International Journal of Theoretical and Mathematical Physics 2013, 3(5): 147-150
DOI: 10.5923/j.ijtmp.20130305.03

Some Definite Integrals Involving Algebraic, Trigonometric and Bessel Functions

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Abstract Some definite analytic integrals involving algebraic, trigonometric and Bessel functions which are needed in some electromagnetic calculations are proposed. The mathematical expansions are based on Mathematica (Wolfram) software and on some well known formulas. The agreement between the present expansion results and the numerical evaluation results is excellent.

Keywords Integrals, Bessel functions, Bessel transforms

1. Introduction

Often in the solution of electromagnetic problems in cylindrical coordinates Bessel functions are encountered. Integrals and mathematical expressions dealing with Bessel functions are continuously under research. For example[1] presents a tutorial that deals with Bessel functions and numerical evaluation of Bessel integrals. Reference[2] shows some Bessel function identities and indefinite integrals. Reference[3] evaluates some definite integrals where the integrand contains double Bessel function and reference[4] presents some integral forms of Bessel functions, relevant to scattering from cylinders.

In this work we deal with some Bessel transforms which are needed in cases of frequency dependent Moment Method (MM) basis functions[5], where the frequency is expressed in terms of the wavenumber k_0. The integrals are defined in the range R to infinity and other parameters are k and k_0. Different expansions are needed in the cases $k > k_0$ and $k < k_0$. The same integrals for $k = k_0$ are known. Similar integral but for a finite integration range is presented in[6]. In our case R, k and k_0 are real and positive.

The structure of the paper is as follows: Section 2 presents detailed proofs and final expressions for the integrals. Section 3 is devoted to the conclusions, and finally a comparison between the analytic expressions of Section 2 with numerical integrations is described in the Appendix.

2. Evaluations of the Integrals

2.1. The Integral I_1 ($k > k_0$)

We apply the well known formula]7]

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta) \quad (1)$$

where J( ) is the Bessel function of the first kind.. We use this formula in order to equate the arguments of the trigonometric and the Bessel functions. Then we get

$$\begin{aligned} I_1 &= \int_R^\infty \frac{\cos(k_0\rho)}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\ &= \int_R^\infty \frac{\cos\left[\rho k(k_0/k)\right]}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\ &= \int_R^\infty \frac{J_0(k\rho) + 2\sum_{n=1}^{\infty} J_{2n}(k\rho)\cos(2n\theta)}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\ &= \frac{1}{2kR} G_{2,4}^{2,1}\left(k^2R^2 \,\middle|\, \begin{matrix} 1, & 1 \\ 1/2, & 1/2, & 1/2, & 1/2 \end{matrix}\right) \\ &\quad + \frac{1}{kR}\sum_{n=1}^{\infty} (-)^n \cos\left[2n\sin^{-1}(k_0/k)\right] \\ &\quad \cdot G_{2,4}^{2,1}\left(k^2R^2 \,\middle|\, \begin{matrix} 1, & 1 \\ n+\frac{1}{2}, & n+\frac{1}{2}, & \frac{1}{2}-n, & \frac{1}{2}-n \end{matrix}\right) \end{aligned} \quad (2)$$

where the last integral is based on Mathematica (Wolfram) result, and G( ) is the Meijer G function[8].

2.2. The Integral I_1 ($k < k_0$)

In this case we apply the known integral]7]

$$J_0(k\rho) = \frac{2}{\pi}\int_0^1 \frac{\cos(k\rho t)}{\sqrt{1-t^2}}dt \quad (3)$$

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Published online at http://journal.sapub.org/ijtmp

then we get

$$\begin{aligned}
& I_1 \\
= & \int_R^\infty \frac{\cos(k_0\rho)}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\
= & \frac{2}{\pi}\int_0^1\int_R^\infty \frac{\cos(k_0\rho)\cos(k\rho t)}{\sqrt{\rho^2-R^2}\sqrt{1-t^2}}dt\,d\rho \\
= & -\frac{1}{2}\int_0^1 \frac{\{Y_0\left[(k_0-kt)R\right]+Y_0\left[(k_0+kt)R\right]\}}{\sqrt{1-t^2}}dt \\
= & -\int_0^1 \frac{\sum\limits_{n=-\infty}^{\infty} Y_n(k_0R)J_n(ktR)+Y_{-n}(k_0R)J_n(ktR)}{2\sqrt{1-t^2}}dt \\
= & -\int_0^1 \frac{1}{\sqrt{1-t^2}} Y_0(k_0R)J_0(ktR) \\
& -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-t^2}} 4\sum_{n=1}^{\infty} Y_{2n}(k_0R)J_{2n}(ktR) \\
= & -\frac{\pi}{2}Y_0(k_0R)J_0^2(kR/2)-\pi\sum_{n=1}^{\infty} Y_{2n}(k_0R)J_n^2(kR/2)
\end{aligned} \tag{4}$$

where Y_0 is the Bessel function of the second kind, and we used the addition theorem of Bessel functions[7].

2.3. The Integral I_2 ($k>k_0$)

$$I_2=\int_R^\infty \frac{\sin(k_0\rho)}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \tag{5}$$

In this case we use[7]

$$\sin(z\sin\theta)=2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin\left[(2k+1)\,\theta\right] \tag{6}$$

then we have

$$\begin{aligned}
& I_2 \\
= & 2\int_R^\infty \frac{\sum\limits_{n=0}^{\infty} J_{2n+1}(k\rho)\sin\left[(2n+1)\theta\right]}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\
= & \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^2}\cdot\sin\left[(2n+1)\theta\right]\cdot \\
& {}_3F_4\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{2}-n,\frac{1}{2}-n,\frac{3}{2}+n.\frac{3}{2}+n;-k^2R^2\right)
\end{aligned} \tag{7}$$

where ${}_3F_4$ ( ) is the generalized hypergeometric function[8].

2.4. The Integral I_2 ($k<k_0$)

Applying the same method as in 2.2, we have

$$\begin{aligned}
& I_2 \\
= & \int_R^\infty \frac{\sin(k_0\rho)}{\sqrt{\rho^2-R^2}} J_0(k\rho)\,d\rho \\
= & \int_R^\infty \frac{\sin(k_0\rho)}{\sqrt{\rho^2-R^2}}\frac{2}{\pi}\int_0^1 \frac{\cos(k\rho t)}{\sqrt{1-t^2}}dtd\rho \\
= & \int_0^1 \frac{\{J_0\left[(k_0-kt)\,R\right]+J_0\left[(k_0+kt)\,R\right]\}}{2\sqrt{1-t^2}}dt \\
= & \sum_{n=-\infty}^{\infty}\int_0^1 \frac{J_n(k_0R)J_n(ktR)+J_{-n}(k_0R)J_n(ktR)}{2\sqrt{1-t^2}}dt \\
= & \int_0^1\left[\frac{J_0(k_0R)J_0(ktR)}{\sqrt{1-t^2}}+2\sum_{n=1}^{\infty}\frac{J_{2n}(k_0r)J_{2n}(ktR)}{\sqrt{1-t^2}}\right]dt \\
= & \frac{\pi}{2}J_0^2\left(\frac{kR}{2}\right)J_0(k_0R)+\pi\sum_{n=1}^{\infty}J_n^2\left(\frac{kR}{2}\right)J_{2n}(k_0r)
\end{aligned} \tag{8}$$

3. Conclusions

Some integrals involving algebraic, trigonometric and Bessel functions were presented. Similar integrals with higher orders of the Bessel functions can be evaluated using the same method. These integrals are needed in some MM solutions for electromagnetic problems where the basis functions are frequency dependent. These basis functions are efficient for high frequency electromagnetic problems.

APPENDIX

A.1 I_1, $k > k_0$

```
k0 := 200

k := 300

R := 0.05

II := NIntegrate [Cos[k0 ro] BesselJ[0, k ro] / Sqrt[ro^2 - R^2], {ro, R, 100}]

GG := MeijerG[{{1}, {1}}, {{1 / 2, 1 / 2}, {1 / 2, 1 / 2}}, (k R)^2] / (2 k R) +
  Sum[MeijerG[{{1}, {1}}, {{1 / 2 + n, 1 / 2 + n}, {1 / 2 - n, 1 / 2 - n}}, (k R)^2]
     (-1)^n Cos[2 n ArcSin[k0 / k]], {n, 1, 20}] / (k R)

NII := N[II]

NGG := N[GG]

NII

0.0373049

NGG

0.0373028
```

A.2 I_1, $k < k_0$

```
R := 0.05

k0 := 70

k := 20

II := NIntegrate [Cos[k0 ro] BesselJ[0, k ro] / Sqrt[ro^2 - R^2], {ro, R , 100}]

GG := - (Pi / 2) BesselY[0, k0 R] (BesselJ[0, k R / 2])^2 -
  Pi Sum[BesselY[2 n, k0 R] (BesselJ[n, k R / 2])^2, {n, 1, 50}]

NII := N[II]

NGG := N[GG]

II

-0.267869

NGG

-0.267869
```

A.3 I_2, $k > k_0$

```
In[1]:= II := NIntegrate [Sin[k0 ro] BesselJ[0, k ro] / Sqrt[ro^2 - R^2], {ro, R, 250}]

In[2]:= k0 := 20

In[3]:= R := 0.05

In[4]:= k := 40

In[5]:= GG :=
         (4 / Pi) NSum[(-1) ^n HypergeometricPFQ[{1 / 2, 1 / 2, 1}, {1 / 2 - n, 1 / 2 - n, 3 / 2 + n, 3 / 2 + n},
             - (k R) ^2] Sin[(2 n + 1) ArcSin[k0 / k]] / (1 + 2 n) ^2, {n, 0, 20}]

In[6]:= II

Out[6]= -0.143806

In[7]:= GG

Out[7]= -0.143816
```

A.4 I_2, $k < k_0$

```
k0 := 90

k := 3

R := 0.05

II := NIntegrate [Sin[k0 ro] BesselJ[0, k ro] / Sqrt[ro^2 - R^2], {ro, R, 300}]

GG := (Pi / 2) (BesselJ[0, k R / 2]) ^2 BesselJ[0, k0 R] +
   Pi NSum[(BesselJ[n, k R / 2]) ^2 BesselJ[2 n , k0 R], {n, 1, 10}]

II

-0.501131

GG

-0.501131
```

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