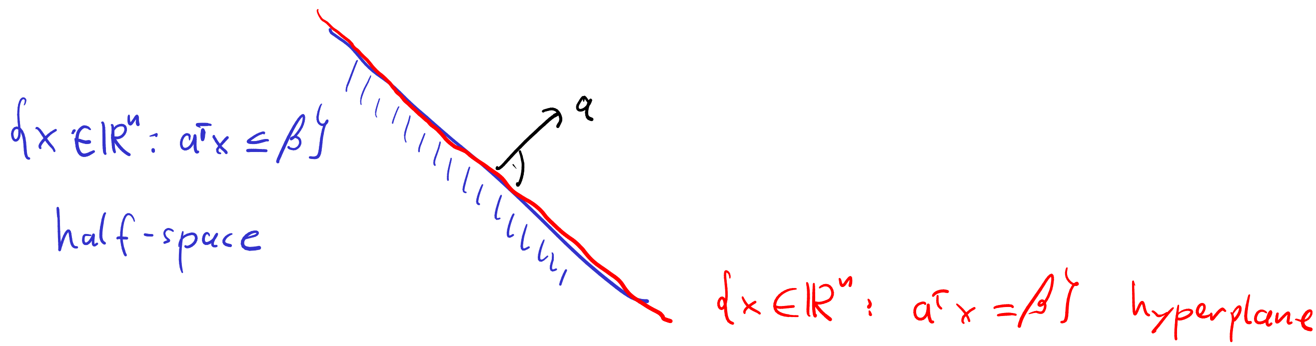
1.2 Polyhedra and basic convex geometry

1.2.1 Basic notions

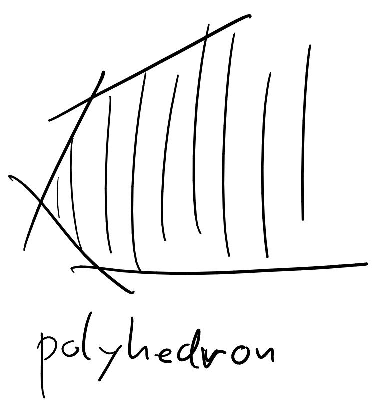
Definition 1.4: Half-space & hyperplane

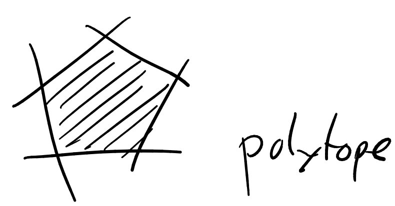
A *half-space* in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^\top x \leq \beta\}$ for $a \in \mathbb{R}^n \setminus \{0\}$ and $\beta \in \mathbb{R}$. Moreover, $\{x \in \mathbb{R}^n : a^\top x = \beta\}$ is called a *hyperplane*.



Definition 1.5: Polyhedron & polytope

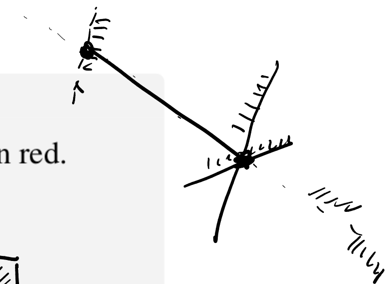
A *polyhedron* $P \subseteq \mathbb{R}^n$ is a finite intersection of half-spaces. Moreover, a bounded polyhedron is called a *polytope*.





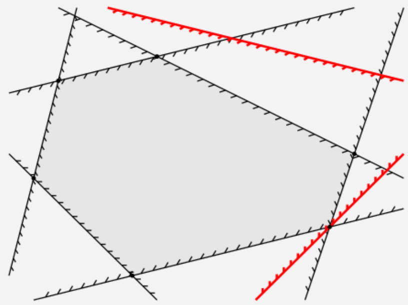
Definition 1.6: Redundancy

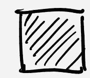
A linear inequality or equality of an inequality description of a polyhedron is called *redundant* if removing it from the description does not change the polyhedron.



Example 1.7: Redundant constraints

The picture below shows a polytope with two redundant constraints, highlighted in red.





$0 \leq x_1 \leq 1$

$0 \leq x_2 \leq 1$

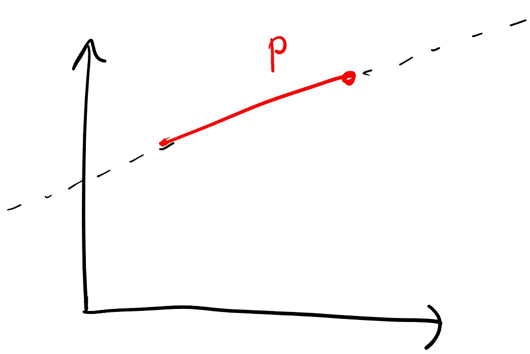
$2x_2 \leq 2$

Definition 1.8: Dimension of a polyhedron

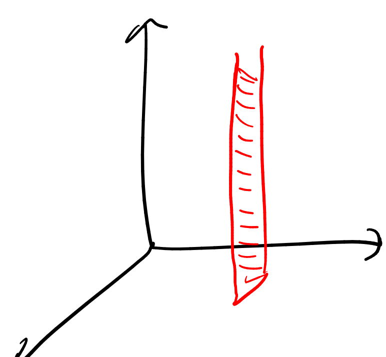
The dimension $\dim(P)$ of a polyhedron $P \subseteq \mathbb{R}^n$ is the dimension of a smallest-dimensional affine subspace containing P, i.e.,

$$\dim(P) := \min\{k \in \mathbb{Z}_{\geq 0} \colon \exists A \in \mathbb{R}^{n \times n} \text{ with } \operatorname{rank}(A) = n - k \;\&\; Ax = Ay \;\forall x, y \in P\} \;.$$

In particular, P is called *full-dimensional* if $\dim(P) = n$.



P is 1-dimensional polytope in 2-dimensional space

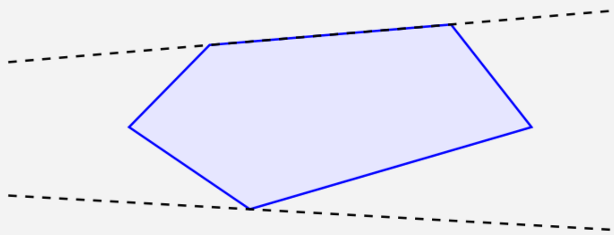


2-dimensional polyhedron in 3 dimensions

> **Definition 1.9: Supporting hyperplane**
>
> Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A hyperplane $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ is called P-*supporting*—or simply *supporting*, if P is clear from context—if $P \cap H \neq \emptyset$ and P is contained in one of the two half-spaces defined by H, i.e., either $P \subseteq \{x \in \mathbb{R}^n : a^\top x \leq \beta\}$ or $P \subseteq \{x \in \mathbb{R}^n : a^\top x \geq \beta\}$.

Example 1.10

The figure below shows a 2-dimensional polytope with two supporting hyperplanes.

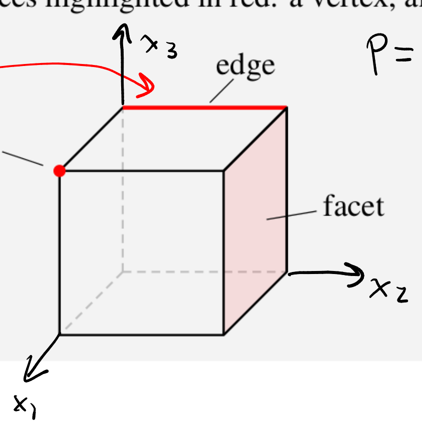


> **Definition 1.11: Face, vertex, edge, and facet**
>
> Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron.
>
> (i) A *face* of P is either P itself or the intersection of P with a supporting hyperplane.
> (ii) A *vertex* of P is a 0-dimensional face of P.
> (iii) An *edge* of P is a 1-dimensional face of P.
> (iv) A *facet* of P is a $(\dim(P) - 1)$-dimensional face of P.
>
> The empty polyhedron has only one face, which is the empty set. We denote by $\mathrm{vertices}(P)$ the set of all vertices of P.

Example 1.12

Below is a cube with three of its faces highlighted in red: a vertex, an edge, and a facet.

$\{x \in P : x_1 = 0, x_3 = 1\}$



$$P = \left\{ x \in \mathbb{R}^3 : \begin{array}{l} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \\ 0 \leq x_3 \leq 1 \end{array} \right\}$$

Proposition 1.13

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq \mathbb{R}^n$ be a non-empty polyhedron with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and let $F \subseteq P$. Then the following statements are equivalent.

(i) F is a face of P.
(ii) $\exists c \in \mathbb{R}^n$ such that $\delta := \max\{c^\top x : x \in P\}$ is finite and $F = \{x \in P : c^\top x = \delta\}$.
(iii) $F = \{x \in P : \overline{A}x = \overline{b}\} \neq \emptyset$ for a subsystem $\overline{A}x \leq \overline{b}$ of $Ax \leq b$.

Proof

We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

F face of P $\Rightarrow$ $F = P \cap \{x \in \mathbb{R}^n : a^T x = \beta\}$, where $\{x \in \mathbb{R}^n : a^T x = \beta\}$ is supporting hyperplane

$\rightarrow$ $P \subseteq \{x \in \mathbb{R}^n : a^T x \leq \beta\}$

$\rightarrow$ $\max\{a^T x : x \in P\} = \beta$ and F are its maximizers.

(ii) $\Rightarrow$ (iii)

We need property that $\delta := \max\{c^T x : x \in P\}$ being finite implies $\exists\ \bar{x} \in P$ with $c^T \bar{x} = \delta$.

$\Rightarrow$ $F := \{x \in P : c^T x = \delta\}$ is non-empty.

See problem sets

$$\underbrace{\begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix}}_{A} x \le \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

Choose numbering such that, for some $k \in \{0, \dots, m\}$, we have

$$a_i^T x = b_i \quad \forall x \in F \quad \forall i \in [k],$$

and for all other constraints, i.e., with row indices $i \in \{k+1, \dots, m\}$, this does not hold.

will show $\rightsquigarrow$ $\boxed{F = \{x \in P : a_i^T x = b_i \;\; \forall i \in [k]\}}$ $\Rightarrow$ (iii)

Claim

$\exists\, z \in F$ s.t. $a_i^T z < b_i \quad \forall i \in \{k+1, \dots, m\}$

Proof of claim

If $k = m$: $z = \bar{x}$

Otherwise: For each $i \in \{k+1, \dots, m\}$ $\exists\, z_i \in F$ with $a_i^T z_i < b_i$.

Choose $z = \frac{1}{m-k} \sum_{i=k+1}^{m} z_i$

$$c^T z = \frac{1}{m-k} \sum_{i=k+1}^{m} \underbrace{c^T z_i}_{= \delta} = \delta \qquad a_i^T z = \frac{1}{m-k} \sum_{j=k+1}^{m} \overbrace{a_i^T z_j}^{\le b_i} \le b_i \;\; \left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow z \in F$$

$\forall i \in [m]$

Moreover, if $i \in \{k+1, \dots, m\} \Rightarrow a_i^T z_i < b_i$.

$\Rightarrow a_i^T z < b_i$. $\#_{\text{claim}}$