Advanced Topics in Control 2020: Large-Scale Convex Optimization

Solution to Exercise 1

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1 Convex Sets

(a) It is a convex set since it can be written as an intersection of two halfspaces. Indeed,

$$\{x \in \mathbb{R}^n : a_1^\top x \leq b_1,\, a_2^\top x \leq b_2\} = \bigcap_{i=1}^{2} \{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}.$$

(b) We will show convexity by using the definition. Let $x, x' \in B(x_0, r)$ and let $\theta \in [0,1]$. We then have

$$\begin{aligned}
\|(\theta x + (1-\theta)x') - x_0\| &= \|\theta(x - x_0) + (1-\theta)(x' - x_0)\| \\
&\leq \|\theta(x - x_0)\| + \|(1-\theta)(x' - x_0)\| \\
&\leq \theta\|(x - x_0)\| + (1-\theta)\|(x' - x_0)\| \\
&\leq \theta r + (1-\theta) r = r,
\end{aligned}$$

where we used the triangle inequality, the homogeneity property of norms and that $x, x' \in B(x_0, r)$ in the last three lines, respectively .

(c) We have $\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} = \cap_{y \in S} C_y$, where $C_y := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq \|x - y\|_2\}$. We will show that $\{C_y\}_{y \in S}$ is a family of halfspaces and thus $\cap_{y \in S} C_y$ is a convex set. Indeed, for a fixed $y \in S$, we have

$$\begin{aligned}
\|x - x_0\|_2 \leq \|x - y\|_2 \quad &\Leftrightarrow \quad \|x - x_0\|_2^2 \leq \|x - y\|_2^2 \\
&\Leftrightarrow \quad (x - x_0)^\top (x - x_0) \leq (x - y)^\top (x - y) \\
&\Leftrightarrow \quad 2x^\top (y - x_0) \leq y^\top y - x_0^\top x_0.
\end{aligned}$$

Therefore, $C_y = \{x \in \mathbb{R}^n : s_y^\top x \leq r_y\}$, where $s_y := 2(y - x_0) \in \mathbb{R}^n$ and $r_y := y^\top y - x_0^\top x_0 \in \mathbb{R}$.

2 Convex Combinations and Convex Hulls

(a) The condition is sufficient, since convex combinations of two elements $x, x' \in C$ just make up the line segment joining them, i.e., the set $[x, x'] = \{\theta x + (1-\theta)x' : \theta \in [0,1]\}$. To prove necessity, let $x_1, \ldots, x_k \in C$ and let $a = (a_1, \ldots, a_k) \in \mathbb{R}_+$ with $\sum_{i=1}^k a_i = 1$. We want to show that if C is convex, then $\sum_{i=1}^k a_i x_i \in C$. To this end, we will use mathematical induction on the number of elements k. The claim holds trivially for k=1,2 by the definition of convexity. Assume that it holds for $k-1$. One at least of the a_i's is positive, say $a_1 > 0$. Then, we can write

$$\sum_{i=1}^{k} a_i x_i = \left(\sum_{i=1}^{k-1} a_i\right) \underbrace{\frac{\sum_{i=1}^{k-1} a_i x_i}{\sum_{i=1}^{k-1} a_i}}_{=:y_{k-1}} + a_k x_k.$$

By the induction hypothesis we get $y_{k-1} \in C$ and so by the convexity of C, we conclude that $\sum_{i=1}^{k} a_i x_i \in C$.

(b) Let $A := \bigcap\{C : C \,\text{is convex and}\, S \subset C\}$ and $B := \{\sum_{i=1}^{k} a_i x_i : k \in \mathbb{N}, \{a_i\}_{i=1}^{k} \subset [0,\infty), \{x_i\}_{i=1}^{k} \subset S, \sum_{i=1}^{k} a_i = 1\}$. We will show that $A = B$. It is direct that B is convex. Moreover, $S \subset B$. Since A is the smallest convex set containing S, we get $A \subset B$. For the inverse inclusion, let $x_1, \ldots, x_k \in S$ and $a_1, \ldots, a_k \geq 0$ with $\sum_{i=1}^{k} a_i = 1$. Since $S \subset A$, we have in particular that $\{x_i\}_{i=1}^{k} \subset A$. Therefore, by question (2a) and the convexity of A, we get $\sum_{i=1}^{k} a_i x_i \in A$. This proves that $B \subset A$.

(c) We have that $\operatorname{conv} S = \{\sum_{i=1}^{k} a_i x_i : k \in \mathbb{N}, \{a_i\}_{i=1}^{k} \subset [0,\infty), \{x_i\}_{i=1}^{k} \subset S, \sum_{i=1}^{k} a_i = 1\}$. Consider a convex combination $\sum_{i=1}^{k} a_i x_i$. It may happen that several of the x_i's belong to the same C_j. To simplify notation suppose that $x_{k-1}, x_k \in C_1$; assume also $a_k > 0$. Then set $(\beta_i, y_i) = (a_i, x_i)$, for $i = 1, \ldots, k-2$ and

$$\beta_{k-1} := a_{k-1} + a_k, \qquad y_{k-1} = \frac{1}{\beta_{k-1}}(a_{k-1}x_{k-1} + a_k x_k) \in C_1,$$

so that $\sum_{i=1}^{k} a_i x_i = \sum_{i=1}^{k-1} \beta_i y_i$. That is, our initial convex combination can also be found among those with $k-1$ elements.

3 Polar Cone and Separation of Convex Sets

(a) Let $s \in K^\circ$. Then, $s^\top x \leq 0$, for all $x \in K$. In particular, since $x_i \in K$, we get $s^\top x_i \leq 0$, for all $i = 1, \ldots, m$. Thus, $K^\circ \subset \{s \in \mathbb{R}^n : s^\top x_j \leq 0 \,\text{for}\, j = 1, \ldots, m\}$. On the other hand, if $s^\top x_i \leq 0$, for all $i = 1, \ldots, m$, then $s^\top(\sum_{i=1}^{m} a_i x_i) \leq 0$, for all $a_i \geq 0$, proving that $s \in K^\circ$.

(b) One can see that $K \subset K^{\circ\circ}$. Indeed, if $x \in K$, then $x^\top s \leq 0$, for all $s \in K^\circ$. Thus, $x \in K^{\circ\circ}$. Moreover. since $K^{\circ\circ}$ is closed (a polar cone is always closed), we get $\operatorname{cl} K \subset K^{\circ\circ}$.

We will now show the inverse inclusion $K^{\circ\circ} \subset \operatorname{cl} K$. Equivalently, if $x \notin \operatorname{cl} K$, then $x \notin K^{\circ\circ}$. Let $x \notin \operatorname{cl} K$. Since $\operatorname{cl} K$ is closed and convex, there exists a separating hyperplane associated to a nonzero vector $s \in \mathbb{R}^n$ and a scalar $r \in \mathbb{R}$, such that

$$s^\top k < r < s^\top x, \;\text{ for all } k \in \operatorname{cl} K.$$

Since $0 \in \operatorname{cl} K$, we get $r > 0$. Moreover, $s^\top(\lambda k) < r$, for all $\lambda \in \mathbb{N}$ and $k \in K$. Therefore $s^\top k < \frac{r}{\lambda} \to 0$, as $\lambda \to \infty$. Thus, $s^\top k \leq 0$, for all $k \in K$ and so $s \in K^\circ$. Since, $s^\top x > r \geq 0$, we get $x \notin K^{\circ\circ}$.

(c) We have that $K = \{\sum_{j=1}^{m} a_j x_j : a_j \geq 0 \,\text{for}\, j = 1, \ldots, m\}$ is closed. Therefore, by (a) and (b) the polar of $\{s \in \mathbb{R}^n : s^\top x_j \leq 0 \,\text{for}\, j = 1, \ldots, m\}$ is the bipolar $K^{\circ\circ} = K$.

4 Normal Cone and Tangent Cone

(a) The result follows since $N_C(x)$ can be written as an intersection of closed halfspaces. Indeed, we have $N_C(x) = \bigcap_{y \in C}\{s \in \mathbb{R}^n : s^\top(y - x) \le 0\}$, where each halfspace is a closed convex set. In addition, one can see easily that $N_C(x)$ is a cone.

(b) For every $x \in C$ we have $N_C(x) = \text{cone}\{s_i : i \in I(x)\} = \{\sum_{i \in I(x)} a_i s_i : a_i \ge 0\}$, where $I(x) = \{i = 1, \ldots, m : s_i^\top x = r_i\}$ is the index set of *active constraints* at $x \in C$. Moreover, $T_C(x) = N_C(x)^\circ = \{y \in \mathbb{R}^n : s_i^\top y \le 0, \text{for}\, i \in I(x)\}$, where we used results from Task 3. In particular, for any interior point x of C, we have $N_C(x) = \{0\}$ and $T_C(x) = \mathbb{R}^n$.