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Mathematical Optimization – Solutions to problem set 12

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Problem 1: Laminar Cut Families and Total Unimodularity

We show that M is totally unimodular using the Ghouila-Houri characterization (Theorem 5.9 in the script). First, note that every subset of the rows of M corresponds to a laminar subfamily $\mathcal{F}' \subseteq \mathcal{F}$. Thus, in order to prove the claim, it suffices to reveal a partition $\mathcal{F} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ such that

$$\sum_{S \in \mathcal{F}_1} \chi^{\delta^+(S)} - \sum_{S \in \mathcal{F}_2} \chi^{\delta^+(S)} \in \{-1, 0, 1\}^A \quad . \tag{1}$$

To this end, define the *level* $\ell(S)$ of a set $S \in \mathcal{F}$ to be the number of sets (including S itself) in $\mathcal{F}$ that contain S, i.e.,

$$\ell(S) \coloneqq |\{W \in \mathcal{F} \colon S \subseteq W\}| \quad .$$

Using this level function, we can define $\mathcal{F}_1 = \{S \in \mathcal{F} \colon \ell(S) \text{ even}\}$ and $\mathcal{F}_2 = \{S \in \mathcal{F} \colon \ell(S) \text{ odd}\}$, and we claim that with this choice, 1 holds. To see this, fix an arc $a = (u, v) \in A$, and let $U_1 \supsetneq U_2 \supsetneq \ldots \supsetneq U_k$ be all sets in $\mathcal{F}$ that contain u but not v. Observe that since $\mathcal{F}$ is laminar, these sets must indeed form a chain. Also note that in the sum

$$\sum_{S \in \mathcal{F}_1} \chi^{\delta^+(S)}(a) - \sum_{S \in \mathcal{F}_2} \chi^{\delta^+(S)}(a) \quad ,$$

the non-zero terms are precisely those where $S \in \{U_1, \ldots, U_k\}$. Moreover, by definition of $\mathcal{F}_1$ and $\mathcal{F}_2$, we have that

$$\sum_{S \in \mathcal{F}_1} \chi^{\delta^+(S)}(a) - \sum_{S \in \mathcal{F}_2} \chi^{\delta^+(S)}(a) = \sum_{i=1}^{k} (-1)^{\ell(U_i)} \quad ,$$

and since by definition of the sets U_i, we have $\ell(U_{i+1}) = \ell(U_i) + 1$ for all $i \in \{1, \ldots, k-1\}$, we have

$$\sum_{i=1}^{k} (-1)^{\ell(U_i)} \in \{-1, 0, 1\} \quad ,$$

which finishes the proof.

Problem 2: Uncrossing Directed Cuts

(a) For any binary vector $x \in \{0,1\}^A$ and subset $U \subseteq A$ of arcs such that $x = \chi^U$, the condition $x(\delta^+(C)) \geq 1 \ \forall C \in \mathcal{C}$ is equivalent to the condition $|U \cap \delta^+(C)| \geq 1 \ \forall C \in \mathcal{C}$. Thus, P describes the correct set of integral points, i.e., $P \cap \{0,1\}^A = \{\chi^D \colon D \in \mathcal{D}\}$.

(b) Assume towards contradiction that for some $S \in \mathcal{F}$, the constraint $x(\delta^+(S)) = 1$ is not implied by the constraints $x(\delta^+(C)) = 1 \ \forall C \in \mathcal{H}$. In other words,

$$\chi^{\delta^+(S)} \notin Q \coloneqq \operatorname{span}\left(\left\{\chi^{\delta^+(C)} \colon C \in \mathcal{H}\right\}\right) \quad .$$

Among all such constraints, choose $S \in \mathcal{F}$ such that

$$\mathcal{H}_S = \left\{ H \in \mathcal{H} \colon \begin{array}{c} H \cap S \neq \emptyset \\ H \setminus S \neq \emptyset \\ S \setminus H \neq \emptyset \end{array} \right\}$$

has smallest cardinality. Notice that $\chi^{\delta^+(S)} \notin Q$ implies $S \notin \mathcal{H}$ and, since $\mathcal{H}$ is maximal, $\{S\} \cup \mathcal{H}$ is not laminar. Hence, it holds that $|\mathcal{H}_S| \geq 1$. Let H be an element of $\mathcal{H}_S$. We have

$$\chi^{\delta^+(H)} + \chi^{\delta^+(S)} = \chi^{\delta^+(H \cup S)} + \chi^{\delta^+(H \cap S)} + \chi^{A(H \setminus S, S \setminus H)} + \chi^{A(S \setminus H, H \setminus S)} = \chi^{\delta^+(H \cup S)} + \chi^{\delta^+(H \cap S)} ,$$

where the second equality holds since both $A(H \setminus S, S \setminus H)$ and $A(S \setminus H, H \setminus S)$ are empty due to the fact that H and S are directed cuts. Since $\chi^{\delta^+(H)} + \chi^{\delta^+(S)} \notin Q$, we get that $\chi^{\delta^+(H \cup S)} \notin Q$ or $\chi^{\delta^+(H \cap S)} \notin Q$.

Furthermore, both $H \cup S$ and $H \cap S$ are in $\mathcal{F}$. On the one hand, $H \cap S$ is in $\mathcal{C}$ because it is clearly nonempty as $H \in \mathcal{H}_S$, different from V, and directed. On the other hand, in order to show that $H \cup S$ is in $\mathcal{C}$, recall that there exists a vertex $r \in V$ such that $(v, r) \in A$ for every $v \in V \setminus \{r\}$. Thus, $r \notin H$ and $r \notin S$ implies $H \cup S \neq V$ and $H \cup S \in \mathcal{C}$ follows directly by noticing that it is non-empty and directed. In order to show that $H \cup S$ and $H \cap S$ are in $\mathcal{F}$, it remains to prove that $y(\delta^+(H \cup S)) = 1$ and $y(\delta^+(H \cap S)) = 1$, which follows from

$$2 = y(\delta^+(H)) + y(\delta^+(S)) = \underbrace{y(\delta^+(H \cup S)))}_{\geq 1} + \underbrace{y(\delta^+(H \cap S))}_{\geq 1} \geq 2 \ .$$

However, we know (problem set 11, Problem 3) that $|\mathcal{H}_{H \cup S}| < |\mathcal{H}_S|$ and $|\mathcal{H}_{H \cap S}| < |\mathcal{H}_S|$. Since this contradicts the choice of S, the claim follows.

(c) Notice that the matrix corresponding to the system

$$x(\delta^+(C)) = 1 \qquad \forall C \in \mathcal{H}$$

is totally unimodular by Problem 1 of this problem set. Moreover, the matrices corresponding to the systems

$$x(a) = 0 \qquad \forall a \in A_0$$

and

$$x(a) = 1 \qquad \forall a \in A_1$$

are submatrices of the identity matrix. Hence, the matrix corresponding to the whole system is totally unimodular as well. Noting that the right-hand side of all the constraints is integral, this implies that the vertex y of P is integral (by Theorem 5.8 in the script). Since y was an arbitrary vertex of P, we conclude that P is indeed integral.

Problem 3: Cutting integral polyhedra

(a) Let $Q := P \cap \{x \in \mathbb{R}^n : x \leq c\}$, and let R be the convex hull of the integer vectors in Q. It is clear that $R \subseteq Q$, and we will show that $Q \subseteq R$. Together, these imply $Q = R$, hence in particular, we see that R is integral.

To show $Q \subseteq R$, let $x \in Q$. As P is integral and Q contains all vertices of P, we have $P = R + \mathbb{R}^n_{\geq 0}$, and hence there exists $y \in R$ with $y \leq x$. Choose such a y with $y_1 + \ldots + y_n$ maximal. We claim that $y = x$, which immediately proves the desired.

To prove the claim, assume for contradiction that there exists a coordinate $i \in [n]$ such that $y_i < x_i$. Since $y \in R$, y is a convex combination of integral vectors in Q. Since $y_i < x_i \leq c_i$, at least one of these integral vectors, say $z \in Q$, has $z_i < c_i$. But then, the vector $z' = z + \chi^i$ belongs to R, and thus by replacing z by z' in the combination that gives y, we can increase y_i, contradicting the extremal choice of y.

(b) As P is a $\{0,1\}$-polytope, $\text{dom}(P)$ is an integral polytope in $\mathbb{R}^n_{\geq 0}$. From part (a), we thus get that $\text{dom}(P) \cap [0,1]^n = \text{dom}(P) \cap \{x \in \mathbb{R}^n_{\geq 0} : x \leq 1\}$ is integral, and an integral polytope in $[0,1]^n$ is a $\{0,1\}$-polytope. This proves the desired.

Problem 4: Minimum-Volume Ellipsoid Containing Half-Ball

(a) As $S \subseteq E(0, I) \cap H_B$, any set containing $E(0, I) \cap H_B$ must also contain S. This also implies that if the minimum-volume ellipsoid containing S happens to contain $E(0, I) \cap H_B$, it must also be minimum-volume ellipsoid containing $E(0, I) \cap H_B$.

(b) We first claim that if (p, D) is a feasible solution of the semi-definite convex program

$$\begin{array}{lrcll} \min & -\log\det X & & & \\ & \|Xs - x\|_2^2 & \leq & 1 & \forall s \in S \\ & X & \succeq & 0 & \\ & x & \in & \mathbb{R}^n & \\ & X & \in & \mathbb{R}^{n\times n} & \end{array} , \tag{2}$$

then the ellipsoid $E(a, A)$ given by $A = (D^\top D)^{-1}$ and $a = D^{-1}p$ is an ellipsoid that contains S. Indeed, we know that for all $s \in S$, we have

$$\begin{aligned}(s-a)^\top A^{-1}(s-a) &= (s - D^{-1}p)^\top (D^\top D)(s - D^{-1}p) \\ &= \big(D(s - D^{-1}p)\big)^\top \big(D(s - D^{-1}p)\big) = \|Ds - p\|_2^2 \leq 1 \ ,\end{aligned}$$

hence $s \in E(a, A)$ and thus $S \subseteq E(a, A)$. On the other hand, for every ellipsoid $E(a, A)$, we can find a positive definite matrix D such that $A = (D^\top D)^{-1}$, and with $p = Da$, the pair (p, D) is feasible for the convex program in (2) if $E(a, A)$ contains S.[1] Thus, we have seen that all ellipsoids containing S are captured by feasible solutions of (2).

Moreover, the volume of the resulting ellipsoid $E(a, A)$ is proportional to $\det(A)$, and we know that $\det(A) = \det\big((D^\top D)^{-1}\big) = \det(D)^{-2}$. Thus, minimizing the volume of $E(a, A)$, i.e., minimizing $\det(A)$, is equivalent to maximizing $\det(D)$ among feasible solutions of (2). As $x \mapsto -\log x$ is a decreasing function, we further see that maximizing $\det(D)$ is equivalent to minimizing $-\log\det D$, which is precisely what the program (2) does.

It remains to show that the program that we wrote has the desired properties, but this is immediate as $X \to -\log\det X$ is convex as given by the hint, and for any fixed s, $(x, X) \mapsto \|Xs - x\|_2^2$ is convex, too.

(c) For the sake of contradiction, assume that (2) has two distinct optimal solutions (x, X) and (y, Y). Let us examine the solution $(z, Z) = \frac{1}{2}(x, X) + \frac{1}{2}(y, Y)$ (which is feasible by convexity). As $M \mapsto -\log\det(M)$ is strictly convex, we would obtain $-\log\det Z < -\frac{1}{2}(\log\det X + \log\det Y)$ if $X \neq Y$, which is impossible, thus $X = Y$.

In this case, we must have $x \neq y$, hence the two (optimal) ellipsoids that we are considering are just translated versions of each other (translated by $y - x$). Thus, if we consider the same ellipsoid with midpoint $\frac{1}{2}(x + y)$ (i.e., the midpoint of the previous two midpoints), we see that all points of S lie in the interior of the new ellipsoid. Thus, we could decrease shrink it while still containing all points of S, contradicting the optimality assumption.

Therefore, (2) cannot have two distinct optimal solutions.

(d) Let us first calculate the gradients that we need. Note that we use $f(x, X) = -\log\det X$, and we have

$$\begin{aligned}\frac{\partial f}{\partial x_i}(x, X) &= 0 \ , \quad \text{and} \\ \frac{\partial f}{\partial X_{ij}}(x, X) &= -\frac{1}{\det X}\cdot\frac{\partial}{\partial X_{ij}}\det X = -\frac{1}{\det X}\cdot\frac{\partial}{\partial X_{ij}}\sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}(\sigma)}\prod_{k\in[n]}X_{k\sigma(k)} = \\ &= -\frac{1}{\det X}\cdot\sum_{\substack{\sigma\in S_n,\\ \sigma(i)=j}}(-1)^{\operatorname{sgn}(\sigma)}\prod_{k\in[n]\setminus\{i\}}X_{k\sigma(k)} = -\frac{1}{\det X}\cdot\det(X^{ij}) \\ &= -(X^{-1})_{ji} \ ,\end{aligned}$$

where X^{ij} is the matrix X with column j replaced by the i^{th} unit vector e_i, ans we used Cramer's rule in the last step. In other words, we get $\nabla f(x, X) = \big(0, (X^{-1})^\top\big)$.

[1] To see that there is a matrix D with $A^{-1} = D^\top D$, recall that the symmetric real matrix A^{-1} admits an orthogonal diagonalization, i.e., A^{-1} can be written as $A^{-1} = M^T \operatorname{diag}(\lambda_1, \ldots, \lambda_n)M$ for a matrix M consisting of an orthonormal basis and with $\lambda_1, \ldots, \lambda_n > 0$ being the eigenvalues of A^{-1}. From this decomposition, we see that we can use $D = M^\top \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})M$. For convenience, we may also write $D = A^{-1/2}$.

Moreover, the constraints have functions of the form $g_k^+(x, X) = \|Xe_k - x\|_2^2$ for $k \in \{1, \ldots, n\}$ and $g_k^-(x, X) = \|-Xe_k - x\|_2^2$ for $k \in \{2, \ldots, n\}$. We have

$$\frac{\partial g_k^\pm}{\partial x_i}(x, X) = \frac{\partial}{\partial x_i} \sum_{\ell=1}^n (\pm X_{\ell k} - x_\ell)^2 = \mp 2X_{ik} + 2x_i \ , \quad \text{and}$$
$$\frac{\partial g_k^\pm}{\partial X_{ij}}(x, X) = \frac{\partial}{\partial X_{ij}} \sum_{\ell=1}^n (\pm X_{\ell k} - x_\ell)^2 = \begin{cases} 0 & \text{if } k \neq j \\ 2X_{ij} \mp 2x_i & \text{else} \end{cases} \ .$$

Thus, we can write $\nabla g_k^\pm(x, X) = (\mp 2Xe_k + 2x, 2Xe_k e_k^\top \mp 2xe_k^\top)$.

Letting λ_k for $k \in \{1, \ldots, n\}$, and μ_k for $k \in \{2, \ldots, n\}$ be the multipliers for ∇g_k^+ and ∇g_k^- and using the above derivatives, KKT conditions for a feasible point (x, X) can be written as

$$\begin{aligned} \sum_{k=1}^n \lambda_k(-2Xe_k + 2x) + \sum_{k=2}^n \mu_i(2Xe_k + 2x) &= 0 \\ -(X^{-1})^\top + \sum_{k=1}^n \lambda_k(2Xe_k e_k^\top - 2xe_k^\top) + \sum_{i=2}^n \mu_i(2Xe_k e_k^\top + 2xe_k^\top) &= 0 \\ \lambda_k \cdot (1 - \|Xe_k - x\|_2^2) &= 0 \qquad \forall k \in \{1, \ldots, n\} \\ \mu_k \cdot (1 - \|-Xe_k - x\|_2^2) &= 0 \qquad \forall k \in \{2, \ldots, n\} \ . \end{aligned}$$

We want to check that for the point $(\overline{x}, \overline{X})$ corresponding to our candidate solution $(\overline{a}, \overline{A})$, there exist λ_k and μ_k such that the above constraints are valid. To this end, we calculate

$$\overline{X} = \overline{A}^{-1/2} = \begin{pmatrix} \frac{n+1}{n} & 0 & \cdots & 0 \\ 0 & \frac{\sqrt{n^2-1}}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sqrt{n^2-1}}{n} \end{pmatrix} \qquad \text{and} \qquad \overline{x} = -\overline{X}\overline{a} = \begin{pmatrix} -\frac{1}{n} & 0 & 0 & \cdots & 0 \end{pmatrix}^T \ .$$

As $\|\pm\overline{X}e_k - \overline{x}\|_2^2 = 1$ for all k, we conclude that the last two lines in the above constraints are satisfied for any λ_k and μ_k. Furthermore, one can see that a solution for the first two constraints is given by $\lambda_1 = \frac{n}{2n+2} \geq 0$ and $\lambda_i = \mu_i = \frac{n^2}{4n^2-4} \geq 0$ for $i \in \{2, \ldots, n\}$. Thus, we conclude that $\overline{X}$ and $\overline{x}$ are indeed the solution to the problem (2). Thus, the ellipsoid $E(\overline{a}, \overline{A})$ is indeed the minimum-volume ellipsoid containing S, and it also contains $E(0, I) \cap H_B$, hence it is the minimum-volume ellipsoid containing $E(0, I) \cap H_B$.