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Mathematical Optimization – Solutions to problem set 10

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Bipartite matching with fixed cardinality

(a) For an integral vector $x \in P \cap \{0,1\}^E$, the condition $x(\delta(v)) \leq 1$ is precisely that at most one of the edges in $\delta(v)$ is contained in $M := \{e \in E \colon x(e) = 1\}$, i.e., v is incident to at most one edge in M for every vertex $v \in V$. This is equivalent to M being a matching, and hence x being the incidence vector of a matching. Moreover, $x(E) = k$ is equivalent to $|M| = k$, i.e., to x being the incidence vector of a matching of size k.

Consequently, we get that the conditions in P_k and those in P are equivalent for integral vectors, hence $P \cap \{0,1\}^E = P_k \cap \{0,1\}^E$.

(b) To show integrality of the polytope P via total unimodularity, we look at the inequality description of P and show that the constraint matrix is totally unimodular, and the right hand side is integral, to then exploit Theorem 5.8 from the script.

Note that in an inequality description, the equality constraint $x(E) = k$ is represented by the two inequality constraints $x(E) \leq k$ and $-x(E) \leq -k$, giving a row with all entries equal to 1 and one row with all entries equal to -1.

To show that the constraint matrix is totally unimodular we use the Ghouila-Houri criterion (Theorem 5.9 in the script). Our constraint matrix is composed of the constraint matrix corresponding to the bipartite matching polytope and the two additional rows mentioned above. Note that duplicating a row or multiplying all entries in a row with -1 preserves total unimodularity. Thus, it is enough to consider the submatrix A consisting of the constraint matrix corresponding to the bipartite matching polytope and one additional row with all entries equal to 1.

Consider any subset R of the rows of the matrix. Each row, apart from the all-ones row, which is potentially also included in the subset, corresponds to a vertex $u \in V_1 \cup V_2$ (where $V = V_1 \dot\cup V_2$ is a bipartition of the vertex set V such that every edge in G has one endpoint in V_1 and one endpoint in V_2). To obtain a partition $R = R_1 \dot\cup R_2$ of the rows in R, we distinguish two cases. In the first case, the all-ones rows is not part of the subset. Here, we can simply put every row corresponding to a vertex in V_1 into R_1, and every row corresponding to a vertex in V_2 into R_2. Since columns correspond to edges, and the graph is bipartite, we obtain that

$$\sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\} \quad \text{for all } e \in E.$$

In the second case, the all-ones row is part of the subset. Here, we put all rows but the all-ones row into R_1, and we put the all-ones row into R_2. Since every column of the matrix contains one entry equal to 1 in the all-ones row and tow entries equal to 1 in all the other rows, we obtain that the corresponding sum in the Ghouila-Houri criterion can only be in $\{-1, 0, 1\}$, as well.

From the Ghouila-Houri criterion, it thus follows that the constraint matrix is totally unimodular. This finishes the proof.

(c) Let $x \in [0,1]^E$ be a fractional point in P. We show that x can be written as a convex combination $x = \frac{1}{2}(x^1 + x^2)$ of two distinct points $x^1, x^2 \in P$. This implies that P is integral.

We call a vertex $v \in V$ *tight* if $x(\delta(v)) = 1$, and *loose* if $x(\delta(v)) < 1$. We call an edge $e \in E$ *fractional* if $x_e \in (0,1)$. Consider the graph H obtained from G by keeping all vertices and only fractional edges (i.e., by removing all edges $e \in E$ with $x_e \in \{0,1\}$). This graph contains some edges, since x is fractional by assumption.

Let $C = (U, F)$ be a non-trivial connected component in H, i.e., a connected component that is not a single vertex. We distinguish two cases. In the first case, assume that there exists a component C that contains a cycle $D \subseteq E$. Since G is bipartite, the cycle contains an even number of edges. Let these edges be $e_1, \ldots, e_{2k}$, numbered according to their order on D (where $k \in \mathbb{Z}_{>0}$). Define

$$\varepsilon := \min_{e \in E \colon x(e) \in (0,1)} \min\{x_e, 1 - x_e\} \ ,$$

and note that $\varepsilon > 0$. Then, we can define x^1, x^2 as follows:

$$x^1_e = \begin{cases} x_e & \text{if } e \in E \setminus D \\ x_e - \varepsilon & \text{if } e = e_i \text{ for } i \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i \text{ for } i \text{ even} \end{cases} \quad \text{and} \quad x^2_e = \begin{cases} x_e & \text{if } e \in E \setminus D \\ x_e + \varepsilon & \text{if } e = e_i \text{ for } i \text{ odd} \\ x_e - \varepsilon & \text{if } e = e_i \text{ for } i \text{ even} \end{cases} .$$

We note that $x^1, x^2 \in P$ indeed holds: First of all, by choice of ε, we have $x^i_e \in [0,1]$ for every $i \in \{1,2\}$ and $e \in E$. Furthermore, all constraints corresponding to vertices are not violated by x^i for $i \in \{1,2\}$; for vertices that are not contained in the cycle D, this follows from feasibility of x, while for vertices contained in D, it follows additionally from the fact that we added ε to exactly one incident edge, and subtracted ε from exactly one incident edge in both x^1 and x^2. Lastly, the constraint $x(E) = k$ is also satisfied, as the total number of edges for which we added ε is the same as the number of edges from which we subtracted ε, namely k. Finally, noticing that $x = \frac{1}{2}(x^1 + x^2)$, the claim is proved for this case.

In the remaining case, assume that there is no component C that contains a cycle, i.e., every connected component of H is a tree. In this case, notice that every leaf must be a loose vertex. If C is not a path, then it is easy to verify that it contains a path R between two leaves that has an even number of edges. To see this, consider, for example, any vertex u of degree at least three in C and any three edge-disjoint paths leading from u to three leaves in C. Since either the lengths of at least two of them are odd, or the lengths of at least two of them are even, the path in C connecting the corresponding leaves has even length. Given such a path, we can define, similarly to the case of an even cycle, a solution x^1 (resp., x^2) by adding (resp., subtracting for x^2) ε to every edge in an even position on the path and subtracting (resp., adding for x^2) ε from every edge in an odd position on the path. If $\varepsilon > 0$ is chosen as above, we can similarly show that the resulting x^1 and x^2 satisfy all required conditions, i.e., they are feasible, and $x = \frac{1}{2}(x^1 + x^2)$. Here we additionally use the fact that the endpoints of the path are loose, implying that adding ε to any incident edge of such a vertex does not violate the degree constraint at that vertex.

By the above, we can reduce our attention to the case where all components C of H are odd paths, whose only loose vertices are the endpoints of the paths. Notice that the number of such paths in H is at least two. Indeed, if there was only one path in H with edges $e_1, e_2, \ldots, e_{2r+1}$ (for some $r \in \mathbb{Z}_{\geq 0}$), then, using the fact that all middle vertices are tight it follows that $x_{e_j} = 1 - x_{e_{j+1}}$ for all $j = 1, \ldots, 2r$, implying that for some $\alpha \in (0,1)$ it holds that

$$x(\{e_1, \ldots, e_{2r+1}\}) = (r+1)\alpha + r(1-\alpha) = r + \alpha \ ,$$

which is fractional. Since these are the only edges in H, it follows that

$$k = x(E) = x(\{e_1, \ldots, e_{2r+1}\}) + x(E \setminus \{e_1, \ldots, e_{2r+1}\}) = r + \alpha + t$$

for some integer t, contradicting the fact that k is an integer. Thus, let Q_1 and Q_2 be two odd paths with loose endpoints in H, corresponding to two distinct connected components. We can now define x^1 by adding ε to every even edge on Q_1 and every odd edge on Q_2, and subtracting ε from every odd edge on Q_1 and every even edge on Q_2. x^2 is defined analogously, except that ε is subtracted (resp., added) whenever ε is added (resp., subtracted) in some edge in x^1. Clearly, since both paths are odd, we now have $x^1(E) = x^2(E) = x(E) = k$, and all other constraints are satisfied as well. Since $x = \frac{1}{2}(x^1 + x^2)$, the proof is complete.

Problem 2: Properties of vertices of the perfect matching polytope

We first show that if we have $y \in \mathbb{R}^{E[W]}_{\geq 0}$ and $z \in \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ with $y \in P(G[W])$ and $z \in P(G[V \setminus W])$, then the vector $x = (y, z) \in \mathbb{R}^{E[W]}_{\geq 0} \times \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ satisfies $x \in P(G)$. Since $y \in P(G[W])$, $z \in P(G[V \setminus W])$, and $\delta(W) = \emptyset$, it is easy to see that $x(\delta(v)) = 1$ for every $v \in V$. Moreover, let $S \subseteq V$ with $|S|$ odd. Then, either $|S \cap W|$ or $|S \cap (V \setminus W)|$ must be odd. We assume here that $|S \cap W|$ is odd, the other case being analogous. Again using that $y \in P(G[W])$ and that $\delta(W) = \emptyset$, we obtain

$$x(\delta(S)) = x(\delta(S \cap W)) + x(\delta(S \cap (V \setminus W))) \geq x(\delta(S \cap W)) = y(\delta(S \cap W)) \geq 1 \ ,$$

as desired.

On the other hand, given $x \in P(G)$, we show that $y = x|_{E[W]} \in \mathbb{R}^{E[W]}_{\geq 0}$ and $z = x|_{E[V \setminus W]} \in \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ are points in the perfect matching polytopes of $G[W]$ and $G[V \setminus W]$, respectively. We show this for y only, as the proof for z is identical. Consider the constraints in the definition of the perfect matching polytope corresponding to $G[W]$. We show that all constraints are satisfied by y, hence it is a point in this polytope. The constraints $y(\delta(v)) = 1$ hold for every $v \in W$ since there are no edges between a vertex in W and a vertex in $V \setminus W$ (because $\delta(W) = \emptyset$), and y is a restriction of $x \in P(G)$ to $E[W]$ (in other words, all edges incident to vertices in W connect two vertices in W). Similarly, the constraint $y(\delta(S)) \geq 1$ is also satisfied for every odd set $S \subseteq W \subseteq V$ since all edges incident to some vertex in W connect two vertices of W.

With the above at hand, we can now proceed to proving the statement of this problem. Let $y \in \mathbb{R}^E_{\geq 0}$ be a vertex of $P(G)$, and let $y_1 = y|_{E[W]} \in \mathbb{R}^{E[W]}_{\geq 0}$ and $y_2 = y|_{E[V \setminus W]} \in \mathbb{R}^{E[V \setminus W]}_{\geq 0}$. From above, we know that y_1 and y_2 are points in the perfect matching polytopes of $G[W]$ and $G[V \setminus W]$, respectively. It remains to show that they are extreme points. Assume for contradiction that y_1 (the same argument goes through for y_2) is not an extreme point of the perfect matching polytope corresponding to $G[W]$, i.e., there exist distinct $z_1, z_2 \in \mathbb{R}^{E[W]}_{\geq 0}$ in this polytope satisfying $y_1 = \frac{1}{2}(z_1 + z_2)$. Using our result from above, we get that the vectors $x_1 = (z_1, y_2) \in \mathbb{R}^{E[W]}_{\geq 0} \times \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ and $x_2 = (z_2, y_2) \in \mathbb{R}^{E[W]}_{\geq 0} \times \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ are distinct points in $P(G)$, and that $y = \frac{1}{2}(x_1 + x_2)$. This contradicts the assumption that y is an extreme point of $P(G)$.

The opposite direction is very similar. Given vertices y_1 and y_2 of the perfect matching polytopes corresponding to $G[W]$ and $G[V \setminus W]$, we show that $y = (y_1, y_2) \in \mathbb{R}^{E[W]}_{\geq 0} \times \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ is a vertex of $P(G)$, the perfect matching polytope of G. As established above, we have that $y \in P(G)$. To finish the proof, we assume for contradiction that y is not an extreme point, i.e., there exist distinct $x_1, x_2 \in P(G)$ such that $y = \frac{1}{2}(x_1 + x_2)$. From above, we know that $x_1|_{E[W]}$ and $x_2|_{E[W]}$ are points in the perfect matching polytope of $G[W]$, and that $x_1|_{E[V \setminus W]}$ and $x_2|_{E[V \setminus W]}$ are points in the perfect matching polytope of $G[V \setminus W]$. Since $x_1 \neq x_2$, we either have $x_1|_{E[W]} \neq x_2|_{E[W]}$ or $x_1|_{E[V \setminus W]} \neq x_2|_{E[V \setminus W]}$. Since $y_1 = \frac{1}{2}(x_1|_{E[W]} + x_2|_{E[W]})$ and $y_2 = \frac{1}{2}(x_1|_{E[V \setminus W]} + x_2|_{E[V \setminus W]})$, we reached a contradiction to the assumption that both y_1 and y_2 are extreme points.

Alternative proof: An alternative proof can be obtained by using the fact that $P(G)$ is the perfect matching polytope corresponding to the graph G, i.e., the convex hull of the characteristic vectors of all perfect matchings in G.

To this end, let $y \in \mathbb{R}^E_{\geq 0}$ be a vertex of $P(G)$. We know that $y \in \{0,1\}^E$, and that $M = \{e \in E \mid y(e) = 1\}$ is a perfect matching in G. Since there are no edges with one endpoint in W and the other in $V \setminus W$, the perfect matching M can be partitioned into $M[W]$ and $M[V \setminus W]$, where $M[W]$ is a perfect matching in $G[W]$ and $M[V \setminus W]$ is a perfect matching in $G[V \setminus W]$. Consequently, $y|_{E[W]}$ is the characteristic vector of a perfect matching in $G[W]$, so it follows immediately that $y|_{E[W]}$ is an extreme point of $P(G[W])$. Analogously, we see that $y|_{E[V \setminus W]}$ corresponds to an extreme point of $P(G[V \setminus W])$.

The other direction follows similarly. If $y|_{E[W]}$ and $y|_{E[V \setminus W]}$ are extreme points of $P(G[W])$ and $P(G[V \setminus W])$, respectively, they are characteristic vectors of perfect matchings in $G[W]$ and $G[V \setminus W]$. Together, the two matchings form a perfect matching in G, so $y = (y|_{E[W]}, y|_{E[V \setminus W]}) \in \mathbb{R}^{E[W]}_{\geq 0} \times \mathbb{R}^{E[V \setminus W]}_{\geq 0}$ is the characteristic vector of a perfect matching in G, and hence an extreme point of $P(G)$.

Problem 3: Perfect matchings in three-regular graphs

Let $G = (V, E)$ be a 3-regular bridgeless graph. To prove that a perfect matching exists, we find a fractional point $x \in P$, where

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V,\ |S| \text{ odd} \end{array} \right\} .$$

is the perfect matching polytope of G. Since P is integral, it follows that there also exists an integral vector $x' \in P$, where x' will be equal to the characteristic vector of a perfect matching in G.

Let $x = (1/3, 1/3, \ldots, 1/3) \in \mathbb{R}^E_{\geq 0}$. We claim that $x \in P$. Let us verify the constraints of P. First, since G is 3-regular, we have $x(\delta(v)) = 1$ for every $v \in V$, as required. The non-negativity constraints are obviously satisfied as well. Next, consider any odd set $S \subseteq V$. We need to show that $\delta(S)$ contains at least three edges.

Since G is 3-regular, we have

$$2 \cdot |E[S]| + |\delta(S)| = \sum_{v \in S} |\delta(v)| = 3 \cdot |S| \ .$$

As $|S|$ is odd, this implies that $|\delta(S)|$ is odd as well. Hence, if we can show that $|\delta(S)| \neq 1$, we are done. This, however, follows immediately from the fact that G is bridgeless. Indeed, if we had $|\delta(S)| = 1$, then the single edge in $\delta(S)$ would be a bridge.

Thus, we conclude that $\delta(S)$ contains at least three edges, implying that $x(\delta(S)) \geq 1$. This finishes the proof.

Remark: The graph in Figure 1 shows that the assumption of being bridgeless cannot be dropped. Indeed, note that the three connected components that we get after deleting the middle vertex all have an odd number of vertices. This implies that in order for a matching to cover one such component, the edge connecting it to the middle vertex has to be a matching edge. A perfect matching would have to cover all three components, and hence by the above reasoning, it would have to contain all three edges incident to the middle vertex, which is impossible for a matching.

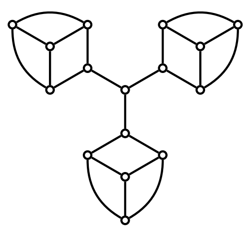

Figure 1: A 3-regular graph that does not admit a perfect matching.

Problem 4: Facets of the spanning tree polytope

(a) We prove that the property holds for the complete graph $K_n = (V, E)$ for every $n \geq 4$.

Consider first any constraint $x(E[S]) \leq |S| - 1$ corresponding to a subset $S \subsetneq V$ of the vertices with $|S| \geq 2$, and consider the polytope P' defined exactly as P, but without the constraint for S. We first assume that $|S| \geq 3$. Let $A \subseteq E$ be a set of edges such that $H \coloneqq (V, A)$ has the following properties:

- The graph $H[S]$ induced on S is a spanning cycle.
- The graph $H[V \setminus S]$ is a spanning tree.
- No edge of A connects a vertex from S to a vertex from $V \setminus S$.

Consider $x = \chi^A$. Clearly $x \notin P$, as $x(E[S]) = |S| > |S| - 1$. We prove that $x \in P'$. It is easy to see that $x(E) = |A| = n - 1$ and $x \geq 0$, thus we only need to check the constraints for sets $T \subsetneq V$ with $|T| \geq 2$ and $T \neq S$. Let T be such a set. If $S \not\subseteq T$, then the graph $H[T]$ is acyclic and the corresponding constraint is satisfied. If, otherwise, $S \subsetneq T$, we have the following property: By removing one edge from the cycle in the subgraph of $H[T]$, we obtain a forest with at least two connected components. This follows from the fact that there is exactly one cycle in H, and that T contains at least one vertex in $V \setminus S$. Consequently, the resulting forest on T has at most $|T|-2$ edges, so by adding the deleted edge, we see $x(E[T]) \leq |T|-1$, and the constraint corresponding to T is satisfied.

If $|S| = 2$, let e be the edge connecting the two vertices in S, and let $A \subseteq E$ be a set of edges such that $H := (V, A)$ has the following properties:

- $H[S]$ equals $(S, \{e\})$.
- $H[V \setminus S]$ is a spanning tree.
- No edge of A connects a vertex from S to a vertex of $V \setminus S$.

Let $x = \chi^A + \chi^{\{e\}}$. Since $x(E[S]) = x_e = 2 > |S| - 1$, we have that $x \notin P$. Again, we show that $x \in P'$. First of all, x satisfies the non-negativity constraints, and $x(E) = |A| + 1 = |V| - 1$. Moreover, let $T \subsetneq V$ with $|T| \geq 2$ and $T \neq S$. If $S \not\subseteq T$, then e is not contained in the graph induced by T. Since H is acyclic, the constraint corresponding to T is satisfied. If $S \subsetneq T$, then $x(E[T]) = x(E[S]) + x(E[T \setminus S]) = 2 + x(E[T \setminus S]) \leq |T| - 1$, because the graph $H[T \setminus S]$ induced on $T \setminus S$ is acyclic.

A similar argument shows that the constraint $x(E) = n - 1$ is non-redundant. To this end, we can take the characteristic vector $x = \chi^A$ of any Hamiltonian cycle $A \subseteq E$ (a simple cycle visiting every node exactly once) as an example of a vector in $P' \setminus P$, where P' is defined as P but without the constraint $x(E) = n - 1$.

Yet another very similar argument proves that non-negativity constraints are not redundant. Let P' now denote the polytope defined as P, but without the constraint $x_e \geq 0$ for some edge $e = \{u, v\}$. Let $A \subseteq E$ be the edges of any Hamiltonian cycle that does not contain e (such a cycle exists if $n \geq 4$), and let $x = \chi^A - \chi^{\{e\}}$. Then $x \in P' \setminus P$.

This completes the proof that no constraints in the description of the spanning tree polytope are redundant.

(b) First of all, since P is contained in the hyperplane defined by $x(E) = |V| - 1$, we have that $\dim(P) \leq |E| - 1$. We now consider the point $\bar{x} = (2/n, 2/n, \ldots, 2/n) \in P$ again. Notice that for any $S \subsetneq V$ with $|S| \geq 2$ we have

$$\bar{x}(E[S]) = \binom{|S|}{2} \cdot \frac{2}{n} = \frac{|S| \cdot (|S| - 1)}{n} < |S| - 1 \ ,$$

since $|S| < n$. Moreover, $x_e > 0$ for any $e \in E$. As all inequalities are strictly satisfied by $\bar{x}$, there exists $\varepsilon > 0$ such that the ball $B^* = B(\bar{x}, \varepsilon) \cap \{x \in \mathbb{R}^E : x(E) = |V| - 1\}$ is contained in the polytope P. Clearly, $\dim(B^*) = |E| - 1$, and hence $\dim(P) \geq |E| - 1$, which shows that $\dim(P) = |E| - 1$.

(c) Let $a^T x \leq b$ be a non-redundant inequality in the description of P, for which a is not orthogonal to H. The following shows that this inequality is facet-defining. The analysis for the case $a^T x \geq b$ follows in the same way, as it is equivalent to $-a^T x \leq -b$.

Since H is the minimal affine subspace that contains P and a is not orthogonal to H, there exist two points $x_1, x_2 \in P$ such that $a^T(x_1 - x_2) \neq 0$. This guarantees that for at least one of them, without loss of generality say x_1, $a^T x_1 < b$ holds.

Let P^* be the polytope defined without the constraint $a^T x \leq b$. Thus, $P = P^* \cap \{x \in \mathbb{R}^n : a^T x \leq b\}$. Let us denote $Y = P^* \setminus P$ and let $y \in Y$ be a point for which $a^T y > b$. Clearly, y exists because $a^T x \leq b$ is not redundant for P.

Since x_1 is in P and $a^T x_1 < b$, there exists a small enough $\lambda \in (0, 1]$ for which $(1 - \lambda)x_1 + \lambda y$ is in P and $a^T((1 - \lambda)x_1 + \lambda y) < b$. Indeed, recall that y violates only the constraint $a^T x \leq b$ in the description of P, thus any other constraint will be satisfied by any convex combination of x_1 and y. However, since x_1 and $(1 - \lambda)x_1 + \lambda y$ are in $P \subseteq H$, also y is in H, as affine combination of two points in H.

Consider then the set $B^* = B(x_1, \varepsilon) \cap P \subseteq H$, for a small enough $\varepsilon > 0$ such that $B^* \subseteq \{x \in \mathbb{R}^n \colon a^T x < b\}$. Clearly, it holds that $\dim(B^*) = \dim(P) = \dim(H) =: d$, and, since y is in H, $\dim(\mathrm{conv}(B^* \cup \{y\})) = \dim(H) = d$. Moreover, the hyperplane $\{x \in \mathbb{R}^n \colon a^T x = b\}$ separates B^* from y, which leads to $\dim(\mathrm{conv}(B^* \cup \{y\}) \cap \{x \in \mathbb{R}^n \colon a^T x = b\}) = d - 1$.

Finally, observe that

$$\mathrm{conv}(B^* \cup \{y\}) \cap \{x \in \mathbb{R}^n \colon a^T x = b\} \subseteq P \cap \{x \in \mathbb{R}^n \colon a^T x = b\} \subseteq H \cap \{x \in \mathbb{R}^n \colon a^T x = b\} \ ,$$

from which $\dim(P \cap \{x \in \mathbb{R}^n \colon a^T x = b\}) = d - 1$ follows directly from

$$\dim(\mathrm{conv}(B^* \cup \{y\}) \cap \{x \in \mathbb{R}^n \colon a^T x = b\}) = \dim(H \cap \{x \in \mathbb{R}^n \colon a^T x = b\}) = d - 1 \ .$$

(d) According to part (c), it is sufficient to show that all the inequalities of the spanning tree polytope for K_n are not redundant and defined by vectors that are not orthogonal to the minimal affine subspace that contains P.

In part (a), we already showed non-redundancy. Part(b) shows that $\{x \in \mathbb{R}^E \colon x(E) = |V| - 1\}$ is the minimal affine subspace that contains P. Finally, since χ^E is linearly independent to any vector $\chi^{E[S]}$ for $S \subsetneq V$ with $|S| \geq 2$, for both non-negativity and anti-cycle constraints the property from part (c) holds, and thus each of these constraints is facet-defining.

Problem 5: Degeneracy and the spanning tree polytope

Consider the complete graph K_n, its spanning tree polytope P, and let $x = \chi^T$ be a characteristic vector of a spanning *star* $T \subseteq E$, i.e., a spanning tree with $n-1$ leaves, and one center vertex r with degree $n - 1$ (see Figure 2 for an example with $n = 4$).

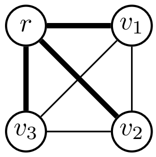

Figure 2: The complete graph K_4 and a spanning star T (thick edges) with center vertex r.

Consider the set of constraints of P that are tight at x. We show that every inequality constraint corresponding to a set of vertices $S \subsetneq V$ with the properties that $r \in S$ and $|S| \geq 2$ corresponds to a tight constraint. Indeed, notice that the subgraph of (V, T) induced on S is a star with $|S| - 1$ edges. It follows that $x(E[S]) = \chi^T(E[S]) = |S| - 1$ for every such S. Now it remains to note that the number of such subsets is $2^{n-1} - 2$. This proves the claim (choosing, for example, $c = \frac{1}{3}$).

Problem 6: Description for the dominant of the r-arborescence polytope

(a) Let $T \subseteq A$ be an r-arborescence in G, and let $x = \chi^T \in \{0,1\}^A$ be its characteristic vector. Take a set $S \subseteq V \setminus \{r\}$ with $S \neq \emptyset$. Since every vertex $v \in S$ can be reached from $r \notin S$ using a directed path in T (and $S \neq \emptyset$), the r-arborescence T must contain an arc that enters S, i.e., $T \cap \delta^-(S) \neq \emptyset$. We conclude that

$$x(\delta^-(S)) = \chi^T(\delta^-(S)) = |T \cap \delta^-(S)| \geq 1 \ ,$$

as desired. As $S \subseteq V \setminus \{r\}$ with $S \neq \emptyset$ was arbitrary (and clearly, we have that $x \in \mathbb{R}^A_{\geq 0}$), it follows that $x = \chi^T \in P$.

We now show that P is up-closed. Since $0 \in \mathbb{R}^A_{\geq 0}$, it is clear that $P = P + \{0\} \subseteq P + \mathbb{R}^A_{\geq 0}$. On the other hand, let $z = x + y \in P + \mathbb{R}^A_{\geq 0}$, where $x \in P$ and $y \in \mathbb{R}^A_{\geq 0}$. Since $x \in P \subseteq \mathbb{R}^A_{\geq 0}$, we get that $z = x + y \in \mathbb{R}^A_{\geq 0}$. Moreover, for $S \subseteq V \setminus \{r\}$ with $S \neq \emptyset$, we have that

$$z(\delta^-(S)) = x(\delta^-(S)) + \underbrace{y(\delta^-(S))}_{\geq 0} \geq x(\delta^-(S)) \geq 1 \ ,$$

where in the last inequality we used that $x \in P$. Consequently, we have that $z \in P$, proving that $P = P + \mathbb{R}^A_{\geq 0}$.

We know that the r-arborescence polytope $P_{r\text{-arb}}$ equals the convex hull of the characteristic vectors of all r-arborescences in G, i.e., $P_{r\text{-arb}} = \operatorname{conv}(\{\chi^T : T \text{ is } r\text{-arborescence in } G\})$. Moreover, we have shown above that $\{\chi^T : T \text{ is } r\text{-arborescence in } G\} \subseteq P$ and that $P = P + \mathbb{R}^A_{\geq 0}$. Using that P (being a polyhedron) is convex, it therefore follows that

$$P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0} = \operatorname{conv}\left(\{\chi^T : T \text{ is } r\text{-arborescence in } G\}\right) + \mathbb{R}^A_{\geq 0} \subseteq P + \mathbb{R}^A_{\geq 0} = P \ .$$

(b) Let x be a vertex of P. Using integrality of P, we have that $x \in \mathbb{Z}^A_{\geq 0}$. We first show that $x \in \{0,1\}^A$. Assume, for the sake of contradiction, that there is $a \in A$ with $x_a \geq 2$. We then define the two vectors $x^- = x - \chi^{\{a\}}$ and $x^+ = x + \chi^{\{a\}}$. Since $x \geq 0$ and $x_a \geq 2$, we easily see that $x^-, x^+ \in P$. The fact that $x = \frac{1}{2}(x^- + x^+)$, however, yields a contradiction to x being a vertex of P. Thus, we conclude that $x \in \{0,1\}^A$, i.e., x equals the charecteristic vector χ^T of some arc set $T \subseteq A$.

We now show that T is an r-arborescence (see Definition 10 in the lecture notes). To see this, let $R \subseteq V$ be the set of all vertices in G that can be reached from r using a directed path in T, and let $S = V \setminus R$. By definition of S, it holds that $T \cap \delta^-(S) = \emptyset$, implying that $x(\delta^-(S)) = \chi^T(\delta^-(S)) = 0$. Since $S \subseteq V \setminus \{r\}$ and $x \in P$, we must have $S = \emptyset$. This shows $R = V$, proving part (ii) of Definition 10.

We observe (for example, by running BFS starting at r in the directed graph (V,T)) that the above also implies that T contains an r-arborescence $F \subseteq T$. We then define the two vectors $y^- = \chi^F$ and $y^+ = \chi^F + 2\chi^{T \setminus F}$. Since F is an r-arborescence and $P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0} \subseteq P$ (using part (a)), we have that $y^-, y^+ \in P$. Moreover, it clearly holds that $x = \chi^T = \frac{1}{2}(y^- + y^+)$. The fact that x is an extreme point of P now implies that $y^- = y^+$, i.e., $T \setminus F = \emptyset$. This means that $T = F$ is indeed an r-arborescence, as we wanted to show.

(c) Let $z \in P$ be arbitrary. By Proposition 1.38 from the script, we can write $P = Q + C$ for a polytope Q and a cone C, and we can choose $Q = \operatorname{conv}(\operatorname{vertices}(P))$. Thus, we can write z as $z = x + y$, where $x \in Q$ and $y \in C$.

In part (b), we have seen that every vertex of P is the characteristic vector of an r-arborescence, hence $Q \subseteq P_{r\text{-arb}}$ because $P_{r\text{-arb}}$ is convex. In particular, we get that $x \in P_{r\text{-arb}}$. Moreover, we observe that $P \subseteq \mathbb{R}^A_{\geq 0}$ implies that $C \subseteq \mathbb{R}^A_{\geq 0}$, and hence $y \in \mathbb{R}^A_{\geq 0}$.

Combining the above, we conclude that $z = x + y \in P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0}$, proving $P \subseteq P_{r\text{-arb}} + \mathbb{R}^A_{\geq 0}$.