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Mathematical Optimization – Problem set 4

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Simplex Algorithm

Consider the following LP in canonical form:

$$\begin{array}{llcrcrcr}
\max & & x_1 & + & x_2 & & \\
\text{s.t.} & & x_1 & & & \leq & 4 \\
& & & & x_2 & \leq & 4 \\
& & x_1 & + & x_2 & \leq & 7 \\
& - & x_1 & - & x_2 & \leq & -3 \\
& & x_1 & & & \geq & 0 \\
& & & & x_2 & \geq & 0
\end{array}$$

(a) Transform the given linear program into standard form, using slack variables $y_1, \dots, y_4$ corresponding to the four constraints (excluding non-negativity constraints).

(b) Write down the short tableau with basis $B = (y_1, y_2, y_3, y_4)$. Is it feasible or infeasible? Why? What is the basic solution in the original space corresponding to this tableau?

(c) Run phase I of the Simplex Method to certify feasibility of the problem. From the tableau you obtain, extract a feasible tableau for the given LP.

(d) Starting from the tableau obtained in (c), run phase II of the Simplex Method to obtain an optimal tableau for the given LP, and argue why it is optimal. Read the corresponding solution and its value from the tableau.

(e) Draw the feasible region of the initial canonical linear program, mark all solutions that you visited when executing phase I and phase II of the Simplex Method. Use the graphic solution method to double-check that phase II ended at an optimal solution.

(f) In the drawing from (e), you might have observed that there exist infinitely many optimal solutions. Can you argue that this is indeed true by only looking at the optimal tableau obtained in part (d)?

Problem 2: Reading simplex tableaus

Consider the short tableaus (a)–(d) given below. For each of them, answer the following questions and argue why your answer is correct.

- Mark all entries that are legal pivots according to the rules of phase II of the Simplex Method.
- What is the basic solution corresponding to the tableau? Is it feasible? If yes, can you determine if it is optimal by looking at the tableau only?
- Can you decide if the objective function of the linear program corresponding to the tableau is unbounded by looking at the tableau only? If it is unbounded, provide a feasible ray along which the objective function is strictly increasing.

(a)

	x_1	x_0	x_3	1
z	$\frac{3}{8}$	0	1	-12
x_4	1	0	-2	6
x_2	$-\frac{3}{4}$	7	-3	1
x_5	$\frac{125}{8}$	10	-3	0

(b)

	y_1	y_2	x_s^+	x_s^-	x_b	x_t	1
z	0	$-\frac{4}{19}$	0	5	-3	-1	12
y_3	1	2	$-\frac{5}{2}$	7	0	-2	2
x_a	-3	-1	$\frac{3}{2}$	0	-3	0	0
y_4	0	0	$\frac{1}{2}$	0	$-\frac{1}{10}$	-1	5

(c)

	J	A	B	G	D	1
z	-4	6	0	-1	$-\frac{1}{2}$	0
C	6	-2	-5	0	-7	0
F	-3	$-\frac{17}{2}$	3	2	0	4
H	3	91	-3	4	-20	12
E	0	14	$\frac{7}{2}$	$\frac{1}{2}$	2	1

(d)

	y_1	y_2	x_3	1
z	3	-5	0	4
y_3	2	0	-2	2
x_1	3	7	3	1
x_2	1	-1	-3	$\frac{9}{2}$
y_4	0	10	3	5

Problem 3: Characterizing potential choices of free variables
Consider a linear program in canonical form, i.e.,

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. When applying the simplex method, we introduce slack variables $y \in \mathbb{R}^m$ and write the equivalent standard form linear program

$$\begin{array}{rrcrcl} \max & c^\top x & & & & \\ & Ax & + & y & = & b \\ & x & & & \geq & 0 \\ & & & y & \geq & 0\ . \end{array}$$

Writing the latter system in tableau form, we can naturally use the variables x as free variables and the variables y as the basic variables. We are interested in characterizing choices of variables that can be used as free variables in an equivalent linear system in tableau form. To this end, let $a_1^\top, \ldots, a_m^\top$ be the rows of A, denote by e_i the i^{th} unit vector in $\mathbb{R}^n$, and let $M \subseteq [m]$ and $N \subseteq [n]$ with $|M| + |N| = n$.

Prove the following equivalence:

There exists a linear system equivalent to $Ax + y = b$ in tableau form that uses $\{x_i \colon i \in N\} \cup \{y_j \colon j \in M\}$ as free variables.	$\iff$	The square matrix with rows $\{e_i^\top : i \in N\} \cup \{a_j^\top : j \in M\}$ has full rank.

Problem 4: Short questions on the Simplex Method
For each of the following statements, decide if it is true or false and argue why (i.e., give a proof or find a counterexample).

(a) If in a legal pivoting step of the Simplex Method, a variable leaves the basis, there is no legal pivoting step that makes the same variable re-enter the basis in the very next step.

(b) If in every potential pivot column of a feasible simplex tableau, there is a legal pivot, then the objective function is bounded.

(c) If there exists a legal pivot in a feasible simplex tableau, the current basic solution is not optimal.

(d) Consider a feasible simplex tableau whose corresponding basic solution is optimal. Then, the entries of the tableau that correspond to coefficients of the objective function are all non-negative.

(e) The number of basic feasible solutions of a system of linear equalities is finite.

Problem 5: Lexicographic pivoting
We have seen in class that in some cases, the Simplex Method might not terminate if the pivots are chosen in an unfortunate way. Bland's rule for selecting pivots was one way to avoid this. We will see

another one in this problem, the *lexicographic pivoting rule.* To this end, let us define the lexicographic order relation on $\mathbb{R}^n$.

Definition. *Let $a, b \in \mathbb{R}^n$ be distinct. We say that a is lexicographically larger than b, denoted by $a \succ b$, if for the minimum $i \in [n]$ such that $a_i \neq b_i$, we have $a_i > b_i$.*

For carrying out the pivoting operations, we use the long tableau with the following conventions: We write the column containing the objective function value and the right-hand sides at the very beginning of the tableau instead of at the end, i.e., in the leftmost column, and we never change the ordering of the columns. A tableau corresponding to the standard form linear program $\max\{c^\top x \colon Ax = b,\ x \in \mathbb{R}^n_{\geq 0}\}$ with $A \in \mathbb{R}^{m\times(m+n)}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^{m+n}$ would thus look as follows.

$$
\begin{array}{c|c}
 & x_1 \ \dots \ x_{m+n} \\
\hline
0 & -c^\top \\
\hline
 & \\
b & A \\
 &
\end{array}
$$

Assume that the above tableau is feasible and in tableau form, so that we can run phase II of the Simplex Method. Furthermore, we assume that we start with a tableau such that all rows except the objective row are lexicographically positive, i.e., $\begin{pmatrix} b_i & A_{i.} \end{pmatrix} \succ 0$ for all $i \in [m]$, where $A_{i.}$ denotes the i^{th} row of A. Note that the latter assumption is easy to achieve for a feasible system in tableau form by arranging the columns such that A contains the identity matrix corresponding to the basic variables in its left part.

If none of the stopping criteria from phase II of the Simplex Method apply, the lexicographic pivoting rule selects the pivot column k and the pivot row j for the next pivoting step as follows:

- Choose an arbitrary pivot column k among the columns corresponding to non-basic variables such that $c_k < 0$.
- Let j be the row index such that the scaled row vector $\frac{1}{A_{jk}}\begin{pmatrix} b_j & A_{j.} \end{pmatrix}$ of the tableau is lexicographically minimal among the scaled tableau row vectors in the set $\left\{\frac{1}{A_{ik}}\begin{pmatrix} b_i & A_{i.} \end{pmatrix} \colon i \in [m],\ A_{ik} > 0\right\}$.

Prove that if phase II of the Simplex Method is applied to a starting tableau satisfying the assumptions stated earlier, and the lexicographic pivoting rule is used to determine the pivot elements, then the following statements hold true.

(a) In each step, the pivot is uniquely determined by the given rules, and it is a legal pivot for phase II of the Simplex Method.

(b) The tableau row vectors are always lexicographically positive, i.e., $\begin{pmatrix} b_i & A_{i.} \end{pmatrix} \succ 0$ for all $i \in [m]$.

(c) In each pivoting step, the objective function row vector of the tableau strictly increases with respect to the lexicographic order relation.

(d) Phase II of the Simplex Method terminates after finitely many steps if the lexicographic pivoting rule is applied.

Programming exercise

Work through the notebook `04_pivoting.ipynb`, where you implement a function for carrying out pivoting steps in short tableaus.