5.3 Total unimodularity

One way to prove that $P = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ is integral, is by proving properties about constraint matrix A.

One strong and influential property: total unimodularity.

5.3.1 Definition and basic properties

Definition 5.3

A matrix is *totally unimodular* (TU) if the determinant of any square submatrix of it is either 0, 1, or -1.

example:

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Remark 5.4

$A \in \mathbb{R}^{m\times n}$ is TU $\Rightarrow A \in \{-1, 0, 1\}^{m\times n}$.

Remark 5.5

A is TU $\Leftrightarrow$ $A^\top$ is TU.

Remark 5.6

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A \ -A]$, i.e., the $\mathbb{R}^{m\times 2n}$ matrix obtained by appending the columns of $-A$ to the columns of A.

Remark 5.7

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A \ I]$, i.e., the $\mathbb{R}^{m\times 2n}$ matrix obtained by appending the columns of an $m \times m$ identity matrix I to the columns of A.

5.3.2 Integrality of polyhedra with TU constraint matrices

Theorem 5.8

Let $A \in \mathbb{Z}^{m \times n}$. Then,

$$A \text{ is TU} \quad \Leftrightarrow \quad P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \text{ is integral } \forall\, b \in \mathbb{Z}^m.$$

Proof

5.3.3 Characterization of Ghouila-Houri

Theorem 5.9: Characterization of Ghouila-Houri

A matrix $A \in \mathbb{R}^{m \times n}$ is TU if and only if for every subset of the rows $R \subseteq [m]$, there is a partition $R = R_1 \dot\cup R_2$ such that

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\} \quad \forall j \in [n] \ . \tag{5.6}$$

Proof

Remark 5.10

Because A is TU if and only if $A^\top$ is TU, one can exchange the roles of rows and columns in Theorem 5.9.

Example 5.11

Consecutive-ones matrices are TU.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$