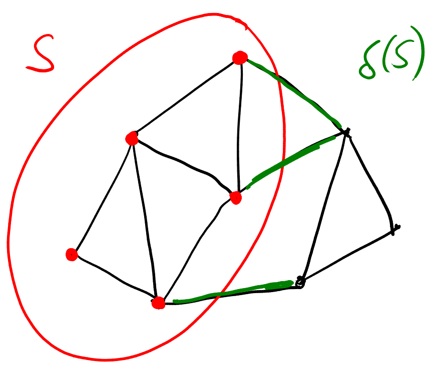
5.7 Non-bipartite matchings

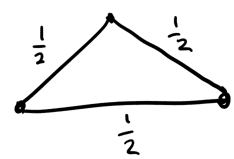
5.7.1 Perfect matching polytope

Theorem 5.21

The perfect matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\}.$$





Proof

- P contains the correct set of integral points.

$F \subseteq E$. $\quad F$ is perfect matching $\iff |F \cap \delta(v)| = 1 \quad \forall\, v \in V$

$\iff |F \cap \delta(v)| = 1 \quad \forall\, v \in V$
$\quad\; |F \cap \delta(S)| \geq 1 \quad \forall\, S \subseteq V,\ |S|$ odd

$\iff \chi^F \in P.$

By sake of contradiction, assume $\exists$ graph $G=(V,E)$ s.t. P is not integral. Among all such graphs take one where $|V|+|E|$ is smallest.

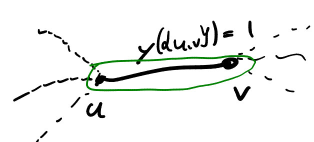
Let $y \in \text{vertices}(P) \setminus \{0,1\}^E$

due to constraint $x(\delta(V)) \geq 1$.

Some basic observations:

- $|V|$ is even. $\rightarrow$ Otherwise $P = \emptyset$ is integral.
- $y(e) > 0 \quad \forall\, e \in E$. $\rightarrow$ Otherwise, delete e to get smaller bad example with fractional vertex $y|_{E \setminus \{e\}}$.
- G is connected $\rightarrow$ Otherwise $\exists$ connected component with at least one fractional y-value.
  $\hookrightarrow$ smaller bad example.
- $y(e) < 1 \quad \forall\, e \in E \rightarrow$ If $y(\{u,v\}) = 1$.

Because $y(\delta(u)) = y(\delta(v)) = 1$, any edge $f \in (\delta(u) \cup \delta(v)) \setminus \{e\}$ satisfies $y(f) = 0$.



$\rightarrow$ There are no such edges.

G connected $\Rightarrow$ Graph G consists only of edge $\{u,v\}$.

$\Rightarrow \nexists$ fractional y-value ↯

$y \in \text{vertices}(P) \implies y$ is unique solution to $|E|$ linearly independent y-tight constraints of P.

3 types of constraints

degree constraints

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non-neg. constraints

cut constraints

we can drop constraints $x(\delta(S)) \geq 1$ for $|S| \in \{1, |V|-1\}$ as they are implied by degree constraints

$y \in (0,1)^E \implies$ no non-neg. constraint is y-tight.

$\implies y$ is unique solution to a square system

$$\circledast \begin{cases} x(\delta(v)) = 1 & \forall v \in W \\ x(\delta(S)) = 1 & \forall S \in \mathcal{F} \end{cases}, \text{ where}$$

$W \subseteq V$, $\mathcal{F} \subseteq \{S \subseteq V : |S| \text{ odd}, |S| \notin \{1, |V|-1\}\}$,

$|E| = |W| + |\mathcal{F}|$.

square system

Case $\mathcal{F} = \emptyset$:

General observation: $\deg(v) \geq 2 \quad \forall v \in V.$

$\rightarrow$ For otherwise if $\delta(v) = \{e\} \Rightarrow \overset{y \in P}{\downarrow} y(\delta(v)) = 1$

$\Rightarrow y(e) = 1$ ↯

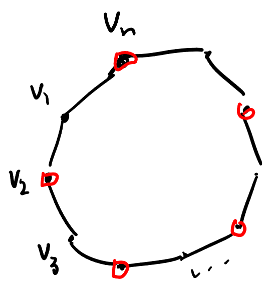
$$\underset{\substack{\uparrow \\ W \subseteq V}}{|V|} \geq \underbrace{|\mathcal{F}|}_{=0} + |W| \geq |E| = \frac{1}{2} \sum_{v \in V} \underbrace{\deg(v)}_{\geq 2} \geq |V|$$

Equality must hold throughout.

$\Rightarrow$ • $\deg(v) = 2 \quad \forall v \in V$

• $W = V$

$\Rightarrow$ G is one (even) cycle going through all vertices.
(Hamiltonian cycle)



$G = (V, E)$

$$\sum_{\substack{i \in [n]: \\ i \text{ even}}} \chi^{\delta(v_i)} = \chi^E = \sum_{\substack{i \in [n]: \\ i \text{ odd}}} \chi^{\delta(v_i)}$$

$x(\delta(v_i)) = 1$

these rows of ⊛ are linearly dependent

↯

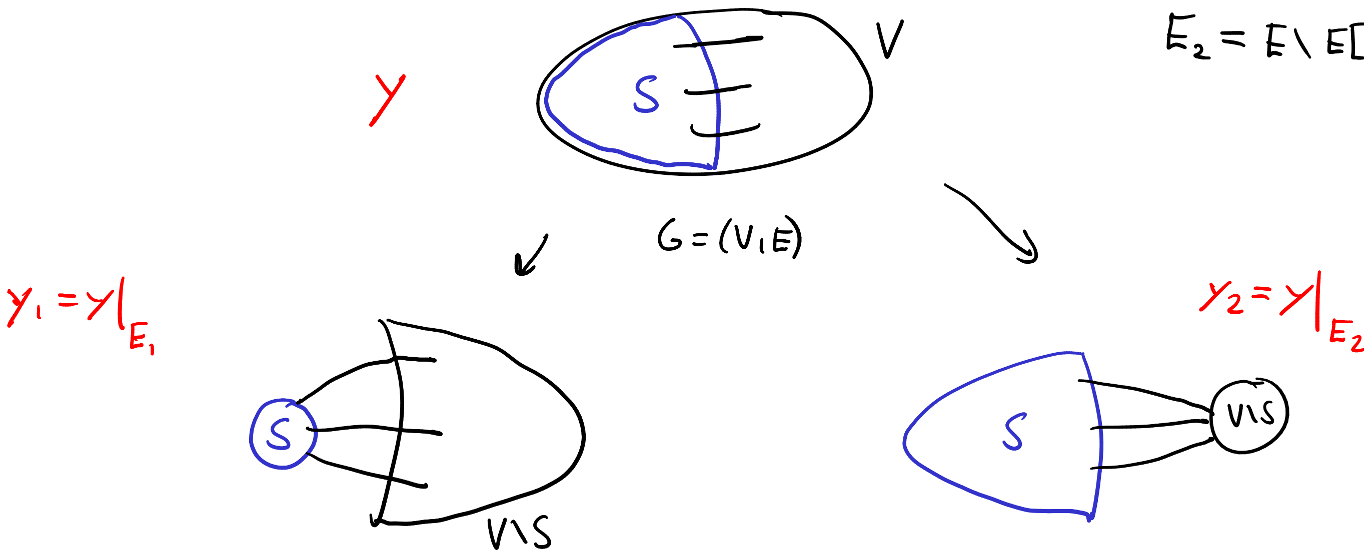
$n := |V|$

Case $\mathcal{F} \neq \emptyset$:

Let $S \in \mathcal{F}$ (Recall $|S| \notin \{1, |V|-1\}$)

$E_1 = E \setminus E[S]$

$E_2 = E \setminus E[V \setminus S]$



$G_1 = G/S =: (V_1, E_1)$ $\qquad$ $G_2 = G/(V \setminus S) =: (V_2, E_2)$

Hence $E_1 \cap E_2 = \delta(S)$.

Let P_{G_i} for $i \in \{1,2\}$ be the polytope given in Thm. 5.21 for G_i.

↳ Because G is smallest bad example, P_{G_i} for $i \in \{1,2\}$ is the perfect matching polytope for G_i.

Moreover, we can observe that

$$y_1 \in P_{G_1}$$
$$y_2 \in P_{G_2}$$

$\Longrightarrow$ y_i is convex combination of perfect matchings in G_i for $i \in \{1,2\}$.

$$y_1 = \sum_{j=1}^{q} \lambda_j \chi^{M^j}$$
$$\sum_{j=1}^{q} \lambda_j = 1$$
$$\lambda_j \geq 0 \quad \forall j \in [q]$$

$\exists\, N \in \mathbb{Z}_{>0}$

$\Longrightarrow$ $$y_i = \frac{1}{N} \sum_{j=1}^{N} \chi^{M_i^j}, \text{ where}$$

$M_i^j \subseteq E$ is a perfect matching in G_i for $i \in \{1,2\}$, $j \in [N]$.

- Notice that $|M_i^j \cap \delta(S)| = 1 \quad \forall i \in [2],\ j \in [N]$.
- For any $e \in \delta(S)$

$$|\{j \in [N] : e \in M_1^j\}| = |\{j \in [N] : e \in M_2^j\}|$$
$$\| \qquad\qquad\qquad\qquad \|$$
$$N \cdot y_1(e) = N \cdot y(e) = N \cdot y_2(e)$$

We can choose numbering $M_1^1, M_1^2, M_1^3, \ldots, M_1^N$
$M_2^1, \ldots, M_2^N$ s.t.

$$M_1^j \cap \delta(S) = M_2^j \cap \delta(S)$$

$$\Longrightarrow\quad y = \frac{1}{N} \sum_{j=1}^{N} \chi^{\boxed{M_1^j \cup M_2^j}}$$

is perfect matching in G.

y is convex comb. of perfect matchings.

$\Rightarrow$ $y \notin$ vertices$(P) \setminus \{0,1\}^E$ $\lightning$

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