see script

Corollary 4.20: Correctness of Algorithm 4

Algorithm 4 correctly determines the maximum number of arc-disjoint s-t paths. Furthermore, by Theorem 4.19, one can find a maximum set of arc-disjoint s-t paths in linear time once an integral maximum s-t flow is found.

Theorem 4.21: Menger's Theorem

Let $G = (V, A)$ be a directed graph with $s, t \in V, s \neq t$. Then the maximum number of arc-disjoint s-t paths is equal to the number of arcs in a minimum cardinality s-t cut.

Proof

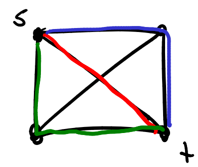
Immediate consequence of Corollary 4.20 and strong max-flow min-cut theorem.

#

Undirected connectivity

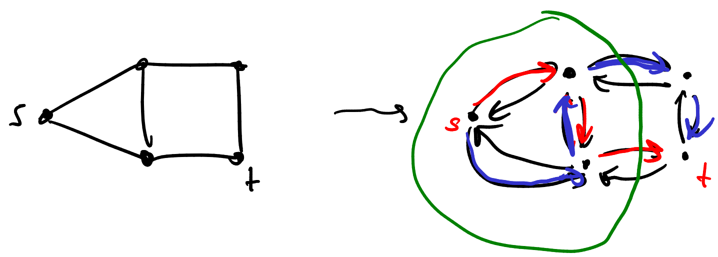
Definition 4.22: k-edge-connectivity in undirected graphs

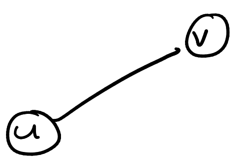
An undirected graph $G = (V, E)$ is *k-edge-connected* if for any two vertices $s, t \in V$ with $s \neq t$ there exist at least k edge-disjoint s-t paths in G.



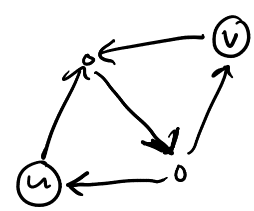
Exercise 4.23: Checking k-edge-connectivity in undirected graphs

Reduce the problem of checking k-edge-connectivity in undirected graphs to the problem of checking k-arc-connectivity in directed graphs.





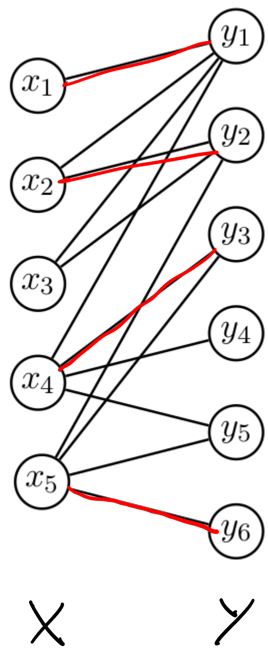
replace each edge by following gadget



4.4.2 Maximum cardinality bipartite matchings

Definition 4.24: Bipartite graph

An undirected graph $G = (V, E)$ is called *bipartite* if there is a bipartition of its vertices $V = X \dot\cup Y$ such that each edge has one endpoint in X and the other in Y.



Definition 4.26: Matching

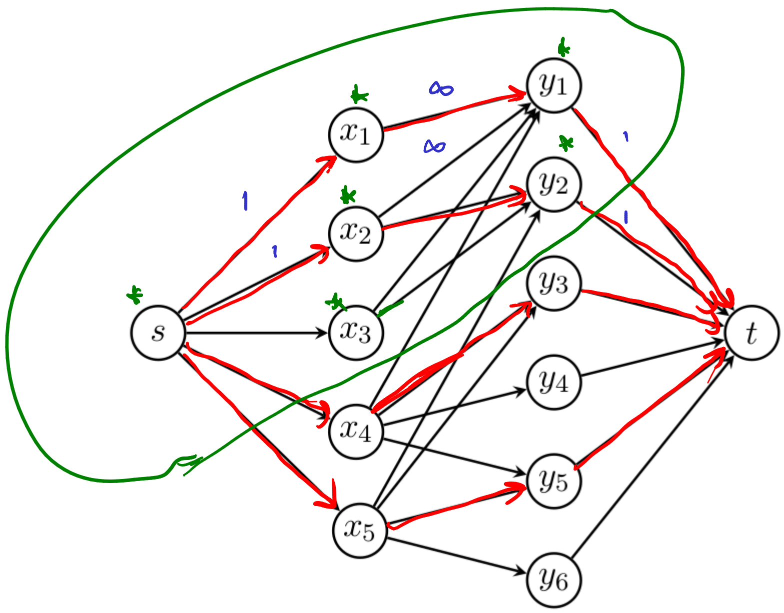
Let $G = (V, E)$ be an undirected graph. A set $M \subseteq E$ is a *matching* in G if M does not contain loops and no two edges of M share a common endpoint.

Maximum cardinality matching problem in bipartite graphs

Input: An undirected bipartite graph $G = (V, E)$.

Task: Find a maximum cardinality matching $M \subseteq E$ in G.

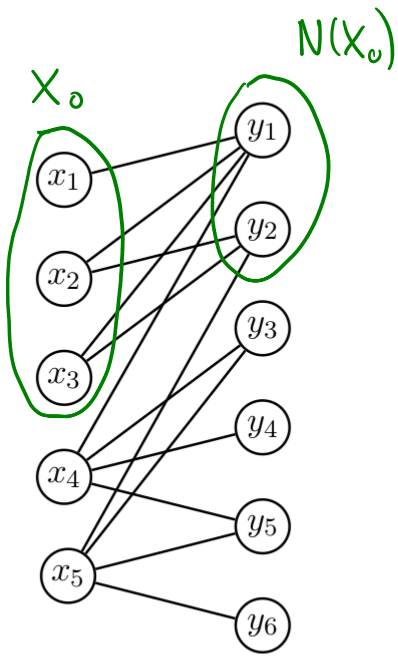
Determining a maximum cardinality bipartite matching via flows

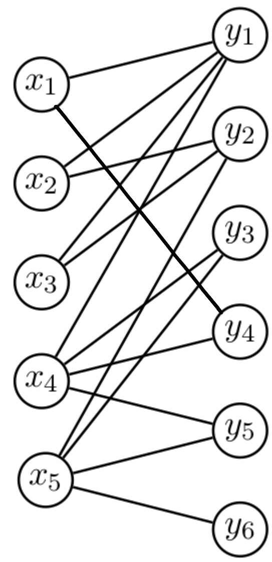


Theorem 4.29: Hall's Theorem

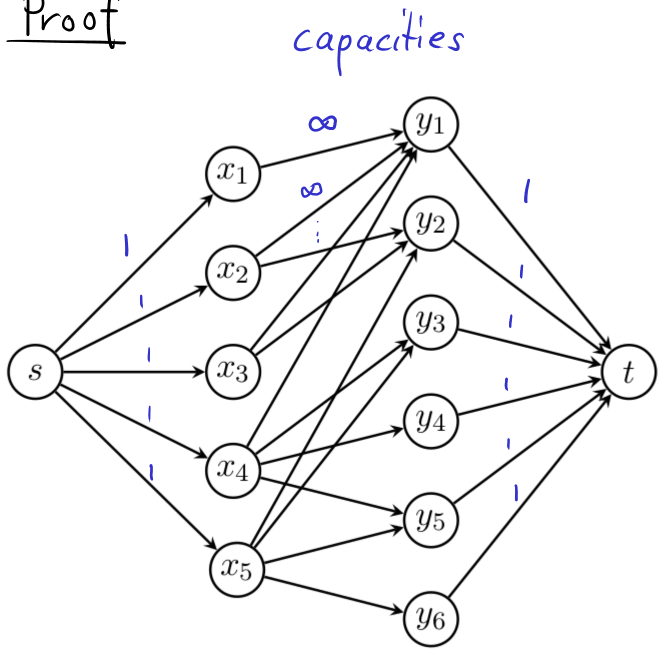
Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \,\dot\cup\, Y$. Then there exists a matching $M \subseteq E$ in G that touches all vertices in X if and only if

$$|N(X_0)| \geq |X_0| \qquad \text{for all } X_0 \subseteq X \ .$$





Proof



$\exists$ matching M touching all vertices in X.

$$\Longleftrightarrow$$

Value of max s-t flow in graph to the left $= |X|$

$$\Longleftrightarrow$$

strong max-flow min-cut theorem

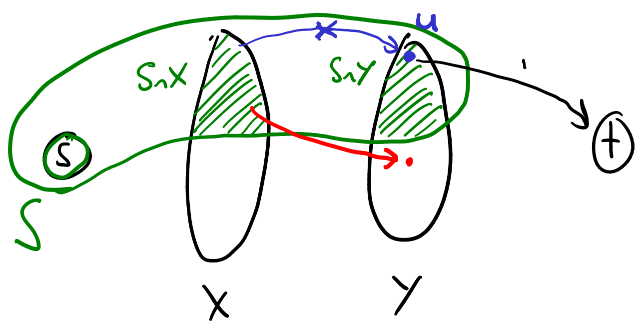
Value of min s-t cut $= |X|$.

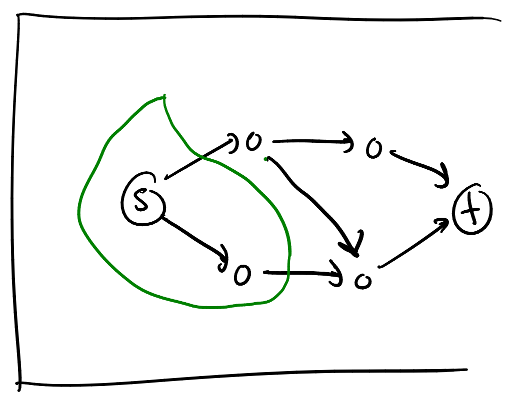
$$\Longleftrightarrow$$

$$u(\delta^+(S)) \geq |X| \quad \forall \text{ s-t cut } S \subseteq V$$

Claim

Any minimum s-t cut $S \subseteq V$ satisfies: $N(S \cap X) = S \cap Y$





Proof of claim

- $N(S \cap X) \subseteq S \cap Y$: For otherwise, there is an ∞-arc in $\delta^+(S)$.
- $N(S \cap X) \supseteq S \cap Y$: For otherwise, $\exists\, u \in (S \cap Y) \setminus N(S \cap X)$ and

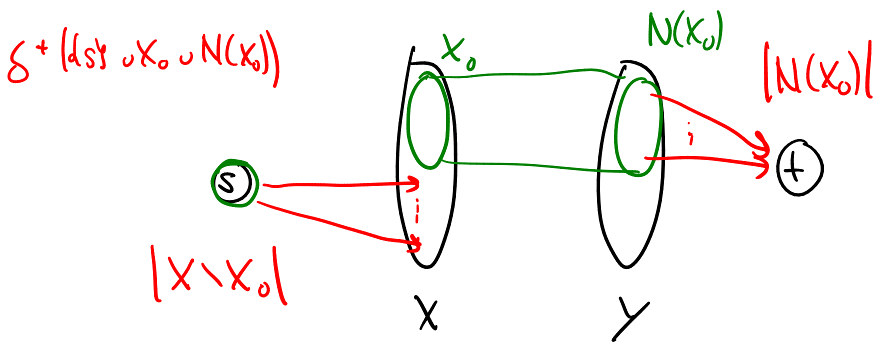
$$u(\delta^+(S \setminus \{u\})) = u(\delta^+(S)) - 1 \Rightarrow S \text{ is not a min s-t cut}$$

$\#_{\text{claim}}$

Hence :

$$u(\delta^+(S)) \geq |X| \quad \forall \overset{\text{minimum}}{\vee}\ s\text{-}t \text{ cut } S \subseteq V$$

$$\Longleftrightarrow$$



$$\underbrace{u(\delta^+(\{s\} \cup X_0 \cup N(X_0)))}_{= |X \setminus X_0| + |N(X_0)|} \geq |X| \quad \forall \ X_0 \subseteq X$$

$$\Longleftrightarrow$$

$$|N(X_0)| \geq |X_0| \quad \forall \ X_0 \subseteq X$$

#

4.3.3 Multiple sources and sinks

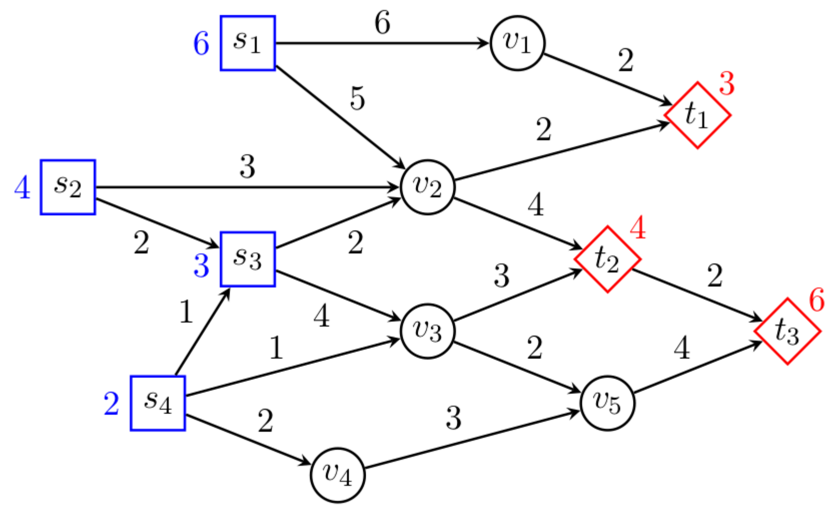
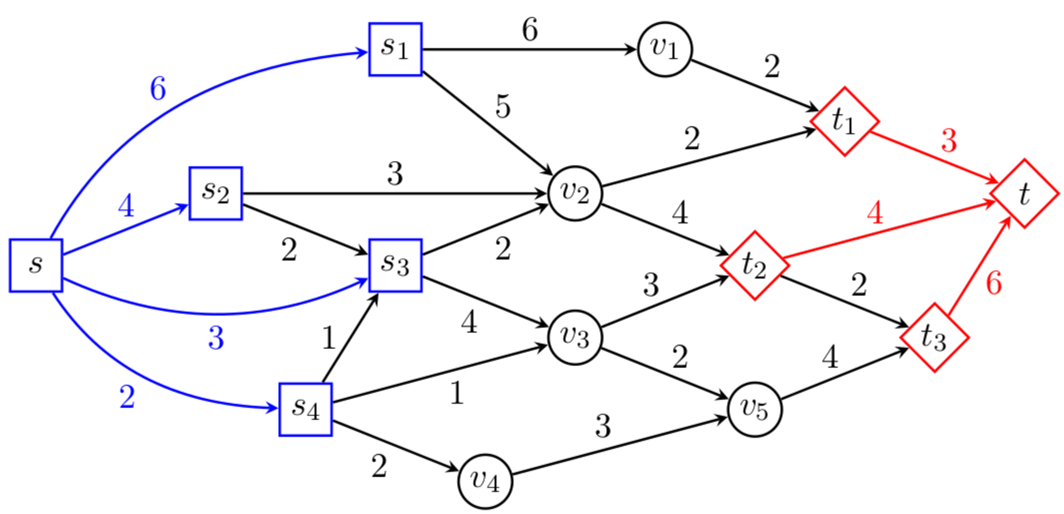
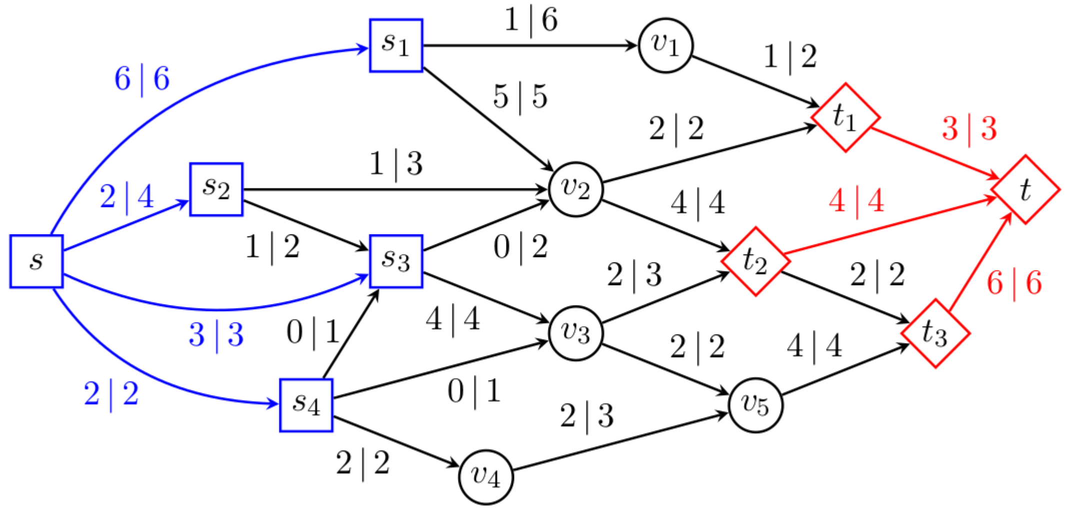


Figure 4.6: A flow problem with four sources $\{s_1, s_2, s_3, s_4\}$ and three sinks $\{t_1, t_2, t_3\}$. The supply of each source is indicated in blue, the demand of the sinks in red. The numbers on the arcs correspond to the respective capacities.





s
s_1
s_2
s_3
s_4
v_1
v_2
v_3
v_4
v_5
t_1
t_2
t_3
t
6 | 6
2 | 4
3 | 3
2 | 2
1 | 6
5 | 5
1 | 3
1 | 2
0 | 2
0 | 1
4 | 4
0 | 1
2 | 2
1 | 2
2 | 2
4 | 4
2 | 3
2 | 2
2 | 3
4 | 4
2 | 2
3 | 3
4 | 4
6 | 6

4.4.4 Roster planning

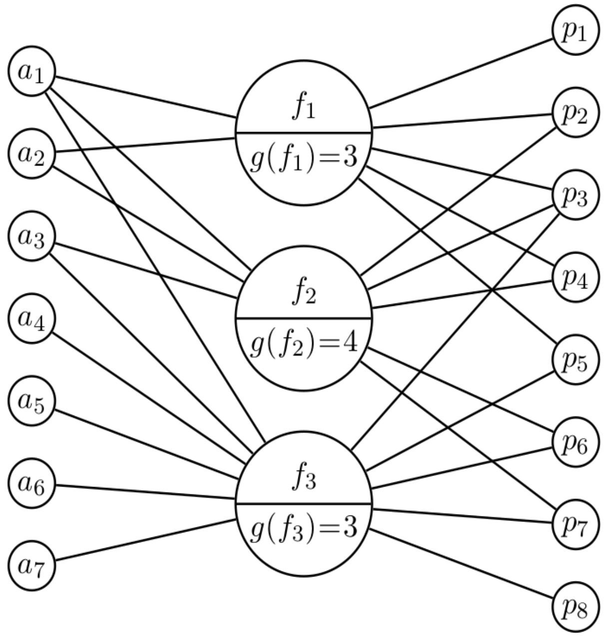


Figure 4.9: The graph $G = (V, E)$ representing a roster planning problem with 7 workers, 3 vehicle types, and 8 projects. The number of available vehicles of each type is specified in the corresponding vertex.

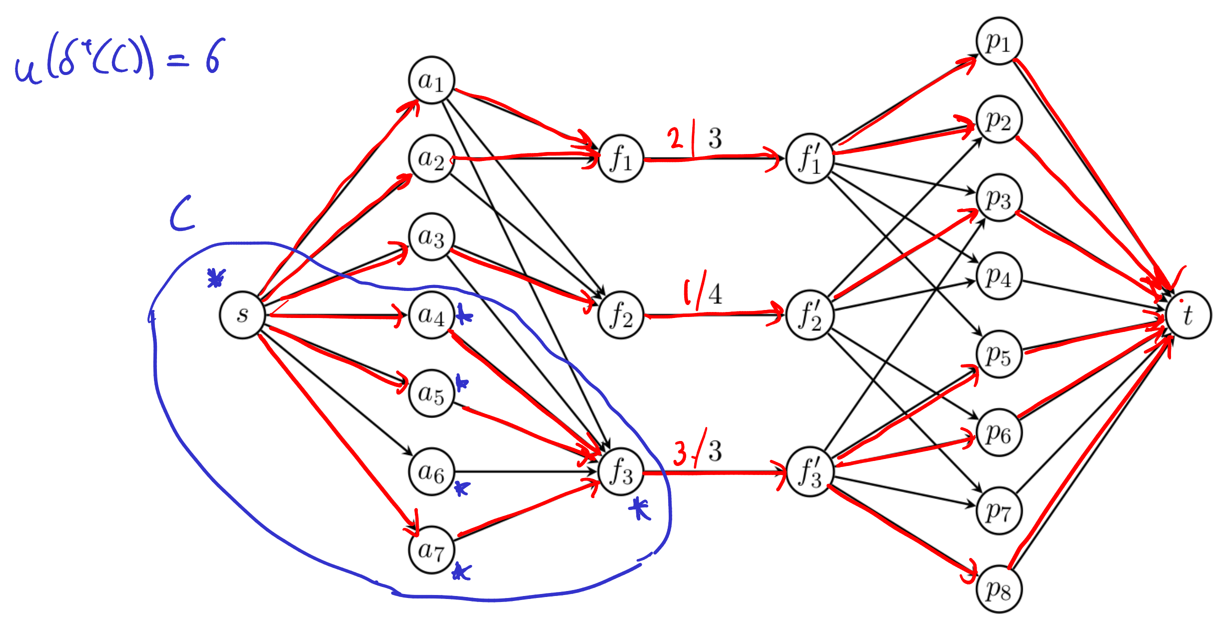


Figure 4.10: The auxiliary network H for the graph G from Figure 4.9. Arcs without explicitly specified capacity have capacity 1.