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Mathematical Optimization – Problem set 5

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Linear programm with unique optimal solution

For a given LP, suppose that there exists a feasible short tableau of the form

$$
\begin{array}{c|ccc|c}
 & x_1 & \dots & x_n & 1 \\
\hline
z & c_1 & \dots & c_n & q \\
\hline
x_{n+1} & A_{11} & \dots & A_{1n} & b_1 \\
\vdots & \vdots & & \vdots & \vdots \\
x_{n+m} & A_{m1} & \dots & A_{mn} & b_m
\end{array}
$$

such that $c_i > 0$ for every $i \in \{1, \dots, n\}$, i.e., all tableau entries that are objective function coefficients (excluding the current basic solution value q) are strictly positive. Prove that in this case, the linear program has a unique optimal solution.

Problem 2: Certifying infeasibility from phase I of the Simplex Method

When solving a linear program of the form

$$
\begin{array}{rrcl}
\max & c^\top x & & \\
 & Ax & \leq & b \\
 & x & \in & \mathbb{R}^n_{\geq 0}
\end{array}
\tag{1}
$$

using the Simplex Method, we start with phase I, where the goal is to decide feasibility of the above linear program, and—if feasible—find a feasible vertex solution to start phase II from. To this end, we solve the auxiliary problem

$$
\begin{array}{rrcrcl}
\max & -x_0 & & & & \\
 & -1 \cdot x_0 & + & Ax & \leq & b \\
 & x_0 & & & \in & \mathbb{R}_{\geq 0} \\
 & & & x & \in & \mathbb{R}^n_{\geq 0}
\end{array}
, \tag{2}
$$

and we know that if and only if the optimal value of the latter auxiliary problem is 0, then the initial linear program is feasible, and we can continue with phase II. The goal of this problem is to show that if the value of the auxiliary problem is strictly negative, then we can obtain a certificate of infeasibility from the corresponding optimal simplex tableau.

(a) Show that a vector $\mu \in \mathbb{R}^m_{\geq 0}$ with $\mu^\top A \geq 0$ and $\mu^\top b < 0$ is a certificate of infeasibility of the linear program (1). In other words, prove that if such a vector μ exists, then (1) is infeasible.

(b) Consider the case where the auxiliary problem (2) has a strictly negative optimal value. What can you say about an optimal short tableau of the corresponding standard form problem?

(c) Use properties of the optimal tableau considered in (b) to find a certificate vector $\mu \in \mathbb{R}^m_{\geq 0}$ as discussed in (a).

Hint: Exploit that the equation corresponding to the objective row of the optimal tableau is a linear combination of the equation corresponding to the objective row of the initial tableau and constraints of the standard form problem corresponding to (2).

Problem 3: The dual of a general linear program
In class, the dual problem of an linear program in canonical form was defined. The goal of this exercise is to derive the dual of a linear program given in the following most general form.

$$
\begin{array}{rrcrcrcl}
\max & d^\top x & + & e^\top y & + & f^\top z & & \\
& Ax & + & By & + & Cz & \leq & a \\
& Dx & + & Ey & + & Fz & = & b \\
& Gx & + & Hy & + & Kz & \geq & c \\
& & & & & x & \in & \mathbb{R}^{n_d}_{\geq 0} \\
& & & & & y & \in & \mathbb{R}^{n_e} \\
& & & & & z & \in & \mathbb{R}^{n_f}_{\leq 0}
\end{array}
\tag{P}
$$

(a) Transform the linear program (P) into an linear program in canonical form and write down the corresponding dual program.

(b) We say that two linear programs are equivalent if every feasible solution of one of the problems can be transformed into a feasible solution of the other problem with the same objective value.

Prove that the dual linear program from (a) is equivalent to the following linear program (D).

$$
\begin{array}{rrcrcrcl}
\min & a^\top u & + & b^\top v & + & c^\top w & & \\
& A^\top u & + & D^\top v & + & G^\top w & \geq & d \\
& B^\top u & + & E^\top v & + & H^\top w & = & e \\
& C^\top u & + & F^\top v & + & K^\top w & \leq & f \\
& & & & & u & \in & \mathbb{R}^{m_a}_{\geq 0} \\
& & & & & v & \in & \mathbb{R}^{m_b} \\
& & & & & w & \in & \mathbb{R}^{m_c}_{\leq 0}
\end{array}
\tag{D}
$$

Problem 4: Primal and dual pivot
Consider the following linear program in canonical form.

$$
\begin{array}{rrcl}
\max & c^\top x & & \\
& Ax & \leq & b \\
& x & \in & \mathbb{R}^n_{\geq 0}
\end{array}
\tag{P_1}
$$

In class, it was shown that performing an exchange step on a non-zero pivot in the tableau corresponding to (P_1) leads to a new tableau corresponding to a linear program equivalent to (P_1). In this problem, we show that the two dual problems corresponding to the primal tableaus before and after the pivot step are equivalent, as well. In other words, we show that the following diagram commutes:

$$
\begin{tikzcd}
P_1 \arrow[r, "\text{simplex pivot}"] \arrow[d, "\text{dualization}"'] & P_2 \arrow[d, "\text{dualization}"] \\
D_1 \arrow[r, "\text{simplex pivot}"'] & D_2
\end{tikzcd}
$$

Recall that we have seen in the lecture that pivoting steps in simplex tableaus transform the underlying primal system into an equivalent system of equations. Commutativity of the above diagram implies that this is also true for the systems that we get from the dual reading of the tableaus.

In the following, assume that the entry A_{ij} in the i^{th} row and j^{th} column of A is non-zero.

(a) Write down the short tableau corresponding to (P_1) and perform a pivot step on A_{ij}. Write down the new tableau, i.e., calculate its entries in terms of the entries of the old tableau.

(b) Formulate the dual of the problem that corresponds to the tableau you got in (a), transform it to canonical form, and write down the short tableau for this problem.

(c) Now first dualize (P_1), transform it to canonical form, and write down the short tableau for this problem.

(d) In the tableau obtained in (c), perform a pivot step on the entry in the j^{th} row and i^{th} column. Write down the new tableau, i.e., calculate its entries in terms of the entries of the old tableau.

(e) Observe that in (b) and (d), you obtained the same tableaus, and conclude that indeed, the diagram given above commutes.

Problem 5: Strong LP duality and Farkas' Lemma
In this problem, we show that strong linear programming duality as seen in class is equivalent to the following important result known as Farkas' Lemma.

Theorem (Farkas' Lemma). *Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$. Precisely one of the following is true.*

(i) *There exists $x \in \mathbb{R}^n$ such that $Ax \leq 0$ and $c^\top x > 0$.*

(ii) *There exists $y \in \mathbb{R}^m$ such that $A^\top y = c$ and $y \geq 0$.*

We will prove the equivalence by considering certain primal/dual linear program pairs. For the sake of simplicity, we will use the pair

$$\begin{array}{lrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \in & \mathbb{R}^n \end{array} \qquad (P') \qquad\qquad \begin{array}{lrcl} \min & b^\top y & & \\ & A^\top y & = & c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad , \qquad (D')$$

and not the canonical form primal/dual linear program pair

$$\begin{array}{lrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \in & \mathbb{R}^n_{\geq 0} \end{array} \qquad (P) \qquad\qquad \begin{array}{lrcl} \min & b^\top y & & \\ & A^\top y & \geq & c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad . \qquad (D)$$

(a) Show that having strong LP duality for primal/dual linear program pairs of the form (P') and (D') is equivalent to having strong LP duality for primal/dual linear program pairs of the form (P) and (D).

(b) Prove Farkas' Lemma using strong linear programming duality.

(c) Conversely, show that strong linear programming duality follows from Farkas' Lemma.

Hint: Given an optimal solution x^* of (P'), use Farkas' Lemma to show that there exists a feasible solution y^* of (D') which has a 0 in every entry corresponding to a constraint in (P') that is not tight at x^*.

Programming exercise
Work through the notebook `05_separationForPolyhedra.tex` on finding separating hyperplanes for two polyhedra using linear programming methods.