Let's take another look at Example 1.88.

Example 1.88

primal

$$
\begin{array}{lrcrcrcrcl}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & = & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \le & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & = & 3 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \ge & 0
\end{array}
$$

$\bar{x}_1 = 0$ $\quad$ $\bar{x}_3 = 0$

dual

$$
\begin{array}{lrcrcrcl}
\min & y_1 & + & 55y_2 & + & 3y_3 & & \\
 & y_1 & + & 5y_2 & - & y_3 & \ge & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \ge & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \ge & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \ge & 3 \\
 & & & & & y_1, y_2, y_3, y_4 & \ge & 0
\end{array}
$$

$\xrightarrow[\text{tableau}]{\text{optimal}}$

	$x_1\,(y_1^s)$	$x_3\,(y_3^s)$	$x_1^s\,(y_1)$	$x_3^s\,(y_3)$	$1\,(w)$
$(1)\ z$	1	-2	-11	6	29
$(y_2^s)\ x_2$	2	4	5	3	14
$(y_4^s)\ x_4$	1	1	2	1	-5
$(y_2)\ x_2^s$	-5	-9	-21	-11	1

optimal primal solution : $(x_1, x_2, x_3, x_4) = (0, 14, 0, 5)$

optimal dual solution : $(y_1, y_2, y_3) = (11, 0, 6)$

primal objective value $= 4\cdot 0 + 1\cdot 14 + 5\cdot 0 + 3\cdot 5 = 29$

dual objective value $= 1\cdot 11 + 55\cdot 0 + 3\cdot 6 = 29$

1.4.5 Complementary slackness

Theorem 1.90: Complementary slackness theorem

Consider a pair of primal and dual linear programs with finite optima:

$$\begin{array}{rcl} \max\ c^\top x & & \\ Ax & \le & b \\ x & \ge & 0 \end{array} \qquad\qquad \begin{array}{rcl} \min\ b^\top y & & \\ A^\top y & \ge & c \\ y & \ge & 0\ . \end{array}$$

Let $\overline{x}$ be a feasible primal solution and $\overline{y}$ a feasible dual solution. Then both $\overline{x}$ and $\overline{y}$ are optimal solutions (for the primal and dual, respectively) if and only if

(i) $(b - A\overline{x})^\top \overline{y} = 0$, and

(ii) $(A^\top \overline{y} - c)^\top \overline{x} = 0$.

Proof

$$\begin{array}{c} A^\top \bar{y} \ge c \\ \bar{x} \ge 0 \end{array} \qquad\qquad \begin{array}{c} A\bar{x} \le b \\ \bar{y} \ge 0 \end{array}$$

$$c^\top \bar{x} \le \bar{y}^\top A \bar{x} \le \bar{y}^\top b = b^\top \bar{y}$$

strong duality

above chain of inequalities is satisfied with equality

Both $\bar{x}$ and $\bar{y}$ are optimal $\iff c^\top \bar{x} = b^\top \bar{y} \iff c^\top \bar{x} = \bar{y}^\top A \bar{x}$, and $\bar{y}^\top A \bar{x} = \bar{y}^\top b$

$$\iff \begin{array}{c} (A^\top \bar{y} - c)^\top \bar{x} = 0 \\ (b - A\bar{x})^\top \bar{y} = 0 \end{array}$$

#

Notice that the complementary slackness relations (i) & (ii) hold for any pair of basic primal/dual solution that are obtained from same tableau.

	$x_1\,(y_1^s)$	$x_3\,(y_3^s)$	$x_1^s\,(y_1)$	$x_3^s\,(y_3)$	$1\,(w)$
$(1)\,z$	1	2	11	6	29
$(y_2^s)\,x_2$	2	4	5	3	14
$(y_4^s)\,x_4$	1	1	2	1	5
$(y_2)\,x_2^s$	-5	-9	-21	-11	1