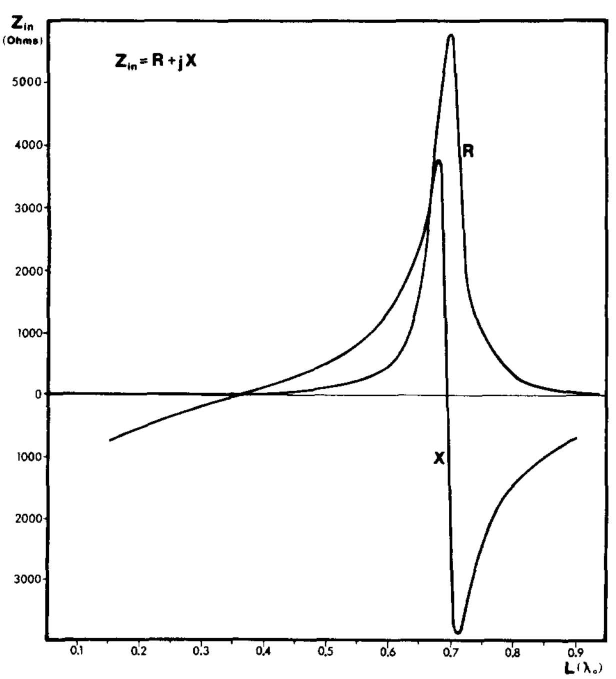

FIG. 5. Input impedance vs dipole length for $\epsilon_r = 2.35$ and $b = 0.1016\lambda_0$.

APPENDIX A: TECHNIQUE FOR THE EXTRACTION OF THE SINGULARITIES

It has been mentioned that the interval $[k_0, k]$ includes a finite set of discrete essential singularities which correspond to surface waves and contribute to the integration along this interval. Their contribution affects seriously the input impedance and radiation characteristics of the printed antenna. The technique which is described below transforms the integral given by

$$I_{ik} = \int_{k_0}^{k} \mathscr{L}_{ik}\{ J_0(\lambda\, |\vec{r} - \vec{r}'|)\} \left[\frac{A(\lambda, b)}{f_1(\lambda, b)\cdot f_2(\lambda, b)} \right] d\lambda \quad \text{(A1)}$$

into a finite series plus an integral of a slowly varying function.

If it is assumed that S is the set of the singularities mentioned above, then S is given by

$$S=\{x_i/[x_i=\text{zero of } f_1(\lambda,b)]\,U\,[x_i=\text{zero of } f_2(\lambda,b)]\,; \quad i=1,2,3,\cdots,N\}\,. \tag{A2}$$

A partition $P[t_{n+1}]$ of the interval $[k_0,k]$ is considered such that

$$P[t_{N+1}]=\{t_0=k_0,\ t_1<x_1,\ldots,t_n<x_N,\ t_{N+1}=k\}, \tag{A3}$$

and the integral (A1) can be written as

$$I_{ik}=\sum_{r=0}^{N}\int_{t_r}^{t_{r+1}} d\lambda\,\frac{\phi_r^{ik}(\lambda,b,\epsilon_r)}{\lambda-x_r} \tag{A4}$$

with $\phi_r^{ik}(\lambda,b,\epsilon_r)$ given by

$$\phi_r^{ik}(\lambda,b,\epsilon_r)=\mathscr{L}_{ik}\{J_0(\lambda|\vec r-\vec r'|)\}\left[\frac{A(\lambda,b)(\lambda-x_r)}{f_1(\lambda,b)\cdot f_2(\lambda,b)}\right]. \tag{A5}$$

In (A5) the quantity $(\phi_r^{ik}(x_r,b,\epsilon_r)/(\lambda-x_r))$ is added and subtracted giving

$$I_{ik}=\sum_{r=0}^{N}\int_{t_r}^{t_{r+1}} d\lambda\,\frac{\phi_r^{ik}(\lambda,b,\epsilon_r)-\phi_r^{ik}(x_r,b,\epsilon_r)}{\lambda-x_r}+\sum_{r=0}^{N}\phi_r^{ik}(x_r,b,\epsilon_r)\left[\ln\left|\frac{t_{r+1}-x_r}{x_r-t_r}\right|-j\pi\right].$$

In the equation above, the integrands are slowly varying functions in the intervals $[t_r,t_{r+1}]$ and the integrals can be evaluated using a Gaussian-Quadrature integration method with four fixed points.

APPENDIX B: FORMULATION OF THE INTEGRAL FOR THE TAIL CONTRIBUTION

If we choose A in such a way that

$$\coth[(A^2-k^2)^{1/2}b]\doteq 1, \tag{B1}$$

the integral given below

$$Z_{ik}^{(2)}=\int_A^\infty \mathscr{L}_{ik}\{J_0(\lambda|\vec r-\vec r'|)\}\left[\frac{A(\lambda,b)}{f_1(\lambda,b)\cdot f(\lambda,b)}\right]d\lambda \tag{B2}$$

can be approximated as follows

$$Z_{ik}^{(2)}=\mathscr{L}_{ik}\left\{\int_A^\infty J_0(\lambda|\vec r-\vec r'|)\left[\frac{C(\lambda,b)}{g_1(\lambda,b)\cdot g_2(\lambda,b)}\right]d\lambda\right\}, \tag{B3}$$

where

$$g_1(\lambda,b)=u_0+u, \tag{B4}$$

$$g_2(\lambda,b)=\epsilon_r u_0+u, \tag{B5}$$

and

$$C(\lambda,b)=\left[C_1\frac{\sinh[u(b-h)]}{\sinh(ub)}+C_2\right](D_1u_0+D_2u)E_2\,. \tag{B6}$$

If one considers that

$$\sinh[u(b-h)]=\sinh(ub)\cosh(uh)-\cosh(ub)\sinh(uh),$$

then

$$C(\lambda,b)=(C_1e^{-uh}+C_2)(D_1u_0+D_2)E_2\lambda, \tag{B7}$$

and since the relation

$$\frac{D_1u_0+D_2u}{u+u_0}\cdot\frac{\lambda}{\epsilon_r u_0+u}=\frac{\lambda}{\epsilon_{r-1}}\left\{\frac{D_1-D_2}{u+u_0}+\frac{D_2\epsilon_r-D_1}{\epsilon_r u_0+u}\right\} \tag{B8}$$

is true Eq. (B3) can be written as

$$Z_{ik}^{(2)}=\mathscr{L}_{ik}\left\{\int_A^\infty J_0(\lambda|\vec r-\vec r'|)(C_1e^{-uh}+C_2)\frac{E_2}{\epsilon_r-1}\times\left(\frac{D_1-D_2}{1+(u_0/u)}+\frac{D_2\epsilon_r-D_1}{1+\epsilon_r(u_0/u)}\right)\frac{\lambda}{u}d\lambda\right\}, \tag{B9}$$

where

$$u=[\lambda^2-k^2]^{1/2}, \tag{B10}$$

$$u_0=[\lambda^2-k_0^2]^{1/2}. \tag{B11}$$

Because of (B10) and (B11) the functions $1+u_0/u$ and $1+\epsilon_r u_0/u$ in (B9) can be written as

$$1+\frac{u_0}{u}=2\{1-e_1(\lambda)\}^{-1} \tag{B12}$$

and

$$1+\epsilon_r\frac{u_0}{u}=(1+\epsilon_r)\left\{1-\frac{\epsilon_r}{2(\epsilon_r+1)}e_2(\lambda)\right\}^{-1}, \tag{B13}$$

where

$$0\leqslant e_1(\lambda)\leqslant\frac{k^2-k_0^2}{4A^2-3k^2-k_0^2},$$

$$0\leqslant e_2(\lambda)\leqslant\frac{k^2-k_0^2}{A^2-k^2+(\epsilon_r/2(\epsilon_r+1))(k^2-k_0^2)} \tag{B14}$$

with

$$\lambda\epsilon[A,\infty).$$

By substituting (B12) and (B13) into (B9) after some manipulations $Z_{ik}^{(2)}$ can be put in the final form

$$\begin{aligned}Z_{ik}^{(2)}={}&\frac{2E_2}{\epsilon_r-1}(D_1-D_2)[1-e_1(A)]\\&\times\mathscr{L}_{ik}\left\{C_1\int_A^\infty J_0(\lambda|\vec r-\vec r'|)e^{-uh}\frac{\lambda}{u}d\lambda+C_2[1-e_3(A)]\int_A^\infty J_0(\lambda|\vec r-\vec r'|)\,d\lambda\right\}+\tilde{\mathscr{R}}_1\\&+E_2\frac{\epsilon_r+1}{\epsilon_r-1}(D_2\epsilon_r-D_1)\left[1-\frac{\epsilon_r}{2(\epsilon_r+1)}e_2(A)\right]\\&\times\mathscr{L}_{ik}\left\{C_1\int_A^\infty J_0(\lambda|\vec r-\vec r'|)e^{-uh}\frac{\lambda}{u}d\lambda\right\}+\frac{\epsilon_r}{2(\epsilon_r+1)}\tilde{\mathscr{R}}_2,\end{aligned} \tag{B15}$$

with

$$\begin{aligned}|\tilde{\mathscr{R}}_1|<A^{1/2}\frac{k^2-k_0^2}{A^2-k^2}\cdot\Bigg[&\frac{2E_2}{\epsilon_r-1}(D_1-D_2)[1-e_1(A)]\\&+\mathscr{L}_{ik}\left\{C_1\int_A^\infty J_0(\lambda|\vec r-\vec r'|)e^{-uh}\frac{\lambda}{u}d\lambda+C_2[1-e_3(A)]\int_A^\infty J_0(\lambda|\vec r-\vec r'|)\,d\lambda\right\}\Bigg]\end{aligned} \tag{B16}$$

and

$$|\tilde{\mathscr{R}}_2| < A^{1/2}\frac{k^2-k_0^2}{A^2-k^2}\left[E_2\frac{\epsilon_r+1}{\epsilon_r-1}(D_2\epsilon_r-D_1)\right.$$
$$\times\left[1-\frac{\epsilon_r}{2(\epsilon_r+1)}e_2(A)\right]$$
$$\times\mathscr{L}_{ik}\left\{C_1\int_A^\infty J_0(\lambda\,|\vec{r}-\vec{r}'|)e^{-uh}\frac{\lambda}{u}\,d\lambda\right.$$
$$\left.\left.+C_2[1-e_3(A)]\int_A^\infty J_0(\lambda\,|\vec{r}-\vec{r}'|)\,d\lambda\right\}\right]. \qquad \text{(B17)}$$

From (B16) and (B17) it can be shown that

$$\left|\tilde{\mathscr{R}}_1+\frac{\epsilon_r}{2(\epsilon_r+1)}\tilde{\mathscr{R}}_2\right| < \frac{k^2-k_0^2}{A^2-k^2}Z_{ik}^{(2)}\times\frac{A^{1/2}}{2}.$$

With A being of the order of 100 the error made by ignoring

$\left[\tilde{\mathscr{R}}_1+\frac{\epsilon_r}{2(\epsilon_r+1)}\tilde{\mathscr{R}}_2\right]$ is given by

$$\text{Error} = O[10^{-4}\times Z_{ik}].$$

This approximation gives an error 0.1% in the input impedance.

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