7 Equivalence Between Optimization and Separation

Loosely speaking, the Ellipsoid Method shows that if one can separate (over a polyhedron) then one can also optimize (a linear function over it).

$$\begin{tikzcd} \text{separation} \arrow[r, "\text{Ellipsoid}", "\text{Method}"'] & \text{optimization} \arrow[l, bend left] \end{tikzcd}$$

It turns out that there is also a reverse connection, which is based on polarity.

Definition

Let $X \subseteq \mathbb{R}^n$. The polar $X^\circ \subseteq \mathbb{R}^n$ of X is given by

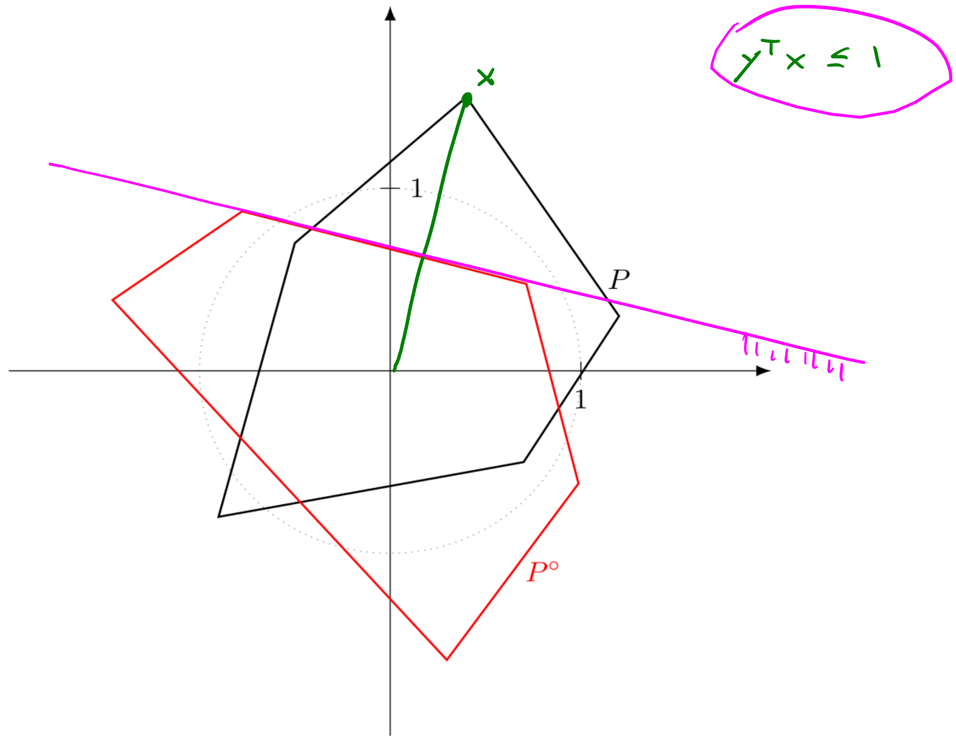
$$X^\circ = \{ y \in \mathbb{R}^n : x^T y \leq 1 \quad \forall x \in X \}.$$

separation over X°: Given $y \in \mathbb{R}^n$.

Clearly, if $A \subseteq B \subseteq \mathbb{R}^n$, then $B^\circ \subseteq A^\circ$.

$\max_{x \in X} y^T x$

Example 1



Example 2

Let $r \in \mathbb{R}_{>0}$ and consider $B(0,r) := \{x \in \mathbb{R}^n : \|x\|_2 \le r\}$.

Then $(B(0,r))^\circ = B(0, \frac{1}{r})$.

Indeed: $(B(0,r))^\circ = \{y \in \mathbb{R}^n : x^T y \le 1 \;\; \forall\, x \in B(0,r)\}$

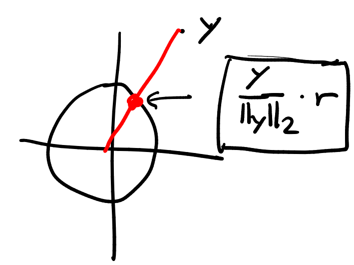
$$= \{y \in \mathbb{R}^n : \max_{x \in B(0,r)} \underbrace{x^T y}_{\le \|x\|_2 \cdot \|y\|_2} \le 1\}$$

Cauchy-Schwarz

$$= \{y \in \mathbb{R}^n : r \frac{y^T}{\|y\|_2} \cdot y \le 1\}$$

$$= \{y \in \mathbb{R}^n : \|y\|_2 \le \frac{1}{r}\}$$

$$= B(0, \tfrac{1}{r})$$



Lemma 7.1

Let $X \subseteq \mathbb{R}^n$ be a compact (i.e., closed and bounded) convex set, containing the origin in its interior. Then

(a) X° is a compact convex set with the origin in its interior.
(b) $(X^\circ)^\circ = X$.

Proof

(a) $0 \in \operatorname{int}(X) \iff \exists\, r \in \mathbb{R}_{>0}$ s.t. $B(0,r) \subseteq X$

$\implies X^\circ \subseteq (B(0,r))^\circ = B(0, \frac{1}{r})$

X is bounded $\iff \exists\, R \in \mathbb{R}_{>0}$ s.t. $X \subseteq B(0,R)$

$\implies B(0,\frac{1}{R}) = (B(0,R))^\circ \subseteq X^\circ \iff 0 \in \operatorname{int}(X^\circ)$

Remains to observe that X° is closed and convex.

$\boxed{X^\circ \text{ is intersection of closed half-spaces}} \implies X^\circ$ is closed.

intersection of closed sets is closed set

$\implies X^\circ$ is convex

intersection of convex sets is convex

(b) $X \subseteq (X^\circ)^\circ$:

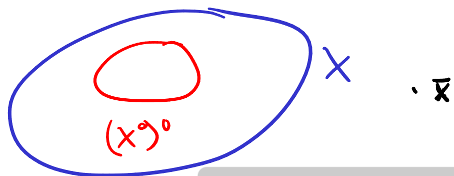
$X^\circ := \{ y \in \mathbb{R}^n : x^T y \leq 1 \quad \forall\, x \in X \}$

$(X^\circ)^\circ := \{ z \in \mathbb{R}^n : y^T z \leq 1 \quad \forall\, y \in X^\circ \}$

Let $\bar{x} \in X$ and let $y \in X^\circ$. We have to show that $y^T \bar{x} \leq 1$.

$\rightarrow$ This holds because y fulfills constraints of X°.

$(X^\circ)^\circ \subseteq X$:



Let $\bar{x} \in \mathbb{R}^n \setminus X$, and we will show $\bar{x} \notin (X^\circ)^\circ$.

Theorem 1.47

Let $Y, Z \subseteq \mathbb{R}^n$ be two disjoint closed convex sets with at least one of them being compact, then there exists a strictly (Y, Z)-separating hyperplane.

$$\boxed{\begin{array}{l}\text{Because both } X \text{ and } \{\bar{x}\} \text{ are convex and compact,} \\ \exists \; c \in \mathbb{R}^n \text{ s.t. } \underbrace{\max\{c^T x : x \in X\}}_{=:\,\alpha} < c^T \bar{x}.\end{array}}$$

Because $0 \in \operatorname{int}(X) \Rightarrow \alpha > 0$

Let $y := \frac{1}{\alpha} c$.

$y \in X^\circ$ because for any $x \in X$: $y^T x = \frac{1}{\alpha} \underbrace{c^T x}_{\leq \alpha} \leq 1$

However, $y^T \bar{x} = \frac{1}{\alpha} \underbrace{c^T \bar{x}}_{> \alpha} > 1$

$\Rightarrow \bar{x} \notin (X^\circ)^\circ$

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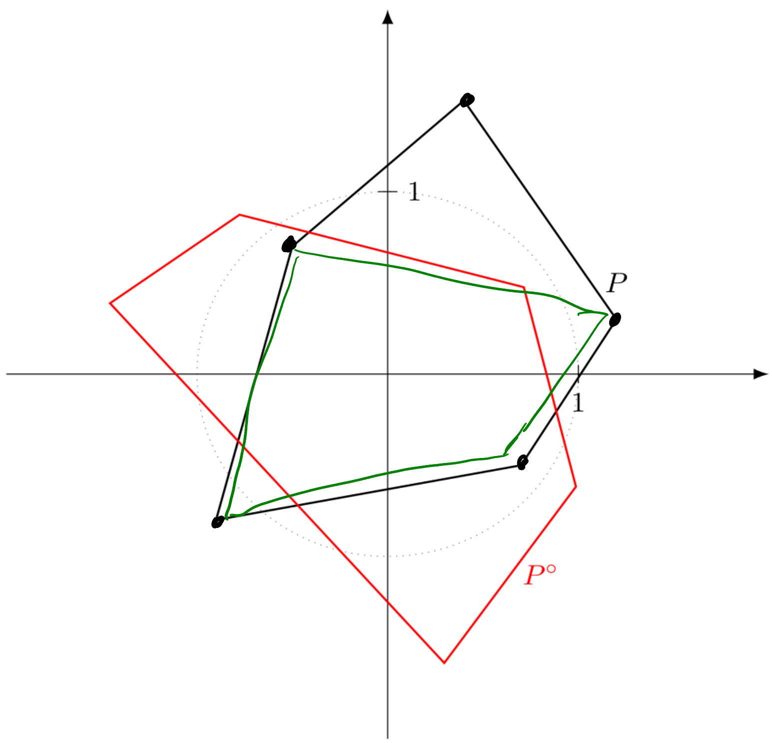
Lemma 7.2

Let $P \subseteq \mathbb{R}^n$ be a polytope containing the origin in its interior. Then P° is a polytope. Moreover, for any $x \in \mathbb{R}^n$, we have

$$x \text{ is a vertex of } P \quad \Leftrightarrow \quad \{y \in \mathbb{R}^n \colon x^\top y \leq 1\} \text{ is facet-defining for } P^\circ.$$



Proof (of Lemma 7.2)

Let $Q := \{ y \in \mathbb{R}^n : x^T y \leq 1 \quad \forall\, x \in \text{vertices}(P) \}$.

We first show $P^\circ = Q$.

This already implies that P° is a polytope.

$P^\circ \subseteq Q$.

To prove $Q \subseteq P^\circ$, let $q \in Q$ and let $x \in P$.

$\xrightarrow{\text{to show}}$ $x^T q \leq 1$ which implies $q \in P^\circ$.

Recall: $\boxed{\text{Proposition 1.32: A polytope is convex hull of its vertices.}}$

$$\Rightarrow \quad x = \sum_{i=1}^{k} \lambda_i x_i \quad , \text{where} \quad k \in \mathbb{Z}_{>0}$$

$x_i \in \text{vertices}(P) \quad \forall\, i \in [k]$,

$\lambda \in \mathbb{R}^k_{\geq 0}$,

$\|\lambda\|_1 = 1$.

$$x^T q = \sum_{i=1}^{k} \lambda_i \underbrace{x_i^T q}_{\leq 1} \leq \sum_{i=1}^{k} \lambda_i = 1.$$

$$\Rightarrow \quad q \in P^\circ \Rightarrow Q \subseteq P^\circ \quad \Rightarrow Q = P^\circ.$$

If $x \in \mathbb{R}^n$ is such that $y^\top x \leq 1$ is facet-defining for P° $\Rightarrow$ $x \in \text{vertices}(P)$.

Proof of $\Leftarrow$)

Consequence of $P^\circ = Q := \{y \in \mathbb{R}^n : x^\top y \leq 1 \ \forall x \in \text{vertices}(P)\}$

Proof of $\Rightarrow$)

Let $\bar{x} \in \text{vertices}(P)$. We will show that $\bar{x}^\top y \leq 1$ is facet-defining for $P^\circ = Q$ by showing that

$$\bar{Q} = \{y \in \mathbb{R}^n : x^\top y \leq 1 \ \forall x \in \text{vertices}(P) \setminus \{\bar{x}\}\} \overset{?}{\neq} Q.$$

This shows that $\bar{x}^\top y \leq 1$ is not redundant in description of Q and it is therefore facet-defining by Lemma 1.20.

Lemma 1.20

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full-dimensional polyhedron, then each inequality $a^\top x \leq \beta$ of $Ax \leq b$ that is not facet-defining for P is redundant.

$\bar{x} \in \text{vertices}(P) \implies \exists\ c \in \mathbb{R}^n \setminus \{0\}$ s.t. $c^T\bar{x} > c^T x \ \forall x \in P \setminus \{\bar{x}\}$

Because $0 \in \text{int}(P)$, we have $c^T\bar{x} > 0$. Hence, we can scale c s.t.

(i) $c^T\bar{x} > 1$

(ii) $c^T x \leq 1 \quad \forall\ x \in \text{vertices}(P) \setminus \{\bar{x}\}$

$$\left.\begin{array}{l} \text{(ii)} \implies c \in \bar{Q} \\ \text{(i)} \implies c \notin Q \end{array}\right\} \implies \bar{Q} \setminus Q \neq \emptyset.$$

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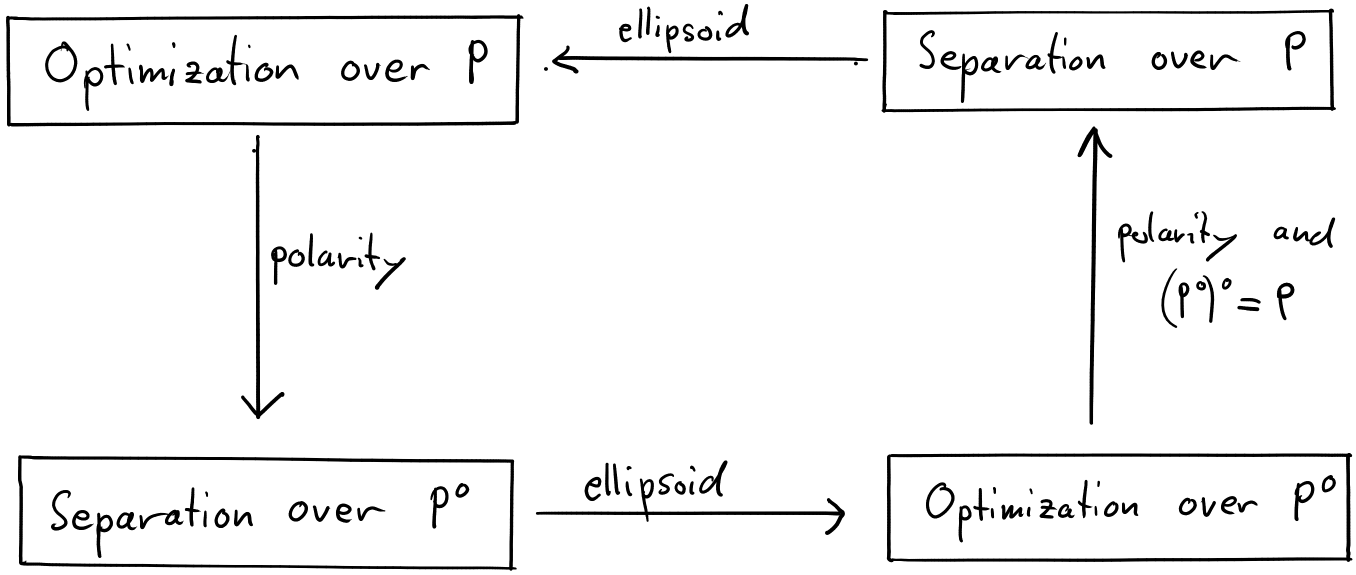
Optimization over P
ellipsoid
Separation over P
polarity
polarity and
$(P^\circ)^\circ = P$
Separation over P°
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Optimization over P°