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Mathematical Optimization – Solutions to problem set 11

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Problem 1: An alternative description of the perfect matching polytope

(a) (i) We start by proving that any integral $x \in \overline{P}$ is the incidence vector of a perfect matching. By definition of $\overline{P}$, $x \in \overline{P} \cap \mathbb{Z}^E$ implies $x \in P \cap \mathbb{Z}^E$ and $x(E) = \frac{|V|}{2}$. As P is the matching polytope of G, the first property gives that x is the incidence vector of a matching in G. From the second property, we get that the cardinality of this matching is $\frac{|V|}{2}$, so it is indeed a perfect matching. For the other direction, let x be the incidence vector of a perfect matching in G, i.e., a matching of cardinality $\frac{|V|}{2}$. In other words, $x \in P$ and $x(E) = \frac{|V|}{2}$, so $x \in \overline{P}$.

(ii) It suffices to see that the inequality $x(E) \leq \frac{|V|}{2}$ is a valid inequality for P. We know that every $x \in P$ satisfies $x(\delta(v)) \leq 1$, and summing these constraints up, we get

$$\sum_{v \in V} x(\delta(v)) \leq |V| \ .$$

Note that for every edge $e \in E$, the term $x(e)$ appears precisely twice in the above sum, once for each of its endpoints. Consequently, the above can be rewritten as

$$|V| \geq \sum_{v \in V} x(\delta(v)) = \sum_{e \in E} 2x(e) = 2x(E) \ ,$$

which implies that $x(E) \leq \frac{|V|}{2}$ is indeed a valid inequality for P. If the polytope $\overline{P}$, which is by definition equal to the intersection of P and the hyperplane $\left\{x \in \mathbb{R}^E \colon x(E) = \frac{|V|}{2}\right\}$, satisfies $\overline{P} \neq \emptyset$, we get that $\overline{P}$ is a face of P.

To conclude from (i) and (ii) that $\overline{P}$ describes the perfect matching polytope, note that by part (i), $\overline{P}$ can only be empty if there do not exist any perfect matchings, so the description is right in this case. If in the other case, $\overline{P}$ is non-empty, then by part (i), we know that it contains the right integral points. Moreover, by part (ii), $\overline{P}$ is a face of the integral polytope P, hence it is integral itself. Together, this shows that $\overline{P}$ describes the perfect matching polytope.

(b) (i) We first show that for all $v \in V$, the equality constraints $x(\delta(v)) = 1$ are implied by the constraints in $\overline{P}$. Using the constraints $x(E) = \frac{|V|}{2}$ and $x(\delta(v)) \leq 1$ from $\overline{P}$, we get

$$|V| = 2x(E) = \sum_{e \in E} 2x(e) = \sum_{v \in V} x(\delta(v)) \leq \sum_{v \in V} 1 = |V| \ ,$$

which is an equality, so all inequalities $x(\delta(v)) \leq 1$ must in fact be equalities. This shows that $x(\delta(v)) = 1$ is implied by the constraints in $\overline{P}$ for every $v \in V$.

We now show that for every odd subset $S \subseteq V$, the constraint $x(\delta(S)) \geq 1$ is implied by the constraints in $\overline{P}$. To do so, we fix an arbitrary such set S. Summing up the constraints $x(\delta(v)) = 1$ over all $v \in S$ (note that we already know that these equality constraints are implied by the constraints in $\overline{P}$), we get

$$|S| = \sum_{v \in S} x(\delta(v)) = \sum_{e \in E[S]} 2x(e) + \sum_{e \in \delta(S)} x(e) = 2x(E[S]) + x(\delta(S)) \leq 2 \cdot \frac{|S|-1}{2} + x(\delta(S)) \ , \quad (1)$$

where we used the constraint $x(E[S]) \leq \frac{|S|-1}{2}$ from $\overline{P}$ in the last step. Rearranging terms, we get the desired inequality $x(\delta(S)) \geq 1$, so this inequality is implied by the constraints in $\overline{P}$ as well.

We also note that the non-negativity constraints are trivially implied.

(ii) The constraints $x(\delta(v)) \leq 1$ for all $v \in V$ in $\overline{P}$ are obviously implied by the corresponding equality constraints in P_{perf}. The constraint $x(E) = \frac{|V|}{2}$ is obtained from the constraints in P_{perf} by summing $x(\delta(v)) = 1$ over all $v \in V$, which indeed yields

$$x(E) = \frac{1}{2} \sum_{v \in V} x(\delta(v)) = \frac{1}{2}|V| \ .$$

Consequently, we only need to show that for every odd subset $S \subseteq V$, the constraint $x(E[S]) \leq \frac{|S|-1}{2}$ is implied by constraints in P_{perf}. To do so, we use the same equality as in (1), but complement it with the inequality $x(\delta(S)) \geq 1$ from P_{perf} to get

$$|S| = 2x(E[S]) + x(\delta(S)) \geq 2x(E[S]) + 1 \ ,$$

which is equivalent to $x(E[S]) \leq \frac{|S|-1}{2}$, as desired.

As before, we again note that the non-negativity constraints are trivially implied.

Note that part (i) proves $P_{perf} \supseteq \overline{P}$, while part (ii) shows $P_{perf} \subseteq \overline{P}$. Together, this therefore gives another proof of $P_{perf} = \overline{P}$.

Problem 2: Properties of polytopes and linear systems

(a) (i) Assume for contradiction that $y = x|_{N\setminus\{e\}}$ is not a vertex of P'. Since $x \in P$ and $x(e) = 0$ we have $y \in P'$. Let $y_1, y_2 \in P'$ be such that $y_1 \neq y_2$ and $y = \frac{1}{2}(y_1 + y_2)$. Such points exist since $y \in P'$ is not a vertex. We can extend y_1, y_2 to vectors $x_1, x_2 \in [0,1]^N$ by setting $x_i(f) := y_i(f)$ for $f \neq e$ and $x_i(e) := 0$ for $i \in \{1,2\}$. Observe that $x_1, x_2 \in P$ holds, as well as $x = \frac{1}{2}(x_1 + x_2)$. Now, since $x_1 \neq x_2$, we reached the desired contradiction to the fact that x is a vertex of P.

(ii) First note that by definition of P', we have $x \in P$. Assume for contradiction that x is not a vertex of P, and hence not an extreme point of P. Let $x_1, x_2 \in P$ be points with $x_1 \neq x_2$ and the property that $x = \frac{1}{2}(x_1 + x_2)$. Since $x(e) = 0$ and $P \subseteq [0,1]^N$, we must have $x_1(e) = 0$ and $x_2(e) = 0$. It follows that $y_1 = x_1|_{N\setminus\{e\}}$ and $y_2 = x_2|_{N\setminus\{e\}}$ are both in P'. Also, note that $y_1 \neq y_2$ and that $y = \frac{1}{2}(y_1 + y_2)$. This contradicts the fact that y is a vertex of P'.

(b) We can express the fact that the rows of A are linear combinations of the rows of C by the existence of a matrix $T \in \mathbb{R}^{m\times k}$ satisfying $A = TC$. Since y is a solution of both systems, we have

$$b = Ay = TCy = Td \ ,$$

i.e., $Td = b$. Finally, to show the desired inclusion, let $x \in \mathbb{R}^n$ such that $Cx = d$. Using the above, we get

$$Ax = TCx = Td = b \ ,$$

which shows that x satisfies $Ax = b$, as well. Consequently, $\{x \in \mathbb{R}^n : Ax = b\} \supseteq \{x \in \mathbb{R}^n : Cx = d\}$, as desired.

Problem 3: Properties of laminar families

(a) Let $K \in \mathcal{L}_{S\cup L}$ be some set intersecting with $S \cup L$. We claim that K is also intersecting with S. Note that $K \not\subseteq L$, otherwise it would not intersect with $S \cup L$. It follows from laminarity of $\mathcal{L}$ that either $L \subseteq K$ or $K \cap L = \emptyset$. In the first case, note that $S \setminus K \neq \emptyset$, as otherwise $L \cup S \subseteq K$. Hence, it follows that K is intersecting with S. In the second case, we have that $K \subseteq V \setminus L$, and thus K is intersecting with S in this case as well.

Finally, there exists at least one set in $\mathcal{L}$ which intersects with S, but not with $S \cup L$, namely L, i.e., $L \in \mathcal{L}_S$ and $L \notin \mathcal{L}_{S\cup L}$. This finishes the proof.

(b) Again, we show that every set intersecting with $S \cap L$ is also intersecting with S, while the set $L \in \mathcal{L}$ is only intersecting with S but not with $S \cap L$. Let $K \in \mathcal{L}_{S \cap L}$ be a set intersecting with $S \cap L$. Since $S \cap L \subseteq L$ and $\mathcal{L}$ is laminar, we have $K \subsetneq L$. Now it easily follows that K is intersecting with S.

(c) The set $L \in \mathcal{L}$ clearly does not intersect with $S \setminus L$, while it does with S. Moreover, we observe that all sets $K \in \mathcal{L}$ with $K \subseteq L$ do not intersect with $S \setminus L$, and all sets $K \in \mathcal{L}$ with $K \subseteq N \setminus L$ are intersecting with S if they are intersecting with $S \setminus L$. Moreover, every set $K \in \mathcal{L}$ with $L \subseteq K$ is intersecting with S if it is intersecting with $S \setminus L$.

(d) Again, $L \in \mathcal{L}$ does not intersect with $L \setminus S$, while it does with S. Now, if some $K \in \mathcal{L}$ intersects with $L \setminus S$, it follows from laminarity of $\mathcal{L}$ that $K \subseteq L$. Clearly $K \nsubseteq S$ and $K \nsubseteq L \setminus S$, so K intersects with S.

Problem 4: Properties of edges in cuts

(a) Consider any edge $e = \{u, v\} \in E$. We claim that the equality holds in the coordinate corresponding to e, that is,

$$\chi^{\delta(A)}(e) + \chi^{\delta(B)}(e) = \chi^{\delta(A \cup B)}(e) + \chi^{\delta(A \cap B)}(e) + 2 \cdot \chi^{E(A \setminus B, B \setminus A)}(e) \ ,$$

which implies the desired vector equality. We distinguish different edge types as follows.

- $e \in \delta(A \cap B)$ and $e \in \delta(A \cup B)$: In this case, one endpoint, say u, is in $A \cap B$. The other endpoint v then cannot be in $A \cup B$, otherwise we would have $e \notin \delta(A \cup B)$. It follows that $e \in \delta(A)$ and $e \in \delta(B)$, but $e \notin E(A \setminus B, B \setminus A)$, thus the claim is proved for this case.
- $e \in \delta(A \cap B)$ and $e \notin \delta(A \cup B)$: In this case, we have one endpoint, say u, in $A \cap B$. The other endpoint v is then contained in $(A \setminus B) \cup (B \setminus A)$. Assume without loss of generality that $v \in A \setminus B$. Then, we have $e \in \delta(B)$, $e \notin \delta(A)$ and $e \notin E(A \setminus B, B \setminus A)$, which proves the claim for this case.
- $e \notin \delta(A \cap B)$ and $e \in \delta(A \cup B)$: In this case, we have one endpoint, say u, in $V \setminus (A \cup B)$. The other endpoint v then satisfies that $v \in (A \cup B) \setminus (A \cap B)$. Assume without loss of generality that $v \in A \setminus (A \cap B)$. Then, we have $e \in \delta(A)$, $e \notin \delta(B)$ and $e \notin E(A \setminus B, B \setminus A)$, which proves the claim for this case.
- In the only remaining case, i.e., $e \notin \delta(A \cap B)$ and $e \notin \delta(A \cup B)$, the corresponding coordinate is either zero in all terms that appear in the equation, or $e \in E(A \setminus B, B \setminus A)$. But then, we also have $e \in \delta(A)$ and $e \in \delta(B)$, proving the equality for this case.

(b) From part (a), we know that $\chi^{\delta(C)} + \chi^{\delta(D)} = \chi^{\delta(C \cup D)} + \chi^{\delta(C \cap D)} + 2 \cdot \chi^{E(C \setminus D, D \setminus C)}$ holds for every pair of subsets $C, D \subseteq V$. Moreover, observe that for every set $U \subseteq V$, it holds that $\delta(U) = \delta(V \setminus U)$. We now choose $C = A$ and $D = V \setminus B$. Since $C \cup D = V \setminus (B \setminus A)$, $C \cap D = A \setminus B$, $C \setminus D = A \cap B$, and $D \setminus C = V \setminus (A \cup B)$, we obtain

$$\begin{aligned}
\chi^{\delta(A)} + \chi^{\delta(B)} &= \chi^{\delta(A)} + \chi^{\delta(V \setminus B)} \\
&= \chi^{\delta(C)} + \chi^{\delta(D)} \\
&= \chi^{\delta(C \cup D)} + \chi^{\delta(C \cap D)} + 2 \cdot \chi^{E(C \setminus D, D \setminus C)} \\
&= \chi^{\delta(V \setminus (B \setminus A))} + \chi^{\delta(A \setminus B)} + 2 \cdot \chi^{E(A \cap B, V \setminus (A \cup B))} \\
&= \chi^{\delta(B \setminus A)} + \chi^{\delta(A \setminus B)} + 2 \cdot \chi^{E(A \cap B, V \setminus (A \cup B))} \ ,
\end{aligned}$$

proving the desired.

Problem 5: The size of laminar families

(a) We prove the claim by induction on $n := |N|$. The claim is obviously true for $n = 1$. Assuming that the claim is true for all $n' \in \{1, \dots, n-1\}$, we now prove the claim for n. The claim is trivially satisfied if $|\mathcal{L}| \in \{0, 1\}$, so let us assume that $|\mathcal{L}| \geq 2$. Let $L \in \mathcal{L} \setminus \{N\}$ be a maximal set in the laminar family different from N (i.e., a set that is not contained in any other set in $\mathcal{L} \setminus \{N\}$; also note that N might not necessarily be contained in $\mathcal{L}$). Define

$$\mathcal{R} = \{K \in \mathcal{L} \setminus \{N\} \colon K \subseteq L\} \quad \text{and} \quad \mathcal{S} = \{K \in \mathcal{L} \setminus \{N\} \colon K \subseteq N \setminus L\} \ .$$

Since $\mathcal{L}$ is laminar, no set intersects both L and $N \setminus L$. Thus, we have $\mathcal{L} \setminus \{N\} = \mathcal{R} \cup \mathcal{S}$, and $\mathcal{R} \cap \mathcal{S} = \emptyset$. This implies that $|\mathcal{L}| \leq |\mathcal{R}| + |\mathcal{S}| + 1$. By the inductive assumption, as $L \subsetneq N$ and $N \setminus L \subsetneq N$, we have $|\mathcal{R}| \leq 2|L| - 1$ and $|\mathcal{S}| \leq 2|N \setminus L| - 1 = 2(n - |L|) - 1$, hence

$$|\mathcal{L}| \leq |\mathcal{R}| + |\mathcal{S}| + 1 \leq 2|L| - 1 + 2(n - |L|) - 1 + 1 = 2n - 1 \ ,$$

as desired.

(b) The statement is trivially true if $\mathcal{L} = \emptyset$, so assume that $\mathcal{L} \neq \emptyset$. We show how to map each set $L \in \mathcal{L}$ to an element $i_L \in N$ in an injective way. This will clearly imply $|\mathcal{L}| \leq |N| = n$.

For every set $L \in \mathcal{L}$, we know that $\mathcal{L}$ does not contain a non-trivial partition of L, hence

$$S(L) \coloneqq \bigcup_{L' \in \mathcal{L} \colon L' \subsetneq L} L'$$

is strictly contained in L, and we can choose $i_L \in L \setminus S(L)$. To prove that the map $\mathcal{L} \to N$ defined by $L \mapsto i_L$ for $L \in \mathcal{L}$ is injective, we let $L_1, L_2 \in \mathcal{L}$ with $L_1 \neq L_2$ and show that $i_{L_1} \neq i_{L_2}$.

If $L_1 \cap L_2 = \emptyset$, this is immediate, because $i_L \in L$ for every L. Thus, assume that $L_1 \cap L_2 \neq \emptyset$. Then, laminarity of $\mathcal{L}$ implies that either $L_1 \subsetneq L_2$ or $L_2 \subsetneq L_1$. In the first case, i.e., if $L_1 \subsetneq L_2$, we have $i_{L_2} \in L_2 \setminus L_1$, and as $i_{L_1} \in L_1$, we get $i_{L_1} \neq i_{L_2}$. The other case, $L_2 \subsetneq L_1$, is analogous. This finishes the proof.

Remark: Both bounds proved above are tight. To see this, let $S_i \coloneqq \{1, \ldots, i\}$ for $i \in \{1, \ldots, n\}$, and let $N = \{1, \ldots, n\}$. The family

$$\mathcal{L}_1 = \{\{i\} \colon i \in [n]\} \cup \{S_i \colon i \in [n]\} \subseteq 2^N \setminus \emptyset$$

is clearly laminar and satisfies $|\mathcal{L}_1| = 2n - 1$, hence we see that the bound from part(a) is tight. Moreover, the laminar family

$$\mathcal{L}_2 = \{S_i \colon i \in [n]\} \subseteq 2^N \setminus \emptyset$$

is laminar, it does not contain a non-trivial partition of some set $L \in \mathcal{L}_2$, and we have that $|\mathcal{L}_2| = n$, matching the bound from part (b).

Illustrations of the two families $\mathcal{L}_1$ and $\mathcal{L}_2$ are shown in Figure 1 and Figure 2, respectively.

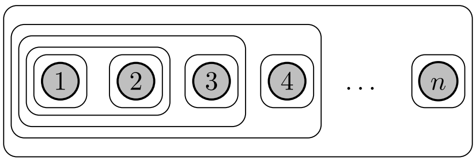

Figure 1: The family $\mathcal{L}_1$.

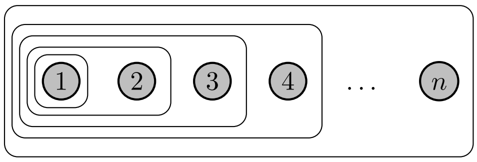

Figure 2: The family $\mathcal{L}_2$.

Problem 6: Bounded degree spanning trees

(a) For the same technical reasons as encountered in class, we write Q as

$$Q = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{rl} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 1 \\ x(\delta(v)) \leq B_v & \forall v \in V \end{array} \right\}$$

(note that we added sets S of cardinality 1, corresponding to trivially true constraints $0 \leq 0$ in the description). By Problem 2 (a), we can without loss of generality delete edges $e \in E$ with $y(e) = 0$ from the graph G, and y reduced to the non-zero entries will be a vertex of the polytope

Q corresponding to the new graph. Let $F = \{e \in E \colon y(e) > 0\}$. Thus, it is enough to consider $y|_F \in \mathbb{R}^F$, as $\operatorname{supp}(y) = \operatorname{supp}(y|_F)$.

In this reduced setting, consider the family of $y|_F$-tight constraints in the description of Q. Some of them are spanning tree constraints, some are degree constraints. By the uncrossing technique seen in class, there exists a laminar subfamily of 2^V such that the constraints (equalities) corresponding to sets in the laminar subfamily imply all y-tight spanning tree constraints. Thus, y is uniquely defined by these spanning tree constraints and the tight degree constraints. By Problem 5 of this problem set, the laminar family can have at most $2|V| - 1$ many elements. Note that we don't need the constraints corresponding to singleton elements to define y (but they will always be in a maximal laminar subfamily), hence there are at most $|V| - 1$ many spanning tree constraints remaining. Together with the at most $|V|$ many tight degree constraints, we arrive at a system of at most $2|V| - 1$ tight constraints that uniquely define $y|_F$. This implies that $\dim(y|_F) \leq 2|V| - 1$, hence $\operatorname{supp}(y) = \operatorname{supp}(y|_F) \leq 2|V| - 1$.

To arrive at $|\operatorname{supp}(y)| \leq 2|V| - 2$, one additional observation is necessary. Observe that the previous analysis is tight and only if all $|V|$ many degree constraints are tight, else we immediately get the better bound $\operatorname{supp}(y) \leq 2|V|-2$. Also note that the full set V always appears in a maximal laminar subfamily of tight spanning tree constraints. However, the linear dependency

$$\frac{1}{2} \cdot \sum_{v \in V} \chi^{\delta(v)} = \chi^E$$

proves that the spanning tree constraint corresponding to the full set is implied by the degree constraints (Problem 2 (b) shows that because of the common feasible point y, it is indeed enough to check a dependency among the coefficient vectors χ^E and $\chi^{\delta(v)}$ for $v \in V$). Consequently, $|V| - 2$ many tight spanning tree constraints and $|V|$ many degree constraints are enough to uniquely define y in this case, hence $\dim(y|_F) \leq 2|V| - 1$, and thus $\operatorname{supp}(y) = \operatorname{supp}(y|_F) \leq 2|V| - 2$, as desired.

(b) Assume for the sake of contradiction that $y(e) < \frac{1}{2}$ for all $e \in E$. We know from part (a) that $|\operatorname{supp}(y)| \leq 2|V| - 2$, hence

$$y(E) = \sum_{e \in E} y(e) < (2|V| - 2) \cdot \frac{1}{2} = |V| - 1 \ ,$$

but $y \in P$ implies $y(E) = |V| - 1$, a contradiction. Consequently, there exists $e \in E$ with $y(e) \geq \frac{1}{2}$.