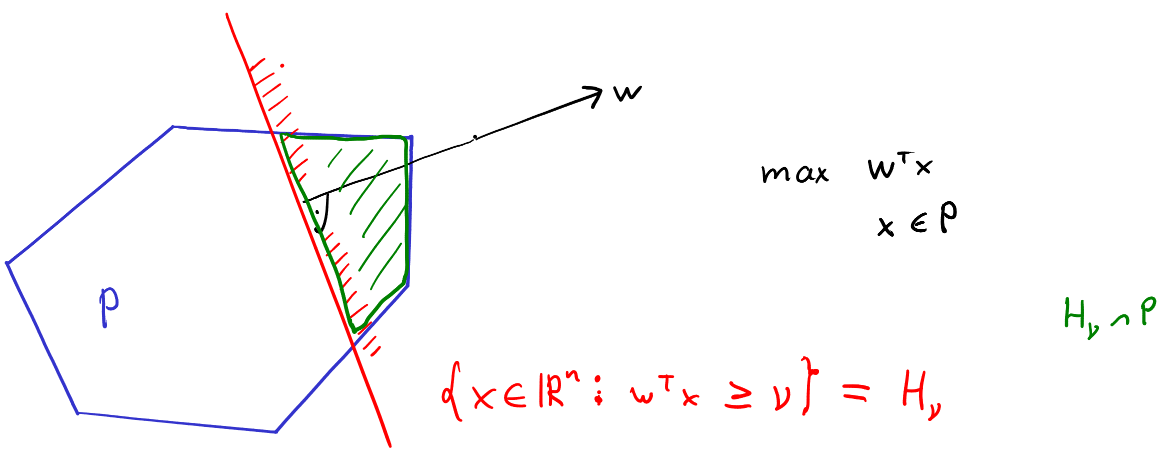
6.4 Ellipsoid Method for finding point in full-dimensional $\{0,1\}$-polytope

We start with simpler question (checking feasibility):

Given a separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, find a point $x \in P$.

Checking feasibility is closely related to optimization



$$\max \; w^T x$$
$$x \in P$$

Basics on ellipsoids

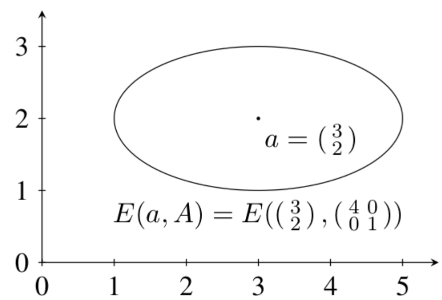
Definition 6.3: Ellipsoid

An ellipsoid in $\mathbb{R}^n$ is a set

$$E(a, A) := \{x \in \mathbb{R}^n \colon (x-a)^\top A^{-1}(x-a) \leq 1\} \ ,$$

where $a \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a positive definite matrix. The point a is called the *center* of the ellipsoid $E(a, A)$.

this implies that A is symmetric



$$\boxed{x^\top A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}}$$

Equivalently, an ellipsoid is the image of the unit ball under an affine bijection:

$A \in \mathbb{R}^{n\times n}$ positive definite $\iff$ $A = QQ^\top$ for some full-rank matrix $Q \in \mathbb{R}^{n \times n}$.

$$\boxed{\begin{aligned} A^{-1} &= (QQ^\top)^{-1} \\ &= (Q^\top)^{-1} Q^{-1} \end{aligned}}$$

$$\begin{aligned}
E(a, A) &= \{x \in \mathbb{R}^n : (x-a)^\top A^{-1}(x-a) \leq 1\} \\
&= \{x \in \mathbb{R}^n : (x-a)^\top (Q^{-1})^\top Q^{-1}(x-a) \leq 1\} \\
&= \{x \in \mathbb{R}^n : \|Q^{-1}(x-a)\|_2^2 \leq 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n, \|y\|_2^2 \leq 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n, \|y\|_2 \leq 1\}
\end{aligned}$$

$y = Q^{-1}(x-a)$

$x = Qy + a$

6.4.1 (High-level) description of Ellipsoid Method

Algorithm 8: Ellipsoid Method

Input : Separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, and an ellipsoid $E_0 = E(a_0, A_0)$ with $P \subseteq E_0$.

Output: A point $y \in P$.

$i = 0$.

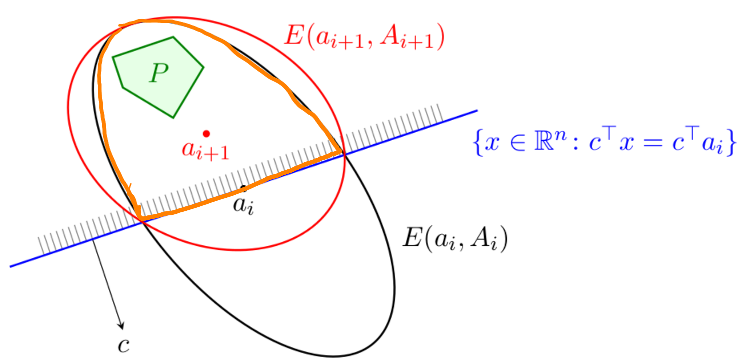
while $a_i \notin P$ *(checked with separation oracle)* **do**

  Get $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top a_i\}$, using separation oracle.

  Find min. volume ellipsoid $E_{i+1} = E(a_{i+1}, A_{i+1})$ containing $E_i \cap \{x \in \mathbb{R}^n : c^\top x \leq c^\top a_i\}$.

  $i = i + 1$.

return a_i.



Two key questions:

- (How quickly) does the Ellipsoid Method terminate?
- How to compute $E_{i+1} = E(a_{i+1}, A_{i+1})$?

6.4.2 Getting a bound on the number of iterations

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} \ .$$

Before proving Lemma 6.4, we show that it implies following bound on number of iterations.

Lemma 6.5

The Ellipsoid Method will stop after at most $2(n+1)\ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right)$ iterations.

Proof

Let $L \in \mathbb{Z}_{\geq 0}$ be last iteration of Ellipsoid Method, i.e., value of i when it terminates.

$$P \subseteq E_L \implies \mathrm{vol}(P) \leq \mathrm{vol}(E_L) \overset{\text{Lemma 6.4}}{\leq} \mathrm{vol}(E_0) \cdot e^{-\frac{L}{2(n+1)}}$$

$$\implies L \leq 2(n+1) \ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right) .$$

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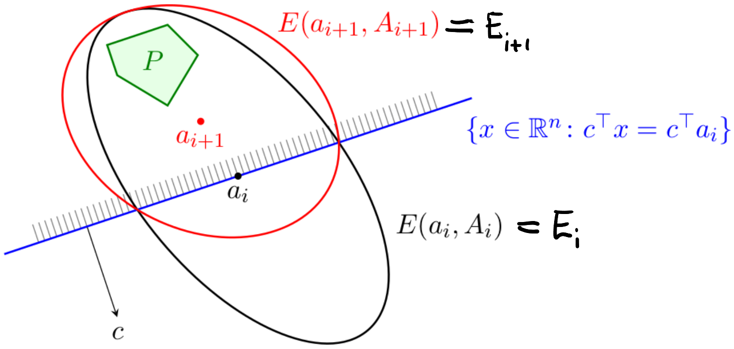
Proof of Lemma 6.4 and explicit description for E_{i+1}



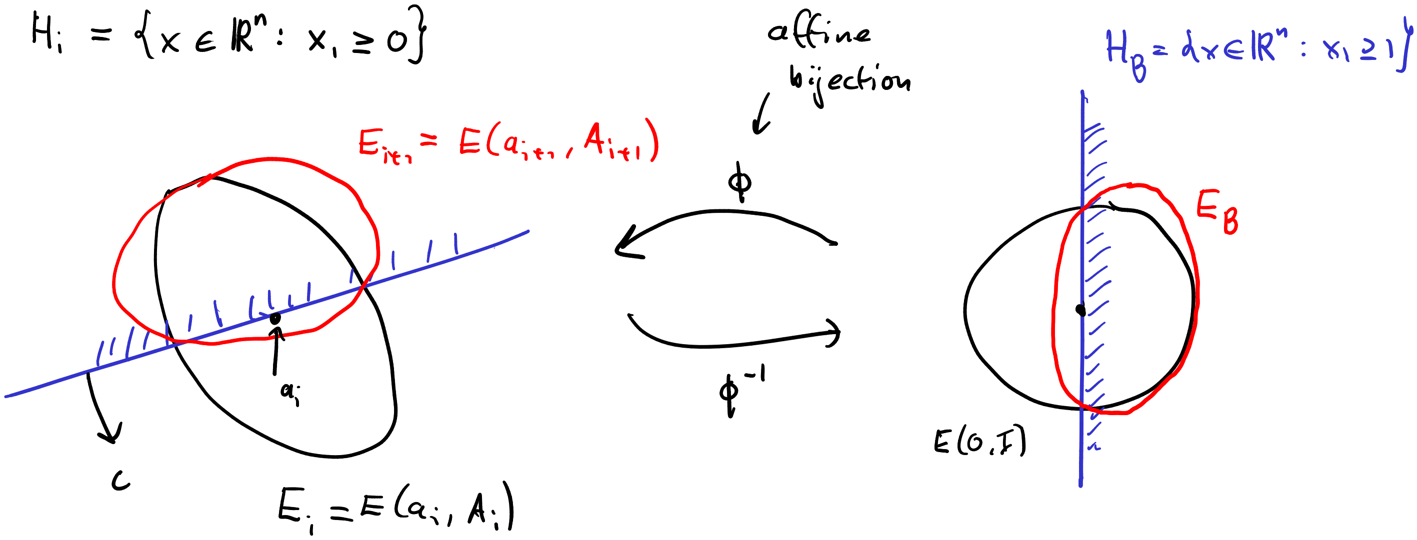
$H_i = \{x \in \mathbb{R}^n : c^T x \leq c^T a_i\}$

What is ratio between $\mathrm{vol}(E_{i+1})$ and $\mathrm{vol}(E_i)$?

This question can be reduced to the case:

$$E_i = E(0, I)$$

$$H_i = \{x \in \mathbb{R}^n : x_1 \geq 0\}$$



$H_i = \{x \in \mathbb{R}^n : c^T x \leq c^T a_i\}$