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Mathematical Optimization – Problem set 13

`https://moodle-app2.let.ethz.ch/course/view.php?id=4844`

Problem 1: A separation oracle for the perfect matching polytope
We have seen that the perfect matching polytope P of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V,\, |S| \text{ odd} \end{array} \right\} .$$

In order to optimize over P using the ellipsoid method, we need a separation oracle for P, but separating over the odd-cut constraints $x(\delta(S)) \geq 1$ for odd subsets $S \subseteq V$ is a non-trivial task. A closely related problem is the minimum T-odd cut problem: Given a graph $G = (V, E)$, edge weights $w\colon E \to \mathbb{R}_{\geq 0}$ and a non-empty set $T \subseteq V$ of even cardinality, this problem is to solve

$$\min\{w(\delta(S))\colon S \subsetneq V,\, |S \cap T| \text{ odd}\} \ . \tag{1}$$

A cut $S \subseteq V$ such that $|S \cap T|$ is odd is also called a T-odd cut.

(a) Assume that we can efficiently find a solution to the minimum T-odd cut problem as given in (1). Show that separation over P can be done efficiently.

For solving the minimum T-odd cut problem, we propose the following algorithm.

Algorithm 1. An algorithm for the odd cut problem.

Input: Graph $G = (V, E)$, $w\colon E \to \mathbb{R}_{\geq 0}$, non-empty $T \subseteq V$ with $|T|$ even.
Output: Solution to the minimum T-odd cut problem on (G, w).

1. For all $\{s, t\} \subseteq T$ with $s \neq t$, let
$$C_{\{s,t\}} \in \arg\min\{w(\delta(C))\colon C \subsetneq V,\, |C \cap \{s, t\}| = 1\} \ ,$$
and let $C \in \arg\min\{w(\delta(C_{\{s,t\}}))\colon \{s, t\} \subseteq T \text{ with } s \neq t\}$.

2. If $|C \cap T|$ is odd, return C. Else, return a cut in
$$\arg\min\left\{y(\delta(Q))\colon Q \in \{\text{ALG}({}^{G}/_{C}, w|_{E\setminus E[C]}, T \setminus C), \text{ALG}({}^{G}/_{\overline{C}}, w|_{E\setminus E[\overline{C}]}, T \setminus \overline{C})\}\right\} \ .$$

Here, we denote $\overline{S} \coloneqq V \setminus S$. Also recall that for a set $S \subseteq V$, ${}^{G}/_{S}$ denotes the graph G with the vertex set S contracted. Moreover, we denote the output of Algorithm 1 on input (G, w, T) by $\text{ALG}(G, w, T)$.

(b) Prove that Algorithm 1 is correct, i.e., show that $\text{ALG}(G, w, T)$ is an optimal solution of (1).

(c) Show that there is an implementation of Algorithm 1 with strongly polynomial running time.

Problem 2: Minimum-volume ellipsoid containing rotated half-ball
Recall that an ellipsoid in $\mathbb{R}^n$ is a set of the form

$$E(a, A) = \{x \in \mathbb{R}^n\colon (x - a)^\top A^{-1}(x - a) \leq 1\} \ ,$$

where $a \in \mathbb{R}^n$ is the center of the ellipsoid and $A \succ 0$ is a positive definite matrix in $\mathbb{R}^{n\times n}$. In particular, $E(0, I)$ is the unit ball. In this problem, we would like to find the minimum-volume ellipsoid containing the half-ball

$$R(c) = \{x \in E(0, I)\colon c^\top x \geq 0\} \ ,$$

where $c \in \mathbb{R}^n$ is a vector such that $\|c\|_2 = 1$.

Prove that for $n \geq 2$, the minimum-volume ellipsoid $E(a, A)$ containing $R(c)$ is defined by

$$a = \frac{1}{n+1} c \quad \text{and} \quad A = \frac{n^2}{n^2 - 1} \left(I_n - \frac{2}{n+1} c c^\top \right) .$$

Hint: Use the results of Problem 4 from Problem set 12, namely that the minimum-volume ellipsoid $E(\overline{a}, \overline{A})$ *containing the half-ball*

$$R(e_1) = \{x \in E(0, I) \colon x_1 \geq 0\}$$

is defined by

$$\overline{a} = \begin{pmatrix} \frac{1}{n+1} & 0 & 0 & \cdots & 0 \end{pmatrix}^\top \quad \textit{and} \quad \overline{A} = \begin{pmatrix} \left(\frac{n}{n+1}\right)^2 & 0 & \cdots & 0 \\ 0 & \frac{n^2}{n^2-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n^2}{n^2-1} \end{pmatrix} .$$

Problem 3: Volume of the standard simplex

For $i \in \{1, \ldots, n\}$, let $e_i \in \mathbb{R}^n$ denote the i^{th} unit vector in $\mathbb{R}^n$. The goal of this problem is to compute the volume of the standard simplex $\Delta \coloneqq \operatorname{conv}(\{0, e_1, e_2, \ldots, e_n\})$. To this end, define

$$\Delta(\sigma) \coloneqq \{x \in \mathbb{R}^n \colon 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)} \leq 1\}$$

for every permutation $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, and let $B = [0, 1]^n$ be the n-dimensional hypercube.

(a) Prove that $\Delta(\sigma)$ is a simplex for every permutation σ.

(b) Prove that for every $x \in B$, there exists a permutation σ with $x \in \Delta(\sigma)$.

(c) Prove that $\operatorname{vol}(\Delta(\sigma_1) \cap \Delta(\sigma_2)) = 0$ for any two distinct permutations σ_1, σ_2.

(d) Prove that $\operatorname{vol}(\Delta) = \operatorname{vol}(\Delta(\sigma))$ for every permutation σ.

(e) Combine the previous steps to compute $\operatorname{vol}(\Delta)$.