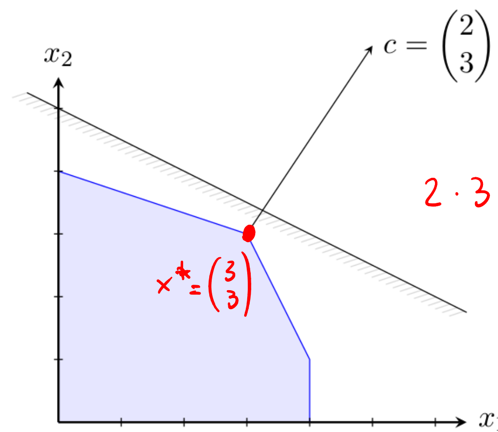
1.4 Linear duality

1.4.1 Motivation: finding bounds on optimal value

$$
\begin{array}{lrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
 & x_1 & + & 3x_2 & \leq & 12 \\
 & 2x_1 & + & x_2 & \leq & 9 \\
 & x_1 & & & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$



$2 \cdot 3 + 3 \cdot 3 = 15$

Assume we want to find a valid upper bound for optimal value of above LP.

One option: "2 *" $\quad x_1 + 2x_2 \leq 10$

$\rightsquigarrow 2x_1 + 4x_2 \leq \underline{20}$

This is upper bound on optimal solution

For any feasible solution (x_1, x_2):

objective value of (x_1, x_2) $\longrightarrow$ $2x_1 + 3x_2 \leq 2x_1 + 4x_2 \leq 20$

$x_1, x_2 \geq 0$

Better option: Add up following 2 constraints:

$$
\begin{array}{rcl}
x_1 & \leq & 4 \\
x_1 + 3x_2 & \leq & 12 \\
\hline
2x_1 + 3x_2 & \leq & 16
\end{array}
$$

How to get best bound of this type? $\longrightarrow$ This leads to the dual LP

$\longrightarrow$ Introduce non-negative variables $y_1, y_2, y_3, y_4 \geq 0$ representing how we add up the 4 constraints.

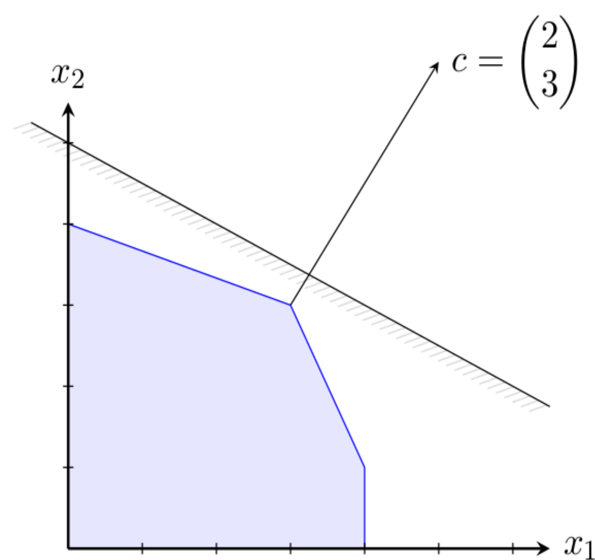
$$\min \; 12y_1 + 9y_2 + 4y_3 + 10y_4$$

$$3y_1 + y_2 + 0 \cdot y_3 + 2y_4 \geq 3$$

$$y_1 + 2y_2 + y_3 + y_4 \geq 2$$

$$
\begin{array}{rrcrcrcl}
 & \max & 2x_1 & + & 3x_2 & & & \\
 & y_1 \times (& x_1 & + & 3x_2 & \leq & 12 &) \\
+ & y_2 \times (& 2x_1 & + & 1 \cdot x_2 & \leq & 9 &) \\
+ & y_3 \times (& x_1 & & 0 \cdot x_2 & \leq & 4 &) \\
+ & y_4 \times (& x_1 & + & 2x_2 & \leq & 10 &) \\
 & & x_1 & & & \geq & 0 & \\
 & & & & x_2 & \geq & 0 &
\end{array}
$$

$y_1, y_2, y_3, y_4 \geq 0$



We need to make sure that resulting coefficients for x_1 and x_2 are at least as large as their coefficients in the objective, i.e., $\underset{\uparrow x_1}{2}$ and $\underset{\uparrow x_2}{3}$, respectively.

Moreover, to obtain a bound as strong as possible, we minimize the rhs of the obtained constraint.

This leads to following LP

$$
\begin{array}{lrcrcrcrcl}
\min & 12y_1 & + & 9y_2 & + & 4y_3 & + & 10y_4 & & \\
 & y_1 & + & 2y_2 & + & y_3 & + & y_4 & \geq & 2 \\
 & 3y_1 & + & y_2 & & & + & 2y_4 & \geq & 3 \\
 & y_1 & & & & & & & \geq & 0 \\
 & & & y_2 & & & & & \geq & 0 \\
 & & & & & y_3 & & & \geq & 0 \\
 & & & & & & & y_4 & \geq & 0
\end{array}
$$

dual LP

For this example, above LP has a unique optimal solution:

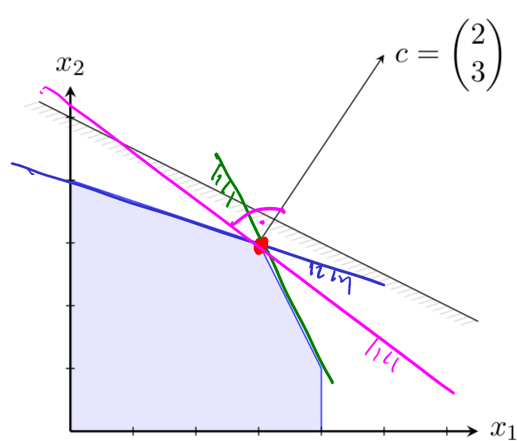
$$(y_1, y_2, y_3, y_4, y_5) = \left(\tfrac{4}{5}, \tfrac{3}{5}, 0, 0, 0\right)$$

$$\xrightarrow{\text{value}} \quad 12 \cdot \frac{4}{5} + 9 \cdot \frac{3}{5} = \underline{15}$$

This is optimal value of original LP!

→ Hence, optimal dual solution certifies optimality of the primal solution $\binom{3}{3}$.

$$
\begin{array}{lrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
 & x_1 & + & 3x_2 & \leq & 12 \\
 & 2x_1 & + & x_2 & \leq & 9 \\
 & x_1 & & & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$



1.4.2 Dual of a linear program

$Ax \leq b$

$y^T A x \leq y^T b$

primal problem		dual problem
$\max\ c^\top x$ $Ax \leq b$ $x \geq 0$	$\xrightarrow[\text{dualization}]{\text{(LP)}}$	$\min\ b^\top y$ $A^\top y \geq c$ $y \geq 0$

$c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

$y \in \mathbb{R}^m_{\geq 0}$

$\bar{y} \leq 0$ $\quad a^T x \geq 2$

Dualization of LPs not in canonical form

To dualize LP that is not in canonical form, one can

(i) First transform it into canonical form and then dualize.

(ii) Adjust the dualization rules by employing an analogous reasoning to the one we applied in introductory example.

example

$$\underbrace{y^T}A x = y^T b$$

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \max\ & b^T y \\ & A^T y \leq c \\ & y \in \mathbb{R}^m \end{aligned}$$

$(c \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m)$

$$c^T x \geq y^T A x = y^T b$$

Dual of dual is primal

$$\begin{array}{lrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

primal LP

$$\begin{array}{lrcl} \min & b^\top y & & \\ & A^\top y & \geq & c \\ & y & \geq & 0 \end{array}$$

dual LP

Bring dual LP in canonical form:

$$\begin{array}{lrcl} -\max & -b^\top y & & \\ & -A^\top y & \leq & -c \\ & y & \geq & 0 \end{array}$$

Apply LP dualization rules:

$$\begin{array}{lrcl} -\min & -c^\top x & & \\ & -Ax & \geq & -b \\ & x & \geq & 0 \end{array}$$

Change signs:

$$\begin{array}{lrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

1.4.3 Weak and strong linear duality

Theorem 1.84: Weak duality

Let x, y be feasible solutions to (PLP) and (DLP), respectively. Then

$$c^\top x \le b^\top y \ .$$

Proof

Recall:

$$\text{(PLP)} \quad \begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \le & b \\ & x & \ge & 0 \end{array} \qquad\qquad \text{(DLP)} \quad \begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & \ge & c \\ & y & \ge & 0 \end{array}$$

$$c^\top x \underset{\substack{A^\top y \ge c \\ x \ge 0}}{\le} y^\top A x \underset{\substack{Ax \le b \\ y \ge 0}}{\le} y^\top b = b^\top y$$

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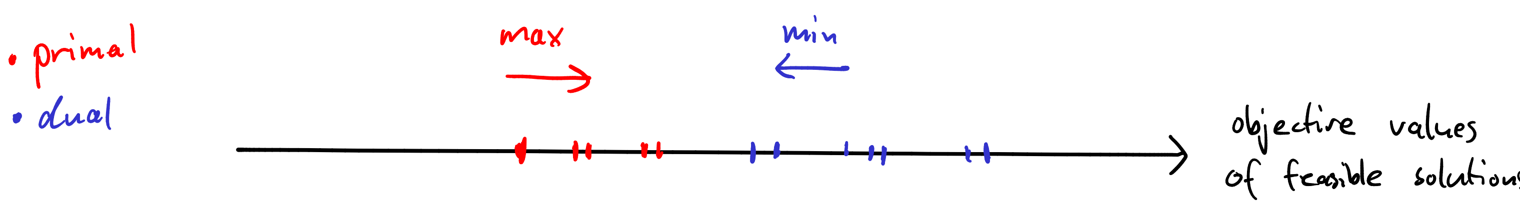

Some implications of weak duality

primal unbounded $\Longrightarrow$ dual infeasible

dual unbounded $\Longrightarrow$ primal infeasible

Moreover, if one finds a primal feasible solution x and a dual feasible solution y with $c^T x = b^T y$, then

x is optimal for primal and y is optimal for dual.

Example 1.85

There are pairs of primal/dual LPs that are both infeasible.

$$\begin{array}{lrcrcr}
\max & 2x_1 & - & x_2 & & \\
 & x_1 & - & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & -2 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{lrcrcr}
\min & y_1 & - & 2y_2 & & \\
 & y_1 & - & y_2 & \geq & 2 \\
 & -y_1 & + & y_2 & \geq & -1 \\
 & y_1 & & & \geq & 0 \\
 & & & y_2 & \geq & 0
\end{array}$$

$x_1 - x_2 \geq 2$

$y_1 - y_2 \leq 1$

Theorem 1.86: Strong duality

If the primal has a finite optimum, then also the corresponding dual problem has a finite optimum and their values are equal, i.e., there exists a feasible solution x for (PLP) and a feasible solution y for (DLP) such that $c^\top x = b^\top y$.

Hence, whenever we have a finite (primal) LP, then the dual LP can always be used to certify optimality of an optimal primal solution and vice-versa.

Possible combos of primal/dual LP types

primal \ dual	finite	unbounded	infeasible
finite	✓	✗	✗
unbounded	✗	✗	✓
infeasible	✗	✓	✓

implied by strong duality

implied by weak duality

implied by example

1.4.4 Dual interpretation of simplex tableau

standard form

$$\boxed{\begin{array}{l} \max \; c^\top x \\ x^s + Ax = b \\ x^s, x \geq 0 \end{array}}$$

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

primal LP

$$\begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & \geq & c \\ & y & \geq & 0 \end{array}$$

dual LP

$\xrightarrow[\text{canonical form}]{\text{dual in}}$

$$\begin{array}{rrcl} -\max & -b^\top y & & \\ & -A^\top y & \leq & -c \\ & y & \geq & 0\,, \end{array}$$

$\xrightarrow[\text{standard form}]{\text{dual in}}$

$$\begin{array}{rrrcr} \max & -w & & = & -b^\top y \\ & y^s & -A^\top y & = & -c \\ & & y, y^s & \geq & 0 \end{array}$$

Dual reading of tableau

	x (y^s)	1 (w)	
			(II)
z (1)	$-c^\top$	0	
			(+)
x^s (y)	A	b	

$$\begin{array}{rclcl} y^s & = & -c & + & A^\top y \\ w & = & 0 & + & b^\top y \end{array}$$

Relation between primal/dual variables and slacks				
x_j^s	primal slack variable	$\longleftrightarrow$	y_j	dual variable
x_i	primal variable	$\longleftrightarrow$	y_i^s	dual slack variable

z primal objective $\leftrightarrow$ 1 dual constant

1 primal constant $\leftrightarrow$ w dual objective

Lemma 1.87

For any primal tableau, the dual reading of the tableau leads to an equivalent equation system for the dual (including the objective function).

Example 1.88

$y_3^s = -5 \cdot 1 - 1 \cdot y_1 + 3 \cdot y_2 + 3 \cdot y_3$

$$\begin{array}{lrcrcrcrcl}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

tableau :

	x_1 (y_1^s)	x_2 (y_2^s)	x_3 (y_3^s)	x_4 (y_4^s)	1 (w)
(1) z	-4	-1	-5	-3	0
(y_1) x_1^s	1	-1	-1	3	1
(y_2) x_2^s	5	1	3	8	55
(y_3) x_3^s	-1	2	3	-5	3

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optimal tableau :

	x_1 (y_1^s)	x_3 (y_3^s)	x_1^s (y_1)	x_3^s (y_3)	1 (w)
(1) z	1	2	11	6	29
(y_2^s) x_2	2	4	5	3	14
(y_4^s) x_4	1	1	2	1	5
(y_2) x_2^s	-5	-9	-21	-11	1

Proof of Lemma 1.87

see script. (problem sets)

$c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

primal LP in standard form:

$$\begin{array}{lrcrcl} \max & z & & & = & c^\top x \\ & & x^s & + & Ax = & b \\ & & & x, x^s & \geq & 0\,, \end{array}$$

dual LP in standard form:

$$\begin{array}{lrcrcl} \max & -w & & & = & -b^\top y \\ & & y^s & -A^\top y & = & -c \\ & & & y, y^s & \geq & 0\,. \end{array} \qquad (1.30)$$

Consider a primal tableau (with respect to an arbitrary basis):

$$\begin{array}{c|c|c} & x_N & 1 \\ \hline z & -\bar{c}^\top & \bar{q} \\ \hline x_B & \bar{A} & \bar{b} \end{array} \qquad (1.31)$$

To show: Dual reading of (1.31) leads to equation system equivalent to (1.30).

Dual reading of (1.31) leads to full row rank system with same number of equations as (1.30).

$\Longrightarrow$ It suffices to show that each equation obtained by dual reading (1.31) is implied by (1.30).

Consider any equality obtained from dual reading (1.31).

This equality either stems from a column that

corresponds to a non-basic primal variable $(x_N)_i$, or

corresponds to the constant term 1.

	x_N	1
z	$-\overline{c}^\top$	$\overline{q}$
x_B	$\overline{A}$	$\overline{b}$

To have a convenient way to talk about primal and dual values in the same space, we define

$$\begin{array}{cccccccc} Q := & \mathbb{R} & \times & \mathbb{R}^m & \times & \mathbb{R}^n & \times & \mathbb{R} \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{corresponds to} & z\ (1) & & x^s\ (y) & & x\ (y^s) & & 1\ (w) \end{array}$$

Define η:

$(x_N)_i$

	x_N	1
z	$-\overline{c}^\top$	$\overline{q}$
x_B	$\overline{A}$	$\overline{b}$

$$\eta = \begin{array}{c} z \\ x^s \\ x \\ 1 \end{array} \begin{pmatrix} \square \\ \hline \square \\ \square \\ 1 \end{pmatrix}$$

- Set x_B/z-entries to values in considered column.
- Entry corresponding to $(x_N)_i$ is set to -1.
- Entry corresponding to 1 is set to 1.
- All other entries are set to 0.

$$\xrightarrow[\text{construction}]{\eta \text{ satisfies by}} \quad \begin{pmatrix} 1 & 0^\top & -c^\top & 0 \\ 0 & I & A & -b \end{pmatrix} \cdot \eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\operatorname{kernel}\left(\begin{pmatrix} 1 & 0^\top & -c^\top & 0 \\ 0 & I & A & -b \end{pmatrix} \right) = \operatorname{span}\left(\begin{pmatrix} c^\top & 0 \\ -A & b \\ I & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$\Longrightarrow \quad \exists\, \lambda \in \mathbb{R}^{n+1} \text{ s.t. } \quad \eta = \begin{pmatrix} c^\top & 0 \\ -A & b \\ I & 0 \\ 0 & 1 \end{pmatrix} \cdot \lambda$$

	x_N	1
z	$-\overline{c}^\top$	$\overline{q}$
x_B	$\overline{A}$	$\overline{b}$

Equation coming from dual reading:

$$\eta^\top \begin{pmatrix} 1 \\ y \\ y^s \\ -w \end{pmatrix} = 0 \qquad (1.33)$$

Recall: Our goal is to show that (1.33) is implied by the equation system stemming from original dual in standard form:

$$\begin{pmatrix} c & -A^\top & I & 0 \\ 0 & b^\top & 0^\top & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y \\ y^s \\ -w \end{pmatrix} = 0 \qquad (1.34)$$

To show: If $\begin{pmatrix} 1 \\ y \\ y^s \\ -w \end{pmatrix}$ fulfills (1.34), then it fulfills (1.33).

This holds, because:

$$\eta^\top \begin{pmatrix} 1 \\ y \\ y^s \\ -w \end{pmatrix} = \lambda^\top \begin{pmatrix} c & -A^\top & I & 0 \\ 0 & b^\top & 0^\top & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ y^s \\ -w \end{pmatrix} = 0$$

#

Recall:

> **Theorem 1.86: Strong duality**
>
> If the primal has a finite optimum, then also the corresponding dual problem has a finite optimum and their values are equal, i.e., there exists a feasible solution x for (PLP) and a feasible solution y for (DLP) such that $c^\top x = b^\top y$.

Proof of Theorem 1.86

Consider an optimal LP tableau for primal LP.

↳ Exists because Simplex Method terminates

$$\begin{array}{c|ccc|c} & & x_N^T & & 1 \\ \hline z & \oplus & \cdots & \oplus & q \\ \hline x_B & & & & \oplus \\ & & & & \vdots \\ & & & & \oplus \end{array}$$

Dual reading lead to dual solution y^* with same objective value as optimal primal solution x^*.

→ Moreover, y^* is feasible because of $\oplus$.

$\Rightarrow$ $c^T x^* = b^T y^*$, which implies the statement by weak duality.

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