An Efficient Pole Extraction Technique for the Computation of Green's Functions in Stratified Media Using a Sine Transformation

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***Abstract*—An efficient and automatic algorithm for the extraction of surface wave poles of the multilayered spectral-domain Green's function is presented. With the sine transformation, we have managed to locate any pole, no matter how close to the branch point singularity lies, something that has not been feasible with techniques proposed till now. The aforementioned transformation maps the k_ρ-plane to the w-plane, where the branch point is removed, thus enabling the application of a fast numerical contour integration technique proposed in [6], which is inaccurate when dealing with poles very close to the branch point.**

***Index Terms*—Closed-form Green's functions, discrete complex image method (DCIM), pole extraction, sine transformation, surface wave poles.**

I. Introduction

In the modeling of printed circuit elements, such as microstrip interconnects terminated by complex loads, patch antennas and printed dipoles, it is widely accepted that algorithms based on integral equations (IE), using the method of moments (MoM) are the most efficient and rigorous methods for the analysis of small to medium-sized (in terms of wavelengths) multilayered structures. More specifically, we prefer the mixed potential integral operator formulation because we want to exploit the efficiency of the integral operators for open geometries of planar stratified media, in conjunction with a less singular kernel. A critical factor for the IE analysis is the fast computation of Green's functions for a multilayer medium, which are expressed in terms of the well known Sommerfeld integrals (SI) [1]. Generally, the analytical solution of the SI is not available, and the numerical integration is time consuming, since the integrand is both highly oscillating and slowly decaying. Among several methods that have been proposed in order to tackle this problem, stands the discrete complex image method (DCIM) [2] or, in other words, the closed-form Green's functions (CFGF) method.

The basic idea of the DCIM is to approximate the spectral-domain Green's functions in terms of complex exponentials, using the generalized pencil of function (GPOF) method and cast the integral representation into closed-form expressions via an integral identity, namely Sommerfeld identity [1]. The approximating functions represent spherical waves with complex distances, referred to as complex images, and these dominant wave constituents of the fields of a dipole source are spherical in nature. However, in a layered structure there are other wave constituents due to a dipole source, like cylindrical and lateral waves, which may not be approximated in terms of complex images, unless their contributions are explicitly accounted for. It has been shown [3], [4], that if the surface wave poles are not extracted prior to the exponential fitting, the DCIM approximation seems to deteriorate violently, even for moderate distances from the source. This behavior can be easily understood by means of the asymptotic behavior of the Green's functions. For example, assuming that the spectral-domain Green's function $\tilde{G}(k_\rho)$ does contain at least one surface wave pole (SWP) then, for large values of ρ, the spatial-domain Green's function will be dominated by the Cauchy residues of these poles

$$\lim_{\rho\to\infty}\frac{1}{4\pi}\int_{\mathrm{SIP}}\tilde{G}(k_\rho)H_n^{(2)}(k_\rho\rho)k_\rho^{n+1}dk_p = -\frac{j}{2}\sum_{i=1}^{N}\alpha_i p_i^{n+1}H_n^{(2)}(p_i\rho) \quad (1)$$

where "SIP" is the Sommerfeld integration path and α_i are the residues of the poles located at the points p_i of the complex k_ρ-plane. Hence, knowing that the surface wave contribution approaches its theoretical limit $(1/\sqrt{\rho})$, even for moderate distances from the source, it is obvious that for such cases, if these poles are not extracted and represented in terms of Hankel functions, the discrete complex images are not sufficient, since they exhibit exponential decay $(1/\rho)$. Therefore, for a general purpose MoM analysis of electrically large structures, the guided modes have to be extracted. The difficulty of locating the poles of the spectral-domain Green's function is responsible for the rather small number of results available in the literature for the multilayered structures, except for the single-layered or double-layered cases [2] which have been extensively studied.

Recently, Ling *et al.* [5] and Teo *et al.* [6] proposed two methods for the automatic extraction of the poles. The first one extracts the poles by recursively performing contour integrals to find the location of these poles. The main drawback of this method is that it requires a large number of sampling points in the integral when the contour lies too close to a pole, which is inherently inevitable as the contour subdivision continues until the desired accuracy is obtained. On the other hand, the second method does not need to integrate near the poles and the computation requirement is equivalent to that of a single contour integral. A complete comparison between them is given in [6], where it is made clear that the technique for the approximation of the poles by Teo *et al.* is much more efficient. At this point, we have to mention that care should be taken, in both methods, to avoid the branch cut of the Green's function that arises for the open problem of radiation since the mode wavenumber

$$k_{z0}=\sqrt{k_0^2-k_\rho^2} \quad (2)$$

is a double-valued complex function. It is the existence of the branch point $k_\rho = k_0$, though, that prevents finding the location of the poles, when they are placed very close to k_0. And, as we know from theory [1], it is very common for the TM surface wave poles to be located very close to the branch point, since there is no low cutoff frequency for them.

In this paper, a proper sine transformation will be embodied in the existing pole extraction algorithm [6], in order to remove the branch cut of the Green's function and locate all the existing poles with great accuracy, no matter how close to the branch point they are placed. We mention, though, that the branch point removal by a suitable transformation was actually suggested in [6], with no further elaboration. Moreover, the sine transformation was previously employed in [7], in the context of a Delves-Lyness pole search procedure. Finally, a suitable form of the working variable was embodied in the pole search procedure [8] giving very good results.

The performance of the proposed modified algorithm and the improvements achieved will be presented in Section III. For this reason, we find it more appropriate to introduce a pole proximity to branch point (PPB) parameter

$$\mathrm{PPB}_i=\frac{p_i-k_0}{k_0}. \quad (3)$$

We will demonstrate that even in cases where a pole is extremely close to the branch point, which are very common, the overall performance of the proposed technique is remarkable.

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II. Pole Extraction With Sine Transformation

Following complex analysis, we introduce a new complex variable w via the transformation [9]

$$k_\rho = k_0 \sin w \tag{4}$$

which identifies w as a complex angle variable and makes $k_i = (k_0^2 - k_\rho^2)^{1/2} = 0$ a regular point in the w-plane. The transcendental function $\sin w$ is single-valued. In the mapping from the k_ρ-plane to the w-plane, the Riemann sheets in the former one appear as adjacent regions of "width" equal to 2π in the latter one. The inverse sine function is multiple-valued in the k_ρ-plane, implying the existence of the branch points on that plane. Moreover, assuming an $\exp(j\omega t)$ time dependence, we select proper branch cuts ($\Im(k_{z0}) = 0$ branch cuts) in the k_ρ-plane, in order to ensure that on the top Riemann sheet the radiation condition $\Im(k_{z0}) \leq 0$ is satisfied. The removal of the branch point allows us to perform the complex integration along the contour, which crosses the regular point $w = \pi/2$. We have to be careful when we want to evaluate the critical points in the w-plane, since the inverse sine function of a complex number is given by the following formula

$$\begin{aligned} w &= \sin^{-1}\left(\frac{k_\rho}{k_0}\right) \\ &= -j \log\left[j\left(\frac{k_\rho}{k_0}\right) + \left(1 - \left(\frac{k_\rho}{k_0}\right)^2\right)^{1/2}\right] \end{aligned} \tag{5}$$

or even better

$$\begin{aligned} w &= -j(\mathrm{Log}|z| + j\arg(z)) \\ &= -j(\mathrm{Log}|z| + j\mathrm{Arg}(z) \pm j2n\pi), \\ &\qquad n = 0, 1, 2, 3, \ldots \end{aligned} \tag{6}$$

where $z = j(k_\rho/k_0) + (1-(k_\rho/k_0)^2)^{1/2}$. Hence, in order to satisfy the radiation condition, we take the principal value of the logarithm ($n = 0$) under the condition that the square root function is also properly defined on the top Riemann sheet.

Having considered the issues of correct mapping and complex plane topology, we proceed in accordance with [6], by constructing the following series:

$$T_n = \int_C \tilde{G}(k_\rho(w)) \exp[nk_\rho(w)] \frac{dk_\rho}{dw} dw \tag{7}$$

for $n = 0, 1, 2, \ldots, M-1$, where M is greater than twice the number of poles that are located within the contour. The computational cost of evaluating the M integrals can be greatly reduced, since the contour for every integral is the same and the only term that changes within the integrand is the exponential one. At this point, it is important to mention that the scaling factor s and the constant offset q introduced in [6] have not been used in this modified method, since the results showed that such a numerical optimization is useless in the w-plane.

By applying the Cauchy's residue theorem, we obtain

$$T_n = 2\pi j \sum_{i=1}^{N} a_i \exp[nP_i] \tag{8}$$

where P_i are the locations of the poles in the w-plane, a_i are the residues of the poles, and N is the total number of poles enclosed by the proper defined contour. The poles p_i in the k_ρ-plane come via the transformation (4), $p_i = k_0 \sin(P_i)$. Since T_n is a series consisting of exponential terms, the GPOF method seems to be the best choice for the accurate calculation of the desired values a_i and p_i, as long as $M \geq 2N$ [6]. The poles obtained using the GPOF method are further refined with the help of the modified Newton-Raphson method and, finally, the algorithm is to be reiterated with the known poles extracted until no pole is to be found.

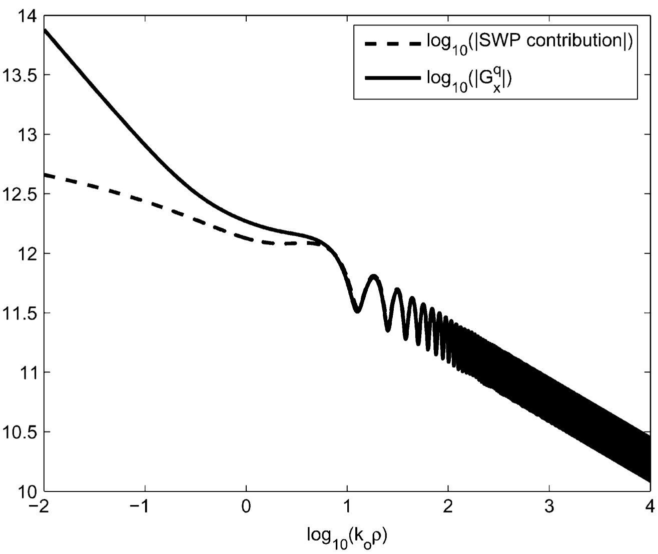

Fig. 1. Magnitude of the scalar Green's function and the SWP contribution for a single-layer geometry. $f = 10.0$ GHz; HED is at the interface, $\varepsilon_r = 4.0, h = 0.5$ cm. Green's function is obtained by first extracting the SWPs in the spectral domain and adding their contribution in the spatial domain analytically.

III. Numerical Results and Discussion

In this section, we will present some numerical results for the validation of the proposed modified pole extraction algorithm's accuracy. For this reason, we choose a combination of geometries and frequencies where the methods in [5] and [6] seem to fail. In order to extract both the TE and TM poles, the spectral-domain Green's function of the scalar potential $\tilde{G}_x^q$ is used [2] (horizontal electric dipole is at the interface of the air layer and the dielectric slab). But, before proceeding with the validation of the proposed technique's achievements, it is of paramount importance to show how these poles, actually, affect the evaluation of the spatial-domain Green's function via the two-level DCIM.

Hence, picking the one-layer geometry of the example first presented in [4], with dielectric parameters: $\varepsilon_r = 4, h = 0.5$ cm and operation frequency: $f = 1$ GHz, we evaluate, first, the Green's function without the extraction of the TM pole at $p = 0.21$ rad/cm, and then we evaluate the SWP contribution analytically. The results, as demonstrated in [4], clearly reveal that the SWP contribution is responsible for the dramatic deterioration of accuracy even for moderate distances from the source ($\log_{10}(k_0\rho) > 2$). By switching the operation frequency at 10 GHz we locate an extra TE pole at $p = 2.22$ rad/cm and evaluate the contribution of the SWPs. Obviously, if the pole extraction technique fails to locate the TE pole, the SWP contribution will still exhibit a cylindrical wave like decay $(1/\sqrt{\rho})$, but the interaction effect between the two poles is completely lost (Fig. 1).

The proposed algorithm can handle, as well, the case where losses are introduced. As it was stated above, the surface waves are the dominant contribution in the far field region, but this is only true for strictly lossless structures. A nonzero loss tangent will push the poles away from the real axis, into the fourth quadrant, and will introduce a small exponentially decreasing behavior in the Hankel functions. As a result, the surface wave behavior dominates up to a certain distance. Then, the terms devoid of exponential attenuation takes over, as it is clearly shown in Fig. 2. For the specific loss tangent, we locate two poles close to the real axis: $p_{\mathrm{TE}} = 2.222599 - j0.023825$ rad/cm and $p_{\mathrm{TM}} = 3.265261 - j0.040184$ rad/cm.

Next to the discussion about the importance of the accurate approximation of the surface wave poles, it is time to demonstrate the efficiency of the proposed technique. First, a three-layer geometry, as shown in

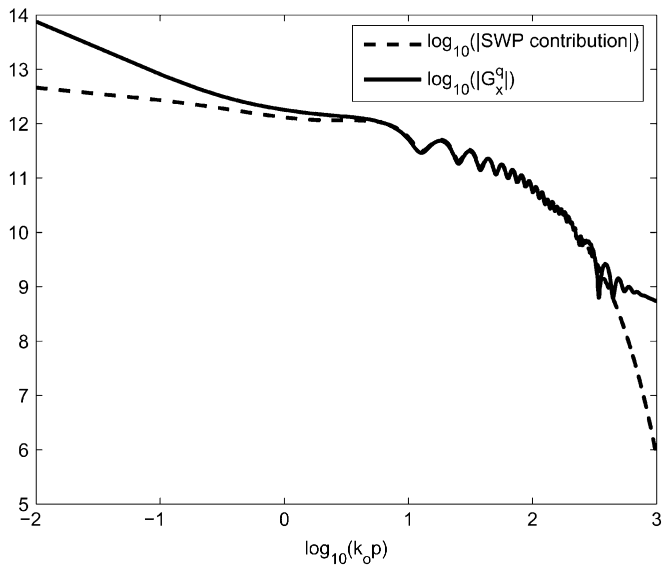

Fig. 2. Magnitude of the scalar Green's function and the SWP contribution for a single-layer geometry. $f = 10.0$ GHz; HED is at the interface, $\varepsilon_r = 4.0, h = 0.5$ cm. Lossy case: $\tan(\delta) = 0.02$. Green's function is obtained by first extracting the SWPs in the spectral domain and adding their contribution in the spatial domain analytically.

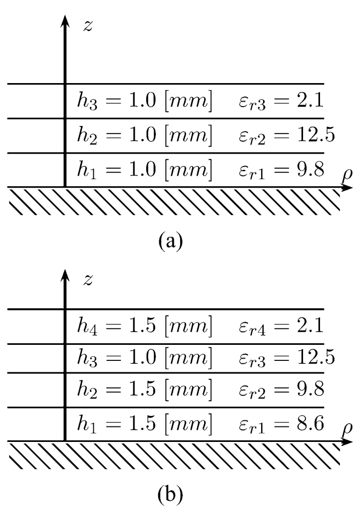

Fig. 3. More complicated geometries that arise in state-of-the-art applications. (a) Three-layer grounded structure geometry. (b) Four-layer grounded structure geometry.

Fig. 3(a), is studied at the frequency of 33.72 GHz. We have to say that the aforementioned frequency is carefully selected, in order to demonstrate the accuracy of the proposed method. The TE and TM poles of the mixed potential Green's functions are as follows:

$$\mathrm{TM}\colon p = \{10.5210, 21.6628\}\ [\mathrm{rad/cm}] \tag{9}$$
$$\mathrm{TE}\colon p = \{7.0671, 20.6318\}\ [\mathrm{rad/cm}]. \tag{10}$$

It is clear, that in this paper, we are interested in the pole that lies close to the branch point, the TE_1 pole

$$\begin{aligned} \mathrm{TE}_1\colon p &= 7.0671\ [\mathrm{rad/cm}] \\ \mathrm{PPB}_{\mathrm{TE}_1} &= 4.5585 \cdot 10^{-7}. \end{aligned} \tag{11}$$

The PPB parameter above stands for the worst case scenario of the examples presented and reveals the unique efficiency of the sine transformation modification of the pole extraction technique. As for a final example, we consider the four-layer structure [Fig. 3(b)], a complex geometry that reveals the general purpose character of the proposed method. The frequency is chosen to be 30 GHz, for which the TM and TE poles located on the real axis of the k_ρ-plane, between the minimum and maximum wavenumbers are as follows:

$$\mathrm{TM}\colon p = \{8.6019, 16.1638, 18.863\}\ [\mathrm{rad/cm}] \tag{12}$$
$$\mathrm{TE}\colon p = \{6.2893, 14.8323, 19.1407\}\ [\mathrm{rad/cm}]. \tag{13}$$

Again, examining the pole proximity to the branch point, we evaluate the PPB parameter

$$\begin{aligned} \mathrm{TE}_1\colon p &= 6.2893\ [\mathrm{rad/cm}], \\ \mathrm{PPB}_{\mathrm{TE}_1} &= 2.9607 \cdot 10^{-4}. \end{aligned} \tag{14}$$

A significant feature of the proposed method is that it can handle a large number of surface wave poles, a case that arises when dealing with complicated geometries at high frequencies, with only a slight increase of the computation time. Therefore, the pole extraction algorithm can easily be embodied in a CFGF method, like the DCIM [2], improving substantially its efficiency without significant increase of the computation time needed.

IV. Conclusion

We have presented a robust, automatic method for the extraction of the surface wave poles of the spectral-domain Green's function. The difficulty in extracting surface wave poles for multilayer media, especially when these poles lie too close to the branch point singularity, is overcome via a sine transformation. The merit of the proposed algorithm is that it can be embodied in the DCIM in a fully automatic way, and ultimately, the spatial-domain Green's function of multilayered structures can be cast into closed form expressions with great accuracy. The proposed scheme works for general multilayer media, whether open or shielded and lossy or lossless.

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