6 Ellipsoid Method

How to solve linear programs over polyhedra with exponentially many constraints?

For example:

Theorem 5.17

The spanning tree polytope of an undirected loopless graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{ x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \} .$$

Theorem 5.21

The perfect matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\} .$$

Ellipsoid method can solve LPs over above polyhedra.

Its main ingredient is:

A <u>separation oracle</u> for the polyhedron over which to optimize.

6.1 Separation problem

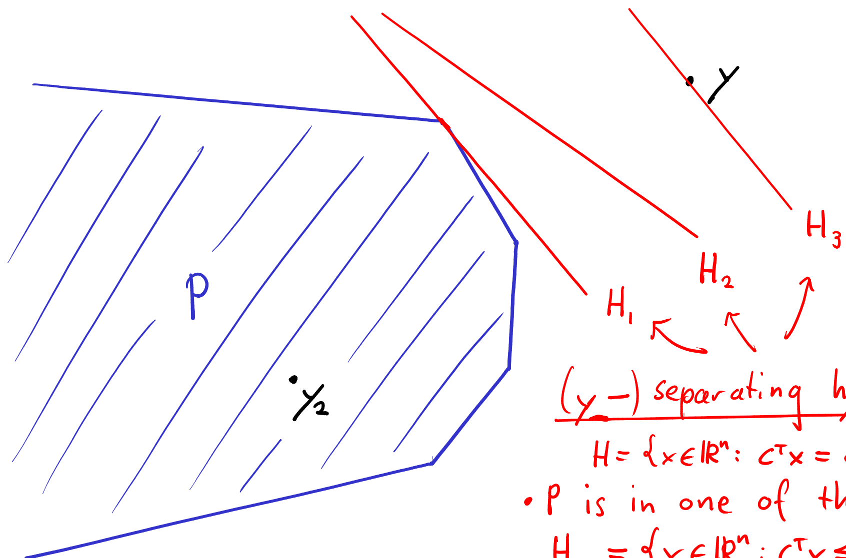
Separation problem for a polyhedron $P \subseteq \mathbb{R}^n$:

Definition 6.1: Separation problem & separation oracle

Given a point $y \in \mathbb{R}^n$:

- Decide whether $y \in P$, and if this is not the case,
- find $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top y\}$.

A procedure that solves the separation problem (for P) is often called a *separation oracle* (for P).



(y-) separating hyperplanes:

$H = \{x \in \mathbb{R}^n : c^\top x = \alpha\}$ s.t.

- P is in one of the closed halfspaces $H_\leq = \{x \in \mathbb{R}^n : c^\top x \leq \alpha\}$ & $H_\geq = \{x \in \mathbb{R}^n : c^\top x \geq \alpha\}$,
- y is in the other one, and
- either $P \cap H = \emptyset$ or $y \notin H$.

6.2 Optimization results based on Ellipsoid Method

Theorem 6.2

Let $P \subseteq \mathbb{R}^n$ be a $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x : x \in P\}$ using a polynomial number (in n) of operations and calls to the separation oracle for P.

$\rightarrow$ Even polyhedra with exponentially many facets can admit efficient separation oracles.

6.3 Example application

Minimum weight r-arborescence

Let $G = (V, A)$ be a directed graph with arc weights $w: A \to \mathbb{Z}_{\geq 0}$.

Our goal: Find r-arborescence $T \subseteq A$ minimizing $w(T)$.

$\to$ Reduces to minimizing w over dominant P of r-arborescence polytope.

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset\} \quad .$$

$$\min \; w^T x$$
$$x \in P$$