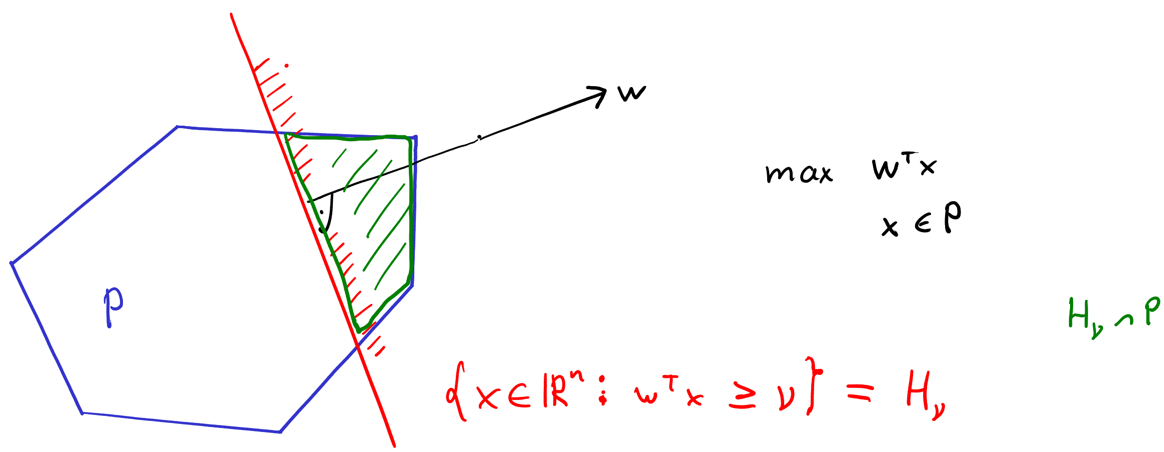
6.4 Ellipsoid Method for finding point in full-dimensional $\{0,1\}$-polytope

We start with simpler question (checking feasibility):

Given a separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, find a point $x \in P$.

Checking feasibility is closely related to optimization



$$\max \; w^T x \quad x \in P$$

Basics on ellipsoids

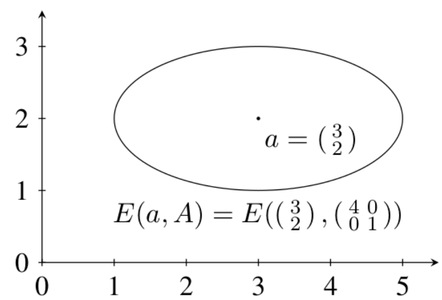
Definition 6.3: Ellipsoid

An ellipsoid in $\mathbb{R}^n$ is a set

$$E(a, A) := \{x \in \mathbb{R}^n \colon (x-a)^\top A^{-1}(x-a) \le 1\} \ ,$$

where $a \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. The point a is called the *center* of the ellipsoid $E(a, A)$.

this implies that A is symmetric



$$x^T A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$$

Equivalently, an ellipsoid is the image of the unit ball under an affine bijection:

$A \in \mathbb{R}^{n\times n}$ positive definite $\iff$ $A = QQ^T$ for some full-rank matrix $Q \in \mathbb{R}^{n\times n}$.

$$\begin{aligned}
E(a, A) &= \{x \in \mathbb{R}^n : (x-a)^T A^{-1}(x-a) \le 1\} \\
&= \{x \in \mathbb{R}^n : (x-a)^T (Q^{-1})^T Q^{-1}(x-a) \le 1\} \\
&= \{x \in \mathbb{R}^n : \|Q^{-1}(x-a)\|_2^2 \le 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n, \|y\|_2^2 \le 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n, \|y\|_2 \le 1\}
\end{aligned}$$

$y = Q^{-1}(x-a)$

$x = Qy + a$

$$\begin{aligned}
A^{-1} &= (QQ^T)^{-1} \\
&= (Q^T)^{-1} Q^{-1}
\end{aligned}$$

6.4.1 (High-level) description of Ellipsoid Method

Algorithm 8: Ellipsoid Method

Input : Separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, and an ellipsoid $E_0 = E(a_0, A_0)$ with $P \subseteq E_0$.

Output: A point $y \in P$.

$i = 0$.

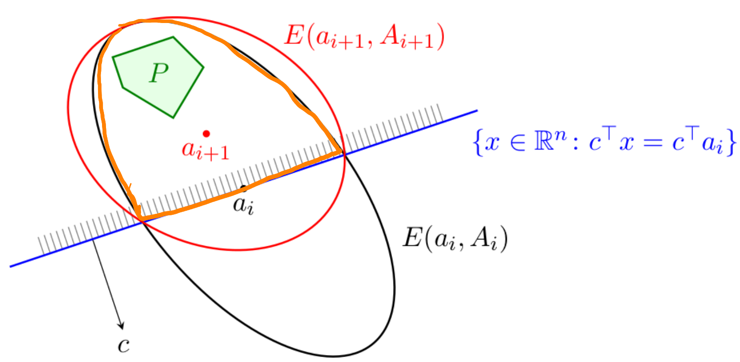
while $a_i \notin P$ *(checked with separation oracle)* **do**

  Get $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top a_i\}$, using separation oracle.

  Find min. volume ellipsoid $E_{i+1} = E(a_{i+1}, A_{i+1})$ containing $E_i \cap \{x \in \mathbb{R}^n : c^\top x \le c^\top a_i\}$.

  $i = i + 1$.

return a_i.



Two key questions:

- (How quickly) does the Ellipsoid Method terminate?
- How to compute $E_{i+1} = E(a_{i+1}, A_{i+1})$?

6.4.2 Getting a bound on the number of iterations

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} .$$

Before proving Lemma 6.4, we show that it implies following bound on number of iterations.

Lemma 6.5

The Ellipsoid Method will stop after at most $2(n+1)\ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right)$ iterations.

Proof

Let $L \in \mathbb{Z}_{\geq 0}$ be last iteration of Ellipsoid Method, i.e., value of i when it terminates.

$$P \subseteq E_L \implies \mathrm{vol}(P) \leq \mathrm{vol}(E_L) \overset{\text{Lemma 6.4}}{\leq} \mathrm{vol}(E_0) \cdot e^{-\frac{L}{2(n+1)}}$$

$$\implies L \leq 2(n+1) \ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right) .$$

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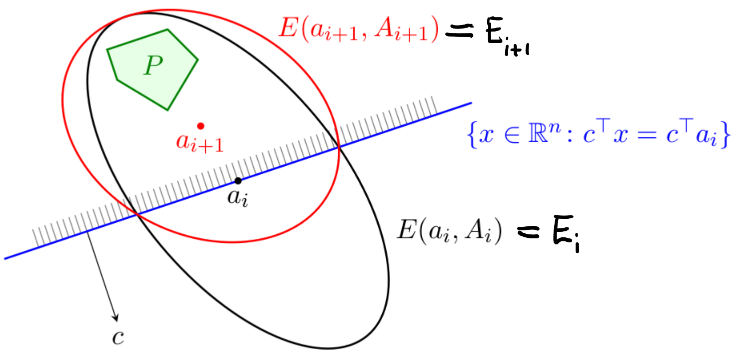
Proof of Lemma 6.4 and explicit description for E_{i+1}



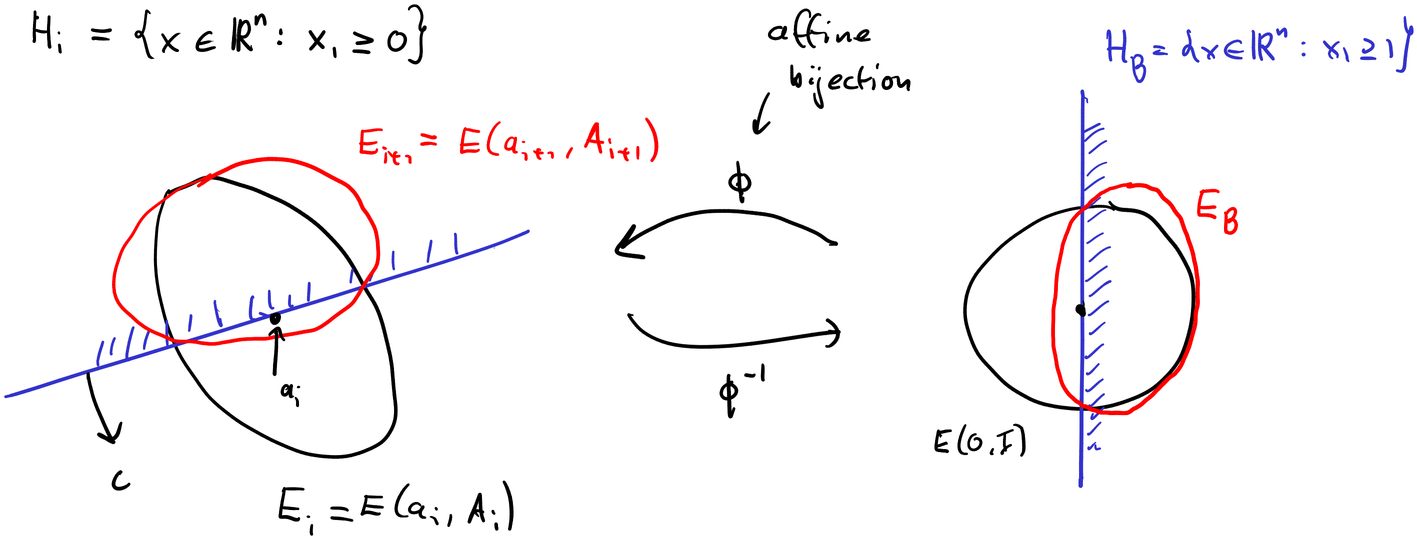
$H_i = \{x \in \mathbb{R}^n : c^\top x \le c^\top a_i\}$

What is ratio between $\mathrm{vol}(E_{i+1})$ and $\mathrm{vol}(E_i)$?

This question can be reduced to the case:

$E_i = E(0, I)$

$H_i = \{x \in \mathbb{R}^n : x_1 \ge 0\}$



$H_i = \{x \in \mathbb{R}^n : c^\top x \le c^\top a_i\}$

Lemma 6.7

Let $H_B = \{x \in \mathbb{R}^n \colon x_1 \geq 0\}$. Then the ellipsoid

$$E_B = \left\{ x \in \mathbb{R}^n \;\middle|\; \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{j=2}^{n} x_j^2 \leq 1 \right\} \tag{6.7}$$

contains $E(0, I) \cap H_B$.

<u>Proof</u>

Proof of Lemma 6.4

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} .$$

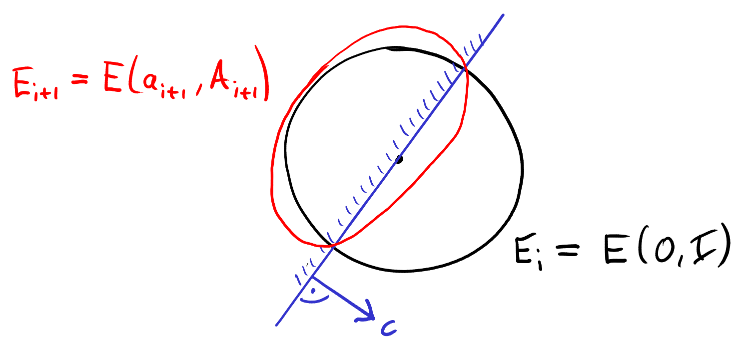
6.4.3 From the unit ball to the general case

We obtained explicit description of E_{i+1} if

- $E_i = E(0, I)$, and
- $H_i = \{x \in \mathbb{R}^n : x_1 \geq 0\}$

From this we obtain explicit description of E_{i+1} for the general case by transforming description of this special case through an approroviate affine bijection.

General half-space cutting $E(0, I)$



$E_i = E(0, I)$

$H_i = \{x \in \mathbb{R}^n : c^T x \leq 0\}$, with $\|c\|_2 = 1$

$\xrightarrow[\text{see problem sets}]{}$

$E_{i+1} = E(a_{i+1}, A_{i+1})$, where

$$a_{i+1} = -\frac{1}{n+1} c$$

$$A_{i+1} = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} c c^T\right)$$

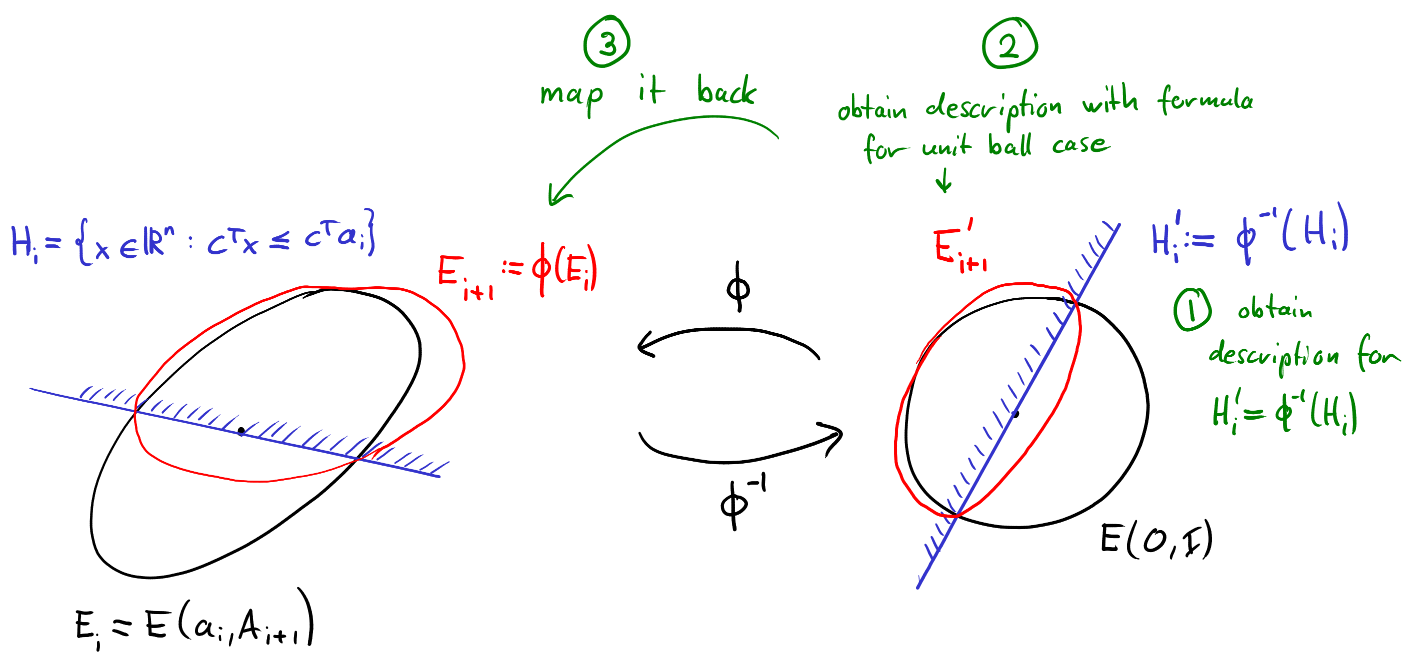
General case

Let $E_i = E(a_i, A_i)$ be a general ellipsoid, and let

$H_i = \{x \in \mathbb{R}^n : c^T x \leq c^T a_i\}$ be a general hyperplane containing a_i.

$$\left.\begin{array}{l} \text{Let } Q_i \in \mathbb{R}^{n \times n} \text{ s.t. } A_i = Q_i Q_i^T. \\ \phi(x) := Q_i x + a_i . \end{array}\right\} \Rightarrow \phi(E(0, I)) = E_i .$$

Plan:



Hence, we can now write more explicitely how an iteration of the Ellipsoid Method looks.

Algorithm 9: Ellipsoid Method

Input : Separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, and an ellipsoid $E_0 = E(a_0, A_0)$ with $P \subseteq E_0$.

Output: A point $y \in P$.

$i = 0$.

while $a_i \notin P$ *(checked with separation oracle)* **do**

- Get $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top a_i\}$, using separation oracle.
- Let $b = \frac{A_i c}{\sqrt{c^\top A_i c}}$.
- Let $a_{i+1} = a_i - \frac{1}{n+1} b$.
- Let $A_{i+1} = \frac{n^2}{n^2-1}(A_i - \frac{2}{n+1} bb^\top)$.
- $i = i + 1$.

return a_i.

6.4.4 From checking feasibility to optimization over $\{0,1\}$-polytopes

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional $\{0,1\}$-polytope.

We want to solve :

$$\max_{x \in P} w^T x$$

for some $w \in \mathbb{Z}^n$.

Getting optimal LP value $\nu^* = \max\{w^T x : x \in P\}$

Starting ellipsoid

Bounding number of iterations

Recall:

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} .$$

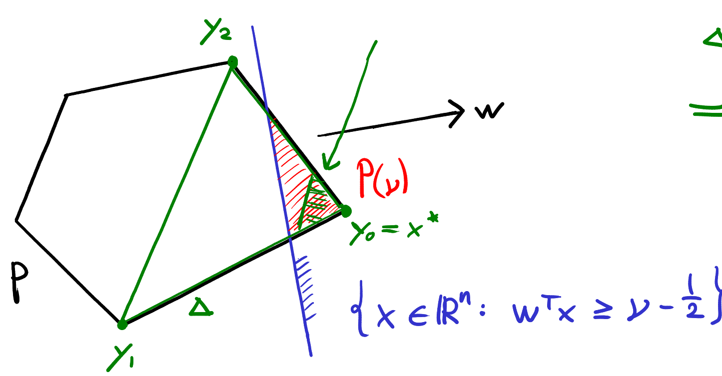
Assuming $P(\nu) \neq \emptyset$, we need lower bound on $\mathrm{vol}(P(\nu))$.

Plan

$\Delta' \leftarrow$ scaled-down version of Δ s.t.

$\Delta' \subseteq P(\nu)$

$\Rightarrow \mathrm{vol}(\Delta') \leq \mathrm{vol}(P(\nu))$



Determining an optimal $\{0,1\}$-solution x^*

One can determine x^* coordinate-wise from x_1^*, x_2^* to x_n^*, by repeatedly solving LPs over P with slightly modified objectives.

To check whether there exists optimal solution x^* to $\max\{w^T x : x \in P\}$ with $x^* = 1$, ...

$\Rightarrow$

Theorem 6.9

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x \colon x \in P\}$ using a polynomial number of elementary operations and calls to the separation oracle for P.

6.5 Comments on the non-full-dimensional case

Theorem 6.2

Let $P \subseteq \mathbb{R}^n$ be a $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x \colon x \in P\}$ using a polynomial number (in n) of elementary operations and calls to the separation oracle for P.