Advanced Topics in Control 2020: Large-Scale Convex Optimization

Exercise 4: Duality

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Please submit your solutions via Moodle as a PDF with filename `Ex04_Surname.pdf`, replacing `Surname` with your surname.

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1 Problem 1

Consider the convex optimization problem:

$$\min_{x} \quad c^\top x + \frac{1}{\mu}\sum_{i=1}^{m} \log\left(1 + e^{\mu(a_i^\top x - b_i)}\right), \tag{1}$$

where $x \in \mathbb{R}^n$, $\mu > 0$.

(a) Show that problem (1) is equivalent to:

$$\begin{aligned} \min_{x} \quad & c^\top x + \frac{1}{\mu}\sum_{i=1}^{m} \log\left(1 + e^{\mu y_i}\right), \\ \text{s.t.} \quad & Ax - b \leq y, \end{aligned} \tag{2}$$

where the matrix $A \in \mathbb{R}^{m\times n}$ has rows $a_i^\top$, $b \in \mathbb{R}^m$ has entries b_i, and $y \in \mathbb{R}^m$ has entries y_i.

(b) Derive the Lagrange dual function and the Lagrange dual problem of (2).

(c) Consider now the pair of primal and dual linear programs (LP):

$$\begin{aligned} \min_{x} \quad & c^\top x, \\ \text{s.t.} \quad & Ax \leq b, \end{aligned} \qquad\qquad \begin{aligned} \max_{z} \quad & -b^\top z, \\ \text{s.t.} \quad & A^\top z + c = 0, \\ & z \geq 0, \end{aligned}$$

where A, b, and c are the same matrices and vectors used in problem (2).
Knowing that the primal LP has a finite optimal value $p^\star$ and the optimizer $z^\star$ of the dual LP has all the entries smaller than 1 (i.e. $z_i \leq 1$ $\forall i = 1, .., m$), show that

$$p^\star \leq q^\star \leq p^\star + \frac{m\log(2)}{\mu}, \tag{3}$$

where $q^\star$ denotes the optimal value of the original problem (1).
Hint: Plug in the optimizer $x^\star$ of the primal LP in the objective function of the original problem (1), and the optimizer $z^\star$ of the dual LP in the cost of the dual problem obtained in part (b). Express these two costs as a function of $p^\star$ and then try to bound them with respect to $q^\star$.

2 Problem 2

Consider the optimization problem:

$$\begin{aligned} \min_{x_1,x_2} \quad & e^{-x_2}, \\ \text{s.t.} \quad & \sqrt{x_1^2+x_2^2} - x_1 \leq 0, \end{aligned} \tag{4}$$

where $x_1, x_2 \in \mathbb{R}$.

(a) Verify that the optimization problem is convex, find its feasible set, and determine the optimal value.

(b) What is the duality gap?

(c) Can you prove that there is a nonzero duality gap without solving the dual problem? Explain why.

3 Problem 3

Consider the optimization problem:

$$\begin{aligned} \min_{x,y} \quad & \|y\|_2 + \gamma\|x\|_1, \\ \text{s.t.} \quad & Ax - b = y, \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the optimization variables, and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\gamma > 0$ are given.

(a) Derive the dual problem of (5).
Hint: When you perform minimization of the Lagrangian $L(x, y, z)$ with respect to the primal variables x, y in order to determine the dual function $g(z)$, you will have to determine two constraints on z such that the dual function remains bounded. It might be helpful to recall, from Problem 4 of Exercise 2, that the conjugate function of a norm (in that problem you examined the case of Euclidean norm) is the indicator function of the dual norm unit ball.

(b) Denote by $x^\star$ an optimal solution of (5). Assume that $Ax^\star - b \neq 0$ and define $r = \frac{Ax^\star - b}{\|Ax^\star - b\|_2}$. Show that the following holds:

$$\begin{aligned} &\|A^\top r\|_\infty \leq \gamma, \\ &r^\top A x^\star + \gamma\|x^\star\|_1 = 0. \end{aligned} \tag{6}$$

Hint: Since $x^\star$ is an optimal solution, it will fulfill the KKT conditions for the problem.

4 Problem 4

Let $g(x) = f(Lx + c)$ where $f : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is closed convex, $L \in \mathbb{R}^{m \times n}$, and $c \in \mathbb{R}^m$. Knowing that relint dom $g \neq \emptyset$, show that:

$$g^*(s) = \inf_{\mu}(f^*(\mu) - \mu^\top c : s = L^\top \mu), \tag{7}$$

where g^* denotes the conjugate function of g.
Hint: Derive an expression for g^* as the optimal value of a convex composite minimization problem, and then use Fenchel duality to obtain an expression which depends on the conjugate function f^*.