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Analytic methods and free-space dyadic Green's functions

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A number of mathematical techniques are presented which have proven successful in obtaining analytic solutions to the differential equations for the dyadic Green's functions of electromagnetic theory. The emphasis is on infinite-medium (or free-space) time-harmonic solutions throughout, thus putting the focus on the physical medium in which the electromagnetic process takes place. The medium's properties enter Maxwell's equations through the constitutive relations, and a comprehensive listing of dyadic Green's functions for which closed-form solutions exist, is given. Presently, the list of media contains (achiral) isotropic, biisotropic (including chiral), generally uniaxial, electrically (or magnetically) gyrotropic, diffusive and moving media as well as certain plasmas. A critical evaluation of the achievements, successes, limits, and failures of the analytic techniques is provided, and a prognosis is put forward about the future place of analytic methods within the general context of the search for solutions to electromagnetic field problems.

1. INTRODUCTION

Throughout the field of electromagnetic theory the ultimate goal is to find solutions of Maxwell's equations. It goes without saying that in general this constitutes a very difficult task. Analytic solutions will be obtainable only for simple problems, while in most cases approximative or numerical solutions must be considered. With the availability of increasingly powerful and fast computers there is an understandable tendency nowadays, not just in electromagnetic theory but in most areas of the physical and technical sciences, to emphasize the numerical approach to a given physical problem.

In this paper an effort will be made to convince the reader that in this age of supercomputers and connection machines, analytic techniques can still contribute a great deal to foster our understanding of the physical processes under consideration. The main purpose of this contribution is therefore to present an overview of the achievements of analytic techniques applied to the dyadic Green's functions. During the development of all kinds of (linear) field theories, Green's functions have played an important role. They permit a general representation of the fields without an ab initio specification of the fields' sources (which in case of electromagnetic theory are the electric and magnetic current density distribution). It is this feature together with the adaptability of Green's functions to different boundary conditions which constitutes the main attraction of the method. See *Felsen and Marcuvitz* [1973], *Marcuvitz* [1969], *Tai* [1971], for detailed introductions to dyadic Green's functions.

The main obstacles to finding analytic solutions of Maxwell's equations mainly stem from three different types of mathematical complications: radiating sources of complicated structures; nontrivial boundary geometry; and complex media.

In choosing the Green's function method the first problem has been eliminated by definition because the technique assumes a unit (dyadic) source term. It should be kept in mind, however, that even after obtaining the dyadic Green's functions a considerable effort may have to be expended to calculate the electromagnetic fields from the integrals involving the dyadic Green's functions and the actual source distributions (see equation (7) or (10)).

The emphasis of this paper will be on infinite-medium (or free-space) dyadic Green's functions; that is, boundaries are of no interest here (it is noted that the infinite-medium dyadic Green's functions can be adapted easily to the presence of perfectly conducting half-spaces by a reflection transformation). Therefore the only boundary condition imposed will be a radiation condition at infinity (demanding that solutions are outward moving waves). It is therefore the media that constitute the main focus in the following sections. The influence of the media in which the electromagnetic process takes place enters Maxwell's equations through the constitutive relations.

The purpose of this paper is twofold. On the one hand, its purpose is to draw attention to the importance of analytic techniques. It also will hopefully serve as an up-to-date compendium for representations of dyadic Green's

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functions for a number of isotropic, anisotropic, and other electromagnetic media. Section 2 will lay down the mathematical foundation of the method of dyadic Green's functions. It will be followed in section 3 by a short description of some of the mathematical techniques which have proven useful in obtaining analytic solutions for dyadic Green's functions. In section 4, such analytic solutions for dyadic Green's functions will be written down for a variety of different media. Section 5 will go on to critically evaluate the successes and failures of the analytic techniques as discussed here in order to put its achievements into proper perspective. Some conclusions are drawn up in the final section 6.

2. MATHEMATICAL FORMULATION

Throughout the paper a time dependence of $\exp(i\omega t)$ is assumed; that is, investigations are carried out in the time-harmonic regime; t stands for time, and ω is the frequency (the ω dependence will be supressed in all equations). In order to obtain results in the time domain, various Fourier integrals have to be evaluated. This procedure can be a complicated one due to the effect of frequency dispersion: in general, the constitutive parameters in the frequency domain will depend on the frequency ω. Restricting attention to time-harmonic effects eliminates this problem. Thus Maxwell's equations can be written as

$$\begin{aligned} -i\omega\,\mathbf{D}(\mathbf{x}) \;+\; \nabla\times\mathbf{H}(\mathbf{x}) \;&=\; +\mathbf{J}(\mathbf{x}), \\ \nabla\times\mathbf{E}(\mathbf{x}) \;+\; i\omega\,\mathbf{B}(\mathbf{x}) \;&=\; -\mathbf{M}(\mathbf{x}). \end{aligned} \tag{1}$$

The vector fields $\mathbf{E}(\mathbf{x}), \mathbf{D}(\mathbf{x}), \mathbf{H}(\mathbf{x})$, and $\mathbf{B}(\mathbf{x})$ stand for electric field, dielectric displacement, magnetic field, and magnetic induction; $\mathbf{J}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})$ represent electric and magnetic current density distribution; ∇ is the derivative operator, and $\mathbf{x}$ is the radius vector.

The notation is dyadic throughout the paper. Vectors appear in boldface and dyadic quantities (including dyadic differential operators) appear in boldface with an overbar (with the exception of 0 which can stand for constant, vector, or dyadic). Contraction of indices is symbolized by a dot; that is, $\mathbf{a}\cdot\mathbf{b}$ stands for $\sum_i a_i b_i$, whereas $\overline{\mathbf{A}} = \mathbf{a}\,\mathbf{b}$ is a dyadic $A_{ij} = a_i\, b_j$.

The complete information about the structure of the physical medium is contained in the constitutive relations. For the most general form of a linear medium, these are

$$\begin{aligned} \mathbf{D}(\mathbf{x}) \;&=\; \overline{\boldsymbol{\epsilon}}(\mathbf{x},\nabla)\cdot\mathbf{E}(\mathbf{x}) \;+\; \overline{\boldsymbol{\xi}}(\mathbf{x},\nabla)\cdot\mathbf{H}(\mathbf{x}), \\ \mathbf{B}(\mathbf{x}) \;&=\; \overline{\boldsymbol{\zeta}}(\mathbf{x},\nabla)\cdot\mathbf{E}(\mathbf{x}) \;+\; \overline{\boldsymbol{\mu}}(\mathbf{x},\nabla)\cdot\mathbf{H}(\mathbf{x}). \end{aligned} \tag{2}$$

In equations (2), $\overline{\boldsymbol{\epsilon}}$ and $\overline{\boldsymbol{\mu}}$ symbolize the well-known permittivity and permeability dyadics, whereas $\overline{\boldsymbol{\xi}}$ and $\overline{\boldsymbol{\zeta}}$ are the magnetoelectric dyadics, responsible for the bianisotropy of the medium. All these dyadics may depend on $\mathbf{x}$; that is, inhomogeneous media are included. Rather than being simple dyadics, they can be dyadic differential operators in itself, indicated by the dependence on ∇.

In full generality, the currents $\mathbf{J}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})$ in the equations in (1) should be the total currents, the sum of impressed and induced currents. Therefore the constitutive relations in (2) would have to be supplemented by similar functional relations for $\mathbf{J}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})$. So, for example,

$$\mathbf{J}(\mathbf{x}) \;=\; \mathbf{J}_s(\mathbf{x}) \;+\; \overline{\mathbf{s}}(\mathbf{x},\nabla)\cdot\mathbf{E}(\mathbf{x}) \;+\; \overline{\mathbf{t}}(\mathbf{x},\nabla)\cdot\mathbf{H}(\mathbf{x}), \tag{3}$$

where $\mathbf{J}_s(\mathbf{x})$ contains the externally applied sources with new dyadics $\overline{\mathbf{s}}$ and $\overline{\mathbf{t}}$ ($\overline{\mathbf{s}}$ is usually labeled conductivity), and a similar expression for $\mathbf{M}(\mathbf{x})$. However, it can be seen quite easily that any dyadics $\overline{\mathbf{s}}$ and $\overline{\mathbf{t}}$ can be incorporated by substituting $i\omega\overline{\boldsymbol{\epsilon}} \longrightarrow i\omega\overline{\boldsymbol{\epsilon}}+\overline{\mathbf{s}}$ and $i\omega\overline{\boldsymbol{\xi}} \longrightarrow i\omega\overline{\boldsymbol{\xi}}+\overline{\mathbf{t}}$ in equations (1) and (2). Without loss of generality, the vector fields $\mathbf{J}(\mathbf{x})$ and $\mathbf{M}(\mathbf{x})$ in equations (1) are therefore interpreted as the externally applied sources.

Equations (1) and (2) can be written in the compact form

$$\overline{\mathbf{M}}(\mathbf{x},\nabla)\cdot\mathbf{F}(\mathbf{x}) = \Big(i\omega\,\overline{\mathbf{K}}(\mathbf{x},\nabla)+\overline{\mathbf{D}}(\nabla)\Big)\cdot\mathbf{F}(\mathbf{x}) = -\mathbf{Q}(\mathbf{x}), \tag{4}$$

where, as can be seen in the notation of *Suchy et al.* [1985], the dyadics $\overline{\mathbf{K}}(\mathbf{x},\nabla)$ and $\overline{\mathbf{D}}(\nabla)$ are given by

$$\begin{aligned} \overline{\mathbf{K}}(\mathbf{x},\nabla) \;&=\; \begin{pmatrix} \overline{\boldsymbol{\epsilon}}(\mathbf{x},\nabla) & \overline{\boldsymbol{\xi}}(\mathbf{x},\nabla) \\ \overline{\boldsymbol{\zeta}}(\mathbf{x},\nabla) & \overline{\boldsymbol{\mu}}(\mathbf{x},\nabla) \end{pmatrix}, \\ \overline{\mathbf{D}}(\nabla) \;&=\; \begin{pmatrix} 0 & -\nabla\times\overline{\mathbf{I}} \\ +\nabla\times\overline{\mathbf{I}} & 0 \end{pmatrix}, \end{aligned} \tag{5}$$

whereas the field vector $\mathbf{F}(\mathbf{x})$ and the source vector $\mathbf{Q}(\mathbf{x})$ are simply

$$\mathbf{F}(\mathbf{x}) \;=\; \begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{x}) \;=\; \begin{pmatrix} \mathbf{J}(\mathbf{x}) \\ \mathbf{M}(\mathbf{x}) \end{pmatrix}, \tag{6}$$

and $\overline{\mathbf{I}}$ is the unit dyadic in three dimensions. Because of the linearity of the dyadic differential equation (4), its solution can be represented in the following form

$$\mathbf{F}(\mathbf{x}) \;=\; \int_{V'} d^3\mathbf{x}'\;\overline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}')\cdot\mathbf{Q}(\mathbf{x}'), \tag{7}$$

where the integration extends over the volume in which the sources are nonzero. As a consequence of (7) the dyadic (super) Green's function $\overline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}')$ must be a solution of the dyadic differential equation:

$$\overline{\mathbf{M}}(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}') \;=\; -\,\delta\,(\mathbf{x}-\mathbf{x}')\,\overline{\mathbf{I}}_S, \tag{8}$$

where $\overline{\mathbf{M}}(\mathbf{x},\nabla) = i\omega\overline{\mathbf{K}}(\mathbf{x},\nabla)+\overline{\mathbf{D}}(\nabla)$ as above; $\delta\,(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function and $\overline{\mathbf{I}}_S$ is the 6×6 unit dyadic.

While the above compact notation is useful for formal purposes, for actual calculations it is easier to revert to standard 3 × 3 dyadics. Then

$$\begin{aligned} \overline{\mathbf{G}}_S(\mathbf{x},\mathbf{x}') &= \begin{pmatrix} \overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') & \overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') \\ \overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') & \overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') \end{pmatrix}, \\ \overline{\mathbf{I}}_S &= \begin{pmatrix} \overline{\mathbf{I}} & 0 \\ 0 & \overline{\mathbf{I}} \end{pmatrix}. \end{aligned} \tag{9}$$

In equation (9), $\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')$, $\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')$, $\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')$, and $\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}')$ are the (standard 3 × 3) dyadic Green's functions of the electric type, the magnetic type and the mixed (or sometimes called hybrid) type; from now on $\overline{\mathbf{I}}$ is the unit dyadic in three dimensions. With (9) and (6), equation (7) reads

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \int_{V'} d^3x' \Big(\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}') + \\ &\qquad \overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')\cdot\mathbf{M}(\mathbf{x}')\Big), \\ \mathbf{H}(\mathbf{x}) &= \int_{V'} d^3x' \Big(\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}') + \\ &\qquad \overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')\cdot\mathbf{M}(\mathbf{x}')\Big). \end{aligned} \tag{10}$$

The differential equations for $\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')$ and $\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}')$ are then given by

$$\begin{aligned} & i\omega\,\overline{\boldsymbol{\epsilon}}\cdot\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') + \\ &\quad (i\omega\,\overline{\boldsymbol{\xi}} - \nabla\times\overline{\mathbf{I}})\cdot\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') = -\overline{\mathbf{I}}\,\delta(\mathbf{x}-\mathbf{x}'), \\ & (i\omega\,\overline{\boldsymbol{\zeta}} + \nabla\times\overline{\mathbf{I}})\cdot\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') + i\omega\,\overline{\boldsymbol{\mu}}\cdot\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') = 0, \end{aligned} \tag{11}$$

and those for $\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')$ and $\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')$ are

$$\begin{aligned} & i\omega\,\overline{\boldsymbol{\epsilon}}\cdot\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') + (i\omega\,\overline{\boldsymbol{\xi}} - \nabla\times\overline{\mathbf{I}})\cdot\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') = 0, \\ & (i\omega\,\overline{\boldsymbol{\zeta}} + \nabla\times\overline{\mathbf{I}})\cdot\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') + \\ &\quad i\omega\,\overline{\boldsymbol{\mu}}\cdot\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') = -\overline{\mathbf{I}}\,\delta(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{12}$$

Equations (11) and (12) form two systems of dyadic partial differential equations of first order. Decoupling leads to the following dyadic differential equations of second order

$$\begin{aligned} & \overline{\mathbf{L}}_e(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') = -i\omega\,\overline{\mathbf{I}}\,\delta(\mathbf{x}-\mathbf{x}'), \\ & \overline{\mathbf{L}}_m(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') = (\nabla\times\overline{\mathbf{I}} + i\omega\,\overline{\boldsymbol{\zeta}})\cdot\overline{\boldsymbol{\epsilon}}^{-1}\,\delta(\mathbf{x}-\mathbf{x}'), \\ & \overline{\mathbf{L}}_e(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') = -(\nabla\times\overline{\mathbf{I}} - i\omega\,\overline{\boldsymbol{\xi}})\cdot\overline{\boldsymbol{\mu}}^{-1}\,\delta(\mathbf{x}-\mathbf{x}'), \\ & \overline{\mathbf{L}}_m(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') = -i\omega\,\overline{\mathbf{I}}\,\delta(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{13}$$

where the differential operators $\overline{\mathbf{L}}_e(\mathbf{x},\nabla)$ and $\overline{\mathbf{L}}_m(\mathbf{x},\nabla)$ have been introduced according to

$$\begin{aligned} \overline{\mathbf{L}}_e(\mathbf{x},\nabla) &= (\nabla\times\overline{\mathbf{I}} - i\omega\,\overline{\boldsymbol{\xi}})\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot(\nabla\times\overline{\mathbf{I}} + i\omega\,\overline{\boldsymbol{\zeta}}) - \omega^2\,\overline{\boldsymbol{\epsilon}}, \\ \overline{\mathbf{L}}_m(\mathbf{x},\nabla) &= (\nabla\times\overline{\mathbf{I}} + i\omega\,\overline{\boldsymbol{\zeta}})\cdot\overline{\boldsymbol{\epsilon}}^{-1}\cdot(\nabla\times\overline{\mathbf{I}} - i\omega\,\overline{\boldsymbol{\xi}}) - \omega^2\,\overline{\boldsymbol{\mu}}. \end{aligned} \tag{14}$$

If $\overline{\mathbf{L}}_e$ and $\overline{\mathbf{L}}_m$ are symbolically written as $\overline{\mathbf{L}}_e(\mathbf{x},\nabla;\overline{\boldsymbol{\epsilon}},\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\xi}},\overline{\boldsymbol{\zeta}})$ and $\overline{\mathbf{L}}_m(\mathbf{x},\nabla;\overline{\boldsymbol{\epsilon}},\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\xi}},\overline{\boldsymbol{\zeta}})$, then the functional relationship $\overline{\mathbf{L}}_e(\mathbf{x},\nabla;\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\epsilon}},-\overline{\boldsymbol{\zeta}},-\overline{\boldsymbol{\xi}}) = \overline{\mathbf{L}}_m(\mathbf{x},\nabla;\overline{\boldsymbol{\epsilon}},\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\xi}},\overline{\boldsymbol{\zeta}})$ holds.

It would of course be tedious having to solve all four equations in (13) independently. In practical situations, after having obtained $\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')$, say, it is straightforward to derive $\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}')$ from the second of equations in (11):

$$\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') = -\frac{1}{i\omega}\,\overline{\boldsymbol{\mu}}^{-1}\cdot(\nabla\times\overline{\mathbf{I}} + i\omega\,\overline{\boldsymbol{\zeta}})\cdot\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}'), \tag{15}$$

and similarly, $\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')$ follows from $\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')$ via the first equation of (12):

$$\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') = +\frac{1}{i\omega}\,\overline{\boldsymbol{\epsilon}}^{-1}\cdot(\nabla\times\overline{\mathbf{I}} - i\omega\,\overline{\boldsymbol{\xi}})\cdot\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}'). \tag{16}$$

Additionally, it is sometimes possible to exploit symmetry properties of the constitutive dyadics $\overline{\boldsymbol{\epsilon}},\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\xi}},\overline{\boldsymbol{\zeta}}$, such that the magnetic type Green's dyadics $\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')$ and $\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')$ can be established directly from their electric counterparts $\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')$ and $\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}')$ (or vice versa).

It is emphasized that the only assumptions about the structure of $\overline{\boldsymbol{\epsilon}},\overline{\boldsymbol{\mu}},\overline{\boldsymbol{\xi}}$, and $\overline{\boldsymbol{\zeta}}$ are that their inverses exist or can be satisfactorily dealt with. Otherwise, the formulation is fully general.

3. ANALYTIC TECHNIQUES

In this section a small number of techniques will be discussed which may be used to obtain analytic solutions of the differential equations of the dyadic Green's functions in (13). The list of methods is of course not to be understood as comprehensive, but it reflects to a certain extent the author's own preferences.

One method, among the most successful ones used in the past, is Fourier transformation. The dyadics $\overline{\mathbf{G}}$ are transformed from $\mathbf{x}$ space into $\mathbf{k}$ space ($\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') \rightarrow \overline{\mathbf{G}}(\mathbf{k},\mathbf{x}')$) and the resulting algebraic problem is solved in Fourier ($\mathbf{k}$) space. The main complication arises when the inverse transformation (from $\mathbf{k}$ to $\mathbf{x}$ space) has to be carried out. This method is mentioned here only briefly because it is covered in detail in most textbooks on advanced electromagnetic theory. However, Fourier transformation has been used recently to find a representation of the dyadic Green's function (of the electric type) of a special anisotropic medium, a so-called biaxial medium [*Kaklamani and Uzunoglu*, 1992]. The obtained solution is in terms of an integral representation but closed-form expressions for the near and far field were successfully extracted [see also *Weiglhofer and Lakhtakia*, 1993].

In the following our attention will be focused on four techniques: simple matrix methods, scalarization, dyadic technique, and decomposition.

3.1. *Simple matrix methods*

Within the framework of this method, which is simple to use and therefore appealing, one adopts a strategy where a dyadic differential operator is treated as if it were a simple dyadic or matrix [*Weiglhofer*, 1988]. Then it is straightforward to apply results from matrix/dyadic inversion theory to find its inverse. However, the inverse of the dyadic differential operator is nothing else but the dyadic Green's function.

To exemplify the method, let $\overline{\mathbf{N}}$ be a dyadic given by $\overline{\mathbf{N}} = \lambda\overline{\mathbf{I}} + \mathbf{c}\,\mathbf{d}$, where $\mathbf{c}$ and $\mathbf{d}$ are vectors, λ is a constant, and $\overline{\mathbf{I}}$ is the unit dyadic. Then in order to find the dyadic $\overline{\mathbf{A}}$ satisfying $\overline{\mathbf{N}}\cdot\overline{\mathbf{A}} = \overline{\mathbf{I}}$, $\overline{\mathbf{N}}$ is inverted to give [*Chen*, 1983]

$$\overline{\mathbf{A}} = \overline{\mathbf{N}}^{-1} = \frac{\operatorname{adj}\overline{\mathbf{N}}}{\det\overline{\mathbf{N}}}, \tag{17}$$

where $^{-1}$ refers to the inverse and the adjoint and determinant of $\overline{\mathbf{N}}$ are

$$\begin{aligned} \operatorname{adj}\overline{\mathbf{N}} &= \lambda\left[(\lambda+\mathbf{c}\cdot\mathbf{d})\overline{\mathbf{I}} - \mathbf{c}\,\mathbf{d}\right], \\ \det\overline{\mathbf{N}} &= \lambda^2(\lambda+\mathbf{c}\cdot\mathbf{d}). \end{aligned} \tag{18}$$

Therefore

$$\overline{\mathbf{A}} = \overline{\mathbf{N}}^{-1} = \frac{1}{\lambda}\overline{\mathbf{I}} - \frac{\mathbf{c}\,\mathbf{d}}{\lambda(\lambda+\mathbf{c}\,\mathbf{d})}. \tag{19}$$

In trying to find a solution to the dyadic Green's function equation $\overline{\mathbf{L}}\cdot\overline{\mathbf{G}} = \overline{\mathbf{I}}\,\delta$ (this definition of $\overline{\mathbf{G}}$ differs by a factor of $-i\omega$ from that in section 2), where

$$\overline{\mathbf{L}} = \nabla\times(\nabla\times\overline{\mathbf{I}}) - k^2\,\overline{\mathbf{I}} = -(\nabla^2+k^2)\overline{\mathbf{I}} + \nabla\nabla, \tag{20}$$

k is a constant (this operator corresponds to an isotropic, homogeneous medium), one would then simply identify λ with the (negative) Helmholtz operator $-(\nabla^2+k^2)$ and $\mathbf{c} = \mathbf{d}$ with ∇. This leads, in complete analogy to the above derviation to

$$\overline{\mathbf{G}} = \overline{\mathbf{L}}^{-1}\delta = \frac{adj\,\overline{\mathbf{L}}}{det\,\overline{\mathbf{L}}}\,\delta = -\left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right)(\nabla^2+k^2)^{-1}\,\delta. \tag{21}$$

Upon interpreting $(\nabla^2+k^2)^{-1}\,\delta$ as a scalar Green's function $g(\mathbf{x},\mathbf{x}')$ fulfilling

$$(\nabla^2+k^2)\,g(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \tag{22}$$

one obtains

$$\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{x},\mathbf{x}'), \tag{23}$$

the complete representation of $\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ in terms of a scalar Green's function g which itself is well known (see section 4.1.). In equation (21), *adj* and *det* have been typeset in italics to indicate that the terms adjoint and determinant are not to be understood literally.

Applying the simple matrix method is very much limited to homogeneous media because it depends crucially on the commutativity of scalar and dyadic differential operators. In a sense the technique is related to the Fourier transformation method, albeit applying the algebraic techniques in $\mathbf{x}$ rather than $\mathbf{k}$ space. It can certainly be useful but may be criticized on the basis of a certain lack of mathematical rigor.

3.2. *Scalarization*

In this technique, the differential equation for a dyadic Green's function,

$$\overline{\mathbf{L}}(\mathbf{x},\nabla)\cdot\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \overline{\mathbf{I}}\,\delta(\mathbf{x}-\mathbf{x}'), \tag{24}$$

say, is manipulated by vector and dyadic operations until it can be recast into the form

$$H\,\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \overline{\mathbf{W}}(\mathbf{x},\nabla)\,\delta(\mathbf{x}-\mathbf{x}'), \tag{25}$$

where H is now a scalar differential operator and $\overline{\mathbf{W}}$ is a dyadic differential operator. Equation (25) suggests a representation of the dyadic Green's function in the form

$$\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \overline{\mathbf{W}}'g(\mathbf{x},\mathbf{x}'), \tag{26}$$

with a new dyadic operator $\overline{\mathbf{W}}'$. The scalar Green's function $g(\mathbf{x},\mathbf{x}')$ is a solution of

$$H'\,g(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'), \tag{27}$$

with a new scalar operator H'. Putting (26) and (27) into (25) leads to an operator condition for H' and $\overline{\mathbf{W}}'$,

$$H\,\overline{\mathbf{W}}' = \overline{\mathbf{W}}\,H'. \tag{28}$$

For homogeneous media, that is, when the constitutive dyadics do not depend on $\mathbf{x}$, it can be seen that the simplification

$$H' = H, \qquad \overline{\mathbf{W}}' = \overline{\mathbf{W}}, \tag{29}$$

applies and thus

$$\begin{aligned} \overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') &= \overline{\mathbf{W}}(\mathbf{x},\nabla)\,g(\mathbf{x},\mathbf{x}'), \\ H\,g(\mathbf{x},\mathbf{x}') &= \delta(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{30}$$

The main achievement of this technique lies in the reduction of the dyadic problem for the dyadic Green's function $\overline{\mathbf{G}}$ to a scalar problem for g; because now a full solution to the dyadic differential equation (24) can be established if the scalar differential equation (27) (or the second equation of (30)) can be solved. This method has been used successfully for a number of different media [*Weiglhofer*, 1989*a*, *b*, 1990, 1991, 1992, 1993].

3.3. *Dyadic technique*

Another method discussed briefly in this paper uses the dyadic techniques originally developed by Gibbs which were subsequently applied to the equations of electromagnetic theory by *Lindell* [1973] (a more detailed description is given by *Lindell* [1992]). Starting from a dyadic differential equation $\overline{\mathbf{L}} \cdot \overline{\mathbf{G}} = \overline{\mathbf{I}}\,\delta$ as usual, a new dyadic operator $\tilde{\overline{\mathbf{L}}}$ is defined such that

$$\tilde{\overline{\mathbf{L}}}^T \cdot \overline{\mathbf{L}} = (det\,\overline{\mathbf{L}})\,\overline{\mathbf{I}}\,, \tag{31}$$

(^{T}: transposed). Thus the dyadic Green's function $\overline{\mathbf{G}}$ can be represented as

$$\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}') = \tilde{\overline{\mathbf{L}}}^T g(\mathbf{x},\mathbf{x}')\,, \tag{32}$$

and the scalar Green's function $g(\mathbf{x},\mathbf{x}')$ must be a solution of

$$(det\,\overline{\mathbf{L}})\,g(\mathbf{x},\mathbf{x}') = \delta\,(\mathbf{x}-\mathbf{x}')\,. \tag{33}$$

A scalarization has therefore been achieved as well. It is evident that this technique has some parallels with the matrix and scalarization methods. Unlike in the scalarization procedure, however, where the dyadic differential equation for $\overline{\mathbf{G}}$ is manipulated, here an algorithm exists such that for any given operator $\overline{\mathbf{L}}$ the operators $\tilde{\overline{\mathbf{L}}}$ and $det\,\overline{\mathbf{L}}$ can be constructed directly. *Lindell* [1973] showed that

$$\tilde{\overline{\mathbf{L}}} = \frac{1}{2}\left(\overline{\mathbf{L}}\,{}^{\times}_{\times}\,\overline{\mathbf{L}}\right)\,, \qquad det\,\overline{\mathbf{L}} = \frac{1}{6}\left(\overline{\mathbf{L}}\,{}^{\times}_{\times}\,\overline{\mathbf{L}} : \overline{\mathbf{L}}\right)\,. \tag{34}$$

The double-cross and double-scalar product for dyadics which appear in equation (34) are defined in the following way: $(\overline{\mathbf{A}}\,{}^{\times}_{\times}\,\overline{\mathbf{B}})_{im} = \sum_{jknp} \epsilon_{ijk}\epsilon_{mnp} A_{jn} B_{kp}$ holds for two dyadics $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$. (ϵ_{ijk} is the completely antisymmetric tensor of rank 3) and $\overline{\mathbf{A}} : \overline{\mathbf{B}} = \sum_{ij} A_{ij} B_{ij}$. See *Lindell* [1992] for more details.

Equations (34) provide a recipe to construct, in principle, the complete solution for a general, homogeneous, anisotropic medium. From a technical point of view, the calculations, similar to the other techniques discussed earlier, very quickly become unmanageable as the complexity of the medium increases.

3.4. *Decomposition*

If a medium displays a certain symmetry, for example, a distinguished axis such as in the case of uniaxial and gyrotropic media, it can be useful to decompose the dyadic Green's function accordingly. If we let $\mathbf{c}$ be a unit vector in direction of a distinguished axis, then $\overline{\mathbf{G}}(\mathbf{x},\mathbf{x}')$ can be decomposed into

$$\overline{\mathbf{G}} = \overline{\mathbf{G}}_t + \mathbf{c}\,\mathbf{G}_{\mathbf{c}} + \mathbf{G}_{\cdot\mathbf{c}}\,\mathbf{c} + G\,\mathbf{c}\,\mathbf{c}\,, \tag{35}$$

where $\overline{\mathbf{G}}_t$ is the (transversal) 2×2 dyadic ($\overline{\mathbf{G}}_t \cdot \mathbf{c} = \mathbf{c} \cdot \overline{\mathbf{G}}_t = 0$); $\mathbf{G}_{\mathbf{c}}$ and $\mathbf{G}_{\cdot\mathbf{c}}$ are transversal two-component vectors ($\mathbf{G}_{\cdot\mathbf{c}} \cdot \mathbf{c} = \mathbf{G}_{\mathbf{c}\cdot} \cdot \mathbf{c} = 0$). Decomposing the dyadic Green's function equation along the same lines then leads to coupled equations for $\overline{\mathbf{G}}_t, \mathbf{G}_{\mathbf{c}\cdot}, \mathbf{G}_{\cdot\mathbf{c}}$ and G for which analytic solutions may be found. Reassembling the components according to (35) then provides the full solution [*Weiglhofer*, 1987].

4. SOLUTIONS

In this section those dyadic Green's functions will be presented for which analytic solutions or analytic representations have been obtained. For any detailed derivations the literature should be consulted. It is noted that most authors have concentrated on the dyadic Green's functions of the electric type, whereas the magnetic type (and the mixed type) have received much less attention. Among the exceptions are *Lakhtakia et al.* [1988*a*, *b*; 1989*a*], who have emphasized the importance of the magnetic-type dyadic Green's functions by pointing out that a magnetic field formulation of certain scattering and propagation problems may be more advantageous than an electric field formalism. Here electric, magnetic, and mixed-type dyadic Green's functions will be presented.

The list of media with their constitutive relations, from isotropic to special classes of anisotropic, bianisotropic and diffusive, is presented in Table 1. All medium parameters that occur in the constitutive dyadics $\overline{\epsilon}, \overline{\mu}, \overline{\xi}$, and $\overline{\zeta}$ are assumed to be constants here; that is, the media are homogeneous throughout. As far as the author is aware, the media listed in Table 1 are the only ones for which closed-form solutions for the dyadic Green's functions have been found.

(Note added in proof: Meanwhile, closed-form solutions for the dyadic Green's functions of certain classes of bianisotropic media have become available, see *Lindell and Weiglhofer* [1993*a*, *b*], and *Weiglhofer and Lindell* [1993].)

4.1. *Isotropic media*

4.1.1. *Isotropic (achiral) medium.* The homogeneous isotropic (achiral) medium is the simplest possible one. The constitutive relations are

$$\overline{\epsilon} = \epsilon\,\overline{\mathbf{I}}\,, \qquad \overline{\mu} = \mu\,\overline{\mathbf{I}}\,, \qquad \overline{\xi} = \overline{\zeta} = 0\,. \tag{36}$$

Therefore the differential operators $\overline{\mathbf{L}}_e$ and $\overline{\mathbf{L}}_m$ are

$$\mu\,\overline{\mathbf{L}}_e(\nabla) = \epsilon\,\overline{\mathbf{L}}_m(\nabla) = \nabla \times (\nabla \times \overline{\mathbf{I}}) - k^2\,\overline{\mathbf{I}}\,, \tag{37}$$

where $k^2 = \omega^2 \epsilon\mu$. The Green's functions have been obtained in the form [e.g. *Felsen and Marcuvitz*, 1973]

$$\begin{aligned}
\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu\left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right)\frac{\exp(-ikr)}{4\pi r}\,,\\
\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= (\nabla \times \overline{\mathbf{I}})\,\frac{\exp(-ikr)}{4\pi r} = -\,\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}')\,,\\
\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= (\epsilon/\mu)\,\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')\,,
\end{aligned} \tag{38}$$

where $r = |\mathbf{x}-\mathbf{x}'|$. Care must be taken when differentiating $\exp(-ikr)/r$ twice in order to obtain $\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')$ (and

TABLE 1. Constitutive Dyadics for Classes of Media for Which Closed-Form Solutions of the Dyadic Green's Functions Are Known

Medium	$\overline{\epsilon}$	$\overline{\mu}$	$\overline{\xi}$	$\overline{\zeta}$
Isotropic	$\epsilon\,\overline{\mathbf{I}}$	$\mu\,\overline{\mathbf{I}}$	0	0
Biisotropic	$\epsilon\,\overline{\mathbf{I}}$	$\mu\,\overline{\mathbf{I}}$	$\xi\,\overline{\mathbf{I}}$	$\zeta\,\overline{\mathbf{I}}$
Isotropic chiral	$\epsilon\,(\overline{\mathbf{I}}+\beta\nabla\times\overline{\mathbf{I}})$	$\mu\,(\overline{\mathbf{I}}+\beta\nabla\times\overline{\mathbf{I}})$	0	0
Uniaxial	$\epsilon_1\overline{\mathbf{I}}+(\epsilon-\epsilon_1)\mathbf{c}\,\mathbf{c}$	$\mu_1\overline{\mathbf{I}}+(\mu-\mu_1)\mathbf{c}\,\mathbf{c}$	0	0
Electrically gyrotropic	$\epsilon_1\overline{\mathbf{I}}+(\epsilon-\epsilon_1)\mathbf{c}\,\mathbf{c}\,-i\epsilon_2\mathbf{c}\times\overline{\mathbf{I}}$	$\mu\,\overline{\mathbf{I}}$	0	0
Magnetically gyrotropic	$\epsilon\,\overline{\mathbf{I}}$	$\overline{\mu}=\mu_1\overline{\mathbf{I}}+(\mu-\mu_1)\mathbf{c}\,\mathbf{c}\,-i\mu_2\mathbf{c}\times\overline{\mathbf{I}}$	0	0
Moving isotropic	$\epsilon\nu\overline{\mathbf{I}}+\epsilon(1-\nu)\mathbf{c}\,\mathbf{c}$	$\mu\nu\overline{\mathbf{I}}+\mu(1-\nu)\mathbf{c}\,\mathbf{c}$	$\beta\mathbf{c}\times\overline{\mathbf{I}}$	$-\beta\mathbf{c}\times\overline{\mathbf{I}}$
Diffusive	$\epsilon\,(\overline{\mathbf{I}}+\alpha^{-2}\nabla\nabla)$	$\mu\,(\overline{\mathbf{I}}+\beta^{-2}\nabla\nabla)$	0	0
Isotropic plasmas	$\epsilon\,\overline{\mathbf{I}}+\epsilon\delta\,(\nabla^2+\alpha^2)^{-1}\nabla\nabla$	$\mu\,\overline{\mathbf{I}}$	0	0

$\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}')$). This process generates a delta function singularity at $\mathbf{x}=\mathbf{x}'$ (see *Weiglhofer* [1989*c*] for details).

4.1.2. *Biisotropic medium.* The constitutive relations are

$$\overline{\epsilon} = \epsilon\overline{\mathbf{I}}, \quad \overline{\mu} = \mu\overline{\mathbf{I}}, \quad \overline{\xi} = \xi\overline{\mathbf{I}}, \quad \overline{\zeta} = \zeta\overline{\mathbf{I}}. \tag{39}$$

Then

$$\mu\,\overline{\mathbf{L}}_e = \epsilon\,\overline{\mathbf{L}}_m = \nabla\times(\nabla\times\overline{\mathbf{I}}) - i\omega(\xi-\zeta)\,(\nabla\times\overline{\mathbf{I}}) + (\omega^2\xi\zeta-k^2)\,\overline{\mathbf{I}}. \tag{40}$$

The Green's functions are [*Monzon*, 1990; *Lakhtakia*, 1992]

$$\begin{aligned}
\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu\,\left(\overline{\mathbf{G}}_1+\overline{\mathbf{G}}_2\right),\\
\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') &= (\gamma_1+i\omega\xi)\,\overline{\mathbf{G}}_1-(\gamma_2-i\omega\xi)\,\overline{\mathbf{G}}_2,\\
\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= -(\gamma_1-i\omega\zeta)\,\overline{\mathbf{G}}_1+(\gamma_2+i\omega\zeta)\,\overline{\mathbf{G}}_2,\\
\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= (\epsilon/\mu)\,\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') = -i\omega\epsilon\,\left(\overline{\mathbf{G}}_1+\overline{\mathbf{G}}_2\right),
\end{aligned} \tag{41}$$

where

$$\gamma_{1,2} = \frac{\omega}{2}\left(\pm i(\zeta-\xi)+\sqrt{4(\epsilon\mu-\xi\zeta)-(\zeta-\xi)^2}\right), \tag{42}$$

and

$$\overline{\mathbf{G}}_{1,2}(\mathbf{x},\mathbf{x}') = \left(\gamma_{1,2}\,\overline{\mathbf{I}}+\frac{\nabla\nabla}{\gamma_{1,2}}\pm(\nabla\times\overline{\mathbf{I}})\right)\frac{\exp(-i\gamma_{1,2}r)}{4\pi(\gamma_1+\gamma_2)r}. \tag{43}$$

The general biisotropic medium contains an important special case which has received much recent attention. By setting $\xi=-\zeta$, the medium becomes reciprocal, and the constitutive relations describe so-called isotropic chiral media. Phenomena associated with isotropic chiral media are natural optical activity and electromagnetic activity. Isotropic chiral media can be studied using a number of different sets of constitutive relations, among those is the form specified in Table 1, which are all equivalent to each other [*Lakhtakia et al.*, 1989*b*]. For derivations of Green's functions of chiral media the reader is referred to the literature [*Bassiri et al.*, 1986; *Varadan et al.*, 1987; *Weiglhofer*, 1989*a*, 1991; *Lindell*, 1990; *Lakhtakia*, 1991].

4.2. *Anisotropic media*

Anisotropic media, as opposed to the more general bianisotropic ones, are defined such that $\overline{\xi}=\overline{\zeta}=0$. Then the equations in (13) lead to

$$\begin{aligned}
\overline{\mathbf{L}}_e(\nabla) &= (\nabla\times\overline{\mu}^{-1})\cdot(\nabla\times\overline{\mathbf{I}}) - \omega^2\overline{\epsilon},\\
\overline{\mathbf{L}}_m(\nabla) &= (\nabla\times\overline{\epsilon}^{-1})\cdot(\nabla\times\overline{\mathbf{I}}) - \omega^2\overline{\mu};
\end{aligned} \tag{44}$$

symbolically, $\overline{\mathbf{L}}_e(\nabla;\overline{\epsilon},\overline{\mu})=\overline{\mathbf{L}}_m(\nabla;\overline{\mu},\overline{\epsilon})$.

4.2.1. *General uniaxial media.* The constitutive relations are [*Chen*, 1983]

$$\overline{\epsilon} = \epsilon_1\overline{\mathbf{I}}+(\epsilon-\epsilon_1)\mathbf{c}\,\mathbf{c}, \qquad \overline{\mu} = \mu_1\overline{\mathbf{I}}+(\mu-\mu_1)\mathbf{c}\,\mathbf{c}, \tag{45}$$

$\mathbf{c}$ being a unit vector. The dyadic Green's functions have been given by *Weiglhofer* [1990] (note that a time dependence of $\exp(-i\omega t)$ was used there)

$$
\begin{aligned}
\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu_1 \left(\epsilon\,\overline{\boldsymbol{\epsilon}}^{-1} + \frac{\nabla\nabla}{k_u^2}\right) g_e + i\omega\mu_1\,\overline{\mathbf{T}}, \\
\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= \epsilon\mu_1\,\overline{\boldsymbol{\mu}}^{-1}\cdot(\nabla\times\overline{\boldsymbol{\epsilon}}^{-1})\,g_e - \mu_1\,\overline{\boldsymbol{\mu}}^{-1}\cdot(\nabla\times\overline{\mathbf{T}}), \\
\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') &= -\mu\epsilon_1\overline{\boldsymbol{\epsilon}}^{-1}\cdot(\nabla\times\overline{\boldsymbol{\mu}}^{-1})\,g_m - \epsilon_1\overline{\boldsymbol{\epsilon}}^{-1}\cdot(\nabla\times\overline{\mathbf{T}}), \\
\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= -i\omega\epsilon_1 \left(\mu\,\overline{\boldsymbol{\mu}}^{-1} + \frac{\nabla\nabla}{k_u^2}\right) g_m - i\omega\epsilon_1\,\overline{\mathbf{T}}.
\end{aligned} \tag{46}
$$

In equations (46) the following abbreviations have been used: the wavenumber k_u of the uniaxial medium $k_u^2 = \omega^2\epsilon_1\mu_1$; the scalar Green's functions $g_e(\mathbf{x},\mathbf{x}')$ and $g_m(\mathbf{x},\mathbf{x}')$ are given below:

$$
g_{e,m}(\mathbf{x},\mathbf{x}') = \frac{\exp(-ik_u r_{e,m})}{4\pi r_{e,m}}, \tag{47}
$$

with $r_e^2 = \epsilon\,\mathbf{r}\cdot\overline{\boldsymbol{\epsilon}}^{-1}\cdot\mathbf{r}$ and $r_m^2 = \mu\,\mathbf{r}\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot\mathbf{r}$; $(\mathbf{r} = \mathbf{x} - \mathbf{x}')$. The dyadic $\overline{\mathbf{T}}$ is given by

$$
\overline{\mathbf{T}}(\mathbf{x},\mathbf{x}') = \Big((\epsilon/\epsilon_1)\,g_e - (\mu/\mu_1)\,g_m\Big)\frac{(\mathbf{r}\times\mathbf{c})(\mathbf{r}\times\mathbf{c})}{(\mathbf{r}\times\mathbf{c})^2} + \left(\overline{\mathbf{I}} - \mathbf{c}\,\mathbf{c} - \frac{2(\mathbf{r}\times\mathbf{c})(\mathbf{r}\times\mathbf{c})}{(\mathbf{r}\times\mathbf{c})^2}\right)\frac{(r_e\,g_e - r_m\,g_m)}{ik(\mathbf{r}\times\mathbf{c})^2}. \tag{48}
$$

4.2.2. *Electrically gyrotropic media.* Gyrotropic media can be characterized by a distinguished axis with medium parameters showing rotational invariance around this axis. The constitutive relations are

$$
\overline{\boldsymbol{\epsilon}} = \epsilon_1\overline{\mathbf{I}} + (\epsilon - \epsilon_1)\mathbf{c}\,\mathbf{c} - i\epsilon_2\,\mathbf{c}\times\overline{\mathbf{I}}, \quad \overline{\boldsymbol{\mu}} = \mu\overline{\mathbf{I}}. \tag{49}
$$

The dyadic Green's functions are [*Weiglhofer*, 1993]

$$
\begin{aligned}
\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= \frac{1}{i\omega\epsilon_1}\overline{\mathbf{W}}_e\,G_e(\mathbf{x},\mathbf{x}'), \\
\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= -\frac{1}{i\omega\mu}\left(\nabla\times\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}')\right).
\end{aligned} \tag{50}
$$

The dyadic operator $\overline{\mathbf{W}}_e$ is given by

$$
\begin{aligned}
\overline{\mathbf{W}}_e = {} & H\nabla\nabla + \omega^2\mu\epsilon_1 H_e\,\overline{\mathbf{I}} + \omega^2\mu(\epsilon_1 - \epsilon)\frac{\partial}{\partial x_c}(\nabla\,\mathbf{c} + \mathbf{c}\,\nabla) + \\
& i\epsilon_2\mu\omega^2\Big((\nabla_t^2 + \omega^2\epsilon\mu)(\mathbf{c}\times\overline{\mathbf{I}}) + \\
& \frac{\partial}{\partial x_c}\left(\mathbf{c}(\nabla\times\mathbf{c}) - (\nabla\times\mathbf{c})\mathbf{c}\right)\Big) + \omega^4\mu^2({\epsilon_1}^2 - {\epsilon_2}^2 - \epsilon\epsilon_1)\mathbf{c}\,\mathbf{c},
\end{aligned} \tag{51}
$$

whereas the various scalar differential operators are

$$
\begin{aligned}
H &= \nabla^2 + \omega^2\mu\epsilon, \qquad \nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial x_c^2}, \\
H_e &= \nabla_t^2 + \frac{\epsilon}{\epsilon_1}\frac{\partial^2}{\partial x_c^2} + \omega^2\mu\epsilon, \\
H_m &= \nabla^2 + \omega^2\mu\frac{(\epsilon_1^2 - \epsilon_2^2)}{\epsilon_1},
\end{aligned} \tag{52}
$$

and the scalar Green's function $G_e(\mathbf{x},\mathbf{x}')$ must be a solution of

$$
\left(H_e H_m + \omega^2\mu\epsilon\frac{\epsilon_2^2}{\epsilon_1^2}\frac{\partial^2}{\partial x_c^2}\right) G_e(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'). \tag{53}
$$

No analytic solution to (53) is known. Applying a Fourier transformation leads to a one-dimensional integral representation for the scalar Green's function $G_e(\mathbf{x},\mathbf{x}')$, which involves cylinder functions (see *Weiglhofer* [1993] for more details).

4.2.3. *Magnetically gyrotropic media.* The constitutive relations are

$$
\overline{\boldsymbol{\epsilon}} = \epsilon\overline{\mathbf{I}}, \quad \overline{\boldsymbol{\mu}} = \mu_1\overline{\mathbf{I}} + (\mu - \mu_1)\mathbf{c}\,\mathbf{c} - i\mu_2\mathbf{c}\times\overline{\mathbf{I}}. \tag{54}
$$

Weiglhofer [1992] found an analytic representation of the dyadic Green's functions in the following form

$$
\begin{aligned}
i\omega\epsilon\,\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= (\nabla\times\overline{\mathbf{I}})\cdot\overline{\mathbf{W}}_m\cdot(\nabla\times\overline{\mathbf{I}})\,G_m(\mathbf{x},\mathbf{x}') - \delta(\mathbf{x}-\mathbf{x}')\overline{\mathbf{I}}, \\
\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') &= -\left(\nabla\times\overline{\mathbf{W}}_m\right)G_m(\mathbf{x},\mathbf{x}'), \\
\overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= \overline{\mathbf{W}}_m\cdot(\nabla\times\overline{\mathbf{I}})\,G_m(\mathbf{x},\mathbf{x}'), \\
\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= -i\omega\epsilon\,\overline{\mathbf{W}}_m\,G_m(\mathbf{x},\mathbf{x}').
\end{aligned} \tag{55}
$$

The operator $\overline{\mathbf{W}}_m$ is given by

$$
\begin{aligned}
\overline{\mathbf{W}}_m = {} & H_m\overline{\mathbf{I}} + \frac{H}{\omega^2\epsilon\mu_1}\nabla\nabla + \frac{\mu_1 - \mu}{\mu_1}\frac{\partial}{\partial x_c}(\nabla\,\mathbf{c} + \mathbf{c}\,\nabla) + \\
& \frac{\omega^2\epsilon}{\mu_1}(\mu_1^2 - \mu_2^2 - \mu\mu_1)\,\mathbf{c}\,\mathbf{c} + \\
& \frac{i\mu_2}{\mu_1}\left((\nabla_t^2 + \omega^2\epsilon\mu)(\mathbf{c}\times\overline{\mathbf{I}}) + \frac{\partial}{\partial x_c}\left(\mathbf{c}(\nabla\times\mathbf{c}) - (\nabla\times\mathbf{c})\mathbf{c}\right)\right),
\end{aligned} \tag{56}
$$

where $\nabla^2 = \nabla_t^2 + \partial^2/\partial x_c^2$ is the Laplace operator and x_c is the coordinate in the direction of the distinguished axis $(\partial/\partial x_c = \nabla\cdot\mathbf{c})$. The various scalar differential operators are all of the Helmholtz type:

$$
\begin{aligned}
H &= \nabla^2 + \omega^2\mu\epsilon, \qquad H_1 = \nabla^2 + \omega^2\mu_1\epsilon, \\
H_e &= \nabla^2 + \omega^2\epsilon\frac{(\mu_1^2 - \mu_2^2)}{\mu_1}, \\
H_m &= \nabla_t^2 + \frac{\mu}{\mu_1}\frac{\partial^2}{\partial x_c^2} + \omega^2\mu\epsilon.
\end{aligned} \tag{57}
$$

The scalar Green's function $G_m(\mathbf{x},\mathbf{x}')$ is a solution of

$$
\left(H_e H_m + \omega^2\mu\epsilon\frac{\mu_2^2}{\mu_1^2}\frac{\partial^2}{\partial x_c^2}\right) G_m(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'). \tag{58}
$$

The same statement about the solvability of (58) applies as for the corresponding equation (53) of the electrically gyrotropic medium of section 4.2.2.

4.3. *Plasmas, moving and diffusive media*

4.3.1. *Isotropic warm plasmas.* Under certain conditions a fluid model can be employed to describe a homogeneous, collisionless one-component plasma. Eliminating the nonelectromagnetic fields (that is, the velocity field and the pressure field of the plasma) leads to an effective formalism, where the plasma properties are contained within an effective dyadic permittivity operator

$$\begin{aligned} \overline{\boldsymbol{\epsilon}} &= \epsilon\left(\overline{\mathbf{I}} + \delta(\nabla^2+\alpha^2)^{-1}\nabla\nabla\right), \\ \overline{\boldsymbol{\mu}} &= \mu\overline{\mathbf{I}}, \qquad \overline{\boldsymbol{\xi}} = \overline{\boldsymbol{\zeta}} = 0. \end{aligned} \tag{59}$$

Felsen and Marcuvitz [1973] have established the dyadic Green's functions in the following form

$$\begin{aligned} \overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu\left(\overline{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)g_k + \frac{i\omega\mu\delta}{k^2}\nabla\nabla\, g_\alpha, \\ \overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= (\nabla\times\overline{\mathbf{I}})\,g_k = -\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}'), \\ \overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= -i\omega\epsilon\left(\overline{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)g_k. \end{aligned} \tag{60}$$

The two scalar Green's functions in (60) are

$$g_\gamma(\mathbf{x},\mathbf{x}') = \frac{\exp(-i\gamma r)}{4\pi r}, \qquad (\gamma = k,\alpha), \tag{61}$$

$(r = |\mathbf{x}-\mathbf{x}'|)$.

4.3.2. *Moving isotropic media.* A simple transformation of the electromagnetic fields from a stationary to a moving frame of reference shows that in the stationary frame of reference a moving isotropic (achiral) medium can be described by constitutive relations of the following form:

$$\begin{aligned} \overline{\boldsymbol{\epsilon}} &= \epsilon\left(\nu\overline{\mathbf{I}}+(1-\nu)\mathbf{c}\,\mathbf{c}\right), \qquad \overline{\boldsymbol{\xi}} = \beta(\mathbf{c}\times\overline{\mathbf{I}}), \\ \overline{\boldsymbol{\mu}} &= \mu\left(\nu\overline{\mathbf{I}}+(1-\nu)\mathbf{c}\,\mathbf{c}\right), \qquad \overline{\boldsymbol{\zeta}} = -\beta(\mathbf{c}\times\overline{\mathbf{I}}). \end{aligned} \tag{62}$$

The dyadic Green's functions have been studied in detail by many authors (see for example *Chen* [1983] or *Tai* [1971]), and see their books for a detailed analysis. Here only the case of velocity of the moving medium smaller than the phase velocity of a wave in the stationary medium is reproduced. The dyadic Green's functions are

$$\begin{aligned} \overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu\nu^{3/2}\left(\overline{\mathbf{I}}+\frac{\overline{\boldsymbol{\epsilon}}\cdot\nabla\nabla}{k^2\nu^2\epsilon}\right)\frac{\exp(-ik_\nu r_\nu)}{4\pi r_\nu}, \\ \overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= -\frac{1}{i\omega}\overline{\boldsymbol{\mu}}^{-1}\cdot\left(\nabla\times\overline{\mathbf{I}} - i\omega(\mathbf{c}\times\overline{\mathbf{I}})\right)\cdot\overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}'), \\ \overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= -i\omega\epsilon\nu^{3/2}\left(\overline{\mathbf{I}}+\frac{\overline{\boldsymbol{\mu}}\cdot\nabla\nabla}{k^2\nu^2\mu}\right)\frac{\exp(-ik_\nu r_\nu)}{4\pi r_\nu}, \\ \overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}') &= \frac{1}{i\omega}\overline{\boldsymbol{\epsilon}}^{-1}\cdot\left(\nabla\times\overline{\mathbf{I}} - i\omega\,(\mathbf{c}\times\overline{\mathbf{I}})\right)\cdot\overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}'), \end{aligned} \tag{63}$$

where $r_\nu^2 = \epsilon\,\mathbf{r}\cdot\overline{\boldsymbol{\epsilon}}^{-1}\cdot\mathbf{r} = \mu\,\mathbf{r}\cdot\overline{\boldsymbol{\mu}}^{-1}\cdot\mathbf{r}$.

4.3.3. *Diffusive media.* A diffusive medium is one for which the constitutive relations become differential operators itself. On the basis of a simple physical model (and a symmetry argument as far as $\overline{\boldsymbol{\mu}}$ is concerned) *Monzon and Lakhtakia* [1991] used the following representations for the permittivity and permeability dyadics:

$$\overline{\boldsymbol{\epsilon}} = \epsilon\left(\overline{\mathbf{I}}+\frac{1}{\alpha^2}\nabla\nabla\right), \qquad \overline{\boldsymbol{\mu}} = \mu\left(\overline{\mathbf{I}}+\frac{1}{\beta^2}\nabla\nabla\right). \tag{64}$$

The complete dyadic Green's functions were then obtained by the same authors as

$$\begin{aligned} \overline{\mathbf{G}}_{ee}(\mathbf{x},\mathbf{x}') &= -i\omega\mu\left(\overline{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)g_k + \frac{i\omega\mu}{k^2}\nabla\nabla\, g_\alpha, \\ \overline{\mathbf{G}}_{me}(\mathbf{x},\mathbf{x}') &= (\nabla\times\overline{\mathbf{I}})\,g_k = -\overline{\mathbf{G}}_{em}(\mathbf{x},\mathbf{x}'), \\ \overline{\mathbf{G}}_{mm}(\mathbf{x},\mathbf{x}') &= -i\omega\epsilon\left(\overline{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)g_k + \frac{i\omega\epsilon}{k^2}\nabla\nabla\, g_\beta, \end{aligned} \tag{65}$$

and the scalar Green's functions g_k, g_α, g_β are

$$g_\gamma(\mathbf{x},\mathbf{x}') = \frac{\exp(-i\gamma r)}{4\pi r}, \qquad (\gamma = k,\alpha,\beta), \tag{66}$$

$(r = |\mathbf{x}-\mathbf{x}'|)$. A comparison of the solutions for the isotropic warm plasma in section 4.3.3 with the results for the diffusive media here shows a close relation between the two. However, the permittivity operator for the plasma also contains an inverse scalar differential operator, so that one set of constitutive relations cannot simply be regarded as a special case of the other.

5. A CRITICAL ASSESSMENT OF ANALYTIC TECHNIQUES

In section 4 those time-harmonic dyadic Green's functions for which closed-form solutions exist have been comprehensively listed. The analytic expressions are the result of a considerable amount of research carried out over a number of decades. This shows that finding analytic solutions to the dyadic equations is very difficult indeed. It is therefore appropriate to examine the achievements thoroughly.

There is no doubt that those dyadic Green's functions which are already known have proven extremely useful tools for a variety of problems in electromagnetic theory. On the other hand, the classes of media for which analytic solutions exist are still very restricted. First of all, all representations in section 4 refer to homogeneous media, that is, all constitutive parameters are complex constants. Second, as far as anisotropy is concerned, uniaxial and gyrotropic media are thus far the only types of media which have been successfully tackled with analytic methods. It is therefore reasonable to ask from which direction generalizations of the results can be expected.

Inhomogeneous media. General inhomogeneity, that is, the constitutive dyadics are arbitrary functions of $\mathbf{x}$: $\overline{\boldsymbol{\epsilon}}(\mathbf{x})$, $\overline{\boldsymbol{\mu}}(\mathbf{x})$, $\overline{\boldsymbol{\xi}}(\mathbf{x})$ and $\overline{\boldsymbol{\zeta}}(\mathbf{x})$, pose problems which are probably insurmountable by any analytic technique. A simpler case concerns inhomogeneity in just one spatial dimension, that is, a stratification of the medium. Here it can be said with some confidence that some of the mathematical methods discussed in section 3 may succeed in deriving solutions for the dyadic Green's functions.

As an example, *Weiglhofer* [1987] has used decomposition for an inhomogeneous, general gyrotropic medium (gyrotropic axis c), where $\overline{\epsilon}(x_c), \overline{\mu}(x_c)$; $x_c = \mathbf{x} \cdot \mathbf{c}$. The dyadic structure of $\overline{\epsilon}$ is as in (49), that of $\overline{\mu}$ as in (54).

General bianisotropic media. The success of the methods for uniaxial and gyrotropic media is largely based on specific symmetries in the constitutive dyadics. Take a general anisotropic medium, where the permittivity dyadic $\overline{\epsilon}$, say, is given by

$$\overline{\epsilon} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}, \tag{67}$$

with arbitrary constants ϵ_{ij}. If no symmetries or relations between the ϵ_{ij} exist, which can be exploited by the mathematical technique, it is very difficult to see how an analytic solution can be obtained. In these general circumstances, Fourier transformation is probably still the best option.

Earlier in this paper it was sometimes distinguished between an analytic solution for the dyadic Green's functions and an analytic representation. The relations written down for the different types of isotropic media are certainly closed-form analytic solutions in the true sense; that is, the dyadic Green's functions are given as sums/differences/-products/quotients of known analytic functions. The results for gyrotropic media fall somewhat short of this goal. On the one hand, the analytic methods helped to find a representation of the dyadic Green's functions in terms of a single scalar Green's function, G_e, for the electrically gyrotropic medium and G_m for magnetic gyrotropy (see (50) and (55)). The problem lies in the fact that thus far no analytic solution to the differential equation of the scalar Green's functions G_e or G_m, (53) and (58) has been established. Therefore, from a computational point of view one would have to find an approximative or numerical solution for the scalar Green's function and then, in order to obtain the dyadic Green's functions, apply a dyadic differential operator to that solution.

One can still argue that the analytic methods have at least achieved a reduction of the complexity of the problem from a system of dyadic partial differential equations to one single scalar partial differential equation. This argument shows that the more complicated the media are, the more difficult it becomes for an analytic procedure to succeed all the way through to the desired goal: the closed-form solution. Considering their limitations in the face of complexity, can the major incentives be found to pursue attempts to establish analytic solutions for dyadic Green's functions?

As discussed in the introduction, complexity in a given electromagnetic field problem can be due to the geometry of the radiating sources, the boundary geometry, and the medium under consideration. If all three complications are present at the same time, it will become very difficult or even impossible to gain a thorough understanding of the way in which different processes contribute to the solution. That means that even if a numerical solution would become available, an interpretation of the results would be an extremely involved task. By concentrating on the dyadic Green's functions, which assume a unit (dyadic) source, one eliminates the source problem and by putting the focus on free-space solutions, one eliminates the boundaries. As a consequence, the characteristics of the medium are studied on their own. Of course, the sources and the boundaries may provide just the effect one is particularly interested in, in which case the infinite-medium dyadic Green's functions cannot be seen as anything more than a first step toward the complete solution of the problem. However, it is undoubtedly an invaluable first step toward such a solution. By reducing the complications, it is often possible to get a good insight into the mathematical intricacies, like singularities, or into the underlying physical processes, like radiation patterns, etc.

The issue of singularities in electromagnetic theory is a fairly complicated one. It is clear that electromagnetic fields must have singularities in source regions [*Van Bladel*, 1991]. A knowledge of the Green's functions usually permits the extraction of these singularities from the equations. Two consequences follow: First, in some important applications the singularity structure of the dyadic Green's functions is all that is required to establish the result. A good example is the long-wavelength scattering approximation [*Lakhtakia and Weiglhofer*, 1992]. Second, even if a full analytic solution is not forthcoming, the extraction of singularities can lead to considerable simplifications as far as any numerical approach is concerned (for example, in evaluating inverse Fourier integrals).

6. CONCLUSION

Analytic methods are important tools in solving electromagnetic field problems. Analytic solutions for Green's functions and for electromagnetic fields, where available, can provide important insights into mathematical structure and the physics behind the processes.

A few methods were presented that have been useful in obtaining infinite-medium dyadic Green's functions in the time-harmonic formulation. Applying these techniques has led to establishing closed-form analytic solutions for the dyadic Green's functions for a number of selected homogeneous media, ranging from isotropic to anisotropic, and diffusive and moving ones to certain types of plasmas.

Where the media under consideration become more complicated than gyrotropic and when inhomogeneities have to be considered, analytic methods very quickly reach their limits, not so much in principle but in terms of feasibility. The future will undoubtedly see an increased amount of direct numerical simulations in electromagnetics. The author believes, however, that analytic methods will still have an important role to play in our understanding of the fundamentals of complex electromagnetic field problems. With the increased availability of symbolic computer software like REDUCE or MATHEMATICA, for example, one can hope that a fruitful symbiosis between analytic and numerical methods may develop.

(Note added in proof: Meanwhile, closed-form solutions for the dyadic Green's functions of certain classes of bianisotropic media have become available, see *Lindell and Weiglhofer* [1993*a*, *b*], and *Weiglhofer and Lindell* [1993].)

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