1.3.7 Simplex Method : phase II

↪ Getting from a basic feasible solution to an optimal one.

$q \in \mathbb{R}$: constant term

$$\begin{array}{rlcl} \max & c^\top x & + & q \\ & Ax + y & = & b \\ & x & \in & \mathbb{R}^n_{\geq 0} \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array}$$

Introduce variable z for objective:

$$\begin{array}{rlcl} \max & z & & \\ & z - c^\top x & = & q \\ & Ax + y & = & b \\ & x & \in & \mathbb{R}^n_{\geq 0} \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array}$$

Corresponding LP tableau:

$$\begin{array}{c|ccc|ccc|c} z & y_1 & \dots & y_m & x_1 & \dots & x_n & 1 \\ \hline 1 & 0 & \dots & 0 & -c_1 & \dots & -c_n & q \\ \hline 0 & & I & & & A & & b \end{array}$$

We will work with short tableau:

$$\begin{array}{c|ccc|c} & x_1 & \dots & x_n & 1 \\ \hline z & -c_1 & \dots & -c_n & q \\ \hline y_1 & A_{11} & \dots & A_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ y_m & A_{m1} & \dots & A_{mn} & b_m \end{array}$$

Definition 1.62: Feasible tableau

The tableau (1.17) is called *feasible* if $b_1, \ldots, b_m \geq 0$, i.e., if the corresponding basic solution is feasible.

Definition 1.63: Value of tableau

The entry q in a tableau (see 1.17) is called the *value of the tableau*. This is the objective value of the basic solution that corresponds to the tableau. We use this terminology both for feasible and infeasible tableaus.

Example 1.64: Feasible tableau

The tableau below corresponds to the linear program shown in (1.12). It is a feasible tableau.

	x_1	x_2	1
z	-400	-900	0
y_1	1	4	40
y_2	2	1	42
y_3	1.5	3	36

$$\begin{aligned} \max\ & 400x_1 + 900x_2 \\ & x_1 + 4x_2 \leq 40 \\ & 2x_1 + x_2 \leq 42 \\ & 1.5x_1 + 3x_2 \leq 36 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Simplex phase II starts with a feasible tableau and performs exchange steps to obtain further feasible tableaus until a feasible tableau is obtained whose basic solution is an optimal solution (in case of finite LP).

Example 1.65

(a)

	x_1	x_2	1
z	-400	-900	0
y_1	1	4	40
y_2	$\boxed{2}$	1	42
y_3	1.5	3	36

(b)

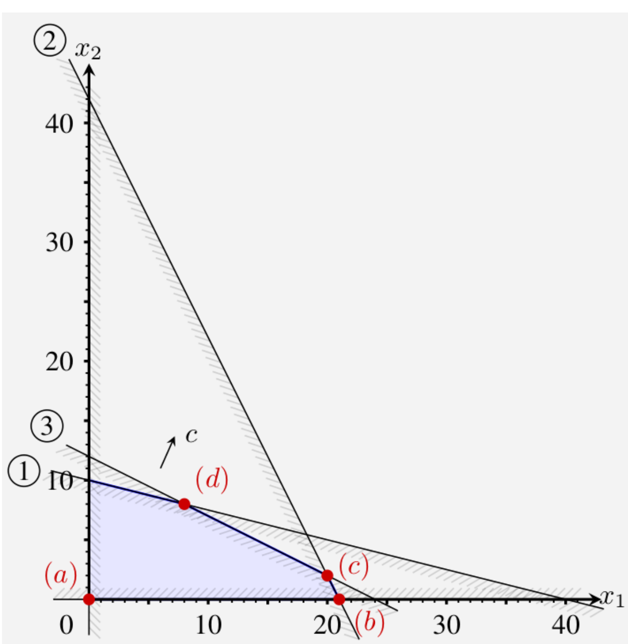
	y_2	x_2	1
z	200	-700	8400
y_1	$-\frac{1}{2}$	$\frac{7}{2}$	19
x_1	$\frac{1}{2}$	$\frac{1}{2}$	21
y_3	$-\frac{3}{4}$	$\boxed{\frac{9}{4}}$	$\frac{9}{2}$

(c)

	y_2	y_3	1
z	$-\frac{100}{3}$	$\frac{2800}{9}$	9800
y_1	$\boxed{\frac{2}{3}}$	$-\frac{14}{9}$	12
x_1	$\frac{2}{3}$	$-\frac{2}{9}$	20
x_2	$-\frac{1}{3}$	$\frac{4}{9}$	2

(d)

	y_1	y_3	1
z	50	$\frac{700}{3}$	10400
y_2	$\frac{3}{2}$	$-\frac{7}{3}$	18
x_1	-1	$\frac{4}{3}$	8
x_2	$\frac{1}{2}$	$-\frac{1}{3}$	8



The basic solution of (d) is optimal because:

$$z = 10400 - 50\overbrace{y_1}^{\geq 0} - \frac{700}{3}\overbrace{y_3}^{\geq 0} \leq 10400$$

Every feasible solution satisfies

$y_1, y_3 \geq 0$

How to choose the pivot element

basic terminology:

non-basic variables

	$x_N^\top$			1
z	c_1	$\ldots$	c_n	q
x_B	A_{11}	$\ldots$	A_{1n}	b_1
	$\vdots$		$\vdots$	$\vdots$
	A_{m1}	$\ldots$	A_{mn}	b_m

basic variables

• boundary of tableau

We want to choose pivot element s.t.

(i) Objective value does not decrease.

(ii) New tableau remains feasible.

Definition 1.66: Legal pivot element for phase II of Simplex Method

Given a feasible tableau, as shown in (1.18), a tuple $(i, k) \in [m] \times [n]$ corresponds to a legal pivot element A_{ik} for phase II of the Simplex Method if

(i) $c_k < 0$, and

(ii) $i \in \operatorname{argmin}\left\{\frac{b_j}{A_{jk}} : j \in [m] \text{ with } A_{jk} > 0\right\}$.

$$z = 0 + 400 \cdot \varepsilon$$

$x_2 = 0$

$x_1 = \varepsilon \geq 0$

$y_1 = 40 - 1 \cdot \varepsilon \geq 0$

$y_2 = 42 - 2 \cdot \varepsilon \geq 0$

$y_3 = 36 - 1.5\,\varepsilon \geq 0$

	x_1	x_2	1
z	-400	-900	0
y_1	1	4	40
y_2	$\boxed{2}$	1	42
y_3	1.5	3	36

$\varepsilon \leq \frac{40}{1} = 40$

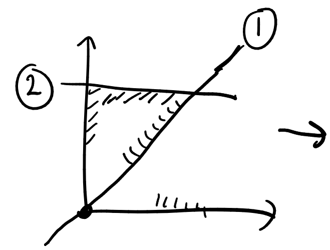
$\varepsilon \leq \frac{42}{2} = 21$

$\varepsilon \leq \frac{36}{1.5} = 24$

Theorem 1.67

Consider a feasible tableau with a legal pivot element A_{ik} for phase II of the Simplex Method. Then the new tableau obtained after pivoting on A_{ik} satisfies the following:

(i) The new tableau is feasible.
(ii) The value of the new tableau is no less than that of the original one.
(iii) If $b_i > 0$, the value of the new tableau is strictly larger than that of the original one.



Proof

(i) Need to check that rhs b' obtained after exchange step is ≥ 0.

pivot row i: $\quad b_i' \underset{\text{pivot rules}}{=} \dfrac{b_i}{A_{ik}} \geq 0$

$b_i \geq 0$ (feasibility of tableau)

$A_{ik} > 0$ (A_{ik} is legal pivot element)

non-pivot row $j \in [m] \setminus \{i\}$: $\quad \boxed{b_j' = b_j - \dfrac{A_{jk} b_i}{A_{ik}}}$

If $A_{jk} \leq 0 \implies b_j' \geq b_j \geq 0$

If $A_{jk} > 0 \implies \dfrac{b_j}{A_{jk}} \underset{\text{quotient rule}}{\geq} \dfrac{b_i}{A_{ik}} \implies b_j' \geq 0$

(ii) Because objective value of basic sol is part of rhs of tableau, the same pivot rules apply as for b.

$$\Longrightarrow q' = q - \frac{c_k b_i}{A_{ik}} \geq q$$

$c_k < 0$, $A_{ik} > 0$ } A_{ik} is legal pivot element

$b_i \geq 0$ } feasible tableau

(iii) If $b_i > 0$, then above reasoning implies $q' > q$.

#

What if there is no legal pivot element?

	$x_N^\top$			1
z	c_1	$\dots$	c_n	q
	A_{11}	$\dots$	A_{1n}	b_1
x_B	$\vdots$		$\vdots$	$\vdots$
	A_{m1}	$\dots$	A_{mn}	b_m

Theorem 1.68

Consider a feasible tableau (given as in (1.18)).

(i) If $c_k \geq 0 \ \forall k \in [n]$, then the basic solution corresponding to (1.18) is an *optimal solution* to the corresponding LP.

(ii) If $\exists k \in [n]$ with $c_k < 0$ and $A_{jk} \leq 0 \ \forall j \in [m]$, then the underlying LP is unbounded.

Moreover, a certificate of unboundedness can be obtained as follows. For $\lambda \in \mathbb{R}_{\geq 0}$, let x_B^λ, x_N^λ be the following assignments to the basic and non-basic variables, respectively:

$$\begin{aligned} x_B^\lambda &= b - \lambda A_{\cdot k} \\ (x_N^\lambda)_k &= \lambda \\ (x_N^\lambda)_\ell &= 0 \quad \forall \ell \in [n] \setminus \{k\} \ , \end{aligned}$$

where $A_{\cdot k}$ is the k-th column of A and $(x_N^\lambda)_k$ is the value of the non-basic variable corresponding to that column. Then, for any $\lambda \in \mathbb{R}_{\geq 0}$, $x^\lambda := \begin{pmatrix} x_B^\lambda \\ x_N^\lambda \end{pmatrix}$ is feasible with objective value going to ∞ as $\lambda \to \infty$.

	y_3	y_2	1
z	1	−1	3
y_1	1	0	3
x_2	2	−1	3
x_1	1	−1	1

By Theorem 1.68 (ii), an unboundedness certificate is

$$\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{with } \lambda \in \mathbb{R}_{\geq 0}$$

Proof

See script.

Remark

Theorem 1.68 (ii) may apply even though there is a legal pivot element (in a different column than column k).

		$x_N^\top$		1
z	c_1	$\dots$	c_n	q
	A_{11}	$\dots$	A_{1n}	b_1
x_B	$\vdots$		$\vdots$	$\vdots$
	A_{m1}	$\dots$	A_{mn}	b_m

Definition 1.69: Optimal and unbounded tableaus

A feasible tableau (as in 1.18) is called *optimal* if $c_k \geq 0 \; \forall k \in [n]$. It is called *unbounded* if $\exists k \in [n]$ with $c_k < 0$ and $A_{jk} \leq 0 \; \forall j \in [m]$.

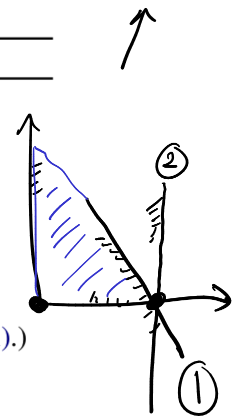
optimal tableau:

	y_3	y_1	1
z	$\frac{700}{3}$	50	10400
x_2	$-\frac{1}{3}$	$\frac{1}{2}$	8
y_2	$-\frac{7}{3}$	$\frac{6}{4}$	18
x_1	$\frac{4}{3}$	-1	8

unbounded tableau:

	y_3	y_2	
$\overline{z}$	2	-1	3
y_1	1	0	3
x_2	2	-1	3
x_1	1	-1	1

Description of phase II of Simplex Method



Algorithm 1: Phase II of Simplex Method

Input: Feasible tableau as shown in (1.18).

1. **Choice of pivot:**
   (a) *Choice of pivot column* (variable entering basis):
   If $c_k \geq 0\ \forall k \in [n]$, then **stop**. (The current basic solution is optimal due to Theorem 1.68 (i).) Otherwise, choose $k \in [n]$ with $c_k < 0$.
   (b) *Choice of pivot row* (variable leaving basis):
   If $A_{jk} \leq 0\ \forall j \in [m]$, then stop. (The problem is unbounded, see Theorem 1.68 (ii).) Otherwise, choose

$$i \in \operatorname{argmin}\left\{ \frac{b_j}{A_{jk}} : j \in [m] \text{ with } A_{jk} > 0 \right\} .$$

2. **Exchange step:**
   Perform an exchange step on pivot element A_{ik} and **go back** to step 1.

Example 1.71

LP in canonical form

$$\begin{array}{lrcrcr}
\max & -3x_1 & + & 2x_2 & & \\
 & x_1 & - & x_2 & \leq & 1 \\
 & -2x_1 & + & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & 2 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

LP in standard form

$$\begin{array}{llllllcrcrcl}
\max & z = & & & & - & 3x_1 & + & 2x_2 & & \\
 & & y_1 & & & + & x_1 & - & x_2 & = & 1 \\
 & & & y_2 & & - & 2x_1 & + & x_2 & = & 1 \\
 & & & & y_3 & - & x_1 & + & x_2 & = & 2 \\
 & & & & & & & & x & \in & \mathbb{R}^2_{\geq 0} \\
 & & & & & & & & y & \in & \mathbb{R}^3_{\geq 0}
\end{array}$$

tableau (a)

	x_1	x_2	1	quotients
z	3	-2	0	
y_1	1	-1	1	—
y_2	-2	$\boxed{1}$	1	1/1
y_3	-1	1	2	2/1

$\rightarrow$

tableau (b)

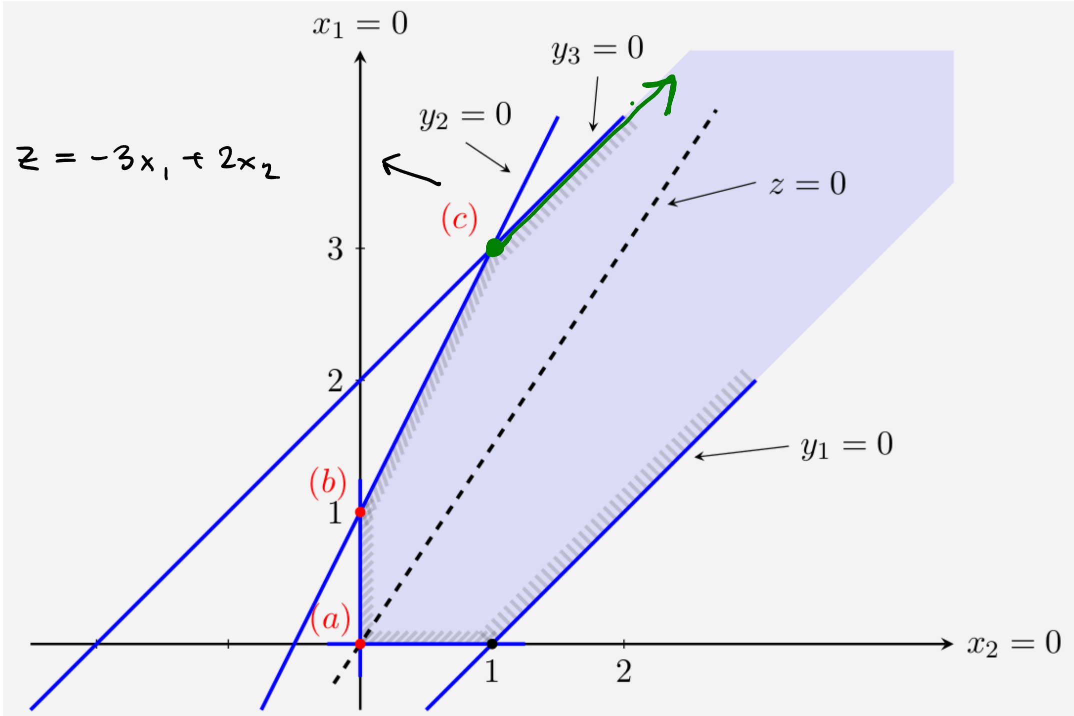
	x_1	y_2	1
z	-1	2	2
y_1	-1	1	2
x_2	-2	1	1
y_3	$\boxed{1}$	-1	1

$\rightarrow$

tableau (c) ← optimal tableau

	y_3	y_2	1
z	1	1	3
y_1	1	0	3
x_2	2	-1	3
x_1	1	-1	1

optimal solution: $(x_1, x_2, y_1, y_2, y_3) = (1, 3, 3, 0, 0)$



If we modify objective to $\max \bar{z} = x_2$ instead of $\max -3x_1 + 2x_2$, then final tableau is:

tableau (c')

	y_3	y_2	
$\bar{z}$	2	-1	3
y_1	1	0	3
x_2	2	-1	3
x_1	1	-1	1

unboundedness certificate:

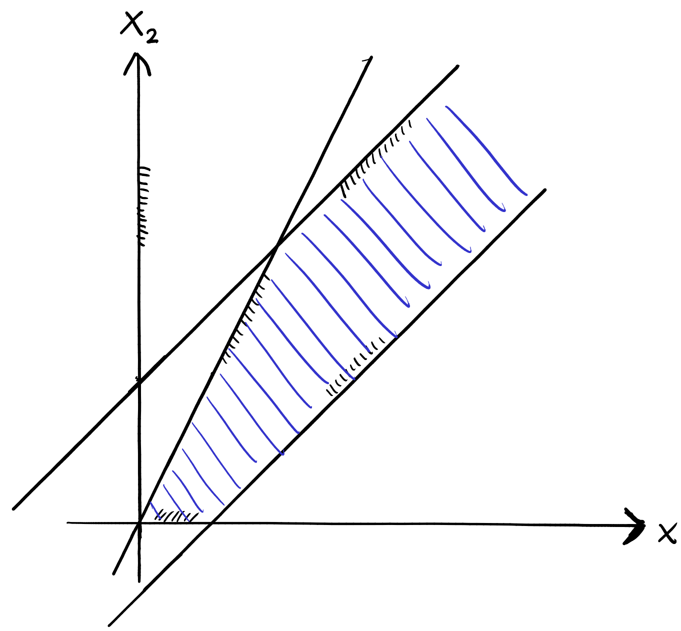
$$\begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Degeneracy

Definition 1.73

The basic solution to a tableau (given as in 1.18) is called *degenerate*, if there is an index $i \in [m]$ such that $b_i = 0$. In this case, the tableau is also called *degenerate*.

	x_1	x_2	
z	3	-2	0
x_3	1	-1	1
x_4	-2	1	0
x_5	-1	1	2



Proposition 1.74

A feasible tableau is degenerate if and only if its basic solution is a degenerate vertex of the feasible region of the problem (with respect to either the canonical ~~and~~ or standard form).

<u>Sketch of proof</u> $\to$ See script.

canonical form:

$$\boxed{\begin{array}{lrcl} \max & c^T x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array} \qquad \begin{array}{l} c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n} \\ b \in \mathbb{R}^m \end{array}}$$

standard form:

$$\boxed{\begin{array}{lrcl} \max & c^T x & & \\ & Ax + y & = & b \\ & x & \geq & 0 \\ & y & \geq & 0 \end{array}}$$

<u>Reasoning for canonical form</u>

Reasoning for standard form

Theorem 1.75: Finiteness of Simplex Method in non-degenerate case

If phase II of the Simplex Method never encounters a degenerate tableau, then it will stop after a finite number of steps. (This holds no matter how the pivot column and row is chosen if there are several legal options.)

<u>Proof</u> $\rightarrow$ see script.