5 Polyhedral Approaches in Combinatorial Optimization

Combinatorial optimization problems can often be described by:

(i) A finite set N, called <u>ground set</u>,

(ii) a family $\mathcal{F} \subseteq 2^N$ of <u>feasible sets</u>, also called <u>solutions</u>, and

(iii) an objective function $w: N \to \mathbb{R}$ to maximize or minimize.

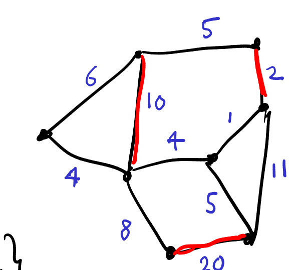
$\xrightarrow[\text{problem}]{\text{corresponding}}$

$$\boxed{\begin{array}{c} \max / \min \quad w(F) := \sum_{e \in F} w(e) \\ F \in \mathcal{F} \end{array}}$$

<u>Examples</u>

Maximum weight matchings:

given is undirected graph $G=(V,E)$ with non-negative edge weights $w: E \to \mathbb{R}_{\geq 0}$



$G=(V,E)$

$w: E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$

(ii) Feasible sets: $\mathcal{F} = \{ M \subseteq E : M \text{ is a matching} \}$

(iii) Objective: maximize w

Well-known special cases:

- Maximum cardinality matching $\rightsquigarrow$ $w(e) = 1 \quad \forall e \in E$.
- Maximum cardinality/weight bipartite matchings $\rightsquigarrow$ G is bipartite.

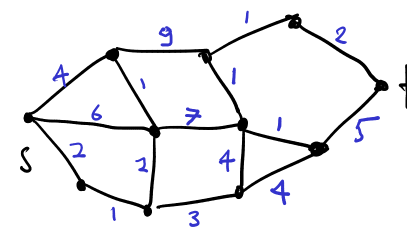
Shortest s-t path (in undir. graphs)

Given: • undir. graph $G=(V,E)$
• vertices $s,t \in V$
• non-neg. edge lengths $\ell: E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$.

(ii) Feasible sets: $\mathcal{F} = \{P \subseteq E : P \text{ is } s\text{-}t \text{ path}\}$.

(iii) Objective: minimize $w = \ell$.



$\ell: E \to \mathbb{R}_{\geq 0}$

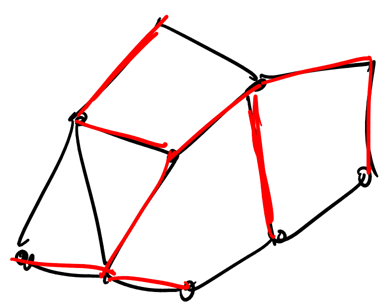
Minimum weight spanning tree

Given: • undir. graph $G=(V,E)$
• edge weights $w: E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$.

(ii) Feasible sets: $\mathcal{F} = \left\{ F \subseteq E : F \text{ is a spanning tree in } G \right\}$.

(iii) Objective: minimize w.



5.1 Polyhedral descriptions of combinatorial optimization problems

Let N be a finite (ground) set.

$\mathbb{R}^{|N|}$

$\chi^U \in \mathbb{R}^N$

Definition

For $U \subseteq N$, we denote by χ^U its <u>characteristic vector</u> (also called <u>incidence vector</u>):

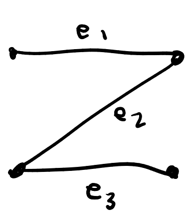
$$\chi^U(e) = \begin{cases} 1 & \text{if } e \in U. \\ 0 & \text{if } e \in N \setminus U. \end{cases}$$

Let $\mathcal{F} \subseteq 2^N$ be all feasible sets to a combinatorial optimization problem.

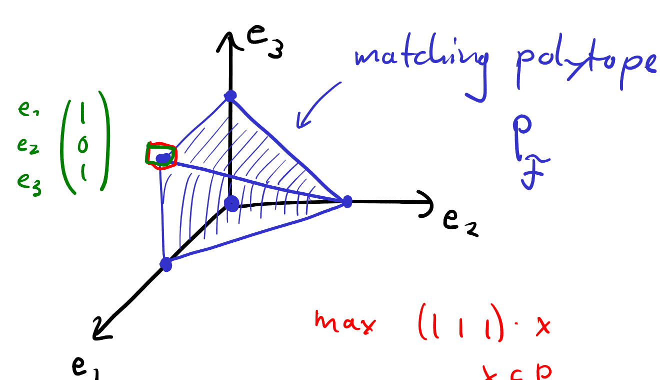
The <u>(combinatorial) polytope that corresponds to $\mathcal{F}$</u> is the polytope $P_{\mathcal{F}} \subseteq [0,1]^N$ whose vertices are precisely $\{\chi^F : F \in \mathcal{F}\}$, i.e.,

$$P_{\mathcal{F}} = \operatorname{conv}\left(\{\chi^F : F \in \mathcal{F}\}\right).$$

$G = (V, E)$



$\mathcal{F} = \{ M \subseteq E : M \text{ is a matching} \}$



$\max \ (1\ 1\ 1) \cdot x$

$x \in P_{\mathcal{F}}$

The combinatorial polytope allows for casting a combinatorial optimization problem into a linear program (and can be used for much more):

$$\boxed{\begin{array}{ll} \max/\min & w(F) \\ & F \in \mathcal{F} \end{array}} \quad \rightsquigarrow \quad \boxed{\begin{array}{ll} \max/\min & w^T x \\ & x \in P_{\mathcal{F}} \end{array}}$$

Optimal vertex solution to this LP is characteristic vector of optimal solution of combinatorial optimization problem.

Key challenge: Find explicit inequality description of $P_{\mathcal{F}}$.

$$\rightarrow \quad P_{\mathcal{F}} = \{ x \in \mathbb{R}^N : Ax \leq b \}$$

Some benefits of getting an inequality description: (let $n := |N|$)

- Often, $\#$ facets of $P_{\mathcal{F}} = O(\text{poly } n)$.
  $\#$ vertices of $P_{\mathcal{F}} = 2^{\Omega(n)}$.

- If we can solve LPs over $P_{\mathcal{F}}$, then we can optimize any linear objective.

- Even when $P_{\mathcal{F}}$ has exponentially many facets, one can often get a description of them and even solve LPs over $P_{\mathcal{F}}$.

  for example, by using the Ellipsoid Method

- Being able to solve LPs over $P_{\mathcal{F}}$ often allows for solving related problems, for example by adding some extra constraints.

- The <u>LP dual</u> of $\max\{w^T x : x \in P_{\mathcal{F}}\}$ can often be interpreted combinatorially. Possible implications:
  - Natural optimality certificates through strong duality.
  - Fast algorithms based on dual such as primal-dual methods.

- Elegant polyhedral proof techniques.

$\vdots$

5.2 Meta-recipe for finding inequality-descriptions

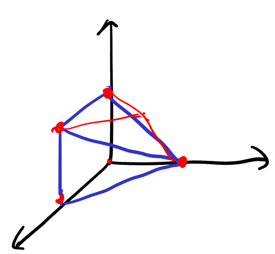
① Determine candidate description $P = \{x \in \mathbb{R}^N : Ax \le b\} \subseteq [0,1]^N$.

This shows $P = P_{\mathcal{F}}$

② Prove
$$P \cap \{0,1\}^N = \{\chi^F : F \in \mathcal{F}\}.$$

③ Prove that $\underline{P \text{ is integral}}$.

↑ i.e., $\text{vertices}(P) \subseteq \mathbb{Z}^N$, which, because $P \subseteq [0,1]^N$, is same as $\text{vertices}(P) \subseteq \{0,1\}^N$



(i) P integral and $P \subseteq [0,1]^N$

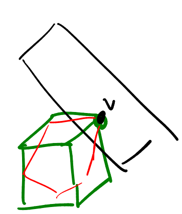
$\Rightarrow \text{vertices}(P) \subseteq P \cap \{0,1\}^N$

(ii) Moreover if $\underline{v \in P \cap \{0,1\}^N} \Rightarrow v \in \text{vertices}(P)$

(i) & (ii) $\Rightarrow \text{vertices}(P) = P \cap \{0,1\}^N \overset{②}{=} \{\chi^F : F \in \mathcal{F}\}$

$\Rightarrow P = \text{conv}(\text{vertices}(P)) = \text{conv}(\{\chi^F : F \in \mathcal{F}\}) =: P_{\mathcal{F}}$

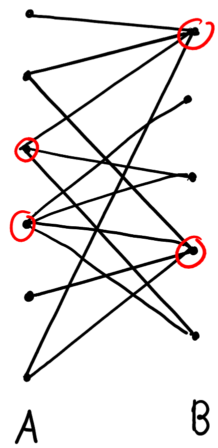
$v \in \text{vertices}([0,1]^N)$



5.2.1 Example: bipartite vertex cover

Definition 5.1: Vertex cover

Let $G = (V, E)$ be an undirected graph. A vertex cover of G is a subset $S \subseteq V$ such that for every edge $e \in E$, at least one of its endpoints is in S.



$G = (V, E)$

$V = A \,\dot\cup\, B$

We follow the recipe:

① candidate description

Theorem 5.2

The vertex cover polytope of a bipartite graph $G = (V, E)$ can be described by

$$P = \left\{x \in [0,1]^V : x(u) + x(v) \geq 1 \ \forall \{u, v\} \in E\right\} \ .$$

Proof

(2) Let $S \subseteq V$.

$$S \text{ is a vertex cover} \iff \{u,v\} \cap S \neq \emptyset \quad \forall \{u,v\} \in E$$
$$\iff \chi^S(u) + \chi^S(v) \geq 1 \quad \forall \{u,v\} \in E$$
$$\iff \chi^S \in P.$$

(3) By sake of contradiction, assume has fractional vertex $y \in \text{vertices}(P) \setminus \{0,1\}^V$.

Let $V = A \cup B$ be bipartition of the bipartite graph.

$$W_A := \{u \in A : y(u) \in (0,1)\}$$
$$W_B := \{u \in B : y(u) \in (0,1)\}$$

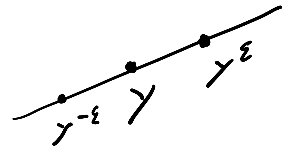
y is not integral $\Rightarrow$ $W_A \cup W_B \neq \emptyset$

For $\delta \in \mathbb{R}$,

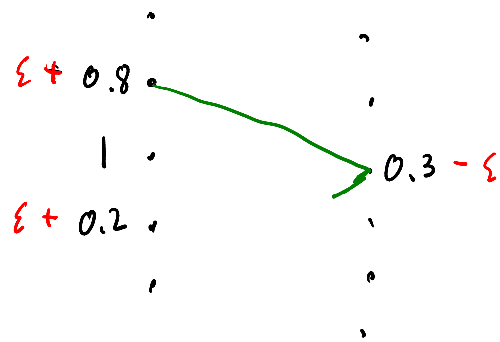
$$y^\delta := y + \delta\,(\chi^{W_A} - \chi^{W_B})$$

Let $\varepsilon := \min\{ \min\{y(u), 1 - y(u)\} : u \in W_A \cup W_B \} > 0$

$y^{\varepsilon}, y^{-\varepsilon} \in P.$



$$y = \frac{1}{2}(y^{\varepsilon} + y^{-\varepsilon})$$



$\to$ see script.