General Analysis of Narrow Strips and Slots

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***Abstract*—The general problems of TE- and TM-excited slots in screens and conducting strips whose widths are narrow relative to the wavelength are investigated. The narrow width approximation is imposed upon the slotted-screen and strip equations from which result *approximate* equations that are solved *exactly*. For general excitation, solutions are presented for both polarizations as products of series of Chebyshev polynomials and appropriate singular functions exhibiting the known edge condition. Special attention is afforded the problem of the slot/strip excited by illumination which can be represented by two terms of a Taylor series due to the practical importance of this situation.**

I. INTRODUCTION

DIFFRACTION BY a conducting strip and by a slot in a conducting screen has been the subject of intensive research since the problem was first addressed by Lord Rayleigh [1] in 1897. For strips or slots that are narrow relative to the wavelength, one- and two-term series approximations of the strip current or slot electric field are available under the condition that the excitation is a uniform plane wave. Morse and Rubenstein [2] employed differential equation methods in elliptic coordinates and expressed solutions to the strip problem in terms of a series of Mathieu functions. Sommerfeld [3], Rayleigh [1], and Bouwkamp [4] have presented techniques for obtaining the first few terms of a series solution of integral equations for the narrow strip and narrow slot problems. In more recent times, Millar [5] has obtained additional series terms and has solved the equations appropriate for cylindrical-wave excitation. Houlberg [6] extended the work of Millar so that it is applicable to the problem of a slot in a screen separating different media. The method of Morse and Rubenstein [2] has been extended by Barakat [7] to the two-media problem, which also has been solved numerically by Butler and Umashankar [8]. The literature pertinent to the narrow strip and slot problems is far too voluminous to allow even a modest review here, but fortunately, excellent coverage of past contributions is available to the interested reader [9]-[11].

The purpose of this paper is to present exact solutions to the problems of the narrow strip and slot, excited by general TE and TM (transverse to strip or slot axis) illumination with the single restriction that the excitation be invariant along the strip or slot axis. Solutions are given for the single-medium slot problem in which the slotted screen separates two half-spaces filled with the same medium as well as for the two-media problem [8] in which the slotted screen separates half-spaces of different media. However, in the strip problem we consider only the case of the strip in a homogeneous medium of infinite extent, because otherwise a different class of equations would be introduced by the presence of a medium inhomogeneity, inconsistent with the commonality of features exhibited by the equations characterizing the aforementioned narrow strip and slot problems.

Employing integral relationships developed in the Appendix, the authors obtain exact solutions to the equations for the narrow strip and slot problems. One equation is a first-kind integral equation (TM strip and TE slot) and the other is an integro-differential equation (TE strip and TM slot). First, with the excitation approximated as a two-term Taylor series about the slot axis, solutions are obtained for the special case of the two-media slot (both polarizations). These solutions are restated in a general setting and a table is given so that one can apply them to all the cases discussed. Second, the general equations are solved exactly for any arbitrary excitation that can be expanded in a series of Chebyshev polynomials.

II. SLOTTED SCREEN SEPARATING HALF-SPACES OF DIFFERENT MEDIA

The equations for the problems of TE- and TM-excitation of a slot of width $2w$ in a planar conducting screen of infinite extent, separating different media, are [8]

TE Case:

$$\frac{1}{2}\int_{x'=-w}^{w} M_{s_y}(x')\left[\frac{k_-}{\eta_-}H_0^{(2)}(k_-|x-x'|) + \frac{k_+}{\eta_+}H_0^{(2)}(k_+|x-x'|)\right]dx' \tag{1}$$
$$= [H_y^{sc-}(x) - H_y^{sc+}(x)], \qquad x \in (-w, w),$$

and

TM Case:

$$\frac{1}{2}\left(\frac{d^2}{dx^2}+k_-^2\right)\int_{x'=-w}^{w} M_{s_x}(x')\frac{1}{k_-\eta_-}H_0^{(2)}(k_-|x-x'|)\,dx'$$
$$+\frac{1}{2}\left(\frac{d^2}{dx^2}+k_+^2\right)\int_{x'=-w}^{w} M_{s_x}(x')\frac{1}{k_+\eta_+}H_0^{(2)}$$
$$\cdot(k_+|x-x'|)\,dx' = [H_x^{sc-}(x) - H_x^{sc+}(x)],$$
$$x \in (-w, w), \tag{2}$$

where $\overline{M}_s$ is the unknown equivalent[1] magnetic surface current to be determined. M_{s_y} becomes unbounded as x approaches $\pm w$ while M_{s_x} approaches zero at these edges. In (1) and (2)

[1] $M_{s_x} = -E_y^a$ and $M_{s_y} = E_x^a$, where E_x^a and E_y^a are components (parallel to the screen) of the total electric field in the slot. $\overline{M}_s$ is the equivalent magnetic surface current for $x \in (-w, w)$ on the side of the shorted screen facing the region $z < 0$ of Fig. 1.

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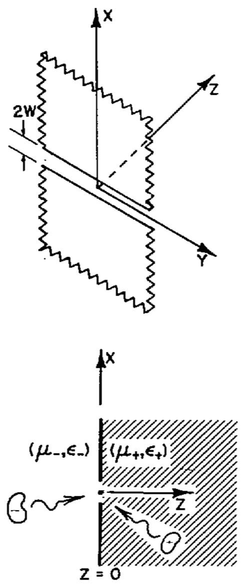

Fig. 1. Slotted conducting screen separating half-spaces having different electromagnetic properties.

$H_0^{(2)}(\cdot)$ is the Hankel function of the second kind and of zero order, $\bar{H}^{sc}$ is the known short-circuit magnetic field, and ω is the angular frequency of the suppressed harmonic variation with time $e^{j\omega t}$. The subscripts and superscripts "–" and "+" designate a quantity peculiar to the left and right half-spaces, respectively, as suggested in Fig. 1, where geometric and media parameters are illustrated. Electromagnetic properties of materials are denoted in the usual way by $(\mu_\pm, \epsilon_\pm)$, and it is convenient to introduce $k_\pm = \omega\sqrt{\mu_\pm \epsilon_\pm}$ and $\eta_\pm = \sqrt{\mu_\pm/\epsilon_\pm}$. The short-circuit magnetic field in the above equations is that magnetic field which would exist on the surface of the screen, due to the specified source, when the slot is closed (shorted) with a perfect conductor, i.e., when the screen has no slot. If the media on the two sides of the slotted screen were the same, then with minor modifications (1) and (2) could be extended to cover the general TE and TM cases allowing y variation.

Narrow TE-Excited Slot

If the slot is narrow relative to the wavelength in both media ($|k_\pm w| \ll 1$), then $|k_\pm(x - x')| \ll 1$ and the Hankel functions in (1) can be replaced by their small argument approximations [12]:

$$-j\frac{2}{\pi}\frac{k_e}{\eta_e}\int_{x'=-w}^{w} M_{s_y}(x')\ln|x-x'|\,dx'$$

$$\doteq j\frac{2}{\pi}\frac{k_e}{\eta_e}C_e\int_{x'=-w}^{w} M_{s_y}(x')\,dx' + [H_y{}^{sc-}(x) - H_y{}^{sc+}(x)],$$

$$|k_\pm w| \ll 1, \tag{3}$$

where

$$C_e = \gamma + j\frac{\pi}{2} + \frac{1}{2}\frac{\epsilon_-}{\epsilon_e}\ln\left(\frac{k_-}{2}\right) + \frac{1}{2}\frac{\epsilon_+}{\epsilon_e}\ln\left(\frac{k_+}{2}\right), \tag{4}$$

where γ ($\doteq 0.5772$) is Euler's constant, and where the definitions of effective quantities are

$$k_e = \omega\sqrt{\mu_e \epsilon_e}, \tag{5a}$$

$$\eta_e = \sqrt{\frac{\mu_e}{\epsilon_e}}, \tag{5b}$$

and

$$\epsilon_e = \tfrac{1}{2}(\epsilon_- + \epsilon_+), \tag{6a}$$

$$\frac{1}{\mu_e} = \frac{1}{2}\left(\frac{1}{\mu_-} + \frac{1}{\mu_+}\right). \tag{6b}$$

With $|k_\pm w|$ sufficiently small, the short-circuit magnetic field does not vary greatly over $x \in (-w, w)$ and it can be expanded in a Taylor series as

$$[H_y{}^{sc-}(x) - H_y{}^{sc+}(x)]$$

$$= H_y{}^{sc+}(0) + x\frac{\partial}{\partial x}H_y{}^{sc-}(0) - H_y{}^{sc+}(0) - x\frac{\partial}{\partial x}H_y{}^{sc+}(0)$$

$$= [H_y{}^{sc-}(0) - H_y{}^{sc+}(0)] + j\frac{k_e}{\eta_e}\left[\frac{\epsilon_-}{\epsilon_e}E_z{}^{sc-}(0) - \frac{\epsilon_+}{\epsilon_e}E_z{}^{sc+}(0)\right]x \tag{7}$$

in which only the first two terms are retained and in which $E_z{}^{sc}$ is the short-circuit electric field normal to the conducting screen. Subject to the approximation of (7), the forcing function of (3) comprises an even function (constant term) and an odd function (linear term). In view of this and the symmetry properties of the argument of the kernel, it is convenient to represent M_{s_y} as the sum of an even and an odd function,

$$M_{s_y}(x) = M_{s_y}{}^e(x) + M_{s_y}{}^0(x), \tag{8a}$$

where

$$M_{s_y}{}^{e,0}(x) = \tfrac{1}{2}[M_{s_y}(x) \pm M_{s_y}(-x)], \tag{8b}$$

which enables one to partition (3) into two equations:

$$-j\frac{2}{\pi}\frac{k_e}{\eta_e}\int_{x'=-w}^{w} M_{s_y}{}^e(x')\ln|x-x'|\,dx'$$

$$= j\frac{2}{\pi}\frac{k_e}{\eta_e}C_e\int_{x'=-w}^{w} M_{s_y}{}^e(x')\,dx' + [H_y{}^{sc-}(0) - H_y{}^{sc+}(0)] \tag{9a}$$

and

$$-\frac{2}{\pi}\int_{x'=-w}^{w} M_{s_y}{}^0(x')\ln|x-x'|\,dx'$$

$$= \left[\frac{\epsilon_-}{\epsilon_e}E_z{}^{sc-}(0) - \frac{\epsilon_+}{\epsilon_e}E_z{}^{sc+}(0)\right]x. \tag{9b}$$

Now we wish to show that (9a) and (9b) can be solved exactly. The solutions must exhibit the properties below:

1) $M_{s_y}{}^e(x) = M_{s_y}{}^e(-x)$

2) $M_{s_y}{}^e \xrightarrow[x \to \pm w]{} \dfrac{\alpha^{e'}}{\sqrt{w - |x|}}$

($\alpha^{e'}$ = constant)

3) $\displaystyle\int_{-w}^{w} M_{s_y}{}^e(x') \ln |x - x'| \, dx'$ must be a constant

$$M_{s_y}(x) \doteq \frac{1}{2} \left\{ j \frac{\eta_e}{k_e} \frac{[H_y{}^{sc-}(0) - H_y{}^{sc+}(0)]}{\left(\gamma + j\dfrac{\pi}{2} + \dfrac{1}{2}\left[\dfrac{\epsilon_-}{\epsilon_e} \ln\left(\dfrac{k_- w}{4}\right) + \dfrac{\epsilon_+}{\epsilon_e} \ln\left(\dfrac{k_+ w}{4}\right)\right]\right)} + \left[\frac{\epsilon_-}{\epsilon_e} E_z{}^{sc-}(0) - \frac{\epsilon_+}{\epsilon_e} E_z{}^{sc+}(0)\right] x \right\} \frac{1}{\sqrt{w^2 - x^2}}. \tag{12}$$

1) $M_{s_y}{}^0(x) = -M_{s_y}{}^0(-x)$

2) $M_{s_y}{}^0 \xrightarrow[x \to \pm w]{} \pm \dfrac{\alpha^{0'}}{\sqrt{w - |x|}}$

($\alpha^{0'}$ = constant)

3) $\displaystyle\int_{-w}^{w} M_{s_y}{}^0(x') \ln |x - x'| \, dx'$ must vary linearly with x.

The first property in each case is simply the even or odd function behavior of the solution. The second property is the required edge condition[2] [13], while the third property simply says that the solution must cause the leftside of the integral equation to vary as the rightside does with x. Reviewing the expressions (A6a) and (A6b) in the Appendix, we postulate

$$M_{s_y}{}^e(x) = \alpha^e \frac{1}{\sqrt{w^2 - x^2}} \tag{10a}$$

and

$$M_{s_y}{}^0(x) = \alpha^0 \frac{x}{\sqrt{w^2 - x^2}} \tag{10b}$$

where α^e and α^0 are constants. By substitution of (10a) into (9a) and (10b) into (9b), one can readily determine α^e and α^0 by appealing to (A6a) and (A6b) of the Appendix. Thus

$$M_{s_y}{}^e(x) = j \frac{\eta_e}{2k_e} \cdot \frac{[H_y{}^{sc-}(0) - H_y{}^{sc+}(0)]}{C_e + \ln\left(\dfrac{w}{2}\right)} \cdot \frac{1}{\sqrt{w^2 - x^2}} \tag{11a}$$

and

$$M_{s_y}{}^0(x) = \frac{1}{2}\left[\frac{\epsilon_-}{\epsilon_e} E_z{}^{sc-}(0) - \frac{\epsilon_+}{\epsilon_e} E_z{}^{sc+}(0)\right] \frac{x}{\sqrt{w^2 - x^2}}. \tag{11b}$$

$M_{s_y}{}^e$ and $M_{s_y}{}^0$ of (11) satisfy the integral equations (9) and the required properties so they are the unique solutions sought. Combining (11a) and (11b) according to (8a), one has

It should be noted that $M_{s_y}{}^e$ and $M_{s_y}{}^0$ of (11) are the exact solutions of (9a) and (9b), respectively, while M_{s_y} of (12) is the approximate solution of (1), valid for a narrow slot whose excitation can be adequately represented by (7).

Narrow TM-Excited Slot

In the case of TM-excitation and under the assumption that $|k_\pm w| \ll 1$, we replace the Hankel functions in (2) by their small argument approximations:

$$-j \frac{2}{\pi k_e \eta_e} \left\{ \left(\frac{d^2}{dx^2} + k_e{}^2\right) \int_{x'=-w}^{w} M_{s_x}(x') \ln |x - x'| \, dx' + k_e{}^2 C_e \int_{x'=-w}^{w} M_{s_x}(x') \, dx' \right\} \doteq [H_x{}^{sc-}(x) - H_x{}^{sc+}(x)], \quad |k_\pm w| \ll 1. \tag{13}$$

We also approximate the forcing function by a two-term Taylor series, let $M_{s_x} = M_{s_x}{}^e + M_{s_x}{}^0$, and partition (13) into its even and odd constituents:

$$-j \frac{2}{\pi k_e \eta_e} \left\{ \left(\frac{d^2}{dx^2} + k_e{}^2\right) \int_{x'=-w}^{w} M_{s_x}{}^e(x') \ln|x - x'| dx' + k_e{}^2 C_e \int_{x'=-w}^{w} M_{s_x}{}^e(x') \, dx' \right\} = [H_x{}^{sc-}(0) - H_x{}^{sc+}(0)], \quad |k_\pm w| \ll 1 \tag{14a}$$

[2] The edge condition at the slot/screen edge is the same in the two-media case as it is in the single-medium case [13].

and

$$-j\frac{2}{\pi k_e\eta_e}\left(\frac{d^2}{dx^2}+k_e^2\right)\int_{x'=-w}^{w} M_{s_x}{}^0(x')\ln|x-x'|dx' \tag{14b}$$

$$=[H_x{}^{sc-}(0)-H_x{}^{sc+}(0)]'x, \qquad |k_\pm w|\ll 1$$

where, of course,

$$[H(0)]'=\left[\frac{d}{dx}H(x)\right]_{x=0}.$$

It is demonstrated below that since $|k_\pm w|\ll 1$ those terms of (14a) and (14b) which contain $k_e{}^2$ as a factor are insignificant compared to the term involving the derivative operator. Accepting this for the present, one obtains

$$-j\frac{2}{\pi k_e\eta_e}\frac{d^2}{dx^2}\int_{x'=-w}^{w}\begin{bmatrix}M_{s_x}{}^e(x')\\ M_{s_x}{}^0(x')\end{bmatrix}\ln|x-x'|\,dx'$$

$$=\begin{bmatrix}H_x{}^{sc-}(0)-H_x{}^{sc+}(0)\\ [H_x{}^{sc-}(0)-H_x{}^{sc+}(0)]'x\end{bmatrix}. \qquad \begin{matrix}(15\text{a})\\(15\text{b})\end{matrix}$$

Now we solve (15a) and (15b). One observes that the solutions must exhibit the following properties:

1) $M_{s_x}{}^e(x)=M_{s_x}{}^e(-x)$

2) $M_{s_x}{}^e \xrightarrow[x\to\pm w]{} \beta^{e\prime}\sqrt{w-|x|}$

($\beta^{e\prime}$ = constant)

3) $\frac{d^2}{dx^2}\int_{-w}^{w} M_{s_x}{}^e(x')\ln|x-x'|\,dx'$ must be a constant

1) $M_{s_x}{}^0(x)=-M_{s_x}{}^0(-x)$

2) $M_{s_x}{}^0 \xrightarrow[x\to\pm w]{} \pm\beta^{0\prime}\sqrt{w-|x|}$

($\beta^{0\prime}$ = constant)

3) $\frac{d^2}{dx^2}\int_{-w}^{w} M_{s_x}{}^0(x')\ln|x-x'|\,dx'$ must vary linearly with x.

A study of the above properties, of (15a) and (15b), and of the results (A6c) and (A6d) of the Appendix suggests that

$$\begin{pmatrix}M_{s_x}{}^e\\ M_{s_x}{}^0\end{pmatrix}=\begin{pmatrix}\beta^e\\ \beta^0 x\end{pmatrix}\sqrt{w^2-x^2}. \qquad \begin{matrix}(16\text{a})\\(16\text{b})\end{matrix}$$

Appealing to the integrals of the Appendix, one can evaluate the constants β^e and β^0 and synthesize from (16) the following exact solution of (13) subject to the edge condition:

$$M_{s_x}(x)=j\frac{k_e\eta_e}{2}\left[[H_x{}^{sc-}(0)-H_x{}^{sc+}(0)]+\tfrac{1}{2}[H_x{}^{sc-}(0)-H_x{}^{sc+}(0)]'x\right]\sqrt{w^2-x^2}. \tag{17}$$

To conclude this section we return to (14) and justify deletion of terms involving $k_e{}^2$. With the even-function part of (17) in (14a), the latter reduces to

$$1+\left\{\frac{(k_ew)^2}{2}\left[\gamma-\tfrac{1}{2}+\left(\frac{x}{w}\right)^2+j\frac{\pi}{2}+\frac{1}{2}\frac{\epsilon_-}{\epsilon_e}\ln\left(\frac{k_-w}{4}\right)+\frac{1}{2}\frac{\epsilon_+}{\epsilon_e}\ln\left(\frac{k_+w}{4}\right)\right]\right\}=1, \qquad x\in(-w,w) \tag{18}$$

while the odd-function part of (17) in (14b) leads to

$$1+\left\{(k_ew)^2\left[\frac{1}{6}\left(\frac{x}{w}\right)^2-\frac{1}{4}\right]\right\}=1, \qquad x\in(-w,w), \tag{19}$$

where once again use is made of the integrals of the Appendix. The terms in the braces in (18) and (19) are those quantities ignored in simplifying (14a) to (15a) and (14b) to (15b), respectively. Since $|k_ew|\ll 1$ whenever $|k_-w|\ll 1$ and $|k_+w|\ll 1$, the terms in the braces of (18) and (19) can each be ignored relative to unity and deletion of terms in (14) that involve $k_e{}^2$ is justified.

III. CONDUCTING STRIPS AND SLOTTED SCREENS

In this section we consider the general problems of TE- and TM-excitation of narrow strips and slots as well as their common features. The infinite perfectly conducting strip to be considered is in an unbounded medium (μ, ϵ), is vanishingly thin, is of uniform width ($2w$), and is in the xy plane with its axis along the y coordinate axis.

If the strip and slot equations [8], [9] are subjected to the approximation $|kw|\ll 1$ ($|k_ew|\ll 1$ for the two-media slot problem), they can be represented by

$$\int_{x'=-w}^{w} u(x')\ln|x-x'|\,dx'+C\int_{x'=-w}^{w} u(x')\,dx'$$

$$=f(x), \qquad x\in(-w,w), \qquad |k_ew|\ll 1 \tag{20}$$

in which the edge condition is $|u(x)|\to \alpha^\pm/\sqrt{w-|x|}$ as $x\to\pm w$ for the TM-excited strip and TE-excited slot, and by

$$\frac{d^2}{dx^2}\int_{x'=-w}^{w} u(x')\ln|x-x'|\,dx'=f(x),$$

$$x\in(-w,w), \qquad |k_ew|\ll 1 \tag{21}$$

TABLE I
TERMS IN (20) AND (21) FOR STRIP AND SLOT PROBLEMS

Problem	Equation	Excitation $f(x)$	Unknown $u(x)$	Constant C
TM Strip	(20)	$E_y{}^i(x)$	$J_{s_y}(x)/[j2\pi/k\eta]$	$\gamma + j(\pi/2) + \ln(k/2)$
TE Strip	(21)	$E_x{}^i(x)$	$J_{s_x}(x)/[j2\pi k/\eta]$	–
TE Slot (Single Medium)	(20)	$H_y{}^{sc-}(x) - H_y{}^{sc+}(x)$	$M_{s_y}(x)/[j\pi\eta/2k]$ $(M_{s_y} = E_x{}^a)$	$\gamma + j(\pi/2) + \ln(k/2)$
TM Slot (Single Medium)	(21)	$H_x{}^{sc-}(x) - H_x{}^{sc+}(x)$	$M_{s_x}(x)/[j\pi k\eta/2]$ $(M_{s_x} = -E_y{}^a)$	–
TE Slot (Two Media)	(20)	$H_y{}^{sc-}(x) - H_y{}^{sc+}(x)$	$M_{s_y}(x)/[j\pi\eta_e/2k_e]$ $(M_{s_y} = E_x{}^a)$	C_e (Equation (4))
TM Slot (Two Media)	(21)	$H_x{}^{sc-}(x) - H_x{}^{sc+}(x)$	$M_{s_x}(x)/[j\pi k_e\eta_e/2]$ $(M_{s_x} = -E_y{}^a)$	–

with the edge behavior $|u(x)| \to \beta^{\pm}\sqrt{w - |x|}$ as $x \to \pm w$ for the TE-excited strip and TM-excited slot, where $\alpha^{\pm}$ and $\beta^{\pm}$ are constants. Table I provides the specific quantities represented by u, f, and C for the narrow strip and slot problems.

For the forcing function given by

$$f(x) = f(0) + f'(0)x, \tag{22}$$

we find from Section II that the solution of (20) is

$$u(x) = \frac{1}{\pi}\left[\frac{f(0)}{C + \ln\left(\frac{w}{2}\right)} - xf'(0)\right]\frac{1}{\sqrt{w^2 - x^2}}, \tag{23}$$

while that of (21) is

$$u(x) = \frac{1}{\pi}\left[f(0) + \tfrac{1}{2}xf'(0)\right]\sqrt{w^2 - x^2}. \tag{24}$$

From (23), (24), and Table I, one can determine narrow slot and strip solutions whenever the excitation can be approximated sufficiently well by the first two terms of its Taylor series about $x = 0$.

IV. GENERALIZATIONS

Taking advantage of the results presented in the Appendix, one can solve[3] (20) and (21) in general for any forcing function f which can be expanded in a series of Chebyshev polynomials. In each case the solution is the product of a polynomial (analytic) and a factor embodying the appropriate edge condition. The solution method is simple: one expands u and f in a Chebyshev polynomial series, invokes transformations given in the Appendix, and determines the series coefficients of u in terms of those of f, which are known.

Solution of (20)

The forcing function f of (20) is represented by a series of Chebyshev polynomials of the first kind T_n as

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n T_n(x/w), \tag{25a}$$

where the coefficients

$$f_n = \frac{2}{\pi}\int_{x=-w}^{w} \frac{f(x)T_n(x/w)}{\sqrt{w^2 - x^2}}\,dx \tag{25b}$$

are readily determined from the well-known orthogonality properties of $\{T_n\}$. The unknown u is expressed as

$$u(x) = \frac{1}{\sqrt{w^2 - x^2}}\left[\frac{u_0}{2} + \sum_{n=1}^{\infty} u_n T_n(x/w)\right], \tag{26a}$$

which exhibits the desired edge condition and where the coefficients u_n are

$$u_n = \frac{2}{\pi}\int_{x=-w}^{w} u(x)T_n(x/w)\,dx. \tag{26b}$$

With f and u in (20) replaced by (25a) and (26a), respectively, one makes use of (A5) and the orthogonality properties of $\{T_n\}$ to arrive at

$$\pi\left(\frac{u_0}{2}\left[C - \ln\left(\frac{2}{w}\right)\right]\right) - \pi\sum_{n=1}^{\infty}\frac{u_n}{n}T_n(x/w)$$

$$= \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n T_n(x/w). \tag{27}$$

In view of the uniqueness of the coefficients of a Chebyshev polynomial series, like coefficients can be equated, so one obtains

$$u_0 = \frac{2}{\pi^2\left[C - \ln\left(\frac{2}{w}\right)\right]}\int_{x=-w}^{w}\frac{f(x)}{\sqrt{w^2 - x^2}}\,dx \tag{28}$$

and

$$u_n = -\frac{2n}{\pi^2}\int_{x=-w}^{w}\frac{f(x)}{\sqrt{w^2 - x^2}}T_n(x/w)\,dx,$$

where use is made of (25b). Since f is known, u_n of (28) can be determined in principle and with these coefficients in the series (26a) one has a solution of (20) with a general forcing function f.

[3] A reviewer has pointed out that these equations can be solved by singular integral equation methods [14], [15].

Solution of (21)

Equation (21) can be solved by a method which differs from that applied to (20) only in the selection of series and transformations. In this case we expand f and u of (21) in a series of Chebyshev polynomials of the second kind U_n, with the expansion for u satisfying the edge condition *a priori*:

$$f(x) = \sum_{n=0}^{\infty} f_n' U_n(x/w), \tag{29a}$$

where

$$f_n' = \frac{2}{\pi w^2} \int_{x=-w}^{w} \sqrt{w^2 - x^2}\, f(x) U_n(x/w)\, dx, \tag{29b}$$

and

$$u(x) = \sqrt{w^2 - x^2} \sum_{n=0}^{\infty} u_n' U_n(x/w), \tag{30a}$$

where

$$u_n' = \frac{2}{\pi w^2} \int_{x=-w}^{w} u(x) U_n(x/w)\, dx. \tag{30b}$$

With (29a) and (30a) in (21) we invoke (A13) and (29b) to find that the coefficients u_n' are

$$u_n' = \frac{2}{\pi^2 w^2 (n+1)} \int_{x=-w}^{w} \sqrt{w^2 - x^2}\, f(x) U_n(x/w)\, dx$$

for the series solution of (21) (series (30a)) due to a general forcing function f.

V. SUMMARY

In this paper are presented *exact* solutions to the *approximate* integral equations for narrow slots in screens and narrow strips excited by TE and TM illumination. Each general solution is expressed as a product of a series of Chebyshev polynomials and a function exhibiting the edge condition for the particular polarization. A table is provided as an aid in the interpretation of duality of narrow slots and strips and as a guide to the application of the solutions.

APPENDIX

CHEBYSHEV POLYNOMIALS AND INTEGRAL OPERATORS WITH LOGARITHMIC KERNELS

The first integral we wish to evaluate is

$$I_1 = \int_{\xi'=-1}^{1} \frac{T_n(\xi')}{\sqrt{1-\xi'^2}} \ln|\xi - \xi'|\, d\xi', \qquad \xi \in (-1, 1), \tag{A1}$$

where $T_n(\xi)$ is the Chebyshev polynomial of the first kind. Gladwell and Coen [16] evaluate this integral by making use of a finite Hilbert transform pair which arises in the theory of singular integral equations [15]. We present here a more direct procedure.

Making use of [14]

$$\ln|\cos\phi - \cos\phi'| = -\ln 2 - \sum_{m=1}^{\infty} \frac{2}{m} \cos m\phi \cos m\phi', \tag{A2}$$

and substituting $\xi = \cos\phi$, $\xi' = \cos\phi'$, and $T_n(\cos\phi') = \cos n\phi'$ [17], we obtain

$$I_1 = \int_{\phi'=0}^{\pi} \cos n\phi' \left\{ -\ln 2 - \sum_{m=1}^{\infty} \frac{2}{m} \cos m\phi \cos m\phi' \right\} d\phi' \tag{A3}$$

which reduces readily to

$$I_1 = \begin{cases} -\pi (\ln 2) T_0(\xi), & n = 0 \\ -\dfrac{\pi}{n} T_n(\xi), & n > 0. \end{cases} \tag{A4}$$

Hence, with ξ and ξ' replaced by x/w and x'/w, respectively, one arrives at the desired transformation:

$$\int_{x'=-w}^{w} \frac{T_n(x'/w)}{\sqrt{w^2 - x'^2}} \ln|x - x'|\, dx' = \begin{cases} -\pi \ln\left(\dfrac{2}{w}\right) T_0(x/w), & n = 0 \\ -\dfrac{\pi}{n} T_n(x/w), & n > 0. \end{cases} \tag{A5}$$

Combining appropriately the transformations (A5) for $n = 0, 1, 2, 3$, one can obtain the useful integral relationships below:

$$\int_{x'=-w}^{w} \frac{1}{\sqrt{w^2 - x'^2}} \ln|x - x'|\, dx' = \pi \ln\left(\frac{w}{2}\right), \tag{A6a}$$

$$\int_{x'=-w}^{w} \frac{x'}{\sqrt{w^2 - x'^2}} \ln|x - x'|\, dx' = -\pi x, \tag{A6b}$$

$$\int_{x'=-w}^{w} \frac{x'^2}{\sqrt{w^2 - x'^2}} \ln|x - x'|\, dx' = -\pi\left[\tfrac{1}{2}x^2 - \left(\frac{w}{2}\right)^2 - \tfrac{1}{2} w^2 \ln\left(\frac{w}{2}\right)\right], \tag{A6c}$$

$$\int_{x'=-w}^{w} \frac{x'^3}{\sqrt{w^2 - x'^2}} \ln|x - x'|\, dx' = -\pi\left[\tfrac{1}{3}x^3 + \tfrac{1}{2} w^2 x\right]. \tag{A6d}$$

In (A6), x must fall on the real line between $-w$ and $+w$.

Next, we wish to evaluate

$$I_2 = \frac{d^2}{d\xi^2} \int_{\xi'=-1}^{1} U_n(\xi')\sqrt{1-\xi'^2} \ln|\xi - \xi'|\, d\xi', \qquad \xi \in (-1, 1). \tag{A7}$$

where $U_n(\xi)$ is the Chebyshev polynomial of the second kind. One makes the same substitutions as before and invokes the definition $U_n(\cos\phi) = \sin(n+1)\phi/\sin\phi$ [17] to convert the intermediate integral

$$I_3 = \int_{\xi'=-1}^{1} U_n(\xi')\sqrt{1-\xi'^2}\ln|\xi-\xi'|\,d\xi' \tag{A8}$$

to

$$I_3 = \int_{\phi'=0}^{\pi} \sin\phi' \sin(n+1)\phi' \cdot \left[-\ln 2 - \sum_{m=1}^{\infty} \frac{2}{m}\cos m\phi \cos m\phi'\right] d\phi', \tag{A9}$$

which can be reduced to

$$I_3 = \begin{cases} -\dfrac{\pi}{2}(\ln 2) + \dfrac{\pi}{4}\cos 2\phi, & n = 0 \\ \dfrac{\pi}{2}\left[\dfrac{1}{n+2}\cos[(n+2)\phi] - \dfrac{1}{n}\cos n\phi\right], & n > 0. \end{cases} \tag{A10}$$

We observe that

$$\frac{d}{d\xi} I_3 = -\frac{1}{\sin\phi}\frac{d}{d\phi} I_3 = \pi\cos[(n+1)\phi] \tag{A11}$$

and, subsequently, that

$$I_2 = \frac{d^2}{d\xi^2} I_3 = -\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\frac{d}{d\xi} I_3\right) = \pi(n+1)U_n(\xi), \tag{A12}$$

from which it follows that

$$\frac{d^2}{dx^2}\int_{x'=-w}^{w} U_n(x'/w)\sqrt{w^2-x'^2}\ln|x-x'|\,dx' = \pi(n+1)\,U_n(x/w), \qquad x \in (-w, w), \tag{A13}$$

the second transformation needed in Section IV.

Though ancillary to the present paper, we point out that the transformations (A5) and (A13) are explicit identifications of eigenvalues/eigenfunctions of the given operators.

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