Fall 2019

Mathematical Optimization – Problem set 11

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: An alternative description of the perfect matching polytope

Let $G = (V, E)$ be a graph. As seen in class, the matching polytope P associated to G is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) \leq 1 & \forall v \in V \\ x(E[S]) \leq \frac{|S|-1}{2} & \forall S \subseteq V,\ |S| \text{ odd} \end{array} \right\} .$$

Let $\overline{P}$ be defined by

$$\overline{P} = P \cap \left\{ x \in \mathbb{R}^E \colon x(E) = \frac{|V|}{2} \right\} .$$

(a) Prove that $\overline{P}$ satisfies the following two properties.

(i) $\overline{P}$ contains the right integral points, and

(ii) if $\overline{P} \neq \emptyset$, then $\overline{P}$ is a face of the matching polytope P.

From (i) and (ii), conclude that $\overline{P}$ describes the perfect matching polytope of G.

In class, we showed that the perfect matching polytope P_{perf} can also be described by

$$P_{perf} = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V,\ |S| \text{ odd} \end{array} \right\} .$$

Do not use the results from part (a) for the following subproblem.

(b) Prove the following two relations for $\overline{P}$ and P_{perf}.

(i) Show that all constraints in P_{perf} are implied by the constraints in $\overline{P}$.

(ii) Show that all constraints in $\overline{P}$ are implied by the constraints in P_{perf}.

From (i) and (ii), conclude that $\overline{P} = P_{perf}$.

Problem 2: Properties of polytopes and linear systems

(a) Let N be a finite set, $P \subseteq [0,1]^N$ be a polytope, and $e \in N$ an element. Let $P' \subseteq [0,1]^{N \setminus \{e\}}$ be the polytope defined by

$$P' \coloneqq \left\{ x|_{N \setminus \{e\}} \colon x \in P,\ x(e) = 0 \right\} .$$

Prove the following two statements, which relate vertices of P with $x(e) = 0$ and vertices of P'.

(i) Show that if $x \in P$ is a vertex with $x(e) = 0$, then $x|_{N \setminus \{e\}}$ is a vertex in P'.

(ii) Show that if $y \in P'$ is a vertex, then the point $x \in [0,1]^N$ given by

$$x(f) \coloneqq \begin{cases} y(f) & \text{if } f \neq e \\ 0 & \text{if } f = e \end{cases}$$

is a vertex of P.

(b) Let $Ax = b$ and $Cx = d$ be two linear systems with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$, and $d \in \mathbb{R}^k$. Let $y \in \mathbb{R}^n$ be a common solution to both systems. Show that if every row of A is a linear combination of the rows of C, then

$$\{x \in \mathbb{R}^n \colon Ax = b\} \supseteq \{x \in \mathbb{R}^n \colon Cx = d\} .$$

Note that the above implies the following statement, which is used in the lecture: The equalities in the system $Ax = b$ are implied by those in the system $Cx = d$ if the two systems have a common solution and the rows of A are linear combinations of the rows of C. Observe that in particular, there is no need to consider the right-hand sides in the linear combinations.

Problem 3: Properties of laminar families
Let $\mathcal{L}$ be a laminar family on a finite ground set N. For any set $S \subseteq N$, define

$$\mathcal{L}_S = \{L \in \mathcal{L} \colon L \text{ and } S \text{ are intersecting}\} \ ,$$

where we recall that two sets $A, B \subseteq N$ are *intersecting* if $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$, and $B \setminus A \neq \emptyset$. Let $S \subseteq N$ be such that $\mathcal{L}_S \neq \emptyset$, and let $L \in \mathcal{L}_S$. Prove the following four inequalities.

(a) $|\mathcal{L}_{S\cup L}| < |\mathcal{L}_S|$.

(b) $|\mathcal{L}_{S\cap L}| < |\mathcal{L}_S|$.

(c) $|\mathcal{L}_{S\setminus L}| < |\mathcal{L}_S|$.

(d) $|\mathcal{L}_{L\setminus S}| < |\mathcal{L}_S|$.

Problem 4: Properties of edges in cuts
Let $G = (V, E)$ be a graph and let $A, B \subseteq V$.

(a) Show that $\chi^{\delta(A)} + \chi^{\delta(B)} = \chi^{\delta(A\cup B)} + \chi^{\delta(A\cap B)} + 2 \cdot \chi^{E(A\setminus B, B\setminus A)}$.

(b) Derive $\chi^{\delta(A)} + \chi^{\delta(B)} = \chi^{\delta(A\setminus B)} + \chi^{\delta(B\setminus A)} + 2 \cdot \chi^{E(A\cap B, V\setminus(A\cup B))}$ from part (a).

Problem 5: The size of laminar families
Let $N = [n]$ and let $\mathcal{L} \subseteq 2^N \setminus \{\emptyset\}$ be a laminar family, i.e., for any $L, L' \in \mathcal{L}$, either $L \subseteq L'$, $L' \subseteq L$, or $L \cap L' = \emptyset$.

(a) Show that $|\mathcal{L}| \leq 2n - 1$.

(b) Assume that $\mathcal{L}$ does not contain a set $L \in \mathcal{L}$ with the property that there exist $k \geq 2$ disjoint sets $L_1, \ldots, L_k \in \mathcal{L}$ with $L = \bigcup_{i=1}^k L_i$. In other words, $\mathcal{L}$ contains no non-trivial partition of some set $L \in \mathcal{L}$. Prove that $|\mathcal{L}| \leq n$.

Problem 6: Bounded degree spanning trees
Let $G = (V, E)$ be an undirected graph with $E \neq \emptyset$. Recall that the spanning tree polytope of G is defined as

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} \ .$$

Given an integer bound $B_v \in \mathbb{Z}_{\geq 0}$ for every vertex $v \in V$, we consider the polytope

$$Q = \{x \in P \colon x(\delta(v)) \leq B_v \text{ for all } v \in V\} \ .$$

Notice that the integer points in this polytope correspond to spanning trees of G respecting the given degree bounds at every vertex. Let y be a vertex of Q.

(a) Show that $|\operatorname{supp}(y)| \leq 2|V| - 2$.

(b) Show that there exists an edge $e \in E$ with $y(e) \geq \frac{1}{2}$.