Convex Sets

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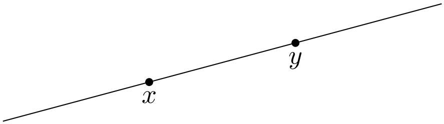
Affine set

- **line** through $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$:

$$\{\theta x + (1-\theta)y \mid \theta \in \mathbb{R}\}$$



- a set $V \subseteq \mathbb{R}^n$ is **affine** if it contains the line through any two distinct points in the set, *i.e.*, for every $x, y \in V$ and $\theta \in \mathbb{R}$:

$$\theta x + (1-\theta)y \in V$$

- **example:** solution set of linear equations $\{x \in \mathbb{R}^n \mid Ax = b\}$

  (conversely, every affine set can be expressed as solution set of system of linear equations)

Convex set

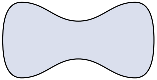
- **line segment** between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$:

$$\{\theta x + (1-\theta)y \mid \theta \in [0,1]\}$$

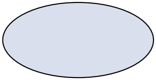
- a set $\mathcal{C} \subseteq \mathbb{R}^n$ is **convex** if it contains the line segment through any two distinct points in the set, *i.e.*, for every $x, y \in \mathcal{C}$ and $\theta \in [0,1]$:

$$\theta x + (1-\theta)y \in \mathcal{C}$$

- **examples:**

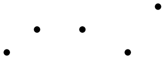


nonconvex



convex

nonconvex



nonconvex

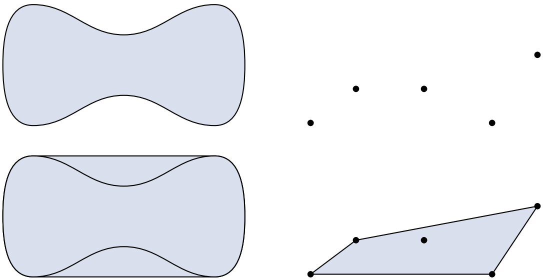
- we will assume that all sets are nonempty and closed

Convex combination and convex hull

- **convex combination** of $x_1, \ldots, x_k$ are all points of the form

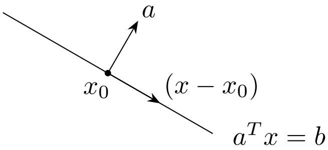
$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

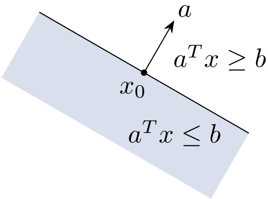
  where $\theta_1 + \ldots + \theta_k = 1$ and $\theta_i \geq 0$
- **convex hull** $(\operatorname{conv} \mathcal{S})$ is the set of all convex combinations of points in $\mathcal{S} \subseteq \mathbb{R}^n$
- **examples:**



Hyperplanes and halfspaces

- **hyperplane** is a set of the form $\{x \in \mathbb{R}^n \mid a^T x = b\}$ $(a \neq 0)$
- **halfspace** is a set of the form $\{x \in \mathbb{R}^n \mid a^T x \leq b\}$ $(a \neq 0)$





- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex
- dimension of hyperplane in $\mathbb{R}^n$ is $n - 1$

  (conversely, every affine set in $\mathbb{R}^n$ of dimension $n - 1$ is a hyperplane)

Euclidean balls and ellipsoids

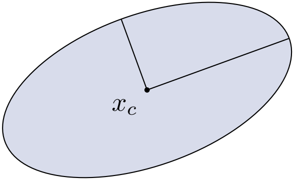
- **(Euclidean) ball** with center $x_c \in \mathbb{R}^n$ and radius $r > 0$:

$$B(x_c, r) = \{x \in \mathbb{R}^n \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

- **ellipsoid:**

$$\{x \in \mathbb{R}^n \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

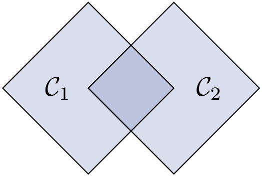
  with $P \in \mathbb{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



- other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$

Intersection and Cartesian product

- intersection of (any number of) convex sets is convex
- union of convex sets need not be convex



- Cartesian product of convex sets is convex
- **example:** $\mathcal{C} = [-1, 1] \times [0, 1]$

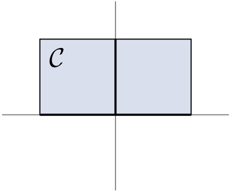


Image and inverse image of set

- let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be an affine map:

$$f(x) = Ax + b$$

  with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

- the image of a convex set under f is convex

$$\mathcal{S} \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(\mathcal{S}) = \{f(x) \mid x \in \mathcal{S}\} \text{ convex}$$

- the inverse image of a convex set under f is convex

$$\mathcal{C} \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(\mathcal{C}) = \{x \in \mathbb{R}^n \mid f(x) \in \mathcal{C}\} \text{ convex}$$

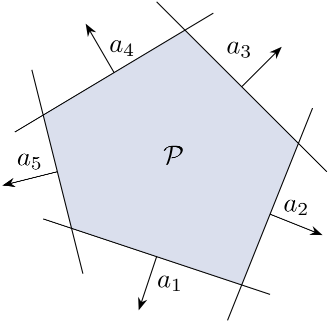
- **examples:**
  - scaling, translation, rotation, projection
  - ellipsoid is the image of a ball under an affine map

Polyhedra

- solution set of finitely many inequalities and equalities:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \preceq b,\ Cx = d\}$$

($A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality)

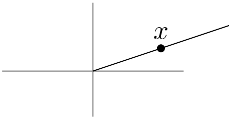


- polyhedron is intersection of finite number of halfspaces and hyperplanes

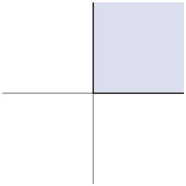
Cones

- a set $\mathcal{K} \subseteq \mathbb{R}^n$ is a **cone** if it contains the full ray through any point in the set, *i.e.*, for every $x \in \mathcal{K}$ and $\theta \geq 0$:

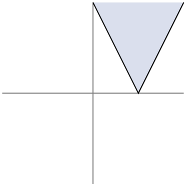
$$\theta x \in \mathcal{K}$$



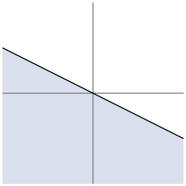
- which of the following figures are cones? which are convex cones?



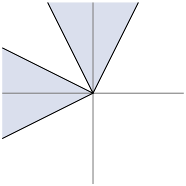
convex cone



not a cone



convex cone



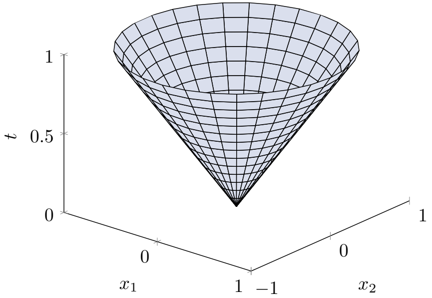
nonconvex cone

Norm balls and norm cones

- **norm** is a function $\|\cdot\|$ that satisfies
  - $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
  - $\|tx\| = |t|\,\|x\|$ for $t \in \mathbb{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$
- **norm ball** with center $x_c \in \mathbb{R}^n$ and radius $r > 0$:

$$\{x \in \mathbb{R}^n \mid \|x - x_c\| \leq r\}$$

- **norm cone**: $\{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$
- norm balls and norm cones are convex
- Euclidean norm cone is called second-order cone



Dual and polar cones

- **dual cone** of a cone $\mathcal{K} \subseteq \mathbb{R}^n$:

$$\mathcal{K}^* = \{y \in \mathbb{R}^n \mid y^T x \geq 0 \text{ for all } x \in \mathcal{K}\}$$

- if $\mathcal{K}$ is convex, then $\mathcal{K}^{**} = \mathcal{K}$
- a cone is **self-dual** if $\mathcal{K} = \mathcal{K}^*$
- **polar cone** of a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$:

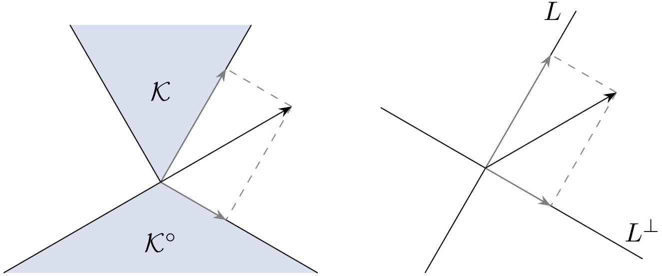
$$\mathcal{K}^\circ = \{y \in \mathbb{R}^n \mid y^T x \leq 0 \text{ for all } x \in \mathcal{K}\} = -\mathcal{K}^*$$

Moreau decomposition

- let $\Pi_{\mathcal{C}}$ denote the (Euclidean) projection of $x \in \mathbb{R}^n$ onto a set $\mathcal{C}$
- **Moreau decomposition:** given a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, we can decompose any $x \in \mathbb{R}^n$ as:

$$x = \Pi_{\mathcal{K}}(x) + \Pi_{\mathcal{K}^\circ}(x)$$

  where $\Pi_{\mathcal{K}}(x) \perp \Pi_{\mathcal{K}^\circ}(x)$
- Moreau decomposition generalizes orthogonal decomposition induced by a subspace L

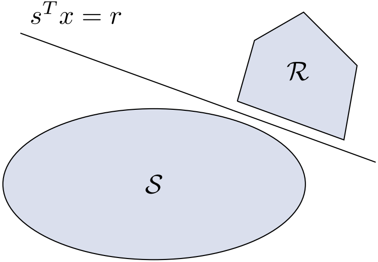


Separating hyperplane

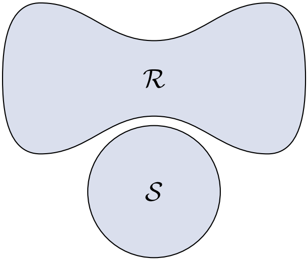
- suppose that $\mathcal{R}, \mathcal{S} \subseteq \mathbb{R}^n$ are two non-intersecting convex sets
- then there exists $s \neq 0$ and r such that

$$\begin{aligned} s^T x \leq r & \qquad \text{for all } x \in \mathcal{R} \\ s^T x \geq r & \qquad \text{for all } x \in \mathcal{S} \end{aligned}$$

- the hyperplane $\{x \in \mathbb{R}^n \mid s^T x = r\}$ is called **separating hyperplane**



example

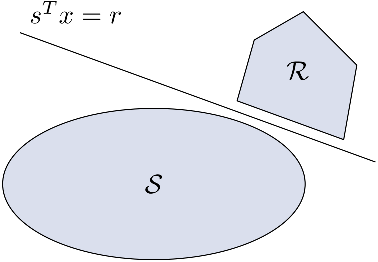


counterexample
($\mathcal{R}$ nonconvex)

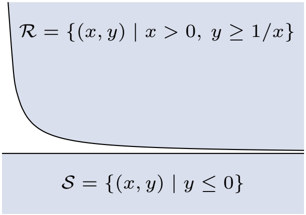
Strictly separating hyperplane

- suppose that $\mathcal{R}, \mathcal{S} \subseteq \mathbb{R}^n$ are two non-intersecting convex sets and that one of them is bounded
- then there exists $s \neq 0$ and r such that

$$s^T x < r \qquad \text{for all } x \in \mathcal{R}$$
$$s^T x > r \qquad \text{for all } x \in \mathcal{S}$$



example



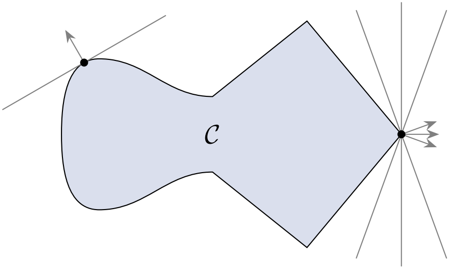
counterexample
($\mathcal{R}, \mathcal{S}$ not bounded)

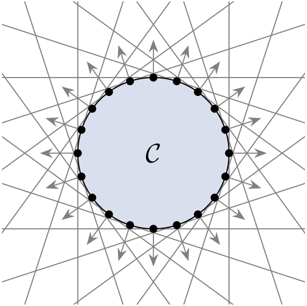
Supporting hyperplane theorem

- the hyperplane $\{x \in \mathbb{R}^n \mid s^T x = r\}$ supports $\mathcal{C} \subseteq \mathbb{R}^n$ at boundary point x_0 if

$$s^T x \leq r \text{ for all } x \in \mathcal{C} \qquad \text{and} \qquad s^T x_0 = r$$

- such hyperplane is called **supporting hyperplane**
- if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of $\mathcal{C}$



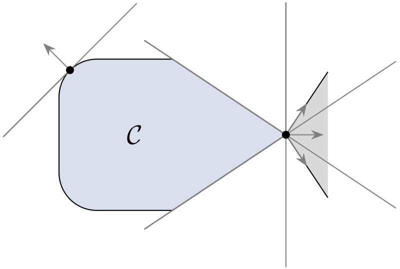


Normal cone operator

- the normal cone operator to a set $\mathcal{C} \subseteq \mathbb{R}^n$:

$$N_{\mathcal{C}}(x) = \{s \in \mathbb{R}^n \mid s^T(y - x) \leq 0 \text{ for all } y \in \mathcal{C}\}$$

  (the operator is defined on $\mathcal{C}$)

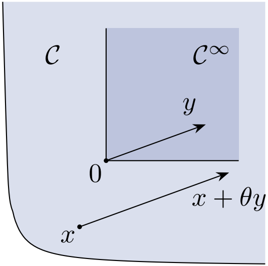


- each element in $N_{\mathcal{C}}(x)$ defines a supporting hyperplane to $\mathcal{C}$ at x
- the polar cone of a normal cone is called the *tangent cone*

Recession cone

- a vector $y \in \mathbb{R}^n$ is a **direction of recession** of a set $\mathcal{C} \subseteq \mathbb{R}^n$ if $\mathcal{C}$ includes all rays in the direction y that start at any $x \in \mathcal{C}$
- **recession cone** of a set $\mathcal{C} \subseteq \mathbb{R}^n$ is the set of all directions of recession, *i.e.*,

$$\mathcal{C}^\infty = \{y \in \mathbb{R}^n \mid x + \theta y \in \mathcal{C} \text{ for all } x \in \mathcal{C} \text{ and } \theta \geq 0\}$$



- recession cone of a convex set is a convex cone
- the polar cone of a recession cone is called the *barrier cone*

Generalized inequalities

- a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ is a **proper cone** if it is
  - solid (has nonempty interior)
  - pointed (contains no line)
- **generalized inequality** defined by a proper cone $\mathcal{K}$:
  - $x \leq_{\mathcal{K}} y$ means that $y - x \in \mathcal{K}$
  - $x <_{\mathcal{K}} y$ means that $y - x \in \operatorname{int} \mathcal{K}$
- many properties of $\leq_{\mathcal{K}}$ are similar to $\leq$ on $\mathbb{R}$, *e.g.*,

$$x \leq_{\mathcal{K}} y, \quad u \leq_{\mathcal{K}} v \quad \Longrightarrow \quad x + u \leq_{\mathcal{K}} y + v$$

- $\leq_{\mathcal{K}}$ is not in general a *linear ordering*
  - we can have $x \not\leq_{\mathcal{K}} y$ and $y \not\leq_{\mathcal{K}} x$

References

- these lecture notes are based to a large extent on the following material:
  - Stanford EE364a class developed by Stephen Boyd
  - Lund course on Large-Scale Convex Optimization developed by Pontus Giselsson

- the original slides can be downloaded from

  `https://web.stanford.edu/class/ee364a/lectures.html`

  `https://archive.control.lth.se/ls-convex-2015/`