4.4.5 Optimal project selection

$G = (P, A)$

$g : P \to \mathbb{Z}$

$g(v) \geq 0$: profit

$g(v) < 0$: cost

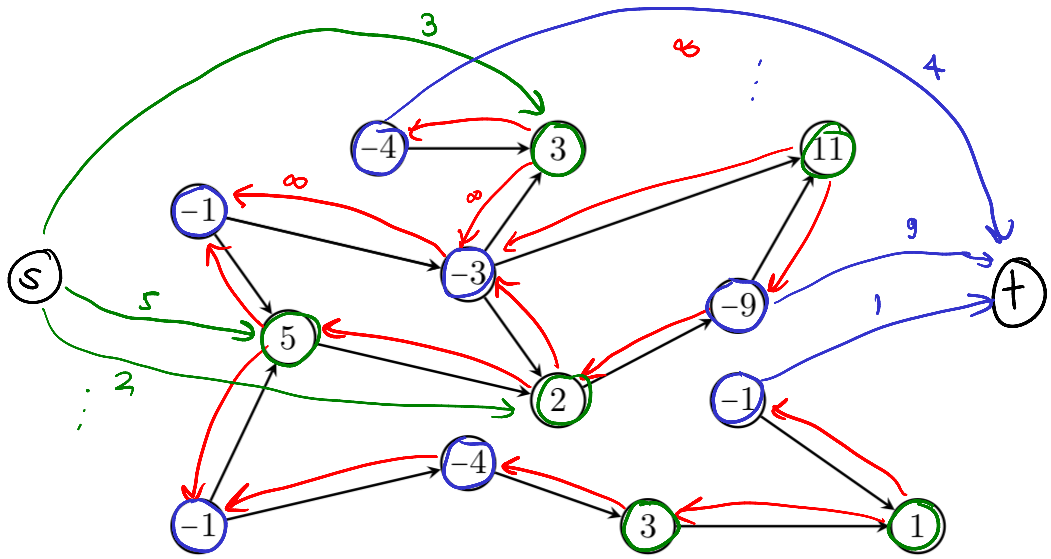
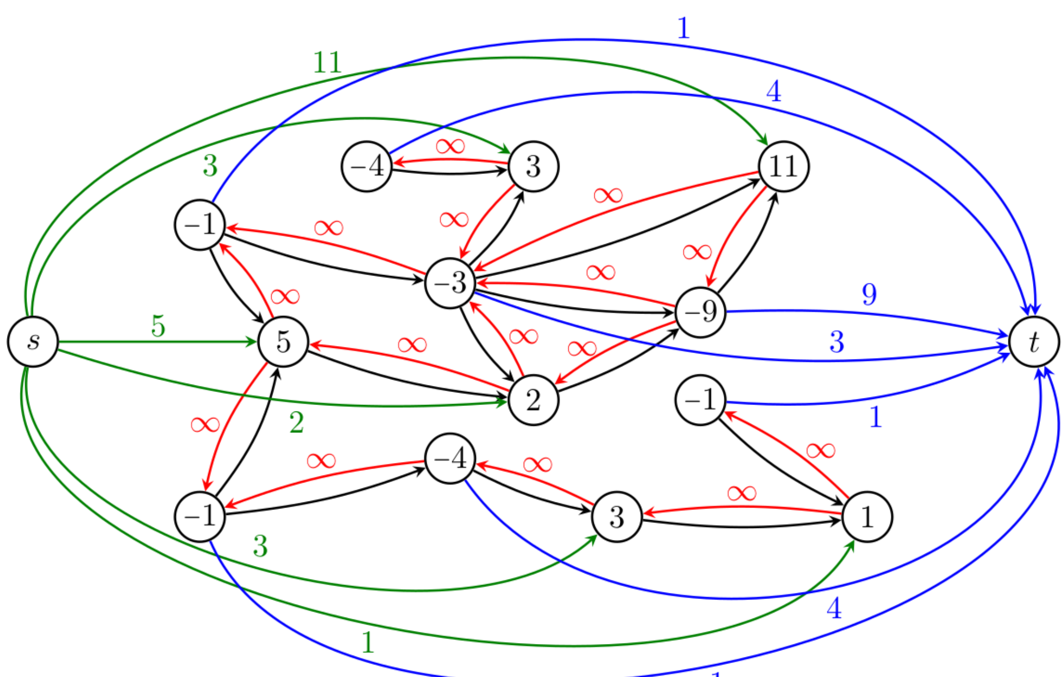
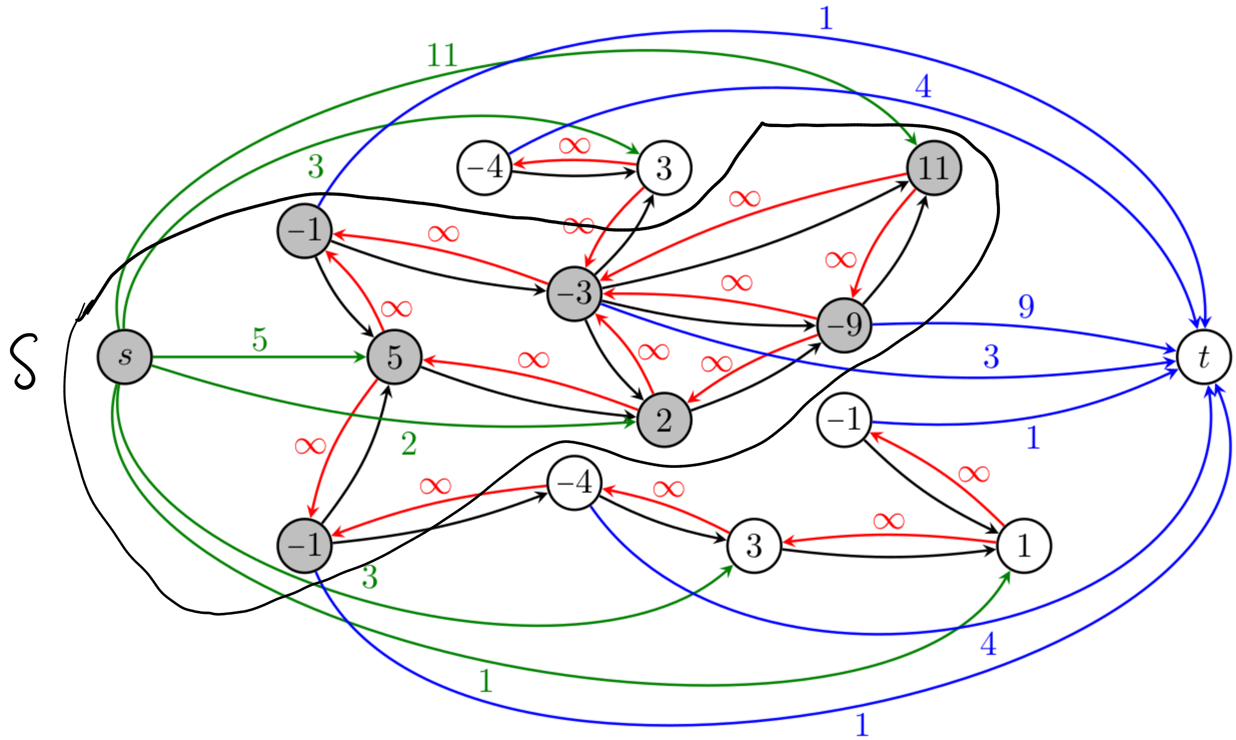


Figure 4.11: A graph G with projects and their precedence constraints. Profits (or costs) are indicated in the corresponding vertices.

Modeling as a minimum s-t cut problem





A minimum s-t cut in auxiliary graph is indeed an optimal solution

Let $P = P^+ \cup P^-$ with $P^+ = \{v \in P : g(v) \geq 0\}$
$P^- = \{v \in P : g(v) < 0\}$

Let S be an s-t cut s.t. $\delta^+(S)$ does not contain ∞-arcs.

$$u(\delta^+(S)) = g(P^+ \setminus S) - g(P^- \cap S)$$
$$= g(P^+) - g(P^+ \cap S) - g(P^- \cap S)$$
$$= g(P^+) - g(S)$$

$\Rightarrow$ minimizing $u(\delta^+(S))$ is same as maximizing $g(S)$. ✓

4.4.6 Open pit mining

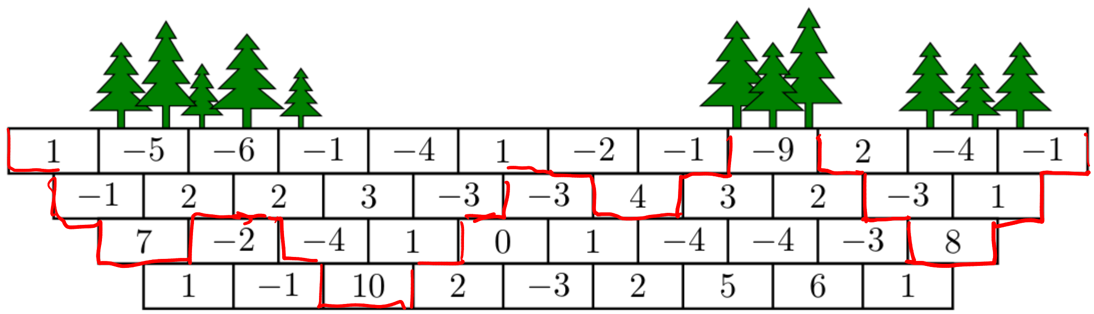


Figure 4.14: A possible soil profile with respective profits.

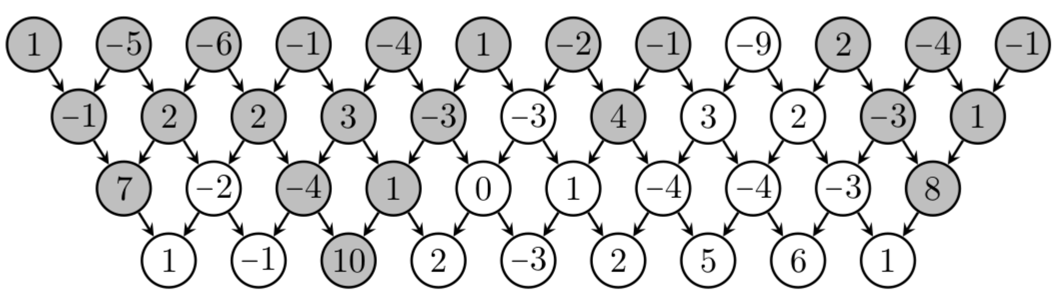


Figure 4.15: Reduction of the open pit mining problem shown in Figure 4.14 to an optimal project selection problem. The gray vertices correspond to an optimal solution.

4.4.7 Image segmentation

(a) The Mona Lisa of Leonardo da Vinci together with a manual selection.

(b) The foreground of the Mona Lisa, extracted due to color differences and manual selection.

Figure 4.16: Extraction of the foreground from an image.

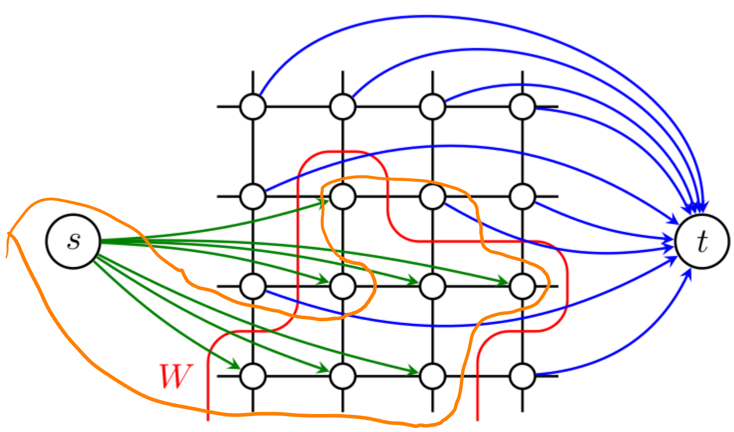
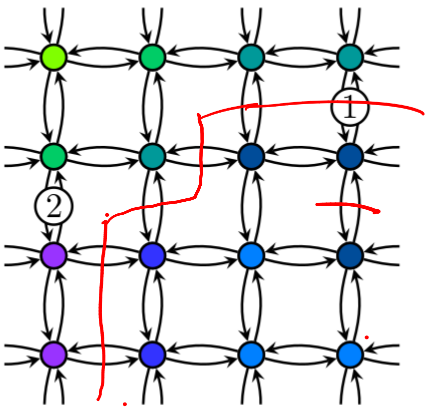


Figure 4.17: Excerpt of an image with manual segmentation W shown in red. The arcs (s, p) are shown in green and arcs (p, t) are highlighted in blue. Each of these colored arcs has equal capacity $x \in \mathbb{Z}_{\geq 0}$.

$$u((p_1, p_2)) \coloneqq 765 - \|(r_1, g_1, b_1) - (r_2, g_2, b_2)\|_1$$



① $= (0, 153, 153)$ $\quad$ $= (0, 76, 153)$

$$\begin{aligned} u((\bigcirc, \bigcirc)) &= 765 - |0 - 0| - |153 - 76| - |153 - 153| \\ &= 765 - 0 - 77 - 0 = 688 \ . \end{aligned}$$

② $= (0, 204, 102)$ $\quad$ $= (153, 51, 255)$

$$\begin{aligned} u((\bigcirc, \bigcirc)) &= 765 - |0 - 135| - |204 - 51| - |102 - 255| \\ &= 765 - 135 - 153 - 153 = 324 \ . \end{aligned}$$

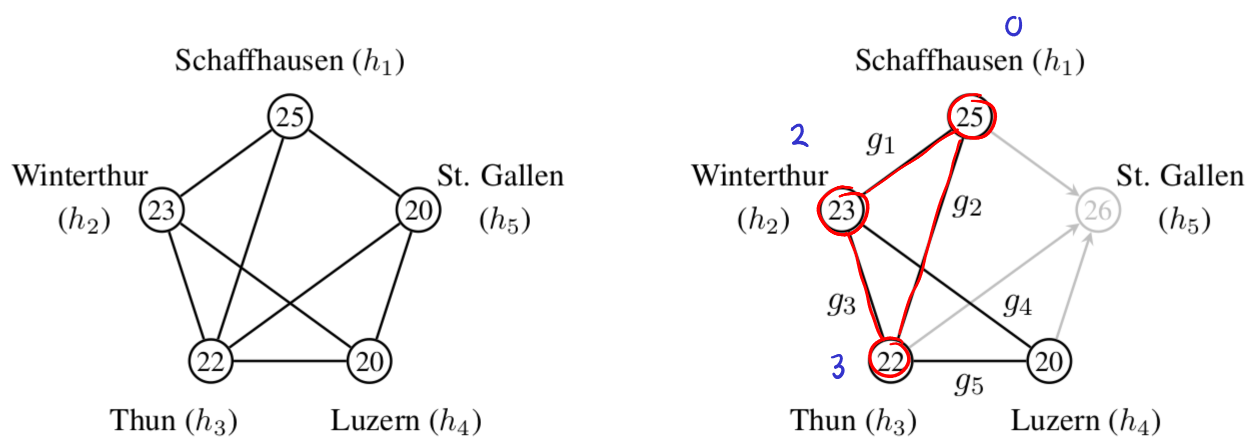
Figure 4.18: Calculation of color difference of adjacent pixels using two examples. The greater the color difference, the smaller the capacity u on the corresponding arcs.

4.4.8 Theoretical winning possibilities in sports competitions

rank	team	remaining games	points
1.	Schaffhausen	3	25
2.	Winterthur	3	23
3.	Thun	4	22
4.	Luzern	3	20
5.	St. Gallen	3	20
⋮	⋮	⋮	⋮

Table 4.1: A possible (partial) handball table.

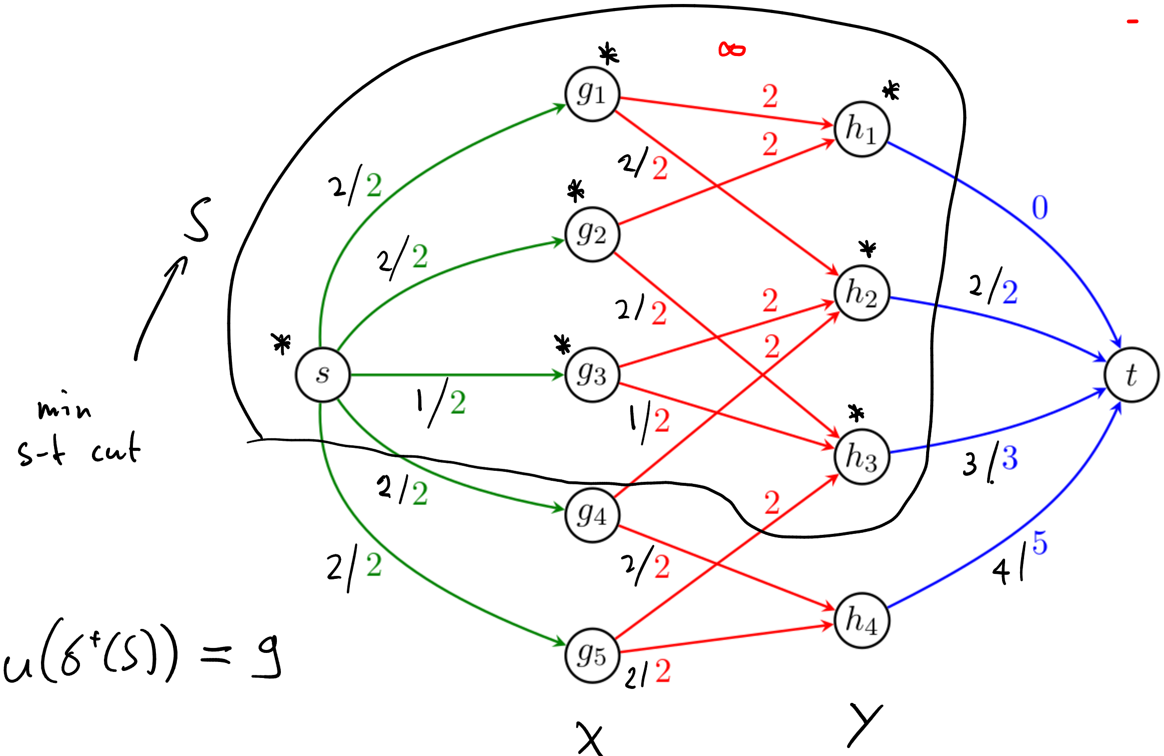
6



(a) Remaining matches between teams: Each edge corresponds to one match.

(b) If we assume that St. Gallen wins all its games, then the 5 games $g_1, \ldots, g_5$ remain.

Figure 4.19: Remaining games displayed as a graph.



$u(\delta^+(S)) = 9$

When using ∞-capacities on red arcs:

$\Rightarrow \nexists$ No arc from $S \cap X$ to $Y \setminus S$. $\Leftrightarrow$ $N^+(S \cap X) \subseteq S \cap Y$

Moreover, we can assume $S \cap Y \subseteq N^+(S \cap X)$. For otherwise,

$\exists\, y \in (S \cap Y) \setminus N^+(S \cap X)$. $\Rightarrow u(\delta^+(S \setminus \{y\})) = u(\delta^+(S)) - u((y,t)) \leq u(\delta^+(S))$

$\Rightarrow u(\delta^+(S \setminus \{y\}))$ is a minimum s-t cut.

$\Rightarrow \underline{N^+(S \cap X) = S \cap Y}$

Assume, $u(\delta^+(S)) < 2|X|$. ← St. Gallen cannot become sole leader anymore

$\Rightarrow \underbrace{2 \cdot |X \setminus S| + u(\delta^+(S \cap Y))}_{u(\delta^+(S))} < 2|X|$

$\Rightarrow u(\delta^+(S \cap Y)) < 2|X \cap S| \Leftrightarrow u(\delta^+(N^+(S \cap X))) < 2|S \cap X|$

$\Rightarrow |S \cap X|$ is set of games, whose game points $2|S \cap X|$ are strictly higher than the teams playing them, $N^+(S \cap X)$ can absorb before at least one of them has at least as many points as St. Gallen.