Coordinate Descent Methods

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Coordinate minimization

- we want to minimize a convex function $f\colon \mathbb{R}^n \mapsto \overline{\mathbb{R}}$
- in coordinate descent, we optimize over one variable at a time
- consider

$$f(x) = f(x_1, x_2, x_3, \ldots, x_n)$$

- the *coordinate minimization* (Gauss-Seidel) algorithm is

$$\begin{aligned}
x_1^{k+1} &\in \operatorname*{argmin}_{x_1} f(x_1, x_2^k, x_3^k, \ldots, x_n^k) \\
x_2^{k+1} &\in \operatorname*{argmin}_{x_2} f(x_1^{k+1}, x_2, x_3^k, \ldots, x_n^k) \\
x_3^{k+1} &\in \operatorname*{argmin}_{x_3} f(x_1^{k+1}, x_2^{k+1}, x_3, \ldots, x_n^k) \\
&\vdots \\
x_n^{k+1} &\in \operatorname*{argmin}_{x_n} f(x_1^{k+1}, x_2^{k+1}, x_3^{k+1}, \ldots, x_n)
\end{aligned}$$

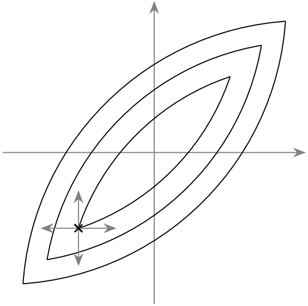
Coordinatewise optimality

- assume f is differentiable and $\bar{x}$ is a fixed-point of the coordinate minimization algorithm, *i.e.*, $0 = \frac{\partial f}{\partial x_i}(\bar{x})$ for all i
- then $\bar{x}$ is a minimizer of f since

$$\nabla f(\bar{x}) = \left(\frac{\partial f}{\partial x_1}(\bar{x}), \ldots, \frac{\partial f}{\partial x_n}(\bar{x}) \right) = 0$$

Nondifferentiable case

- coordinate optimality does not necessarily imply optimality if f is nondifferentiable
- **example:** $f(x_1, x_2) = |x_1 - x_2| + \frac{1}{2}(x_1^2 + x_2^2)$



Separable case

- consider the problem

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- g is convex and differentiable
- h is convex, not necessarily differentiable, and has the form

$$h(x) = \sum_{i=1}^{n} h_i(x_i)$$

- is every fixed-point of the algorithm a minimizer of f?

Separable case – optimality

- let $\mathbf{x}_i^k = (x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \ldots, x_n^k)$
- we first show that x_i optimizes ith update if

$$\langle \nabla_i g(\mathbf{x}_i^k), y_i - x_i \rangle + h_i(y_i) - h_i(x_i) \geq 0, \quad \forall y_i \in \mathbb{R}$$

- for $\mathbf{y}_i^k = (x_1^{k+1}, \ldots, x_{i-1}^{k+1}, y_i, x_{i+1}^k, \ldots, x_n^k)$ we have

$$g(\mathbf{y}_i^k) - g(\mathbf{x}_i^k) \geq \langle \nabla g(\mathbf{x}_i^k), \mathbf{y}_i^k - \mathbf{x}_i^k \rangle = \langle \nabla_i g(\mathbf{x}_i^k), y_i - x_i \rangle$$

- then, if condition holds, we have for all $\mathbf{y}_i^k$

$$\begin{aligned} f(\mathbf{y}_i^k) - f(\mathbf{x}_i^k) &= g(\mathbf{y}_i^k) - g(\mathbf{x}_i^k) + h_i(y_i) - h_i(x_i) \\ &\geq \langle \nabla_i g(\mathbf{x}_i^k), y_i - x_i \rangle + h_i(y_i) - h_i(x_i) \\ &\geq 0 \end{aligned}$$

- therefore, $f(\mathbf{x}_i^k)$ has the lowest value along ith coordinate

Separable case – optimality

- assume that we have reached a fixed-point of the algorithm, *i.e.*, $\mathbf{x}_i^k = \mathbf{x}_j^k$ for all $i \neq j$
- then, for any y and all $\mathbf{x}_j^k$, we have

$$\begin{aligned} f(y) - f(\mathbf{x}_j^k) &= g(y) - g(\mathbf{x}_j^k) + \sum_{i=1}^{n} \left(h_i(y_i) - h_i(x_i)\right) \\ &\geq \langle \nabla g(\mathbf{x}_j^k), y - \mathbf{x}_j^k \rangle + \sum_{i=1}^{n} \left(h_i(y_i) - h_i(x_i)\right) \\ &= \sum_{i=1}^{n} \big(\langle \nabla_i g(\mathbf{x}_j^k), y_i - x_i \rangle + h_i(y_i) - h_i(x_i) \big) \\ &\geq 0 \end{aligned}$$

- therefore, $\mathbf{x}_i^k$ minimizes f

Convergence

- strong convergence results require additional assumptions
- we know that the function value is nonincreasing, *i.e.*,

$$f(\mathbf{x}_{i+1}^k) \leq f(\mathbf{x}_i^k)$$

- note that the minimizers in the updates may not be unique
- therefore, arguments for convergence of iterates become tricky
- **variations:**
  - *block coordinate descent*: extension to the case where $x_i \in \mathbb{R}^{n_i}$ are subvectors of x
  - order of updates can be changed (random schemes exist as well)

Parallelization

- the *parallel coordinate minimization* (Jacobi) algorithm is

$$x_i^{k+1} \in \operatorname*{argmin}_{x_i} f(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_n^k)$$

- each component can be updated simultaneously
- unfortunately, the algorithm does not necessarily converge, even when f is differentiable
- *regularized Jacobi algorithm* can be used instead

$$x_i^{k+1} \in \operatorname*{argmin}_{x_i} f(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_n^k) + \tfrac{c}{2}\|x_i - x_i^k\|_2^2$$

- requires Lipschitz smoothness of f and appropriate choice of the regularization parameter $c > 0$ to converge
- there exist *asynchronous* variants

Coordinate gradient descent

- in coordinate gradient descent we solve

$$\text{minimize} \quad f(x)$$

- assume f is block-smooth
  - let

$$\mathbf{x}_i = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$
$$\mathbf{y}_i = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

  - f satisfies

$$f(\mathbf{y}_i) \leq f(\mathbf{x}_i) + \langle \nabla f(\mathbf{x}_i), \mathbf{y}_i - \mathbf{x}_i \rangle + \tfrac{L_i}{2} \|\mathbf{y}_i - \mathbf{x}_i\|_2^2$$

  for some $L_i \geq 0$, all $\mathbf{x}_i, \mathbf{y}_i$ and all $i = \{1, \ldots, n\}$
- equivalent condition:

$$f(\mathbf{y}_i) \leq f(\mathbf{x}_i) + \langle \nabla_i f(\mathbf{x}_i), y_i - x_i \rangle + \tfrac{L_i}{2} \|y_i - x_i\|_2^2$$

- if f is L-smooth, then $L_i \leq L$

Coordinate gradient descent

- the algorithm performs the following updates (*e.g.*, in a cyclic fashion):

$$\begin{aligned} x_i^{k+1} &\in \operatorname*{argmin}_{x_i} \left\{ f(\mathbf{x}_i^k) + \langle \nabla_i f(\mathbf{x}_i^k), x_i - x_i^k \rangle + \tfrac{L_i}{2} \|x_i - x_i^k\|_2^2 \right\} \\ &= x_i^k - \tfrac{1}{L_i} \nabla_i f(\mathbf{x}_i^k) \end{aligned}$$

- can be extended to the case where $f(x) = g(x) + h(x)$, where g is block-smooth and h is separable
- the updates have the following form:

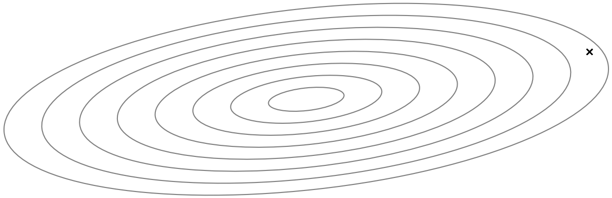
$$\begin{aligned} x_i^{k+1} &\in \operatorname*{argmin}_{x_i} \left\{ g(\mathbf{x}_i^k) + \langle \nabla_i g(\mathbf{x}_i^k), x_i - x_i^k \rangle + \tfrac{L_i}{2} \|x_i - x_i^k\|_2^2 + h_i(x_i) \right\} \\ &= \operatorname*{argmin}_{x_i} \left\{ \tfrac{L_i}{2} \|x_i - x_i^k + \tfrac{1}{L_i} \nabla_i g(\mathbf{x}_i^k)\|_2^2 + h_i(x_i) \right\} \end{aligned}$$

Coordinate gradient descent – example

- consider the following L-smooth problem

$$\text{minimize} \quad \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- choose $L_i = L$ for all i



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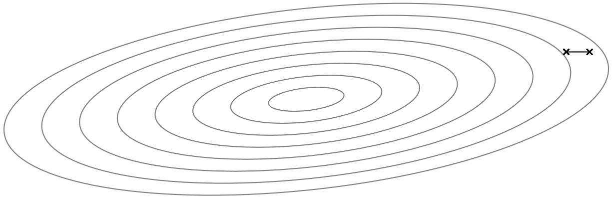

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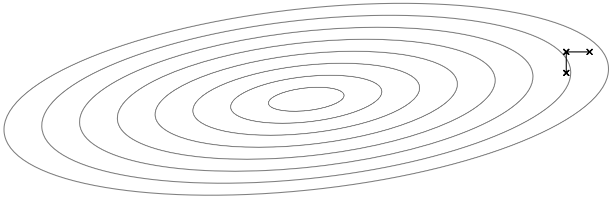

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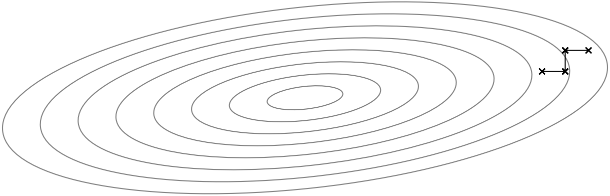

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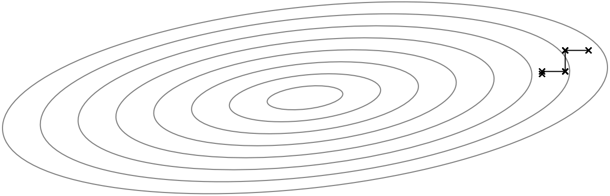

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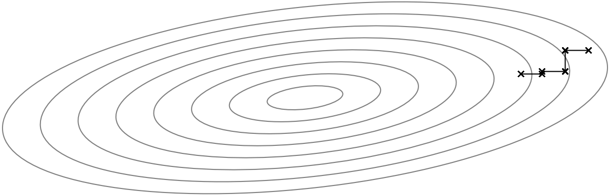

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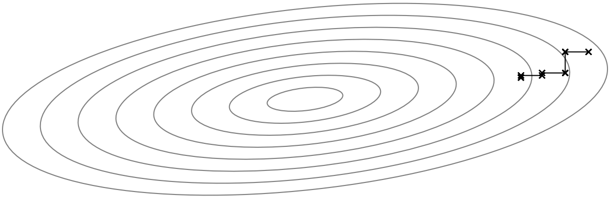

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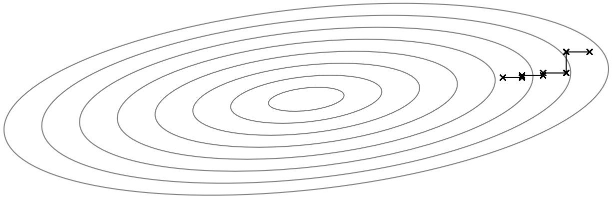

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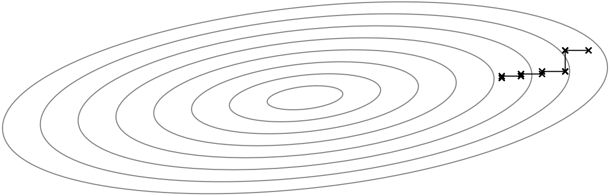

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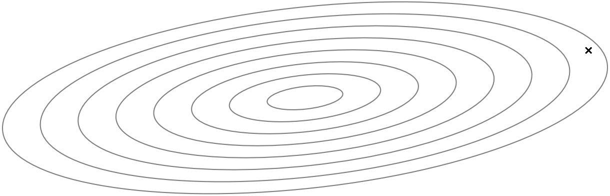
- now choose $L_1 = 0.1$ and $L_2 = 1$



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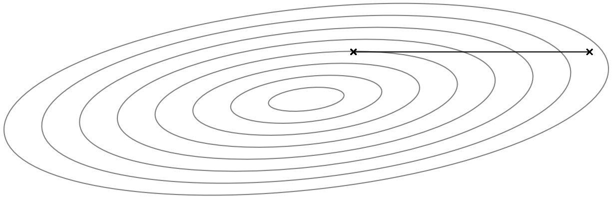

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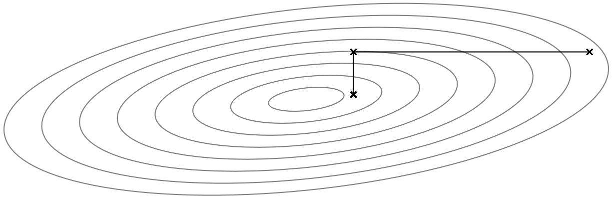

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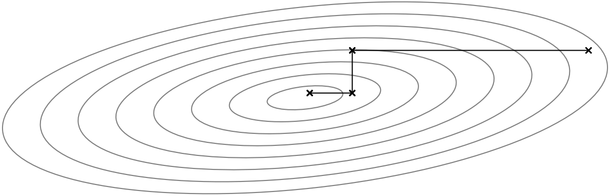

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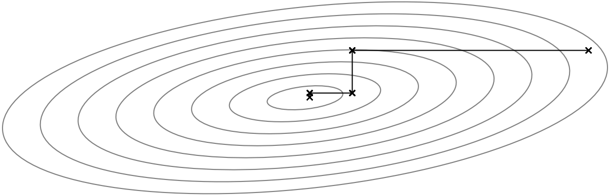

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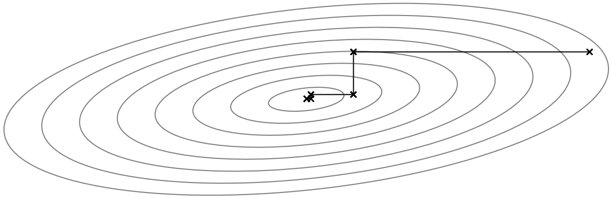


Finite sum problems

- consider *finite sum problems* of the form:

$$\text{minimize} \quad f(x) = \tfrac{1}{N} \sum_{i=1}^{N} f_i(x)$$

  where all f_i are differentiable
- for large problems gradient can be expensive to compute
- can be replaced by unbiased stochastic approximation of gradient

Unbiased stochastic gradient approximation

- stochastic gradient:
  - estimator $\widehat{\nabla} f(x)$ outputs $\mathbb{R}^n$-valued random variable
  - realization $\widetilde{\nabla} f(x)$ outputs a realization in $\mathbb{R}^n$
- an unbiased stochastic gradient approximator $\widehat{\nabla} f$ satisfies

$$\mathbb{E}\widehat{\nabla} f(x) = \nabla f(x)$$

- if x is random variable, then an unbiased estimator satisfies

$$\mathbb{E}\big[\widehat{\nabla} f(x) \mid x\big] = \nabla f(x)$$

Stochastic gradient descent

- the following iteration generates a sequence of *random* variables:

$$x^{k+1} = x^k - \gamma_k \widehat{\nabla} f(x^k)$$

- *stochastic gradient descent* finds a realization of this sequence:

$$x^{k+1} = x^k - \gamma_k \widetilde{\nabla} f(x^k)$$

- sloppy notation when x^k is *random variable* vs *realization*
- efficient if realizations $\widetilde{\nabla} f$ much cheaper to evaluate than ∇f
- analyze former and draw conclusions of (almost) all realizations

Stochastic gradient for finite sum problems

$$\text{minimize} \quad f(x) = \tfrac{1}{N} \sum_{i=1}^{N} f_i(x)$$

- select f_i at random and take gradient step
- realization: let i be drawn from I:

$$\widetilde{\nabla} f(x) = \nabla f_i(x)$$

  where I is the uniform probability distribution

$$p_i = p(I = i) = \tfrac{1}{N}$$

- stochastic gradient is unbiased:

$$\mathbb{E}\big[\widehat{\nabla} f(x) \mid x\big] = \sum_{i=1}^{N} p_i \nabla f_i(x) = \tfrac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \nabla f(x)$$

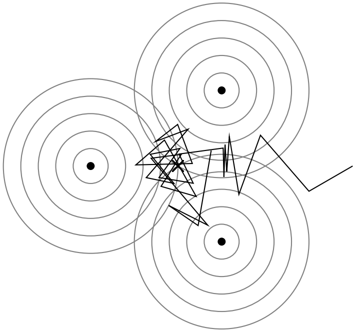
- *mini-batch stochastic gradient*: extension to the case where $\widetilde{\nabla} f(x)$ is obtained from K gradients ∇f_i

Stochastic gradient descent – example

- consider the following finite sum problem:

$$\text{minimize} \quad \tfrac{1}{2}\|x - c_1\|_2^2 + \tfrac{1}{2}\|x - c_2\|_2^2 + \tfrac{1}{2}\|x - c_3\|_2^2$$

- stochastic gradient descent with $\gamma_k = 1/3$



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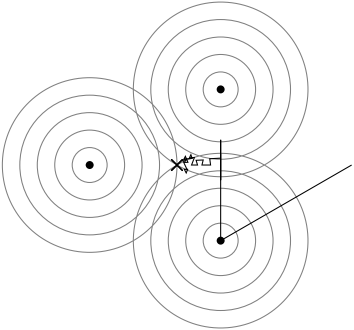
- stochastic gradient descent with $\gamma_k = 1/k$



Assumptions for convergence

- f is L-smooth for all $x, y \in \mathbb{R}^n$
- stochastic gradient of f is unbiased: $\mathbb{E}\big[\widehat{\nabla} f(x) \mid x\big] = \nabla f(x)$
- bounded variance: $\mathbb{E}\big[\|\widehat{\nabla} f(x) - \nabla f(x)\|_2^2 \mid x\big] \leq \sigma^2$
- step sizes satisfy

$$\sum_{k=0}^{\infty} \gamma_k = +\infty, \qquad \sum_{k=0}^{\infty} \gamma_k^2 < +\infty$$

References

- these lecture notes are based to a large extent on the following courses developed by Pontus Giselsson at Lund:
  - Large-Scale Convex Optimization
  - Optimization for Learning

- the original slides can be downloaded from

`https://archive.control.lth.se/ls-convex-2015/`

`http://www.control.lth.se/education/engineering-program/frtn50-optimization-for-learning/`