1.3.9 Simplex Method : phase I

How to get a feasible tableau to start phase II of the Simplex Method?

Original canonical LP

$$
\begin{aligned}
\max \quad & c^\top x \\
Ax &\leq b \\
x &\in \mathbb{R}^n_{\geq 0}
\end{aligned}
\tag{1.23}
$$

Auxiliary LP.

$$
\begin{aligned}
\max \quad & -x_0 \\
Ax - \mathbf{1}\cdot x_0 &\leq b \\
x &\in \mathbb{R}^n_{\geq 0} \\
x_0 &\in \mathbb{R}_{\geq 0}
\end{aligned}
\tag{1.25}
$$

Observation 1.80

LP (1.23) is feasible $\Leftrightarrow$ LP (1.25) has optimal value 0.

We will see:

- There is a simple way to get feasible tableau for auxiliary LP.
- Optimal tableau for auxiliary LP allows for obtaining feasible tableau of original LP.

Wlog, assume that at least one entry of b is < 0; for otherwise phase II of Simplex Method can be started immediately.

write auxiliary LP in standard form

$$\begin{array}{lrcl} \max & -x_0 & & \\ & x_s + Ax - \mathbf{1} \cdot x_0 & = & b \\ & x & \in & \mathbb{R}^n_{\geq 0} \\ & x_0 & \in & \mathbb{R}_{\geq 0} \\ & x_s & \in & \mathbb{R}^m_{\geq 0} \end{array}$$

corresponding tableau

$$\begin{array}{c|c|r|c} & x & x_0 & 1 \\ \hline \tilde{z} & 0 & 1 & 0 \\ \hline & & -1 & \\ x_s & A & -1 & b \\ & & -1 & \end{array}$$

With a single, well-chosen exchange step, we can obtain a feasible tableau:

(i) Choose x_0 as the variable entering the basis, i.e., as the pivot column.
(ii) Choose as basis-leaving variable a row with most negative b-value, i.e., a row with index $i \in \operatorname{argmin}\{b_\ell \colon \ell \in [m]\}$.

Proposition 1.81

Performing an exchange step on tableau (1.26) using a pivot element as described above leads to a feasible tableau.

Proof

	x	x_0	1
$\tilde{z}$	0	1	0
		-1	
x_s	A	-1	b
		-1	

all-ones vector

$\bar{A} = [A \ \ -\mathbb{1}]$

Let $\bar{A}_{ik} = -1$ be pivot element according to above rule.

New rhs b' (after exchange step), satisfies by pivoting rules:

$$b_i' = \frac{b_i}{\bar{A}_{ik}} = -b_i > 0 .$$

And, for $j \in [m] \setminus \{i\}$:

holds because of choice of pivot row

$$b_j' = b_j - \frac{\bar{A}_{jk}}{\bar{A}_{ik}} \cdot b_i = b_j - b_i \geq 0$$

$\bar{A}_{jk} = \bar{A}_{ik} = -1$

#

Example 1.82

negative entries $\Rightarrow$ basis corresponding to slack variables is not feasible.

original LP

$$\begin{array}{rrrrrrrl}
\max & x_1 & - & x_2 & + & x_3 & & \\
 & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

auxiliary LP

$$\begin{array}{rrrrrrrrrl}
\max & & & & & & - & x_0 & & \\
 & 2x_1 & - & x_2 & + & 2x_3 & - & x_0 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & - & x_0 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & - & x_0 & \leq & -1 \\
 & & & & & & & x_0, x_1, x_2, x_3 & \geq & 0
\end{array}$$

	x_1	x_2	x_3	x_0	1
$\tilde{z}$	0	0	0	1	0
x_4	2	-1	2	-1	4
x_5	2	-3	1	$\boxed{-1}$	-5
x_6	-1	1	-2	-1	-1

most negative rhs

	x_1	x_2	x_3	x_5	1
$\tilde{z}$	2	-3	1	1	-5
x_4	0	2	1	-1	9
x_0	-2	3	-1	-1	5
x_6	-3	$\boxed{4}$	-3	-1	4

next pivot element

We continue with phase II of the Simplex Method.

	x_1	x_6	x_3	x_5	1
$\tilde{z}$	-0.25	0.75	-1.25	0.25	-2
x_4	1.5	-0.5	2.5	-0.5	7
x_0	0.25	-0.75	$\boxed{1.25}$	-0.25	2
x_2	-0.75	0.25	-0.75	-0.25	1

$\rightsquigarrow$

	x_1	x_6	~~x_0~~	x_5	1
~~$\tilde{z}$~~	~~0~~	~~0~~	~~1~~	~~0~~	~~0~~
x_4	1	1	~~-2~~	0	3
x_3	0.2	-0.6	~~0.8~~	-0.2	1.6
x_2	-0.6	-0.2	~~0.6~~	-0.4	2.2

optimal tableau

We can now remove column corresponding to x_0 and row corresponding to auxiliary objective.

$\rightsquigarrow$ This leads to feasible tableau of original problem with missing objective row.

Objective row is obtained by expressing original objective

$$\max \; x_1 - x_2 + x_3$$

in terms of current non-basic variables x_1, x_5, x_6, by substituting

$$x_2 = 2.2 + 0.6x_1 + 0.2x_6 + 0.4x_5$$

$$x_3 = 1.6 - 0.2x_1 + 0.6x_6 + 0.2x_5$$

$$\Rightarrow \; z = -0.6 + 0.2x_1 + 0.4x_6 - 0.2x_5$$

	x_1	x_6	x_5	1
z	-0.2	-0.4	0.2	-0.6
x_4	1	1	0	3
x_3	0.2	-0.6	-0.2	1.6
x_2	-0.6	-0.2	-0.4	2.2

feasible tableau for original LP.

Remark

If x_0 is basic in obtained optimal tableau of auxiliary LP and has value 0 in basic solution (otherwise, the original problem is infeasible), then another optimal tableau for auxiliary LP with x_0 being non-basic can be obtained by performing an exchange step on any non-zero element in row of x_0.

	x_1	x_6	x_3	x_5	1
z	0.5	0.75	1.25	0.25	0
x_4	-5.6	0.25	7	-1.25	12
x_2	1.6	1	0.4	-1	11
x_0	0.75	-0.75	-1.25	-0.25	0

Keeping track of objective row during phase I of Simplex Method

One can simply add an additional row during phase I of the Simplex Method that keeps track of original objective.

auxiliary column

$\downarrow$

			x	x_0	1		
auxiliary row	$\rightarrow$	$\widetilde{z}$	0	1	0	$\leftarrow$	objective function for phase I
objective function	$\rightarrow$	z	$-c$	0	0	$\leftarrow$	objective function for phase II
				-1			
		x_s	A	$\vdots$	b		
				-1			