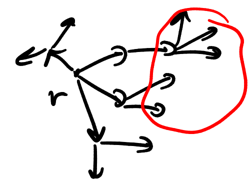
5.8.2 Integrality of dominant of r-arborescence polytope

Recall, for a directed graph $G=(V,A)$, we have:

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset\} \ .$$



Proof of integrality of P

$y \in$ vertices(P)

Wlog: $y(a) > 0 \quad \forall\ a \in A \quad \rightsquigarrow$ arcs $a \in A$ with $y(a)=0$ can be removed.

$$\mathcal{F} = \{S \subseteq V \setminus \{r\} : \ y(\delta^-(S)) = 1\}$$

describes y-tight constraints

$\Rightarrow$ y is unique sol. to

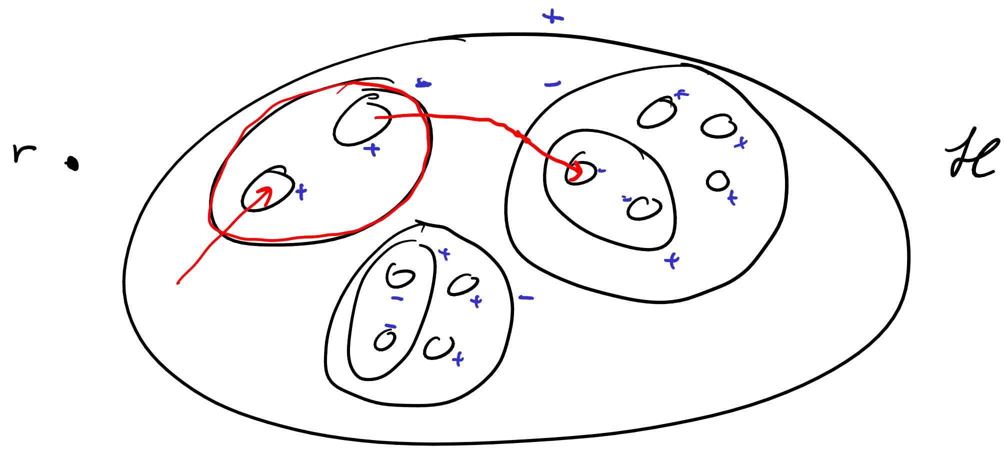
$$(*) \qquad x(\delta^-(S)) = 1 \quad \forall\ S \in \mathcal{F}$$

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$ and consider

$$(\square) \qquad x(\delta^-(H)) = 1 \qquad \forall\, H \in \mathcal{H}$$

Notice that $(\square)$ is TU system:

We can use Ghouila-Houri:



Remains to show that every equation of $(*)$ is implied by $(\square)$.

Lemma 5.27

For any two sets $S_1, S_2 \subseteq V$, we have

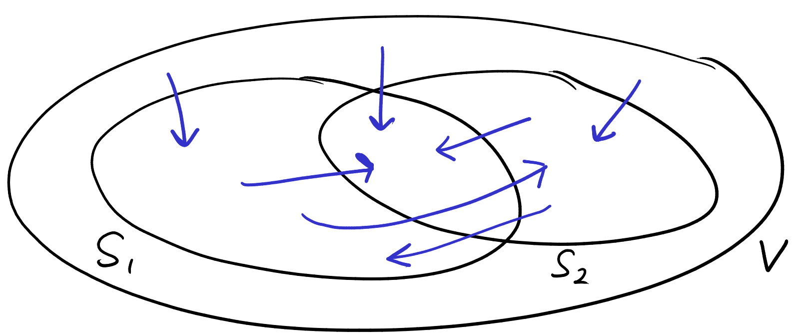
$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} + \chi^{A(S_1 \setminus S_2, S_2 \setminus S_1)} + \chi^{A(S_2 \setminus S_1, S_1 \setminus S_2)} ,$$

which implies in particular

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} \geq \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} .$$

Proof

Idea: Check all different "arc types":



Lemma 5.28

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $A(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, $A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset$. In particular, this implies by Lemma 5.27

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} \;.$$

Proof

By lemma 5.27:

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} + \chi^{A(S_1\setminus S_2,\, S_2 \setminus S_1)} + \chi^{A(S_2 \setminus S_1,\, S_1 \setminus S_2)}$$

$$2 = \underbrace{y(\delta^-(S_1))}_{=1} + \underbrace{y(\delta^-(S_2))}_{=1} = \underbrace{y(\delta^-(S_1 \cup S_2))}_{\geq 1} + \underbrace{y(\delta^-(S_1 \cap S_2))}_{\geq 1} + \overbrace{y(A(S_1\setminus S_2, S_2 \setminus S_1))}^{\geq 0} + \underbrace{y(A(S_2 \setminus S_1, S_1 \setminus S_2))}_{\geq 0} \geq 2$$

$S_1, S_2 \in \mathcal{F}$ $\qquad$ $y \in P$

$\Rightarrow$ Equality holds throughout.

$\Rightarrow$
- $y(\delta^-(S_1 \cup S_2)) = 1 \Rightarrow S_1 \cup S_2 \in \mathcal{F}$
- $y(\delta^-(S_1 \cap S_2)) = 1 \Rightarrow S_1 \cap S_2 \in \mathcal{F}$
- $\left.\begin{array}{l} y(A(S_1 \setminus S_2, S_2 \setminus S_1)) = 0 \\ y(A(S_2 \setminus S_1, S_1 \setminus S_2)) = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} A(S_1 \setminus S_2, S_2, S_1) = \emptyset \\ A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset \end{array}\right.$

#

Back to : $\boxed{}$ implies $*$.