Advanced Topics in Control 2020: Large-Scale Convex Optimization

Exercise 1: Convex Sets

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Please submit your solutions via Moodle as a PDF with filename `Ex01_Surname.pdf`, replacing `Surname` with your surname.

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1 Convex Sets

Show that the following sets are convex.

(a) A *wedge*, i.e., a set of the form $\{x \in \mathbb{R}^n : a_1^\top x \leq b_1,\, a_2^\top x \leq b_2\}$, where $a_1, a_2 \in \mathbb{R}^n$ and $b_1, b_2 \in \mathbb{R}$.

(b) A norm ball with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$, i.e., the set

$$B(x_0, r) := \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\},$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.
Hint: Use the definition of convexity.

(c) The set of points closer to a given point than a given set, i,e.,

$$\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq \|x - y\|_2 \,\text{for all}\, y \in S\},$$

where $S \subset \mathbb{R}^n$.
Hint: First show that for fixed $y \in S$, the set $C_y := \{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq \|x - y\|_2\}$ is an affine halfspace.

2 Convex Combinations and Convex Hulls

Prove the following statements.

(a) A set $C \subset \mathbb{R}^n$ is convex if and only if it contains every convex combination of its elements.
Hint: Use mathematical induction on the number of elements k.

(b) The convex hull of a set $S \subset \mathbb{R}^n$ is the intersection of all convex sets containing S, i.e.,

$$\operatorname{conv} S = \bigcap \{C : C \,\text{is convex and}\, S \subset C\}.$$

(c) Let $S = C_1 \cup \ldots \cup C_k$, where $C_1, \ldots, C_k$ are convex sets in $\mathbb{R}^n$. Then,

$$\operatorname{conv} S = \{\sum_{i=1}^{k} a_i x_i : x_i \in C_i,\ a_i \geq 0,\ i = 1, \ldots, k,\ \sum_{i=1}^{k} a_i = 1\}.$$

3 Polar Cone and Separation of Convex Sets

(a) Consider the conical hull of m points $x_1, \ldots, x_m$ in $\mathbb{R}^n$, i.e.,

$$K = \{\sum_{j=1}^{m} a_j x_j :\ a_j \geq 0 \,\text{for}\, j = 1, \ldots, m\}.$$

Show that $K^\circ = \{s \in \mathbb{R}^n : s^\top x_j \leq 0 \,\text{for}\, j = 1, \ldots, m\}$.

(b) Let K be a convex cone (not necessarily closed) with polar K°. Show that the polar $K^{\circ\circ}$ of K° is the closure of K.
Hint: For the inclusion $\operatorname{cl} K \subset K^{\circ\circ}$ use the fact that a polar cone is always closed (why?). For the inverse inclusion, assume for the sake of contradiction that there exists $x_0 \in K^{\circ\circ}$ with $x_0 \notin \operatorname{cl} K$. Since $\operatorname{cl} K$ is closed and convex (why?) you can use the strict separation theorem.

(c) Using your results in (a) and (b) find the polar of $\{s \in \mathbb{R}^n : s^\top x_j \leq 0 \,\text{for}\, j = 1, \ldots, m\}$.

4 Normal Cone and Tangent Cone

(a) Show that the normal cone of a set C at $x \in C$ is a closed convex cone (with no assumptions on C).

(b) Give a simple description of the normal cone and tangent cone of a polyedron $C = \{x \in \mathbb{R}^n : s_j^\top x \leq r_j \,\text{for}\, j = 1, \ldots, m\}$ at a point $x \in C$.