**Proposition 1.36**

If $C \subseteq \mathbb{R}^n$ is a non-empty polyhedral cone, then

$$C = \{x \in \mathbb{R}^n : Ax \leq 0\} \ , \tag{1.7}$$

for some matrix $A \in \mathbb{R}^{m \times n}$, where $m \in \mathbb{Z}_{\geq 0}$. Vice-versa, any set $C$ with a description as in (1.7) is a polyhedral cone.

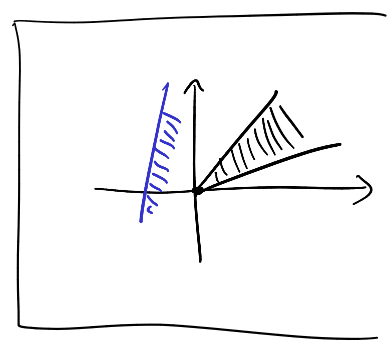
## Proof

Any set $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$ clearly is a polyhedral cone.

$x \in C, \lambda \geq 0 \qquad A(\lambda x) = \lambda \underbrace{Ax}_{\leq 0} \leq 0$

$x \in C$ $\qquad$ $\lambda \geq 0$

Conversely, let $C \subseteq \mathbb{R}^n$ be a polyhedral cone.



$C$ is polyhedron $\Rightarrow C = \{x \in \mathbb{R}^n : Ax \leq b\}$

wlog assume that all inequalities $Ax \leq b$ are non-redundant.

<u>We will show: $b = 0$</u>

$0 \in C \Rightarrow A \cdot 0 \leq b \Rightarrow 0 \leq b.$

Assume by sake of contradiction that $\exists$ inequality $a^T x \leq \beta$ in $Ax \leq b$ with $\beta > 0$.

$a^T x \leq \beta$ is not redundant $\Rightarrow \exists\, y \in C$ s.t. $a^T y = \beta$

$C$ is a cone $\Rightarrow 2y \in C \Rightarrow \underbrace{a^T(2y)}_{2\beta} \leq \beta$ $\lightning$

\#

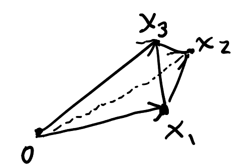
**Proposition 1.37**

If $C \subseteq \mathbb{R}^n$ is a polyhedral cone, then

$$C = \left\{ \sum_{i=1}^{k} \lambda_i x_i \colon \lambda_i \geq 0 \ \forall i \in [k] \right\} \quad , \tag{1.8}$$

for some finite set of points $x_1, \ldots, x_k \in \mathbb{R}^n$. The points $x_1, \ldots, x_k$ are called a *set of generators* of $C$. Vice-versa, any set $C$ as described in (1.8) is a polyhedral cone.

See problem sets for proof.




**Proposition 1.38**

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then

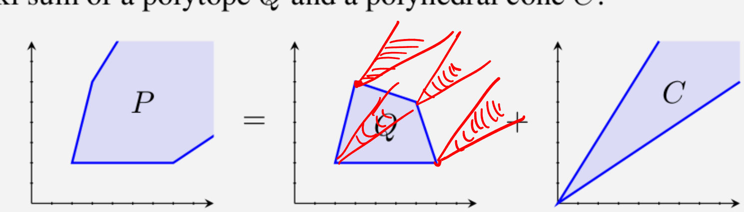
$$P = Q + C \ ,$$

where $Q \subseteq \mathbb{R}^n$ is a polytope and $C \subseteq \mathbb{R}^n$ is a polyhedral cone. Vice-versa, the Minkowski sum of a polytope and a polyhedral cone is always a polyhedron.

See problem sets for proof.

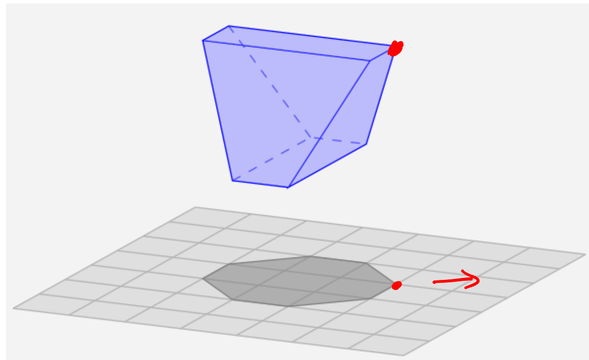
**Example 1.39**

The graphic below shows an unbounded 2-dimensional polyhedron $P$ and how it can be written as the Minkowski sum of a polytope $Q$ and a polyhedral cone $C$.



**Proposition 1.40**

An affine image of a polyhedron is a polyhedron, i.e., for any polyhedron $P \subseteq \mathbb{R}^n$ and any affine function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$, the set $\varphi(P) := \{\varphi(x) : x \in P\}$ is a polyhedron.



Proof

We first observe that it suffices to prove the statement for linear functions $\varphi: \mathbb{R}^n \to \mathbb{R}^m$. Indeed, assume statement holds for linear functions.

If $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ is affine, then $\varphi(x) = \phi(x) + t$, where $\phi: \mathbb{R}^n \to \mathbb{R}^m$ is linear and $t \in \mathbb{R}^m$.

By assumption, this is a polyhedron, i.e., $\phi(P) = \{x \in \mathbb{R}^m : Ax \leq b\}$

$$
\begin{aligned}
\varphi(P) &= \boxed{\phi(P)} + t \\
&= \{x + t : x \in \mathbb{R}^m,\ Ax \leq b\} \\
&= \{y \in \mathbb{R}^m : A(y - t) \leq b\} \qquad y = x + t \\
&= \underbrace{\{y \in \mathbb{R}^m : Ay \leq b + At\}}
\end{aligned}
$$

This is finite intersection of half-spaces

$\Rightarrow$ it is a polyhedron.

Hence, assume $\ell$ is linear.

By Proposition 1.38,

$$\underset{\substack{\nearrow \\ \text{polytope}}}{P = Q} + \underset{\substack{\nwarrow \\ \text{polyhedral cone}}}{C}$$

$$\begin{aligned}
\ell(P) &= \ell(Q + C) \\
&= \{\ell(q+c) : q \in Q,\ c \in C\} \\
&\overset{\ell \text{ linear}}{=} \{\ell(q) + \ell(c) : q \in Q,\ c \in C\} \\
&= \ell(Q) + \ell(C)
\end{aligned}$$

Plan: We finish proof by showing that

(a) $\ell(Q)$ is polytope

(b) $\ell(C)$ is polyhedral cone

$\xrightarrow{\text{Proposition 1.38}}$ $\overbrace{\ell(Q) + \ell(C)}^{= \ell(P)}$ is polyhedron

<u>(a) $\ell(Q)$ is a polytope</u>

By Proposition 1.32 : $Q$ polytope $\Rightarrow$ $Q = \mathrm{conv}(\mathrm{vertices}(Q))$

Let $\{q_1, q_2, \ldots, q_k\} = \mathrm{vertices}(Q)$

$$\begin{aligned}
\ell(Q) &= \{\ell(x) : x \in \mathrm{conv}(\mathrm{vertices}(Q))\} \\
&= \left\{ \ell\left( \sum_{i=1}^{k} \lambda_i q_i \right) : \lambda \in \mathbb{R}^k_{\geq 0},\ \sum_{i=1}^{k} \lambda_i = 1 \right\}
\end{aligned}$$

$$\overset{\ell \text{ linear}}{=} \left\{ \sum_{i=1}^{k} \lambda_i \, \ell(q_i) : \lambda \in \mathbb{R}^k_{\geq 0}, \ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

$$= \operatorname{conv}\left(\{\ell(q_1), \ell(q_2), \dots, \ell(q_k)\}\right)$$

$\Rightarrow$ $\ell(Q)$ is convex hull of finitely many points.

$\xRightarrow{\text{Proposition 1.32}}$ $\ell(Q)$ is a polytope.

<u>(b) $\ell(C)$ is a polyhedral cone</u>

By Proposition 1.37 : $\exists \, x_1, \dots, x_\ell \in \mathbb{R}^n$ s.t.

$$C = \left\{ \sum_{i=1}^{\ell} \lambda_i x_i : \lambda \in \mathbb{R}^\ell_{\geq 0} \right\}$$

$$\ell(C) = \left\{ \ell\left(\sum_{i=1}^{\ell} \lambda_i x_i\right) : \lambda \in \mathbb{R}^\ell_{\geq 0} \right\}$$

$$\overset{\ell \text{ linear}}{=} \left\{ \sum_{i=1}^{\ell} \lambda_i \, \ell(x_i) : \lambda \in \mathbb{R}^\ell_{\geq 0} \right\}$$

$\xRightarrow{\text{Proposition 1.37}}$ $\ell(C)$ is a polyhedral cone.

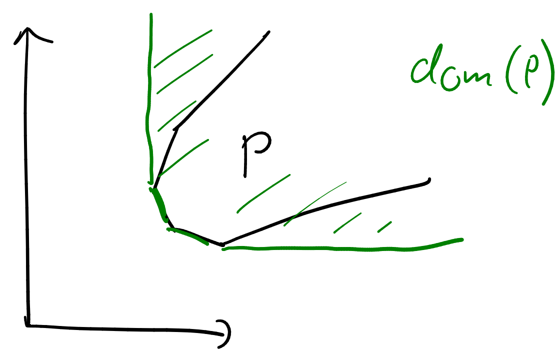
$\#$

**Proposition 1.42**

The dominant of a polyhedron is a polyhedron.

Proof

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.




$$\mathrm{dom}(P) = P + \mathbb{R}^n_{\geq 0}$$

$$\xRightarrow{\text{Proposition 1.38}} \quad P = \underbrace{Q}_{\text{polytope}} + \underbrace{C}_{\text{polyhedral cone}}$$

$$\mathrm{dom}(P) = (Q + C) + \mathbb{R}^n_{\geq 0} = Q + (C + \mathbb{R}^n_{\geq 0})$$

Plan: Show that $C + \mathbb{R}^n_{\geq 0}$ is a polyhedral cone.

Then, Proposition 1.38 implies that $\mathrm{dom}(P)$ is a polyhedron.

$$\xRightarrow{\text{Proposition 1.37}} \exists\, x_1, \dots, x_k \in \mathbb{R}^n \quad \text{s.t.}$$

$$C = \left\{ \sum_{j=1}^{k} \mu_j x_j \;:\; \mu \in \mathbb{R}^k_{\geq 0} \right\}$$

$$\Rightarrow C + \mathbb{R}^n_{\geq 0} = \left\{ \sum_{j=1}^{k} \mu_j x_j \;:\; \mu \in \mathbb{R}^k_{\geq 0} \right\} + \left\{ \sum_{i=1}^{n} \lambda_i e_i \;:\; \lambda \in \mathbb{R}^n_{\geq 0} \right\}$$

i-th unit vector in $\mathbb{R}^n$

$$= \left\{ \sum_{j=1}^{k} \mu_j x_j + \sum_{i=1}^{n} \lambda_i e_i \;:\; \mu \in \mathbb{R}^k_{\geq 0},\ \lambda \in \mathbb{R}^n_{\geq 0} \right\}$$

is a polyhedral cone due to Proposition 1.37.

#

### 1.2.3 Convex separation theorems

**Definition 1.43: (Strictly) separating hyperplanes**

Let $Y, Z \subseteq \mathbb{R}^n$ be two sets. A hyperplane $H = \{x \in \mathbb{R}^n : a^\top x = \beta\}$ is called a $(Y, Z)$-*separating hyperplane*, or simply *separating hyperplane*, if $Y$ is contained in one of the half-spaces defined by $H$ and $Z$ in the other one, i.e., either

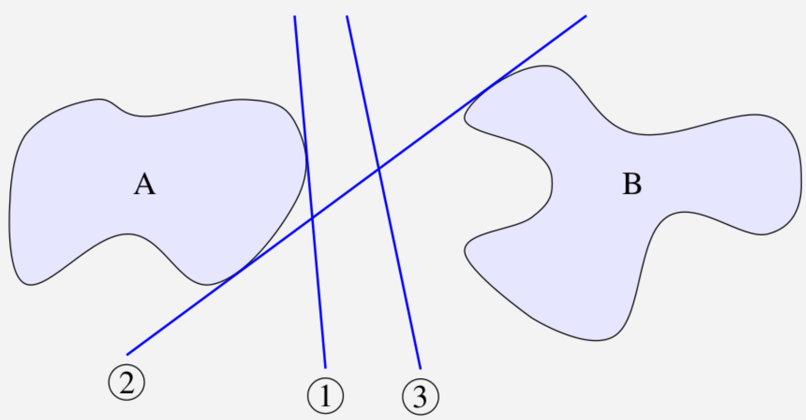
$$\begin{aligned} a^\top y \leq \beta \leq a^\top z \qquad & \forall y \in Y, z \in Z \text{ , or} \\ a^\top y \geq \beta \geq a^\top z \qquad & \forall y \in Y, z \in Z \text{ .} \end{aligned}$$

The hyperplane is called *strictly* $(Y, Z)$*-separating*, or simply *strictly separating*, if the above inequalities are strict.

For $Y = \{y\}$, we also use $(y, Z)$-separating for $(\{y\}, Z)$-separating.

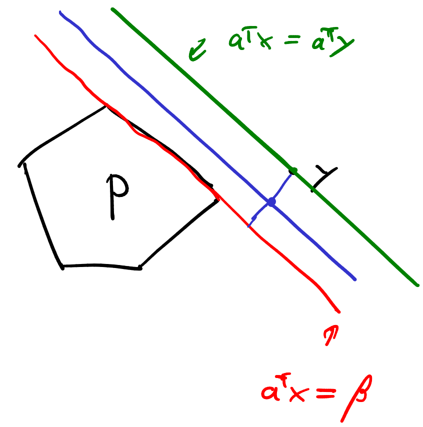
**Example 1.44: Separating two sets**

The illustration below shows two sets $A, B \subseteq \mathbb{R}^2$ together with three separating hyperplanes. Hyperplane 3 is strictly separating the sets whereas hyperplanes 1 and 2 do not separate $A$ and $B$ in a strict sense.



**Theorem 1.45: Separating a point from a polyhedron**

Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $y \in \mathbb{R}^n \setminus P$. Then there is a strictly $(y, P)$-separating hyperplane.




Proof

$P$ polyhedron $\Rightarrow P = \{x \in \mathbb{R}^n : Ax \leq b\}$

$y \in P \setminus \mathbb{R}^n \Rightarrow \exists$ constraint $a^T x \leq \beta$ in $Ax \leq b$ s.t.

$$\boxed{a^T y > \beta}$$

Then, the following hyperplan is strictly $(y, P)$-separating:

$$\{x \in \mathbb{R}^n : a^T x = \tfrac{1}{2}(\beta + a^T y)\}$$

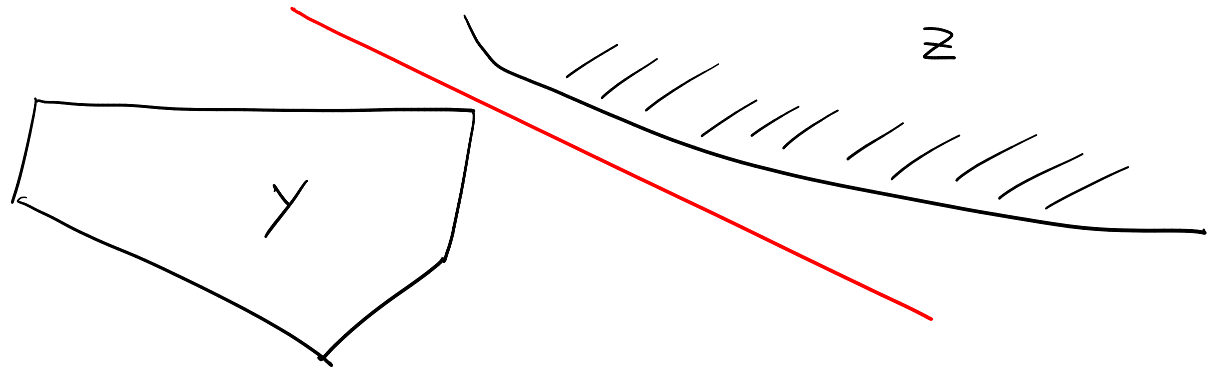
Indeed, for $x \in P$ $\quad a^T x \leq \beta < \tfrac{1}{2}(\beta + a^T y)$.

Moreover, $\quad a^T y > \tfrac{1}{2}(\beta + a^T y)$

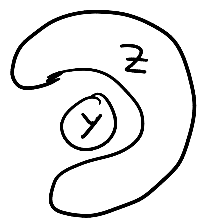
\#

**Theorem 1.47**

Let $Y, Z \subseteq \mathbb{R}^n$ be two disjoint closed convex sets with at least one of them being compact, then there exists a strictly $(Y, Z)$-separating hyperplane.

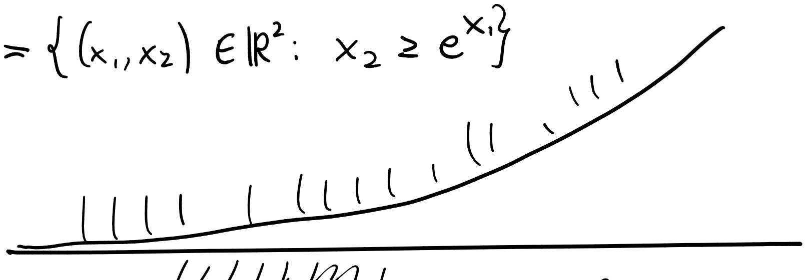



This fails if one set is not convex.




It also fails if none of the sets is compact.

$Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq e^{x_1}\}$



$Z = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$