Convex Optimization.

1. Convex sets.

(a) Assume $x, y \in \{x \in \mathbb{R}^n : a_1^T x \le b_1,\ a_2^T x \le b_2\} =: S$

$\forall \theta \in [0,1]$ check $\theta x + (1-\theta) y \in S$.

$$a_1^T(\theta x + (1-\theta)y) = \theta a_1^T x + (1-\theta) a_1^T y \le \theta b_1 + (1-\theta) b_1 = b_1$$

$$a_2^T(\theta x + (1-\theta)y) = \theta a_2^T x + (1-\theta) a_2^T y \le \theta b_2 + (1-\theta) b_2 = b_2$$

Hence $\theta x + (1-\theta) y \in S$.

Hence S is convex.

(b). Assume $x, y \in B(x_0, r)$

$\forall \theta \in [0,1]$.

$$\begin{aligned}\|\theta x + (1-\theta) y - x_0\| &= \|\theta(x-x_0) + (1-\theta)(y-x_0)\| \\ &\le \|\theta(x-x_0)\| + \|(1-\theta)(y-x_0)\| \\ &= \theta\|x-x_0\| + (1-\theta)\|y-x_0\| \\ &\le \theta \cdot r + (1-\theta) r \\ &= r\end{aligned}$$

Hence $\theta x + (1-\theta) y \in B(x_0, r)$

$\Rightarrow$ $B(x_0, r)$ is convex.

(c) For fixed y. the set
$$\begin{aligned} C_y :&= \{x \in \mathbb{R}^n : \|x - x_0\|_2 \le \|x-y\|_2\} \\ &= \{x \in \mathbb{R}^n : (x-x_0)^T(x-x_0) \le (x-y)^T(x-y)\} \\ &= \{x \in \mathbb{R}^n : 2(y-x_0)^T x \le y^T y - x_0^T x_0\}\end{aligned}$$
is an affine halfspace.

$\forall x_1, x_2 \in C := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \le \|x-y\|_2 \text{ for all } y \in S\}.$
$$= \{x \in \mathbb{R}^n : 2(y-x_0)^T x \le y^T y - x_0^T x_0,\ \forall y \in S\}.$$

$\forall \theta \in [0,1]$. check $\theta x_1 + (1-\theta) x_2$.

$\forall y \in S.$
$$\begin{aligned} 2(y-x_0)^T[\theta x_1 + (1-\theta) x_2] &= \theta \cdot 2(y-x_0)^T x_1 + (1-\theta) \cdot 2(y-x_0)^T x_2 \\ &\le \theta \cdot (y^T y - x_0^T x_0) + (1-\theta) \cdot (y^T y - x_0^T x_0) \\ &= y^T y - x_0^T x_0\end{aligned}$$

Hence $\theta x_1 + (1-\theta) x_2 \in C$.

Hence C is convex.

2. Convex Combinations and Convex Hulls.

(a) By induction.

$k=1$ is trivial. as C contains every its elements.

$k=2$.. $\forall x, y \in C$. $\theta x + (1-\theta) y \in C$, $\theta \in [0,1]$.

Assume for $k=n$. it holds.

then $\forall x_1, x_2 .. x_n$,

$$\theta_1 x_1 + \theta_2 x_2 + .. + \theta_n x_n \in C, \quad \sum_{i=1}^{n} \theta_i = 1$$

Consider $k=n+1$, $\forall x_1, x_2 .. x_n, x_{n+1}$

$\forall \lambda \in [0,1]$. we have

$$\lambda x_n + (1-\lambda) x_{n+1} \in C.$$

Let $y = \lambda x_n + (1-\lambda) x_{n+1} \in C$.

Consider $x_1, x_2 ... x_{n-1}, y$.

then. $\forall \sum_{i=1}^{n} \theta_i = 1$, $\theta_i \geq 0$.

we have by induction

$$\theta_1 x_1 + \theta_2 x_2 + .. + \theta_{n-1} x_{n-1} + \theta_n y \in C.$$

$$\Leftrightarrow \theta_1 x_1 + \theta_2 x_2 + ... + \theta_{n-1} x_{n-1} + \lambda \theta_n x_n + (1-\lambda)\theta_n x_{n+1} \in C.$$

and we have

$$\sum_{i=1}^{n-1} \theta_i + \lambda \theta_n + (1-\lambda)\theta_n = \sum_{i=1}^{n} \theta_i = 1.$$

Let $\tilde{\theta}_i = \theta_i$. $i = 1, \cdots n-1$

$\tilde{\theta}_n = \lambda \theta_n$

$\tilde{\theta}_{n+1} = (1-\lambda)\theta_n$

$$\Rightarrow \sum_{i=1}^{n+1} \tilde{\theta}_i = 1, \quad \tilde{\theta}_i \geq 0.$$

Hence. for $k=n+1$, it holds.

By induction. the original statement holds true. $\square$

(b) Denote $Q := \cap \{C : C \text{ is convex and } S \subset C\}$.

As Q is intersection of convex sets. $\Rightarrow$ Q is ~~convex~~.

Also it's obvious. $S \subset Q$. as all its element contains S.

... $S \subset Q$ and $Q \subset \text{Conv } S$.

(b) Denote $Q := \cap\{C : C \text{ is convex and } S \subset C\}$

As C is convex and $S \subset C$, Q is intersection of all such Cs.

we have $S \subset Q$, and Q is convex,

We first show that $\operatorname{Conv} S \subset Q$

$\forall x \in \operatorname{Conv} S$.

$$x = \sum_{i=1}^{k} \theta_i x_i, \quad x_i \in S, \quad \theta_i \geq 0, \quad \sum_{i=1}^{k} \theta_i = 1.$$

Because Q is convex and $S \subset Q$

by (a) we know $x = \sum_{i=1}^{k} \theta_i x_i \in Q$.

Hence $\operatorname{Conv} S \subset Q$

Then assume $\exists y \in Q \setminus \operatorname{Conv} S$.

Let $\tilde{C} = \operatorname{Conv} S$. as $S \subset \tilde{C}$ and $\tilde{C}$ is convex.

~~$\tilde{Q} = Q \cap$~~ $\tilde{C}$. ~~As~~ Hence $\tilde{C}$ is also in C_s that constructs Q

Hence $Q \subset \tilde{C} = \operatorname{Conv} S$

Hence $\nexists\, y \in Q \setminus \operatorname{Conv} S$

therefore $\operatorname{Conv} S = Q$ $\square$

(C) 1° $\operatorname{Conv} S \subset Q := \{\sum_{i=1}^{k} a_i x_i : x_i \in C_i, a_i \geq 0, i = 1, \cdots k, \sum_{i=1}^{k} a_i = 1\}$

By (a). We know $\forall x \in \operatorname{Conv} S$.

$$x = \sum_{i=1}^{n} \theta_i x_i, \quad x_i \in S. \quad \theta_i \geq 0, \quad \sum_{i=1}^{n} \theta_i = 1.$$

wlog, rearrange the order of x_i

Let $$x = \sum_{j=1}^{K} \sum_{x_i \in C_j} \theta_i x_j$$

denote $a_j = \sum_{x_i \in C_j} \theta_i$. $y_j = \frac{1}{a_j} \sum_{x_i \in C_j} \theta_i x_i$.

Because $\sum \frac{\sum_{x_i \in C_j} \theta_i}{a_j} = \frac{\sum_{x_i \in C_j} \theta_i}{\sum_{x_i \in C_j} \theta_i} = 1$, $\frac{\theta_i}{a_j} \geq 0$, $x_i \in C_j$

then we have $y_j \in C_j$.

Hence $x = \sum_{j=1}^{K} a_j y_j$, $\sum_{j=1}^{K} a_j = \sum_{j=1}^{K} \sum_{x_i \in C_j} \theta_i = 1$. $a_j \geq 0$. $y_j \in C_j$

Hence $x \in Q$. $\Rightarrow \operatorname{Conv} S \subset Q$.

Hence by 1° and 2°. $\Rightarrow \operatorname{Conv} S = Q$.

2° $Q \subset \operatorname{Conv} S$.

$\forall x \in Q$. $x = \sum_{i=1}^{k} a_i x_i$; as $x_i \in S$. $\sum a_i = 1$. $a_i \geq 0$.

$\Rightarrow x \in \operatorname{Conv} S$. $\Rightarrow Q \subset \operatorname{Conv} S$. $\square$

3. Polar Cone and Separation of Convex Sets

(a). By definition, we have. $\bar{K}^\circ = \{s \in \mathbb{R}^n : s^T x \le 0, \text{ for all } x \in \bar{K}\}$

We prove the statement by showing $K^\circ \subseteq \bar{K}^\circ$ and $\bar{K}^\circ \subseteq K^\circ$.

1° $\bar{K}^\circ \subseteq K^\circ$.

$\forall s \in \bar{K}^\circ$. s satisfies, $s^T x \le 0$, for all $x \in \bar{K}$.

$\Rightarrow$ $s^T\left(\sum_{i=1}^{m} a_i x_i\right) \le 0$, $a_i \ge 0$.

$\Leftrightarrow$ $\sum_{i=1}^{m} a_i s^T x_i \le 0$ $\forall a_i \ge 0$.

Let $a = (a_1, \dots a_m)^T$.

Choose $a = e_j$, then we have $s^T x_j \le 0$. $j = 1, 2, \dots m$

Hence $s \in K^\circ$.

$\Rightarrow$ $\bar{K}^\circ \subseteq K^\circ$

2° $K^\circ \subseteq \bar{K}^\circ$

$\forall s \in K^\circ$. we have $s^T x_j \le 0$. for $j = 1, \dots, m$

$\Rightarrow \forall a_j \ge 0$, $a_j s^T x_j \le 0$

$\Rightarrow$ $\sum_{j=1}^{m} a_j s^T x_j \le 0$

Hence $s \in \bar{K}^\circ$.

$\Rightarrow$ $K^\circ \subseteq \bar{K}^\circ$.

Hence, we conclude that $\bar{K}^\circ = K^\circ$. $\square$.

(b). $K^\circ = \{s \in \mathbb{R}^n : s^T x \le 0. \text{ for all } x \in K\}$

We first show that K° is always closed.

As K° is the intersection of closed halfspaces. so K° is closed.

Hence. $K^{\circ\circ}$ is also closed.

1° show $\mathrm{cl}\, K \subseteq K^{\circ\circ}$.

$\forall x \in K$. $\forall s \in K^\circ$. $s^T x \le 0$.

$\Rightarrow$ $x \in K^{\circ\circ}$

$\Rightarrow$ $K \subseteq K^{\circ\circ}$

As $K^{\circ\circ}$ is closed. $\Rightarrow \mathrm{cl}\, K \subseteq K^{\circ\circ}$

2° If $\exists x_0 \in K^{\circ\circ} \setminus \mathrm{cl} K$. we show by contradiction that if $x_0 \notin \mathrm{cl} K$, then $x_0 \notin K^{\circ\circ}$

Because $x_0 \notin \mathrm{cl}\, K$. by strict separation theorem. (Because $\mathrm{cl}\, K$ is closed and convex) show later.

$\exists$ halfplane $a^T x = b$ such that $a^T x_0 > b$, $a^T x \le b$. $\forall x \in \mathrm{cl} K$.

As $0 \in K$. $\Rightarrow b \ge 0$

Because $K^\circ = \{s \in \mathbb{R}^n : s^T x \le 0, \text{for all } x \in K\}$.

we can exactly take one of its supporting planes as the separating hyperplane. (at least)

Otherwise $x_0 \in K$.

say. $s_1^T x = 0$, $s_1 \in K^\circ$.

Hence we have $\forall x \in K$. $s_1^T x \le 0$,

$s_1^T x_0 > 0$.

As $s_1 \in K^\circ$. we then conclude $x_0 \notin K^{\circ\circ}$ $\square$

Hence. $\mathrm{Cl} K = K^{\circ\circ}$

- To prove $\mathrm{Cl} K$ is closed and convex.

closeness is trivial.

to show convexity, because K is convex. $\forall x, y \in K$. $\forall \theta \in [0,1]$. $\theta x + (1-\theta) y \in K$.

For any $x, y \in \mathrm{Cl} K$. construct two sequences $\{x_i\}$, $\{y_i\}$.

which satisfies. $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$.

and $x_i \in K$, $y_i \in K$.

Hence. $\theta x_i + (1-\theta) y_i = z_i \in K \subseteq \mathrm{Cl} K$.

Take the limit, and because $\mathrm{Cl} K$ is closed.

we have $\theta x + (1-\theta) y = z \in \mathrm{Cl} K$.

$\Rightarrow$ $\mathrm{Cl} K$ is convex $\square$.

(c). Using results of (a) and (b)

$K^{\circ\circ} = \mathrm{Cl} K$.

Hence $\{ s \in \mathbb{R}^n : s^T x_j \le 0 \text{ for } j = 1, \dots, m \}^\circ = \mathrm{Cl} K$

$$= \mathrm{Cl} \cdot \left\{ \sum_{j=1}^{m} a_j x_j : a_j \ge 0 \text{ for } j = 1, \dots m \right\}$$

$$= \sum_{j=1}^{m} a_j x_j : a_j \ge 0 \text{ for } j = 1, \dots m \}$$

$$= K.$$

4. Normal cone and Tangent Cone

(a) $N_C(x) = \{ s \in \mathbb{R}^n \mid s^T (y - x) \le 0 \text{ for all } y \in C \}$

Because $N_C(x)$ is the intersection of closed convex halfspaces.

so $N_C(x)$ is closed, convex

if $s \in N_C(x)$. then $\forall \theta \ge 0$ $(\theta s)^T (y - x) = \theta s^T (y - x) \le 0$, for all $y \in C$

Hence $N_C(x)$ is a cone.

$\Rightarrow$ $N_C(x)$ is a closed convex cone.

(b) If $x \in \operatorname{int} C$, then $N_C(x) = 0$. $T_C(x) = N_C^\circ(x) = \mathbb{R}^n$.

If $x \in \operatorname{Bd} C$.

denote the tight constraints at point x as $J(x) = \{j : s_j^T x = r_j, j = 1, \dots, m\}$

Let $N_C(x) = \{ \sum_{j \in J(x)} a_j s_j, \ a_j \geq 0 \}$

we show this is the normal cone at the point x.

For one single tight constraint $s_j^T x = r_j$.

the corresponding normal cone, vector is s_j. as

$$s_j^T (y - x) \leq r_j - r_j = 0.$$

When the tight constraint increase by 1, the generator increases by 1 as well.

And ~~$N_C(x)$~~ $\left(\sum_{j \in J(x)} a_j s_j\right)^T (y - x)$, for all $y \in C$

$$= \sum_{j \in J(x)} [a_j \, s_j^T (y - x)]$$

$$\leq \sum_{j \in J(x)} a_j (r_j - r_j)$$

$$= 0.$$

Hence $N_C(x) = \{ \sum_{j \in J(x)} a_j s_j, \ a_j \geq 0 \}$ is the description of the normal cone.

From 3(a) we know

$T_C(x) = N_C^\circ(x) = \{ y \in \mathbb{R}^n : y^T s_j \leq 0 \text{ for } j \in J(x) \}$.