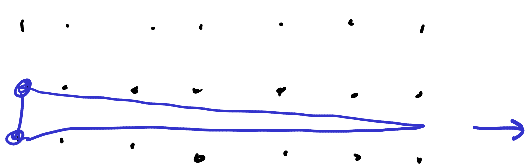
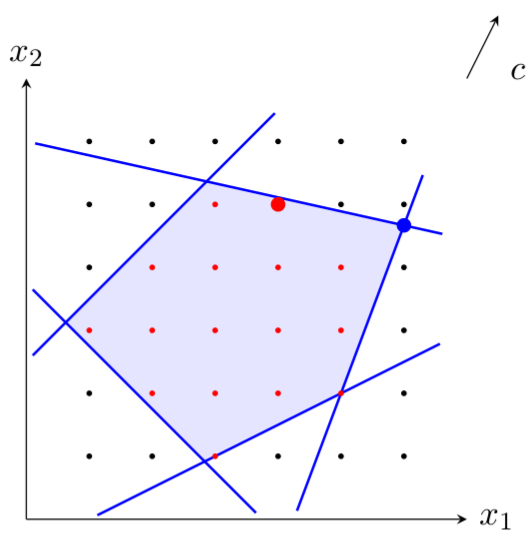
8 Integer Programming

8.1 Introduction to integer programming

Integer programs (IPs) are defined analogously to linear programs with the only difference that all variables take integer values.

$$\max \; c^T x$$
$$Ax \le b$$
$$x \in \mathbb{Z}^n \qquad \left(\mathbb{Z}^n_{\ge 0}\right)$$



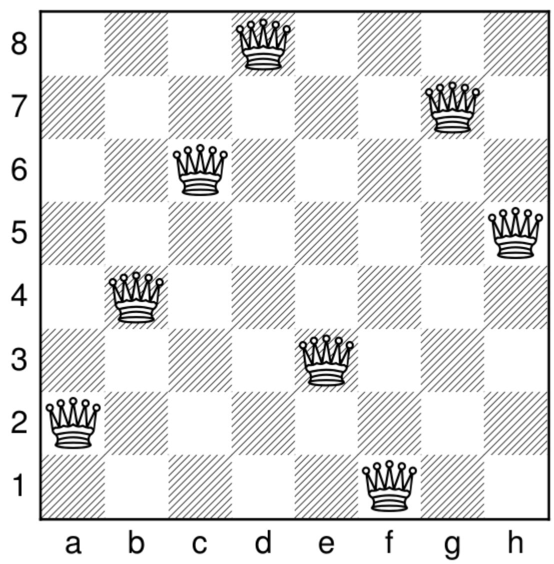


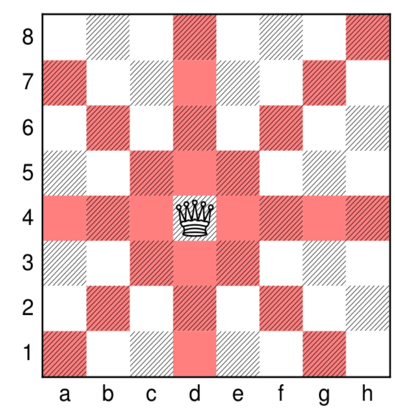
$$c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{lrcrcr}
\max & x_1 & + & 2x_2 & & \\
\text{s.t.} & 2x_1 & + & 9x_2 & \le & 54 \\
 & -2x_1 & + & 2x_2 & \le & 5 \\
 & 4x_1 & + & 4x_2 & \ge & 15 \\
 & x_1 & - & 2x_2 & \le & 1 \\
 & 8x_1 & - & 3x_2 & \le & 34
\end{array}$$

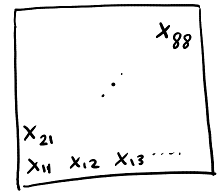
IP example: the eight queens puzzle

Goal: Place 8 queens on a chessboard s.t. no 2 queens threaten each other.





Modeling the problem as an IP



→ Introduce 64 (binary) variables $\{x_{ij}\}_{i\in[8],\, j\in[8]}$, one per square.

Interpretation: $x_{ij} = 1 \rightarrow$ There is a queen on square (i,j).

$x_{ij} = 0 \rightarrow$ " " no " " " ".

$$\max \sum_{i=1}^{8} \sum_{j=1}^{8} x_{ij}$$

≤ 1 queen per row ("rank")
$$\sum_{j=1}^{8} x_{ij} \leq 1 \qquad \forall\, i \in [8]$$

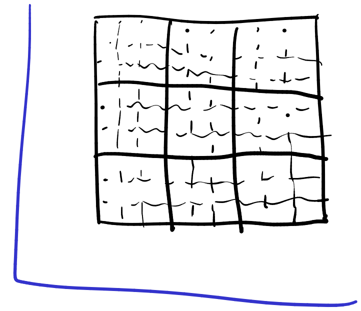
≤ 1 queen per column ("file")
$$\sum_{i=1}^{8} x_{ij} \leq 1 \qquad \forall\, j \in [8]$$

≤ 1 queen per - diagonal
$$\sum_{\substack{i,j\in[8] \\ i+j=k}} x_{ij} \leq 1 \qquad \forall\, k \in \{2, \dots, 16\}$$

≤ 1 queen per - diagonal
$$\sum_{\substack{i,j\in[8] \\ i-j=k}} x_{ij} \leq 1 \qquad \forall\, k \in \{-7, -6, \dots, 7\}$$

$$x \in \{0,1\}^{8\times 8}$$

8.2 Branch & bound

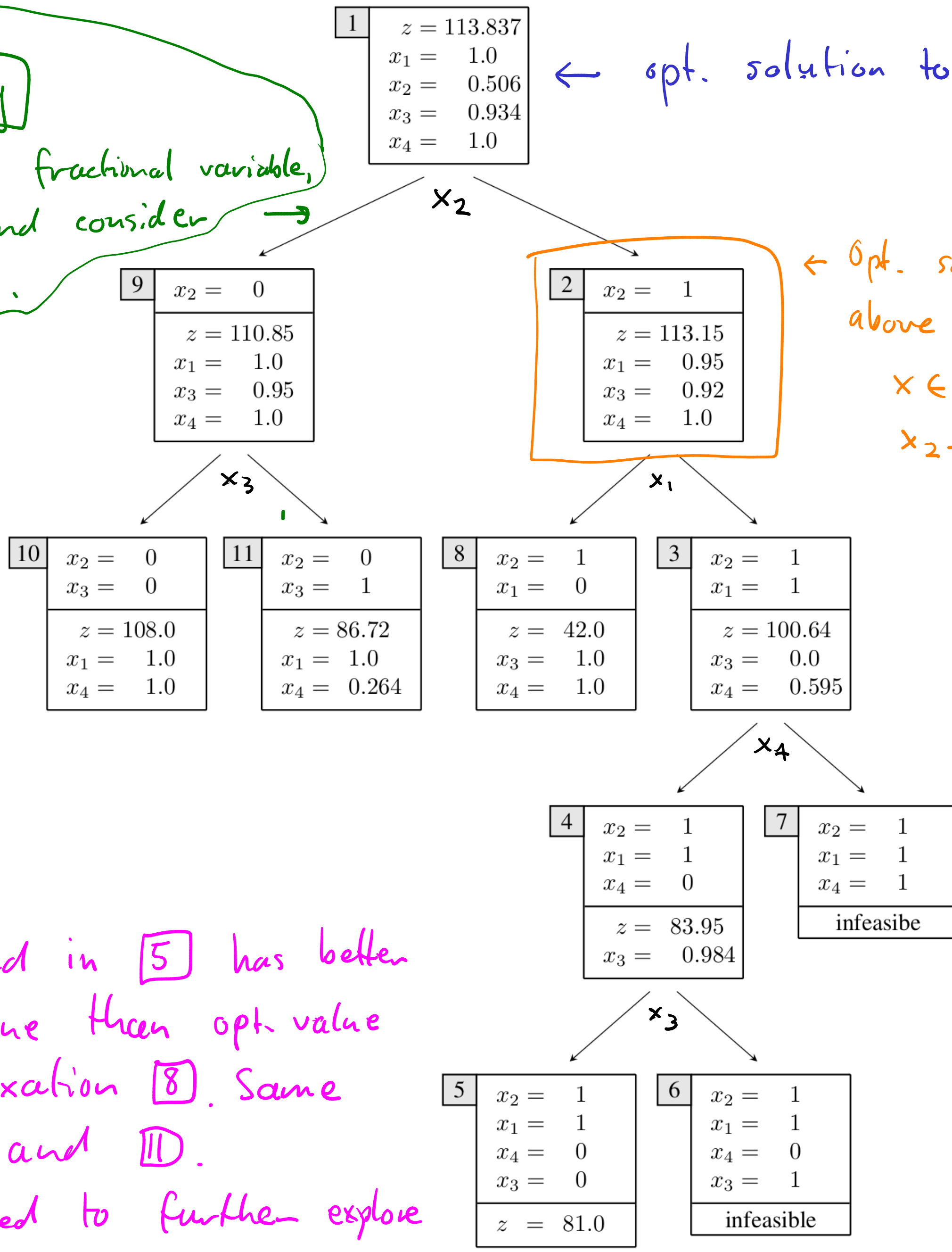


objective value → $z =$

$$
\begin{array}{rrrrrrrrrrr}
\max & 75x_1 & + & 6x_2 & + & 3x_3 & + & 33x_4 & & \\
 & 774x_1 & + & 76x_2 & + & 22x_3 & + & 42x_4 & \leq & 875 \\
 & 67x_1 & + & 27x_2 & + & 794x_3 & + & 53x_4 & \leq & 875 \\
 & & & & & & & x & \in & \cancel{\{0,1\}^4} \quad [0,1]^4
\end{array}
$$

Branching

Choose some fractional variable, say x_2, and consider → $x_2 = 0$, $x_2 = 1$.



← opt. solution to relaxation

← Opt. sol. to LP with above constraints and $x \in [0,1]^4$, and $x_2 = 1$.

Bounding

Solution found in $\boxed{5}$ has better objective value than opt. value of LP relaxation $\boxed{8}$. Same for $\boxed{10}$ and $\boxed{11}$.

$\Rightarrow$ No need to further explore $\boxed{8}$ or $\boxed{11}$.

Remarks on branch & bound

- Best solution found so far is called the <u>incumbent</u>.
- Different branching rules can be applied
  $\Rightarrow$ branch & bound tree is not unique.
- Speed-ups can often be achieved by employing fast techniques to quickly find some good incumbent at the start of the procedure.
- Speed-ups can often be achieved by using stronger relaxations.

8.3 Branch & cut

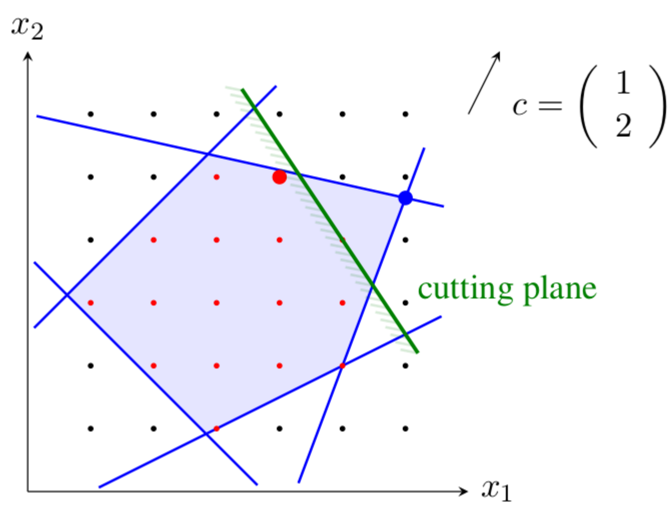
Enhancement of branch & bound using cutting planes.

Consider sub-problem in branch & bound procedure such that:

(i) Its LP relaxation is feasible.

(ii) Value of LP relaxation is strictly better than incumbent.

(iii) Obtained optimal LP solution is not integral.



Instead of branching, one can add a cutting plane to strengthen the LP.

(A cutting plane separates a given optimal and fractional LP solution from feasible solutions of considered problem.)

8.3.1 Generating a Chvátal-Gomory cut via simplex tableau

Consider again:

$$
\begin{array}{rrcrcrcrcl}
\max & 75x_1 & + & 6x_2 & + & 3x_3 & + & 33x_4 & & \\
 & 774x_1 & + & 76x_2 & + & 22x_3 & + & 42x_4 & \leq & 875 \\
 & 67x_1 & + & 27x_2 & + & 794x_3 & + & 53x_4 & \leq & 875 \\
 & & & & & & & x & \in & \{0,1\}^4
\end{array}
$$

Linear relaxation in canonical form:

$$
\begin{array}{rrcrcrcrcl}
\max & 75x_1 & + & 6x_2 & + & 3x_3 & + & 33x_4 & & \\
 & 774x_1 & + & 76x_2 & + & 22x_3 & + & 42x_4 & \leq & 875 \\
 & 67x_1 & + & 27x_2 & + & 794x_3 & + & 53x_4 & \leq & 875 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & & x_2 & & & & & \leq & 1 \\
 & & & & & x_3 & & & \leq & 1 \\
 & & & & & & & x_4 & \leq & 1 \\
 & & & & & & & x & \in & \mathbb{R}^4_{\geq 0}
\end{array}
$$

Starting tableau:

	x_1	x_2	x_3	x_4	1
z	-75	-6	-3	-33	0
y_1	774	76	22	42	875
y_2	67	27	794	53	875
y_3	1	0	0	0	1
y_4	0	1	0	0	1
y_5	0	0	1	0	1
y_6	0	0	0	1	1

Simplex phase II leads to optimal tableau:

	y_3	y_1	y_2	y_6	1
z	14.2289	0.0784	0.0016	29.623	113.8373
x_2	−10.2608	0.0133	−0.0004	−0.5386	0.506
x_3	0.2645	−0.0005	0.0013	−0.0484	0.9337
x_1	1	0	0	0	1
y_4	10.2608	−0.0133	0.0004	0.5386	0.494
y_5	−0.2645	0.0005	−0.0013	0.0484	0.0663
x_4	0	0	0	1	1

This corresponds to box $\boxed{1}$ of previous branch & bound tree.

$\boxed{1}$

$z = 113.837$
$x_1 = 1.0$
$x_2 = 0.506$
$x_3 = 0.934$
$x_4 = 1.0$

$\longrightarrow$ optimal basic solution: $(x_1^*, x_2^*, x_3^*, x_4^*) = (1, 0.506, 0.9337, 1)$.

Consider tableau row corresponding to one of the fractional variables, say x_2:

$$x_2 - 10.2608\, y_3 + 0.0133 y_1 - 0.0004 y_2 - 0.5386 y_6 = 0.506$$

$$\Rightarrow x_2 + \lfloor -10.2608 \rfloor y_3 + \lfloor 0.0133 \rfloor y_1 + \lfloor -0.0004 \rfloor y_2 + \lfloor -0.5386 \rfloor y_6 \leq 0.506$$

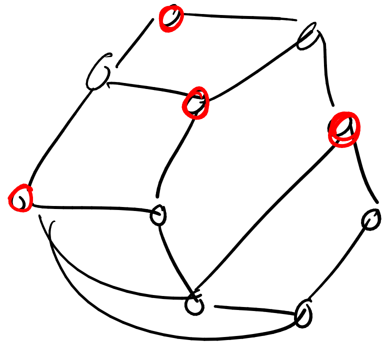
$$\rightarrow \quad x_2 - 11 y_3 - y_2 - y_6 \leq 0.506 .$$

Because all lhs coefficients are integral and all variables take integral values, we have that following constraint is valid for any feasible integral point:

$$(*) \qquad x_2 - 11 y_3 - 2 y_2 - y_6 \leq \lfloor 0.506 \rfloor = 0 .$$

→ This is a cutting plane for current optimal and fractional LP solution $(x_1^*, x_2^*, x_3^*, x_4^*)$.

→ Instead of branching, we can add $(*)$ as additional constraint and solve new LP relaxation.

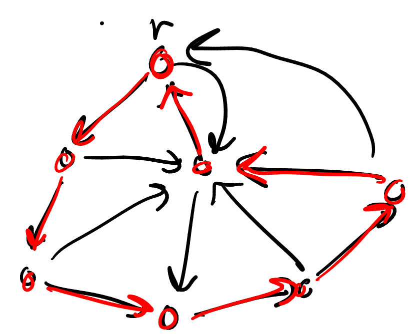


max card. indep. set.

$$\boxed{\begin{aligned} &\max \quad x(V) \\ &x_u + x_v \leq 1 \qquad \forall \{u,v\} \in E \\ &x \in \{0,1\}^V \end{aligned}}$$

(A) TSP

$G = (V, A) \qquad \ell : A \to \mathbb{R}_{\geq 0}$



$$\min \sum_{a \in A} x(a) \cdot \ell(a)$$

$$x(\delta^+(v)) = 1 \quad \forall v \in V$$

$$x(\delta^-(v)) = 1 \quad \forall v \in V$$

$$y_u - y_v + |V| \cdot x_{(u,v)} \leq |V| - 1 \quad \forall (u,v) \in A \setminus \delta^-(r)$$

$$x \in \{0,1\}^A$$

$$y \in \{1, \dots, |V|\}^V$$