Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Prof. Dr. Rico Zenklusen and Assistants
Contact: math.opt@ifor.math.ethz.ch

Institute for Operations Research
Department of Mathematics

Fall 2019

Mathematical Optimization – Problem set 7

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: The algorithm of Ford-Fulkerson and the value of a flow

(a) Consider the graph $G = (V, A)$ with edge capacities $u\colon A \to \mathbb{Z}_{\geq 0}$ given in Figure 1.

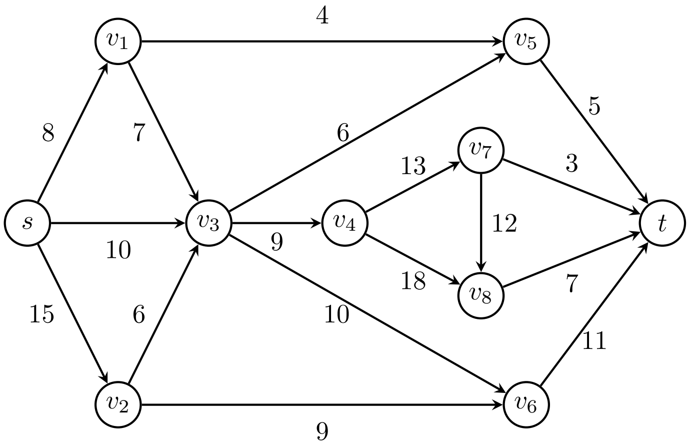

Figure 1: A digraph $G = (V, A)$ with edge capacities $u\colon A \to \mathbb{Z}_{\geq 0}$.

Apply the algorithm of Ford and Fulkerson to obtain a maximal s-t flow. In every iteration, provide the current flow and its value, the corresponding residual graph, as well as an augmenting s-t path together with its increment value γ (or a certificate that there is no augmenting s-t path).

(b) Show that the value of any s-t flow f is equal to the difference between the inflow into t and the outflow at t, i.e., $\nu(f) = f(\delta^-(t)) - f(\delta^+(t))$.

Problem 2: Flow through intermediate vertices

Let $G = (V, A)$ be a directed graph with arc capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, and let $s_1, s_2, \dots, s_\ell \in V$ be distinct. Assume that for every $i \in \{1, \dots, \ell - 1\}$, there is an s_i-s_{i+1} flow f_i with value $\nu(f_i) \geq k$ for some $k \in \mathbb{Z}_{\geq 0}$. Prove that there exists an s_1-s_ℓ flow f with value $\nu(f) \geq k$.

Problem 3: Max-flow min-cut via duality I

Let $G = (V, A)$ be a directed graph with arc capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, and let $s, t \in V$ be two distinct vertices. Consider the following linear program (P).

$$
\begin{array}{rrcll}
\max & \nu & & & \\
& \displaystyle\sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a & = & \begin{cases} \nu & \text{if } v = s \\ -\nu & \text{if } v = t \\ 0 & \text{if } v \in V \setminus \{s, t\} \end{cases} & \\
& f_a & \leq & u_a & \forall a \in A \\
& f_a & \in & \mathbb{R}_{\geq 0} & \forall a \in A \\
& \nu & \in & \mathbb{R}_{\geq 0} &
\end{array} \tag{P}
$$

(a) Prove that the optimal value of (P) equals the value of a maximum s-t flow in G.

(b) Write down the dual of the linear program (P), using variables y_v for $v \in V$ and z_a for $a \in A$.

(c) Let $C \subseteq V$ be an s-t cut in G. Define

$$y_v = \begin{cases} 0 & \text{if } v \in C \\ 1 & \text{if } v \notin C \end{cases} \qquad \text{and} \qquad z_a = \begin{cases} 1 & \text{if } a \in \delta^+(C) \\ 0 & \text{if } a \notin \delta^+(C) \end{cases} .$$

Show that (y, z) is feasible for the dual linear program obtained in part (b) with value $u(\delta^+(C))$.

(d) Use the findings from (a), (b), and (c) together with weak duality (Theorem 1.84) to conclude the weak max-flow min-cut theorem (Theorem 4.5).

Can you also directly deduce the strong max-flow min-cut theorem (Corollary 4.14) if you use strong duality?

(e) Let (y, z) be an optimal solution for the dual linear program obtained in (b) such that $y_s = 0$. For $\theta \in (0, 1)$ chosen uniformly at random, define the random variables

$$\begin{aligned} Y_v &= \begin{cases} 0 & \text{if } y_v < \theta \\ 1 & \text{if } y_v \geq \theta \end{cases} && \text{for all } v \in V, \\ Z_a &= \max\{0, Y_w - Y_v\} && \text{for all } a = (v, w) \in A, \\ \text{and} \qquad C &= \{v \in V : Y_v = 0\} \ . \end{aligned}$$

(i) Show that the assumption $y_s = 0$ can be made without loss of generality.

(ii) Prove that (Y, Z) is a feasible solution for the dual linear program obtained in (b) and prove that its expected value is at most the value of the solution (y, z).

(iii) Prove that for every $\theta \in (0, 1)$, C is an s-t cut, and show that its value $u(\delta^+(C))$ is equal to the value of the solution (Y, Z). Use this to conclude that the value of the dual linear program equals the value of a minimum s-t cut in G.

(iv) Exploit strong linear programming duality to deduce the strong max-flow min-cut theorem (Corollary 4.14).

Problem 4: Matchings and vertex covers in bipartite graphs

For a given graph $G = (V, E)$, a *vertex cover* is a subset $C \subseteq V$ of the vertices of G such that every edge $e \in E$ has at least one endpoint in C. A vertex cover is a *minimum vertex cover* if it is a vertex cover of minimum cardinality. Similarly, a *maximum matching* is a matching of maximum cardinality.

(a) Argue that for every graph $G = (V, E)$, the cardinality of a maximum matching is a lower bound on the cardinality of a minimum vertex cover.

(b) Provide an example of a graph where equality does not hold.

(c) Use the max-flow min-cut theorem to prove that in bipartite graphs, the cardinality of a maximum matching equals the cardinality of a minimum vertex cover.

Problem 5: Vertex connectivity

Consider the following reformulation of Menger's Theorem (Theorem 4.21 in the script), which was proved in the lecture.

Theorem (Menger's Theorem). *Let $G = (V, A)$ be a directed graph with $s, t \in V$, $s \neq t$. There exist k pairwise arc-disjoint s-t paths in G if and only if every s-t cut $C \subseteq V$ satisfies $|\delta^+(C)| \geq k$.*

We are interested in proving a similar result on internally vertex-disjoint s-t paths instead of arc-disjoint s-t paths, where we call two s-t paths *internally vertex-disjoint* if they share no vertices other than s and t. Moreover, we say that a subset of vertices $S \subseteq V \setminus \{s, t\}$ *disconnects* s and t in G if the graph $G[V \setminus S]$ that we obtain from G by removing all vertices in S (and all arcs incident to at least one vertex in S) does not contain an s-t path.

(a) With the help of the max-flow min-cut theorem, prove the following variant of Menger's Theorem.

Theorem (Menger's Theorem for vertex connectivity). *Let $G = (V, A)$ be a directed graph with $s, t \in V$, $s \neq t$, such that there is no arc from s to t. There exist k pairwise internally vertex-disjoint s-t paths in G if and only if every set $S \subseteq V \setminus \{s, t\}$ that disconnects s and t satisfies $|C| \geq k$.*

(b) What happens in the theorem you proved in (a) if the graph G contains arcs from s to t?

Programming exercises

Work through the notebook `07_imageSegmentation.ipynb`, where you implement the image segmentation algorithm seen in class using minimum cuts in a suitable capacitated digraph.