5 Polyhedral Approaches in Combinatorial Optimization

Combinatorial optimization problems can often be described by:

(i) A finite set N, called <u>ground set</u>,

(ii) a family $\mathcal{F} \subseteq 2^N$ of <u>feasible sets</u>, also called <u>solutions</u>, and

(iii) an objective function $w : N \to \mathbb{R}$ to maximize or minimize.

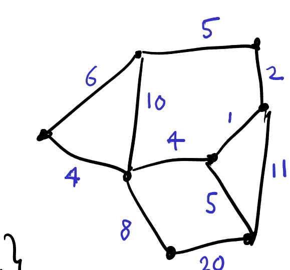
corresponding problem

$$\boxed{\begin{aligned} \max / \min \quad w(F) := \sum_{e \in F} w(e) \\ F \in \mathcal{F} \end{aligned}}$$

<u>Examples</u>

given is undirected graph $G=(V,E)$ with non-negative edge weights $w : E \to \mathbb{R}_{\geq 0}$

<u>Maximum weight matchings</u>:



$G=(V,E)$

$w : E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$

(ii) Feasible sets: $\mathcal{F} = \{ M \subseteq E : M \text{ is a matching} \}$

(iii) Objective: maximize w

Well-known special cases:

- Maximum cardinality matching $\rightsquigarrow$ $w(e) = 1 \ \forall e \in E$.
- Maximum cardinality/weight bipartite matchings $\rightsquigarrow$ G is bipartite.

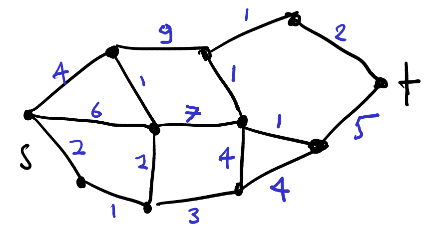
Shortest s-t path (in undir. graphs)

Given:
- undir. graph $G=(V,E)$
- vertices $s,t \in V$
- non-neg. edge lengths $\ell: E \to \mathbb{R}_{\geq 0}$

(i) Ground set : $N = E$.

(ii) Feasible sets : $\mathcal{F} = \{P \subseteq E : P \text{ is } s\text{-}t \text{ path}\}$.

(iii) Objective : minimize $w = \ell$.



$\ell: E \to \mathbb{R}_{\geq 0}$

Minimum weight spanning tree

Given:
- undir. graph $G=(V,E)$
- edge weights $w: E \to \mathbb{R}_{\geq 0}$

(i) Ground set : $N = E$.

(ii) Feasible sets : $\mathcal{F} = \left\{ F \subseteq E : \begin{array}{l} F \text{ is a spanning} \\ \text{tree in } G \end{array} \right\}$.

(iii) Objective : minimize w.

5.1 Polyhedral descriptions of combinatorial optimization problems

Let N be a finite (ground) set.

<u>Definition</u>

For $U \subseteq N$, we denote by χ^U its <u>characteristic vector</u> (also called <u>incidence vector</u>):

$$\chi^F(e) = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{if } e \in N \setminus F. \end{cases}$$

Let $\mathcal{F} \subseteq 2^N$ be all feasible sets to a combinatorial optimization problem.

The <u>(combinatorial) polytope</u> that corresponds to $\mathcal{F}$ is the polytope $P_{\mathcal{F}} \subseteq [0,1]^N$ whose vertices are precisely $\{\chi^F : F \in \mathcal{X}^F\}$, i.e.,

$$P_{\mathcal{F}} = \operatorname{conv}(\{\chi^F : F \in \mathcal{F}\}).$$

The combinatorial polytope allows for casting a combinatorial optimization problem into a linear program (and can be used for much more):

$$\boxed{\begin{array}{ll} \max/\min & w(F) \\ & F \in \mathcal{F} \end{array}} \rightsquigarrow \boxed{\begin{array}{ll} \max/\min & w^T x \\ & x \in P_{\mathcal{F}} \end{array}}$$

Optimal vertex solution to this LP is characteristic vector of optimal solution of combinatorial optimization problem.

Key challenge: Find explicit inequality description of $P_{\mathcal{F}}$.

$$\rightarrow \quad P_{\mathcal{F}} = \{x \in \mathbb{R}^N : Ax \le b\}$$

Some benefits of getting an inequality description: (let $n := |N|$)

- Often, $\#$ facets of $P_{\mathcal{F}} = O(\mathrm{poly}\, n)$.
  $\#$ vertices of $P_{\mathcal{F}} = 2^{\Omega(n)}$.

- If we can solve LPs over $P_{\mathcal{F}}$, then we can optimize any linear objective.

- Even when $P_{\mathcal{F}}$ has exponentially many facets, one can often get a description of them and even solve LPs over $P_{\mathcal{F}}$.
  for example, by using the Ellipsoid Method

- Being able to solve LPs over $P_{\mathcal{F}}$ often allows for solving related problems, for example by adding some extra constraints.

- The LP dual of $\max\{w^T x : x \in P_{\mathcal{F}}\}$ can often be interpreted combinatorially. Possible implications:
  - Natural optimality certificates through strong duality.
  - Fast algorithms based on dual such as primal-dual methods.

- Elegant polyhedral proof techniques.

$\vdots$

5.2 Meta-recipe for finding inequality-descriptions

① Determine candidate description $P = \{x \in \mathbb{R}^N : Ax \leq b\} \subseteq [0,1]^N$.

② Prove

$$P \cap \{0,1\}^N = \{\chi^F : F \in \mathcal{F}\}.$$

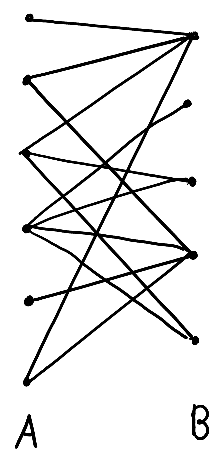
③ Prove that <u>P is integral</u>.

↑ i.e., vertices$(P) \subseteq \mathbb{Z}^N$, which, because $P \subseteq [0,1]^N$, is same as vertices$(P) \subseteq \{0,1\}^N$

5.2.1 Example : bipartite vertex cover

Definition 5.1: Vertex cover

Let $G = (V, E)$ be an undirected graph. A vertex cover of G is a subset $S \subseteq V$ such that for every edge $e \in E$, at least one of its endpoints is in S.



$G = (V, E)$

$V = A \dot\cup B$

Theorem 5.2

The vertex cover polytope of a bipartite graph $G = (V, E)$ can be described by

$$P = \left\{ x \in [0,1]^V : x(u) + x(v) \geq 1 \ \forall \{u, v\} \in E \right\} \quad .$$

Proof