ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Prof. Dr. Rico Zenklusen and Assistants
Contact: math.opt@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Fall 2019

Mathematical Optimization – Problem set 10

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Bipartite matchings with fixed cardinality

Let $G = (V, E)$ be a bipartite graph and let $k \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{M}_k \subseteq 2^E$ be the set of all matchings M in G of size k, i.e., $|M| = k$. Assume we have to find a matching $M \in \mathcal{M}_k$ of maximum weight with respect to some non-negative edge weights $w \in \mathbb{Z}^E_{\geq 0}$. One way to solve this problem efficiently is by first finding a good description of the corresponding polytope

$$P_k = \operatorname{conv}\left(\left\{\chi^M : M \in \mathcal{M}_k\right\}\right) \subseteq [0,1]^E \ ,$$

which can then be used to obtain an optimal vertex solution to the linear program $\max\{w^T x : x \in P_k\}$. A natural candidate for a good description of P_k is

$$P \coloneqq \left\{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \text{ for all } v \in V, \text{ and } x(E) = k\right\} \ .$$

(a) Prove that $P \cap \{0,1\}^E = P_k \cap \{0,1\}^E$.

(b) Prove that P is integral using total unimodularity.

(c) Prove that P is integral by showing that if $x \in P$ has fractional components, then it is not an extreme point of P.

Finally, recall that indeed from (a) and (b), as well as from (a) and (c), it follows that $P = P_k$.

Problem 2: Properties of vertices of the perfect matching polytope

Let $G = (V, E)$ be a graph and consider the corresponding perfect matching polytope

$$P(G) = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V,\ |S| \text{ odd} \end{array} \right\} \ .$$

Let $W \subsetneq V$ be a non-empty set of vertices such that $\delta(W) = \emptyset$, and let $y \in P(G)$. For a subset $F \subseteq E$ of the edges of G and a vector $x \in \mathbb{R}^E$, let $x|_F \in \mathbb{R}^F$ denote the restriction of x to the edges in F.

Prove that y is vertex of $P(G)$ if and only if both $y|_{E[W]}$ and $y|_{E[V \setminus W]}$ are vertices of the perfect matching polytopes $P(G[W])$ and $P(G[V \setminus W])$ associated with the induced subgraphs of G on the vertex sets W and $V \setminus W$, respectively.

Problem 3: Perfect matchings in three-regular graphs

Prove that every 3-regular bridgeless graph admits a perfect matching. To this end, recall that a *bridge* in a graph $G = (V, E)$ is an edge $e \in E$ whose removal from the graph increases the number of connected components by one. A graph is *bridgeless* if it contains no bridge. Moreover, a graph is 3-*regular* if the degree of every vertex is three.

Problem 4: Facets of the spanning tree polytope

Recall that the spanning tree polytope of a graph $G = (V, E)$ admits the description

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} \ .$$

(a) Show that for every $n \geq 4$, there is a graph on n vertices with the property that every constraint (including non-negativity constraints) of its spanning tree polytope P is non-redundant.

(b) Show that the dimension of the spanning tree polytope P for the complete graph $G(V, E) = K_n$ is $|E| - 1$.

(c) Let H be the minimal affine subspace that contains P. Show that any non-redundant inequality $a^T x \leq b$ or $a^T x \geq b$ of the description of P for which a is not orthogonal to H, i.e., there exist two points $x, y \in H$ such that $a^T(x - y) \neq 0$, is facet-defining.

(d) Conclude that all inequalities in the above description of the spanning tree polytope P for the complete graph K_n, $n \geq 4$, are facet-defining (note that this explicitly excludes the equality $x(E) = |V| - 1$).

Problem 5: Degeneracy and the spanning tree polytope
Show that the spanning tree polytope

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

can be highly degenerate. Concretely, show that there exists a constant $c > 0$ such that for every $n \geq 3$, there is a graph $G = (V, E)$ with $|V| = n$, and a spanning tree $T \subseteq E$ of G with the property that at least 2^{cn} inequalities of P are tight at the point χ^T.

Problem 6: Description for the dominant of the r-arborescence polytope
Let $G = (V, A)$ be a directed graph and let $r \in V$. Denote the r-arborescence polytope of G by $P_{r\text{-arb}}$. In class, it was claimed that the dominant of the r-arborescence polytope, i.e., $P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0}$, equals

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \colon x\left(\delta^-(S)\right) \geq 1 \text{ for all } S \subseteq V \setminus \{r\} \text{ with } S \neq \emptyset \right\} .$$

What you will see in class is that the polyhedron P is integral, but it will not be proven that it is indeed equal to $P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0}$. We show this here.

(a) Show that P contains the characteristic vectors of all r-arborescences and that P is up-closed, i.e., $P = P + \mathbb{R}^A_{\geq 0}$. Deduce that $P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0} \subseteq P$.

(b) Show that every vertex of P is the characteristic vector of an r-arborescence.

Hint: You can use that P is integral, even though you haven't seen it in class yet.

(c) Prove that $P \subseteq P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0}$.

Hint: One approach to show the above uses part (b) and Proposition 1.38 from the script.

Programming exercises
Complete the notebook `10_eventPlanning.ipynb`, where you apply knowledge about integral polyhedra to solve a discrete problem using linear programming.