Running time analysis of Ford and Fulkerson's algorithm

Lemma 4.11: Positive integral augmentation volume

Each augmentation step of Ford and Fulkerson's algorithm has an integral augmentation volume of at least one unit.

Proof

- Starting flow f (all-zeros flow) is integer.

$\Rightarrow$ Because capacities $u: A \to \mathbb{Z}_{\geq 0}$ and f are both integer, also the residual capacities $u_f: B \to \mathbb{Z}_{\geq 0}$ are integer.

$$\boxed{\left.\begin{array}{l} u_f(a) = u(a) - f(a) \\ u_f(a^R) = f(a) \end{array}\right\} \forall a \in A}$$

$\Rightarrow$ Increment μ of augmenting path $P \subseteq B$, i.e.,

$$\boxed{\mu = \min\{u_f(b) : b \in P\}}$$

is integer.

$\Rightarrow$ Augmentation of f along P leads to s-t flow f' that is integer.

$$\boxed{\begin{array}{lll} f'(a) = f(a) + \mu & \forall\, a \in A \cap P \\ f'(a) = f(a) - \mu & \forall\, a \in A \text{ s.t. } a^R \in P \\ f'(a) = f(a) & \text{otherwise} \end{array}}$$

$\hookrightarrow$ Repetition of this reasoning leads to statement. (induction)

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Corollary 4.12: Running time bound for Ford and Fulkerson's algorithm

The number of augmentation steps in Ford and Fulkerson's algorithm is bounded by the value $\alpha \in \mathbb{R}_{\geq 0}$ of a maximum s-t flow in G. Hence, the running time of Ford and Fulkerson's algorithm is bounded by $O(\alpha \cdot (m + n))$, assuming $\alpha \geq 1$.

Proof

Each augmentation increases flow by at least one unit. (Lemma 4.11)

$\Longrightarrow$ # augmentations $\leq \lceil \alpha \rceil$

- Time per augmentation : $O(m+n)$ (Lemma 4.10)

$\Longrightarrow$ Total running time : $O(\alpha (m+n))$

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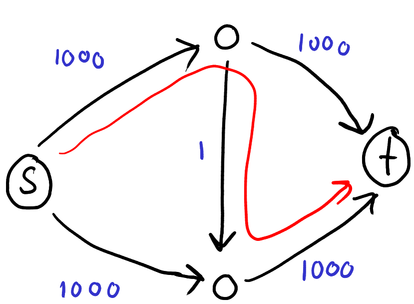
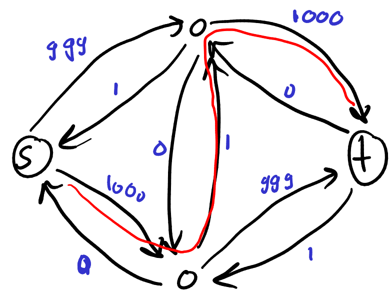
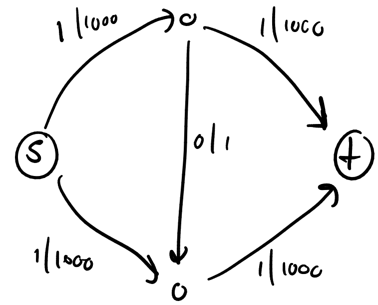
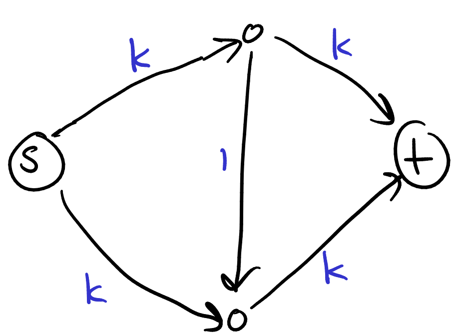
Running time of Ford and Fulkerson is not polynomial in general







Repeating these steps, we get a maximum s-t flow after 2000 iterations.



Input size : $\Theta(\log k)$

Running time : $\Theta(k)$

Max-flow min-cut theorem & correctness of Ford and Fulkerson

Theorem 4.13

Let f be an s-t flow in G. Then the following are equivalent:

(i) f is a maximum s-t flow.
(ii) There does not exist an f-augmenting path in G_f.
(iii) There exists an s-t cut $C \subseteq V$ with $u(\delta^+(C)) \underset{\geq}{=} \nu(f)$.

Furthermore, a minimum s-t cut can be found in linear time given a maximum s-t flow.

Proof   We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

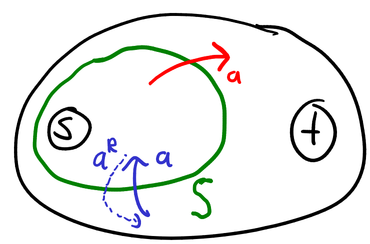
(i) $\Rightarrow$ (ii) : We show contraposition, i.e., $\neg$(ii) $\Rightarrow$ $\neg$(i).

$\neg$(ii) $\Leftrightarrow$ $\exists$ f-augmenting path $\Rightarrow$ f is not a maximum flow, because it can be augmented along the f-augmenting path.

(ii) $\Rightarrow$ (iii) :

$G_f = (V, B)$. Let $U := \{ b \in B : u_f(b) > 0 \}$

Let $S := \{ v \in V : \exists\ s\text{-}v \text{ path in } (V, U) \}$



We have : (a) $s \in S$

(b) $t \notin S$ (because $\nexists$ f-augmenting path)

(c) $\big[f(a) = u(a) \quad \forall\ a \in \delta^+(S)$ ← for otherwise: $u_f(a) > 0 \Rightarrow a \in U$

$\big[f(a) = 0 \quad \forall\ a \in \delta^-(S)$ ← for otherwise: $u_f(a^R) > 0 \Rightarrow a^R \in U$

Lemma 4.3 & S is an s-t cut

$$\nu(f) = \underbrace{f(\delta^+(S))}_{= u(\delta^+(S))} - \underbrace{f(\delta^-(S))}_{= 0} = u(\delta^+(S))$$

(iii) $\Rightarrow$ (i)

By (iii): $\exists$ s-t cut $S \subseteq V$ with $u(\delta^+(S)) = \nu(f)$.

By weak max-flow min-cut theorem, we have for any s-t flow f':

$$\nu(f') \leq u(\delta^+(S)) = \nu(f)$$

$\Rightarrow$ f has maximum value.

\#

Corollary 4.14: Strong max-flow min-cut theorem

The value of a maximum s-t flow in G is equal to the value of a minimum s-t cut in G:

$$\max\{\nu(f) : f \text{ is } s\text{-}t \text{ flow in } G\} \underset{\leq}{=} \min\left\{u(\delta^+(C)) : C \subseteq V, s \in C, t \notin C\right\} .$$

weak max-flow min-cut theorem

Corollary 4.15: Correctness of Ford and Fulkerson's algorithm

Ford and Fulkerson's algorithm returns a maximum s-t flow.

4.3 Integrality of s-t flows

Theorem 4.16: Integral maximum flows

Let $G = (V, A)$ be a directed graph with capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, and let $s, t \in V$, $s \neq t$. Ford and Fulkerson's algorithm finds a maximum s-t flow that is integral.

Proof

The flow f found by Ford and Fulkerson's algorithm is

(i) maximum by Corollary 4.15, and

(ii) integral because all augmentation volumes are integral by Lemma 4.11.

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