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Fall 2019

Mathematical Optimization – Solutions to problem set 14

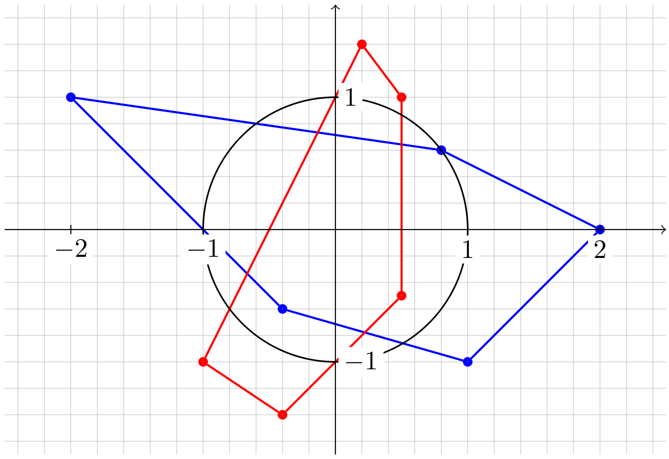
https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: The polar of a polytope

As shown in the lecture, it is enough for the polar P° to consider the inequalities coming from extreme points of P. Hence, we have that

$$P^\circ = \left\{ y \in \mathbb{R}^2 : y^T \begin{pmatrix} 2 & 0.8 & -2 & -0.4 & 1 \\ 0 & 0.6 & 1 & -0.6 & -1 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} \right\}.$$

This polytope P° is given in red in the following picture, while the starting polytope P is given in blue.



Problem 2: Properties of the polar

(a) In part (b), it is shown that every polytope Q with $0 \in Q$ satisfies $(Q^\circ)^\circ = Q$. Hence, our candidate P should not contain the origin.

Trying $P = [1, 2]$, which is a full-dimensional polytope in $\mathbb{R}$, we compute

$$\begin{aligned} P^\circ &= \{y \in \mathbb{R} : y^T x \leq 1 \ \ \forall x \in P\} \\ &= \{y \in \mathbb{R} : y \cdot x \leq 1 \ \ \forall x \in [1, 2]\} \\ &= (-\infty, \tfrac{1}{2}] \ . \end{aligned}$$

Hence, we get that

$$\begin{aligned} (P^\circ)^\circ &= \{x \in \mathbb{R} : x^T y \leq 1 \ \ \forall y \in P^\circ\} \\ &= \{x \in \mathbb{R} : x \cdot y \leq 1 \ \ \forall y \in (-\infty, \tfrac{1}{2}]\} \\ &= [0, 2] \ , \end{aligned}$$

where $(P^\circ)^\circ = [0, 2] \neq [1, 2] = P$.

(b) Since for any $x \in X$ and $y \in X^\circ$, we have $x^T y \leq 1$ (by definition of X°), we clearly have $X \subseteq (X^\circ)^\circ$.

To show that $(X^\circ)^\circ \subseteq X$, assume by contradiction that there exists $z \in (X^\circ)^\circ \setminus X$. We now apply a separation theorem (Theorem 1.47) from the script to the sets X and $\{z\}$. Note that X

and $\{z\}$ are disjoint non-empty (X contains the origin) closed convex sets, where $\{z\}$ is moreover compact. Hence, Theorem 1.47 gives a vector $c \in \mathbb{R}^n \setminus \{0\}$ and a number $b \in \mathbb{R}$ such that $c^T x < b$ for all $x \in X$, and $c^T z > b$. Since $0 \in X$, we have $b > c^T 0 = 0$. Hence, we can define $y = \frac{1}{b}c$. We observe that $y \in X^\circ$ since $x^T y = \frac{1}{b}x^T c < \frac{1}{b}b = 1$ for every $x \in X$. However, we also have $y^T z = \frac{1}{b}c^T z > \frac{1}{b}b = 1$. Since $y \in X^\circ$, this implies that $z \notin (X^\circ)^\circ$, which contradicts the assumption that $z \in (X^\circ)^\circ$.

Remark: Since we always have that $0 \in (X^\circ)^\circ$ (this is very easy to see by just using the definition of the polar), we can conclude that $X \neq (X^\circ)^\circ$ whenever $0 \notin X$. In particular, any full-dimensional polytope $P \subseteq \mathbb{R}^n$ with $0 \notin P$ works for part (a).

Problem 3: Exponentially many facets and polynomially many vertices

Let $Q = [-1,1]^n \subseteq \mathbb{R}^n$ be a scaled and translated version of the n-dimensional unit hypercube $[0,1]^n$ and note that Q contains the origin in its interior. The polytope $Q = [-1,1]^n$ has 2^n vertices (thus, exponentially many) and $2n$ facets (thus, polynomially many).

Note that this is exactly the opposite of what we are looking for. However, the idea is to find a polytope $P \subseteq \mathbb{R}^n$ that contains the origin in its interior and satisfies $P^\circ = Q$, and thus $P = (P^\circ)^\circ = Q^\circ$. In this case, Lemma 7.2in the script implies that every vertex of Q corresponds to a facet of P (and vice-versa) and every facet of Q corresponds to a vertex of P (and vice-versa). Hence, such a polytope P has 2^n facets and $2n$ vertices.

To get P, we compute

$$\begin{aligned} Q^\circ &= \{y \in \mathbb{R}^n \colon y^T x \leq 1 \ \ \forall x \in Q\} \\ &= \{y \in \mathbb{R}^n \colon y^T x \leq 1 \ \ \forall x \in \{-1,1\}^n\} \ , \end{aligned}$$

where we use that the polar of Q is defined by the inequalities given by the vertices of Q. We now let $P = Q^\circ$ and show that this P has the desired properties. Indeed, P is clearly a polytope (P is a polyhedron with $P \subseteq [-1,1]^n$) that contains the origin in its interior (since $B_{1/\sqrt{n}}(0) \subseteq P$). Moreover, Lemma 31 in the lecture notes implies that $(P^\circ)^\circ = P$ and $(Q^\circ)^\circ = Q$. Together with the definition of P, we thus get that $P^\circ = (Q^\circ)^\circ = Q$ and $P = (P^\circ)^\circ = Q^\circ$, as desired.

Problem 4: NP-completeness of integer programming feasibility

(a) As given in the hint, let $y_1, \dots, y_n$ be binary variables for the IP that we are going to write, with the interpretation that

$$y_i = 1 \quad \iff \quad x_i \text{ is } \texttt{true} \ .$$

Note that in this way, every assignment $f \colon X \to \{\texttt{true}, \texttt{false}\}$ corresponds to a point $y \in \{0,1\}^n$, and vice versa. Thus, the goal is to write linear inequalities in y such that the given CNF formula is satisfied by x iff the linear inequalities are satisfied by the corresponding y.

Observe that the CNF formula is satisfied iff every single clause is satisfied. Consider one clause, this is of the form

$$\bigvee_{i \in A} x_i \vee \bigvee_{i \in B} \neg x_i$$

for some sets $A, B \subseteq [n]$. We claim that an assignment is satisfying for this clause iff the corresponding point $y \in \{0,1\}^n$ satisfies the linear inequality

$$\sum_{i \in A} y_i + \sum_{i \in B} (1 - y_i) \geq 1 \ .$$

Indeed, if the assignment x is satisfying, then either $x_i = \texttt{true}$ for some $i \in A$, or $x_i = \texttt{false}$ for some $i \in B$, hence either $y_i = 1$ for some $i \in A$, or $1 - y_i = 1$ for some $i \in B$, and thus the constraint is satisfied. The same is true the other way round.

Thus, requiring the above inequality for each constraint makes sure that a feasible solution y corresponds to an assignment x that is satisfying for each clause, i.e., a satisfying assignment for the given CNF formula. On the other hand, every satisfying assignment of the CNF formula corresponds to a feasible solution of our system.

We conclude that the CNF formula is feasible if and only if our system of linear inequalities is feasible. Moreover, we can construct the solution of one problem from a solution of the other.

(b) First of all, CNF-SAT is obviously in NP: It is a decision problem, and given any assignment of its variables, deciding whether this assignment is satisfying can be done in polynomial time. Moreover, part (a) shows that if we can decide feasibility of a binary integer program, then we can also decide feasibility of CNF-SAT. As CNF-SAT is NP-complete, we thus conclude that CNF-SAT is NP-complete, too.