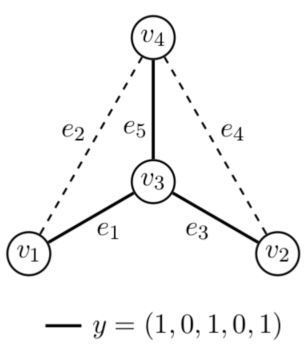
# 5.8 Combinatorial Uncrossing

Main goal :

> Given a heavily overdetermined linear system that uniquely defines a point, find a well-structured full-rank subsystem.

## 5.8.1 Integrality of spanning tree polytope

$$P = \left\{ x \in \mathbb{R}^{E}_{\geq 0} : \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall\, S \subsetneq V,\ |S| \geq 2 \end{array} \right\}$$




spanning tree constraints:

$$\begin{array}{rc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{r} \{v_1,v_2\} \\ \{v_1,v_3\} \\ \{v_1,v_4\} \\ \{v_2,v_3\} \\ \{v_2,v_4\} \\ \{v_3,v_4\} \\ \{v_1,v_2,v_3\} \\ \{v_1,v_2,v_4\} \\ \{v_1,v_3,v_4\} \\ \{v_2,v_3,v_4\} \\ \{v_1,v_2,v_3,v_4\} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} y \overset{=}{\leq} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix} \end{array}$$

non-negativity constraints:

$$\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} y \overset{=}{\geq} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

## Proof of integrality of $P$

**Lemma 5.23**

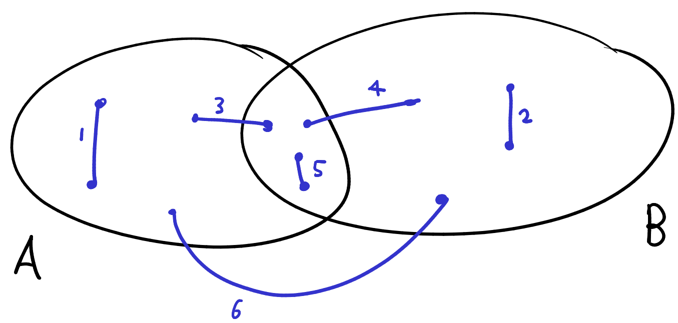
For any sets $A, B \subseteq V$, we have

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A \setminus B, B \setminus A)} = \chi^{E[A \cup B]} + \chi^{E[A \cap B]} ,$$

which implies

$$\chi^{E[A]} + \chi^{E[B]} \leq \chi^{E[A \cup B]} + \chi^{E[A \cap B]} .$$

Proof




→ 6 "edge types" in $E[A \cup B]$.

For each edge type, contribution to lhs and rhs is same.

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A \setminus B, B \setminus A)} \qquad \chi^{E[A \cup B]} + \chi^{E[A \cap B]}$$

| | $\chi^{E[A]}$ | + | $\chi^{E[B]}$ | + | $\chi^{E(A \setminus B, B \setminus A)}$ | = | $\chi^{E[A \cup B]}$ | + | $\chi^{E[A \cap B]}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | 0 | | 0 | | 1 | | 0 |
| 2 | 0 | | 1 | | 0 | | 1 | | 0 |
| 3 | 1 | | 0 | | 0 | | 1 | | 0 |
| 4 | 0 | | 1 | | 0 | | 1 | | 0 |
| 5 | 1 | | 1 | | 0 | | 1 | | 1 |
| 6 | 0 | | 0 | | 1 | | 1 | | 0 |

#

**Lemma 5.24**

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{F}$ and $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$. In particular, this implies by Lemma 5.23

$$\chi^{E[S_1]} + \chi^{E[S_2]} = \chi^{E[S_1 \cup S_2]} + \chi^{E[S_1 \cap S_2]} \ .$$

Proof

Back to : Each equality in $*$ is implied by $\square$.