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Fall 2019

Mathematical Optimization – Solutions to problem set 3

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Carathéodory's theorem

(a) To start with, note that V is finite. Indeed, V is the set of all vertices of the polytope P, which is finite: If $\{x \in \mathbb{R}^n : Ax \leq b\}$ is an inequality description of P, then every vertex of P is the unique solution of a system $\overline{A}x = \overline{b}$, where $\overline{A}x \leq \overline{b}$ is a subsystem of $Ax \leq b$. The number of such subsystems is obviously finite, hence the number of vertices is finite, too. Consequently, Q is a subset of a finite-dimensional space.

Next, observe that Q is not empty. Indeed, we know that $P = \text{conv}(\text{vertices}(P)) = \text{conv}(V)$, thus $x \in P$ can be written as a convex combination of points in V, i.e., there exists $\mu \in \mathbb{R}^V_{\geqslant 0}$ such that

$$x = \sum_{v \in V} \mu_v v \quad \text{and} \quad \sum_{v \in V} \mu_v = 1 \ .$$

This implies $\mu \in Q$, hence $Q \neq \emptyset$.

Finally, as Q is defined by a system of linear inequalities (non-negativity constraints) and equations (the constraints given explicitely), Q is obviously a polyhedron. Moreover, note that $Q \subseteq [0,1]^V$, so Q is also bounded: Any $\lambda \in Q$ satisfies $\lambda \geq 0$ because the description of Q includes non-negativity constraints. Moreover, non-negativity constraints together with the constraint $\sum_{v \in V} \lambda_v = 1$ also imply $\lambda \leq 1$.

Altogether, the above shows that Q is a non-empty polytope.

(b) Q has a vertex because it is a polytope (see Problem 4(a) of Problem set 2). By Proposition 1.19 in the lecture notes, the vertex λ^* is the unique solution of the subsystem of all tight constraints at λ^*. Note that this system of tight constraints has the form

$$\begin{aligned} \lambda_v &= 0 \quad \forall v \in V_0 \\ \sum_{v \in V} \lambda_v v &= x \\ \sum_{v \in V} \lambda_v &= 1 \end{aligned}$$

for some subset $V_0 \subseteq V$. In other words, we have the $n+1$ constraints that are anyways equality constraints, and some tight non-negativity constraints. In order to have a full-rank system, we need at least $|V|$ many constraints, as $\lambda \in \mathbb{R}^V$. Thus, at least $|V| - (n+1)$ many constraints are tight non-negativity constraints. These lead to components of λ^* that equal zero, hence at most $n+1$ many components of λ^* are non-zero.

Consequently, the convex combination

$$x = \sum_{v \in V} \lambda^*_v \cdot v$$

has in fact only at most $n+1$ non-zero terms, thus it is in fact writing x as a convex combination of at most $n+1$ vertices of P, as desired.

Problem 2: Representation of polyhedral cones

(a) Let $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$, where $A \in \mathbb{R}^{n \times m}$, be an inequality description of the given polyhedral cone C. Define $P \coloneqq C \cap [-1,1]^n$, let vertices$(P) = \{x_1, x_2, \ldots, x_k\}$, and denote $Q \coloneqq C(x_1, x_2, \ldots, x_k)$. We prove that $C = Q$ by showing that $Q \subseteq C$ and $C \subseteq Q$.

First, we prove that $Q \subseteq C$. To this end, consider a point $y \in Q$, which can by definition of Q be written as a conic combination of vertices of P, i.e., $y = \sum_{i=1}^k \lambda_i x_i$, where $\lambda \in \mathbb{R}^k_{\geq 0}$. Then

$$Ay = A\left(\sum_{i=1}^k \lambda_i x_i\right) = \sum_{i=1}^k \lambda_i (Ax_i) \leq \sum_{i=1}^k \lambda_i \cdot 0 = 0 \ ,$$

i.e., $y \in C$. Therefore, $Q \subseteq C$.

Now let us show that $C \subseteq Q$. Since $0 \in C \cap Q$, consider an arbitrary point $y \in C \setminus \{0\}$. Note that there exists a coefficient $\alpha > 0$ such that $\alpha y \in P$. Concretely, we may choose

$$\alpha = \frac{1}{\max\{|y_i| : i \in [k]\}} \ ,$$

so that $|\alpha y_i| \leq 1$ for every $i \in [k]$ (see Figure 1). As $P = \operatorname{conv}(\operatorname{vertices}(P))$ (Proposition 1.32 in the lecture notes), we can write $\alpha y = \sum_{i=1}^k \lambda_i x_i$ for some $\lambda \in [0,1]^k$ with $\sum_{i=1}^k \lambda_i = 1$. Denoting $\mu := \frac{\lambda}{\alpha}$, we obtain $y = \sum_{i=1}^k \mu_i x_i$ with $\mu_i \geq 0$ for every $i \in [k]$, i.e., $y \in Q$. Therefore, $C \subseteq Q$.

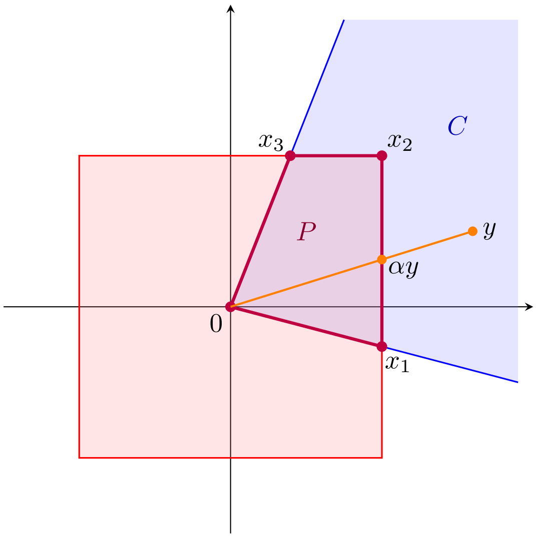

Figure 1: For any point $y \in C$, there exists a scaling factor $\alpha > 0$ such that $\alpha y \in P$.

(b) To simplify notation, we define $C := C(x_1, x_2, \ldots, x_k)$. Note that C is a cone. Indeed, $0 \in C$ by construction and for every $\gamma \in \mathbb{R}_{\geq 0}$ and $x \in C$, which can be written as $x = \sum_{i=1}^k \lambda_i x_i$ for some $\lambda \in \mathbb{R}^k_{\geq 0}$, we have $\lambda x \in C$, since $\lambda x = \sum_{i=1}^k \mu_i x_i$ with $\mu_i = \gamma \lambda_i \geq 0$ for every $i \in [k]$. Thus, to prove that C is a polyhedral cone, it remains to show that C is a polyhedron.

Consider the polyhedron $P = \operatorname{conv}(0, x_1, x_2, \ldots, x_k)$ (the convex hull is indeed a polyhedron due to Proposition 1.33 in the lecture notes). Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be an inequality description of P. Since $0 \in P$, we have $b \geq A \cdot 0 = 0$. Splitting the constraints into those where the right hand side b is 0 and those where it is strictly positive, we can thus write

$$P = \left\{ x \in \mathbb{R}^n : \begin{pmatrix} A^{(1)} \\ A^{(2)} \end{pmatrix} x \leq \begin{pmatrix} 0 \\ b^{(2)} \end{pmatrix} \right\} \ ,$$

where $b^{(2)} > 0$. Define $Q := \{x \in \mathbb{R}^n : A^{(1)} x \leq 0\}$. Our goal is to prove that $C = Q$ by showing that $C \subseteq Q$ and $Q \subseteq C$.

First, we prove that $C \subseteq Q$. Let $x = \sum_{i=1}^k \lambda_i x_i \in C$, where $\lambda \in \mathbb{R}^k_{\geq 0}$. Since $x_i \in P \subseteq Q$ for every $i \in [k]$, we have $A^{(1)} x_i \leq 0$, thus

$$A^{(1)} x = A^{(1)} \left(\sum_{i=1}^k \lambda_i x_i\right) = \sum_{i=1}^k \lambda_i \left(A^{(1)} x_i\right) \leq \sum_{i=1}^k \lambda_i \cdot 0 = 0 \ ,$$

i.e., $x \in Q$. Therefore, $C \subseteq Q$.

Now let us show that $Q \subseteq C$. Since $0 \in Q \cap C$, consider an arbitrary point $y \in Q \setminus \{0\}$. Note that since $b^{(2)} > 0$, there exists a coefficient $\alpha > 0$ such that $A^{(2)}(\alpha y) \leq b^{(2)}$ and hence $\alpha y \in P$ (see Figure 2). Since $P = \operatorname{conv}(0, x_1, x_2, \ldots, x_k)$, we can write $\alpha y = \lambda_0 \cdot 0 + \sum_{i=1}^k \lambda_i x_i$ with coefficients $\lambda_0, \ldots, \lambda_k \in [0,1]$ such that $\sum_{i=0}^k \lambda_i = 1$. Denoting $\mu := \frac{\lambda}{\alpha}$, we obtain $y = \sum_{i=1}^k \mu_i x_i$ with $\mu_i \geq 0$ for every $i \in [k]$, i.e., $y \in Q$. Therefore, $Q \subseteq C$.

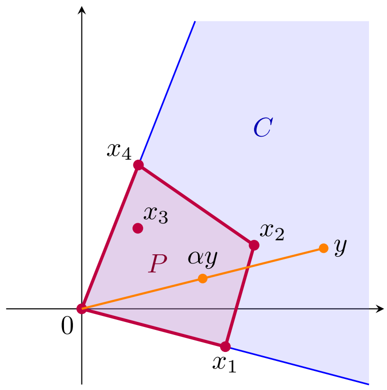

Figure 2: For any point $y \in Q$, there exists a scaling factor $\alpha > 0$ such that $\alpha y \in P$.

Problem 3: Slicing a polyhedron into polytopes

If P is bounded, then $P_\beta \subseteq P$ is bounded for any choice of c and β, so there is nothing to show. Thus, let us assume that P is unbounded.

Since P does not contain a line, we know that P has a vertex, say y (see Problem 2 on Problem set 2; note that P is non-empty because we assumed unboundedness). Thus, $\{y\}$ is a face of P, hence by Proposition 1.13 in the script, there exists a vector c such that y is the unique maximizer of $\max\{c^\top x \colon x \in P\}$. We claim that this c has the desired properties.

First of all, note that c is non-zero. Indeed, if it was zero, any other point $x \in P$ different from y would satisfy $c^\top y = c^\top x = 0$, contradicting the fact that y is a unique maximizer. Note that P does contain a point different from x because we assumed it to be unbounded.

Furthermore, assume for contradiction that there exists $\beta \in \mathbb{R}$ such that P_β is unbounded. Then, there exist vectors $x_0, x_1 \in \mathbb{R}^n$ with $x_1 \neq 0$ such that $x_0 + \lambda x_1 \in P_\beta$ for all $\lambda \geq 0$. We claim that $z := y + x_1 \in P$ and $c^\top z = c^\top y$, contradicting the fact that y is a unique maximizer of $\max\{c^\top x \colon x \in P\}$, and thus finishing the proof.

To prove the claim, let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be an inequality description of P. As $P_\beta \subseteq P$, all points $x_0 + \lambda x_1$ for $\lambda \geq 0$ are contained in P. In other words,

$$\forall \lambda \geq 0 \colon \quad Ax_0 + \lambda \cdot Ax_1 \leq b \ .$$

It is easy to see that this can only be true if $Ax_1 \leq 0$. But then, $Az = Ay + Ax_1 \leq b$, so $z \in P$. Note that $x_0 + \lambda x_1 \in P_\beta$ also implies $c^\top(x_0 + \lambda x_1) = \beta$ for all $\lambda \geq 0$, hence we must have $c^\top x_1 = 0$ and consequently, $c^\top z = c^\top y + c^\top x_1 = c^\top y$, as desired.

Problem 4: Decomposing a polyhedron into a polytope and a cone

(a) Let $H = \{x \in \mathbb{R}^n \colon w^\top x = 0\}$. We claim that

$$P = (P \cap H) + \{\lambda w \colon \lambda \in \mathbb{R}\} \quad \text{and} \quad \dim(P \cap H) < \dim(P) \ . \tag{1}$$

To prove the first equality, let $y \in P$ and define $\lambda = \frac{w^\top y}{w^\top w}$ as well as $z = y - \lambda \cdot w$. Then $w^\top z = 0$, so $z \in P \cap H$. Thus, $y = z + \lambda w \in (P \cap H) + \{\lambda w \colon \lambda \in \mathbb{R}\}$, and hence $P \subseteq (P \cap H) + \{\lambda w \colon \lambda \in \mathbb{R}\}$.

For the other direction, let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be an inequality description of P. As $L(v, w) \subseteq P$, we have $A(v + \lambda w) \leq b$ for all $\lambda \in \mathbb{R}$. It is easy to see that this implies $Aw = 0$. Now let $z \in P \cap H$ and $\lambda > 0$. Then $y \coloneqq z + \lambda w \in P$ as $Ay = Az + \lambda \cdot Aw \leq b$. Thus, $(P \cap H) + \{\lambda w \colon \lambda \in \mathbb{R}\} \subseteq P$.

To obtain $\dim(P \cap H) < \dim(P)$, it is enough to see that $P \cap H$ is contained in the hyperplane H, while we saw above that for any point $y \in P \cap H$, the full line $\{y + \lambda w \colon \lambda \in \mathbb{R}\}$ through y orthogonal to H is contained in P.

Now using (1), if we know the statement for polyhedra of dimension at most $\dim(P) - 1$, we can write

$$P \cap H = Q_{P \cap H} + C_{P \cap H}$$

for a polytope $Q_{P \cap H}$ and a cone $C_{P \cap H}$. Consequently,

$$P = (P \cap H) + \{\lambda w \colon \lambda \in \mathbb{R}\} = Q_{P \cap H} + \big(C_{P \cap H} + \{\lambda w \colon \lambda \in \mathbb{R}\}\big) \ , \tag{2}$$

and if we write $C_{P \cap H} = \left\{\sum_{i=1}^k \lambda_i x_i \colon \lambda_i \geq 0 \ \forall i \in [k]\right\}$ for suitable $x_i \in \mathbb{R}^n$ (Proposition 1.37 in the script), we can rewrite

$$C_{P \cap H} + \{\lambda w \colon \lambda \in \mathbb{R}\} = \left\{\sum_{i=1}^k \lambda_i x_i + \lambda_{k+1} w + \lambda_{k+2}(-w) \colon \lambda_i \geq 0 \ \forall i \in [k+2]\right\} \ ,$$

which is a cone by Proposition 1.37 again. Consequently, the sum in (2) is in fact the sum of a polytope and a cone, which is what we wanted to prove.

(b) Let us first prove that $Q + C \subseteq P$. To this end, let $z \in Q + C$ and write $z = q + v$ with $q \in Q$ and $v \in C$. As by definition, $Q \subseteq P$, we have $Aq \leq b$. Moreover, by definition of C, we also have $Av \leq 0$. Thus $Az = Aq + Av \leq b$, hence $z \in P$. Thus, $Q + C \subseteq P$.

To prove that we also have $P \subseteq Q + C$, let $c \in \mathbb{R}^n$ be a vector guaranteed by Problem 3 of this problem set, and let $p \in P$. We will show that $p \in Q + C$.

The choice of c implies that $\overline{P} \coloneqq \{x \in P \colon c^\top x = c^\top p\}$ is a polytope containing p. Thus, p can be written as a convex combination of vertices of $\overline{P}$, i.e., $p = \sum_{i=1}^k \lambda_i x_i$, where $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, and $\{x_1, \ldots, x_k\} \subseteq \text{vertices}(\overline{P})$. Every vertex x_i of P is defined by a full-rank system of tight constraints of $\overline{P}$. One of these tight constraints is for sure $c^\top x = c^\top p$, but the remaining tight constraints are all tight constraints of P. Thus, every vertex x_i lies on an edge of P. We now distinguish two cases:

- x_i lies on a bounded edge of P: Then, x_i is a convex combination of the vertices of P bounding the edge. In particular, this implies $x_i \in Q$.
- x_i lies on an unbounded edge of P: As P does not contain a line by assumption, the edge must be bounded on one side, i.e., it is a ray $\{q_i + \lambda v_i \colon \lambda \in \mathbb{R}_{\geq 0}\}$, where $q_i \in \text{vertices}(P)$ and hence $q_i \in Q$, and $v_i \in \mathbb{R}^n \setminus \{0\}$. Consequently, we can write $x_i = q_i + \mu_i v_i$ for some $\mu_i \geq 0$. Note that feasibility of the line for P implies $A(q_i + \lambda v_i) \leq b$ for all $\lambda \geq 0$, hence we must have $Av_i \leq 0$, i.e., $v_i \in C$.

To summarize the above two cases, we can write

$$x_i = q_i + \mu_i v_i \ ,$$

where $q_i \in Q$, $v_i \in C$ and $\mu_i \in \mathbb{R}_{\geq 0}$ (note that in the first case above, we use $q_i = x_i$ and $\mu_i = 0$ for this representation). We thus get

$$p = \sum_{i=1}^k \lambda_i x_i = \sum_{i=1}^k \lambda_i (q_i + \mu_i v_i) = \underbrace{\sum_{i=1}^k \lambda_i q_i}_{\in Q} + \underbrace{\sum_{i=1}^k \lambda_i \mu_i v_i}_{\in C} \ ,$$

which proves $p \in Q + C$, and hence $P \subseteq Q + C$.

(c) We proceed by induction on the dimension of P. If P is empty, we can choose Q to be the empty polytope and let C be any cone. Else, if $\dim(P) = 0$, then $P = \{p\}$ consists of a single point, and we can choose $Q = \{p\}$ and $C = \{0\}$.

For the inductive step, assume that we know the statement for all P with $\dim(P) < k$ for some $k \in \mathbb{Z}_{>0}$, and consider a polyhedron P with $\dim(P) = k$. If P contains a line, then by (a), it is enough to show the statement for a polyhedron of strictly smaller dimension, which is true by the inductive assumption. On the other hand, if P does not contain a line, then (b) proves the statement. This concludes the inductive step and completes the proof.

Problem 5: Minkowski sum of a polytope and a cone

(a) If $k := \dim(P) < n$, then P is contained in an affine subspace of dimension k, and there exists an affine bijection that maps this subspace to $\mathbb{R}^k \times \{0\}^{n-k}$. Thus, it is enough to show the statement in $\mathbb{R}^k$ (ignoring the last $n-k$ coordinates) and then transform back to the original space using the inverse of the bijection, exploiting that the image of a polytope under affine bijections is a polytope.

To see the last statement, note that an affine bijection $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is of the form $\Phi(x) = Mx + t$ for an invertible matrix $M \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}^n$. The image of a polyhedron $\{x \in \mathbb{R}^n : Ax \le b\}$ is then given by

$$\begin{aligned}\Phi(\{x \in \mathbb{R}^n : Ax \le b\}) &= \{\Phi(x) : x \in \mathbb{R}^n,\ Ax \le b\} \\ &= \{y \in \mathbb{R}^n : A \cdot M^{-1}(y - t) \le b\} \\ &= \{y \in \mathbb{R}^n : AM^{-1}y \le b + AM^{-1}t\}\ ,\end{aligned}$$

which is itself a polyhedron. Note that we used the substitution $y = \Phi(x) = Mx + t$, which is equivalent to $x = M^{-1}(y-t)$, and also observe that it is important that the map Φ is invertible.

(b) If $\dim(P) = n$, then P contains $n+1$ affinely independent points $x_1, \dots, x_{n+1}$. Let $Q' = \operatorname{conv}(\text{vertices}(Q) \cup \{x_1, \dots, x_{n+1}\})$. Then obviously, $Q + C = Q' + C$, and Q' is a full-dimensional polytope. Thus, in case Q is not full-dimensional, we can equivalently always consider the sum $Q' + C$ with full-dimensional Q'.

(c) Let $L(v,w) := \{v + \lambda w : \lambda \in \mathbb{R}\}$ be a line contained in P, where $v, w \in \mathbb{R}^n$ with $w \ne 0$. Define $H = \{x \in \mathbb{R}^n : w^\top x = 0\}$. We claim that without loss of generality, we can assume $Q \subseteq H$.

To see this, let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\Phi(x) = x - \frac{w^\top x}{w^\top w} w$ for all $x \in \mathbb{R}^n$, i.e., Φ is the orthogonal projection on H. Let $Q' := \Phi(Q)$ be the image of the polytope Q. Note that this is again a polytope: If $\text{vertices}(Q) = \{q_1, \dots, q_k\}$, then

$$\begin{aligned}Q' = \Phi(Q) &= \{\Phi(x) : x \in \operatorname{conv}(\text{vertices}(Q))\} \\ &= \left\{\Phi\left(\sum_{i=1}^k \lambda_i q_i\right) : \lambda \in \mathbb{R}^k,\ \sum_{i=1}^k \lambda_i = 1\right\} \\ &= \left\{\sum_{i=1}^k \lambda_i \Phi(q_i) : \lambda \in \mathbb{R}^k,\ \sum_{i=1}^k \lambda_i = 1\right\} \\ &= \operatorname{conv}(\{\Phi(q) : q \in \text{vertices}(Q)\}),\end{aligned}$$

which is the convex hull of the finitely many points $\Phi(q_1), \dots, \Phi(q_k)$, and thus a polytope by Proposition 1.32 in the script. It is easy to see that $Q' + C = Q + C$, thus we can always replace Q by Q', which is a subset of H, proving the claim.

Thus, assume that $Q \subseteq H$. Note that

$$P = Q + C = Q + (C \cap H) + \{\lambda w : \lambda \in \mathbb{R}\}\ ,$$

where both Q and $C \cap H$ are contained in the hyperplane H. Let Ψ be an affine bijection such that $\Psi(H) = \mathbb{R}^{n-1} \times \{0\}$ and $\Psi(w) = e_n := (0, \dots, 0, 1)$. Then

$$\Psi(P) = \Psi(Q) + \Psi(C \cap H) + \{\lambda e_n : \lambda \in \mathbb{R}\}\ .$$

Note that as $\Psi(Q), \Psi(C \cap H) \subseteq \mathbb{R}^{n-1} \times \{0\}$, we can reduce the last coordinate, thus reducing the ambient dimension by one. Applying the statement in this smaller dimension gives that $\Psi(Q) + \Psi(C \cap H)$ is a polytope in $\mathbb{R}^{n-1} \times \{0\}$, i.e.,

$$\Psi(Q) + \Psi(C \cap H) = \{x \in \mathbb{R}^n \colon A \cdot (x_1, \ldots, x_{n-1})^\top \leq b,\, x_n = 0\}$$

for some matrix $A \in \mathbb{R}^{m \times (n-1)}$. Plugging this into the previous sum, we see that

$$\Psi(P) = \{x \in \mathbb{R}^n \colon A \cdot (x_1, \ldots, x_{n-1})^\top \leq b\} \ ,$$

which is a polyhedron in $\mathbb{R}^n$ (note that the last variable x_n is unconstrained). The image of a polyhedron under an affine bijection is a polyhedron, hence $P = \Psi^{-1}(\Psi(P))$ is a polyhedron, as desired.

(d) (i) As the cone C does not contain a line, we can apply the result of Problem 3 from this problem set to get a vector $c \in \mathbb{R}^n \setminus \{0\}$ such that all sets $C_\beta := \{x \in C \colon c^\top x = \beta\}$ are polytopes.

As C is non-trivial, there exist $x_1, \ldots, x_k \in \mathbb{R}^n \setminus \{0\}$ for some $k \in \mathbb{Z}_{>0}$ such that $C = \left\{\sum_{i=1}^k \lambda_i x_i \colon \lambda \in \mathbb{R}^k_{\geq 0}\right\}$. Observe that $c^\top x_i$ must all have the same sign: If, say, $c^\top x_i < 0$ and $c^\top x_j > 0$, then there is a non-zero convex combination x of x_i and x_j such that $c^\top x = 0$, and $x \in C$. But then, no non-empty slice C_β is bounded, as for any point $y \in C$, we also have that $y + \lambda x \in C_\beta$ for all $\lambda \in \mathbb{R}$, contradicting the construction of c. For the same reason, we also have that $c^\top x_i \neq 0$.

Let $v = \pm c$, where the sign is chosen such that $v^\top x_i > 0$ for all $i \in [k]$, and fix $\beta > \max\{v^\top x \colon x \in Q\}$. We claim that these v and β are what we are looking for.

First of all, we have $Q \subseteq \{x \in \mathbb{R}^n \colon v^\top x \leq \beta\}$ by definition of β. Moreover, $P \cap \{x \in \mathbb{R}^n \colon v^\top x = \beta\}$ is non-empty for the following reason: Let $q \in Q$ and $\lambda := \frac{\beta - v^\top q}{v^\top x_1}$, then $\lambda \geq 0$, so $y = q + \lambda x_1 \in P$. Moreover, $v^\top y = \beta$, hence $y \in P \cap \{x \in \mathbb{R}^n \colon v^\top x = \beta\}$, so the intersection is non-empty.

(ii) As seen in the previous point, we have $v^\top q \leqslant \beta$ for all $q \in \text{vertices}(Q)$, and $v^\top x_i > 0$ for all $i \in [k]$. Thus, there is a unique $\lambda \in \mathbb{R}_{\geq 0}$ such that $v^\top (q + \lambda x_i) = \beta$, and the intersection will contain precisely the point $q + \lambda x_i$.

(iii) We start by observing that the constraint $v^\top x \leq \beta$ is valid for P_β. This is easily seen as $v^\top q < \beta$ for all $q \in \text{vertices}(Q)$ and $v^\top w_{q,i} = \beta$ for all $q \in \text{vertices}(Q)$ and $i \in [k]$.

Note that this also implies

$$\begin{aligned} P_\beta \cap \{x \in \mathbb{R}^n \colon v^\top x = \beta\} &= \text{conv}(\{w_{q,i} \colon q \in \text{vertices}(Q),\, i \in [k]\}) \\ &= (Q + C) \cap \{x \in \mathbb{R}^n \colon v^\top x = \beta\} \ . \end{aligned}$$

As $Q + C$ is full-dimensional, the intersection with the hyperplane $\{x \in \mathbb{R}^n \colon v^\top x = \beta\}$ is $(n-1)$-dimensional, thus the face $P_\beta \cap \{x \in \mathbb{R}^n \colon v^\top x = \beta\}$ is indeed a facet, as desired.

(iv) As $Q \subseteq P_\beta$, we for sure also have $Q \subseteq \overline{P}$, as indicated by the hint. If furthermore, $Ax_i \leq 0$ for all $i \in [k]$, then for any $q \in Q$ and $c = \sum_{i=1}^k \lambda_i x_i \in C$, we have

$$A(q + c) = Aq + Ac = Aq + \sum_{i=1}^k \lambda_i A x_i \leq b \ ,$$

so $q + c \in \overline{P}$, and hence $P = Q + C \subseteq \overline{P}$.

Thus, it remains to show $Ax_i \leq 0$. Assume for contradiction that there exists a row $a_j^\top$ of A and an index $i \in [k]$ such that $a_j^\top x_i > 0$. As $a_j^\top x_i \leq b_j$ is a facet-defining constraint for the full-dimensional polytope P_β, there are $n+1$ affinely independent vertices on the corresponding facet. Not all of them lie on the hyperplane $\{x \in \mathbb{R}^n \colon v^\top x = \beta\}$ as otherwise, the vertices would define the facet corresponding to the constraint $v^\top x \leq \beta$ (and the description has precisely one constraint per facet). Thus, there is a vertex q of P_β with $a_j^\top q = b_j$ and $v^\top q < \beta$. By definition of P_β, this vertex must be a vertex of Q.

Now consider the point $x_\varepsilon \coloneqq q + \varepsilon x_i$. Note that for small enough $\varepsilon > 0$, this point is a convex combination of q and $w_{q,i}$, hence lies in P_β. But on the other hand, $a_j^\top x_\varepsilon = a_j^\top q + \varepsilon \cdot a_j^\top x_i > b_j$, which is a contradiction, as $a_j^\top x \leq b_j$ is a valid constraint for P_β.

(v) As P is convex and closed, and y is compact, there exists a separating hyperplane, i.e., $d \in \mathbb{R}^n \setminus \{0\}$ and $\delta \in \mathbb{R}$ such that $P \subseteq \{x \in \mathbb{R}^n \colon d^\top x \leq \delta\}$ and $d^\top y > \delta$. Let $q_0 \in Q$ and $c \in C$ such that $q_0 + c \in \arg\max\{d^\top x \colon x \in P\}$.

Note that we must have $d^\top c \leq 0$, as by definition of P, all points $q_0 + \lambda c$ for $\lambda \geq 0$ lie in P; and if $d^\top c > 0$, then $d^\top(q_0 + \lambda c) \to \infty$ for $\lambda \to \infty$, contradicting that the maximum is at most δ. Thus, $d^\top(q_0 + c) \leq d^\top q_0$, so $q_0 \in Q$ is a maximizer for $\max\{d^\top x \colon x \in P\}$, as well. But then, $\max\{d^\top x \colon x \in P\} = \max\{d^\top x \colon x \in Q\}$, so the maximum is also attained by a vertex q of Q.

As $d^\top y > \delta > d^\top q = \max\{d^\top x \colon x \in P\}$, all points of the form $z_\mu \coloneqq q + \mu(y - q)$ for $\mu \in (0, 1]$ satisfy $z_\mu \notin P$; but as both q and y are contained in $\overline{P}$, we have $z_\mu \in \overline{P}$. Moreover, as q satisfies $v^\top q < \beta$, we will also get $v^\top z_\mu < \beta$ for small enough $\mu > 0$. Thus, we found a point $z \in \overline{P} \setminus P$ with $v^\top z \leq \beta$, as desired.

To conclude, note that $z \in \overline{P}$ and $v^\top z \leq \beta$ imply $z \in P_\beta$. As $P_\beta \subseteq P$, this gives $z \in P$, a contradiction.

(e) First of all, note that if Q is empty or C is trivial (i.e., $C = \{0\}$), then there is nothing to show: In the first case P is empty; in the second, $P = Q$ is a polytope. Thus assume that Q is non-empty and C is non-trivial from now on.

We prove the statement by induction on the dimension of the ambient space. In zero dimensions, there is nothing to show. Thus, assume that we know the statement for ambient spaces of dimension less than some $n \in \mathbb{Z}_{>0}$. If $\dim(P) < n$, then point (a) shows how to reduce to a lower-dimensional ambient space, so we can conclude by induction. Similarly, if C has a line, then point (c) shows the reduction to a lower-dimensional ambient space.

If $\dim(P) = n$ and C does not have a line, let us employ point (b) to assume that $\dim(Q) = n$, as well. By point (d), we conclude that the statement does also hold in this case. This concludes the inductive step, and hence the proof of the initial statement.