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Mathematical Optimization – Solutions to problem set 13

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Problem 1: A separation oracle for the perfect matching polytope

(a) If $|V|$ is odd, then the constraint $x(\delta(V)) \geq 1$ appears in the description of P, causing the polytope to be empty, as $\delta(V) = \emptyset$ and hence $x(\delta(V)) = 0$ independent of x. Consequently, let us assume that $|V|$ is even from now on.

To decide whether for a given graph $G = (V, E)$ and a point $y \in \mathbb{R}^E$, we have $y \in P$ or not, we can first check non-negativity constraints and all degree constraints $x(\delta(v)) = 1$ for $v \in V$ in polynomial time. If none of them separates y from P, we do in particular have $y \geq 0$.

To separate over the odd cut constraints, let $w\colon E \to \mathbb{R}^E_{\geq 0}$ be defined by $w(e) = y(e)$ for all $e \in E$. Let C be a minimum V-odd cut, which we can find efficiently by assumption. If $w(\delta(C)) \geq 1$, then for any V-odd cut S (i.e., for any cut S with $|S|$ odd), we have $y(\delta(S)) = w(\delta(S)) \geq w(\delta(C)) \geq 1$, hence y satisfies all odd-cut constraints, and hence $y \in P$. If, on the other hand, $w(\delta(C)) < 1$, then $y(\delta(C)) = w(\delta(C)) < 1$, hence y and P are separated by the constraint $x(\delta(C)) \geq 1$.

(b) The cut C determined in step 1 of the given algorithm is the minimum cut C in the graph that has non-trivial intersection with T, i.e., $T \cap C \notin \{\emptyset, T\}$. If C is T-odd, we can in particular conclude that C is a minimum T-odd cut, and the algorithm returns the correct result in step 2.

If, on the other hand, $|C \cap T|$ is even, we recursively apply the algorithm in the graphs ${}^{G}/_{C}$ and ${}^{G}/_{\overline{C}}$. First of all, observe that the two recursive calls are well-defined: Indeed, as C has non-trivial and even intersection with T, both $T \setminus C$ and $T \setminus \overline{C}$ are non-empty and of even cardinality.

Moreover, note that cuts in the two graphs ${}^{G}/_{C}$ and ${}^{G}/_{\overline{C}}$ naturally correspond to cuts in the original graph G of the same value, and the two graphs contain precisely all those cuts that do not intersect with C (i.e., cuts S such that $S \subseteq C$, $C \subseteq S$ or $C \cap S = \emptyset$). Thus, to prove correctness, we have to show that there is a minimum T-odd cut that does not intersect with C.

To this end, let Q be a minimum T-odd cut. If Q and C are not intersecting, there is nothing to do, so assume that $C \setminus Q$, $C \cap Q$, and $Q \setminus C$ are all non-empty. We distinguish cases based on the parity of the intersections of these sets with T.

Case 1: $Q \setminus C$ is T-odd. This implies that $C \setminus Q$ and $C \cap Q$ have even intersection with T, and as C has non-trivial intersection with T, at least one of the two sets must have non-trivial intersection with T.

Case 1a: $C \setminus Q$ has non-trivial intersection with T. We have

$$w(\delta(C)) + w(\delta(Q)) \geq w(\delta(C \setminus Q)) + w(\delta(Q \setminus C)) \ .$$

As $C \setminus Q$ has non-trivial intersection with T and because C is the minimum cut with this property, we have $w(\delta(C)) \leq w(\delta(C \setminus Q))$, thus the above inequality gives $w(\delta(Q)) \leq w(\delta(Q \setminus C))$. The cuts Q and $Q \setminus C$ are both T-odd, and Q is a minimum T-odd cut in G, hence we must have $w(\delta(Q)) = w(\delta(Q \setminus C))$, so $Q \setminus C$ is a minimum T-odd cut that does not intersect with C, as desired.

Case 1b: $C \cap Q$ has non-trivial intersection with T. We have

$$w(\delta(C)) + w(\delta(Q)) \geq w(\delta(C \cup Q)) + w(\delta(C \cap Q)) \ .$$

As $C \cap Q$ has non-trivial intersection with T and because C is the minimum cut with this property, we have $w(\delta(C)) \leq w(\delta(C \cap Q))$, thus the above inequality gives $w(\delta(Q)) \leq w(\delta(C \cup Q))$. The cuts Q and $Q \cup C$ are both T-odd, and Q is a minimum T-odd cut in G, hence we must have $w(\delta(Q)) = w(\delta(C \cup Q))$, so $C \cup Q$ is a minimum T-odd cut that does not intersect with C, as desired.

Case 2: $Q \setminus C$ has even intersection with T. This case can be reduced to case 1 above after replacing both Q and C by their complements (note that as we are in an undirected setting, cut values do not change when taking complements).

Consequently, in all cases, there exists a minimum T-odd cut not intersecting with C, hence in at least one of the two subproblems, a minimum T-odd cut for that subproblem will be a minimum T-odd cut for the initial problem.

It remains to see that the algorithm terminates. This, however, is clear, since in every recursive call, the size of the set T reduces by at least two, and it always remains even. Thus, at the latest in a recursive call with $|T| = 2$, step 1 of the algorithm will only have one choice for $\{s,t\} \subseteq T$, and the minimum s-t cut will be T-odd and hence this cut will be returned without going into further recursive calls.

(c) Observe that a single call to the proposed algorithm (without taking the time for recursive calls into account) can be implemented in strongly polynomial time: Step 1 is a series of $\binom{|T|}{2}$ many minimum s-t cut computations, which we know how to realize in strongly polynomial time. Finding the cut C of minimum value among all computed s-t cuts is easy, too, as well as deciding whether $|C \cap T|$ is even or odd. Consequently, we see that we only have to prove that the number of calls to Algorithm 1 (counting recursive calls now) is strongly polynomial.

To prove this, consider a rooted tree $\mathcal{T}$ where every node corresponds to a call to Algorithm 1, and the children of a node are the nodes corresponding to the two recursive calls made in the call corresponding to the parent node. Thus, a node in $\mathcal{T}$ either is a leave (corresponding to the case where $|C \cap T|$ is odd), or it has two children (corresponding to the case where $|C \cap T|$ is even and recursive calls are issued).

As observed in the previous subtask, Algorithm 1 is finite for sure, hence $\mathcal{T}$ is finite, too. Now label all nodes with the vertex sets T that are used in the respective calls. At the root node, this label is the full (initial) set T. Note that whenever two recursive calls are made, one uses the set $T \setminus C$, while the other uses $T \setminus \overline{C} = T \cap C$. Thus, the labels of two child nodes form a partition of the label of their parent node, and consequently, the labels of all leaves form a partition of T. Moreover, note that the sets used as labels are always of even cardinality, hence there can be at most $|T|/2$ many leaves. But then, the whole tree can have at most $|T|$ many vertices, proving the desired.

Problem 2: Minimum-Volume Ellipsoid Containing Rotated Half-Ball
First observe that $R(c)$ is a linear transformation of the half-ball

$$S = E(0, I) \cap \{x \in \mathbb{R}^n \colon x_1 \geq 0\} \ .$$

The linear transformation ϕ is of the form $x \mapsto Ux$, where U is an orthogonal matrix with the property that $Ue_1 = c$, i.e., with the first column of U being the vector c (e_1 denotes the first unit vector in $\mathbb{R}^n$). Such an orthogonal matrix U exists since $\|c\|_2 = 1$.

Note that transforming an ellipsoid $E(a', A')$ that contains S using the transformation ϕ results in an ellipsoid that contains $R(c)$. Moreover, any ellipsoid that contains $R(c)$ can be transformed back to one that contains S by $\phi^{-1} : x \mapsto U^{-1}x$.

We know that the minimum-volume ellipsoid $E(\overline{a}, \overline{A})$ containing S is given by

$$\overline{a} = \begin{pmatrix} \frac{1}{n+1} & 0 & 0 & \cdots & 0 \end{pmatrix}^\top \qquad \text{and} \qquad \overline{A} = \begin{pmatrix} \left(\frac{n}{n+1}\right)^2 & 0 & \cdots & 0 \\ 0 & \frac{n^2}{n^2-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n^2}{n^2-1} \end{pmatrix} \ .$$

Moreover, we know that for any set $S \subseteq \mathbb{R}^n$ and and matrix $V \in \mathbb{R}^{n \times n}$, we have $\mathrm{vol}(\{Vx \mid x \in S\}) = |\det(V)| \cdot \mathrm{vol}(S)$, hence $\phi(E(\overline{a}, \overline{A}))$ is the minimum volume ellipsoid containing $R(c)$, and thus it suffices to show that $E(a, A) = \phi(E(\overline{a}, \overline{A}))$.

Since U is orthogonal, we have $U^{-1} = U^\top$. We get that

$$\begin{aligned}\phi(E(\overline{a}, \overline{A})) = \{Ux \colon x \in E(\overline{a}, \overline{A})\} &= \{y \in \mathbb{R}^n \colon U^\top y \in E(\overline{a}, \overline{A})\} && (y = Ux)\\ &= \{y \in \mathbb{R}^n \colon (U^\top y - \overline{a})^\top \overline{A}^{-1} (U^\top y - \overline{a}) \leq 1\}\\ &= \left\{y \in \mathbb{R}^n \colon (y - U\overline{a})^\top \left(U \cdot \overline{A} \cdot U^\top\right)^{-1} (y - U\overline{a}) \leq 1\right\} \ . && (1)\end{aligned}$$

Now observe that $\overline{A} = \frac{n^2}{n^2-1} \cdot \left(I - \frac{2}{n+1} e_1 e_1^\top\right)$, hence

$$U \cdot \overline{A} \cdot U^\top = \frac{n^2}{n^2-1} \cdot \left(UU^\top - \frac{2}{n+1} U e_1 e_1^\top U^\top\right) = \frac{n^2}{n^2-1} \cdot \left(I - \frac{2}{n+1} cc^\top\right) = A \ ,$$

and $\overline{a} = \frac{1}{n+1} \cdot e_1$ gives $U\overline{a} = \frac{1}{n+1} \cdot Ue_1 = \frac{1}{n+1} \cdot c = a$. Thus, the expression in (1) is indeed equal to $E(a, A)$, as desired.

Problem 3: Volume of Standard Simplex

(a) Define $a_0 = (1, \ldots, 1)^\top \in \mathbb{R}^n$ and let $a_i = a_0 - \sum_{j=1}^{i} e_{\sigma(j)}$ for all $i \in [n]$. In particular, note that this definition implies $a_n = (0, \ldots, 0)^\top \in \mathbb{R}^n$. We claim that $\mathrm{conv}(\{a_0, a_1, \ldots, a_n\}) = \Delta(\sigma)$. As the points $a_0, a_1, \ldots, a_n$ are affinely independent, this implies that $\Delta(\sigma)$ is a simplex.

To see that $\Delta(\sigma) \subseteq \mathrm{conv}(\{a_0, a_1, \ldots, a_n\})$, note that any $x \in \Delta(\sigma)$ can be written in the form

$$x = \sum_{j=0}^{n} \left(x_{\sigma(j+1)} - x_{\sigma(j)}\right) a_j \ ,$$

where we define $x_{\sigma(0)} = 0$ and $x_{\sigma(n+1)} = 1$. This is indeed a convex combination of the points $a_0, \ldots, a_n$ because all coefficients are non-negative (we have $0 \leq x_{\sigma(j)} \leq x_{\sigma(j+1)} \leq 1$ as $x \in \Delta(\sigma)$ by assumption) and their sum equals 1.

For the other inclusion, note that $\Delta(\sigma)$ is a polyhedron described by linear inequalities of the type $x_{\sigma(j)} - x_{\sigma(j+1)} \leq 0$, so in particular, $\Delta(\sigma)$ is convex. Hence, it suffices to check that $a_j \in \Delta(\sigma)$ for $j \in \{0, 1, \ldots, n\}$, which is immediate.

(b) Let $x \in B$. Consider the permutation σ_x defined by letting $\sigma_x(i)$ be the position of the i^{th} coordinate in x after sorting them increasingly (with ties broken arbitrarily). It is easy to see from the definition that $x \in \Delta(\sigma_x)$.

(c) Let σ_1, σ_2 be arbitrary distinct permutations. Let $i, j \in [n]$ be two indices with the property that $\sigma_1^{-1}(i) < \sigma_1^{-1}(j)$ and $\sigma_2^{-1}(i) > \sigma_2^{-1}(j)$. In particular, this implies that $i \neq j$. Such two indices must exist if the permutations are different. Let $x \in B$ such that $x \in \Delta(\sigma_1) \cap \Delta(\sigma_2)$. From the former two conditions it now follows that $x_i \leq x_j$ and $x_j \leq x_i$, i.e., $x_i = x_j$. We conclude that all $x \in B$ contained in both simplices must lie on the hyperplane $H = \{z \in \mathbb{R}^n \mid z_i = z_j\}$, implying that $\Delta(\sigma_1) \cap \Delta(\sigma_2)$ is not full-dimensional. This, in turn, implies that $\mathrm{vol}(\Delta(\sigma_1) \cap \Delta(\sigma_2)) = 0$.

(d) Let $a_0, a_1, \ldots, a_n \in \Delta(\sigma)$ be defined as in part (a). Consider the $n \times n$ matrix A whose j^{th} column is a_{j-1}. Clearly, A is non-singular, and $Ae_j = a_{j-1}$ for all $j \in [n]$. It follows that A is a linear transformation transforming the standard simplex Δ into $\Delta(\sigma)$. To prove that $\mathrm{vol}(\Delta) = \mathrm{vol}(\Delta(\sigma))$, it remains to prove that $|\det A| = 1$. To this end, notice that one can permute the rows of A to arrive at an upper-triangular matrix with all diagonal entries equal to 1, which implies that indeed, $|\det A| = 1$.

(e) We note that $\mathrm{vol}(B) = 1$ and that there are exactly $n!$ permutations of $[n]$, and thus $n!$ many different simplices $\Delta(\sigma)$. They are all pairwise disjoint in their interior by (c), and they cover B by (b). Since all these simplices have the same volume $\mathrm{vol}(\Delta)$ by (d), we conclude that

$$\mathrm{vol}(\Delta) = \frac{1}{n!} \ .$$