Computer Simulation of a Small Buried Loop

Evaluating the magnetic fields from a loop antenna buried in a conducting medium is not trivial; but requires a large amount of computing power. ***Reno Lippold*** *explains his implementation of Wait's equations in the MathCad package, and compares the results with Shope's earlier simulation.*

Introduction

In an issue of the NSS Bulletin[1], Steven Shope (1991) presented a set of equations, originally developed by James Wait (1969, 1971), for determining the magnetic fields above the surface of the ground, which were generated by a buried current loop. Shope also developed a computer program that ran on a PC and produced results for user-defined parameters (see *Figure 1*). He called this program FATE-VMD (Fields above the earth – vertical magnetic dipole). I decided to develop my own solution to these equations (which I'll call the Wait equations) using a programming tool called Mathcad.

My original goal was simply to learn more about Mathcad and especially its built-in Bessel functions. (The Wait equations are actually Sommerfeld integrals, which are an integral of a Bessel function). Now completed, my work serves as a check of the results obtained by Shope, both of the graphs included in his 1991 article, and of the data produced by the FATE-VMD program.

The Mathcad Worksheet

My Mathcad worksheet (program) is included with this article, and shows Wait's equations and their usage. To understand the graphs I have provided, it is important that you know a little about the dimensionless functions P and Q, which the program generates, and the dimensionless parameters D, T and Z, which the user provides. P and Q are proportional to the horizontal and vertical magnetic fields, respectively. D expresses the receiver's offset from 'ground zero', T the transmitter depth and Z the receiver elevation. The derivations of D, T and Z are shown in the worksheet, and the Appendix. In all the graphs Shope and I have produced Z is set to zero. T also incorporates the skin depth, and gets larger for increasing depth, frequency or ground conductivity. Shope emphasised the nature of P and Q by plotting them in various ways. I have followed this precedent to facilitate comparison of results.

1 The National Speleological Society is roughly the equivalent of the BCRA in the USA.

Using MathCad

Mathcad is a PC based maths programming application sold by Mathsoft Inc., **www.mathsoft.com**. It is different to traditional programming tools (BASIC and FORTRAN) and applications such as spreadsheets, in that it utilises a graphical programming environment, which closely mimics the way one would write equations on a scratch-pad. So if you are familiar with standard mathematical notation, you should be able to follow most of the worksheet; however, a couple aspects of the worksheet do require special explanation.

Tolerance Setting – TOL

Mathcad uses the tolerance (TOL) setting in iterative operations such as integration. A small *delta-x* is chosen to numerically integrate the function. Then a smaller *delta-x* is chosen and the integration is re-evaluated and the two results are compared. The process is repeated until the difference between the two calculations is less than the TOL setting. A smaller TOL setting increases the precision of the integration, at the cost of greater processing time.

Upper Limit of Integration - ULI

P and Q are the result of two integrals that are symbolically evaluated from 0 to infinity. The integration to infinity must be approximated by using a finite upper limit of integration (ULI). Through trial-and-error, I determined that a ULI above 50 rarely changed the results significantly. So I used the somewhat arbitrary value of 100 as the ULI to create my graphs.

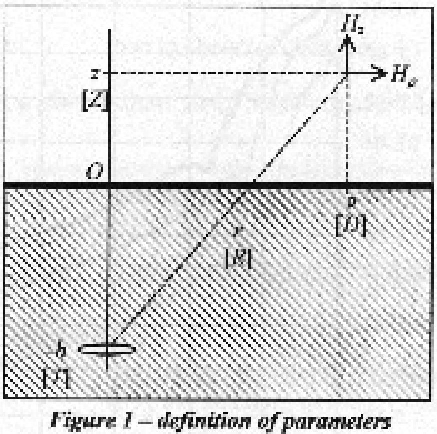


Figure 1 – definition of parameters

The transmitter loop is buried at a depth h. The receiver is a distance z above the surface, at an offset of ρ. The direct loop-to-observer distance is r.
These parameters are all 'normalised' to dimensionless quantities (shown in brackets), as explained in the text. The simulation produces 'normalised' values of the vertical and horizontal magnetic field components (amplitude and phase of both)

Results of Simulations

I reproduced all five of Shope's graphs, of which two are provided with this article. Visually, they appear to me to be an exact match. If you desire to compare them, a reproduction of Shope's equivalent to my graph 1 is given in (Bedford, 1993), p14. I also did some quantitative error analysis between the results of Shope's FATE-VMD program and my Mathcad program. For graphs 1 and 2, the greatest difference was around 1%. Differences of less than 0.01% were more typical. Therefore, for this range of values, I did confirm the graphical results presented by Shope.

Iteration Errors

During development of the Mathcad program, I noted the functions of P and Q ceased to be 'well-behaved' under certain conditions when the offset was large. I eventually realised this was occurring because the integration tolerance setting was greater than the expected value of P and Q. By lowering TOL sufficiently, I was able to eliminate this problem. FATE-VMD suffers from similar problems, but its internal parameters cannot be adjusted.

In Graph 3, Q is plotted over a range of D from 0 to 20, with T at 4, 6, and 10. The FATE results become erratic for the higher values of D, which becomes worse for higher values of T. The Mathcad results remain stable over this entire range. In this case, TOL had to be set to 10^{-11} and the processing time was around ten minutes. FATE is a good program when used where D and/or T are not too large. Keeping D and T less than ten is a rough rule-of-thumb and is the area of most practical applications. The signal strength

Graphical Results

The three graphs to the right were all produced using MathCad. For each graph, the field point is taken to be at an elevation of zero ($Z = 0$). The x axis shows the 'normalised' offset, D, from Ground Zero.

Graph 1 (top right)

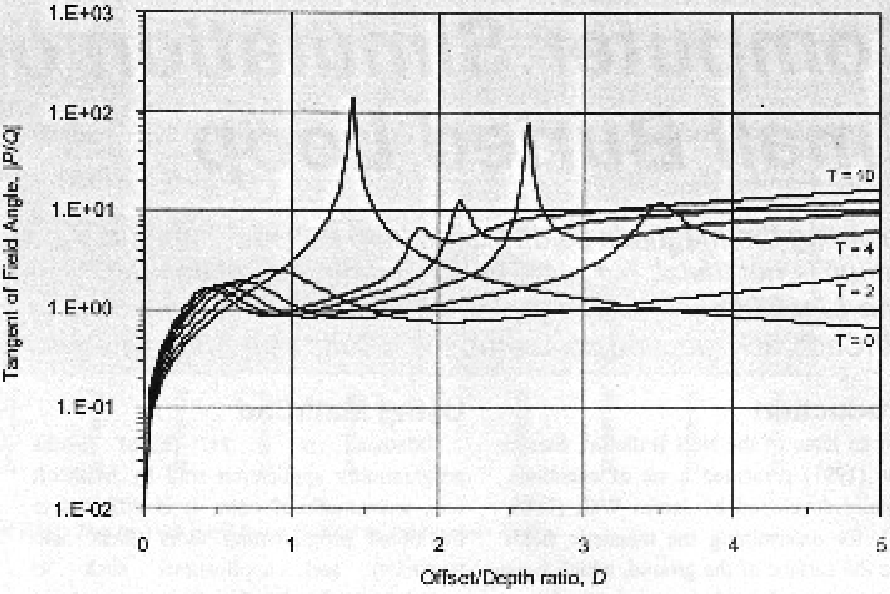
This shows the tangent of the field angle $|P/Q|$ v. normalised offset D, for normalised transmitter depth $T = \{0,2,4,6,8,10\}$. This can be compared with Shope's Figure 5.

Graph 2 (middle right)

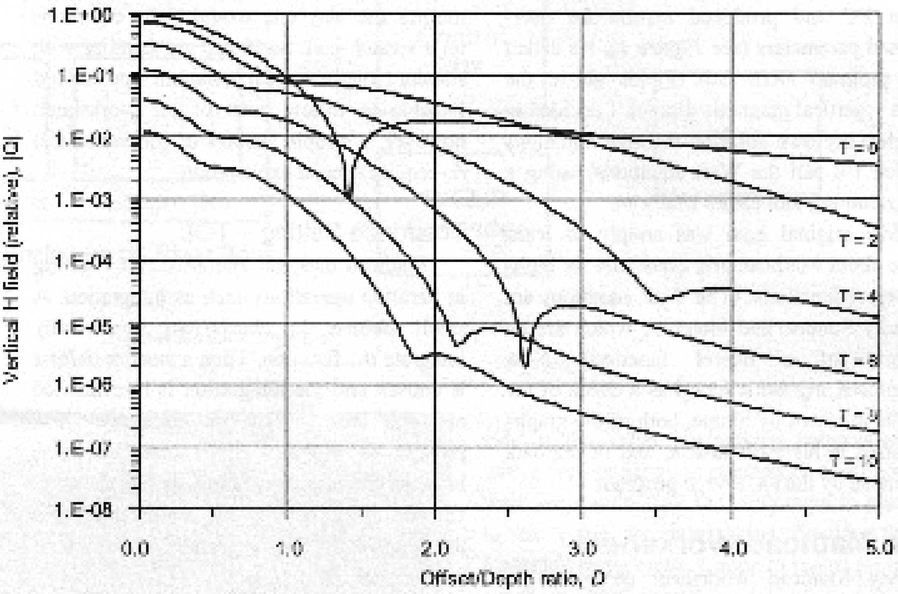
This is similar to Graph 1 but shows the vertical H-field $|Q|$. See Shope's Figure 3.

Graph 3 (bottom right)

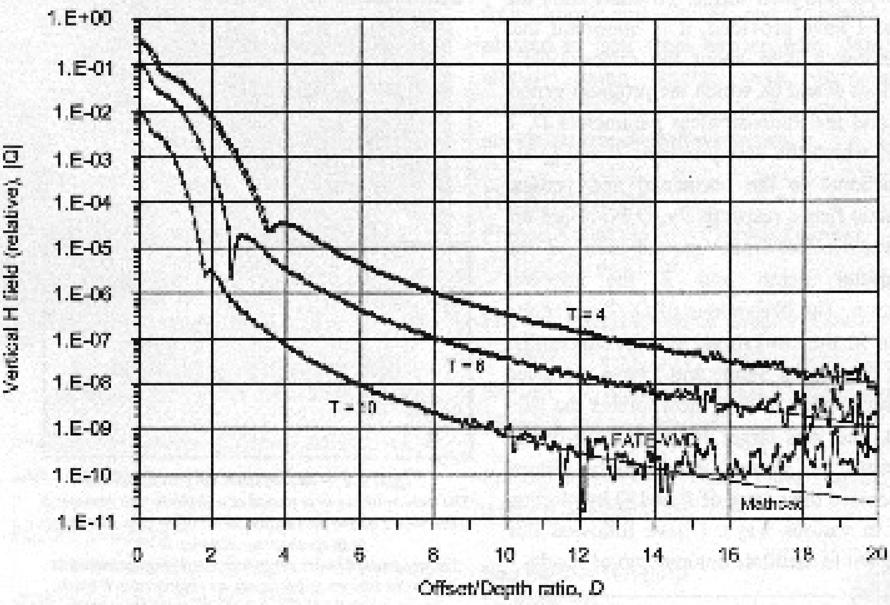
This is a comparison between the FATE-VMD and MathCad results. It shows the vertical H-field $|Q|$ v. D for T = {4,6,10}.



Graph 1



Graph 2



Graph 3

outside of this range would be small, and might be swamped by noise.

Conclusions

The primary advantage of using Mathcad, or something like it, is the ability to customise your inputs and outputs. The FATE program assumes a relative permeability of 1, whereas I have made it a variable that you can set in the program. Also, the Mathcad program allows you to enter the loop current, and its diameter and number of turns, instead of the magnetic moment. You may also note that I added a number of 'tests' throughout the program, that ensure that restrictions on the equations and program variables, are being met. Other desired features could easily be added.

P and Q have both an amplitude and a phase associated with them. Shope and I have emphasised the amplitudes but, if the earth is at all conductive, and the coil current is anything other than DC, the phase of P and Q (and therefore the phase of the vertical and horizontal magnetic fields) will differ by various amounts that depend on the point of observation and frequency. This phenomenon may be of use for cave radio and deserves additional study.

References

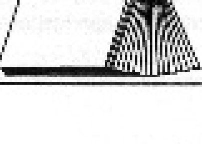
Bedford, Mike (1993), *An Introduction to Radio-Location*, CREGJ **14**, pp16-18,14, Dec. 1993

Shope, Steven (1991), *A Theoretical Model of Radio-Location*, NSS Bulletin 53(2), Dec. 1991. ISSN 0146 9517.

Wait, J.R. (1969), *Electromagnetic fields of Sources in Lossy Media*, Antenna Theory, Part 2, chapter 24, (ed R.E. Collin & F. J. Zucker). McGraw-Hill, New York, 1969, pp 468-471.

Wait, J.R. (1971), *Electromagnetic Induction Technique for Locating a Buried Source*, IEEE Transactions on Geoscience and Electronics GE-9, pp95-98.

See Further Reading on page 16



MathCad Worksheet – Magnetic Fields Due to a Buried Current Loop

1. MATHCAD CONFIGURATION SETTINGS.

$TOL = 0.00001$ The tolerance setting for internal integration. Set lower than the lowest magnitude of Q or P.

2. CONSTANTS.

$\mu_0 := 4\cdot\pi\cdot 10^{-7}\cdot\frac{H}{m}$ permeability of free space (Henrys/meter)

$\varepsilon_0 := 8.854\cdot 10^{-12}\cdot\frac{F}{m}$ free space dielectric constant (permittivity) (Farads/meter)

3. VARIABLE DEFINITIONS (INPUTS).

$ULI := 100$ The upper limit of integration. This approximates infinity. Set high enough so the results stop changing with increasing ULI. A value of 100 is normally sufficient.

$h := 100\cdot m$ height of ground surface above loop (loop depth)

$z := 0\cdot m$ distance above ground of the observer

$y := 100\cdot m$ offset to observer (horizontal distance from vertical axis of current loop)

$\sigma := 0.001\cdot\frac{S}{m}$ conductivity of the rock/soil overburden (Siemens/meter)

$\mu_r := 1$ relative permeability (factor by which the permeability exceeds that of free space)

$\varepsilon_r := 3$ relative permittivity (factor by which the permittivity exceeds that of free space)

$f := 30000\cdot s^{-1}$ operating frequency (Hertz)

$N := 10$ number of turns in the current loop

$I := 1\cdot A$ current in the loop (amps)

$d := 1\cdot m$ diameter of the current loop

Mathcad Worksheet

This printout shows the graphical programming environment of the Mathcad worksheet. This printout is what you see on your computer screen. Values for the parameters are entered exactly as you would if this was a word-processor document, and the re-calculated results appear in the appropriate boxes automatically, a bit like a spreadsheet.

This version is set up to calculate a single specific set of results in section 7, due to the parameters that the user defines in section 3. To produce graphs, a slightly different version of the worksheet is used to calculate a range of results. The graphical results (Figures 1-3) are in the form of 'normalised' dimensionless parameters, rather than specific values of distance, magnetic field and so on, as used in this worksheet.

4. INITIAL CALCULATIONS.

$\omega := 2\cdot\pi\cdot f$ $\omega = 1.885\cdot 10^{5}\cdot rad\cdot s^{-1}$ radian frequency

$\varepsilon := \varepsilon_r\cdot\varepsilon_0$ $\varepsilon = 2.656\cdot 10^{-11}\cdot\frac{F}{m}$ permittivity

$\mu := \mu_r\cdot\mu_0$ $\mu = 1.257\cdot 10^{-6}\cdot\frac{H}{m}$ permeability

$\frac{\sigma}{\omega\cdot\varepsilon} = 199.728$ This ratio should be approximately **100 or greater** for these equations to be valid ($\sigma >> \omega\varepsilon$).

$LTOD := \sqrt{y^2 + (h+z)^2}$ $LTOD = 141.421\cdot m$ loop to observer distance

$\frac{LTOD}{d} = 141.421$ This ratio should be **greater than 10** for these equations to be valid.

$A_1 := \frac{\pi\cdot d^2}{4}$ $A_1 = 0.785\cdot m^2$ area of the current loop

$M := I\cdot N\cdot A_1$ $M = 7.854\cdot A\cdot m^2$ magnetic moment of current loop

5. DERIVE DIMENSIONLESS INPUT PARAMETERS.

$T := h\cdot\sqrt{(\mu\cdot\sigma\cdot\omega)}$ $T = 1.539$ loop depth / skin depth ratio (note: this definition of skin depth is NOT the standard definition – see appendix to article)

$D := \frac{y}{h}$ $D = 1$ offset ratio

$Z := \frac{z}{h}$ $Z = 0$ elevation ratio

MathCad worksheet continued overleaf

MathCad Worksheet – continued

6. CALCULATE DIMENSIONLESS OUTPUT FUNCTIONS FOR PLOTTING.

$$Q(T,D) := \int_0^{ULI} x^3 \cdot \left[\frac{\left[e^{-x-\sqrt{x^2+i\cdot T^2}} \cdot e^{x\cdot(1-Z)} \right]}{\left(x+\sqrt{x^2+i\cdot T^2}\right)} \cdot J0(x\cdot D) \right] dx$$

J0(x,D) is the Mathcad internal function for a 0th order ordinary Bessel function of the first kind, evaluated at argument x·D.

$$P(T,D) := \int_0^{ULI} x^3 \cdot \left[\frac{\left[e^{-x-\sqrt{x^2+i\cdot T^2}} \cdot e^{x\cdot(1-Z)} \right]}{\left(x+\sqrt{x^2+i\cdot T^2}\right)} \cdot J1(x\cdot D) \right] dx$$

J1(x,D) is the Mathcad internal function for a 1st order ordinary Bessel function of the first kind, evaluated at argument x·D.

$|Q(T,D)| \cdot 0.1 = 7.408 \cdot 10^{-3}$

$|P(T,D)| \cdot 0.1 = 0.021$

TOL should be less than 1/10 the smallest of |Q| or |P|.

$TOL = 1 \cdot 10^{-5}$

7. CALCULATE MAGNETIC FIELD RESULTS.

<u>Vertical Magnetic Field</u>

$H_z := \frac{M \cdot Q(T,D)}{\left(2 \cdot \pi \cdot h^3\right)}$ $\quad H_z = -0.047 - 0.08i \frac{\mu A}{m}$ $\quad |H_z| = 0.093 \frac{\mu A}{m}$

at a phase angle of: $\arg\left(H_z\right) = -120.843 \deg$

<u>Horizontal Magnetic Field</u>

$H_y := \frac{M \cdot P(T,D)}{\left(2 \cdot \pi \cdot h^3\right)}$ $\quad H_y = 0.18 - 0.193i \frac{\mu A}{m}$ $\quad |H_y| = 0.264 \frac{\mu A}{m}$

at a phase angle of: $\arg\left(H_y\right) = -47.046 \deg$

Appendix – David Gibson[2]

To interpret the graphs and to understand the implications for radiolocation it would help you to read Shope's observations. It is not necessary to discuss the maths in detail here, but the following notes may be useful. For further detail refer to Wait, and note that Shope used a slightly different notation.

P and *Q*

The horizontal and vertical magnetic fields are related to the magnetic moment of the transmitter ***M*** by the functions *P* and *Q*,

$$H_\rho = \frac{\mathbf{M}}{2\pi h^3} P, \quad H_z = \frac{\mathbf{M}}{2\pi h^3} Q \qquad (1)$$

where H_z is the vertical magnetic field and H_ρ is the horizontal component[3]. For a quasi-static field, the traditional shape of the field lines is defined by

$$H_r = \frac{\mathbf{M}}{2\pi r^3} \cos\theta, \quad H_\theta = \frac{\mathbf{M}}{4\pi r^3} \sin\theta \qquad (2)$$

but note that the formulas in (2) use *r* as a parameter, not *h*; and that they define the radial and tangential fields, not the vertical and horizontal fields. So, even for the quasi-static case, *P* and *Q* are not simple values.

D, *T* and *Z*

The parameters *D*, *T* and *Z* are all dimensionless. *T* represents the transmitter depth below the surface in terms of skin depth δ. *D* and *Z* represent the offset of the receiver from 'ground zero' and its height *above* the surface; both in terms of the loop depth *h*. That is,

$$T = \frac{h}{\delta}\sqrt{2}, \quad D = \frac{\rho}{h}, \quad Z = \frac{z}{h} \qquad (3)$$

With the conventional (and 'sanitary') definition of δ, a factor of √2 has to be included in (4). Shope, with his different definition did not need this – this is important to note; ***beware!***

Quasi-static case

In the quasi-static or d.c. case some geometrical algebra lead to

$$P = \tfrac{3}{2}\frac{D(Z+1)}{R^5}, \quad Q = \tfrac{1}{2}\left(\frac{3(Z+1)^2}{R^5} - \frac{1}{R^3}\right) \qquad (4)$$

with $R^2 = D^2 + (Z+1)^2$

where $R = r/h$ is the distance to the field point.

It is ***important*** to note, should you be merely skimming through this text, that the above approximation is ***only*** true for low frequencies, where the transmission depth and field point are both well within a skin depth (i.e. $T \ll 1$, $DT \ll 1$, $DZ \ll 1$). Normally, of course, this is not the case, which is why we need the full analysis.

It is fairly obvious, from (2), that if *D* and *Z* are both zero, then the quasi-static case ($T \ll 1$) ought to give $P=0$ and $Q=1$, which (3) confirms. Similarly, if the loops are nearly co-planar, with $D \gg 1$, $Z=0$, we would expect $P=0$ and $Q \approx 1/(2D^3)$. It was Brian Pease's testing of this condition that first highlighted a problem with Shope's program at large offsets (although, to be fair, if $D \gg 10$ the simulation may not be useful anyway – the signal level will be very low and affected significantly by local rock conditions).

In the general case, *P* and *Q* are both complex values, indicating that the vertical and horizontal fields have different phases, and can be represented as elliptically-polarised waves. This is, of course, why it is difficult to find a null if the receiver is not close to the transmitter. Shope made some observations on this point and it is a topic worth pursuing.

CREG

2 CREG Technical editor

3 I have used the term 'horizontal' rather than 'radial' to avoid confusion that might arise depending on the spherical co-ordinates in use.