Advanced Topics in Control 2020: Large-Scale Convex Optimization

Exercise 7: Operator Splitting Methods

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Please submit your solutions via Moodle as a PDF with filename `Ex07_Surname.pdf`, replacing `Surname` with your surname.

Problem 1

Compute the proximal mapping $\mathbf{prox}_f(x)$ of the following functions:

a) Convex quadratic function: $f(x) = \frac{1}{2}x^T Ax + b^T x + c, A \in \mathbb{S}^n_+, b \in \mathbb{R}^n$;

b) Indicator function of the Euclidean ball:

$$f(x) = \mathcal{I}_C(x), C = \{x \in \mathbb{R}^n : \|x - c\|_2 \leq r\}, c \in \mathbb{R}^n, r \in \mathbb{R}_+;$$

Hint: You can derive it geometrically.

c) Indicator function of an affine set:

$$f(x) = \mathcal{I}_C(x), C = \{x \in \mathbb{R}^n : Px = q\}, P \in \mathbb{R}^{m\times n} \text{ has full row rank}, q \in \mathbb{R}^m, m \leq n;$$

d) One-dimensional: $f(x) = \lambda|x|, x \in \mathbb{R}, \lambda > 0$;
Hint: You can use the fact that $\mathbf{prox}_{\lambda f} = (\mathrm{Id} + \lambda\partial f)^{-1}(x)$ and the graphical arguments.

e) ℓ_1-norm: $f(x) = \|x\|_1 = \sum_{i=1}^n |x_i|$;
Hint: (it was shown already in the lecture using the Moreau's identity) Now show it using a separability property of the proximal operators.

f) ℓ_2-norm: $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.
Hint: Use Moreau's identity to connect it to one of the previous subtasks. You can use the result that you derived in the Exercise 4.b) of the Exercise set 2.

g) Under which condition on matrix A in the subtask 1.a) is the $\mathbf{prox}_f$ operator contractive? Derive the contraction factor.

Problem 2

a) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, closed, and convex. Prove that

1. $u = \mathbf{prox}_f(x)$ if and only if $x - u \in \partial f(u)$.
2. x is a minimizer of f if and only if $x = \mathbf{prox}_f(x)$.

Hint: Use Fermat's rule. Prove the second statement using the first one.

b) Let f be a proper, closed and convex. Show that the $\mathbf{prox}_f$ operator is nonexpansive, i.e., that for any $x, y \in \mathbb{R}^n$: $\|\mathbf{prox}_f(x) - \mathbf{prox}_f(y)\| \leq \|x - y\|$.
Hint: Let $\mathbf{prox}_f(x) = u$, $\mathbf{prox}_f(y) = v$ and apply for them the result of previous subtask 2.a)1. Then, use the subgradient inequalities, sum them together and derive that $\langle x - y, u - v\rangle \geq \|u - v\|^2$. Next, considering separately the cases when $u = v$ and $u \neq v$, and using the Cauchy-Schwarz inequality, derive the result.

Problem 3

Consider the problem

$$\min_{x \in \mathbb{R}^n} f(x) + g(x) = \min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 + \|x\|_1,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. For this problem, write down the iteration of

a) Douglas-Rachford splitting;

b) Forward-backward splitting.