Advanced Topics in Control 2020: Large-Scale Convex Optimization

Exercise 2: Convex Functions

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Please submit your solutions via Moodle as a PDF with filename `Ex01_Surname.pdf`, replacing `Surname` with your surname.

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1 Convex Functions

Show that the following functions from $\mathbb{R}^n$ to $(-\infty, \infty]$ are convex. You may use convexity preserving operations.

(a) $f_1(x) = \ln(e^{x_1} + \ldots + e^{x_n})$, where $x = (x_1, \ldots, x_n)$.
Hint: f_1 is twice differentiable.

(b) $f_2(x) = \|x\|^p$ with $p \geq 1$.
Hint: Use the composition rule in Slide 17. Consider the function $h(t) = t^p$, if $t \geq 0$ and $h(t) = 0$, elsewhere.

(c) $f_3(x) = \frac{1}{f(x)}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is concave and $f(x) > 0$, for all x.

(d) $f_4(x) = \|Ax - b\|_2^2 + \|x\|_1$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Hint: First show that $x \to \|x\|_2^2$ is convex. Then use convexity preserving operations.

(e) $f_5(x) = \sum_{i=1}^{n}(\max(0, 1 + x_i)) + \|x\|_2^2$, where $x = (x_1, \ldots, x_n)$.
Hint: First show that for each $i = 1, \ldots, n$, the function $x \to \max(0, 1 + x_i)$ is convex.

(f) $f_6(x) = \sup_y (x^\top y - g(y))$, with $g : \mathbb{R}^n \to \mathbb{R}$ not necessarily convex (this is called the *conjugate function* of g and is denoted by g^*).

(g) $\phi_C(x) = \frac{1}{2}(\|x\|_2^2 - d_C^2(x))$, where $C \subset \mathbb{R}^n$ is a nonempty compact (closed and bounded) set and $d_C(x) := \min_{y \in C} \|x - y\|_2$ is the *distance function* to C.
Hint: Note that $d_C^2(x) = \min_{y \in C} \|x - y\|_2^2 = \|x\|_2^2 - \max_{y \in C}\{2x^\top y - \|y\|_2^2\}$.

(h) $g(x) = \inf_{u \in \mathbb{R}^n}\{h_1(u) + h_2(x - u)\}$, where $h_1, h_2 : \mathbb{R}^n \to (-\infty, \infty]$ are proper convex functions that have a *common affine minorant*, i,e. for some $(s, r) \in \mathbb{R}^n \times \mathbb{R}$,

$$h_i(x) \geq s^\top x - r, \text{ for } i = 1, 2 \text{ and all } x \in \mathbb{R}^n.$$

We call g the *infimal convolution* of h_1, h_2.
Hint: Note that since h_1 and h_2 have a common affine minorant, we ensure that $g(x) > -\infty$, for all $x \in \mathbb{R}^n$. Indeed, for arbitrary $x \in \mathbb{R}^n$ and x_1, x_2 such that $x_1 + x_2 = x$, we have by assumption that $h_1(x_1) + h_2(x_2) \geq s^\top x - 2r > -\infty$. So, $g(x) = \inf\{h_1(x_1) + h_2(x_2) : x_1 + x_2 = x\} \geq s^\top x - 2r > -\infty$. Then, show that g is proper. Finally use the partial minimization rule.

2 Strictly Convex Functions and Unique Minimizers

(a) Suppose that f is a *strictly convex* function, i.e., $f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$, for all $x \neq y$ and $\theta \in (0,1)$. Show that if there exists a (global) minimizer $x^\star$ of f, then it is unique.
Hint: Assume for the sake of contradiction that there exists $x \neq x^\star$ that satisfies $f(x) = f(x^\star)$. Upper bound the term $f(\frac{1}{2}x + \frac{1}{2}x^\star)$.

(b) Provide a strictly convex f whose infimum is not attained by any point.

3 Subdifferentials

(a) Suppose that $g(x) = \sum_{i=1}^n g_i(x_i)$, where $x = (x_1, \ldots, x_n)$. Show that $s \in \partial g(x)$ if and only if $s_i \in \partial g_i(x_i)$, where $s = (s_1, \ldots, s_n)$.
Hint: Assume first that $s = (s_1, \ldots, s_n) \in \partial g(x)$. Then $g(y) \geq g(x) + s^\top(y - x)$ $(\star)$, for all $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For arbitrary $j = 1, \ldots, n$, set $x_i = y_i$, for all $i \neq j$ and use $(\star)$ to conclude that $s_j \in \partial g_j(x_j)$. For the other direction suppose that $s_i \in \partial g_i(x_i)$ and write the associated inequality. Summing over i gives the desired result.

(b) Show that the subdifferential of the indicator function of a nonempty set C is the normal cone to C.
Hint: Consider two cases: $x \in C$ and $x \notin C$. For the latter case you have to show that $\partial \mathcal{I}_C(x) = \emptyset$.

4 Conjugate Functions

(a) Let $f(x) = \|x\|_2$. Compute the conjugate f^* via the following steps.

   (1) Show that $f^*(s) \geq 0$, for all s.

   (2) Show that $f^*(s) \leq 0$, for all s with $\|s\|_2 \leq 1$.
   Hint: Use the Cauchy-Schwarz inequality $|s^\top x| \leq \|s\|_2 \|x\|_2$.

   (3) Show that $f^*(s) = \infty$, for all s with $\|s\|_2 > 1$.
   Hint: Set $x = ts$ with $t \geq 0$ to get $f^*(s) = \sup(s^\top x - \|x\|_2) \geq t\|s\|_2(\|s\|_2 - 1)$.

   (4) Combine the results to derive $f^*(s)$.

(b) Use the conjugate to compute the subdifferential of $f(x) = \|x\|_2$.
Hint: Use the formula $\partial f(x) = \arg\max_s (s^\top x - f^*(s)) = \arg\max_{\|s\|_2 \leq 1} (s^\top x)$. For $x = 0$ show that $\partial f(0) = \{s : \|s\|_2 \leq 1\}$. For $x \neq 0$ show that $\partial f(x) = \{\frac{x}{\|x\|_2}\}$. You have to use once more the Cauchy-Schwarz inequality.