On the evaluation of Sommerfeld integrals

Prof. A. Mohsen, B.Sc., B.Sc. (Eng.), M.S., Ph.D.

Indexing terms: *Antennas, Electromagnetics, Microwave systems*

Abstract: Some of the basic recent developments in the evaluation of Sommerfeld integrals are reviewed. The image approximation of the integrals is extended to include both the earth permeability and the effect of the displacement currents. The non-uniqueness of the field representation in terms of Hertz vectors is stressed and a proper representation for horizontal dipoles is used. The semi-analytical evaluation of the integrals using the fast fourier transform (FFT) is extended via proper decomposition of the integrand. The analysis increases the range of applicability of these techniques.

1 Introduction

In recent years, there has been increased interest in the behaviour of electromagnetic systems which are operated near a conducting earth. Many of these systems have been successfully used to measure the electromagnetic properties of the earth. That is, if the field of a transmitting source can be calculated in terms of idealised earth models, then measurements of actual fields near a real earth can be interpreted in terms of these models. This method is known as the induction method of geophysical exploration [1, 2]. The radiation characteristics of an antenna in the presence of a lossy ground can be influenced substantially by the finite ground conductivity and its inhomogeneity [3]. The original work on this subject was begun by Sommerfeld [4] and was later discussed and extended by many authors. Thus, the integrals expressing the interaction of electromagnetic sources with the ground are known as Sommerfeld integrals (SI).

The fields of a dipole in the presence of a conducting half-space may be expressed in terms of Hertz potentials. An account on the development of the problem of these potentials was given by Baños [5] and more recently by Kuester and Chang [6]. Exact expressions for the fundamental integrals can be found only in few particular cases; for example, when both the source and observation points are on the ground as well as when an observation point lies directly above or below the source [6].

For the case of arbitrary source and observer locations, the most direct method for evaluating SI is numerical integration. The direct numerical integration of SI apparently was first done by Siegel and King [7]. Direct integration using Simpson's rule or some other straightforward quadrature was also applied by Tsang *et al.* [8]. The application of Shanks algorithm to accelerate convergence was discussed by Lytle and Lager [9] who also considered various possible contour deformation to give a rapid and accurate integration methods. Since the SI have semi-infinite interval, a semi-analytical quadrature may be achieved by adding and subtracting a term with the appropriate limiting behaviour, as the integration variable tends to infinity, such that the corresponding integrand is known. The remaining integrals can then be evaluated much faster than the original [6].

A direct numerical integration of the integrals is generally useful in near-field regions. In far-field regions, this method consumes a large amount of computer time and the saddle-point technique is then more appropriate, and it leads to the geometrical-optics results. Several approximate analytical expressions have been determined for a wide variety of conditions. The range of applicability of these expressions depends upon the medium parameters (relative dielectric constant, permeability and conductivity) as well as the frequency and the relative locations of source, receiver and ground.

Paper 2020H, first received 10th December 1981 and in revised form 15th March 1982

The author is with the Engineering Mathematics & Physics Department, Faculty of Engineering, Cairo University, Giza, Egypt

Various asymptotic solutions have been developed for different ranges of observation. Recent development concerning far-field evaluation was done by Yokoyama [10], while Chang and Wait [11] considered the near-field approximation. The use of asymptotic methods which assume large refractive index was made by Kuo [12], who transformed the SI, with the aid of several approximations, to fast-convergent forms. Fuller [13] employed an alteration of the contour of integration in conjunction with approximate analytic results. Expressions of some exact and approximate forms of SI in terms of the incomplete Hankel functions were given by Keuster and Chang [6] and Chang and Fisher [14].

An alternative attractive approach is to expand part of the integrand in a special form such that the resulting integrals are known. Lindsay [15] used spline functions to represent part of the integrand and the resultant integrals were then evaluated in terms of Lommel functions. When the reflection function appearing in the integrals is represented in terms of exponentials, the image-theory approximation is obtained [16]. Justification of the image theory and its extension to the general case of an arbitrary quasi-static source was done by Weaver [17]. The accuracy of the theory was demonstrated by Bannister [18] by comparison with available analytical and numerical results. In these applications of the image theory the effect of earth's permeability and, in some cases, displacement currents were neglected. Extension of the theory to these cases is considered in this research for vertical magnetic dipole (VMD), vertical electric dipole (VED) as well as horizontal electric and magentic dipoles. For the horizontal dipole case, the non-uniqueness of the representation of the fields is discussed, and an appropriate representation is given.

A part of the integrand may be Fourier-expanded using the fast Fourier transform (FFT) such that the resulting integrals are known [8]. The advantages of such an approach, particularly with respect to computation time, were discussed by Tsang *et al.* [8]. However, they presented only a particular form of decomposition of the integrand. Other possible decompositions leading to increasing the range of applicability of the technique are considered in this research.

2 Vertical magnetic dipole

An infinitesimal vertical magnetic dipole (VMD), carrying a constant current I and having an area da, is situated at height h along the z-axis in a cylindrical co-ordinate system (ρ, ϕ, z). The conducting earth occupies the lower half-space $(z < 0)$, whereas the air occupies the upper half-space $(z > 0)$. The permittivity, permeability and conductivity of air and those of earth are assumed $(\epsilon_0, \mu_0, 0)$ and (ϵ, μ, σ), respectively. MKS units are employed and a suppressed time factor, $\exp(i\omega t)$, is assumed.

The electromagnetic field in air may be derived from a z-directed Hertz vector $\Pi_0^* = (0, 0, \pi_0^*)$ using the relations:

$$E^0 = -i\omega\mu_0 \nabla \times \Pi_0^*, H^0 = \nabla(\nabla \cdot \Pi_0^*) - \gamma_0^2 \Pi_0^* \quad (1)$$

where $\gamma_0^2 = -\omega^2 \mu_0 \epsilon_0$ and Π_0^* is given by [19]:

0143-7097/82/040177+06 $01.50/0

$$\Pi_0^* = S\int_0^\infty \{e^{-u_0|z-h|} + \Gamma^*(u_0)\, e^{-u_0(z+h)}\} J_0(\lambda\rho)\frac{\lambda}{u_0}\, d\lambda \tag{2}$$

where $S = Ida\,(4\pi i\omega\mu_0)^{-1}$, $\Gamma^*(u_0) = (\mu_r u_0 - u)/(\mu_r u_0 + u)$, $\mu_r = \mu/\mu_0$, $u_0^2 = \lambda^2 + \gamma_0^2$, $u^2 = \lambda^2 + \gamma^2$, $\gamma^2 = i\omega\mu(\sigma + i\omega\epsilon)$ and J_n is a Bessel function of order n.

The reflection function $\Gamma^*(u_0)$ appearing in eqn. 2 is identical to that previously considered by the author [20]. A convenient approximation of $\Gamma^*(u_0)$ is possible by considering the Taylor-series expansion of the function:

$$f(u_0) = \Gamma^*(u_0)\exp(\alpha u_0)$$

about $u_0 = 0$. The arbitrary value of α is determined to yield a good approximation to $\Gamma^*(u_0)$. If α is taken equal to $2\mu_r/c$, where $c^2 = \gamma^2 - \gamma_0^2$, the coefficient of both u_0 and u_0^2 vanish and we get:

$$\Gamma^*(u_0) \simeq -e^{-\alpha u_0}\,[1 + \tfrac{1}{6}(6\mu_r - 4\mu_r^3)\,(u_0/c)^3 + \dots] \tag{3}$$

The introduction of the first term in this approximation of $\Gamma^*(u_0)$ in the exact expression given by eqn. 2 yields:

$$\Pi_0^* \simeq S\,[\psi(R_0) - \psi(R_i)] \tag{4}$$

where the identity [19]:

$$\psi\left(\sqrt{\rho^2 + z^2}\right) = \int_0^\infty e^{-u_0 z} J_0(\lambda\rho)\,\lambda u_0^{-1}\, d\lambda \tag{5}$$

is used, $R_0^2 = \rho^2 + (z-h)^2$, $R_i^2 = \rho^2 + (z+h+\alpha)^2$ and $\psi(R) = \exp(-\gamma_0 R)\cdot R^{-1}$. The first term in eqn. 4 corresponds to the source contribution, while the second term may be attributed to an image at $z = -(h+\alpha)$. This is equivalent to replacing the earth by a perfectly conducting plane located at a depth $\alpha/2$.

The electric and magnetic fields in air for $0 < z < h$, using eqn. 1, are given by:

$$H_z^0 = \left(-\gamma_0^2 + \frac{\partial^2}{\partial z^2}\right)\Pi_0^*$$

$$\simeq S\,[U(z-h, R_0)\,\psi(R_0) - U(z+h+\alpha, R_i)\,\psi(R_i)] \tag{6}$$

where $U(z,R) = -\gamma_0^2\rho^2 R^{-2} - \gamma_0 R^{-1} + 3\gamma_0 z^2 R^{-3} + 3z^2 R^{-4} - R^{-2}$.

$$H_\rho^0 = \partial^2\Pi_0^*/\partial\rho\,\partial z$$

$$\simeq S\,[V(z-h, R_0)\,\psi(R_0) - V(z+h+\alpha, R_i)\,\psi(R_i)] \tag{7}$$

where $V(z,R) = z\rho R^{-2}(\gamma_0^2 + 3\gamma_0 R^{-1} + 3R^{-2})$

$$E_\phi^0 = +\,i\omega\mu_0\frac{\partial\Pi_0^*}{\partial\rho}$$

$$\simeq Ida\,(4\pi)^{-1}\,[W(R_0)\,\psi(R_0) - W(R_i)\,\psi(R_i)] \tag{8}$$

where $W(R) = -\rho(\gamma_0 + R^{-1})R^{-1}$.

3 Vertical electric dipole

The VMD in the previous Section is replaced by an infinitesimal electric dipole carrying a constant current I and of length dl. The electromagnetic field in air may then be derived from a z-directed Hertz vector $\mathbf{\Pi}_0 = (0, 0, \Pi_0)$ using the relations:

$$E^0 = \nabla(\nabla\cdot\mathbf{\Pi}_0) - \gamma_0^2\mathbf{\Pi}_0,\quad H^0 = i\omega\epsilon_0\nabla\times\mathbf{\Pi}_0 \tag{9}$$

Then, Π_0 is given by [19]:

$$\Pi_0 = A\int_0^\infty [e^{-u_0|z-h|} + \Gamma(u_0)e^{-u_0(z+h)}]\, J_0(\lambda\rho)\,\lambda u_0^{-1}\, d\lambda \tag{10}$$

where $A = Idl\,(4\pi i\omega\epsilon_0)^{-1}$, $\Gamma(u_0) = (n^2u_0 - u)/(n^2u_0 + u)$ and $n^2 = (\sigma + i\omega\epsilon)/(i\omega\epsilon_0)$.

The usefulness of the extended image-theory formulation depends mainly on finding a suitable exponential expansion of the reflection function $\Gamma(u_0)$ such that the resulting fields may be easily derived from one or a set of images. Consideration of the limiting values of $\Gamma(u_0)$, as the integration variable λ tends to zero as well as ∞, yields:

$$\Gamma(u_0) \to (n - \sqrt{\mu_r})/(n + \sqrt{\mu_r}) \simeq 1 \text{ as } \lambda \to 0 \tag{11}$$

$$\Gamma(u_0) \to (n^2 - 1)/(n^2 + 1) \simeq 1 \quad \text{as } \lambda \to \infty \tag{12}$$

where, based on most practical situations, the assumption that $n \gg \sqrt{\mu_r} \geqslant 1$ is implemented. Thus, it is more convenient to write $\Gamma(u_0)$ in the form:

$$\Gamma(u_0) = 1 - 2u/[n^2u_0 + u] = 1 - 2f(u_0) \tag{13}$$

The direct application of the exponential approximation of $f(u_0)$ about $u_0 = 0$, as in the previous Section, may not be convenient in this case owing to the usual large value of n^2 which is multiplied by u_0 in eqn. 13. Instead, an expansion of the function:

$$F(\lambda) = f(u_0)\exp(\beta u_0) \tag{14}$$

about $\lambda = 0$ is attempted, where β is determined from reducing to zero the coefficient of λ^2 in the Taylor-series expansion of $F(\lambda)$. Since $F(\lambda)$ is an even function of λ, the next coefficient is that of λ^4, and one obtains:

$$f(u_0) \simeq \frac{b}{2}\exp(-\beta u_0)\left[1 + \frac{c^2}{\gamma_0^2\gamma^2\delta}\left\{\frac{2}{\gamma} + \frac{(\gamma^3 + \gamma_0^3\mu_r)}{\gamma_0^2\gamma\delta}\right\}\frac{\lambda^4}{8} + \dots\right] \tag{15}$$

where $\beta = c^2/[\gamma\gamma_0\delta]$, $b = 2\gamma_0\mu_r\exp(\beta\gamma_0)\delta^{-1}$ and $\delta = (\gamma + \gamma_0\mu_r)$. Retaining only the first term in eqn. 15, the expression for $\Gamma(u_0)$ reduces to:

$$\Gamma(u_0) \simeq 1 - b\exp(-\beta u_0) \tag{16}$$

The introduction of this approximation of $\Gamma(u_0)$ in the exact expression given by eqn. 10 yields:

$$\Pi_0 \simeq A\,[\psi(R_0) + \psi(R_1) - b\psi(R_e)] \tag{17}$$

where $R_1^2 = \rho^2 + (z+h)^2$ and $R_e^2 = \rho^2 + (z+h+\beta)^2$. The first term in eqn. 17 corresponds to the source contribution, while the second term is due to an image at a position corresponding to a perfectly conducting earth, i.e. at $z = -h$. The third term may be attributed to an image at $z = -(h+\beta)$.

The electromagnetic fields in air for $0 < z < h$, using eqn. 9, are given by:

$$E_z^0 = \left(\frac{\partial^2}{\partial z^2} - \gamma_0^2\right)\Pi_0$$

$$\simeq A\,[U(|z-h|, R_0)\,\psi(R_0) + U(z+h, R_1)\,\psi(R_1) - bU(z+h+\beta, R_e)\,\psi(R_e)] \tag{18}$$

$$E^0_\rho = \partial^2 \Pi_0/(\partial\rho\,\partial z)$$
$$\simeq A\,[V(|z-h|,R_0)\,\psi(R_0) + V(z+h,R_1)\,\psi(R_1) - b\,V(z+h+\beta,R_e)\,\psi(R_e)] \quad (19)$$

$$H^0_\phi = -i\omega\epsilon_0\,\partial\Pi_0/\partial\rho$$
$$\simeq -Idl\,(4\pi)^{-1}\,[W(R_0)\,\psi(R_0) + W(R_1)\,\psi(R_1) - bW(R_e)\,\psi(R_e)] \quad (20)$$

In the quasi-static range where the measurement distance is much less than a free-space wavelength, i.e. $\gamma_0 R_0 \ll 1$, the approximation of $\Gamma(u_0)$ as $\lambda \to \infty$ is the appropriate one. In this case,

$$\Gamma(u_0) \cong (n^2-1)/(n^2+1) \simeq 1 - 2\mu_r(\gamma_0/\gamma)^2 \quad (21)$$

The corresponding field expressions are similar to eqns. 18–20, except that the second terms are multiplied by $[1-2\mu_r(\gamma_0/\gamma)^2]$ and the third terms are dropped.

4 Horizontal dipoles

The VED in the previous Section is replaced by a horizontal electric dipole (HED) along the x-direction ($\phi = 0$). It is well known that an x-component Π_x of the Hertz vector alone cannot describe the electromagnetic field everywhere. Sommerfeld [21] thus assumed a z-component Π_z in addition to Π_x, which led to a set of boundary relations for Π_x and Π_z that were consistent with the boundary conditions on $\boldsymbol{E}$ and $\boldsymbol{H}$.

Erteza and Park [22] pointed out that the familiar Sommerfeld resolution of $\boldsymbol{\Pi}$ is not unique and presented a solution in terms of Π_x and Π_y using plane wave expansion. However, their analysis does not cover the whole range of possibilities. This fact may be realised upon investigating such possibilities when different combinations of components of both electric and magnetic Hertz vectors are used.

In general, the electromagnetic fields in air may be represented in terms of both electric and magnetic Hertz vectors in the form:

$$\boldsymbol{E}^0 = \nabla(\nabla\cdot\boldsymbol{\Pi}_0) - \gamma_0^2\boldsymbol{\Pi}_0 - i\omega\mu_0\,\nabla\times\boldsymbol{\Pi}_0^* \quad (22a)$$

$$\boldsymbol{H}^0 = \nabla(\nabla\cdot\boldsymbol{\Pi}_0^*) - \gamma_0^2\boldsymbol{\Pi}_0^* + i\omega\epsilon_0\,\nabla\times\boldsymbol{\Pi}_0 \quad (22b)$$

The governing equations and the required boundary conditions can be fulfilled using z-directed electric and magnetic Hertz vectors. Their components satisfy the Helmholtz equation, and the boundary conditions at $z = 0$ are:

$$\Pi_0 = n^2\Pi, \quad \Pi_0^* = \mu_r\Pi^* \quad (23a)$$

$$\partial\Pi_0/\partial z = \partial\Pi/\partial z, \quad \partial\Pi_0^*/\partial z = \partial\Pi^*/\partial z \quad (23b)$$

Upon application of these requirements, it can be shown that the equations satisfied by Π_0 and Π_0^* for $0 < z < h$ are given by:

$$\Pi_0 = -A\cos\phi\int_0^\infty\{e^{-u_0|z-h|} + \Gamma(u_0)\,e^{-u_0(z+h)}\}\,J_1(\lambda\rho)\,d\lambda \quad (24)$$

and

$$\Pi_0^* = i\omega\epsilon_0 A\sin\phi\int_0^\infty\{e^{-u_0|z-h|} + \Gamma^*(u_0)\,e^{-u_0(z+h)}\}\,J_1(\lambda\rho)\frac{d\lambda}{u_0} \quad (25)$$

It may be pointed out that the first parts of eqns. 24 and 25 are due to the direct fields, while the second parts represent the earth contribution. The reflection functions $\Gamma(u_0)$ and $\Gamma^*(u_0)$ appearing in these equations are identical to those encountered for VED and VMD, respectively. Using the approximate values of these functions, we get the required field estimations, from eqn. 22. In particular, the z-components of the electromagnetic fields in air are given by:

$$E^0_z = (\partial^2/\partial z^2 - \gamma_0^2)\Pi_0$$
$$= -A\cos\phi\int_0^\infty\{e^{-u_0|z-h|} + \Gamma(u_0)\,e^{-u_0(z+h)}\}\,J_1(\lambda\rho)\lambda^2 d\lambda$$
$$\simeq -A\cos\phi\,[V(z-h,R_0)\psi(R_0) + V(z+h,R_1)\psi(R_1) - bV(z+h+\beta,R_e)\psi(R_e)] \quad (26)$$

and

$$H^0_z = (\partial^2/\partial z^2 - \gamma_0^2)\Pi_0^*$$
$$= i\omega\epsilon_0 A\sin\phi\int_0^\infty\{e^{-u_0|z-h|} + \Gamma^*(u_0)\,e^{-u_0(z+h)}\}\,J_1(\lambda\rho)\frac{\lambda^2}{u_0}\,d\lambda$$
$$\simeq -i\omega\epsilon_0 A\sin\phi\{W(R_0)\psi(R_0) - W(R_i)\,\psi(R_i)\} \quad (27)$$

When the HED is replaced by a horizontal magnetic dipole (HMD) at the same location and which has an area da, an analysis similar to the one done for the HED may be performed. On the other hand, the field representation for the HMD may be deduced from the corresponding results for the HED using the duality [23]. This may be obtained by making the transformation $dl \to da$, $\sigma + i\omega\epsilon \to i\omega\mu$, $i\omega\mu \to \sigma + i\omega\epsilon$, $\boldsymbol{E} \to \boldsymbol{H}$ and $\boldsymbol{H} \to -\boldsymbol{E}$.

5 Use of the FFT

Since the solution for the HED combines the basic computational features encountered in the VED and VMD excitation, only the HED is considered next. The transverse field components (normal to z) are usually, e.g. Reference 24, taken as the ρ and ϕ components. Each component consists of two parts: one includes Bessel function while the other includes its derivative. A considerable saving in the computational requirement of these components may be gained if the transverse components are taken as the x and y components. The expressions of these components may be derived either from eqn. 22 or from the ρ and ϕ components [24]. Thus, the transverse-electric magnetic field component in air along x is given by:

$$H_x^{TE} = H_\rho^{TE}\cos\phi - H_\phi^{TE}\sin\phi$$
$$= -\frac{Idl}{4\pi}\sin\phi\cos\phi\int_0^\infty[e^{-u_0|z-h|} + \Gamma(u_0)e^{-u_0(z+h)}]\,\rho^{-1}\times\{\lambda\rho J_1'(\lambda\rho) - J_1(\lambda\rho)\}d\lambda$$
$$= \frac{Idl}{8\pi}\sin 2\phi\int_0^\infty[e^{-u_0|z-h|} + \Gamma(u_0)e^{-u_0(z+h)}]\times J_2(\lambda\rho)\,\lambda d\lambda \quad (28)$$

where the known identity $xJ_1'(x) = J_1(x) - xJ_2(x)$ is used.

Similarly, the transverse-magnetic part is given by:

$$H_x^{TM} = -\frac{Idl}{8\pi}\sin 2\phi\int_0^\infty[e^{-u_0|z-h|} - \Gamma^*(u_0)\,e^{-u_0(z+h)}]\,J_2(\lambda\rho)\,\lambda d\lambda \quad (29)$$

Consequently, we have:

$$H_x^0 = \frac{Idl}{8\pi}\sin 2\phi \int_0^\infty [\Gamma(u_0) + \Gamma^*(u_0)] e^{-u_0(z+h)} J_2(\lambda\rho)\lambda d\lambda \quad (30)$$

This expression clearly exhibits the requirements that H_x^0 satisfies the Helmholtz equation and does not have a direct contribution from the source. It can be shown that other Cartesian transverse field components may be written as the sum of a source term and reflection integral involving $J_0(\lambda\rho)$ and $J_2(\lambda\rho)$. Since these reflection integrals are solutions of the Helmholtz equation, their ϕ-dependence is in terms of sin 2 ϕ or cos 2 ϕ for the $J_2(\lambda\rho)$ part. On the other hand, the longitudinal field components along z has integrals of $J_1(\lambda\rho)$ only, and consequently sin ϕ or cos ϕ dependence. Thus, a typical form of the encountered integral is:

$$I(z,\rho) = \int_0^\infty \bar{g}(z,\lambda) J_n(\lambda\rho)\, d\lambda \quad (31)$$

where $n = 0$, 1 or 2. Although a number of numerical techniques appropriate to semi-infinite intervals are available, the analytic characteristics of the integrand as well as the range of change of z and ρ must be considered if an efficient routine is to be used. These considerations were discussed by Tsang *et al.* [8], Lytle and Lager [9], and more recently by Keuster and Chang [6].

Owing to the computational advantages it provides, Tsang *et al.* [8] used the FFT in order to compute an integral similar to $I(z, \rho)$ given by eqn. 31. The function $\bar{g}$ in the integrand was replaced by:

$$\bar{g}(z,\lambda) = g_1(z,\lambda)\exp(-\nu_R\lambda) \quad (32)$$

where $g_1(z, \lambda) = \bar{g}(z, \lambda) \exp(\nu_R\lambda)$. The positive non-zero value of the real constant ν_R must be properly chosen, depending on $(z + h)$ [8], to avoid the divergence of g_1. A complex Fourier-series expansion of g_1 may then be performed using the FFT, which may be written as [8]:

$$g_1(z,\lambda) \simeq \frac{1}{N\Delta\lambda}\sum_{m=-N/2}^{\frac{N}{2}-1} a_1\left(\frac{m}{N\Delta\lambda}\right)\exp\left(i\,\frac{2\pi m}{N\Delta\lambda}\,\lambda\right);$$

$$0 < \lambda < \left(\frac{N}{2} - 1\right)\lambda \quad (33)$$

The number of samples N must be properly chosen for best performance of the FFT and should be some integral power of 2. The resulting integrals may then be performed exactly using the formula [Reference 25, p. 707]:

$$\int_0^\infty \exp(-\nu\lambda) J_n(\lambda\rho)\, d\lambda = \rho^{-n}\left[\sqrt{\nu^2+\rho^2} - \nu\right]^n / \sqrt{\nu^2+\rho^2}\,; R_e\nu > 0 \quad (34)$$

It may be more appropriate, in certain cases, to write the function $\bar{g}$ in the form:

$$\bar{g}(z,\lambda) = g_2(z,\lambda)\lambda^n \exp(-\nu_R\lambda) \quad (35)$$

The integrals encountered may be evaluated, after performing a complex Fourier series expansion of g_2, using [Reference 25, p. 712]:

$$\int_0^\infty \lambda^n \exp(-\nu\lambda) J_n(\lambda\rho)\, d\lambda = \frac{(2\rho)^n\,\Gamma(n+1/2)}{\sqrt{\pi}\,(\nu^2+\rho^2)^{n+1/2}}$$

$$R_e\nu > 0 \quad (36)$$

Another two possibilities are writing $\bar{g}$ as:

$$\bar{g}(z,\lambda) = g_3(z,\lambda)\lambda^{n+1}\exp(-\nu_R\lambda) \quad (37)$$

or

$$\bar{g}(z,\lambda) = g_4(z,\lambda)\lambda^{-1}\exp(-\nu_R\lambda) \quad (38)$$

The relevant corresponding integrals in these cases are [Reference 25, p. 712]:

$$\int_0^\infty \lambda^{n+1}\exp(-\nu\lambda) J_n(\lambda\rho)\, d\lambda = \frac{(2\nu)(2\rho)^n\,\Gamma(n+3/2)}{\sqrt{\pi}\,(\nu^2+\rho^2)^{n+3/2}};$$

$$R_e\nu > 0 \quad (39)$$

and

$$\int_0^\infty \lambda^{-1}\exp(-\nu\lambda) J_n(\lambda\rho)\, d\lambda = \left(\sqrt{\nu^2+\rho^2} - \nu\right)^n / n\rho^n;$$

$$R_e\nu > 0 \quad (40)$$

respectively. The choice of the appropriate decomposition of the function $\bar{g}$ depends on its form and asymptotic behaviour, so that an expansion in the form of eqn. 33 is adequate for a reasonable sampling number N.

6 Discussion and conclusion

The evaluation of the SI has attracted the interest of many investigators. The long persistent interest in the topic reflects the complex nature of the problem. The integrand of the integrals involved has a complicated behaviour in the complex plane that includes branch cuts and may have simple poles. It is possible to numerically evaluate these integrals, regardless of the parameters involved, by using 'brute force' integration. However, this can be so expensive that it may not be economically feasible to apply the results to the wide variety of problems of interest.

When the integrand is approximated in a particular form, the resulting estimate would apply within a certain range of parameters. The particular advantages of the complex image approximation are its simplicity and wide range of applicability. In particular, Bannister [18] demonstrated that, when the measurement distance R_1 is much greater or much less than an earth skin depth δ, the approximations reduce to previously derived results. Furthermore, when R and δ are comparable, the image theory and previously derived analytical, or numerical integration, results were in good agreement. When $z = h = 0$, the maximum error encountered was 10% when the exact field E_ρ of a VED, E_z of an HED, H_ϕ of a HED and E_ρ of a HMD were compared with the image approximation. When the exact analytical expression for E_ϕ for a HMD and H_ρ for a HED were compared with the image approximation, the error was less than 20%. Numerical results for the mutual impedance between coplanar horizontal elevated loop antennas were compared with the image results, and a maximum error of 10% was encountered. In the quasi-static range, the field approximations given in Sections 2–4 are identical or comparable to those of Bannister when the change in permeability is neglected and upon invoking the assumption that $|\gamma_0| \ll |\gamma|$. Besides having the advantages of Bannister's results, our results do not have the last two limitations and hence apply to a wider range.

Wait and Spies [26] and Bannister [18] have examined the analytical validity of the image-theory technique via consideration of the Taylor-series expansion involved. The higher-order terms in the expansion yield multipoles located at the complex image. The strengths of these multipoles, and consequently their significance, depend on the expansion coefficients which depend on the earth's characteristics. The terms neglected in eqn. 4 are of the order of $O(1/c^3 R^3)$ compared to the terms retained. Thus the image theory yields good results for far-fields evaluation. Such fields are usually estimated using the saddle-point technique [19]. This technique yields the saddle-point at $|u_0| = |\gamma_0(z+h)R_1^{-1}|$, $|\lambda| = |\gamma_0 \rho R_1^{-1}|$. Consequently, expansion for small u_0 is legitimate as long as $|\gamma_0(z+h)| \ll R_1$, while expansion for small λ is acceptable whenever $|\gamma_0 \rho| \ll R_1$. On the other hand, as $|\alpha|$ becomes very large, the approximations are acceptable since the image, and hence the induced currents, are then remote enough that only the source term effectively contributes to the field. When the ground is perfectly conducting, $\sigma \to \infty$, the image approximations given in Sections 2–4 reduce to the known exact solutions.

The image approximation performed in Section 2 is only one possible form. The typical integral encountered may be written as:

$$T = \int_0^\infty \Gamma^*(u_0) e^{-u_0(z+h)} J_0(\lambda\rho) \frac{\lambda}{u_0} d\lambda \tag{41}$$

If $\Gamma^*(u_0)$ can be expressed in the form [6]:

$$\Gamma^*(u_0) = \int_\Omega \Gamma^*(t) e^{-u_0 t} dt \tag{42}$$

where Ω is a suitable contour such that the interchange of integration sequence is permissible upon insertion of eqn. 42 in eqn. 41. Consequently, T reduces to:

$$T = \int_\Omega \bar{\Gamma}^*(t) \psi(R_t) dt \tag{43}$$

where $R_t^2 = \rho^2 + (z+h+t)^2$. Thus T is represented as an integral of $\psi(R_t)$, the field of an image at $z = -(h+t)$, with a suitable weighting function over contour Ω.

Alternatively, $\Gamma^*(u_0)$ may be written in terms of a discrete set of exponentials as:

$$\Gamma^*(u_0) \simeq \sum_{n=1}^{N} A_n \exp(-u_0 \alpha_n) \tag{44}$$

whose number N depends on the required accuracy. The problem then reduces to the well known problem of separation of exponentials [27], whose related Prony's method has been used recently by Van Blaricum and Mittra [28] in connection with the SEM technique. The resultant field may be attributed to N images located at $z = -(h + \alpha_n); n = 1, 2, \ldots N$. The expansion used in Section 2, on the other hand, is a particular case of the more general form:

$$\Gamma^*(u_0) \simeq \sum_{n=1}^{N} \left(\sum_{m=0}^{M} A_{nm} u_0^m \right) \exp(-u_0 \alpha_n) \tag{45}$$

where the field is in terms of a finite set of images and multipoles located at $z = -(h + \alpha_n); n = 1, 2, \ldots N$.

The literature contains a number of generalised continuous image representations for SI. One of the first of such representations was given by Sommerfeld [Reference 21, pp. 246] for large $|n|$ as a continuous set along the z-axis from the image point $z = -h$ to $z = -\infty$. Van der Pol [29] gave a form with images distributed over an entire half space below the source. Image representations distributed over a plane were considered by Briquet and Filippi [30] and Filippi and Habault [31].

The non-uniqueness in the field representation of a horizontal dipole was discussed in Section 4. The main advantage of the form used is that the reflection functions encountered are the same as those for VMD and VED. Consequently, the approximations are consistent and the reciprocity principle, as given by Bannister [32], is automatically satisfied. The use of Π_x and Π_z, as previously done by Bannister [33], does not have these advantages. The interesting aspects of using Cartesian rather than polar transverse field components were discussed in Section 5.

The semi-analytical evaluation of the integrals using the FFT technique was considered in Section 5. For typical cases [8], the computation time using Simpson's rule was 32 min for the 3-layer earth model case and 40 min for the six-layer case, compared to only 2.5 min for each using FFT. The sampling number should be such that outside the range $-N\Delta\lambda/2$ to $N\Delta\lambda/2$ the functions $g_i(\lambda)$, $i = 1, 2, 3, 4$, are sufficiently small [34]. Thus, the speed of computation is related to the asymptotic behaviour of g_i. The required number of samples for a specified accuracy may be reduced by multiplying the expanded part of the integrand with a suitable decaying function [8]. It may be pointed out that the new decomposition forms given by eqns. 35 and 37 have the advantages that the associated functions g_2 and g_3 have better asymptotic behaviours than g_1. Once the FFT is performed, the results may then be used to evaluate the field at a certain z for a large range of the radial distance ρ via summation of the resulting series. Both the type of decomposition of $\bar{g}$ and the order of magnitude of ρ control the convergence rate of the series. In particular, eqn. 39 behaves as $0(1/\nu^{2n+2})$ for large ν, and when $\rho \gg \nu$ has the order of $0(1/\rho^{n+3})$, which makes it suitable for far field evaluation. The use of the FFT technique avoids the difficulty introduced in the direct numerical evaluation of the integrals due to the oscillatory behaviour of the Bessel functions for large arguments. If the integral corresponding to the asymptotic value $\bar{g}_\infty$ of $\bar{g}$ as $\lambda \to \infty$ is known, this value may be added and subtracted from $\bar{g}$ and the remaining integral involving $(\bar{g} - \bar{g}_\infty)$ may then be computed more accurately and far more faster. A proper decomposition of $(\bar{g} - \bar{g}_\infty)$ would then ensure the convergence of the resulting series, increase the accuracy of the results and reduce the computation time. Ways to eliminate the mathematical difficulties which may be caused by the singularities in the integrand were discussed by Tsang *et al.* [8] in connection with the use of the FFT. Using these ways in addition to the present results, there would be no limitation, in theory, on the use of the FFT technique.

In conclusion, it is believed that, in spite of the long history of the subject of evaluation of SI, many interesting aspects can yet be studied. Recent investigations were done by Rahmat-Samii *et al.* [35] and by Burke *et al.* [36]. This paper presents some contributions to the analytical approach, via image theory, as well as the computational aspects using the FFT. Common features between these two techniques are their small computation time and their wide range of applicability, besides depending on exponential approximation of a part of the integrand. It may be appropriate to point out that demanding very high accuracy in the evaluation of SI may not be fully justified in view of the nonuniformity of the earth's surface and the uncertainity in information about its electromagnetic characteristics.

7 Acknowledgment

The author acknowledges the constructive criticism of the reviewers which contributed to improving the presentation. This work was supported by the US National Oceanic & Atmospheric Adminstration (NOAA) under the SFC agreement no. 04-6-158-4403 with Cario University.

References

1 KELLER, G.V., and FRISCHKNECHT, F.C.: 'Electrical methods in geophysical prospecting' (Pergamon Press, 1966)
2 ANNAN, A.P.: 'Radio interferometry depth sounding: Part I. Theoretical discussion', *Geophysics*, 1973, 38, pp. 557–580

3 WAIT, J.R.: 'Characteristic of antennas over lossy earth', in COLLIN, R.E., and ZUCKER, F.J. (Eds): 'Antenna theory: Part II' (McGraw-Hill, 1969), Chap 23
4 SOMMERFELD, A.: 'On the propagation of waves in wireless telegraphy', *Ann. Phys.*, 1909, **25**, pp. 665–736; and 1926, **81**, pp. 1135–1153
5 BANOS, A.: 'Dipole radiation in the presence of a conducting half-space' (Pergamon Press, 1966)
6 KUESTER, E.F., and CHANG, D.C: 'Evaluation of Sommerfeld integrals associated with dipole sources above earth'. Electromagnetics Laboratories, Department of Electrical Engineering, University of Colorado, Boulder, Science Report 43, 1979
7 SIEGEL, M., and KING, R.W.P.: 'Electromagnetic fields in a dissipative half-space, a numerical approach', *J. Appl. Phys.*, 1970, **41**, pp. 2415–2423
8 TSANG, L., BROWN, R., KONG, J.A., and SIMMONS, G.: 'Numerical evaluation of electromagnetic fields due to dipole antennas in the presence of stratified media', *J. Geophys. Res.*, 1974, 79, pp. 2077–2080
9 LYTLE, R.J., and LAGER, D.L.: 'Numerical evaluation of Sommerfeld integrals' Lawrence Livermore Laboratories , Report UCRL-51688, 1974
10 YOKOYAMA, A.: 'Dipole radiation affected by the plane earth', *J. Phys. Soc. Jpn.*, 1969, **27**, pp, 224–229
11 CHANG, D.C., and WAIT, J.R.: 'Appraisal of near-field solutions for a Hertzian dipole over a conducting half-space', *Can. J. Phys.*, 1970, **48**, pp. 737–743
12 KUO, W.C.: 'Numerical treatment of Sommerfeld's integrals and its application to antenna and scattering problems', Ph.D. Thesis, Electrical Engineering Department, University of California, Berkeley, 1973
13 FULLER, A.: 'Electromagnetic radiation and propagation in the presence of dispersive geological media'. Ph.D. Thesis, Electrical Engineering Department, University of Colorado, Boulder, 1974
14 CHANG, D.C., and FISHER, R.J.: 'A unified theory on radiation of a vertical electric dipole above a dissipative earth', *Earth Sci.*, 1974, 9, pp. 1129–1138
15 LINDSAY, J.E. Jr.: 'A direct approach to the numerical evaluation of electromagnetic fields due to sources buried beneath the earth's surface'. Presented at 1973 IEEE-AP International Sympiosium, Boulder, Colorado
16 WAIT, J.R.: 'Image theory of a quasistatic magnetic dipole over a dissipative half-space', *Electron. Lett.*, 1969, **5**, (13) pp. 281–282
17 WEAVER, J.T.: 'Image theory for an arbitrary quasistatic field in the presence of a conducting half-space', *Radio Sci.*, 1971, **6**, pp. 647–653
18 BANNISTER, P.R.: 'Summary of image theory expansions for the quasistatic fields of antennas at or above the earth's surface', *Proc. IEEE*, 1979, **67**, pp. 1001–1008
19 WAIT, J.R.: 'Electromagnetic waves in stratified media' (Pergamon Press, 1970), Chap. 2
20 MOHSEN, A.: 'On the analysis of transmission lines above lossy earth'. Proc. Canadian Conference on Communications & Power, Montreal, Canada, 1980, pp. 196–199
21 SOMMERFELD, A.: 'Partial differential equations' (Academic Press, 1949) pp. 246–267
22 ERTEZA, A., and PARK, B.K.: 'Non-uniqueness of resolution of Hertz vector in presence of a boundary and the horizontal dipole problem', *IEEE Trans.* 1969, **AP-17**, pp. 376–378
23 HARRINGTON, R.F.: 'Time harmonic electromagnetic fields' (McGraw-Hill, 1961), Chap. 3
24 KONG, J.A.: 'Electromagnetic fields due to dipole antennas over stratified anisotropic media', *Geophys.*, 1972, **37**, pp. 985–996
25 GRADSHTEYN, I.S., and RYZHIK, I.W.: 'Tables of integrals, series and products' (Acadamic Press, 1965)
26 WAIT, J.R., and SPIES, K.P.: 'On image representation of the quasistatic field of a line current source above the ground', *Can. J. Phys.*, 1969, **47**, pp. 2731–2733
27 LANCZOS, C.: 'Applied analysis' (Pitman & Sons, 1967), pp. 272–280
28 VAN BLARICUM, V.L., and MITTRA, R.: 'A technique for extracting the poles and residues of a system directly from its transient response', *IEEE Trans.*, 1975, **AP-23**, pp. 777–781
29 VAN DER POL, B.: 'Theory of the reflection of the light from a point source by a finitely conducting flat mirror, with an application to radiotelegraphy', *Physica*, 1935, **2**, pp. 843–853
30 BRIQUET, M., and FILIPPI, P.J.T.: 'Diffraction of a spherical wave by an absorbing plane'. *J. Acoust. Soc. Am.* 1977, **61**, pp. 640–646
31 FILIPPI, P.J.T., and HABAULT, D.: 'Reflexion of a spherical wave by the plane interface between a perfect fluid and a porous medium', *J. Sound & Vib.*, 1978, **56**, pp. 97–103
32 BANNISTER, P.R.: 'The image theory quasistatic fields of antennas above the earth's surface'. Navy Underwater Sound Laboratory, New London, Connecticut, USA, Report no. 1061, 1969
33 BANISTER, P.R.: 'The quasi-near fields of dipole antennas', *IEEE Trans.*, 1967, **AP-15**, pp. 618–626
34 HOWARD, A.Q., Jr.: 'On approximating Fourier integral transforms by their discrete counterparts in certain geophysical applications', *IEEE Trans.*, 1975, **AP-23**, pp. 264–266
35 RAHMAT-SAMII, Y., MITTRA, R., and PARHAMI, P.: 'Evaluation of Sommerfeld integrals for lossy half-space problems', *Electromagnetics*, 1981, **1**, pp. 1–28
36 BURKE, G.J., MILLER, E.K., BRITTINGHAM, J.N., LAGER, D.L., LYTLE, R.J., and OKADA. J.T.: 'Computer modelling of antennas near the ground', *ibid.*, 1981, **1**, pp. 29–49