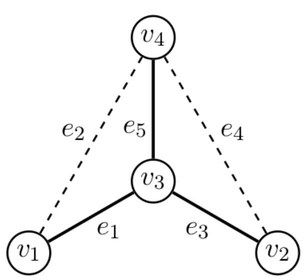
5.8 Combinatorial Uncrossing

Main goal:

> Given a heavily overdetermined linear system that uniquely defines a point, find a well-structured full-rank subsystem.

5.8.1 Integrality of spanning tree polytope

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} : \begin{array}{ll} x(E) & = |V| - 1 \\ x(E[S]) & \leq |S| - 1 \quad \forall\, S \subsetneq V,\ |S| \geq 2 \end{array} \right\}$$



— $y = (1,0,1,0,1)$

spanning tree constraints:

$$\begin{array}{r} \\ \{v_1,v_2\} \\ \{v_1,v_3\} \\ \{v_1,v_4\} \\ \{v_2,v_3\} \\ \{v_2,v_4\} \\ \{v_3,v_4\} \\ \{v_1,v_2,v_3\} \\ \{v_1,v_2,v_4\} \\ \{v_1,v_3,v_4\} \\ \{v_2,v_3,v_4\} \\ \{v_1,v_2,v_3,v_4\} \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} y \overset{=}{\leq} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$

non-negativity constraints:

$$Qy = q$$

$$y = Q^{-1} q$$

$$\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} y \overset{=}{\geq} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Proof of integrality of P

Let $y \in \text{vertices}(P)$.

$\text{supp}(y) = E$

Wlog, assume $y(e) > 0 \quad \forall e \in E$.

We can delete edges $e \in E$ with $y(e) = 0$.

$\to$ Observe $y|_{E \setminus \{e\}}$ is vertex of spanning tree polytope of $(V, E \setminus \{e\})$.

our guess P of the

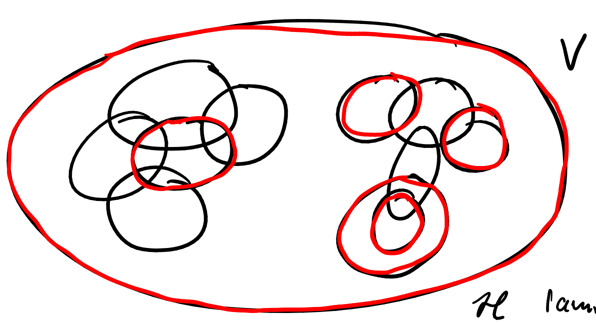
Let $\mathcal{F} = \{ S \subseteq V : y(E[S]) = |S| - 1 \}$

y-tight spanning tree constraints

$y \in \text{vertices}(P) \implies y$ is unique sol. to

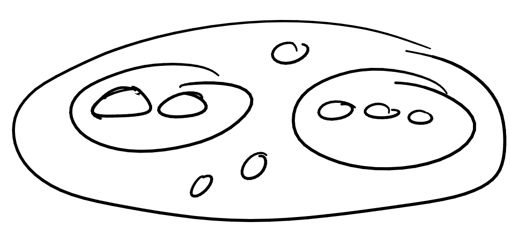
$$(*) \qquad x(E[S]) = |S| - 1 \quad \forall S \in \mathcal{F}$$

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$.



$\mathcal{H} \subseteq \mathcal{F}$

laminar family



$\mathcal{H}$ laminar:

for any $H_1, H_2 \in \mathcal{H}$ either:

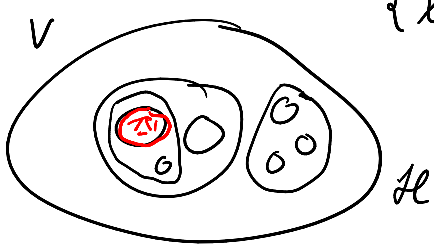
(i) $H_1 \cap H_2 = \emptyset$

(ii) $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Consider the following subsystem of ⊛:

$$\text{⊡} \qquad x(E[H]) = |H| - 1 \quad \forall\, H \in \mathcal{H}$$

Observe: ⊡ is TU system because it is a laminar system:



$\{\chi^{E[H]}\}_{H\in\mathcal{H}}$ are characteristic vectors of a laminar family.

↑ because $\{E[H]\}_{H\in\mathcal{H}}$ is a laminar family.

↑ see problem sets

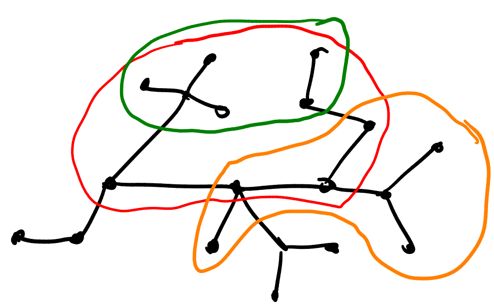
We finish proof by showing $\boxed{\text{⊡ has full column-rank}}$

↑ This implies that y is unique sol. to ⊡, which is TU $\Rightarrow$ y is integral ($y \in \{0,1\}^E$)

We show $\boxed{}$ by showing that each equation of ⊛ is implied by ⊡.

→ To this end, we need better understanding of $\underbrace{\text{structure of } \mathcal{F}}$.

↑ tight spanning tree constraints



Lemma 5.23

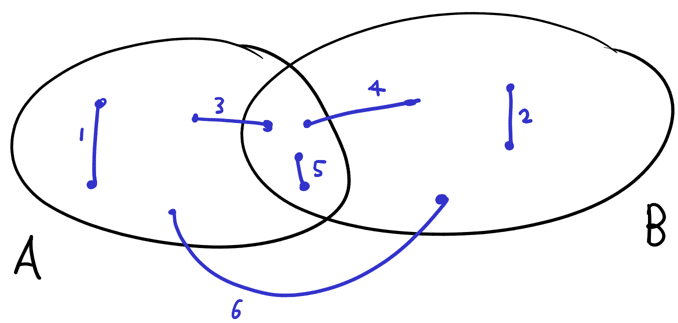
For any sets $A, B \subseteq V$, we have

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A \setminus B, B \setminus A)} = \chi^{E[A \cup B]} + \chi^{E[A \cap B]} ,$$

which implies

$$\chi^{E[A]} + \chi^{E[B]} \le \chi^{E[A \cup B]} + \chi^{E[A \cap B]} .$$

Proof



→ 6 "edge types" in $E[A \cup B]$.

For each edge type, contribution to lhs and rhs is same.

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A \setminus B, B \setminus A)} \qquad \chi^{E[A \cup B]} + \chi^{E[A \cap B]}$$

	$\chi^{E[A]}$	+	$\chi^{E[B]}$	+	$\chi^{E(A \setminus B, B \setminus A)}$	=	$\chi^{E[A \cup B]}$	+	$\chi^{E[A \cap B]}$
1	1		0		0		1		0
2	0		1		0		1		0
3	1	+	0	+	0	=	1	+	0
4	0		1		0		1		0
5	1		1		0		1		1
6	0		0		1		1		0

#

Lemma 5.24

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{F}$ and $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$. In particular, this implies by Lemma 5.23

$$\chi^{E[S_1]} + \chi^{E[S_2]} = \chi^{E[S_1 \cup S_2]} + \chi^{E[S_1 \cap S_2]} .$$

Proof

By Lemma 5.23: $\chi^{E[S_1]} + \chi^{E[S_2]} + \chi^{E(S_1 \setminus S_2, S_2 \setminus S_1)} = \chi^{E[S_1 \cup S_2]} + \chi^{E[S_1 \cap S_2]}$

$$\Rightarrow |S_1| + |S_2| - 2 \leq \underbrace{y(E[S_1])}_{|S_1| - 1} + \underbrace{y(E[S_2])}_{|S_2| - 1} + \underbrace{y(E(S_1 \setminus S_2, S_2 \setminus S_1))}_{\geq 0 \; ③} = \underbrace{y(E[S_1 \cup S_2])}_{① \leq |S_1 \cup S_2| - 1} + \underbrace{y(E[S_1 \cap S_2])}_{② \leq |S_1 \cap S_2| - 1}$$

$y \in P$

$$\leq |S_1 \cup S_2| + |S_1 \cap S_2| - 2$$

$$= |S_1| + |S_2| - 2$$

Hence, we have equality throughout:

① $\cdot$ $y(E[S_1 \cup S_2]) = |S_1 \cup S_2| - 1 \Longrightarrow S_1 \cup S_2 \in \mathcal{F}$

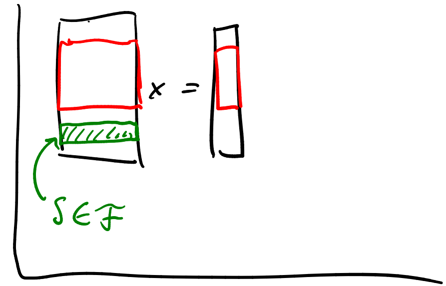
② $\cdot$ $y(E[S_1 \cap S_2]) = |S_1 \cap S_2| - 1 \Longrightarrow S_1 \cap S_2 \in \mathcal{F}$

③ $\cdot$ $y(E(S_1 \setminus S_2, S_2 \setminus S_1)) = 0 \xRightarrow{\text{supp}(y) = E} E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset.$

$\text{supp}(y) := \{e \in E : y(e) > 0\}$

#

Back to : Each equality in ⊛ is implied by $\boxed{\square}$.



Let $Q := \text{span}(\{\chi^{E[H]} : H \in \mathcal{H}\})$

For $S \in \mathcal{F}$

$$x(E[S]) = |S|-1 \text{ is implied by } \boxed{\square} \iff \chi^{E[S]} \in Q.$$

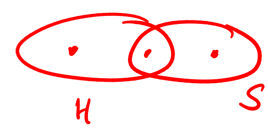
(see problem sets)

Assume by sake of contradiction that $\exists\, S \in \mathcal{F}$ s.t. $\chi^{E[S]} \notin Q$.

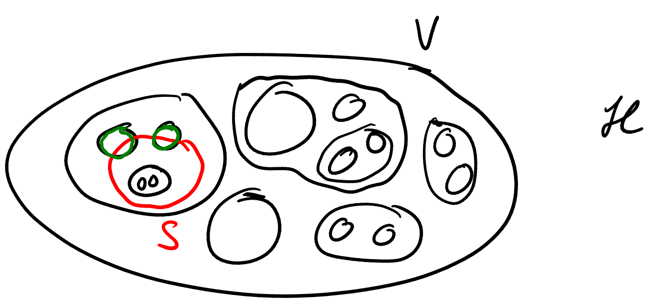
Among all such set $S \in \mathcal{F}$, we choose one for which

$$\mathcal{H}_S := \{H \in \mathcal{H} : \underline{H \text{ and } S \text{ are intersecting}}\}$$

- $H \cap S \neq \emptyset$
- $H \setminus S \neq \emptyset$
- $S \setminus H \neq \emptyset$



has smallest cardinality.



$|\mathcal{H}_S| = 2$

We have that $\mathcal{H}_S \neq \emptyset$; for otherwise, we could have added S to $\mathcal{H}$ $\rightsquigarrow$ contradicts maximality of laminar family $\mathcal{H}$.

Let $H \in \mathcal{H}_S$.

By Lemma 5.24 : $S \cup H, S \cap H \in \mathcal{F}$ and

$$\underbrace{\chi^{E[S]}}_{\notin Q} + \underbrace{\chi^{E[H]}}_{\in Q} = \chi^{E[S \cup H]} + \chi^{E[S \cap H]}$$

$\Rightarrow$ Not possible that both $\chi^{E[S \cup H]} \in Q$ and $\chi^{E[S \cap H]} \in Q$,
because this would imply $\underbrace{\chi^{E[S]}}_{\notin Q} = \underbrace{\chi^{E[S \cup H]}}_{\in Q} + \underbrace{\chi^{E[S \cap H]}}_{\in Q} - \underbrace{\chi^{E[H]}}_{\in Q}$

$\lightning$ because Q is linear space.

However,

$$\mathcal{H}_{S \cup H} \subsetneq \mathcal{H}_S \text{ , and}$$
$$\mathcal{H}_{S \cap H} \subsetneq \mathcal{H}_S$$

(see problem sets)

$\rightsquigarrow$ Contradiction with choice of S:

- If $\chi^{E[S \cup H]} \notin Q$, we could have chosen S to be $S \cup H$.
- If $\chi^{E[S \cap H]} \notin Q$, " " " " S to be $S \cap H$.

#