Approximate Boundary Conditions for Thin Structures

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***Abstract*—In wave propagation problems thin structures are often replaced by boundaries of zero thickness, in order to reduce the numerical mesh. The reduction of thickness has to be incorporated in the approximate boundary condition applied to the interface. This paper presents a modification of an approximate boundary condition originally introduced by Mitzner. The modified Mitzner condition can be applied to the boundary of zero thickness and it is more general and accurate than the commonly used impedance boundary condition. For frequency domain solvers the condition is as easy to implement as the impedance boundary condition. It is tested and compared to the impedance boundary condition for planar and cylindrical structures. The new condition is exact at normal incidence on a planar structure.**

***Index Terms*—Boundary value problems, coatings, nonhomogeneous media, scattering.**

I. Introduction

THIN conductive structures are hard to handle for most numerical methods for electromagnetic wave propagation since a very fine mesh is required to resolve the fields in the structure. One way to get around this problem is to exclude the interior of the structure from the computational domain and apply an approximate boundary condition (BC) to the structure. A simple BC for a thin non-magnetic dielectric structure is the impedance BC

$$\begin{aligned} \hat{\vec{n}} \times \vec{E}^{+} &= \hat{\vec{n}} \times \vec{E}^{-} = \hat{\vec{n}} \times \vec{E} \\ \hat{\vec{n}} \times (\vec{H}^{+} - \vec{H}^{-}) &= Z_L^{-1} \hat{\vec{n}} \times (\vec{E} \times \hat{\vec{n}}) \end{aligned} \quad (1)$$

where $\hat{\vec{n}}$ is the normal to the upper surface of the structure and $+$ and $-$ refer to the upper and lower surface. When the structure is replaced by a boundary of zero thickness the layer admittance is $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$, where ε and d are the complex permittivity and thickness of the layer, respectively. The BC in (1) has been analyzed in the literature, e.g., [1], [4], [6], and [5] and is also utilized in commercial softwares. The condition above is valid when the attenuation depth is much larger than the thickness of the structure and the absolute value of the wavenumber in the structure is much larger than that in the surrounding media.

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The approximate BC presented in this paper is valid even when the thickness is not much smaller than the attenuation depth. The condition is based on the following approximate BC for a single layer structure

$$\begin{aligned} \hat{\vec{n}} \times (\vec{H}^{+} - \vec{H}^{-}) &= -i\omega\varepsilon\kappa\hat{\vec{n}} \times ((\vec{E}^{+} + \vec{E}^{-}) \times \hat{\vec{n}}) \\ \hat{\vec{n}} \times (\vec{E}^{+} - \vec{E}^{-}) &= i\omega\mu\kappa\hat{\vec{n}} \times ((\vec{H}^{+} + \vec{H}^{-}) \times \hat{\vec{n}}) \end{aligned} \quad (2)$$

where ε and μ are the complex permittivity and permeability of the layer. The quantity κ is defined in the next section. This BC was introduced in a paper by Mitzner [2]. A similar condition has been used in [3] for the finite-difference time-domain method (FDTD). It seems that the Mitzner condition has not been frequently used, even though in many cases it is superior to the condition in (1). A plausible explanation is that the Mitzner BC, as well as the condition used in [3], is not compensated for the reduction of the structure to a boundary of zero thickness. This is in contrast to the condition in (1) which is compensated for the reduction of layer thickness by using $\varepsilon - \varepsilon_0$ instead of ε in the impedance.

The merit of this paper is to modify the BC in (2) such that the structure can be reduced to a boundary of zero thickness. The new condition has been tested numerically for a planar slab, a cylindrical one-layer shell, and a stratified cylindrical shell.

II. Prerequisites

Consider a thin structure that consists of one or several layers. The structure is characterized by its complex permittivity ε, permeability μ, and thickness d. The upper surface of the structure is denoted S^{+} and the lower surface S^{-}. The outward directed normal to the upper surface is $\hat{\vec{n}}$ and the thickness is the distance along the straight line that runs in the direction of $\hat{\vec{n}}$ from a point on S^{-} to the corresponding point on S^{+}. The wavenumber in the structure is $k_c = \omega\sqrt{\varepsilon\mu}$. In the analysis a local coordinate system with the z-axis directed in the $\hat{\vec{n}}$ direction is used. The following conditions were assumed by Mitzner, and they are also adopted here.

Condition 1 The radius of the curvatures of the structure are much larger than the attenuation length in the structure.

Condition 2 The attenuation length is much smaller than the distance to the nearest significant source.

Condition 3 The wavenumbers in the surrounding media are much smaller than the wavenumber in the structure.

Condition 4 The thickness of the structure is much smaller than the minimum of the radius of curvature of the structure.

III. Single Layer Structure

A layer with z-independent permittivity and permeability is considered. The structure occupies the region $0 < z < d$. Inside the structure the electric field satisfies

$$\nabla^2 \vec{E}(\vec{r}) + k_c^2 \vec{E}(\vec{r}) = \nabla_t^2 \vec{E}(\vec{r}) + \frac{\partial^2 \vec{E}(\vec{r})}{\partial z^2} + k_c^2 \vec{E}(\vec{r}) = \vec{0}. \quad (3)$$

Based on the conditions above it is assumed that the electric field varies much faster in the z-direction than in the x- and y-direction, and hence the $\nabla_t^2 \vec{E}$ term can be neglected. That leads to

$$\frac{\partial^2 \vec{E}(\vec{r})}{\partial z^2} + k_c^2 \vec{E}(\vec{r}) = \vec{0} \quad (4)$$

with solution

$$\vec{E}(\vec{r}) = \vec{\alpha}(x,y) e^{ik_c z} + \vec{\beta}(x,y) e^{-ik_c z} \quad (5)$$

where the time-dependence $e^{-i\omega t}$ is assumed. From the Ampére and induction laws the Mitzner BC in (2) follows, where

$$\kappa = \frac{e^{ik_c d} + e^{-ik_c d} - 2}{ik_c (e^{ik_c d} - e^{-ik_c d})} = \frac{1}{k_c} \tan\left(\frac{k_c d}{2}\right). \quad (6)$$

An equivalent representation of the BC is

$$\begin{pmatrix} \hat{n} \times (\vec{E}^+ \times \hat{n}) \\ \hat{n} \times \vec{H}^+ \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{n} \times (\vec{E}^- \times \hat{n}) \\ \hat{n} \times \vec{H}^- \end{pmatrix} \quad (7)$$

$$\mathbf{A} = \frac{1}{1 + \omega^2 \varepsilon \mu \kappa^2} \begin{pmatrix} 1 - \omega^2 \varepsilon \mu \kappa^2 & -2i\omega\mu\kappa \\ -2i\omega\varepsilon\kappa & 1 - \omega^2 \varepsilon \mu \kappa^2 \end{pmatrix}. \quad (8)$$

In (7) the original layer has been replaced by a BC that excludes the layer from the computational domain, but keeps the thickness d of the structure. It is often convenient to replace the structure with a layer of zero thickness. One has to keep in mind that this replacement alters the geometry and affects the numerical solution. It is appropriate to compensate the BC for the change in geometry. In this paper the compensation is done by replacing the original structure with a layer of thickness d, with the same material parameters as in the region above S^+, on top of an interface of zero thickness at $z = 0$. Denote the electric and magnetic fields on the side $z = 0^+$ of the zero thickness layer by $\vec{E}_1$ and $\vec{H}_1$. To find the condition between $\hat{n} \times \vec{E}_1$, $\hat{n} \times \vec{H}_1$, $\hat{n} \times \vec{E}^-$, and $\hat{n} \times \vec{H}^-$ one introduces the matrix

$$\mathbf{A}_{\rm ab} = \frac{1}{1 + \omega^2 \varepsilon_1 \mu_1 \kappa_1^2} \begin{pmatrix} 1 - \omega^2 \varepsilon_1 \mu_1 \kappa_1^2 & -2i\omega\mu_1\kappa_1 \\ -2i\omega\varepsilon_1\kappa_1 & 1 - \omega^2 \varepsilon_1 \mu_1 \kappa_1^2 \end{pmatrix} \quad (9)$$

where ε_1, μ_1 and $\kappa_1 = k_1^{-1} \tan(k_1 d/2)$ are the quantities in the region $z > d$. The parameters in the layer $0 < z \leq d$ is changed to ε_1 and μ_1 and according to (7) the relation between the tangential fields at $z = d$ and $z = 0^+$ is

$$\begin{pmatrix} \hat{n} \times (\vec{E}^+ \times \hat{n}) \\ \hat{n} \times \vec{H}^+ \end{pmatrix} = \mathbf{A}_{\rm ab} \begin{pmatrix} \hat{n} \times (\vec{E}^1 \times \hat{n}) \\ \hat{n} \times \vec{H}^1 \end{pmatrix}. \quad (10)$$

When this is combined with (7), the boundary condition for the zero thickness layer at $z = 0$ is obtained

$$\begin{pmatrix} \hat{n} \times (\vec{E}^1 \times \hat{n}) \\ \hat{n} \times \vec{H}^1 \end{pmatrix} = \mathbf{A}_{\rm ab}^{-1} \mathbf{A} \begin{pmatrix} \hat{n} \times (\vec{E}^- \times \hat{n}) \\ \hat{n} \times \vec{H}^- \end{pmatrix}. \quad (11)$$

This is the compensated Mitzner BC. When $k_c d \ll 1$ it is reduced to the impedance BC in (1) with the admittance $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_1)d$.

If the zero thickness boundary is placed at a position $z = z_0$, $z_0 > 0$, the condition reads

$$\begin{pmatrix} \hat{n} \times (\vec{E}^1 \times \hat{n}) \\ \hat{n} \times \vec{H}^1 \end{pmatrix} = \mathbf{A}_{\rm ab}^{-1} \mathbf{A} \mathbf{A}_{\rm bel}^{-1} \begin{pmatrix} \hat{n} \times (\vec{E}^2 \times \hat{n}) \\ \hat{n} \times \vec{H}^2 \end{pmatrix} \quad (12)$$

where in this case $\vec{E}^1$, $\vec{H}^1$ are the fields, for the zero thickness layer, at $z = z_0^+$ and $\vec{E}^2$, $\vec{H}^2$ are the corresponding fields at $z = z_0^-$. Here $\mathbf{A}_{\rm ab}$ is the matrix in (7) with parameters of the region $z > d$ and thickness $d - z_0$ and $\mathbf{A}_{\rm bel}$ is the corresponding matrix with parameters of the region $z < 0$ and thickness z_0.

IV. Approximate Boundary Conditions for Multilayered Structures

Consider a thin structure with N layers. The prerequisites are otherwise the same as before. The lower surface of the structure is at $z = z_0 = 0$ and the upper at $z = z_N = d$. The thickness of layer number n is $d_n = z_n - z_{n-1}$ and $d = \sum_{n=1}^N d_n$. The Mitzner BC for the multilayered structure follows from (7)

$$\begin{pmatrix} \hat{n} \times (\vec{E}_T^+ \times \hat{n}) \\ \hat{n} \times \vec{H}_T^+ \end{pmatrix} = \prod_{n=1}^N \mathbf{A}_n \begin{pmatrix} \hat{n} \times (\vec{E}_T^- \times \hat{n}) \\ \hat{n} \times \vec{H}_T^- \end{pmatrix} \quad (13)$$

where the index n refers to layer number n and

$$\mathbf{A}_n = \frac{1}{1 + \omega^2 \varepsilon_n \mu_n \kappa_n^2} \begin{pmatrix} 1 - \omega^2 \varepsilon_n \mu_n \kappa_n^2 & -2i\omega\mu_n\kappa_n \\ -2i\omega\varepsilon_n\kappa_n & 1 - \omega^2 \varepsilon_n \mu_n \kappa_n^2 \end{pmatrix}$$
$$\kappa_n = \frac{1}{k_n} \tan\left(\frac{k_n d_n}{2}\right). \quad (14)$$

The compensated Mitzner BC for the multilayered structure is still given by (11) with $\mathbf{A}$ replaced by

$$\mathbf{A} = \prod_{n=1}^N \mathbf{A}_n. \quad (15)$$

V. Numerical Examples

Three numerical examples are discussed in this section. In all examples the thin structures are non-magnetic $\mu = \mu_0$ and outside the structures there is vacuum. The structures are replaced by a boundary of zero thickness. The frequency is 1 GHz in all examples. Results referring to different boundary conditions are denoted by **M** for the BC by Mitzner, (7), **CM** for the compensated Mitzner BC, (11), and **IMP** for the impedance BC in (1). The Mitzner BC is in these examples applied to a zero thickness boundary, in order to see the effect of the compensation.

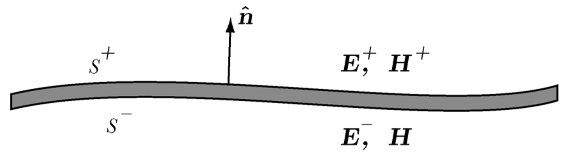

Fig. 1. The thin structure.

TABLE I
RELATIVE ERROR ABSORBED POWER. $\sigma = 10$ S/m, $\varepsilon_r = 5, \alpha = 30°$, $f = 1$ GHz. THE SKIN DEPTH IS $\delta = 5$ mm AND THE IMPEDANCE BC FAILS COMPLETELY FOR $d > \delta$

pol.	method	d=1 mm	5 mm	10 mm	100 mm
TE	CM	$5 \cdot 10^{-4}$	$4 \cdot 10^{-4}$	$6 \cdot 10^{-5}$	$1 \cdot 10^{-5}$
TE	IMP	$7 \cdot 10^{-4}$	0.09	0.5	0.9
TM	CM	$1 \cdot 10^{-4}$	0.001	$5 \cdot 10^{-4}$	$6 \cdot 10^{-4}$
TM	IMP	$4 \cdot 10^{-4}$	0.09	0.4	0.9

A. Plane Wave Incidence at a Planar Layer

The planar layer occupies $0 < z < d$ and the equivalent zero thickness boundary is placed at $z = 0$. A plane wave $\vec{E}(x,y,z) = \vec{E}_0 e^{i\vec{k}\cdot\vec{r}}$ is incident on the structure. The angle of incidence is defined as the angle between $\vec{k}$ and $\hat{\vec{z}}$, i.e., $\alpha = \arccos(\hat{\vec{n}} \cdot \vec{k}/|\vec{k}|)$. The incident plane wave is either a TE-wave, with the electric field parallel to the surface, or a TM-wave, with the magnetic field parallel to the surface. In all examples the impedance BC is quite accurate for very thin layers as long as $|k_c|d \ll 1$ but fails when $|k_c|d > 1$. At normal incidence $(\alpha = 0)$ the compensated Mitzner BC gives the correct values of the reflection and transmission coefficients, R and T, regardless of thickness d, whereas the Mitzner BC gives R and T with correct absolute values but erroneous phases.

The relative error in the absorbed power is defined by

$$\frac{|P_{\text{bc}} - P_{\text{exact}}|}{P_{\text{exact}}} \tag{16}$$

where P_{bc} is the absorbed power calculated from one of the approximate boundary conditions and P_{exact} is the exact absorbed power calculated from the full problem. The relative error is calculated for the three boundary conditions in (1), with $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$, (7), and (11) at different angles of incidence, thicknesses d, permittivities ε_r and conductivities σ. The relative errors are presented in Tables I and II for two cases. The compensated Mitzner BC and the Mitzner BC give the same results for this case. However, the transmission coefficient has large errors in the phase in the Mitzner BC, even at normal incidence, as seen from Fig. 2. The phase error for the Mitzner BC occurs since the Mitzner BC does not compensate for the phase shift due to the reduction of layer thickness. The phase error for the impedance BC is large since the phase shift inside the layer is not included in the BC. In this example the phase error of the compensated Mitzner condition is less than 5 degrees up to $k_0 a = 0.7$. It is very hard to give estimates for the errors in amplitude and phase even though a simple analysis based on Snell's law and rays gives a rough indication of the errors. For this reason an error analysis is not given in this paper.

TABLE II
RELATIVE ERROR ABSORBED POWER. $\sigma = 1$ S/m, $\varepsilon_r = 5, \alpha = 45°$, $f = 1$ GHz. THE SKIN DEPTH IS $\delta = 16$ mm AND THE IMPEDANCE BC FAILS COMPLETELY FOR $d > \delta$

pol.	method	d=1 mm	5 mm	10 mm	100 mm
TE	CM	$7 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	0.002	0.005
TE	IMP	0.001	0.0002	0.02	0.8
TM	CM	0.004	0.01	0.02	0.004
TM	IMP	$4 \cdot 10^{-4}$	0.01	0.05	0.8

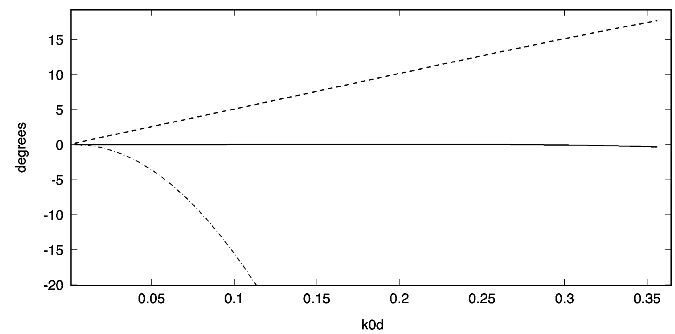

Fig. 2. The error in the phase of the transmitted electric field for the TE-case with parameters as in table I, i.e., $\sigma = 10$ S/m, $\varepsilon_r = 5, \alpha = 30°, f = 1$ GHz. The phase error in the compensated Mitzner BC (solid curve) is very small. The phase errors in the Mitzner BC (dashed) are due to the lack of compensation for the reduction of thickness. The phase errors in the impedance BC (dash-dotted) are due to the phase shifts inside the layer. The horizontal axis depicts $k_0 d$ where k_0 is the vacuum wave number and d is the layer thickness. The vertical axis depicts the error in the phase measured in degrees.

No examples are presented for metallic layers in this paper. Since the conductivity of a metal is very large and the attenuation depth is very small, the Mitzner and compensated Mitzner conditions are almost identical and extremely accurate for all thicknesses and angles of incidence. The impedance BC is only accurate if the thickness of the layer is much thinner than the skin depth.

B. Scattering from Circular Cylindrical Shell

The shell has an inner radius a, outer radius $a + d$, conductivity $\sigma = 0$ and relative permittivity $\varepsilon_r = 5$. The incident plane wave is either a TE-wave, i.e., with the E-field along the symmetry axis, or a TM-wave with the H-field along the symmetry axis. The boundary conditions have been tested for a large number of cases and in general the compensated Mitzner BC gives the best results. Both the Mitzner BC and the impedance BC are much less robust and they sometimes give large errors. The examples, see Figs. 3–5, are chosen since similar examples are used in the papers [1] and [4]. The exact results were calculated using Mie theory. The examples show the radar cross section (RCS).

C. Scattering from Circular Cylindrical Stratified Shell

The shell now consists of two layers. The inner layer has an inner radius a and outer radius $1.05a$, and the outer layer has an inner radius $1.05a$ and an outer radius $1.1a$. In the regions $\rho < a$ and $\rho > 1.1a$ there is vacuum. The inner layer has relative permittivity $\varepsilon_r = 4$ and conductivity $\sigma = 1$ S/m. The corresponding values for the outer layer are $\varepsilon_r = 2$ and $\sigma = 2$ S/m. In Figs. 6 and 7 the RCS is plotted as a function of $k_0 a$ where $k_0 = 2\pi f/c_0$ is the wave number in vacuum. The relative errors

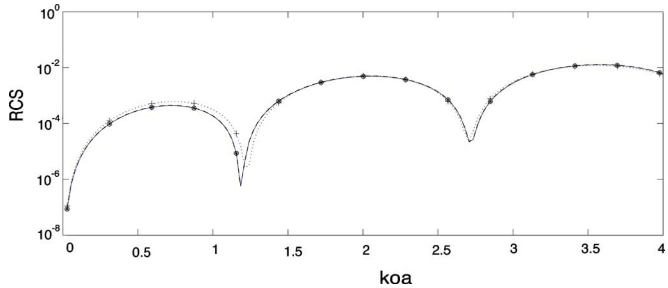

Fig. 3. RCS from a circular cylindrical shell with radius a, thickness $d = 0.01a$, permittivity $\varepsilon_r = 5$ and $\sigma = 0$. k_0 is the vacuum wavenumber. The incident plane wave is a TE-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) and impedance BC (dash-dotted line with $*$) are more accurate than the Mitzner BC (dotted line with $+$). From the relative error presented in the next figure it is seen that the compensated Mitzner BC is the best on for this case.

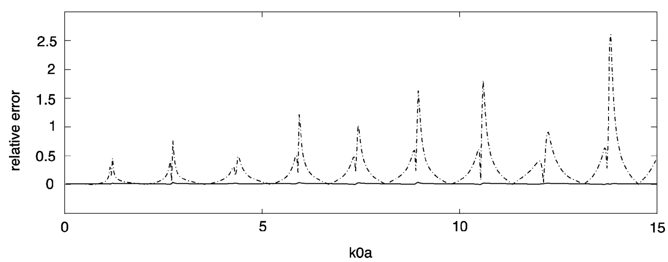

Fig. 4. The relative error of the RCS from the circular cylindrical shell in Fig. 3. The compensated Mitzner (solid line) and impedance BC (dash-dotted line) are depicted. As seen the compensated Mitzner BC is superior for almost all radii. The errors are concealed in Fig. 3 due to the logarithmic scale.

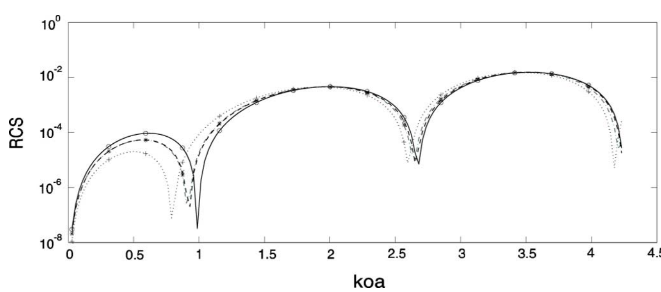

Fig. 5. RCS from a circular cylindrical shell with radius a, thickness $d = 0.01a$, permittivity $\varepsilon_r = 5$ and $\sigma = 0$. k_0 is the vacuum wavenumber. The incident plane wave is a TM-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) and impedance BC (dash-dotted line with $*$) give almost the same results. The error is larger for the Mitzner BC (dotted line with $+$).

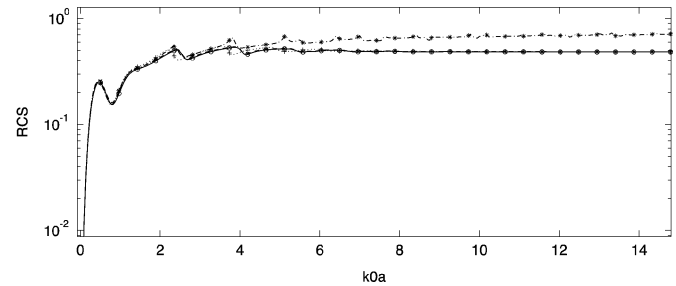

Fig. 6. RCS from a circular cylindrical stratified shell. For $\rho < a$ and $\rho > 1.1a$ there is vacuum, for $a < \rho < 1.05a$ there is a non-magnetic material with $\varepsilon_r = 4$ and $\sigma = 1$, for $1.05 < \rho < 1.1a$ there is a non-magnetic material with $\varepsilon_r = 2$ and $\sigma = 2$. The incident plane wave is a TE-wave with frequency 1 GHz. The compensated Mitzner BC (dashed line with x) gives very good results, the Mitzner BC (dotted line with $+$) is almost as good for large radii, but not as good for small radii. The impedance BC (dash-dotted line with $*$) is only accurate for small radii.

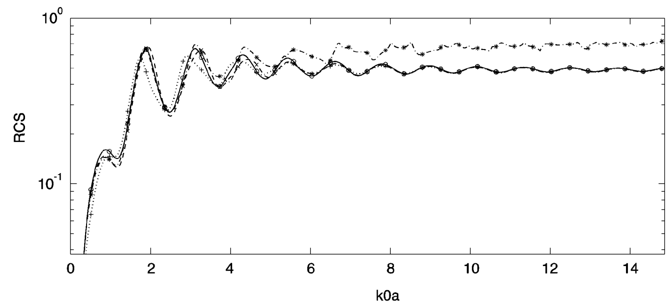

Fig. 7. RCS from a circular cylindrical stratified shell. For $\rho < a$ and $\rho > 1.1a$ there is vacuum, for $a < \rho < 1.05a$ there is a non-magnetic material with $\varepsilon_r = 4$ and $\sigma = 1$, for $1.05 < \rho < 1.1a$ there is a non-magnetic material with $\varepsilon_r = 2$ and $\sigma = 2$. The incident plane wave is a TM-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner BC (dashed line with x) gives very good results, the Mitzner BC (dotted line with $+$) is almost as good for large radii, but not as good for small radii. The impedance BC (dash-dotted line with $*$) is very inaccurate for large radii.

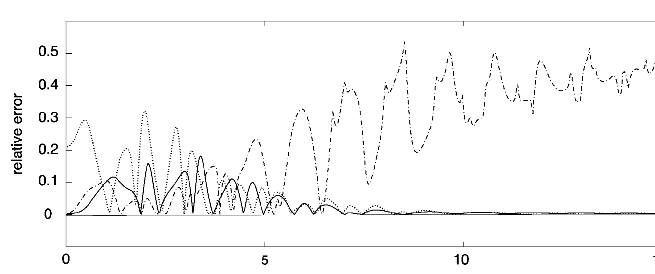

Fig. 8. The relative errors for the graph in Fig. 7. The compensated Mitzner is the solid line, the Mitzner is the dotted line, and the impedance BC is the dash-dotted line. The compensated Mitzner BC gives the best results for high frequencies. Notice the impedance BC is best in the region $1.5 < k_0a < 3.5$.

for the TM-case are given in Fig. 8. It is seen that for $k_0a < 3.5$ the compensated Mitzner and impedance BC are better than the Mitzner BC. It is even so that the impedance BC is slightly better than the compensated Mitzner BC in this interval. This is one of the very few examples found by the author where the impedance BC is superior to the compensated Mitzner BC. For $k_0a > 3.5$ the compensated Mitzner and the Mitzner BC are both much better than the impedance BC. The reason for this is that the impedance BC is only applicable when the layer is thinner than the skin depth.

The absolute value of the electric near fields are depicted in Fig. 9 for $k_0a = 10.47$. The compensated Mitzner and Mitzner BC give very accurate near fields, whereas the impedance BC is quite inaccurate at this radius.

VI. BC SUITABLE FOR FDTD

The time domain versions of the BC in (7) and (11) are complicated and it is not straightforward to use them in the finite difference time domain method, FDTD. In [3] a BC similar to the Mitzner BC was analyzed in the time domain. A technique

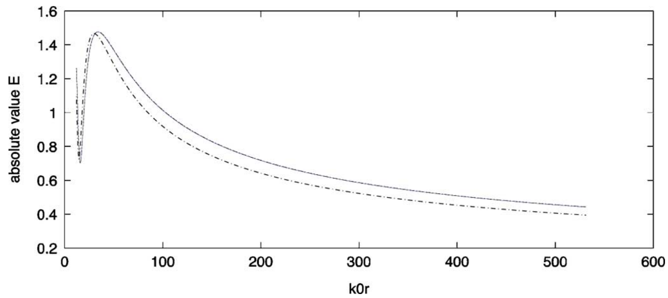

Fig. 9. The electric near field as a function of the radial distance in the forward direction. The radius $a = 0.5$ m which corresponds to $k_0 a = 10.47$. For $\rho < a$ and $\rho > 1.1a$ there is vacuum, for $a < \rho < 1.05a$ there is a non-magnetic material with $\varepsilon_r = 4$ and $\sigma = 1$, for $1.05 < \rho < 1.1a$ there is a non-magnetic material with $\varepsilon_r = 2$ and $\sigma = 2$. The incident plane wave is a TM-wave with frequency 1 GHz. The compensated Mitzner (dashed line) and the Mitzner BC (dotted line) give very accurate results. The impedance BC (dash-dotted line) is less accurate at this radius.

based upon expansions in powers of ω led to ordinary differential equations in the time domain and these equations were implemented in the FDTD method. One can expect that a method similar to the one in [3] can be used to transform the compensated Mitzner BC to the time domain. It has been shown in a number of papers, cf., [6] that the simplified BC in (1) is suitable to use in FDTD when $k_c d \ll 1$. The method suggested in [6] is to adjust the layer thickness such that it equals the thickness of a voxel in the mesh used by the FDTD program. In that paper only examples with $\sigma \gg \omega\varepsilon_0\varepsilon_r$ were considered. When the thickness of the layer was changed from d to d' the conductivity σ was changed to σ' such that the admittance of the layer was unaltered, i.e., $Z^{-1} = \sigma d = \sigma' d'$. From the analysis in this paper it is straightforward to derive the corresponding admittance when the condition $\sigma \gg \omega\varepsilon_0\varepsilon_r$ is not valid. As long as $k_c d \ll 1$ it is seen that if the original structure occupies the region $0 < z < d$ and is replaced by a structure that occupies the region $0 < z < d'$, the relevant boundary condition for the new structure is

$$\begin{pmatrix} \hat{\vec{n}} \times (\vec{E}^1 \times \hat{\vec{n}}) \\ \hat{\vec{n}} \times \vec{H}^1 \end{pmatrix} = \mathbf{A}_{\mathrm{ab}} \mathbf{A} \begin{pmatrix} \hat{\vec{n}} \times (\vec{E}^- \times \hat{\vec{n}}) \\ \hat{\vec{n}} \times \vec{H}^- \end{pmatrix} \tag{17}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & -i\omega\mu d \\ -i\omega\varepsilon d & 1 \end{pmatrix}$$
$$\mathbf{A}_{\mathrm{ab}} = \begin{pmatrix} 1 & -i\omega\mu_1(d' - d) \\ -i\omega\varepsilon_1(d' - d) & 1 \end{pmatrix}. \tag{18}$$

This condition can be transformed to the time domain and leads to a system of ordinary differential equations suitable for FDTD, cf., the technique in [3].

VII. Conclusions and Comments

When approximate boundary conditions for thin structures are used it is often assumed that the structure can be replaced by a layer of zero thickness. This is the case for the most commonly used impedance BC. A BC for thin structures was suggested by Mitzner [2] that was more general than the impedance BC. However, the condition does not take into account a reduction to zero thickness and that leads to errors that increases with thickness of the structure. In this paper the Mitzner BC is modified such that the structure can be replaced by a layer of zero thickness. This makes the Mitzner BC more general than the traditional impedance BC, cf., [1]. Numerical examples show that the compensated Mitzner BC is almost always superior to the Mitzner BC and the traditional impedance BC. It is simple and straightforward to implement in numerical programs for fixed frequqency. It is possible to take into account the curvature of the structure for the Mitzner BC. The technique is discussed in [2].

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