ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22
Prof. Dr. Rico Zenklusen and Assistants
Contact: math.opt@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Fall 2019

Mathematical Optimization – Problem set 9

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Relaxation of the vertex cover polytope

Let $G = (V, E)$ be a (not necessarily bipartite) undirected graph. Define the polytope

$$P := \{x \in \mathbb{R}^V : 0 \leq x(v) \leq 1 \text{ for all } v \in V,\ x(u) + x(v) \geq 1 \text{ for all } \{u, v\} \in E\}\ .$$

(a) Show that P is a relaxation of the vertex cover polytope of G, i.e., prove that $P \cap \{0,1\}^V$ is the set of all incidence vectors of vertex covers of G.

(b) Prove that any vertex y of P satisfies $y \in \{0, \frac{1}{2}, 1\}^V$, i.e., the vertices of P are half-integral.

(c) Give a 2-approximation for the minimum weight vertex cover problem, where we are given non-negative vertex weights $w\colon V \to \mathbb{Z}_{\geq 0}$, and the goal is to find a vertex cover $S \subseteq V$ minimizing $w(S)$. Recall that a 2-approximation for a minimization problem is an efficient algorithm that returns a solution of value within twice the value of an optimal solution.

Hint: You may assume that linear programs over P can be solved efficiently.

Problem 2: Incidence Matrices and Total Unimodularity

Let $G = (V, E)$ be an undirected graph and let A be its incidence matrix. Prove that if A is totally unimodular, then G is bipartite.

Remark: This is the backward direction of Theorem 5.13 in the script.

Problem 3: Max-flow min-cut via duality II

Let $G = (V, A)$ be a directed graph with arc capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, and let $s, t \in V$ be two distinct vertices. Consider the following linear program (P).

$$\begin{array}{rrcll} \max & \nu & & & \\ & \displaystyle\sum_{a\in\delta^+(v)} f_a - \sum_{a\in\delta^-(v)} f_a & = & \begin{cases} \nu & \text{if } v = s \\ -\nu & \text{if } v = t \\ 0 & \text{if } v \in V \setminus \{s,t\} \end{cases} & \\ & f_a & \leq & u_a & \forall a \in A \\ & f_a & \in & \mathbb{R}_{\geq 0} & \forall a \in A \\ & \nu & \in & \mathbb{R}_{\geq 0} & \end{array} \tag{P}$$

The dual linear program of (P) is given by

$$\begin{array}{rrcll} \min & \displaystyle\sum_{a\in A} u_a z_a & & & \\ & -y_s + y_t & \geq & 1 & \\ & y_v - y_w + z_a & \geq & 0 & \forall a = (v,w) \in A \\ & y_v & \in & \mathbb{R} & \forall v \in V \\ & z_a & \in & \mathbb{R}_{\geq 0} & \forall a \in A \end{array} \tag{D}$$

Recall from Problem 3 of Problem set 7 that it is easy to see that the value of the primal (P) equals the value of a maximum s-t flow in G, and that the value of the dual (D) is at most the value of a minimum s-t cut in G. Moreover, we were able to refine these observations to prove the strong max-flow min-cut theorem using the above primal-dual pair. The aim of this problem is to obtain another more direct proof by exploiting integrality properties.

(a) Prove that the dual linear program (D) has an optimal solution $(y,z) \in \{0,1\}^V \times \{0,1\}^A$.

(b) Let $(y,z) \in \{0,1\}^V \times \{0,1\}^A$ be an optimal solution of the dual linear program (D) and define

$$C = \{v \in V : y_v = 0\}.$$

Prove that C is an s-t cut with value equal to the optimal value of the dual linear program (D).

(c) Exploit strong linear programming duality to deduce the strong max-flow min-cut theorem.

Problem 4: Relaxation of the matching polytope

Let $G = (V,E)$ be a simple undirected graph. We are interested in studying a polytope P that may be a candidate for the matching polytope $P_{\mathcal{M}}$. Recall that the family of all matchings is

$$\mathcal{M} = \{M \subseteq E : e_1 \cap e_2 = \emptyset \text{ for all } e_1, e_2 \in M, e_1 \neq e_2\} \ ,$$

and that the matching polytope is, by definition, $P_{\mathcal{M}} \coloneqq \operatorname{conv}(\{\chi^M : M \in \mathcal{M}\})$. Consider the polytope

$$P \coloneqq \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \text{ for all } v \in V\} \ .$$

Show that there are graphs G for which P does not describe the matching polytope, i.e., $P \neq P_{\mathcal{M}}$.

Problem 5: Low Discrepancy Coloring

Let $n \in \mathbb{Z}_{>0}$ be even, and let $\sigma_1, \sigma_2 : [n] \to [n]$ be two permutations of the set $[n] \coloneqq \{1, \ldots, n\}$. We say that $R \subseteq [n]$ has discrepancy k if for every $i \in \{1,2\}$ and every $l, u \in [n]$ with $l < u$, the set $I = \{\sigma_i(l), \sigma_i(l+1), \ldots, \sigma_i(u)\}$ satisfies

$$\big||I \cap R| - |I \setminus R|\big| \leq k.$$

(a) We call a set $R \subseteq [n]$ *good* if we have $|R \cap \{\sigma_i(2t-1), \sigma_i(2t)\}| = 1$ for every $i \in \{1,2\}$ and every $t \in [n/2]$. Let $\mathcal{R} \subseteq 2^{[n]}$ be the family of all good sets. Find an inequality description of the polytope

$$P_{\mathcal{R}} \coloneqq \operatorname{conv}\left(\{\chi^R : R \in \mathcal{R}\}\right) \ .$$

(b) Use part (a) to prove that there always exists a subset R with discrepancy 2.

Problem 6: Laminar Matroids

Let N be a finite set and $\mathcal{L} \subseteq 2^N$ be a family of subsets of N that is *laminar*, i.e., for any $L, L' \in \mathcal{L}$, either $L \subseteq L'$, $L' \subseteq L$, or $L \cap L' = \emptyset$. An example of a laminar family is shown in Figure 1 below.

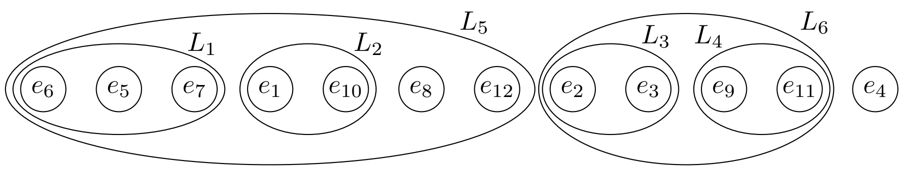

Figure 1: A laminar set family $\mathcal{L} = \{L_1, \ldots, L_6\}$ on the ground set $N = \{e_1, \ldots, e_{12}\}$.

Moreover, for each set $L \in \mathcal{L}$, we are given a bound $b_L \in \mathbb{Z}_{>0}$. Let $\mathcal{F}$ be the family of all subsets $S \subseteq N$ that contain no more than b_L elements of L for every $L \in \mathcal{L}$, i.e.,

$$\mathcal{F} = \{S \subseteq N : |S \cap L| \leq b_L \text{ for all } L \in \mathcal{L}\} \ .$$

Prove that the corresponding polytope $P_{\mathcal{F}} = \operatorname{conv}\left(\{\chi^S : S \in \mathcal{F}\}\right)$ is equal to

$$P = \{x \in [0,1]^N : x(L) \leq b_L \text{ for all } L \in \mathcal{L}\} \ .$$