Fall 2019

Mathematical Optimization – Solutions to problem set 4

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Simplex Algorithm

(a) Transforming the given linear program into standard form yields the following.

$$\begin{array}{lrrrrrrrrrrrrrr}
\max & & x_1 & + & x_2 & & & & & & & & & & \\
\text{s.t.} & & x_1 & & & + & y_1 & & & & & & & = & 4 \\
& & & & x_2 & & & + & y_2 & & & & & = & 4 \\
& & x_1 & + & x_2 & & & & & + & y_3 & & & = & 7 \\
& - & x_1 & - & x_2 & & & & & & & + & y_4 & = & -3 \\
& & x_1 & & & & & & & & & & & \geq & 0 \\
& & & & x_2 & & & & & & & & & \geq & 0 \\
& & & & & & y_1 & & & & & & & \geq & 0 \\
& & & & & & & & y_2 & & & & & \geq & 0 \\
& & & & & & & & & & y_3 & & & \geq & 0 \\
& & & & & & & & & & & & y_4 & \geq & 0
\end{array}$$

(b) The short tableau with basis $B = (y_1, y_2, y_3, y_4)$ is the following.

	x_1	x_2	1
z	-1	-1	0
y_1	1	0	4
y_2	0	1	4
y_3	1	1	7
y_4	-1	-1	-3

This tableau is infeasible, which can be seen from the negative entry -3 in the right-hand side vector; the full basic solution corresponding to the tableau is $(x_1, x_2, y_1, y_2, y_3, y_4) = (0, 0, 4, 4, 7, -3)$, from which it can be seen that y_4 violates the non-negativity constraint. The corresponding solution in the original space is $(x_1, x_2) = (0, 0)$.

(c) In phase I of the Simplex Method, we solve the auxiliary problem

$$\begin{array}{lrrrrrrrrrrrrrrr}
\max & & & & & & & & & & & & & - & x_0 & & \\
\text{s.t.} & & x_1 & & & + & y_1 & & & & & & & - & x_0 & = & 4 \\
& & & & x_2 & & & + & y_2 & & & & & - & x_0 & = & 4 \\
& & x_1 & + & x_2 & & & & & + & y_3 & & & - & x_0 & = & 7 \\
& - & x_1 & - & x_2 & & & & & & & + & y_4 & - & x_0 & = & -3 \\
& & x_1 & & & & & & & & & & & & & \geq & 0 \\
& & & & x_2 & & & & & & & & & & & \geq & 0 \\
& & & & & & y_1 & & & & & & & & & \geq & 0 \\
& & & & & & & & y_2 & & & & & & & \geq & 0 \\
& & & & & & & & & & y_3 & & & & & \geq & 0 \\
& & & & & & & & & & & & y_4 & & & \geq & 0 \\
& & & & & & & & & & & & & & x_0 & \geq & 0
\end{array}$$

with the following corresponding short tableau (including the original objective z and the new objective $\tilde{z}$).

	x_1	x_2	x_0	1
$\tilde{z}$	0	0	1	0
z	-1	-1	0	0
y_1	1	0	-1	4
y_2	0	1	-1	4
y_3	1	1	-1	7
y_4	-1	-1	-1	-3

The first pivoting operation is to make the tableau feasible, and it will make x_0 enter the basis and y_4 leave the basis (see the marked pivot above), as the right-hand side of the row corresponding to y_4 has the most negative entry (see Proposition 1.81 in the script for an explanation of this choice). Performing a pivot operation on this pivot and then continuing until the tableau is optimal, we obtain

	x_1	x_2	y_4	1
$\tilde{z}$	-1	-1	1	-3
z	-1	-1	0	0
y_1	2	1	-1	7
y_2	1	2	-1	7
y_3	2	2	-1	10
x_0	1	1	-1	3

$\rightsquigarrow$

	x_1	x_0	y_4	1
$\tilde{z}$	0	1	0	0
z	0	1	-1	3
y_1	1	-1	0	4
y_2	-1	-2	1	1
y_3	0	-2	1	4
x_2	1	1	-1	3

The latter tableau is optimal because the objective function row has no negative entries, and the objective value is 0. Thus, we found a solution of the auxiliary problem with value 0, which corresponds to a feasible solution of the initial problem. The corresponding tableau can be extracted from the tableau above, it is the following.

	x_1	y_4	1
z	0	-1	3
y_1	1	0	4
y_2	-1	1	1
y_3	0	1	4
x_2	1	-1	3

This tableau corresponds to the basic feasible solution $(0, 3)$ in the original space.

(d) A legal pivot to start pivoting is marked in the tableau above; continuing from there, we obtain the subsequent tableaus.

	x_1	y_2	1
z	-1	1	4
y_1	1	0	4
y_4	-1	1	1
y_3	1	-1	3
x_2	0	1	4

$\rightsquigarrow$

	y_3	y_2	1
z	1	0	7
y_1	-1	1	1
y_4	1	0	4
x_1	1	-1	3
x_2	0	1	4

The last tableau is optimal because again, the objective function coefficient row has no negative entries. Thus, the corresponding basic feasible solution $(x_1, x_2) = (3, 4)$ with value 7 is optimal for the initial linear program.

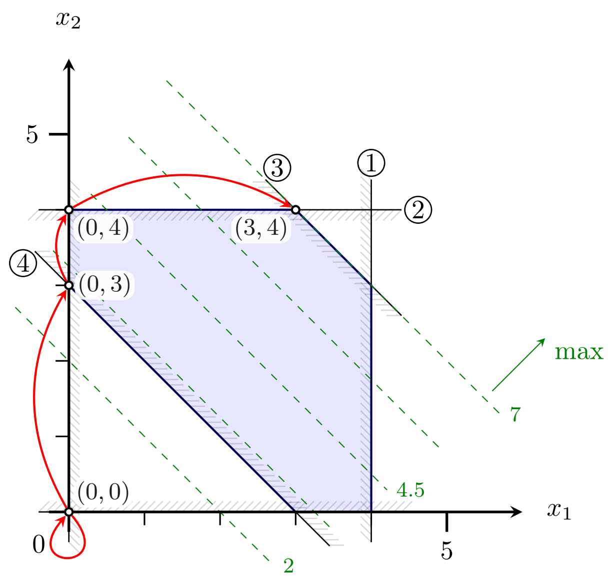

Figure 1: Graphical representation of the given linear program. The constraints are drawn in black and numbered according to the given order, with grey dashes indicating the feasible direction. They result in the blue feasible region. Level curves and their values are shown in green, and the path taken by the Simplex Method (phase I and II).

(e) A graphical representation of the linear program is given in Figure 1. The figure shows that the solution $(3, 4)$ lies on the level curve corresponding to the LP value 7; and no other level curve with larger value intersects the feasible region, thus judging from the picture, the solution is indeed optimal.

The basic solutions visited by the Simplex Method in phase I are $(0, 0)$ (starting tableau and after the first step) and $(0, 3)$, which is the feasible solution obtained from phase I. Phase II then started at $(0, 3)$ and, via $(0, 4)$, ended at the optimal solution $(3, 4)$.

Note that in case you chose a different pivot for the second step of phase I, you might have taken the symmetric path from $(0, 0)$ via $(3, 0)$ and $(4, 0)$ to $(4, 3)$, which is optimal as well.

(f) The picture clearly shows that all points on the line segment connecting $(4, 3)$ and $(3, 4)$ are optimal. Looking at the tableau only, we can see that the extra pivoting step

	y_3	y_2	1
z	1	0	7
y_1	-1	$\boxed{1}$	1
y_4	1	0	4
x_1	1	-1	3
x_2	0	1	4

$\rightsquigarrow$

	y_3	y_1	1
z	1	0	7
y_2	-1	1	1
y_4	1	0	4
x_1	0	1	4
x_2	1	-1	3

leads to another feasible and optimal tableau that corresponds to the different optimal solution $(4, 3)$. Thus, by linearity we know that not only $(3, 4)$ and $(4, 3)$ are optimal, but also all infinitely many convex combinations of $(3, 4)$ and $(4, 3)$.

Problem 2: Legal Pivots in Simplex Tableaus

(a) There are no legal pivots in the first tableau. The tableau is feasible as all right-hand sides are non-negative, and it is optimal since the objective coefficent row does not contain negative entries. The corresponding basic feasible and optimal solution is $(x_0, x_1, x_2, x_3, x_4, x_5) = (0, 0, 1, 0, 6, 0)$. The linear program is bounded an has value 12.

	x_1	x_0	x_3	1
z	$\frac{3}{8}$	0	1	-12
x_4	1	0	-2	6
x_2	$-\frac{3}{4}$	7	-3	1
x_5	$\frac{125}{8}$	10	-3	0

(b) Legal pivots are marked in the tableau below.

	y_1	y_2	x_s^+	x_s^-	x_b	x_t	1
z	0	$-\frac{4}{19}$	0	5	-3	-1	12
y_3	1	2	$-\frac{5}{2}$	7	0	-2	2
x_a	-3	-1	$\frac{3}{2}$	0	-3	0	0
y_4	0	0	$\frac{1}{2}$	0	$-\frac{1}{10}$	-1	5

The point $(x_a, x_b, x_s^+, x_s^-, x_t, y_1, y_2, y_3, y_4) = (0,0,0,0,0,0,0,2,5)$ is a basic solution corresponding to the tableau, and it is feasible because the right hand sides are non-negative. It is not optimal because the LP is in fact unbounded: The columns corresponding to variables x_b and x_t are non-positive (and strictly negative in the objective row), hence we can read a feasible increasing cone from the tableau, namely

$$(0,0,0,0,0,0,0,2,5) + \lambda \cdot \left(3,1,0,0,0,0,0,0,\frac{1}{10}\right) + \mu \cdot (0,0,0,0,1,0,0,2,1)$$

for $\lambda, \mu \geq 0$.

(c) Legal pivots are marked in the tableau below. There are possible legal pivoting steps that strictly increase the objective function value, hence the current basic solution $(A,B,C,D,E,F,G,H,J) = (0,0,0,0,1,4,0,12,0)$ is not optimal. It is feasible, though, as all right-hand sides are non-negative. Since there is no potential pivot column with all non-positive entries, we can in the current state not decide if the linear program is unbounded or not.

	J	A	B	G	D	1
z	-4	6	0	-1	$-\frac{1}{2}$	0
C	6	-2	-5	0	-7	0
F	-3	$-\frac{17}{2}$	3	2	0	4
H	3	91	-3	4	-20	12
E	0	14	$\frac{7}{2}$	$\frac{1}{2}$	2	1

(d) There is one legal pivot, marked below. Pivoting with this choice increases the objective value by 5, so the current basic solution $(x_1, x_2, x_3, y_1, y_2, y_3, y_4) = (1, \frac{9}{2}, 0, 0, 0, 2, 5)$ is not optimal. As all right-hand sides are non-negative, it is feasible. We can not decide from the current tableau whether or not the LP is unbounded, as there is no potential pivot column with all non-positive entries.

	y_1	y_2	x_3	1
z	3	-5	0	4
y_3	2	0	-2	2
x_1	3	7	3	1
x_2	1	-1	-3	$\frac{9}{2}$
y_4	0	10	3	5

Problem 3: Characterizing potential choices of free variables
Let $M \subseteq [m]$ and $N \subseteq [n]$ with $|M|+|N| = n$, as given in the problem statement. Assume that there exists a linear subsystem equivalent to $Ax+y = b$ in tableau form that uses $\{x_i : i \in N\} \cup \{y_j : j \in M\}$ as free variables. For a system in tableau form, we know that when we set the free variables to zero, the system has a unique solution. Thus, by the assumption, the system $Ax + y = b$ has a unique solution if the variables in $\{x_i : i \in N\} \cup \{y_j : j \in M\}$ are set to zero. Denoting $\overline{M} := [m] \setminus M$ and $\overline{N} := [n] \setminus N$, we can write the system consisting of the remaining variables as

$$\underbrace{\left(\begin{array}{c|c} A_{.\overline{N}} & \mathrm{I}_{.\overline{M}} \end{array}\right)}_{=:S} \begin{pmatrix} x_{\overline{N}} \\ y_{\overline{M}} \end{pmatrix} = b \ ,$$

and the fact that this system has a unique solution is equivalent to the coefficient matrix $S \in \mathbb{R}^{m\times m}$ of the system being invertible.

Observe that the above implication also holds the other way round: If the matrix S is invertible, then the system $S^{-1}Ax + S^{-1}y = S^{-1}b$ is a system equivalent to $Ax + y = b$ in tableau form that uses the variables in $\{x_i : i \in N\} \cup \{y_j : j \in M\}$ as free variables.

Consequently, using the notation introduced above, it is enough to show the following.

$$\left(\begin{array}{c|c} A_{.\overline{N}} & \mathrm{I}_{.\overline{M}} \end{array}\right) \text{ is invertible} \qquad \Longleftrightarrow \qquad \left(\begin{array}{c} A_{M.} \\ \hline \mathrm{I}_{N.} \end{array}\right) \text{ is invertible}$$

After rearranging rows of the first matrix to first write the rows corresponding to M and then those corresponding to $\overline{M}$, and rearranging columns of the second matrix to first write those corresponding to $\overline{N}$ and then those corresponding to N, the above statement can be equivalently read as

$$\left(\begin{array}{c|c} A_{M\overline{N}} & 0 \\ \hline A_{\overline{M}\overline{N}} & \mathrm{I}_{\overline{M}\overline{M}} \end{array}\right) \text{ is invertible} \qquad \Longleftrightarrow \qquad \left(\begin{array}{c|c} A_{M\overline{N}} & A_{MN} \\ \hline 0 & \mathrm{I}_{NN} \end{array}\right) \text{ is invertible} \ .$$

This last equivalence is certainly true because both matrices are invertible iff the submatrix $A_{M\overline{N}}$ is invertible. This finishes the proof.

Problem 4: Short questions on the Simplex Method

(a) True. Consider a legal simplex pivoting step where variable x_k enters the basis and variable x_j leaves the basis. At this point, the corresponding short tableau might look as follows:

$$\begin{array}{c|ccc|c} & \dots & x_k & \dots & 1 \\ \hline z & \dots & c_k & \dots & q \\ \hline \vdots & \ddots & & & \vdots \\ x_j & & A_{jk} & & b_j \\ \vdots & & & \ddots & \vdots \end{array}$$

As we assume the exchange step that we are considering is legal, we have $c_k < 0$ and $A_{jk} > 0$. When pivoting, the new objective function coefficient corresponding to x_j will be $-\frac{c_k}{A_{jk}}$, which is positive by the previous inequalities. Thus, the new column corresponding to x_j will not be a legal pivoting column, hence x_j will not immediately reenter the basis.

(b) False. Consider the following tableaus, where the second is reached from the first by doing the indicated pivoting step.

$$
\begin{array}{c|cc|c}
 & x_1 & x_2 & 1 \\
\hline
z & 1 & -4 & -1 \\
\hline
y_1 & -1 & 1 & 3 \\
y_2 & -2 & 3 & 2
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c|cc|c}
 & x_1 & y_2 & 1 \\
\hline
z & -\frac{5}{3} & \frac{4}{3} & \frac{5}{3} \\
\hline
y_1 & -\frac{1}{3} & -\frac{1}{3} & \frac{7}{3} \\
x_2 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}
\end{array}
$$

In the first tableau, every potential pivot column has a legal pivot, but the second tableau shows that the objective function is unbounded (note the negative column for x_1). The reason is that in Simplex Method, we always move along edges of the corresponding polytope. The vertex corresponding to the first tableau above does not have an edge incident to it that is improving and unbounded, while the vertex corresponding to the second tableau does.

(c) False. Consider the following two tableaus.

$$
\begin{array}{c|cc|c}
 & x_1 & x_2 & 1 \\
\hline
z & -2 & 1 & 7 \\
\hline
y_1 & 2 & 3 & 0 \\
y_2 & 4 & 0 & 3
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c|cc|c}
 & y_1 & x_2 & 1 \\
\hline
z & 1 & 4 & 7 \\
\hline
x_1 & \frac{1}{2} & \frac{3}{2} & 0 \\
y_2 & -2 & -6 & 3
\end{array}
$$

The marked pivot is legal, the second tableau is optimal, and both tableaus have the same value. Thus, the vertex corresponding to the first tableau was already optimal. Note that the reason why this happens is degeneracy: If you check the basic solutions for the two tableaus, you will see that they are actually the same.

(d) False. Consider the example given in (c). The first tableau is optimal, as discussed above, but it has a negative entry in the objective row.

(e) True. In a system with m equality constraints and $n+m$ variables, every basic feasible solution is defined by m basic variables. For choosing m variables from $n+m$ variables, there are $\binom{n+m}{m}$ options (note that not all choices must also be a basis!), and this number is finite.

Problem 5: Lexicographic pivoting

(a) By definition of the lexicographic order, for any two different vectors $a, b \in \mathbb{R}^n$, either $a \succ b$ or $b \succ a$. Thus, the only case where the lexicographic minimum that we consider when selecting the pivot row is not unique is the one where two scaled row vectors $\frac{1}{A_{ik}}\begin{pmatrix} b_i & A_{i.} \end{pmatrix}$ are equal. This, though, would imply that the rows are the same up to scaling, i.e., the long tableau coefficient matrix has two linearly dependent rows. Note, however, that a system in tableau form has full row rank, contradicting the previous implication. Thus, the lexicographic minimum is always unique.

Now consider the chosen row j. As $\frac{1}{A_{jk}}\begin{pmatrix} b_j & A_{j.} \end{pmatrix}$ is lexicographically minimal among all scaled row vectors $\frac{1}{A_{ik}}\begin{pmatrix} b_i & A_{i.} \end{pmatrix}$ with $A_{ik} > 0$, we have $\frac{b_j}{A_j k} \leq \frac{b_i}{A_{ik}}$ for all i with $A_{ik} > 0$. Equivalently, $j \in \arg\min \left\{ \frac{b_i}{A_{ik}} : A_{ik} > 0 \right\}$. Additionally, by choice of the pivot row k, the coefficient of variable k in the objective function row is negative. Thus, the two conditions in Definition 1.66 are satisfied, so A_{jk} is a legal pivot.

(b) By assumption, the tableau row vectors are lexicographically positive at the start: By feasibility, we have $b \geq 0$; furthermore, the first m columns of the matrix A form an identity matrix—hence the first non-zero entry of each tableau row is positive, for sure (and there is a positive entry in each row).

Let us assume we are faced with a tableau of the following form, where all rows (except maybe the objective function row) are lexicographically positive.

$$\begin{array}{c|c} & x_1 \;\ldots\; x_{m+n} \\ \hline q & -c^\top \\ \hline \\ b & A \\ \\ \end{array}$$

When doing a simplex step on the pivot in column k and row j of A, the rows change as follows:

- Row j is divided by $A_{j,k}$, which is positive because it is a legal pivot, hence remains lexicographically positive.
- Row i, for $i \neq j$, which is $r_i := \begin{pmatrix} b_i & A_{i.} \end{pmatrix}$ modified by subtracting $\frac{A_{ik}}{A_{jk}}$ times row j, which is $r_j := \begin{pmatrix} b_j & A_{j.} \end{pmatrix}$. In other words, the new row i, say $\tilde{r}_i$, is given by

$$\tilde{r}_i = r_i - \frac{A_{ik}}{A_{jk}} r_j \ .$$

  If $A_{ik} \leq 0$, we are done: Then $-\frac{A_{ik}}{A_{jk}} > 0$, so r_i and $-\frac{A_{ik}}{A_{jk}} r_j$ are both lexicographically positive, and the sum of lexicographically positive vectors is lexicographically positive. If on the other hand, $A_{ik} > 0$, note that we can equivalently rewrite $\tilde{r}_i \succ 0$ in the form

$$\frac{1}{A_{ik}} r_i \succ \frac{1}{A_{jk}} r_j \ ,$$

  which is true by the choice of the row index j such that the lexicographically minimal scaled row $\frac{1}{A_{ik}} \begin{pmatrix} b_i & A_{i.} \end{pmatrix}$ with $A_{ik} > 0$ is obtained for $i = j$.

Thus, tableau row vectors remain lexicographically positive throughout when applying the lexicographic pivoting rule.

(c) In a pivoting step using a legal pivot A_{jk}, the objective row $\begin{pmatrix} q & -c^\top \end{pmatrix}$ is changed by adding $\frac{c_k}{A_{jk}}$ times row j to it. Note that as A_{jk} is a legal pivot, the objective row entry in the pivot column is negative, i.e., $-c_k < 0$. Thus, $\frac{c_k}{A_{jk}}$ is positive, and the pivoting operation actually adds a positive multiple of row j to the objective row. By (b), this row j is lexicographically positive, and so is a positive multiple of it, hence the objective function row is strictly lexicographically increasing.

(d) Every basic solution of the system has a unique corresponding objective function row. As there are only finitely many basic solutions, there are also only finitely many different objective function rows. As the lexicographic pivoting rule guarantees that we see different objective function rows only in strictly lexicographically increasing order, the procedure must terminate after finitely many steps (at the latest when reaching the lexicographically maximal objective function row).