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# EVALUATION OF SOMMERFELD INTEGRALS ASSOCIATED WITH DIPOLE SOURCES ABOVE EARTH

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## FORWARD

Because of the immense interest expressed during the National Radio Science Meeting held during November 1978 in Boulder, Colorado on the evaluation of Sommerfeld integrals associated with dipole sources above a dissipative earth, and in particular, the use of incomplete Hankel functions in the approximation of these integrals, we feel it may be appropriate to publish this review, which basically forms a chapter in our book on the subject of Antennas and Transmission Lines Above the Earth currently under preparation. While the work is still somewhat incomplete, it is our hope that it will bring readers up-to-date on some of the developments made on this subject by us, as well as other research groups in the country. Comments and suggestions on how to improve the presentation would certainly be appreciated.

## 1. STATEMENT OF THE PROBLEM

Let us consider a dipole of dipole moment p radiating in the presence of a plane interface between two media. The problem is depicted in Fig. 1. The dipole is located at a height $z_0$ above the interface along the z-axis of a cylindrical coordinate system. The dipole is located in medium 1, with refractive index $n_1$, which we normally will consider to be the air, above medium 2, with refractive index $n_2$, normally taken to be the earth. Both media are assumed to be non-magnetic, but different permeabilities may be taken into account with little additional effort.

It is well-known that the electromagnetic field in a homogeneous, source-free region of space can be derived entirely from two scalar potentials known as Whittaker potentials, which are the sole components of parallel electric and magnetic Hertz vectors, $\Pi_{ez}$ and $\Pi_{mz}$, for example. We can therefore formulate the problem using $\Pi_{ez}$ and $\Pi_{mz}$ outside the source point only, and include the source as part of the electromagnetic field boundary conditions. In the case of a vertical electric dipole (VED) it is not difficult to show that the electric and magnetic fields, obtained in this manner, are given in the following table, where $G_1$ and $G_2$ are the Green's functions in air and in earth,

$$G_1 = e^{ik_1R_1}/k_1R_1; \quad R_1 = [\rho^2 + (z-z_0)^2]^{\frac{1}{2}}$$

$$G_2 = e^{ik_1R_2}/k_1R_2; \quad R_2 = [\rho^2 + (z+z_0)^2]^{\frac{1}{2}}$$

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TABLE 1

ELECTROMAGNETIC FIELD COMPONENTS OF A VERTICAL ELECTRIC DIPOLE ABOVE HALF-SPACE IN TERMS OF GREEN's FUNCTIONS AND FUNDAMENTAL INTEGRALS

| rce | In the air $(z > 0)$ | In the earth $(z < 0)$ |
|---|---|---|
| D | $E_z = \frac{p}{4\pi}\frac{k_o}{o}(k_1^2 + \frac{\partial^2}{\partial z^2})[\frac{1}{n_1}(G - G_2) + \frac{n_2^2}{n_1^2} V_1(z + z_o;\rho)]$ | $E_z = \frac{p}{4\pi}\frac{k_o}{\varepsilon_o}(k_2^2 + \frac{\partial^2}{\partial z^2})V_2(z,z_o;\rho)$ |
| | $E_x = \frac{p}{4\pi}\frac{k_o}{\varepsilon_o}\frac{\partial^2}{\partial x\partial z}[\frac{1}{n_1}(G - G_2) + \frac{n_2^2}{n_1^2} V_1(z + z_o;\rho)]$ | $E_x = \frac{p}{4\pi}\frac{k_o}{\varepsilon_o}\frac{\partial^2}{\partial x\partial z} V_2(z,z_o;\rho)$ |
| | $E_y = \frac{p}{4\pi}\frac{k_o}{\varepsilon_o}\frac{\partial^2}{\partial y\partial z}[\frac{1}{n_1}(G_1 - G_2) + \frac{n_2^2}{n_1^2} V_1(z + z_o;\rho)]$ | $E_y = \frac{p}{4\pi}\frac{k_o}{\varepsilon_o}\frac{\partial^2}{\partial y\partial z} V_2(z,z_o;\rho)$ |
| | $H_x = -\frac{i\omega p}{4\pi} k_o \frac{\partial}{\partial y}[n_1(G_1 - G_2) + n_2^2 V_1(z + z_o;\rho)]$ | $H_x = -\frac{i\omega p}{4\pi} k_o n_2^2 \frac{\partial}{\partial y} V_2(z,z_o;\rho)$ |
| | $H_y = \frac{i\omega p}{4\pi} k_o \frac{\partial}{\partial x}[n_1(G_1 - G_2) + n_2^2 V_1(z + z_o;\rho)]$ | $H_y = \frac{i\omega p}{4\pi} k_o n_2^2 \frac{\partial}{\partial x} V_2(z,z_o;\rho)$ |

at $(0,0,-z_o)$; $k_{1,2} = k_o n_{1,2}$ and $k_o = \omega(\mu_o\varepsilon_o)^{\frac{1}{2}}$ is the wave number in air. The time factor of $\exp(-i\omega t)$ is assumed and suppressed. Fields due to a vertical magnetic dipole are likewise expressible in terms of $G_1$, $G_2$, and the functions $U_1(z + z_o;\rho)$ and $U_2(z,z_o;\rho)$. The four fundamental integrals $U_1$, $U_2$, $V_1$, and $V_2$ which are corrections to a perfectly-conducting ground, are known as follows:

$$U_1(z;\rho) = 2\int_o^\infty \frac{e^{-k_o u_1 z}}{u_1 + u_2} J_o(k_o\lambda\rho)\lambda d\lambda \tag{1.1}$$

$$U_2(z_1,z_2;\rho) = 2\int_o^\infty \frac{e^{k_o u_2 z_1 - k_o u_1 z_2}}{u_1 + u_2} J_o(k_o\lambda\rho)\lambda d\lambda \tag{1.2}$$

$$V_1(z;\ \rho) = 2\int_0^\infty \frac{e^{-k_o u_1 z}}{n_2^2 u_1 + n_1^2 u_2} \quad J_o(k_o \lambda\rho)\lambda d\lambda \tag{1.3}$$

$$V_2(z_1;z_2;\rho) = 2\int_0^\infty \frac{e^{k_o u_2 z_1 - k_o u_1 z_2}}{n_2^2 u_1 + n_1^2 u_2} \quad J_o(k_o\lambda\rho)\lambda d\lambda \tag{1.4}$$

where

$$u_{1,2} = (\lambda^2 - n_{1,2}^2)^{\frac{1}{2}} \quad ; \quad \mathrm{Re}(u_{1,2}) \geq 0 \tag{1.5}$$

and $J_o$ is the Bessel function of zeroth-order. The objective of this report then is to evaluate and obtain approximate forms of these integrals. Before we can proceed however, it is important to first examine the analytic properties of the integrands.

To start with, it is possible to express the fundamental integrals as integrations over a path which terminates only at infinity:

$$\Omega = \int_{\infty e^{\pi i}}^{\infty} g(\lambda) e^{-au_1 + bu_2} H_o^{(1)}(k_o\lambda\rho)\lambda d\lambda \tag{1.6}$$

where $a$ and $b$ are some nonnegative real quantities, while

$$g(\lambda) = \frac{1}{n_2^2 u_1 + n_1^2 u_2} \qquad \text{for } V_1 \text{ and } V_2 \tag{1.7}$$

$$g(\lambda) = \frac{1}{u_1 + u_2} \qquad \text{for } U_1 \text{ and } U_2 \tag{1.8}$$

It is well-known that the stipulation $\mathrm{Re}(u_1) \geq 0$ leads to a pair of branch cuts in the $\lambda$-plane; the condition $\mathrm{Re}(u_2) \geq 0$ (which was also necessary to assure convergence of the integral and satisfaction of the limiting absorption principle) will likewise lead to a second pair of cuts as shown in Fig. 1.2. Also present is a cut along the negative real axis which is due to the artificially introduced Hankel function.

Since $\rho \geq 0$ and the behavior at infinity of this branch of the Hankel function (or any other branch where the cut proceeds to infinity in the lower half-plane) involves a factor $\exp(ik_0\lambda\rho)$, it will not prove possible to deform the integration contour $\Gamma$ to infinity in the lower half-plane. Thus, we seldom encounter this cut in actual practice, and will ignore it during subsequent manipulations.

The only other possible singularities in the $\lambda$-plane are poles of the function $g(\lambda)$. While it is readily verified that $u_1 + u_2$ is never zero if $n_1 \neq n_2$, the other so-called Sommerfeld denominator can in fact vanish at $\lambda = \pm\lambda_p$, where

$$\lambda_p = \frac{n_1 n_2}{\sqrt{n_1^2 + n_2^2}} \quad ; \quad \mathrm{Im}(\lambda_p) \geq 0 \qquad (1.9)$$

$$n_2^2 u_{1p} + n_1^2 u_{2p} = 0$$

and by $u_{1p}$ and $u_{2p}$ we mean the values of $u_1$ and $u_2$ at $\lambda_p$ respectively. In typical situations where $n_2$ is the refractive index of the earth, we have $|n_2| >> |n_1|$, and the poles are located in close proximity to $\pm n_1$ as shown in Fig. 1.2.

Although it has been established that the denominator of (1.7) can vanish, it might be questioned whether this actually occurs in the proper Riemann sheet defined by $\mathrm{Re}(u_{1,2}) \geq 0$, since $\lambda_p$ must be found by rationalizing the denominator, which could introduce a spurious solution. To resolve this point, we define

$$\left.\begin{aligned} \lambda_p &= |\lambda_p| e^{i\chi_p} \; ; \quad 0 \le \chi_p \\ n_{1,2} &= |n_{1,2}|\, e^{i\chi_{1,2}} \end{aligned}\right\} \tag{1.10}$$

For media whose properties are such that

$0 \le \chi_{1,2} < \pi/4$, it is readily shown that

$$\chi_p = \frac{1}{2} \tan^{-1} \left\{ \frac{\sin 2\chi_2 + |n^2| \sin 2\chi_1}{\cos 2\chi_2 + |n^2| \cos 2\chi_1} \right\} \tag{1.11}$$

where $n^2 = n_2^2/n_1^2$. When $n_2$ is the refractive index of the earth, and $n_1$ that of air, we have $|n^2| > 1$ and $\chi_1 \approx 0$, in which case it follows that

$$0 \le \chi_p < \chi_2 \tag{1.12}$$

The phase angles of $u_{1p}$ and $u_{2p}$ are found by observing that

$$u_{1p} = \pm i \frac{n_1}{n_2} \lambda_p \; ; \qquad u_{2p} = \pm i \frac{n_2}{n_1} \lambda_p$$

where the ± signs (not necessarily the same for $u_{1p}$ and $u_{2p}$) are to be specified so that $\mathrm{Re}(u_{1p}, u_{2p}) \ge 0$ corresponding to the proper sheet of the $\lambda$-plane. Since

$$-\frac{\pi}{2} < \chi_p - \chi_2 < 0$$

and

$$0 < \chi_p + \chi_2 < \frac{\pi}{2}$$

as follows from (1.12), we need

$$\left.\begin{aligned} \arg(u_{1p}) &= \chi_p - \chi_2 + \frac{\pi}{2} \\ \arg(u_{2p}) &= \chi_p + \chi_2 - \frac{\pi}{2} \end{aligned}\right\} \tag{1.13}$$

and thus

$$u_{1p} = +i \frac{n_1}{n_2} \lambda_p \; ; \quad u_{2p} = -i \frac{n_2}{n_1} \lambda_p \tag{1.14}$$

in the proper sheet and therefore $n_2^2 u_{1p} + n_1^2 u_{2p} = 0$ in the proper sheet.

We note that the present choice of branch cuts has been made to assure a bounded integrand as $|z| \to \infty$ for all integrands at all points in the complex $\lambda$-plane. It is in this proper sheet that the pair of "real" poles is located. However, there can arise situations when a different choice of cuts is more appropriate, even though the integrand is no longer bounded for all values of $\lambda$ .

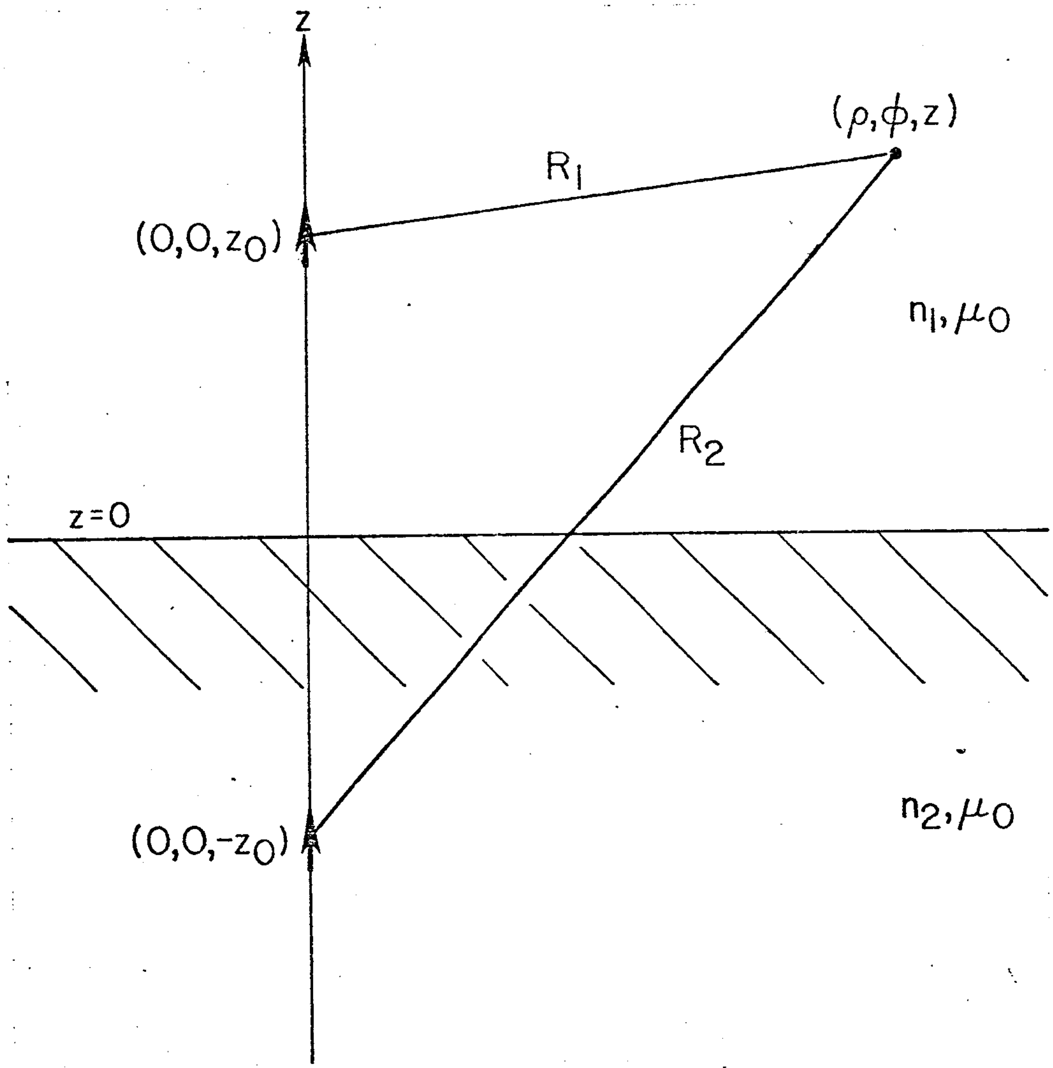


Fig. 1

Vertical dipole above an interface

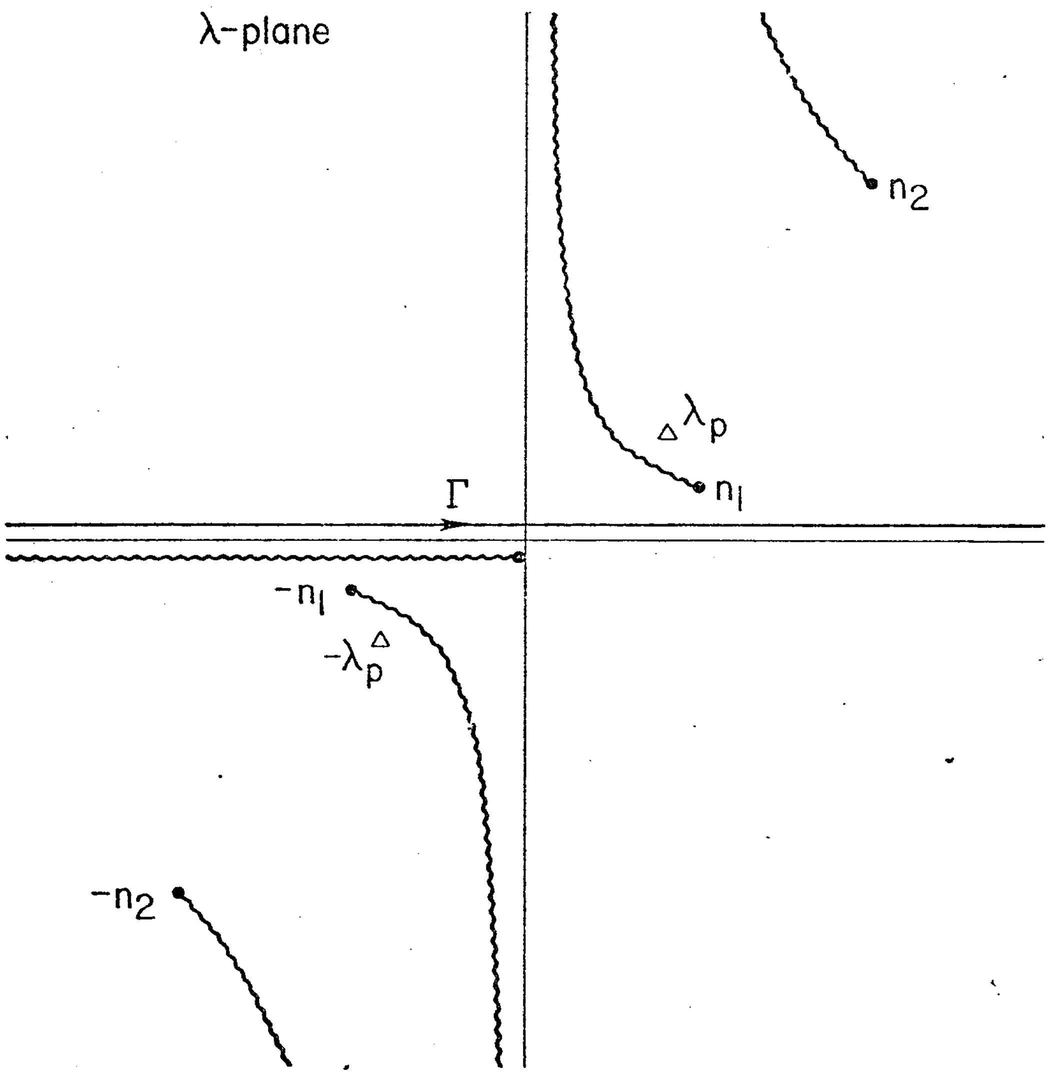


Fig. 2

Singularities and integration path in the complex plane for the fundamental integrals

# CHAPTER 2

# APPROXIMATIONS AND EXPANSIONS FOR THE FUNDAMENTAL INTEGRALS

" 'Let's consider your age to begin with - how old are you?'

'I'm seven and a half, exactly.'

'You needn't say "exactly," the Queen remarked. 'I can believe it without that.' "

- Lewis Carroll, Through the Looking Glass

In this chapter we deal with the problem of evaluating the Sommerfeld integrals $U_1$, $U_2$, $V_1$ and $V_2$ in terms of which the dipole fields are expressed. In general, these integrals are not expressible in finite form in terms of well-known or tabulated functions, and so approximate expressions must be obtained which are valid for certain restricted ranges of the parameters.

## §2.1 Exact Expressions for Special Cases

One of the few cases for which relatively simple exact expressions for the fundamental integrals can be found is when both the source point and observation point lie on the interface ($z = z_0 = 0$). This is far from being merely a trivial special case, since many remote sensing techniques employ electrically small transmitting and receiving antennas on the surface of the earth, and make use of field measurements in an attempt to infer the electrical properties of the ground in the region between the two antennas.

The fundamental integrals $U_1$ and $U_2$ in (1.62) and (1.66) reduce to

$$U_1(o;\rho) = U_2(o,o;\rho) \equiv U(\rho) = 2\int_0^\infty \frac{J_0(k_0\lambda\rho)\lambda d\lambda}{u_1 + u_2} \tag{2.1}$$

when both source and observer are on the interface. Rationalizing the denominator of the integrand by multiplying by $u_1 - u_2$, we have

$$U(\rho) = \frac{2}{N^2}\left\{\int_0^\infty u_1 J_0(k_0\lambda\rho)\lambda d\lambda - \int_0^\infty u_2 J_0(k_0\lambda\rho)\lambda d\lambda\right\} \tag{2.2}$$

where we have abbreviated the quantity

$$N^2 = n_2^2 - n_1^2 \quad ; \qquad \mathrm{Im}(N) \geq 0 \tag{2.3}$$

which will show up frequently in subsequent analysis. The integrals in (2.2) can be evaluated with the aid of (1.38) and (1.43):

$$\int_0^\infty \frac{e^{-k_o uz}}{u} J_o(k_o\lambda\rho)\lambda d\lambda = \frac{e^{ik_o nR}}{k_o R} \tag{2.4}$$

where $z \geq 0$, and u,n stand for $u_1,n_1$ or $u_2,n_2$ as appropriate, while $R^2 = z^2+\rho^2$. The integrals in (2.2) are now obtained by differentiating (2.4) twice by z and setting z = 0:

$$\int_0^\infty u\, J_o(k_o\lambda\rho)\lambda d\lambda = \frac{1}{k_o^2} \frac{\partial^2}{\partial z^2}\left(\frac{e^{ik_o nR}}{k_o R}\right)\Bigg|_{z=0} = \frac{e^{ik_o n\rho}}{k_o^2\rho^2}[in - 1/k_o\rho] \tag{2.5}$$

Thus,

$$U(\rho) = \frac{2}{k_o^2 N^2\rho^2}\left\{[in_1 - \frac{1}{k_o\rho}]\, e^{ik_o n_1\rho} - [in_2 - \frac{1}{k_o\rho}]\, e^{ik_o n_2\rho}\right\}, \tag{2.6}$$

an expression containing only elementary functions.

The integrals $V_1$ and $\overset{*}{V}_2$ in (1.62) and (1.63) become

$$V_1(0;\rho) = V_2(0,0;\rho) \equiv V(\rho) = 2\int_0^\infty \frac{J_o(k_o\lambda\rho)\lambda d\lambda}{n_2^2 u_1 + n_1^2 u_2} \tag{2.7}$$

for source and observer on the interface. The technique for evaluating $U(\rho)$ fails in this case because the denominator in (2.7) cannot be eliminated by a similar rationalization. But (2.7) can, however, be transformed by an extremely clever trick due to van der Pol. Consider the integral

$$\int_{n_1/n_2}^{n_2/n_1} \frac{v dv}{\sqrt{(\lambda^2-\lambda_p^2)v^2-\lambda_p^2}} = \frac{n_2^4-n_1^4}{n_1 n_2}\, \frac{1}{n_2^2 u_1 + n_1^2 u_2} \tag{2.8}$$

where $\lambda_p = n_1 n_2/(n_1^2+n_2^2)^{\frac{1}{2}}$ is the location of the Sommerfeld pole discussed in Chapter 1. Inserting (2.8) into (2.7) and reversing the order of the integrations, we have

$$V(\rho) = \frac{2\lambda_p^2}{N^2 n_1 n_2}\int_{n_1/n_2}^{n_2/n_1} dv\left\{\int_0^\infty \frac{J_o(k_o\lambda\rho)\lambda d\lambda}{\sqrt{\lambda^2-\lambda_p^2(1+1/v^2)}}\right\} = \frac{2\lambda_p^2}{N^2 n_1 n_2 k_o\rho}\int_{n_1/n_2}^{n_2/n_1} e^{ik_o\lambda_p(1+1/v^2)^{\frac{1}{2}}\rho}\, dv \tag{2.9}$$

Changing variables to $t = 1/v$ and integrating once by parts, we arrive at

$$V(\rho) = \frac{2\lambda_p^2}{N^2 n_1 n_2 k_o \rho} \; \frac{n_2}{n_1} e^{ik_o n_1 \rho} - \frac{n_1}{n_2} e^{ik_o n_2 \rho} + \frac{2i}{N^2 \lambda_p} \int_{n_1/n_2}^{n_2/n_1} \frac{e^{ik_o \lambda_p (1+t^2)^{1/2} \rho}}{(1+t^2)^{1/2}} \, dt \qquad (2.10)$$

The last term in (2.10) cannot be expressed in terms of any of the usual elementary or special functions. However, it can be identified as a combination of the so-called <u>incomplete Hankel functions</u>, defined by

$$H_o^{(1)}(a,z) = \frac{2}{i\pi} \int_o^a \frac{\exp[iz(1+w^2)^{1/2}]}{(1+w^2)^{1/2}} \, dw$$

The important properties of this function and some related functions are summarized in Appendix B. From the definition, it is clear that

$$V(\rho) = \frac{2\lambda_p^2}{N^2 n_1 n_2} \left\{ \frac{1}{k_o \rho} \left[ \frac{n_2}{n_1} e^{ik_o n_1 \rho} - \frac{n_1}{n_2} e^{ik_o n_2 \rho} \right] - \frac{\pi}{2} \lambda_p H_o^{(1)}(n_2/n_1, \, k_o \lambda_p \rho) \right.$$

$$\left. + \frac{\pi}{2} \lambda_p H_o^{(1)}(n_1/n_2, \, k_o \lambda_p \rho) \right\} \qquad (2.11)$$

Numerical evaluation of these functions is straightforward through the use of the expansions given in Appendix B. We might note that for the case when $|n_2/n_1| >> 1$ (a diectrically dense or highly conducting earth), the leading terms of expansions (B.10) and (B.15) may be used to give an approximate form for $V(\rho)$ as

$$V(\rho) \simeq \frac{2\lambda_p^2}{N^2 n_1 n_2} \left\{ \frac{1}{k_o \rho} \left[ \frac{n_2}{n_1} e^{ik_o n_1 \rho} - \frac{n_1}{n_2} e^{ik_o n_2 \rho} \right] - \frac{\pi}{2} \lambda_p H_o^{(1)}(k_o \lambda_p \rho) \right.$$

$$\left. - i\lambda_p E_1(-ik_o n_2 \rho) - i\lambda_p \left( \frac{\pi}{-2ik_o \lambda_p \rho} \right)^{1/2} e^{ik_o \lambda_p \rho} \operatorname{erf}(-ip_o^{1/2}) \right\} \qquad (2.12)$$

where

$$p_o = ik_o \lambda_p \rho \left[ \left(1 + \frac{n_1^2}{n_2^2}\right)^{1/2} - 1 \right] \qquad (2.13)$$

is a special case of what is known as Sommerfeld's numerical distance. In (2.12) $H_o^{(1)}(k_o\lambda_p\rho)$ is the ordinary Hankel function of the first kind and zeroth order, $E_1$ is the exponential integral function of first order, and erf (x) is the error function. The important properties of the latter two functions are given in Appendix C.

Unfortunately, only some of the field components can be obtained from these exact expressions, since z does not appear in them, while from Table 1.3, a number of z-derivatives must be taken in order to obtain certain of these components. While $\partial^2/\partial z^2$ can be replaced by

$$- \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial}{\partial\rho}\right) + k^2$$

since $U_{1,2}$ and $V_{1,2}$ satisfy the Helmholtz equation, an odd number of z-derivatives cannot, apparently, be handled by this approach. As a result, the usefulness of (2.6) and (2.11) is somewhat restricted.

One further special case permitting a closed-form expression is that of $U_1(z;0)$ - i.e., for an observation point directly above or below the dipole, above the earth's surface. Here

$$\begin{aligned} U_1(z\,;\,0) &= 2\int_0^\infty \frac{e^{-k_o u_1 z}\lambda d\lambda}{u_1+u_2} \\ &= 2\int_{-in_1}^\infty \frac{e^{-tk_o z}\,tdt}{t+\sqrt{t^2-N^2}} \end{aligned} \tag{2.14}$$

through an obvious change of variable. Rationalizing the denominator as before, we have

$$\begin{aligned} U_1(z\,;\,0) &= \frac{2}{N^2}\int_{-in_1}^\infty e^{-tk_o z}\,[t^2 - t\sqrt{t^2-N^2}]dt \\ &= \frac{2}{N^2}\left\{\frac{1}{k_o^2}\frac{\partial^2}{\partial z^2}\int_{-in_1}^\infty e^{-tk_o z}\,dt + \frac{1}{k_o}\frac{\partial}{\partial z}\int_{-in_1}^\infty \sqrt{t^2-N^2}\,e^{-tk_o z}\,dt\right\} \end{aligned} \tag{2.15}$$

The first integral is elementary, while the second, through the change of variable $t = N(1 + w^2)^{\frac{1}{2}}$, becomes

$$\int_{-in_1}^{\infty} \sqrt{t^2-N^2}\, e^{-tk_oz}\, dt = N^2 \int_{in_2/N}^{\infty} \frac{e^{-Nk_oz(1+w^2)^{\frac{1}{2}}}\, w^2\, dw}{(1+w^2)^{\frac{1}{2}}}$$

$$= \left( \frac{1}{k_o^2} \frac{\partial^2}{\partial z^2} - N^2 \right) \int_{in_2/N}^{\infty} \frac{e^{-Nk_oz(1+w^2)^{\frac{1}{2}}}}{(1+w^2)^{\frac{1}{2}}}\, dw$$

$$= \left( \frac{1}{k_o^2} \frac{\partial^2}{\partial z^2} - N^2 \right) \left[ \frac{\pi i}{2} H_0^{(1)}(ik_oNz) - \frac{\pi i}{2} H_0^{(1)}\left(i \frac{n_2}{N}, ik_oNz\right) \right] \tag{2.16}$$

using the integral representation (B.1) of the incomplete Hankel function. Using (B.7) and (B.21) from Appendix B, we now have

$$U_1(z\,;\,0) = \frac{2}{N^2} \left[ \frac{2}{k_o^3z^3} - \frac{2in_1}{k_o^2z^2} - \frac{n_1^2}{k_oz} + \frac{in_2}{k_oz} \right] e^{ik_on_1z} + \frac{\pi}{Nk_oz} \left[ H_1^{(1)}\left(i \frac{n_2}{N}, ik_oNz\right) - H_1^{(1)}(ik_oNz) \right] \tag{2.17}$$

If $|n_2^2| \gg |n_1^2|$, then the parameter of the incomplete Hankel function is nearly equal to i, and thus from (B.31), we have

$$U_1(z;\,0) \simeq \frac{2}{N^2} \left[ \frac{2}{k_o^3z^3} - \frac{2in_1}{k_o^2z^2} - \frac{n_1^2}{k_o^2} + \frac{in_2}{k_oz} \right] e^{ik_on_1z} + \frac{\pi i}{Nk_oz} \left[ \mathbf{H}_1(ik_oNz) - Y_1(ik_oNz) \right] \tag{2.18}$$

where $\mathbf{H}_1$ is the Struve function of first order (cf. Appendix C). Of course, since $\rho$ does not appear in these formulas, we are again unable to obtain all field components from them, and their utility is somewhat limited.

## §2.2 Numerical Integration for Sommerfeld Integrals

For the case of arbitrary source and observer heights above the interface, the most direct method for evaluating the Sommerfeld integrals is numerical integration. Although a number of techniques appropriate to infinite or semi-infinite intervals are available, the analytic characteristics of the integrands must be considered if an efficient routine is to be written. In this section, we will discuss the integral $V_1(z;\rho)$ only, since this function incorporates all the essential features to be found in the other integrals. The reader is referred to the notes on this section for references where detailed descriptions of computer programs are to be found.

Consider the integral $V_1(z;\rho)$ in either of its alternative forms:

$$V_1(z;\rho) = 2\int_0^\infty \frac{e^{-k_0 u_1 z}}{n_2^2 u_1 + n_1^2 u_2} J_0(k_0\lambda\rho)\,\lambda d\lambda \tag{2.19}$$

$$= \int_{\infty e^{\pi i}}^\infty \frac{e^{-k_0 u_1 z}}{n_2^2 u_1 + n_1^2 u_2} H_0^{(1)}(k_0\lambda\rho)\,\lambda d\lambda \tag{2.20}$$

The first question to arise is: what quadrature formula should be chosen for the actual integration? The simplest approach would be to apply Simpson's rule to (2.19) - breaking up the integration interval from 0 to some large value L into a number of subintervals and apply Simpson's rule to each subinterval. To be sufficiently accurate, a large number of subintervals must necessarily be taken, and L must be increased until the value of the integral remains unchanged to within some specified accuracy. The step sizes must be small compared to the oscillation of the Bessel function and to any other rapid variation of the integrand (i.e., poles or branch points). As a refinement to this, larger subintervals can be taken (say, a complete oscillation of the Bessel function) and a Romberg integration algorithm applied to each. The convergence of the summation over the subintervals in either of these approaches can be improved by algorithms such as that due to Shanks.

If, by a suitable change of variable, the integral under study can be cast into one of the forms

$$\int_0^{\infty} e^{-t} f_1(t)\, dt \qquad \text{or} \qquad \int_{-\infty}^{\infty} e^{-t^2} f_2(t)\, dt$$

then a Gaussian-type of quadrature (Gauss-Laguerre or Gauss-Hermite, respectively) can be used. Several other more specialized quadrature formulas have also been suggested. A sort of "fast Fourier transform (FFT)" method involves approximating the multiplier of $J_o(k_o\lambda\rho)$ by a series

$$\sum_{m=o}^{M} a_m e^{-b_m\lambda}$$

where $a_m$ and $b_m$ are some complex constants. Each term of this series can now be integrated analytically and thereby an approximate value for the integral is obtained. Along somewhat the same lines, we can also approximate the multiplier of $e^{-k_o u_1 z} J_o(k_o\lambda\rho)\lambda$ in (2.19) by a rational function of $\underline{u_1 \text{ only}}$. We shall see in a later section that such integrals as result from this approximation can always be expressed in terms of incomplete Hankel functions, and so the integral is in this way reduced to a number of series evaluations.

A kind of combination of the "brute force" approach (Simpson or Romberg-Shanks) and one of these "semi-analytical quadratures" is achieved by adding and subtracting a term with the appropriate limiting behavior as $\lambda\to\infty$ to the integrand of (2.19). Thus, for example,

$$V_1(z;\rho) = 2\int_o^{\infty} \frac{e^{-k_o u_1 z}}{(n_2^2+n_1^2)u_1} J_o(k_o\lambda\rho)\, \lambda d\lambda$$

$$+ 2\int_o^{\infty} \left[\frac{1}{(n_2^2 u_1+n_1^2)u_2} - \frac{1}{(n_2^2+n_1^2)u_1}\right] e^{-k_o u_1 z} J_o(k_o\lambda\rho)\, \lambda d\lambda \tag{2.21}$$

The first integral is known exactly (see (2.4)), while the term in brackets in the second integral is $O(\lambda^{-3})$ instead of $O(\lambda^{-1})$ as $\lambda \to \infty$. The second integral can thus be numerically integrated much nore rapidly than (2.19), and in fact, if the integration interval is split into $[0,\lambda_0)$ and $[\lambda_0,\infty)$, the second of these can be evaluated approximately if $\lambda_0$ is large by expanding the bracketed term in inverse powers of $\lambda$ and using known integral relationships to compute each term. There then remains only a finite integration on $[0,\lambda_0)$ to perform using any appropriate method from those described above.

Some of these methods, notably the Simpson and Romberg-Shanks algorithms, have efficiencies which are quite sensitive to the choice of path of integration in the complex plane. Although the real axis path does not require evaluation of Bessel functions of complex argument, this advantage can be more than offset by slow decay and/or rapid oscillation of the integrand, particularly if $k_0\rho \gg 1$ and $|k_0 z| \ll 1$. The proximity of the poles (or even branch cuts) to the real axis can also be troublesome. These latter can be avoided by dipping below the real axis as in $\Gamma'$ in Fig. 2.1, but the oscillation problem remains, and in fact, $J_0(k_0\lambda\rho)$ grows exponentially as $\lambda$ acquires a nonzero imaginary part.

Because of this difficulty, it is more convenient for this purpose to work with (2.20). For large argument, the Hankel function becomes

$$H_0^{(1)}(k_0\lambda\rho) \sim \left( \frac{2}{\pi k_0\lambda\rho} \right)^{\frac{1}{2}} e^{i(k_0\lambda\rho - \pi/4)} \qquad (|k_0\lambda\rho| \to \infty)$$

and thus, aside from some algebraic behavior, the integrand behaves as $\exp[-k_0 z\sqrt{\lambda^2 - n_1^2} + ik_0\lambda\rho\,]$, which, by virtue of the definition of the square root, is $\exp[-k_0\lambda(z - i\rho)]$ in the right half-plane, and $\exp[k_0\lambda(z + i\rho)]$ in the left half-plane as $|\lambda| \to \infty$. Therefore, if we were to deform the path of integration from the real axis to $\Gamma_1$ in Fig. 2.2, which forms angles of $\theta = \frac{\pi}{2} - \tan^{-1}(z/\rho) = \frac{\pi}{2} - \chi$ to the real axis, the oscillating part of the exponential term would disappear as $|\lambda| \to \infty$, and only a decaying behavior persists. Unfortunately, this path, as is, crosses the branch cut due to $u_1$ (if $n_1$ is real, or nearly so), and is impermissible because of the exponential growth of the integrand as $|\lambda|$ increases, but if we modify this contour to some path $\Gamma_2$ as shown in Fig. 2.2, we can achieve our purpose with a permissible integration path. This new path passes momentarily into an improper Riemann sheet to avoid the logarithmic singularity of the Hankel function at the origin, then under and around the branch point at $n_1$ and back to $\infty\, e^{i(\pi/2 - \theta)}$ as with $\Gamma_1$. The pole and branch point singularities have been avoided, but for large $|\lambda|$ the decay of the integrand (hence also the convergence) of the integral is maximized.

This contour is still somewhat undesirable if $k_0\rho$ is very large, however, because by the time $\Gamma_2$ rejoins the straight line segments of $\Gamma_1$, the integrand is exceedingly small (due to the exponential factor), and the value of the integral is almost entirely determined by the portion of $\Gamma_2$ which deviates from $\Gamma_1$. Since the oscillations of the integrand have <u>not</u> been suppressed along this portion of the path, we have not yet found the optimum path of integration. To completely suppress the oscillation, we need to choose the coutour so that as $\lambda$ moves along the contour, the term $\exp[G(\lambda)]$, where $G(\lambda) = -k_0 z\sqrt{\lambda^2 - n_1^2} + ik_0\lambda\rho$ always represents a purely decaying factor. Since as $|\lambda| \to \infty$ toward either end of the path, this exponential is decaying, and since the argument of the exponential is, except for the isolated points $\lambda = \pm\, n_1$, analytic, the exponential must reach a maximum absolute value at some point along this path. By specifying the smallest possible magnitude for this maximum, we can minimize the portion of the contour which contributes significantly to the integral. A point $\lambda_s$ which meets these conditions is called a saddle point, and, for a general analytic function $f(\lambda)$, is a solution of

$$f'(\lambda_s) = 0 \tag{2.22}$$

For the function being considered here, this becomes $G'(\lambda_s) = 0$, or

$$-k_o z \frac{\lambda_s}{\sqrt{\lambda_s^2 - n_1^2}} + ik_o\rho = 0$$

or in other words

$$\lambda_s = n_1 \sin\theta \tag{2.23}$$

Near the saddle point, the exponential term behaves like

$$e^{G(\lambda)} \simeq e^{G(\lambda_s)}\, e^{G''(\lambda_s)(\lambda - \lambda_s)^2/2} \tag{2.24}$$

and we see that depending upon how $\lambda$ moves away from $\lambda_s$, the exponential might increase, decrease, or oscillate, and that the path which we seek, upon which this term is <u>purely</u> decaying, will be such that $G''(\lambda_s)(\lambda - \lambda_s)^2$ is real and negative in the vicinity of $\lambda_s$. The entire path can be specified by

$$\left.\begin{aligned} &\mathrm{Im}\,[G(\lambda) - G(\lambda_s)] = 0 \\ &G(\lambda) - G(\lambda_s) < 0 \qquad \text{for } \lambda \neq \lambda_s \end{aligned}\right\} \tag{2.25}$$

Such a contour is known as a path of steepest descent (SDP).

For the present case, $G''(\lambda_s) = -ik_o R/n_1 \cos^2\theta$, where $R = \sqrt{z^2 + \rho^2}$, and so if $n_1$ is real, the SDP will make an angle of $45^o$ with the real axis at the saddle point as shown in Fig. 2.3. From (2.25), the entire path can be defined by

$$G(\lambda) - G(\lambda_s) = -k_o n_1 R\, \tau^2/2 \tag{2.26}$$

where $\tau$ runs through the real numbers, or equivalently,

$$\frac{\tau^2}{2} = \cos\theta \frac{\sqrt{\lambda^2 - n_1^2}}{n_1} - i\frac{\lambda}{n_1}\sin\theta + i \tag{2.27}$$

It can thus be seen that if $n_1$ is real, the SDP will recross the real axis at $\lambda = n_1/\sin\theta$, and evidently it must approach the same asymptotes as in Fig. 2.2, and takes the overall form shown in Fig. 2.3. As with the path of Fig. 2.2, the

SDP also passes momentarily onto an improper Riemann sheet before curving below the branch point at $n_1$. Fig. 2.4 depicts the SDP for three different angles of observation $\theta = 0^o$, $30^o$, $80^o$.

Integration along the SDP obviously has the advantage of truncating the path of integration according to the actual value of R, since the integral now decays exponentially away from the saddle point. However, it can be seen from Fig. 2.4 that the SDP comes close to the Sommerfeld pole for the V integrals when the elevation angle $\chi = \frac{\pi}{2} - \theta$ is small. To circumvent possible numerical difficulties near this point, one can either deform the path to avoid this pole (though this retracts the advantages of going to the steepest-descent path in the first place), or a quadrature formula can be used which takes care of the pole singularity analytically. This can be done by explicitly displaying the pole as $(\tau - \tau_p)^{-1}$, and making use of the formulas of the next section for expressing this contribution as an error function. Numerical integrations for the various derivatives of $U_{1,2}$ and $V_{1,2}$ needed to calculate the field components are done in exactly the same way.

It should, of course, always be remembered to include any contributions from singularities encountered when deforming from the real axis to any other path of integration. Although none of the paths considered here crosses the Sommerfeld pole, the SDP will encounter the second branch cut at $n_2$ if the elevation angle is small enough (see Fig. 2.5). In this case, the additional contribution around the cut must be added on.

## §2.3 Far-Field Expansions (Method of Steepest Descent)

The calculation of the fundamental integrals by numerical integration is usually time-consuming, and although very accurate results can be obtained, little physical insight into the detailed radiation is gained. In the next several sections, we will proceed to study some approximate but explicit formulas for the fundamental integrals which are valid in various regions of observation, or in various ranges of the electrical parameters of the earth.

For the case when the observation distance $R = \sqrt{\rho^2 + z^2}$ is much greater than a wavelength in the air (region 1), an analytical derivation of approximate formulas for $U_1$ and $V_1$ is possible using the so-called method of steepest descent. The idea of the method is based on the steepest-descent path discussed in the previous section. If $k_1R$ is large, and the integration contour has been deformed to the SDP, only a very small portion of the contour near the saddle point contributes appreciably to the integral, unless some singularity comes close to the SDP at some point. Then, while the dominant exponential behavior $\exp[G(\lambda)]$ is retained exactly, the remainder of the integrand can be approximated by one which can be easily integrated, but need only be accurate near the saddle point.

To carry this out, let us consider the integral of general form

$$\Omega = \int_{\infty e^{\pi i}}^{\infty} e^{G(\lambda)}\, h(\lambda)\, d\lambda \tag{2.28}$$

where

$$G(\lambda) = -k_1 R \frac{\sqrt{\lambda^2 - n_1^2}}{n_1} \cos\theta - i\,\frac{\lambda}{n_1} \sin\theta \tag{2.29}$$

and

$$h(\lambda) = \left\{ \begin{array}{ll} e^{-ik_0\lambda\rho}\, H_0^{(1)}(k_0\lambda\rho)\, \dfrac{\lambda}{u_1 + u_2} & \text{for } U_1 \\[2ex] e^{-ik_0\lambda\rho}\, H_0^{(1)}(k_0\lambda\rho)\, \dfrac{\lambda}{n_2^2 u_1 + n_1^2 u_2} & \text{for } V_1 \end{array} \right\} \tag{2.30}$$

Let us introduce the change of variable (2.26) and deform the contour to the SDP to obtain

$$\Omega_{SDP} = e^{G(\lambda_s)} \int_{-\infty}^{\infty} e^{-k_1 R\tau^2/2} \left\{ h\left[\lambda(\tau)\right] \frac{d\lambda}{d\tau} \right\} d\tau \tag{2.31}$$

where the subscript SDP denotes the contribution only from the SDP contour, and not from any singularities which may have been encountered during the deformation. If the bracketed term in (2.31) is slowly-varying for, say, $|\tau| \leq |k_1 R|^{-\frac{1}{2}}$, a first approximation to (2.31) would be obtained by replacing it with its value at $\tau = 0$.

$$\Omega_{SDP} \simeq e^{G(\lambda_s)} \left\{ h(\lambda_s) \right\} \left.\frac{d\lambda}{d\tau}\right|_{\tau=0} \int_{-\infty}^{\infty} e^{-k_1 R\tau^2/2} d\tau \tag{2.32}$$

$$= \left(\frac{2\pi}{k_1 R}\right)^{\frac{1}{2}} h(\lambda_s) \left.\frac{d\lambda}{d\tau}\right|_{\tau=0} e^{G(\lambda_s)}$$

Roughly speaking, (2.32) is valid if $|k_1 R|$ is large enough, and if $\left\{ h\left[\lambda(\tau)\right] \frac{d\lambda}{d\tau} \right\}$ is smooth enough near $\tau = 0$. To make a more accurate assessment of its validity, we expand the bracketed term as a Taylor series in $\tau$:

$$h\left[\lambda(\tau)\right] \frac{d\lambda}{d\tau} = \sum_{m=0}^{\infty} h_m \tau^m \tag{2.33}$$

where

$$h_m = \frac{1}{m!} \frac{d^m}{d\tau^m} \left\{ h(\lambda) \frac{d\lambda}{d\tau} \right\}_{\tau=0} \tag{2.34}$$

Generally, this series converges only for $|\tau| < \tau_c$, where $\tau_c$ is its radius of convergence. However, if we insert (2.33) into (2.31) and interchange the sum and integral anyway, we obtain the formal series

$$\Omega_{SDP} \sim e^{G(\lambda_s)} \sum_{m=0}^{\infty} \left(-\frac{2}{k_1}\right)^m h_{2m} \left[\frac{d^m}{dR^m} \int_{-\infty}^{\infty} e^{-k_1 R\tau^2/2} d\tau\right] \tag{2.35}$$

which can be reworked into the form (using (2.32)):

$$\Omega_{SDP} \sim e^{G(\lambda_s)} \left(\frac{2\pi}{k_1 R}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(2m)!\,h_{2m}}{m!} (2k_1 R)^{-m} \tag{2.36}$$

Because the termwise integration is not valid over the infinite range of $\tau$, if $\tau_c < \infty$ , this formal series will not converge. However, it does represent an asymptotic expansion of $\Omega_{SDP}$ (denoted by the symbol ~) in the sense of Poincaré. This means that if we denote the partial sum $S_M$ of this series by

$$S_M = \sum_{m=o}^{M} \frac{(2m)!h_{2m}}{m!} (2k_1R)^{-m} \tag{2.37a}$$

and

$$\Delta_M = \left(\frac{2\pi}{k_1R}\right)^{\frac{1}{2}} e^{-G(\lambda_s)} \Omega_{SDP} \quad -S_M \tag{2.37b}$$

then

$$\lim_{k_1R \to \infty} (k_1R)^M \Delta_M = 0 \qquad \text{for all } M \tag{2.38}$$

The proof of the asymptotic nature of such a series relies on Watson's lemma, for which the reader is referred to the notes on this section at the end of the chapter.

In general, since the error incurred by using the partial sum $S_M$ of an asymptotic series is bounded by $(k_1R)^{-M}$, retaining only the first few terms may provide an extremely accurate estimate of $\Omega_{SDP}$ if $k_1R$ is large. This occurs essentially because only a few terms of the Taylor series are needed to accurately represent the integrand near the saddle point. To find the leading term of this series, we use (2.19) and (2.21):

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=o} = e^{-i\pi/4} n_1 \cos\theta \tag{2.39}$$

Moreover, from (2.30),

$$h(\lambda_s) = h(n_1\sin\theta) \simeq \left\{\begin{array}{ll} \left(\frac{2n_1}{\pi k_oR}\right)^{\frac{1}{2}} \frac{e^{-i\pi/4}}{u_{1s}(\theta)+u_{2s}(\theta)} & \text{for } U_1 \\ \left(\frac{2n_1}{\pi k_oR}\right)^{\frac{1}{2}} \frac{e^{-i\pi/4}}{n_2^2u_{1s}(\theta)+n_1^2u_{2s}(\theta)} & \text{for } V_1 \end{array}\right\} \tag{2.40}$$

$$\left.\begin{aligned} \text{where} \qquad u_{1s}(\theta) &= -in_1 \cos\theta \\ u_{2s}(\theta) &= (n_1^2 \sin^2\theta - n_2^2)^{\frac{1}{2}} \end{aligned}\right\} \tag{2.41}$$

It will be noticed that $H_0^{(1)}(k_o\lambda_s\rho)$ has been replaced in (2.40) with the leading term of its asymptotic expansion. Strictly speaking, this cannot be done for near vertical observation angles ($\theta \simeq 0$). However, by returning to a double integral in Chapter 1, we can carry out a two-dimensional version of the steepest-descent process and show that the leading term in the far-field expansion remains the same, namely:

$$\Omega_{SDP} \simeq \frac{2}{k_o R} e^{ik_1R} (-in_1 \cos\theta)\, g(n_1 \sin\theta) \tag{2.42}$$

where $g(\lambda) = (u_1+u_2)^{-1}$ or $(n_2^2u_1+n_1^2u_2)^{-1}$ for $U_1$ or $V_1$, respectively.

If it is desired to retain more terms of (2.36), the analysis becomes rather cumbersome since not only must more derivatives (2.34) be evaluated, but more terms in the asymptotic expansion of $H_0^{(1)}(k_o\lambda_s\rho)$ must be kept as well. An easier way to obtain higher order terms in this expansion is to appeal directly to the Helmholtz equation which $U_1$ and $V_1$ satisfy in the region $z > 0$. By azimuthal symmetry there can be no $\phi$ dependence in these expansions, and it is clear from the discussion above and the form of the asymptotic expansion of $H_0^{(1)}(k_o\lambda_s\rho)$ that the complete expansion of $U_1$ or $V_1$ must have the form

$$\frac{e^{ik_1R}}{k_oR} \sum_{m=o}^{\infty} A_m(\theta)(2k_1R)^{-m} \tag{2.43}$$

Inserting (2.43) into the Helmholtz equation,

$$\frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial W}{\partial R}\right) + \frac{1}{R^2\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial W}{\partial\theta}\right) + k_1^2 W = 0 \tag{2.44}$$

where W is either $U_1$ or $V_1$, and then gathering like powers of $2k_oR$, and setting the coefficients of these powers equal to zero individually, we obtain

$$i(m+1)\, A_{m+1} = m(m+1)A_m + \cot\theta \;\; A'_m + A''_m \tag{2.45}$$

where primes denote differentiations with respect to $\theta$. This recursion relation permits $A_1$, $A_2$, etc. to be found once $A_o$ is known, and from (2.42) we have that

$$A_o(\theta) = -2in_1 \cos\theta \; g(n_1 \sin\theta) \tag{2.46}$$

The asymptotic series (2.43) represents the complete asymptotic expansion for $U_1$ or $V_1$ only if no additional contribution to the integral results from deforming the integration contour to the SDP. The reader can easily verify that the contributions from arcs at $\infty$ in the $\lambda$-plane which arise in this connection are zero. Now, for media such as we are considering, it is easily shown that the Sommerfeld pole $\lambda_p$ lies to the left of the line $Re(\lambda) = n_1$, and so as seen in Fig. 2.4, this pole is never picked up during the deformation (the left side of the SDP, for small values of $\chi$, passes "under" the pole on the improper Riemann sheet, where no pole is located.) Clearly, however, the SDP encounters the branch cut from $n_2$ (which we will be thinking of as the earth parameter, and $n_1$ that of the air) for all elevation angles less than some critical angle $\chi_c$ given by

$$\cos \chi_c = \frac{q + p\sqrt{q^2 + p^2 - 1}}{q^2 + p^2} \tag{2.47}$$

where $q = Re(n_2/n_1)$ and $p = Im(\sqrt{n_2^2 - n_1^2}/n_1)$. For a highly conducting earth, $\chi_c = 45^o$. In the range of $\chi < \chi_c$, the SDP passes on to an improper Riemann sheet and must be reconnected to the proper sheet by an additional contour $\Gamma_1$ as shown in Fig. 2.6. Although $\Gamma_1$ is constrained to pass below $n_2$, we are free to choose the remainder of the path so that $G(\lambda)$ falls off most rapidly from $n_2$ into either the proper or improper sheet as shown, <u>i.e.</u>, $\Gamma_1$, is to be a steepest-descent path. Now in this case, the point $n_2$ is not a <u>bona fide</u> saddle point, and the steepest descent method for evaluating this integral must be modified somewhat; this can be done, and it is possible to show that the contribution from $\Gamma_1$ has a leading order term of the form

$$(k_2R)^{-2} \exp\left\{-k_oR\left[(n_2^2 - n_1^2)^{\frac{1}{2}} \cos\theta - in_2 \sin\theta\right]\right\} \tag{2.48}$$

If medium 2 is a good conductor, this wave is heavily attenuated, in addition to being one order smaller in $k_oR$ than the saddle-point contribution. However, in general, this contribution must be added to the series (2.43) to obtain the complete expansion.

Using the dominant term (2.42) only in conjunction with the perfect image and source terms $G_2$ and $G_1$ from Chapter 1, we can construct a table (Table 2.1) of far-field (or so-called space-wave) expressions for the various elementary dipole sources. As can be seen, if only terms of order $(k_1R_1)^{-1}$ and $(k_1R_2)^{-1}$ are retained, the fields have no radial components in a spherical coordinate system. In all cases, there is a direct contribution from the source, and a contribution from an image located at $-z_o$ and modified by a reflection coefficient of either $\Gamma_{TE}$ or $\Gamma_{TM}$, where

$$\Gamma_{TE}(\theta) = \frac{\cos\theta - (n^2 - \sin^2\theta)^{\frac{1}{2}}}{\cos\theta + (n^2 - \sin^2\theta)^{\frac{1}{2}}} \tag{2.49}$$

$$\Gamma_{TM}(\theta) \quad \frac{n^2\cos\theta - (n^2 - \sin^2\theta)^{\frac{1}{2}}}{n^2\cos\theta + (n^2 - \sin^2\theta)^{\frac{1}{2}}} \tag{2.50}$$

are the Fresnel reflection coefficients for a plane wave reflected from the air-earth interface and polarized with either $\overline{E}$ or $\overline{H}$ parallel to the interface, respectively. The reflection process is illustrated schematically in Fig. 2.7. Since we are dealing here with a dipole source problem, it will be seen that for certain orientations (those other than vertical), both polarizations will be excited by the source, and the plane-wave approximation becomes progressively better as $R \to \infty$.

Table 2.1

Space-wave expressions for the fundamental integrals and the electromagnetic fields in the air due to dipoles located above a homogeneous earth, to order $(k_1R)^{-1}$.

**Notation:** $n^2 = n_2^2/n_1^2$ ; $R_1 = \left[(z-z_o)^2 + \rho^2\right]^{1/2}$; $R_2 = \left[(z+z_o)^2 + \rho^2\right]^{1/2}$;

$\zeta_o = (\mu_o/\varepsilon_o)^{1/2}$.

$$\Gamma_{TE}(\theta) = \frac{\cos\theta - (n^2 - \sin^2\theta)^{1/2}}{\cos\theta + (n^2 - \sin^2\theta)^{1/2}} \qquad \cos\theta_1 = (z-z_o)/R_1$$

$$\Gamma_{TM}(\theta) = \frac{n^2\cos\theta - (n^2 - \sin^2\theta)^{1/2}}{n^2\cos\theta + (n^2 - \sin^2\theta)^{1/2}} \qquad \cos\theta_2 = (z+z_o)/R_2$$

$$\theta_1 \simeq \theta_2 \simeq \theta$$

Fundamental Integrals

$$U_1(z;\rho) \quad \frac{e^{ik_1R}}{k_oR}\left[1 + \Gamma_{TE}(\theta)\right] \qquad R \quad (z^2+\rho^2)^{1/2}$$

$$V_1(z;\rho) \quad \frac{e^{ik_1R}}{k_oR}\,\frac{\left[1 + \Gamma_{TM}(\theta)\right]}{n_2^2} \qquad \cos\theta = z/R$$

Elementary Dipole Fields

**VED**

$$E_\theta \simeq (\zeta_o/n_1)H_\phi \simeq Q_E\sin\theta\left[\frac{e^{ik_1R_1}}{k_1R_1} + \Gamma_{TM}(\theta)\,\frac{e^{ik_1R_2}}{k_2R_2}\right]$$

$$E_r \simeq H_r \simeq 0$$

$$Q_E = k_1^2\,\frac{\omega p}{4\pi}\,\frac{\zeta_o}{n_1}$$

$$E_\phi \simeq -(\zeta_o/n_1)H_\theta \simeq 0$$

**HED**

For observation points on the interface, i.e., for $\theta=\frac{\pi}{2}$, it will be noticed that $\Gamma_{TE} = \Gamma_{TM} = -1$, and all the fields from Table 2.1 vanish to the order of accuracy retained. As far as the contributions from $U_1(z;\rho)$ are concerned, this simply means that higher order terms - $O(k_1R)^{-2}$ at least - from the expansion (2.43) must be kept to obtain an estimate of the field strength. For $V_1(z;\rho)$, however, as can be seen from Fig. 2.4, there is a pole in close proximity to the SDP and saddle point, and the assumption that the integrand is slowly varying in this vicinity is no longer valid. Table 2.1 is then strictly valid only away from the interface.

The variation of amplitude and phase for $\Gamma_{TE}$ and $\Gamma_{TM}$ is shown in Figs. 2.8 and 2.9 as a function of $\chi$ for several values of frequency f, and typical earth parameters $\sigma_2 = 12 \times 10^{-3}$ ℧/m, $\varepsilon_2 = 15\ \varepsilon_0$.

So far, our result is valid whenever the singularities of the integrand are sufficiently far away from the saddle point. Obviously, this is always true when $k_1R$ is sufficiently large. However, the presence of the poles at $\pm\lambda_p$ in the integrand of $V_1$, where

$$\lambda_p = \frac{n_1 n_2}{\sqrt{n_1^2+n_2^2}} \tag{2.51}$$

leads, by virtue of the mapping (2.27), to poles in the complex $\tau$-plane at

$$\tau_{p\pm} = \left\{2i\left[1 + \frac{\lambda_p}{n_2}\cos\theta \pm \frac{\lambda_p}{n_1}\sin\theta\right]\right\}^{\frac{1}{2}} \tag{2.52}$$

When $|n| = |n_2/n_1|$ is large,

$$\begin{cases} \tau_{p+} \simeq 2e^{i\pi/4}\left[\sin\frac{\chi}{2}\left(\sin\frac{\chi}{2} + \frac{1}{n}\cos\frac{\chi}{2}\right) + \frac{1}{4n^2}\right]^{\frac{1}{2}} \\ \tau_{p-} \simeq 2e^{i\pi/4}\left[\cos\frac{\chi}{2}\left(\cos\frac{\chi}{2} + \frac{1}{n}\sin\frac{\chi}{2}\right) + \frac{1}{4n^2}\right]^{\frac{1}{2}} \end{cases} \tag{2.53}$$

This means that the radius of convergence $\tau_c$ of (2.33), for near-grazing angles $\chi \simeq 0$, will be equal to $|\tau_{p+}|$ which is $O(n^{-1})$ and thus small. The asymptotic expansion (2.43) can be valid only if the Taylor series (2.33) accurately represents the function $h(\lambda)\ d\lambda/d\tau$ when the exponential term is significant; in other words, we must have $|k_1 R\tau_c^2/2| >> 1$, or, roughly,

$$|ik_1 R/2n^2| >> 1 \tag{2.54}$$

The import of such a condition is demonstrated by the following example. Let us take as typical earth parameters $\sigma_2 = 10^{-3}$ ℧/m, and $\varepsilon_2 = 10\ \varepsilon_0$. At f = 6.4 kHz, $n \simeq 37.5(1+i)$, and at $R = 10^3$ km, the expression in (2.54) has a value of only about .02. For grazing angles, the space-wave solution is thus practically useless at VLF. Even at 640 kHz, R still has to be at least 100 km or so before (2.54) is satisfied.

In order to obtain more useful far-field expressions in this situation, the effect of $\tau_{p+}$ must be accounted for in the saddle point integration. In analogy with (2.13), we can define the Sommerfeld-Norton numerical distance P as

$$p = ik_1 R\left[1 - \frac{\lambda_p}{n_1}\sin\theta + \frac{\lambda_p}{n_2}\cos\theta\right] = k_1 R\tau_{p+}^2/2 \tag{2.55}$$

which reduces to (2.13) at $\theta = \frac{\pi}{2}$. It is now no longer $h(\lambda)$ in (2.30) which is slowly varying near the saddle point, but $\hat{h}(\lambda)$, where

$$h(\lambda) = \frac{\hat{h}(\lambda)}{\tau^2 - \tau_{p+}^2} = \frac{\hat{h}(\lambda)}{\tau^2 - 2p/k_1 R} \tag{2.56}$$

The procedure leading to (2.35) can be repeated, to give

$$\Omega_{SDP} \sim e^{ik_1 R} \sum_{m=o}^{\infty} \left(-\frac{2}{k_1}\right)^m \hat{h}_{2m} \left[\frac{d^m}{dR^m} \int_{-\infty}^{\infty} \frac{e^{-k_1 R\tau^2/2}}{\tau^2 - 2p/k_1 R}\, d\tau\right] \tag{2.57}$$

where $\hat{h}_{2m}$ are the Taylor coefficients of $\hat{h}(\lambda)d\lambda/d\tau$ :

$$\hat{h}\left[\lambda(\tau)\right]\frac{d\lambda}{d\tau} = \sum_{m=o}^{\infty} \hat{h}_m\ \tau^m \qquad |\tau| < \hat{\tau}_c \tag{2.58}$$

and the new radius of convergence $\hat{\tau}_c$ is larger than the old one, $|\tau_{p+}|$. To complete the study of (2.57), we must therefore evaluate the integral function:

$$I(p) = \left(\frac{2p}{k_1R}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-k_1R\tau^2/2}}{\tau^2 - 2p/k_1R}\, d\tau \tag{2.59}$$

Before doing so, however, we should note that if $|p| \gg 1$, this result will reduce to the previous, simpler form, and that (2.57) need be employed only when $|p| = O(1)$. The most common case when this occurs is when $\chi$ is small and $|n|$ is large. By (2.53) and (2.55), we can then obtain an approximate expression for p as

$$p \simeq \frac{ik_1R}{2n^2}\left[1 + n \sin\chi\right]^2 \simeq i\left(1 - \frac{\lambda_p}{n_1}\right)k_1R\left[1 + n \sin\chi\right]^2 \tag{2.60}$$

with $\sin \chi = z/R \simeq \chi$. The latter form has the advantage of reducing to the exact form (2.55) when $\theta = 0$, and makes it clear that p is in fact a measure of the separation between the pole at $\lambda_p$ and the branch point at $n_1$.

To study the function I(p), let us change integration variables to $t = (k_1R/2)^{\frac{1}{2}}\tau$ so that

$$I(p) = p^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t^2 - p}\, dt \tag{2.61}$$

It is readily seen that

$$\left(\frac{d}{dx} + p\right) \int_{-\infty}^{\infty} \frac{e^{-t^2x}}{t^2-p}\, dt = -\int_{-\infty}^{\infty} e^{-t^2x}\, dt = -\left(\frac{\pi}{x}\right)^{\frac{1}{2}} \tag{2.62}$$

so that

$$\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t^2-p}\, dt = -\pi^{\frac{1}{2}}e^{-p} \int_0^1 \frac{e^{px}}{x^{\frac{1}{2}}}\, dx + e^{-p} \int_{-\infty}^{\infty} \frac{dt}{t^2-p}$$

or

$$\begin{aligned} I(p) &= i\pi^{\frac{1}{2}}\, e^{-p}\left[\pi^{\frac{1}{2}} - \gamma\left(\tfrac{1}{2}, -p\right)\right] \\ &= i\pi\, e^{-p}\, \mathrm{erfc}\,(-ip^{\frac{1}{2}}) \end{aligned} \tag{2.63}$$

where $\gamma$ is the incomplete gamma function, which in this case reduces to the complementary error function (Appendix C)

$$\mathrm{erfc}(z) = 2\pi^{-\frac{1}{2}} \int_z^\infty e^{-t^2} dt \tag{2.64}$$

The resulting form of (2.57) is

$$\Omega_{SDP} \approx i\pi\left(\frac{k_1 R}{2p}\right)^{\frac{1}{2}} e^{ik_1 R} \sum_{m=o}^{\infty} \left(-\frac{2}{k_1}\right)^m \hat{h}_{2m} \frac{d^m}{dR^m}\left[e^{-p}\ \mathrm{erfc}(-ip^{\frac{1}{2}})\right] \tag{2.65}$$

To leading order, this is simply

$$\Omega_{SDP} \sim -\frac{2}{k_1 R} e^{ik_1 R} (\pi p)^{\frac{1}{2}} e^{-p} \mathrm{erfc}(-ip^{\frac{1}{2}})\ n_1^2 \cos\theta\ \ g(n_1 \sin\theta\ ) \tag{2.66}$$

or, alternatively,

$$\Omega_{SDP} \sim -\frac{2i}{k_1 R}\ e^{ik_1 R} \left[1 - W(p)\right]\ \ n_1^2 \cos\theta\ g(n_1\ \sin\theta) \tag{2.67}$$

where $W(p)$ is Sommerfeld's attenuation function

$$W(p) = 1 + i(\pi p)^{\frac{1}{2}} e^{-p}\ \mathrm{erfc}\ (-ip^{\frac{1}{2}}) \tag{2.68}$$

which is a measure of the correction to the space-wave expression due to the influence of the pole (compare (2.42)). When $|p|$ is large, the error function may be approximated by its large argument form, giving

$$W(p) \sim\ -\frac{1}{2p} - \frac{1.3}{(2p)^2} + \ldots \tag{2.69}$$

which is indeed a small correction, as we might expect. For $p \to o$, however, $W(p) \to 1$ and a significant modification of the field structure occurs. Norton has published numerical values for $W(p)$ which we have reproduced in Fig. 2.10.

Expression (2.67) can be split into the space-wave (2.42) and a correction known as a <u>ground wave</u>:

$$\Omega_{SDP} = \Omega_{SP} + \Omega_g$$

where

$$\Omega_{SP} = -\frac{2i}{k_1 R} e^{ik_1 R} n_1^2 \cos\theta\ g(n_1\ \sin\theta) \tag{2.70}$$

$$\Omega_g = \frac{2i}{k_1 R} e^{ik_1 R} W(p)\ n_1^2 \cos\theta\ g(n_1\ \sin\theta\ ) \tag{2.71}$$

The ground wave is, in the far-field ($k_1 R >> 1$) only significant near the interface, where the space wave is correspondingly small, as evidenced by Table 2.1. The correction factors to be added to the space wave terms from Table 2.1 in the ground wave region are summarized in Table 2.2. It should be noted that derivatives of $W(p)$ which arise have been neglected compared to $W(p)$, which is justified in the range $|p| \lesssim 1$ and $k_1R \gg 1$ when the ground wave expression is most useful.

Table 2.2

Ground wave expressions for electromagnetic fields in the air due to dipoles located above a homogeneous earth ($k_1R \gg 1$). Notations as in Table 2.1; also

$$W(p) = 1 + i(\pi p)^{\frac{1}{2}} e^{-p} \operatorname{erfc}(-ip^{\frac{1}{2}})$$

$$p = ik_1R \left[1 - \frac{\lambda_p}{n_1} \sin\theta + \frac{\lambda_p}{n_2} \cos\theta\right]$$

| | |
|---|---|
| VED | $E_\theta \simeq (\zeta_o/n_1)H_\phi \simeq Q_E \sin\theta\, (\Gamma_{TM} - 1)\, W(p) \dfrac{e^{ik_1R_2}}{k_1R_2}$ |
| | $E_r \simeq H_r \simeq 0$ |
| | $E_\phi \simeq -(\zeta_o/n_1)H_\theta \simeq 0$ |
| HED | |

It can be seen from (2.65) that the complete expansion for $\Omega_{SDP}$ must have the form

$$\frac{e^{ik_1R}}{k_oR}\left\{\sum_{m=o}^{\infty} \hat{A}_m(\theta)(2k_1R)^{-m} + (\pi p)^{\frac{1}{2}}\, e^{-p}\, \mathrm{erfc}\,(-ip^{\frac{1}{2}}) \sum_{m=o}^{\infty} \hat{B}_m(\theta)(2k_1R)^{-m}\right\} \quad (2.72)$$

We can proceed as with (2.43) to obtain recursion relationships for $\hat{A}_m$ and $\hat{B}_m$ to allow us to compute higher order terms in a reasonably straightforward manner. The details are left as an exercise for the reader.

It is of interest to note that, because the integral function $U_1'$ does not possess any poles in its integrand and hence contains no ground wave contribution, it vanishes completely as we allow both the source and observation points to be on the earth's surface (though only in the asymptotic sense). As a result, the electric field in the far-field zone ($k_1R >> 1$) is vertically-polarized (TM) on the earth's surface for a vertical electric dipole as well as a horizontal electric dipole. It is seen for a given dipole moment. The ratio of the two electric fields is given by

$$\left.\frac{E_\theta\,(HED)}{E_\theta\,(VED)}\right|_{\theta=0} = -\,\frac{\cos\phi}{(n^2-1)^{\frac{1}{2}}} \quad (2.73)$$

where $\phi$ is the angle measured from the horizontal dipole axis. This means for most practical cases where $|n|^2 >> 1$, the electric field produced by a horizontal dipole (HED) on the earth surface is substantially less than the one produced by a vertical electric dipole (VED). The situation is even more dramatic in the case of a magnetic dipole. We see from Table 2.2 that, while the VMD will not produce any significant field strength in the far-field zone (again, to the order of $R^{-1}$), the HMD actually allows the propagation of waves of both polarization. The ratio of the two fields is given in this case by

$$\left.\frac{E_\phi\,(HMD)}{E_\theta\,(HMD)}\right|_{\theta=0} = \frac{\cot\phi}{(n^2-1)^{\frac{1}{2}}} \quad (2.74)$$

Thus, the field strength of the TM wave (i.e., $E_\theta$) is substantially greater than that of the TE wave (i.e., E ), except in the direction along the dipole axis.

## §2.4 Expansions for Earth with Large Refractive Index

The restriction $k_1R >> 1$ of the previous section is often too stringent for applications when near fields of antennas are required, or at ELF or VLF frequencies when the space wave and ground wave solutions are valid only at unrealistically large distances. It is the objective of this section to obtain expansions which assume only that $|n^2| >>1$.

By way of preface to these discussions, let us investigate a general procedure for the transformation of Sommerfeld integrals. Consider the integral

$$\Omega = 2 \int_0^\infty \gamma(u_1) \frac{e^{-k_o u_1 z}}{u_1} J_o(k_o \lambda \rho) \lambda d\lambda \tag{2.75}$$

where $\gamma(u_1) = u_1 g(\lambda) e^{bu_2}$ and $a = k_o z$ (cf. (1.76)). If $\gamma(u_1)$ can be expressed as the Laplace transform of a function $\tilde{\gamma}(t)$ over some contour:

$$\gamma(u_1) = \int_0^{\infty e^{i\xi}} \tilde{\gamma}(t) e^{-k_o u_1 t} dt \tag{2.76}$$

then from (2:4) and (2.75) we have

$$\Omega = \frac{2}{k_o} \int_0^{\infty e^{i\xi}} \tilde{\gamma}(t) \frac{e^{ik_1 R_t}}{R_t} dt \tag{2.77}$$

where $R_t = \sqrt{\rho^2 + (z+t)^2}$, provided the phase angle $\xi$ can be chosen so that the interchange of integrations leading to (2.77) is valid. This transformation is essentially an application of a convolution theorem, and represents $\Omega$ as an integral of $e^{ik_1R_t}/R_t$, the field of a source at $z = -t$, with a suitable weighting function from $t = 0$ to $t = \infty e^{i\xi}$. Thus (2.77) is a generalized image representation of (2.75), which will prove useful in subsequent manipulations.

A special class of functions $\gamma(u_1)$ which leads to closed-form expressions from (2.77) is the set of rational functions of $u_1$. In particular, consider the integral (for Re(b) > 0 to start with):

$$X_\nu(z,\rho;b) \equiv \int_0^\infty \frac{e^{-k_o u_1 z}}{(u_1+b)^\nu u_1} J_o(k_o\lambda\rho)\,\lambda d\lambda \tag{2.78}$$

By expressing

$$(u_1+b)^{-\nu} = \frac{k_o^{\ \nu}}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-k_o t(u_1+b)}\,dt \qquad (\mathrm{Re}\ \nu > 0) \tag{2.79}$$

we obtain the following special case of (2.76):

$$X_\nu(z,\rho;b) = \frac{k_o^{\ \nu-1}}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-k_o bt} \frac{e^{ik_1 R_t}}{R_t}\,dt \tag{2.80}$$

Although $X_\nu$ is an entire function of the index $\nu$, we are primarily interested in the case when $\nu$ is a positive integer, in which case

$$X_m = \frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1} X_1}{\partial b^{m-1}} \tag{2.81}$$

By the successive changes of variable $u = t+z$, and

$$w = u + i\frac{b}{n_1}\sqrt{\rho^2+u^2}$$

whence also

$$\sqrt{\rho^2+u^2} + i\frac{b}{n_1}u = \sqrt{w^2+c^2} \quad ; \qquad u\left(1+\frac{b^2}{n_1^2}\right) = w - i\frac{b}{n_1}\sqrt{w^2+c^2}$$

$$\left(c^2 = \rho^2\left(1+\frac{b^2}{n_1^2}\right)\right)$$

we get

$$X_1 = e^{k_o bz} \int_{z+i\frac{b}{n_1}R}^\infty \frac{e^{ik_1\sqrt{w^2+c^2}}}{\sqrt{w^2+c^2}}\,dw \tag{2.82}$$

By an obvious change of integration variable, this can be identified with the incomplete Hankel function from Appendix B:

$$X_1 = \frac{\pi i}{2}\frac{e^{k_o bz}}{k_o}\left\{H_o^{(1)}\left[k_o\rho(n_1^2+b^2)^{\frac{1}{2}}\right]-H_o^{(1)}\left[\frac{n_1 \cos\theta + ib}{(n_1^2+b^2)^{\frac{1}{2}} \sin\theta}, k_o\rho(n_1^2+b^2)^{\frac{1}{2}}\right]\right\} \qquad (2.83)$$

where $\tan\theta = z/\rho$ as before. From the properties of $H_o^{(1)}(a,z)$ given in Appendix B, it is clear that $X_m$ is expressible as a combination of complete and incomplete Hankel functions of order zero and one, together with elementary functions.

A few relations which are easily verified from (2.78) or (2.80) are:

$$\frac{\partial X_\nu}{\partial z} = k_o b X_\nu - k_o X_{\nu-1} \qquad (2.84)$$

$$\frac{\partial X_\nu}{\partial b} = -\nu X_{\nu+1} \qquad (2.85)$$

To extend the definition of $X_\nu$ to $\mathrm{Re}(b) \leq 0$, we note that as b passes into the left half of its complex plane, a pair of poles (if $\nu$ is an integer) or branch points (if $\nu$ is not an integer) of the integrand of (2.78) will appear in the $\lambda$-plane at $\lambda = \pm(n_1^2 + b^2)^{\frac{1}{2}}$ (for definiteness, let us specify the imaginary part of this square root to be positive). As Re(b) becomes negative, $X_\nu$ as defined by (2.78) will remain analytic until the poles (or branch points) in the $\lambda$-plane reach the contour of integration of (2.78), i.e., the real axis. Let us trace the movement of these singularities in the $\lambda$-plane as b passes into the left half-plane as shown in Fig. 2.11, along the path AA'. At the starting point A, the configuration in the $\lambda$-plane is as shown in Fig. 2.12 (as in Chapter 1, the branch cuts cannot cross the real axis; if $\nu$ is an integer, the points $\pm (n_1^2 + b^2)^{\frac{1}{2}}$ are poles, so of course no branch cuts are present). As b moves from A to A', the singularities in the $\lambda$-plane rotate $180^o$ to the positions indicated in Fig. 2.13. It is clear that the values of the integral $X_\nu$ from (2.78) at the positions A and A' must be different due to the different values taken by $(u_1 + b)^{-\nu}$ along the portion of the contour adjacent to the branch cuts in Figs. 2.12 and 2.13 (if $\nu$ is an integer, the difference will be equal to the residue contribution from the pole). As a result, we conclude that, as a function of b, $X_\nu$ must possess a branch cut at $in_1$ as indicated in Fig. 2.11, and that the definition (2.78) leads consistently to the Riemann sheet specified by $\mathrm{Im}(n_1^2 + b^2)^{\frac{1}{2}} > 0$. In this manner we can analytically continue the result (2.83) to all complex values of b.

By decomposing a rational function $\gamma(u_1)$ into partial fractions, the corresponding Sommerfeld integral (2.75) can now be expressed in closed form using incomplete Hankel functions supplemented by more common functions, eliminating the need for numerical integration. This is especially valuable if fields are required at a number of points, in which case repeated numerical integration can become quite costly. Moreover, if an arbitrary function $\gamma(u_1)$ can be well approximated by a rational function of $u_1$ (particularly one with only a few terms in its partial fraction expansion), then the corresponding Sommerfeld integral can be efficiently evaluated to reasonably high accuracy, again without numerical integration, if a routine for evaluating $H_0^{(1)}(a,z)$ by the series expansions of Appendix B is available.

§2.4.1 Asymptotic Expansion for $|k_2R| >> 1$

We are now in a position to obtain expressions for $U_1$ and $V_1$ when $|k_2R| >> 1$. The integral $U_1$ can be reworked in a straightforward manner, once again by rationalizing the denominator:

$$U_1(z,\rho) = 2\int_0^\infty \frac{e^{-k_o u_1 z}}{u_1+u_2} J_o(k_o\lambda\rho)\,\lambda d\lambda \tag{2.86}$$

$$= \frac{2}{N^2}\int_0^\infty u_1 e^{-k_o u_1 z} J_o(k_o\lambda\rho)\lambda d\lambda - \int_0^\infty u_2 e^{-k_o u_1 z} J_o(k_o\lambda\rho)\,\lambda d\lambda$$

$$= \frac{2}{k_o^2N^2}\frac{\partial^2}{\partial z^2}\frac{e^{ik_1R}}{k_oR} - \frac{2}{N^2}\int_0^\infty u_2\, e^{-k_o u_1 z} J_o(k_o\lambda\rho)\,\lambda d\lambda$$

where $N^2 = n_2^2 - n_1^2$ as in the analysis of §2.1. An asymptotic series for $|k_2R| >>1$ can now be obtained by expanding

$$u_2 = (u_1^2 - N^2)^{\frac{1}{2}} = -iN\left(1 - \frac{u_1^2}{N^2}\right)^{\frac{1}{2}}$$

$$= -iN \sum_{m=o}^{\infty}\binom{\frac{1}{2}}{m}\left(-\frac{u_1^2}{N^2}\right)^m . \tag{2.87}$$

where $\binom{\frac{1}{2}}{m} = \Gamma(\frac{3}{2})/\Gamma(\frac{3}{2} - m)\,\Gamma(m + 1)$ is the binomial coefficient, since the contribution to the integrand comes mainly from small $\lambda$. By integrating the remaining integral in (2.86) term by term, we find that

$$U_1(z;\rho) \sim \frac{2}{k_o^2N^2}\frac{\partial^2}{\partial z^2}\left[\frac{e^{ik_1R}}{k_oR}\right] + \frac{2i}{N}\sum_{m=o}^{\infty}\binom{\frac{1}{2}}{m}\left(-\frac{1}{N^2}\right)^m X_{-2m-1}(z,\rho;0) \tag{2.88}$$

where, because of (2.84) and the fact that

$$X_o(z,\rho;b) = \frac{e^{ik_1R}}{k_oR} \tag{2.89}$$

we have

$$X_{-2m-1}(z,\rho;0) = \left(-\frac{1}{k_o}\right)^{2m+1}\frac{\partial^{2m+1}}{\partial z^{2m+1}}\left[\frac{e^{ik_1R}}{k_oR}\right] \tag{2.90}$$

Since each z derivative of $e^{ik_1R}/R$ increases the power of R in the denominator by one, (2.88) is evidently an asymptotic expansion in inverse powers of $(k_oNR)^2$, which for $|n|^2 >> 1$, is essentially $(k_2R)^2$, and we have achieved the desired series. To leading order we have

$$U_1(z;\rho) \simeq \frac{2n_1}{N} \cos\theta \frac{e^{ik_1R}}{k_oR} \qquad (|k_1R| >> 1) \qquad (2.91)$$

Comparing this with the space-wave expression from Table 2.1, we see that (2.89) will be accurate provided that

$$\frac{n_1}{N} = \frac{1}{(n^2-1)^{\frac{1}{2}}} \simeq \frac{1}{\cos\theta+(n^2-\sin^2\theta)^{\frac{1}{2}}}$$

which is true within the restriction that $|n^2| >> 1$. However, by retaining the terms of (2.88) which cannot be neglected if we assume only that $|k_2R| >> 1$, we have a solution which is valid down to much smaller distances than is the space-wave solution.

The Sommerfeld integral $V_1$ requires somewhat more special handling since its denominator cannot be completely rationalized in the same way as $U_1$. However, this function can be expressed as

$$V_1(z;\rho) = \frac{2n_2^2}{n_2^4-n_1^4}\int_o^\infty \frac{e^{-k_ou_1z}}{u_1-u_{1p}} J_o(k_o\lambda\rho)\ \lambda d\lambda - \frac{2n_1^2}{n_2^4-n_1^4}\int_o^\infty \frac{e^{-k_ou_1z}}{u_2-u_{2p}} J_o(k_o\lambda\rho)\ \lambda d\lambda \qquad (2.92)$$

where, from (1.84):

$$u_{1p} = \frac{in_1^2}{(n_1^2+n_2^2)^{\frac{1}{2}}} \quad ; \qquad u_{2p} = \frac{-in_2^2}{(n_1^2+n_2^2)^{\frac{1}{2}}} \qquad (2.93)$$

This splitting of $V_1$ enables us to treat the first integral exactly as described above, while the second can be asymptotically expanded for large $|k_2R|$ in a manner similar to that of $U_1$. Thus, for $|n^2| >> 1$, the first integral, which is multiplied by a constant $n^2$ times larger than that of the second integral, and clearly is the more important term, is expressible in terms of $X_1$:

$$
\begin{aligned}
V_1^{(o)} &\equiv \frac{2n_2^2}{n_2^4-n_1^4}\int_0^\infty \frac{e^{-k_o u_1 z}}{u_1-u_{1p}} J_o(k_o\lambda\rho)\lambda\, d\lambda \\
&= \frac{2n_2^2}{n_2^4-n_1^4}\int_0^\infty \left[\frac{u_{1p}}{(u_1-u_{1p})u_1}+\frac{1}{u_1}\right] e^{-k_o u_1 z} J_o(k_o\lambda\rho)\lambda\, d\lambda \qquad (2.94)\\
&= \frac{2n_2^2}{n_2^4-n_1^4}\left\{\frac{e^{ik_1R}}{k_oR} + u_{1p}\, X_1(z,\rho;\, -u_{1p})\right\} \\
&= \frac{2n_2^2}{n_2^4-n_1^4}\left\{\frac{e^{ik_1R}}{k_oR} + \frac{\pi i u_{1p}}{2}\, e^{-k_o u_{1p} z}\left[H_o^{(1)}(k_o\lambda_p\rho) - H_o^{(1)}\left(\frac{n_2\cos\theta+\lambda_p}{iu_{2p}\sin\theta},\, k_o\lambda_p\rho\right)\right]\right\}
\end{aligned}
$$

It will be noted that for $z = 0$, the form of the incomplete Hankel function reduces to $H_o^{(1)}(n_1/n_2,\, k_o\lambda_p\rho)$, as was encountered in the exact expression (2.11) for $z = 0$.

We can see that when the parameter of the incomplete Hankel function in (2.94) is small, which will occur for small elevation angles $\theta$, the leading term of expansion (B.10) may be used, giving

$$
V_1^{(o)} \simeq \frac{2n_2^2}{n_2^4-n_1^4}\,\frac{e^{ik_1R}}{k_oR}\left\{1+u_{1p}e^{-p}\left[\frac{\pi i}{2}k_oRe^{-ik_o\lambda_p\rho}\, H_o^{(1)}(k_o\lambda_p\rho)-\left(\frac{i\pi k_1R}{2\lambda_p\sin\theta}\right)^{\frac{1}{2}}\mathrm{erf}(-ip^{\frac{1}{2}})\right]\right\} \qquad (2.95)
$$

where p is the Sommerfeld-Norton numerical distance given by (2.55). In the far field $|k_1R| >> 1$, we can further replace the Hankel function in (2.95) by its large argument form, resulting in

$$
V_1^{(o)} \simeq \frac{2n_2^2}{n_2^4-n_1^4}\,\frac{e^{ik_1R}}{k_oR}\left\{1 + u_{1p}\, e^{-p}\left(\frac{\pi i k_oR}{2\lambda_p\sin\theta}\right)^{\frac{1}{2}}\mathrm{erfc}(-ip^{\frac{1}{2}})\right\} \qquad (2.96)
$$

since $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$. If $|n^2| >>1$, and $\chi <<1$, then it is easily verified (cf. (2.60)) that the term in brackets in (2.96) is approximately equal to $W(p)$, the Sommerfeld attenuation function which arose in the ground wave portion of the far-field analysis of $V_1$.

For higher elevation angles, when the parameter of the incomplete Hankel function in (2.94) is large - especially for $\theta$ near 0, we may use the leading term of (B.15) to obtain the approximation

$$V_1^{(o)} \simeq \frac{2n_2^2}{n_2^4-n_1^4}\left\{\frac{e^{ik_1R}}{k_oR} + u_{1p}\, e^{-k_ou_{1p}z}\, E_1(-ik_1R)\right\} \tag{2.97}$$

where again the inequality $|n^2| >> 1$ has been invoked.

The second integral in (2.92) can be expanded asymptotically for $|k_2R| >> 1$ as was (2.86), by using the expansion

$$\frac{1}{u_2+u_{2p}} = \sum_{m=o}^{\infty} a_m u_1^{2m} \tag{2.98}$$

where

$$a_o = (-iN + u_{2p})^{-1} \tag{2.99}$$

$$a_1 = -ia_o^2/2N \tag{2.100}$$

and

$$u_{1p}^2\, a_m = a_{m-1} + \binom{\frac{1}{2}}{m}(iN)^{1-2m} \qquad m = 1,2,\ldots \tag{2.101}$$

Substituting (2.98) and integrating termwise, we obtain the remainder term in the expansion of $V_1$:

$$V_1^{(1)} \equiv -\frac{2n_1^2}{n_2^4-n_1^4}\int_o^{\infty}\frac{e^{-k_ou_1z}}{u_2+u_{2p}}\, J_o(k_o\lambda\rho)\lambda d\lambda$$
$$\sim -\frac{2n_1^2}{n_2^4-n_1^4}\sum_{m=o}^{\infty} a_m\, X_{-2m-1}\,(z,\rho;o) \tag{2.102}$$

The leading term of (2.102) is

$$V_1^{(1)} \simeq \frac{n_1^2}{n_2^4-n_1^4}\;\frac{2i}{N+iu_{2p}}\left(\frac{ik_1R-1}{k_o^2R^2}\right) e^{ik_1R}\cos\theta \tag{2.103}$$

Since this quantity is $O(|n|^{-3})$ compared to $V_1^{(o)}$, it is possible to neglect $V_1^{(1)}$ under the assumption that $|k_2R| >> 1$.

## §2.42 Asymptotic Expansion for $|k_o NR| \sim O(1)$ and $|n^2| >> 1$

Returning now to (2.86), suppose we expand $u_2$ as a descending series in $u_1$ rather than the ascending series (2.87):

$$u_2 = u_1 \left(1 - \frac{N^2}{u_1^2}\right)^{\frac{1}{2}} = u_1 \sum_{m=o}^{\infty} \binom{\frac{1}{2}}{m} \left(- \frac{N^2}{u_1^2}\right)^m \tag{2.104}$$

Clearly, such an approximation is valid in the case when the contribution to the integral comes mainly from large $\lambda$. Integrating termwise, we obtain in place of (2.88):

$$U_1(z;\rho) \sim - \frac{2}{N^2} \sum_{m=1}^{\infty} \binom{\frac{1}{2}}{m} (-N^2)^m X_{2m-2}(z,\rho;o) \tag{2.105}$$

Roughly, the terms in this series acquire an additional factor of $(k_o NR)^2$ for each successive m, and so constitute a series valid for $|k_o NR| \le O(1)$ if the decrease in successive terms continues sufficiently far. Again, however, since (2.101) does not converge over the entire range of integration, (2.102) is only an asymptotic series for the case we are considering ($|n^2| >> 1$).

A similar procedure can be applied to $V_1^{(1)}$ by writing

$$\frac{1}{u_2+u_{2p}} = \sum_{m=o}^{\infty} \frac{b_m}{u_1^{m+1}} \tag{2.106}$$

where

$$b_o = 1 \tag{2.107}$$

$$b_1 = -u_{2p} \tag{2.108}$$

and

$$b_{m+2} = \begin{cases} u_{1p}^2 b_m & \text{(m odd)} \\ u_{1p}^2 b_m + \binom{\frac{1}{2}}{\frac{m}{2}+1} (-N^2)^{\frac{m}{2}+1} & \text{(m even)} \end{cases} \qquad m = o,1, \ldots \tag{2.109}$$

Termwise integration now gives, in place of (2.102),

$$V_1^{(1)} \sim - \frac{2n_1^2}{n_2^4 - n_1^4} \sum_{m=o}^{\infty} b_m X_m(z,\rho;o) \tag{2.110}$$

## §2.5 Quasi-Static Forms of $U_1$ and $V_1$ (Quasi-Static Image Theory)

An important special case often encountered in practice is when $|k_1R| << 1$, the so-called quasi-static range. This is equivalent to saying that the fields in region 1 need only satisfy the static field equations, but makes no restriction on the magnitude of $k_2R$. One can approach this problem directly and obtain the quasi-static forms of $U_1$ and $V_1$ by letting $u_1 \to \lambda$ in the respective integrands, giving

$$U_1^{QS}(z;\rho) \equiv 2\int_0^\infty \frac{e^{-k_0\lambda z}}{\lambda+\sqrt{\lambda^2-n_2^2}} J_0(k_0\lambda\rho)\lambda d\lambda \tag{2.111}$$

$$V_1^{QS}(z;\rho) \equiv 2\int_0^\infty \frac{e^{-k_0\lambda z}}{n_2^2\lambda + n_1^2\sqrt{\lambda^2-n_2^2}} J_0(k_0\lambda\rho)\lambda d\lambda \tag{2.112}$$

These forms are commonly employed simplifications of $U_1$ and $V_1$, but estimates of the error incurred in this approximation are rarely given. A simple way of providing such an estimate is to evaluate the difference between (2.111) or (2.112) and $U_1$ or $V_1$ respectively in the limit as $z$ and $\rho \to 0$. The resulting integrals are elementary, we find easily that

$$\Delta U_1 \equiv \lim_{z,\rho\to 0}\left[U_1(z;\rho)-U_1^{QS}(z;\rho)\right] = 2\int_0^\infty\left[\frac{1}{u_1+u_2} - \frac{1}{\lambda+\sqrt{\lambda^2-n_2^2}}\right]\lambda d\lambda$$

$$= \frac{2i}{3}\,\frac{n_1^2}{n_1+n_2} \simeq \frac{2in_1}{3n} \qquad (|n| >> 1) \tag{2.113}$$

$$\Delta V_1 \equiv \lim_{z,\rho\to 0}\left[V_1(z;\rho)-V_1^{QS}(z;\rho)\right] = 2\int_0^\infty\left[\frac{1}{n_2^2u_1+n_1^2u_2} - \frac{1}{n_2^2 + n_1^2\sqrt{\lambda^2-n_2^2}}\right]\lambda d\lambda \tag{2.114}$$

$$= \frac{2in_1^2n_2^2}{n_2^4-n_1^4}\left\{\frac{1}{n_1} - \frac{1}{(n_1^2+n_2^2)^{1/2}}\log\left[\frac{n_1}{n_2}\,\frac{n_1+(n_1^2+n_2^2)^{1/2}}{n_2+(n_1^2+n_2^2)^{1/2}}\right] - \frac{n_2}{(n_2^4-n_1^4)^{1/2}}\log\left[\frac{n_2^2+(n_2^4-n_1^4)^{1/2}}{n_1^2}\right]\right\}$$

$$\simeq \frac{2i}{n^2n_1} \qquad (|n| >> 1)$$

Since $U_1^{QS}$ and $V_1^{QS}$ possess 1/R singularities as $R \to 0$, while (2.113) and (2.114) vanish as $n_1/n_2 = 1/n \to 0$, the approximate forms (2.111) and (2.112) should be quite accurate within these restrictions.

Unfortunately, even the simplified integrals $U_1^{QS}$ and $V_1^{QS}$ cannot be expressed in finite form as known functions except in certain special cases. However, if $|k_2R| >> 1$, we may take the asymptotic expansions of §2.4.1 and simplify them under the assumption of $|k_1R| << 1$ to obtain approximate results over part of the quasi-static range. For $U_1$, for instance, we take only the first term and the leading term of the sum from (2.88), and make the quasi-static approximation; the result is

$$U_1^{QS} \simeq \frac{2}{N^2}\left\{\frac{3\cos^2\theta - 1}{k_o^3R^3}\right\} + \frac{2i}{N}\,\frac{\cos\theta}{k_o^2R^2} \tag{2.115}$$

It can be observed that <u>all</u> of the terms of (2.115) are also obtained as the leading terms of the expansion of

$$U_1^{QS} \simeq \frac{1}{k_oR} - \frac{1}{k_oR_d} \tag{2.116}$$

for $|k_oNR| >> 1$, where

$$R_d = \left[\rho^2 + (z+d)^2\right]^{\frac{1}{2}} ; \qquad d = \frac{2i}{k_oN} \tag{2.117}$$

Clearly, the second term of (2.116) represents the effect of an image source at a <u>complex</u> distance d from z = o. If region 2 is a good conductor, so that

$$k_oN \simeq k_2 \simeq (1+i)\delta_2$$

where $\delta_2$ is the skin depth in region 2, then $d \simeq (1+i)\delta_2$. The approximation (2.116) is known as the complex image representation for $U_1^{QS}$ .

For $V_1^{QS}$ we similarly take the quasi-static approximation on (2.94); for $|k_1R| << 1 << |k_2R|$, we thus have

$$V_1^{QS} \simeq \frac{2n_2^2}{n_2^4 - n_1^4}\,\frac{1}{k_oR} \tag{2.118}$$

No complex image interpretation is either possible or necessary for (2.118), which is quite simple as it stands.

Formulas (2.116) and (2.118) are restricted in their range of applicability as has been indicated, but within these ranges they provide quite acceptable accuracy using only elementary functions, and thus are useful when quick calculations are needed.

## §2.6 Unified Approximations for $U_1$ and $V_1$ ($|n^2| >> 1$)

In view of the considerations of the §2.4, it is clear that if the terms involving $u_2$ in the Sommerfeld integrals can be accurately approximated by a rational function of $u_1$ for both small and large $\lambda$, then a single expression can be obtained which is valid in both near and far field. Although this accuracy can be made arbitrarily high by taking a flexible enough rational function, an expression containing only a few terms can be dealt with much more conveniently when it comes to performing the differentiations necessary to compute individual field components.

When $|n^2| >>1$, it is appropriate to write $u_2$ as:

$$\begin{aligned} u_2 &= (u_1^2-N^2)^{\frac{1}{2}} = \left[(u_1-iN)^2 + 2iu_1N\right]^{\frac{1}{2}} \\ &= (u_1-iN)\left[1 + \frac{2iu_1N}{(u_1-iN)^2}\right]^{\frac{1}{2}} \end{aligned} \tag{2.119}$$

The second term inside the square brackets in now small for both $|u_1/N| << 1$ and $|u_1/N| >>1$, and is only equal in magnitude to ½ when $u_1 = -iN$. Thus, over the entire range of integration, we can write

$$u_2 = (u_1-iN) \sum_{m=o}^{\infty} \binom{\frac{1}{2}}{m} \left[\frac{2iu_1N}{(u_1-iN)^2}\right]^m \tag{2.120}$$

A very simple, unified approximation for $U_1$ and $V_1$ should thus be obtained if we retain only the m=o term of (2.120) and replace it in the appropriate integral representation:

$$\begin{aligned} U_1(z;\rho) &\simeq 2\int_0^\infty \frac{e^{-k_ou_1z}}{2u_1-iN} J_o(k_o\lambda\rho)\, \lambda d\lambda \\ &= \frac{e^{ik_1R}}{k_oR} + \frac{iN}{2} X_1(z,\rho;\, -\frac{iN}{2}) \end{aligned} \tag{2.121}$$

On the other hand, $V_1^{(o)}$ has already been expressed exactly in (2.94), so we need only approximate $V_1^{(1)}$:

$$V_1^{(1)} \approx -\frac{2n_1^2}{n_2^4-n_1^4}\int_0^\infty \frac{e^{-k_o u_1 z}}{u_1-iN+u_{2p}}\, J_o(k_o\lambda\rho)\lambda d\lambda$$

$$= -\frac{2n_1^2}{n_2^4-n_1^4}\quad \frac{e^{ik_1R}}{k_oR} + (iN-u_{2p})\; X_1(z,\rho;\, -iN+u_{2p}) \qquad (2.122)$$

These approximate expressions even have the correct behavior as $n_2^2 \to n_1^2$, and therefore might be expected to be accurate for quite a wide range of the physical parameters of the problem. This conjecture is subject, of course, to numerical confirmation by direct comparison with numerically "exact" values, as will be discussed in the next section.

These approximate representations for $U_1$ and $V_1$ can be seen to be similar to those arrived at in the analysis of a dipole over an impedance surface (Problem 1. -). The form of $V_1^{(o)}$ in (2.94) in particular is quite similar to the integral appearing in the problem of a VED over an impedance plane.

## §2.7 Numerical Results and Discussions:

Extensive numerical computation of the exact integrals as well as those based upon various approximate forms will not be included in this report. The readers however are referred to the notes given in the following section for numerical results available in the literature. We should point out that in the work of Chang and Fisher [1974], approximate results based upon the use of the incomplete Hankel function were shown to agree well with the numerically "exact" ones for a surprisingly wide range of earth parameters and observation distances.

Notes on Chapter 2

§2.1 The expression for $U(\rho)$ is undoubtedly nearly as old as the Sommerfeld problem itself. It can be traced back at least as far as the work of Fock (Fock and Bursian [1926]; Fock [1933] ) and can be found mentioned by many authors since then. The more difficult investigation of $V(\rho)$ has been taken up by Wise [1931, 1933] , Fock [1933] , Rice [1937] and Ryazin [1946] , who obtain various series expansions, not always rapidly convergent, but equivalent to the incomplete Hankel function representation. The very clever trick of using (2.8) was suggested by van der Pol [1931], and is known as the vander Pol substitution. The formula (2.10) was acually obtained earlier by Thomas [1930] , but in a much more cumbersome form. However, it was apparently not until 1965 that anyone recognized that the result of this substitution could be expressed in terms of incomplete Hankel functions (Agrest and Maksimov [1971] ). Formulas for $U_1(z;o)$ similar to (2.18) have been given by Wait [1953], Bhattacharyya [1963a] , and Kochmanova and Perov [1974].

§2.2 Only recently has it become reasonably economical to evaluate Sommerfeld integrals by direct numerical integration, the first such work apparently being that of Siegel and King [1970]. Direct numerical integration using Simpson's rule or some other straightforward quadrature are applied in Kong, Tsang and Simmons [1974], Tsang, Brown, Kong and Simmons [1974], Kong, Shen and Tsang [1977], as well as King and Sandler [1977]. The application of Shanks' algorithm to accelerate convergence is discussed by Lytle and Lager [1974] and Lager and Lytle [1975], who also discuss various possible contour deformations. Other semi-analytical quadrature methods as well as contour deformations are to be found in Wait and Fuller [1971] ; Koeffoed, Chosh and Polman [1972] , Fuller and Wait [1976], Haddad and Chang [1977], Bubenik [1977], Sarkar [1977], Kuo and Mei [1978] and Rahmat-Samii, et al. [1978]. For all these techniques, the reader is referred to the papers for programming details and additional literature citations.

§2.3 The method of steepest descents is described in many texts; see, for example, Baños [1966], Bleistein and Handelsman [1975], Brekhovskikh [1960], Budden [1966], or Felsen and Marcuvitz [1973]. The situation is complicated smewhat when evaluating the saddle point for $U_2$ or $V_2$, since the presence of $k_0u_2z_1$ in the exponent requires the solution of an equation of higher order than quadratic for $\lambda_p$. This case can be treated by making use of appropriate approximations (e.g., $|n^2| >> 1$), and the reader is referred to Baños [1966] for a fuller discussion. Two saddle points actually arise in this case, and the method may need to be generalized to account for this (see Budden [1966] or Felsen and Marcuvitz [1973]). The idea for the recursion relation (2.45) is due to Wise [1933] and to Sherman [1973]. Details of the evaluation of (2.48), sometimes called a head wave or lateral wave, can be found in Brekhovskikh [1960] or Baños [1966]. The exact form (2.55) for the numerical distance is not often used because (2.60) is more than adequate for cases of practical interest, and is the form most often quoted (see, e.g., Wait [1970a]). The data in Fig. 2.10 is based on more extensive graphs and tables published by Norton [1936, 1937]. A more formal and in some ways easier approach to finding the leading terms only of asymptotic expansions such as those of this section has been given by Yokoyama [1969, 1972, 1974, 1975], who was able to clarify several approximations implicit in the work of Baños and others. The extension of the method to treat double integrals directly (and thus avoid the rather shaky step of replacing $H_0^{(1)}(k_0\lambda_s\rho)$ by its asymptotic expansion) is done in Baños [1966] (See also Bleistein and Handelsman [1975]).

§2.4 There are a number of generalized image representations for Sommerfeld integrals in the literature. While the images in (2.77) are distributed along a line below the source and perpendicular to the interface, van der Pol [1935] gives a form with images distributed over an entire volume - a half-space below the source. Image representations distributed over a plane have been given by Briquet and Filippi [1977] and Filippi and Habault [1978]. The $X_\nu$ functions of (2.80) are generalizations of those considered by Sommerfeld [1909], which in our notation are essentially $X_\nu(z,\rho;0)$. Subsequent discussions of this special case were given by Wolf [1913], Niessen [1937], Brekhovskikh [1948], Krylov and Makarov [1960], Krylov [1961] and Brittingham et al. [1978]. The integral $X_1(z,\rho;b)$ appeared very early on in the study of the dipole problem, since it can be used in the exact representation of the problem of a dipole source above an impedance surface, which can be a good approximation to the half-space problem if $|n_2^2/n_1^2| >> 1$. Among the authors who discuss the integral form (2.80) of $X_1$ are Weyl [1919], Sommerfeld [1926], van der Pol and Niessen [1931], Murray [1932, 1933], Niessen [1933], Sawicki [1954], Dobrovol'skii [1958], and Furutsu [1959]. Abraham [1922], Malyuzhinets [1940] and Diaz and Ludford [1955] obtain this solution for the impedance surface by a method which avoids the use of Sommerfeld integrals. Expansions for $X_1$ (in the form of (2.78)) in terms of $X_m(z,\rho;0)$ were, in essence, given by Krylov and Makarov [1960]. It was apparently Chang and Fisher [1974], however, who first identified $X_1$ in terms of incomplete Hankel functions explicitly. The convergent series expansions of Sommerfeld integrals given by Krylov and Makarov [1960] and by Brittingham et al. [1978] are essentially expansions in functions similar to $X_m(z,\rho;0)$. The method of Kuo and Mei [1978], on the other hand, consists of expressing the Sommerfeld integral as a finite combination of $X_1(z,\rho;b_i)$ for some set of $b_i$. The approximate form for $V_1^{(o)}$ given by (2.97) was given by Chang and Wait [1970], and, under the additional restriction that $|k_1R/n| << 1$, can be found in somewhat simpler form in Wait [1962, 1969b, 1970b]. The asymptotic expansions (2.88) and (2.102) for $|k_2R| >> 1$ are derived here using the method of Wise [1929]. The splitting of $V_1$ into $V_1^{(o)}$ and $V_1^{(1)}$ to preserve the effect of the pole was a modification suggested by Murray [1932]. Asymptotic expansions (2.105) and (2.110) resemble the convergent series given by Krylov and Makarov [1960] and Brittingham et al. [1978] to some extent, and may in fact converge if $(n^2-1)$ is small enough (Brekhovskikh [1948]). However, the convergence rate of these series is extremely slow, and their useful range would seem to be limited in the same way as (2.105) and (2.110).

§2.5 The quasi-static forms (2.111) and (2.112) go back to the work of Foster [1931] and Fock [1933]; the reader should consult Wait [1961] and Baños [1966] for a more detailed discussion of this approximation, as well as for further references and expansions. Expansions for (2.111) have also been given by Bhattacharyya [1963b], illustrating the nontrivial nature of this function. The complex image idea can be traced back to A.D. Watt in an unpublished work in 1966. A formal derivation of (2.116) directly from (2.111) has been given by Wait [1969a] and Wait and Spies [1969]. The particular choice of d results from the happenstance that so many terms of (2.115) can be matched by the function (2.116) with this value. Wait [1969a] performs a similar matching of terms in an expansion of the integrand of (2.111) in powers of $\lambda$. Much work in the area of complex image theory has been done by Bannister (see Bannister [1967], Bannister and Dube [1978] and Bannister [1978] . In the last of these, the complex image method is generalized to eliminate the restriction $|k_1R| << 1$).

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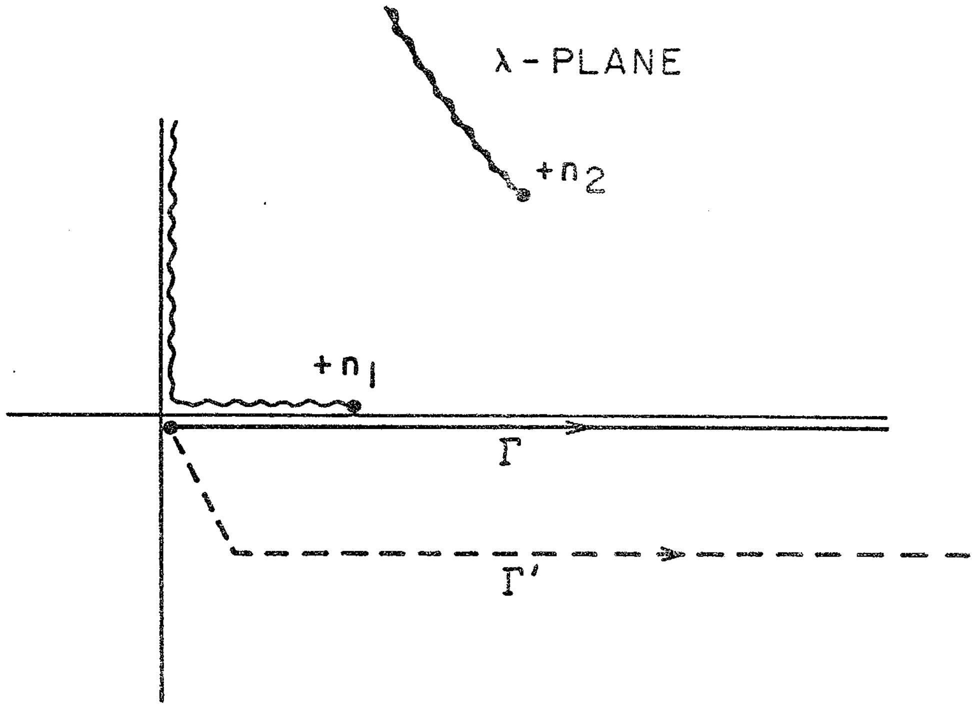


Fig. 2.1. Alternate paths of integration for (2.14).

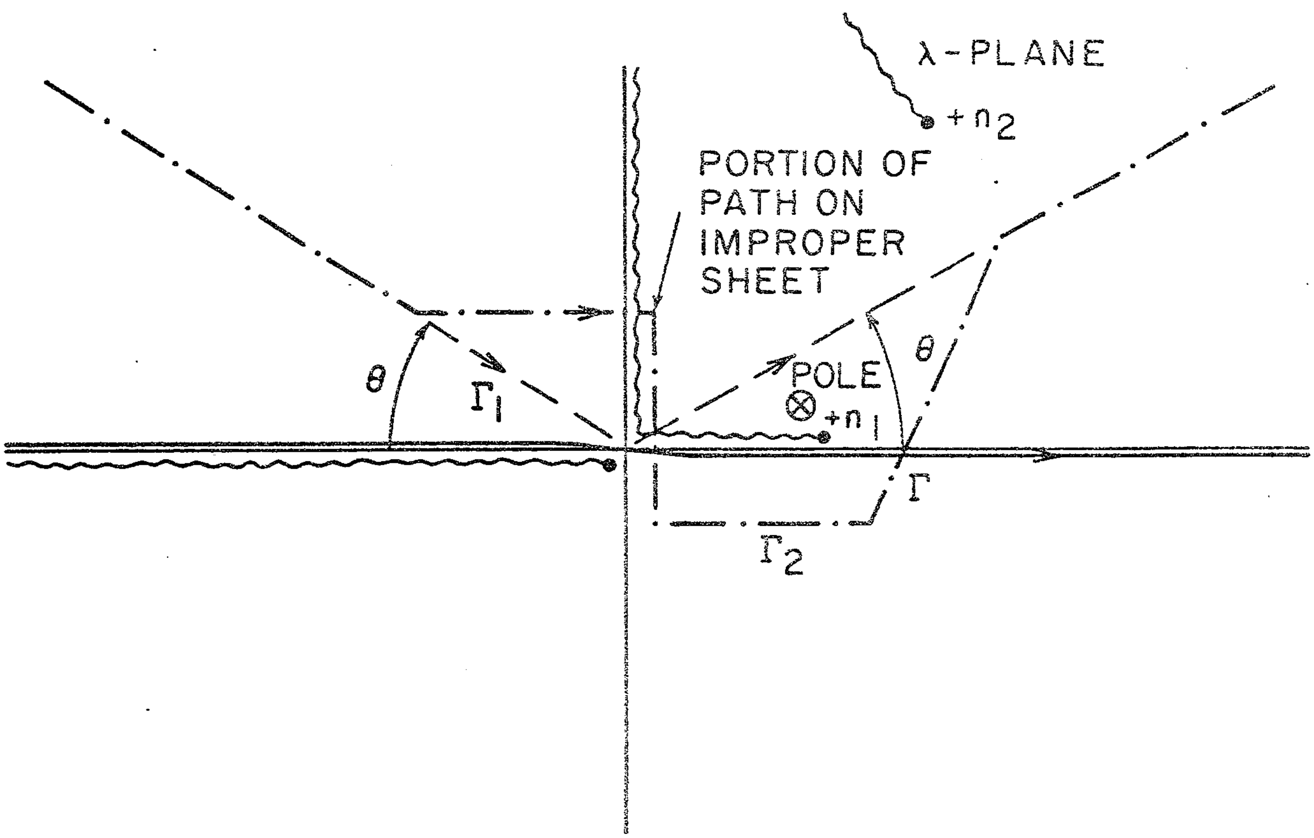


Fig. 2.2. Alternate paths of integration for (2.15). $\theta = \frac{\pi}{2} - \tan^{-1}(z/\rho)$.

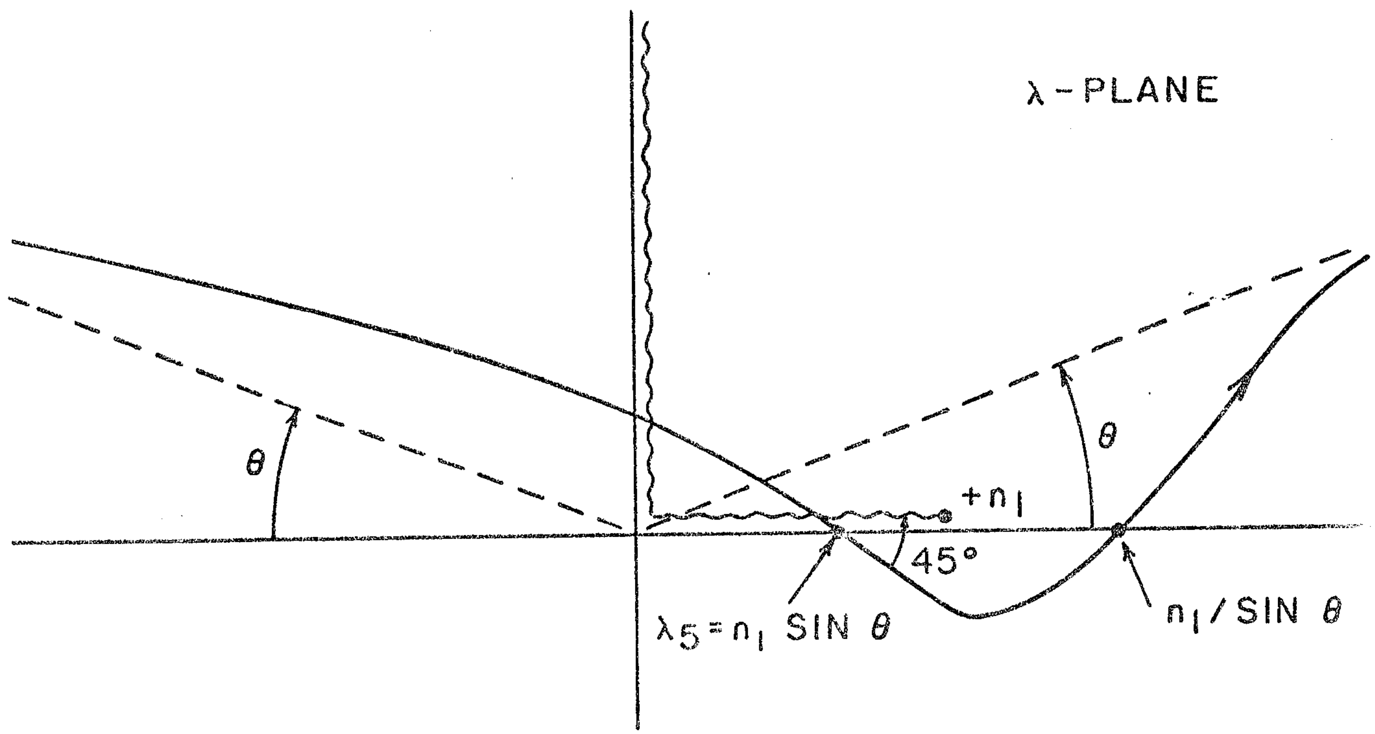


Fig. 2.3. Steepest-descent path in the λ-plane.

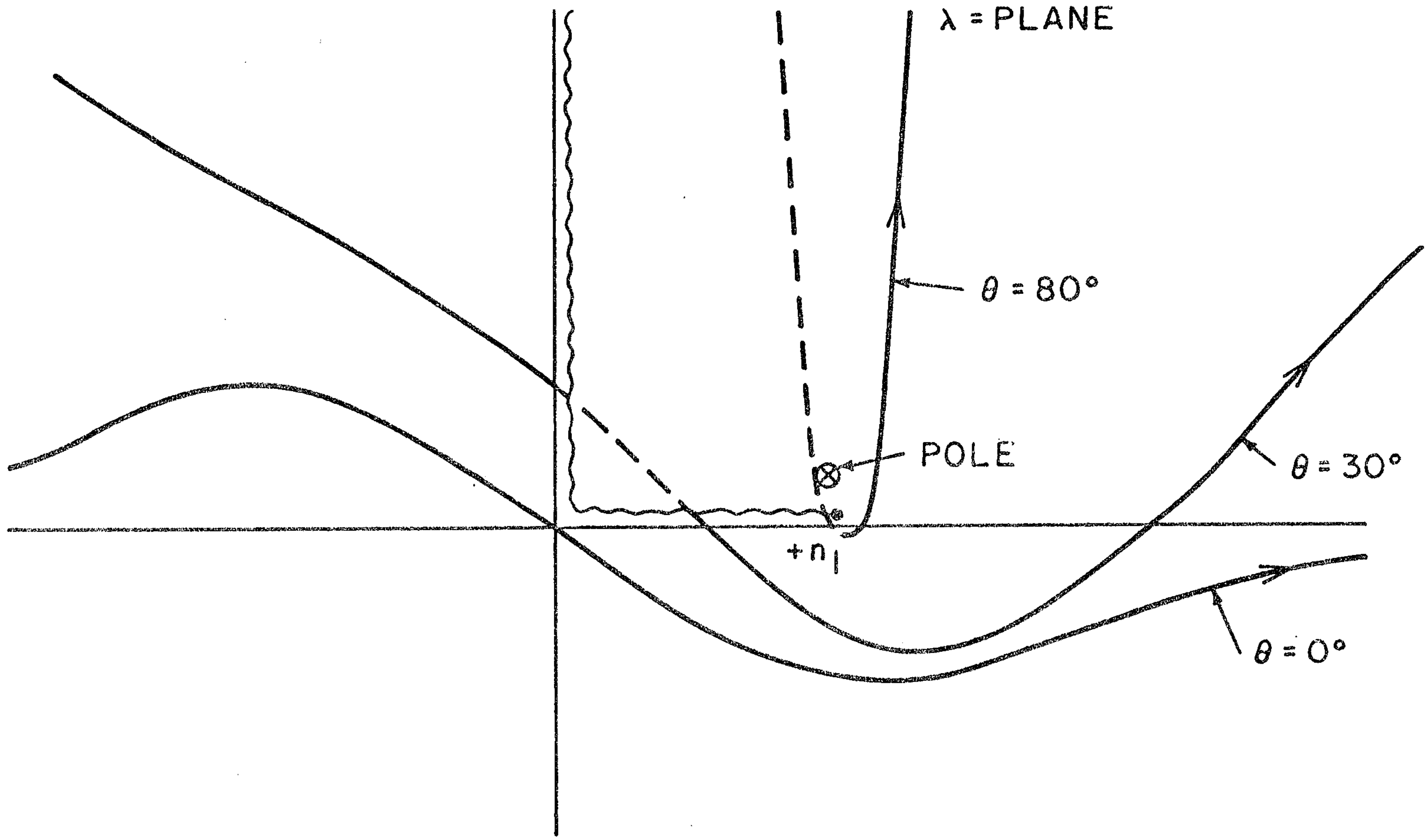


Fig. 2.4. Steepest-descent paths for $\theta = 0°, 30°, 80°$.

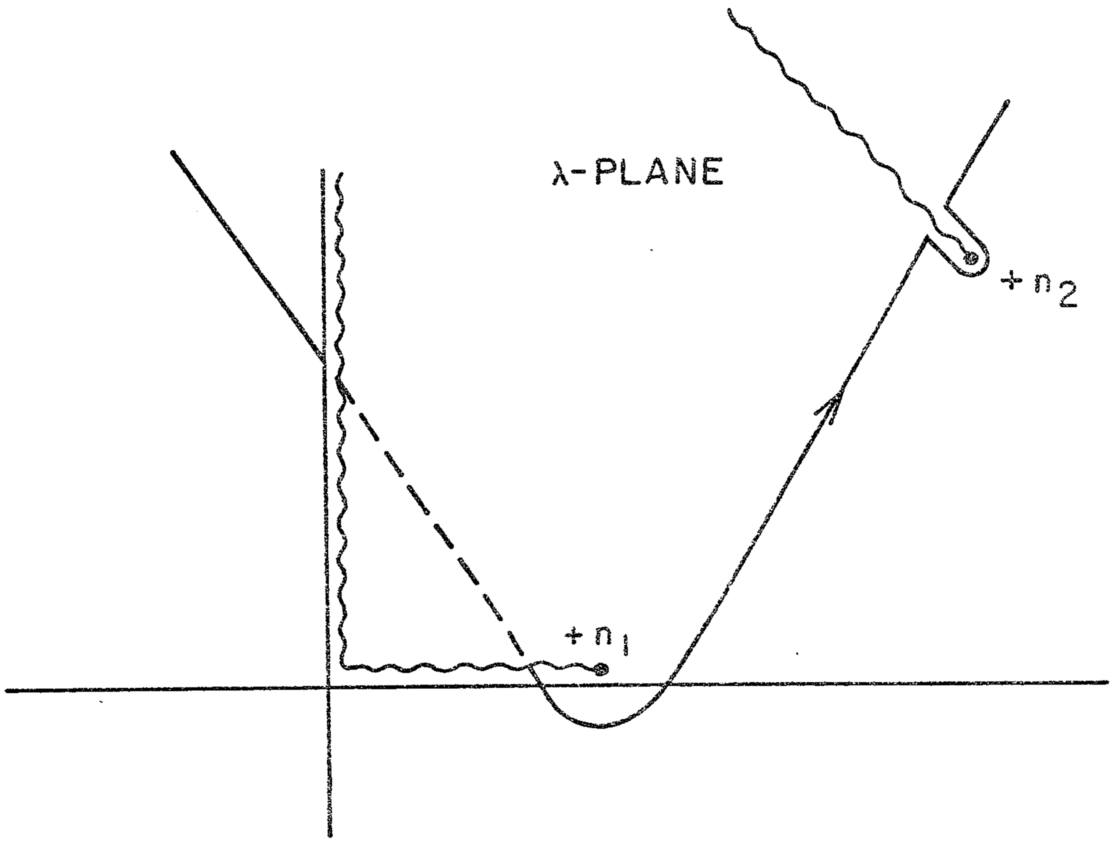


Fig. 2.5. Contribution from branch cut at $n_2$ to SDP integration.

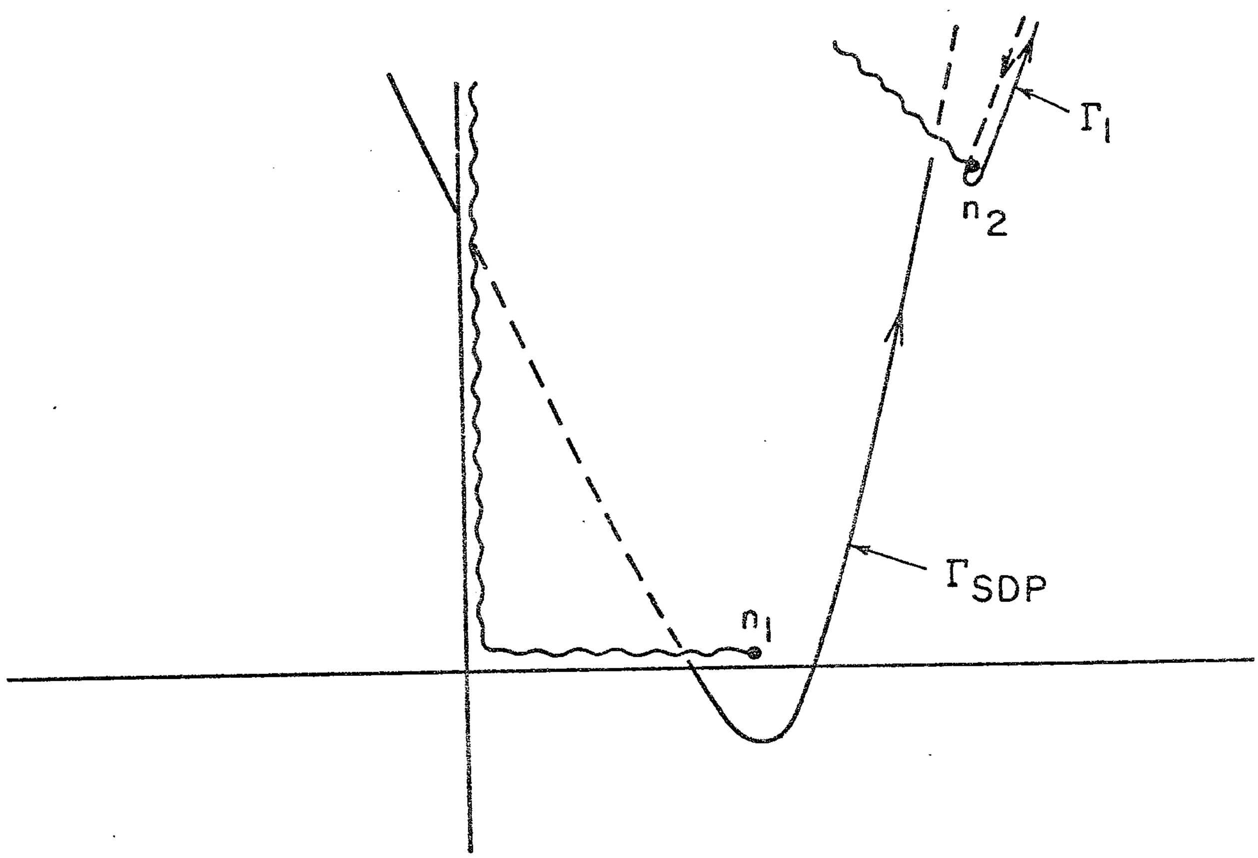


Fig. 2.6. Supplementary contour $\Gamma_1$ to SDP when $\chi < \chi_c$.

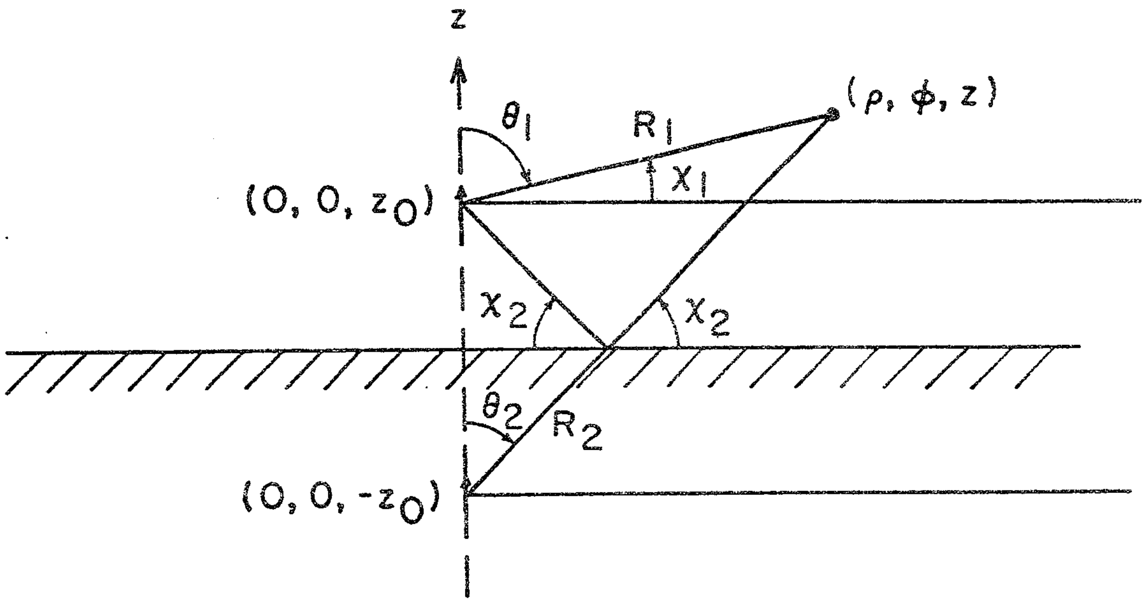


Fig. 2.7. Diagram of reflection process for the space wave.

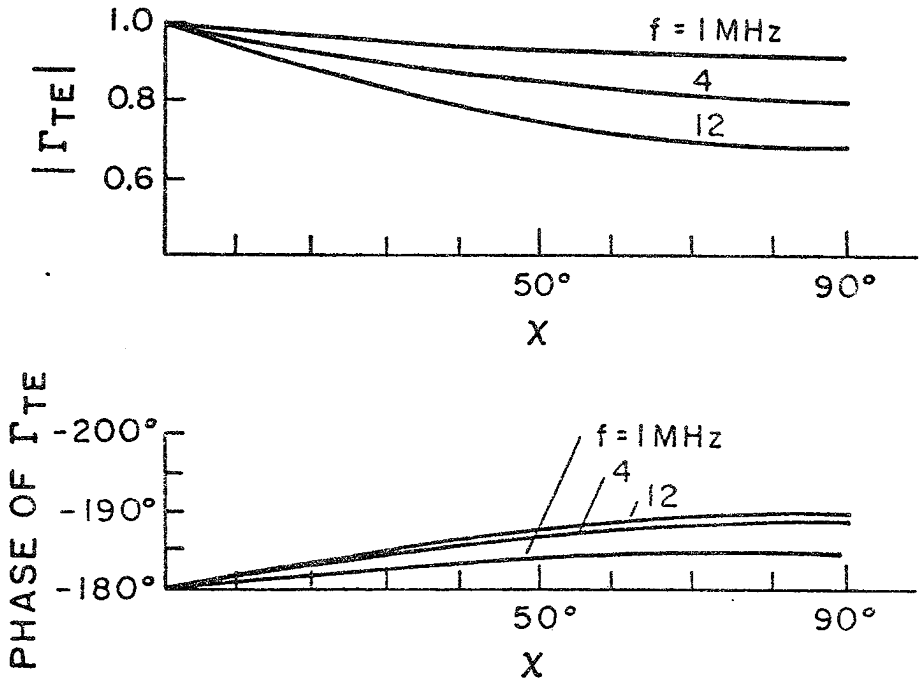


Fig. 2.8. Amplitude and phase of $\Gamma_{TE}(\theta)$ for various values of f. $\chi = \frac{\pi}{2} - \theta$ and $\sigma_2 = 12\times10^{-3}$ ℧/m; $\varepsilon_2 = 15\varepsilon_0$.

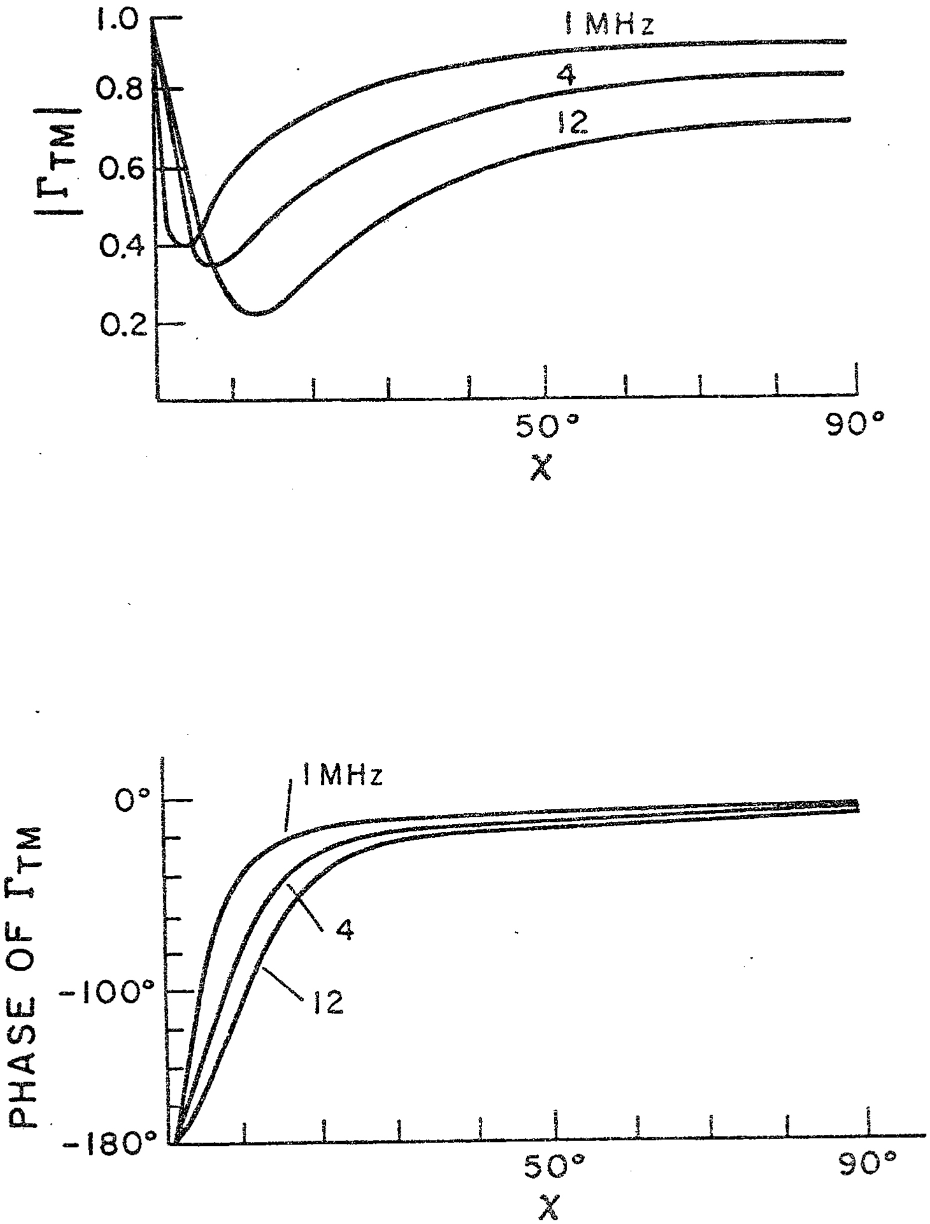


Fig. 2.9. Amplitude and phase of $\Gamma_{TM}(\theta)$ for various values of f. $\chi = \frac{\pi}{2} - \theta$ and $\sigma_2 = 12\times10^{-3}$ ℧/m; $\varepsilon_2 = 15\varepsilon_o$.

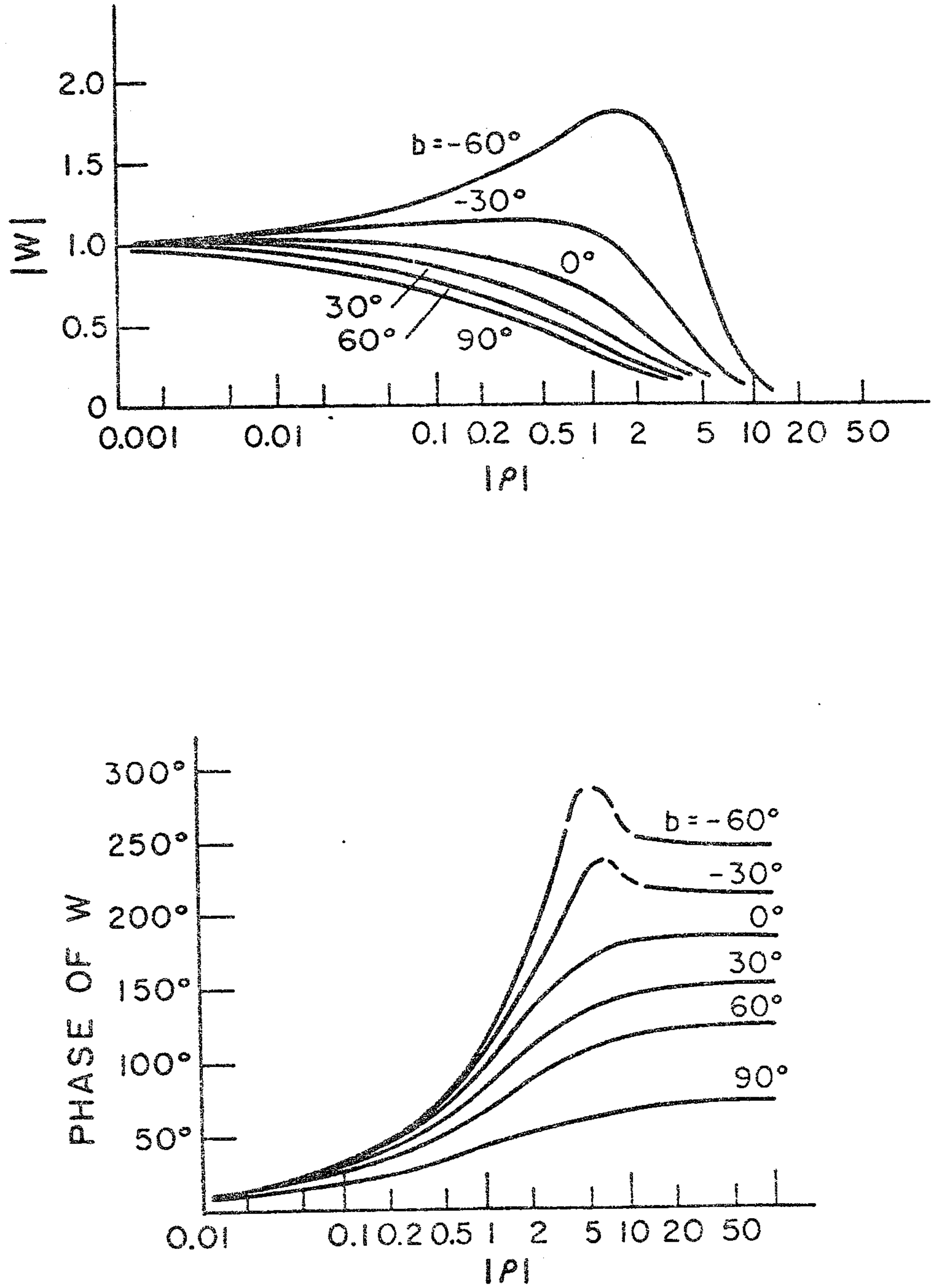


Fig. 2.10. Amplitude and phase of the attenuation function $W(\rho)$ for various values of $p = |p|e^{ib}$ (after Norton [1936-1937]).

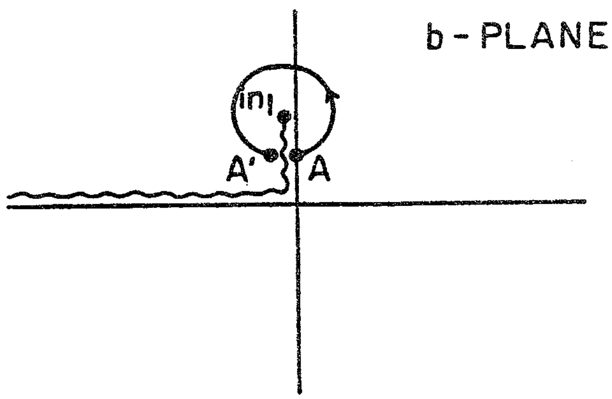


Fig. 2.11. Movement of b into left half of complex plane.

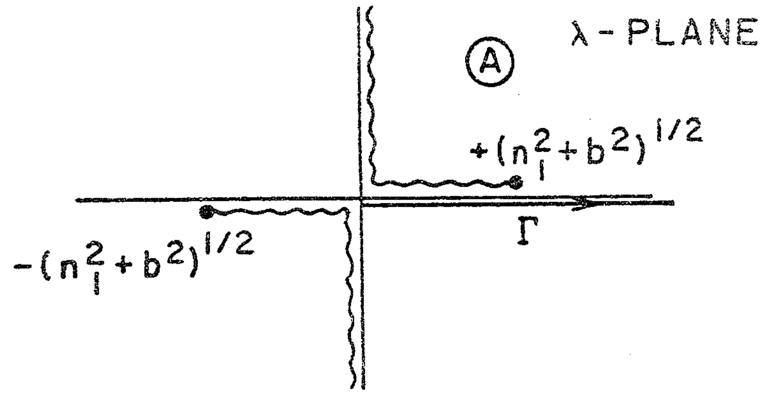


Fig. 2.12. Position of singularities and integration contour at A.

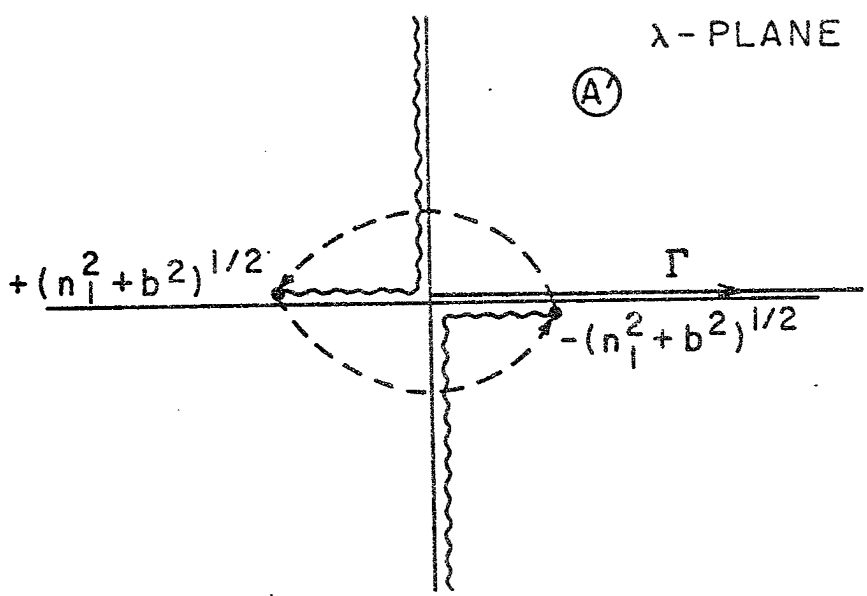


Fig. 2.13. Position of singularities and integration contour after moving to A'.

# Appendix B

## Incomplete Hankel Functions and Related Functions

In this Appendix, we define and derive a number of properties of incomplete Hankel functions and certain allied functions which play an important role in the description of the fields of sources in the vicinity of the earth's interface. The primary source for this Appendix is the book of Agrest and Maksimov [1971], but many of the results can be found only scattered throughout the literature. The references consulted have been collected at the end of this Appendix (see also further references given in these works) but individual results are not attributed to individual papers since in most cases numerious authors have independently obtained identical formulas, and we have rather attempted to collect the most useful of these here.

The incomplete Hankel function (finite form) is defined as

Need frequent use of consistent branch of $\sqrt{\ }$ give notation

$$H_0^{(1)}(a,z) = \frac{2}{i\pi} \int_0^a \frac{\exp\left[iz(1+w^2)^{\frac{1}{2}}\right]}{(1+w^2)^{\frac{1}{2}}}\, dw; \quad \mathrm{Re}(1+w^2)^{\frac{1}{2}} \geq 0 \tag{B.1}$$

The complete (ordinary) Hankel function of the first kind is then readily identified as

$$H_0^{(1)}(z) = H_0^{(1)}(\infty,z) \equiv \lim_{|a|\to\infty} H_0^{(1)}(a,z) \qquad \left[\mathrm{Im}(z) \geq 0\right]$$

through a well-known integral representation of $H_0^{(1)}(z)$. Two convergent series expansions for $H_0^{(1)}(a,z)$ can immediately be obtained through the use of (B.1).

First, we may write the exponential in (B.1) as a Taylor series and integrate term by term, obtaining

$$H_0^{(1)}(a,z) = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} C_k(a) \tag{B.2}$$

where

$$C_k(a) = \frac{2}{i\pi} \int_0^a (1+w^2)^{(k-1)/2}\, dw \tag{B.3}$$

It looks like

$$C_k \sim \frac{2c^{k+1}}{\pi\sqrt{1-c^2}\,(k+1)} - \left(\frac{2}{\pi(k+1)}\right)^{1/2}$$

where $c = (1+a^2)^{1/2}$, as $k \to +\infty$.

Integrating (B.3) by parts gives the recurrence relation

$$C_{k+2}(a) = \frac{k+1}{k+2} C_k(a) + \frac{2}{\pi i} \frac{a(1+a^2)^{(k+1)/2}}{k+2} \tag{B.4}$$

and therefore all these coefficients can be evaluated from a knowledge of the first two, which are elementary integrals:

$$\begin{aligned} C_0(a) &= \frac{2}{i\pi} \sinh^{-1}(a) = \frac{2}{i\pi} \ \ln\left[a+(1+a^2)^{\frac{1}{2}}\right] \\ C_1(a) &= \frac{2a}{i\pi} \end{aligned} \tag{B.5}$$

If a is to be extended into complex values the value of $(1+a^2)^{\frac{1}{2}}$ is to have non-negative real part and the logarithm its principal value. The series (B.2) converges for all a and z, but is most useful if neither $|a|$ nor $|z|$ is large.

We can likewise define an incomplete Hankel function of first order as

$$H_1^{(1)}(a,z) = -\frac{2z}{i\pi} \int_0^a \frac{w^2 \exp\left[iz(1+w^2)^{\frac{1}{2}}\right]}{(1+w^2)^{\frac{1}{2}}} dw \tag{B.6}$$

By differentiating (B.1) and integrating by parts, we have

$$\frac{\partial H_0^{(1)}(a,z)}{\partial z} = -H_1^{(1)}(a,z) + \frac{2a}{\pi} e^{iz(1+a^2)^{\frac{1}{2}}} \tag{B.7}$$

which generalizes the well-known form for $H_0^{(1)'}(z)$. We can obtain an expansion for $H_1^{(1)}(a,z)$ similar to (B.2) either by operating directly on (B.6), or by using (B.7) to express the series in terms of $C_k^0(a)$:

$$H_1^{(1)}(a,z) = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \left[-iC_{k+1}(a) + \frac{2a}{\pi}(1+a^2)^{\frac{1}{2}}\right] \tag{B.8}$$

A second type of series expansion for $H_0^{(1)}(a,z)$ is found by changing the variable of integration in (B.1) to $t = (1+w^2)^{\frac{1}{2}} - 1$:

$$H_0^{(1)}(a,z) = \frac{\pm 2}{i\pi} \int_0^b \frac{\exp\left[iz(t+1)\right]}{t^{\frac{1}{2}}(2+t)^{\frac{1}{2}}} dt \quad \left(\text{as } \mathrm{Re}(a) \gtrless 0;\ \text{take } \mathrm{Re}\sqrt{t} \geq 0,\ \mathrm{Re}\sqrt{t+2} \geq 0\right) \tag{B.9}$$

where $b = (1+a^2)^{\frac{1}{2}} - 1$. By expanding $(1+t/2)^{-\frac{1}{2}}$ as a binomial series in t and integrating term by term, we obtain

$$H_0^{(1)}(a,z) = \frac{\mp 2}{i\pi} e^{iz} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-2iz)^{-k-\frac{1}{2}} \gamma(k+\tfrac{1}{2}, -izb) \tag{B.10}$$

where $\binom{-\frac{1}{2}}{k} = \Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-k)\Gamma(k+1)$ is the binomial coefficient, and $\gamma(a,z)$ is the incomplete gamma function (finite form):

$$\gamma(a,z) = \int_0^z e^{-t}\, t^{a-1}\, dt \tag{B.11}$$

for which we have the recurrence formula (Appendix C)

$$\gamma(k+\frac{3}{2}, x) = (k+\tfrac{1}{2})\gamma(k+\tfrac{1}{2}, x) - x^{k+\frac{1}{2}} e^{-x} \tag{B.12}$$

The initial term is expressible in terms of the error function

$$\gamma(\tfrac{1}{2}, -izb) = \sqrt{\pi}\,\mathrm{erf}\,(e^{-i\pi/4}\sqrt{zb}) \tag{B.13}$$

for which series developments for large and small argument are available (see Appendix C). The series (B.10) converges at least for $|b| < 2$, and thus is most useful when $|a|$ is small. Converges most rapidly for $|zb| \leq 2$ or so; see Bateman HTF II, pp. 135 and 140.

When $|a|$ is large, it is more convenient to work with a complementary form of $H_0^{(1)}(a,z)$. Thus, if $\mathrm{Im}(z) \geq 0$, we have

$$\left.\begin{aligned} H_0^{(1)}(a,z) &= H_0^{(1)}(z) - \frac{2}{i\pi}\int_a^{\infty} \frac{\exp\left[iz(1+w^2)^{\frac{1}{2}}\right]}{(1+w^2)^{\frac{1}{2}}}\, dw \qquad (\mathrm{Re}(a) > 0) \\ &= -H_0^{(1)}(z) + \frac{2}{i\pi}\int_{-\infty}^{a} \frac{\exp\left[iz(1+w^2)^{\frac{1}{2}}\right]}{(1+w^2)^{\frac{1}{2}}}\, dw \qquad (\mathrm{Re}(a) < 0) \end{aligned}\right\}$$

or

$$H_0^{(1)}(a,z) = \pm H_0^{(1)}(z) \mp \frac{2}{i\pi}\int_{\pm a}^{\infty} \frac{\exp\left[iz(1+w^2)^{\frac{1}{2}}\right]}{(1+w^2)^{\frac{1}{2}}}\, dw \tag{B.14}$$

the upper and lower signs corresponding to Re(a) greater or less than zero, respectively. The change of variable $t = (1+w^2)^{\frac{1}{2}}$ in the integral remaining in

(B.14) leads to

$$\frac{2}{i\pi}\int_{(1+a^2)^{\frac{1}{2}}}^{\infty} \frac{e^{izt}}{(t^2-1)^{\frac{1}{2}}}\, dt$$

with $\mathrm{Re}(t^2-1)^{\frac{1}{2}} > 0$. If $|a|$ is large, then we can expand

$$(t^2-1)^{-\frac{1}{2}} = t^{-1}(1-1/t^2)^{-\frac{1}{2}} = t^{-1} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-t^{-2})^k$$

which can be made to converge over the entire path of integration if $|1+a^2| > 1$. Integrating term by term results in:

$$H_0^{(1)}(a,z) = \pm H_0^{(1)}(z) \mp \frac{2}{i\pi} \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 2^{2k}} (-z^2)^k \Gamma(-2k, -iz(1+a^2)^{\frac{1}{2}}) \ , \quad \mathrm{Re}(a) \gtrless 0 \qquad \text{(B.15)}$$

where $\Gamma(a,z) = \Gamma(a) - \gamma(a,z)$ is the complementary form of the incomplete gamma function. The recursion formula is

$$\Gamma(-2k-2,x) = \left[\Gamma(-2k,x) - x^{-2k-2}\, e^{-x}(x-2k-1)\right]/(2k+1)(2k+2) \qquad \text{(B.16)}$$

and the initial term is the exponential integral function

$$\Gamma(0, -iz(1+a^2)^{\frac{1}{2}}) = E_1(-iz(1+a^2)^{\frac{1}{2}}) \qquad \text{(B.17)}$$

Again, small and large argument series expansions are available in Appendix C.
Actually, converges more rapidly if $|z(1+a^2)^{\frac{1}{2}}| >> 1$
This series is clearly most useful for $|a| >> 1$. We can also obtain expansions
see Bateman, HTF II, p. 140, eq. (4).
for $H_1^{(1)}(a,z)$ similar to (B.10) and (B.15) through the use of (B.7) or by working directly on the definition (B.6).

The series derived above permit us to write efficient computer subroutines for calculating $H_0^{(1)}(a,z)$ and $H_1^{(1)}(a,z)$ for all ranges of the parameters except when $|z| >> 1$ and a is arbitrary. In principle, we could deal with this case by starting from the representation (B.9), only now we proceed to integrate by parts. Thus we have

$$I(z) \equiv \int_0^b \frac{e^{izt}}{t^{\frac{1}{2}}(2+t)^{\frac{1}{2}}}\,dt$$

$$= 2^{-\frac{1}{2}} \int_0^b t^{-\frac{1}{2}} e^{izt}\,dt + \int_0^b t^{-\frac{1}{2}} e^{izt}\left[(2+t)^{-\frac{1}{2}} - 2^{-\frac{1}{2}}\right] dt$$

$$= (-2iz)^{-\frac{1}{2}}\,\gamma(\tfrac{1}{2},-izb) + \frac{1}{iz}\left[t^{-\frac{1}{2}} e^{izt}((2+t)^{-\frac{1}{2}} - 2^{-\frac{1}{2}})\right]_0^b$$

$$-\frac{1}{iz}\int_0^b e^{izt}\frac{d}{dt}\left[t^{-\frac{1}{2}}((2+t)^{-\frac{1}{2}} - 2^{-\frac{1}{2}})\right] dt \tag{B.18}$$

$$=(-2iz)^{-\frac{1}{2}}\,\gamma(\tfrac{1}{2},\,-izb) + \frac{b^{-\frac{1}{2}}}{iz}\left[(2+b)^{-\frac{1}{2}} - 2^{-\frac{1}{2}}\right] e^{izb}$$

$$-\frac{1}{iz}\int_0^b e^{izt}t^{-\frac{1}{2}}\left\{-\frac{(2+t)^{-\frac{1}{2}} - 2^{-\frac{1}{2}}}{2t} - \tfrac{1}{2}(2+t)^{-3/2}\right\} dt$$

Clearly, this process can be repeated, adding and subtracting to the factor in the integrand of $e^{izt}t^{-\frac{1}{2}}$ its value at t=o at each step. In this way, we generate an asymptotic expansion for $H_o^{(1)}(a,z)$ of the form

$$H_o^{(1)}(a,z) \sim \frac{2e^{iz}}{i\pi}\left\{\sum_{k=o}^{\infty}\frac{A_k}{(-iz)^{k+\frac{1}{2}}}\,\gamma(\tfrac{1}{2},-izb) + \sum_{k=o}^{\infty}\frac{B_k}{(-iz)^{k+1}}\,e^{izb}\right\} \tag{B.19}$$

where, from (B.18),

$$A_o = 2^{-\frac{1}{2}} \quad ; \qquad B_o = b^{-\frac{1}{2}}\left[2^{-\frac{1}{2}} - (2+b)^{-\frac{1}{2}}\right] \tag{B.20}$$

The expansion (B.19) will simply be a rearrangement of (B.10), with like powers of -iz grouped into one term. Unfortunately, neither this observation nor a continuation of the process of (B.18) seems likely to yield recurrence relations for the $A_k$ and $B_k$ very easily.

A simple technique for obtaining these relations is to use a differential equation satisfied by $H_o^{(1)}(a,z)$ (or more precisely, I(z)). If we apply Bessel's differential operator to (B.1) and make use of (B.6) and (B.7), we find that $H_o^{(1)}(a,z)$ satisfies the inhomogeneous differential equation

$$\left[z \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z} + z\right] H_0^{(1)}(a,z) = \frac{2a}{\pi} e^{iz(1+a^2)^{\frac{1}{2}}} \tag{B.21}$$

It follows at once that

$$z \frac{\partial^2 I}{\partial z^2} + (2iz+1) \frac{\partial I}{\partial z} + iI = iae^{izb} \tag{B.22}$$

By inserting the bracketed formal series from (B.19) for I in (B.22), and equating coefficients of identical powers of $(-iz)$ to zero, we obtain the recurrence formulas

$$(k+\tfrac{1}{2})^2 A_k + 2(k+1)A_{k+1} = 0 \tag{B.23}$$

$$b(b+2)B_{k+1} + k^2 B_{k-1} + (2k+1)(b+1)B_k = \frac{b^{\frac{1}{2}}}{2(k+1)} \left[(k+1)(4k+1) - (k+\tfrac{1}{2})^2(b+2)\right] A_k \tag{B.24}$$

It is to be understood that $A_{-1}$, $B_{-1}$, $A_{-2}$, $B_{-2}$, etc. are zero. From the initial values (B.20) these relations will produce all the coefficients in the expansion (B.19).

The differential equation (B.21) can also be used to express $H_0^{(1)}(a,z)$ in terms of some related special functions which arise in applications. To achieve this, we multiply (B.21) by $H_0^{(1)}(z)$, the analogous homogeneous equation for $H_0^{(1)}(z)$ by $H_0^{(1)}(a,z)$, and subtract the two equations. The result is

$$\frac{\partial}{\partial z} \left[zH_0^{(1)}(z) \frac{\partial H_0^{(1)}(a,z)}{\partial z} - zH_0^{(1)}(a,z) \frac{\partial H_0^{(1)}(z)}{\partial z}\right] = \frac{2a}{\pi} e^{iz(1+a^2)^{\frac{1}{2}}} H_0^{(1)}(z)$$

Upon integrating, we have

$$He_0^{(1)}((1+a^2)^{\frac{1}{2}},z) \equiv \int_0^z e^{it(1+a^2)^{\frac{1}{2}}} H_0^{(1)}(t)\, dt \tag{B.25}$$

$$= \frac{\pi z}{2a} \left[H_0^{(1)}(a,z)H_1^{(1)}(z) - H_1^{(1)}(a,z)H_0^{(1)}(z)\right] + ze^{iz(1+a^2)^{\frac{1}{2}}} H_0^{(1)}(z) + \frac{iC_0(a)}{a}$$

The integral on the left side of (B.25) is known as an incomplete Lipschitz-Hankel integral, or somewhat more simply, a Schwarz function, and we have obtained a representation for it in terms of the incomplete Hankel functions of order zero and one. On the other hand, we could have performed exactly the same set of manipulations with $H_0^{(1)}(z)$ replaced by $H_0^{(2)}(z)$, with the result

$$He_0^{(2)}((1+a^2)^{\frac{1}{2}},z) \equiv \int_0^z e^{it(1+a^2)^{\frac{1}{2}}} H_0^{(2)}(t)\, dt \tag{B.26}$$

$$= \frac{\pi z}{za}\left[H_0^{(1)}(a,z)H_1^{(2)}(z)-H_1^{(1)}(a,z)H_0^{(2)}(z)\right] +ze^{iz(1+a^2)^{\frac{1}{2}}} H_0^{(2)}(z) - \frac{iC_0(a)}{a}$$

The incomplete Hankel function of zeroth order can now be eliminated between (B.25) and (B.26) and we get

$$H_0^{(1)}(a,z) = C_0(a)J_0(z) + \frac{ai}{2}\left[H_0^{(2)}(z)He_0^{(1)}((1+a^2)^{\frac{1}{2}},z)-H_0^{(1)}(z)He_0^{(2)}((1+a^2)^{\frac{1}{2}},z)\right] \tag{B.27}$$

A similar form can be obtained for $H_1^{(1)}(a,z)$.

If we temporarily assume that $\text{Im}(z)$ is strictly greater than zero in (B.25) and (B.26) and let $|z| \to \infty$, we find that

$$He_0^{(1)}((1+a^2)^{\frac{1}{2}}, \infty) = \frac{iC_0(a)}{a} = \frac{2}{\pi}\ \frac{\sinh^{-1}a}{a} \tag{B.28}$$

and, under the additional assumption that $\text{Re}\left\{-iz\left[(1+a^2)^{\frac{1}{2}}-1\right]\right\} > 0$,

$$He_0^{(2)}((1+a^2)^{\frac{1}{2}}, \infty) = -\frac{iC_0(a)}{a} \tag{B.29}$$

and these can be extended by analytic continuation to all values of a in the cut complex plane.

It might be remarked that in the special case when $a = i$, the integral representation (B.1) can be identified as a combination of Bessel and Struve functions (see Appendix C):

$$H_0^{(1)}(i,z) = J_0(z) + i\,\mathbf{H}_0(z) \tag{B.30}$$

From (B.7) it then follows that

$$H_1^{(1)}(i,z) = J_1(z) + i\,\mathbf{H}_1(z) \tag{B.31}$$

To summarize the results of this Appendix, then, we collect the various expansions for $H_0^{(1)}(a,z)$ in Table B.1 together with their regions of utility.

Table B.1

Expansions for the Incomplete Hankel Function

| Equation | Applicability | Leading Term |
|---|---|---|
| (B.2) with (B.4),(B.5) | Moderate $\lvert a\rvert$ and $\lvert z\rvert$ | $H_0^{(1)}(a,z) \simeq \frac{2}{i\pi} \sinh^{-1} a; \quad \lvert z\rvert << 1$ |
| (B.10) with (B.12),(B.13) | $\lvert(1+a^2)^{\frac{1}{2}} - 1\rvert < 2$ | $H_0^{(1)}(a,z) \simeq (-\frac{2i}{\pi z})^{\frac{1}{2}} e^{iz} \mathrm{erf}(e^{-i\pi/4}\sqrt{zb});$ $b = (1+a^2)^{\frac{1}{2}} - 1 \; ; \quad \lvert a\rvert << 1$ |
| (B.15) with (B.16),(B.17) | $\lvert 1+a^2\rvert > 1$ | $H_0^{(1)}(a,z) \simeq \pm H_0^{(1)}(z) \mp \frac{2}{i\pi} E_1\left[-iz(1+a^2)^{\frac{1}{2}}\right];$ $\mathrm{Re}(a) \gtrless 0; \quad \lvert a\rvert >> 1$ |
| (B.19) with (B.20),(B.23), (B.24) | Large $\lvert z\rvert$ | $H_0^{(1)}(a,z) \simeq (-\frac{2i}{\pi z})^{\frac{1}{2}} e^{iz} \mathrm{erf}(e^{-i\pi/4}\sqrt{zb})$ $+ i\left[(2b)^{-\frac{1}{2}} - 1/a\right] e^{iz(1+a^2)^{\frac{1}{2}}}/z \; ;$ $\lvert z\rvert >> 1$ |

The incomplete Hankel functions and a number of related functions have been tabulated for real argument, and in a few cases for purely imaginary argument, in Harvard Computation Laboratory [1949] and in Agrest and Maksimov [1971]. Tabulation for general complex values of the arguments has apparently proved too cumbersome a task, since no such tables seem to exist.

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# Appendix C

## Some Useful Special Functions

In this Appendix, we shall summarize the basic properties of a number of special functions frequently encountered in our investigations. All material is found in Abramowitz and Stegun [1965], to which the reader is referred for further formulas and references.

### §C.1 Incomplete Gamma Functions

The finite and infinite forms of the incomplete gamma function are defined, respectively, as

$$\gamma(a,z) = \int_0^z e^{-t}\, t^{a-1}\, dt \qquad (\mathrm{Re}\ a > 0) \tag{C.1}$$

$$\Gamma(a,z) = \int_z^\infty e^{-t}\, t^{a-1}\, dt \tag{C.2}$$

$$= \Gamma(a) - \gamma(a,z)$$

The recurrence relations

$$\gamma(a+1\ ,z) = a\gamma(a,z) - z^a e^{-z} \tag{C.3}$$

$$\Gamma(a+1\ ,z) = a\ \Gamma(a,z) + z^a e^{-z} \tag{C.4}$$

follow easily by integration by parts. For $\gamma(a,z)$ we have the everywhere convergent series expansions

$$\gamma(a,z) = z^a e^{-z} \sum_{m=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+m+1)}\ z^m \tag{C.5}$$

$$= z^a \sum_{m=0}^{\infty} \frac{(-z)^m}{(a+m)m!} \tag{C.6}$$

while for $\Gamma(a,z)$ we have the asymptotic expansion

$$\Gamma(a,z) \sim z^{a-1} e^{-z} \sum_{m=0}^{\infty} \frac{\Gamma(a)}{\Gamma(z-m)}\ z^{-m} \qquad \left(|z| \to \infty,\ |\arg z| < \frac{3\pi}{2}\right) \tag{C.7}$$

$$= z^{a-1}\ e^{-z} \left\{ 1 + \frac{(a-1)}{z} + \frac{(a-1)(a-2)}{z^2} + \quad \cdots \right\}$$

## §C.2 Exponential Integral Functions

A special case of $\Gamma(a,z)$ is the exponential integral function of first order:

$$E_1(z) = \Gamma(0,z) = \int_z^\infty e^{-t}\, t^{-1}\, dt \tag{C.8}$$

An asymptotic expansion for $E_1$ is obtained by letting $a \to o$ in (C.7), while a convergent power series is given by

$$E_1(z) = -\gamma - \ell n\, z - \sum_{m=1}^{\infty} \frac{(-z)^m}{mm!} \qquad (|\arg z| < \pi\ ) \tag{C.9}$$

## §C.3 Error Functions

Another special case of the incomplete gamma function occurs for $a = \frac{1}{2}$:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}}\, \gamma(\tfrac{1}{2}, z^2) \tag{C.10}$$

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}}\, \Gamma(\tfrac{1}{2}, z^2) \tag{C.11}$$

Small and large argument expansions follow immediately by putting $a = \frac{1}{2}$ in (C.5) - (C.7), and using the definitions (C.1), (C.2), (C.10), and (C.11).

## §C.4. Struve Functions

A function related to the cylindrical functions is $\mathbf{H}_n(z)$, the Struve function of nth order:

$$\mathbf{H}_n(z) = \frac{2}{\sqrt{\pi}} \frac{(z/2)^n}{\Gamma(n+\frac{1}{2})} \int_0^1 (1-t^2)^{n-\frac{1}{2}} \sin\,(zt)dt \tag{C.12}$$

$$= Y_n(z) + \frac{2}{\sqrt{\pi}} \frac{(z/2)^n}{\Gamma(n+\frac{1}{2})} \int_0^\infty e^{-zt}\, (1+t^2)^{n-\frac{1}{2}}\, dt \qquad (|\arg z| < \frac{\pi}{2}\ ) \tag{C.13}$$

There are a variety of recursion relations for the Struve functions; in particular,

$$H_0'(z) = \frac{2}{\pi} - \mathbf{H}_1(z) \tag{C.14}$$

We have the convergent series

$$\mathbf{H}_n(z) = (z/2)^{n+1} \sum_{m=o}^{\infty} \frac{(-z^2/4)^m}{\Gamma(m+\frac{3}{2})\Gamma(m+n+\frac{3}{2})} \tag{C.15}$$

The asymptotic behavior of the Struve functions is

$$\mathbf{H}_n(z) \sim Y_n(z) + \frac{1}{\pi} \sum_{m=o}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(n+\frac{1}{2}-m)} (z/2)^{n-1-2m} \tag{C.16}$$

## Reference

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