ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

IFOR
Institute for Operations Research
Department of Mathematics

Exam

Mathematical Optimization

January 22, 2020

Surname:	
Name:	
Legi.-Nr.:	

General Remarks:

- ☺ Place your student identity card on the table.
- ☺ This is a closed book exam. No extra material is allowed.
- ☺ Use a pen to write down your solutions. Pencils and red pens are not allowed.
- ☺ Turn off all your technical devices and put them away.
- ☺ Write the answers for Problem 1 on the exam. For the remaining questions, use the paper provided. Use different sheets of paper for different problems and always indicate which problem you are solving. Do not use your own paper, we will hand out more if needed.
- ☺ You have 180 minutes to solve the exam. 15 minutes before the end of the exam, no premature submission is allowed anymore.
- ☺ For all problems, provide a complete solution in English, including all explanations and implications in a mathematically clear and well-structured way. Please hand in a readable and clean solution. Cross out any invalid solution attempts.
- ☺ Unless you are explicitly asked to prove them, you may use results from the lecture without proof (provided that you reference or state them clearly). Results from the problem sets may only be used if you prove them.
- ☺ Ask any questions that you might have immediately and during the exam.

Good luck!

Problem	1	2	3	4	5	Total
Points	/ 10	/ 10	/ 10	/ 10	/ 10	/ 50

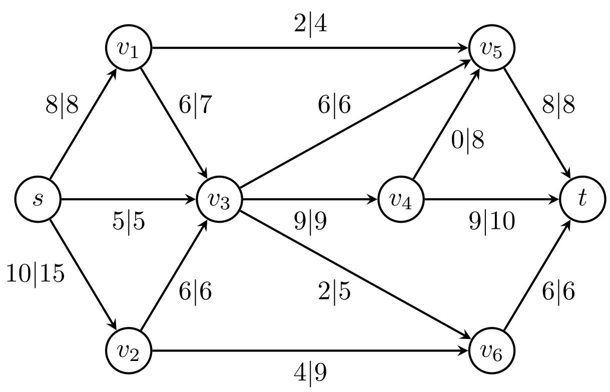
Problem 1: Short questions (4×2.5 points)

(a) Determine all legal pivot elements in the short tableau given below. For each of them, fill in a line of the given table by stating the variables leaving and entering the basis, respectively, as well as the change in the objective value if an exchange step on the corresponding pivot is performed.

	x_1	x_2	x_3	x_4	1
z	-1	0	-2	3	4
x_5	-2	3	-1	-1	-2
x_6	1	4	0	2	0
x_7	3	2	5	-1	3

Pivot element value	*Variable leaving the basis*	*Variable entering the basis*	*Change in objective value*

(b) In the directed graph $G = (V, A)$ below, arc capacities $u \colon A \to \mathbb{Z}_{\geq 0}$ and values of a maximal s-t flow $f \colon A \to \mathbb{R}_{\geq 0}$ are given in the form $f(a)|u(a)$ on every arc a. Provide a minimum s-t cut and prove that it is indeed a minimum s-t cut.



The following vertices form a minimum s-t cut:

Proof:

(c) Consider the integer linear program

$$
\begin{array}{lccccccccccccc}
\max & & & & & - & 2x_5 & - & x_6 & - & 3x_7 & & \\
& x_1 & & & & + & x_5 & - & \frac{5}{4}x_6 & + & \frac{1}{2}x_7 & = & \frac{19}{4} \\
& & x_2 & & & + & x_5 & - & x_6 & + & x_7 & = & 7 \\
& & & x_3 & & - & \frac{3}{2}x_5 & + & x_6 & - & \frac{3}{4}x_7 & = & \frac{7}{2} \\
& & & & x_4 & + & \frac{5}{4}x_5 & - & \frac{7}{2}x_6 & + & x_7 & = & \frac{21}{4} \\
& & & & & & & & & & x & \in & \mathbb{Z}^7_{\geq 0}\ .
\end{array}
$$

The point $x^* = \left(\frac{19}{4}, 7, \frac{7}{2}, \frac{21}{4}, 0, 0, 0\right)$ is an optimal solution of the linear relaxation of the above problem, where the constraint $x \in \mathbb{Z}^7_{\geq 0}$ is replaced by $x \in \mathbb{R}^7_{\geq 0}$.

Provide a cutting plane that cuts off x^* from the feasible region of the relaxation without cutting off any solution of the integer program. You do not have to indicate how you found the cutting plane, and you do not have to prove that it has the desired properties.

Cutting plane:

(d) Let $f, g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be two functions. Does $\log f = O(\log g)$ imply $f = O(g)$? Write your answer below and prove that it is correct.

Answer:

Proof:

Problem 2: Facets of polars and satisfying distance requirements $(4+6$ points)

(a) Let $n \in \mathbb{Z}_{>0}$ and let $P_1, P_2 \subseteq \mathbb{R}^n$ be two polytopes that both contain the origin in their interior. Prove that

$$\left|\text{facets}\left(\left(\text{conv}(P_1 \cup P_2)\right)^\circ\right)\right| \leq |\text{facets}(P_1^\circ)| + |\text{facets}(P_2^\circ)| \ .$$

Here, facets(P) denotes the set of all facets of a polytope P, and P° denotes the polar of P.

(b) Let $G = (V, A)$ be a directed graph, and let $s \in V$. For a given $r: V \setminus \{s\} \to \mathbb{R}_{\geq 0}$, we would like to define arc lengths $\ell: A \to \mathbb{R}_{\geq 0}$ such that the following statement is true for all $v \in V \setminus \{s\}$:

The shortest s-v path in G with respect to ℓ has length at least $r(v)$.

Among all such ℓ, we want to find one such that the total arc length (i.e., the sum of all arc lengths in the graph with respect to ℓ) is minimized. Provide a polynomial-time algorithm that solves this problem. Prove that your algorithm is correct and argue that its running time is polynomial in the input size.

Hint: You can use without proof that linear programs can be solved in running time polynomial in the input size of the LP.

Problem 3: Polyhedral description of linearly independent subsets $(2+3+5$ points)

Let $n \in \mathbb{Z}_{>0}$ and let $N \subseteq \mathbb{R}^n$ be a finite set of vectors. Define

$$\mathcal{I} := \{S \subseteq N \colon \text{the vectors in } S \text{ are linearly independent}\} \ ,$$

and let $r: 2^N \to \mathbb{Z}$ be given by $r(S) = \dim(\text{span}(S))$ for all $S \subseteq N$. It is known that this function has the property that

$$r(S_1) + r(S_2) \geq r(S_1 \cap S_2) + r(S_1 \cup S_2) \quad \text{for all } S_1, S_2 \subseteq N \ ,$$

and you can use this inequality without proving it. Define the polyhedron

$$P := \left\{x \in \mathbb{R}^N_{\geq 0} \colon x(S) \leq r(S) \text{ for all } S \subseteq N\right\} \ ,$$

and let $y \in \mathbb{R}^N$ be a vertex of P and assume that $\text{supp}(y) = N$.

(a) Prove that $P \cap \{0,1\}^N = \left\{\chi^S \colon S \in \mathcal{I}\right\}$.

(b) A set $S \subseteq N$ is called *y-tight* if $y(S) = r(S)$. Prove that if $S_1, S_2 \subseteq N$ are both y-tight, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are y-tight, too.

(c) Prove that there exists a laminar family $\mathcal{L} \subseteq 2^N$ with the following two properties:

(i) $y(L) = r(L)$ for all $L \in \mathcal{L}$.

(ii) The equation system

$$x(L) = r(L) \quad \text{for all } L \in \mathcal{L}$$

has rank $|N|$.

Hint: You can use the following two statements without proving them.

(1) Let $\mathcal{F}$ be a laminar family on a finite ground set G. For any set $S \subseteq G$, define

$$\mathcal{F}_S = \{F \in \mathcal{F} \colon F \text{ and } S \text{ are intersecting}\} \ ,$$

where we recall that two sets $A, B \subseteq N$ are intersecting if $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$, and $B \setminus A \neq \emptyset$. Let $S \subseteq G$ be such that $\mathcal{F}_S \neq \emptyset$, and let $F \in \mathcal{F}_S$. Then

$$|\mathcal{F}_{S \cup F}| < |\mathcal{F}_S| \qquad \text{and} \qquad |\mathcal{F}_{S \cap F}| < |\mathcal{F}_S| \ .$$

(2) Let $Ax = b$ and $Cx = d$ be linear systems with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$, and $d \in \mathbb{R}^k$. If the two systems have a common solution and the rows of A are linear combinations of the rows of C, then $Ax = b$ is implied by $Cx = d$.

Problem 4: r-arborescences and separation $(2 + 8$ points)

(a) Let $G = (V, A)$ be a directed graph, and let $r \in V$. Provide a linear inequality description of the dominant of the r-arborescence polytope. You do not have to prove that it is correct.

(b) Provide a strongly polynomial-time separation oracle for the dominant of the r-arborescence polytope. Prove that it is correct and argue that its running time is indeed strongly polynomial.

Problem 5: Two equivalent polyhedral statements (10 points)
Let $m, n \in \mathbb{Z}_{>0}$, let $A \in \mathbb{R}^{m\times n}$, and let $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n \colon Ax \le b\}$ is a non-empty polytope. For $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, prove that the following two statements are equivalent:

(i) $P \subseteq \{x \in \mathbb{R}^n \colon a^\top x \le \beta\}$.

(ii) There exist $\lambda \in \mathbb{R}^m_{\ge 0}$ and $\Delta \in \mathbb{R}_{\ge 0}$ such that

$$a = A^\top \lambda \qquad \text{and} \qquad \beta = b^\top \lambda + \Delta \ .$$