5.8 Combinatorial Uncrossing

Main goal :

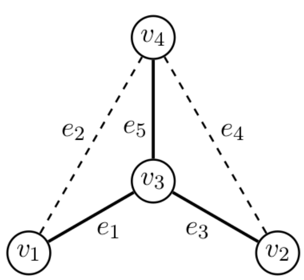
> Given a heavily overdetermined linear system that uniquely defines a point, find a well-structured full-rank subsystem.

5.8.1 Integrality of spanning tree polytope

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} : \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall\, S \subsetneq V,\ |S| \geq 2 \end{array} \right\}$$

spanning tree constraints:

$$\begin{array}{rc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{r} \{v_1,v_2\} \\ \{v_1,v_3\} \\ \{v_1,v_4\} \\ \{v_2,v_3\} \\ \{v_2,v_4\} \\ \{v_3,v_4\} \\ \{v_1,v_2,v_3\} \\ \{v_1,v_2,v_4\} \\ \{v_1,v_3,v_4\} \\ \{v_2,v_3,v_4\} \\ \{v_1,v_2,v_3,v_4\} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} y \overset{=}{\leq} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$



— $y = (1,0,1,0,1)$

non-negativity constraints:

$$\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} y \overset{=}{\geq} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Qy = q$$

$$y = Q^{-1} q$$

Proof of integrality of P

Let $y \in \text{vertices}(P)$.

Wlog, assume $\overbrace{y(e) > 0 \quad \forall\, e \in E}^{\mathrm{supp}(y) = E}$.

We can delete edges $e \in E$ with $y(e) = 0$.

→ Observe $y|_{E \setminus \{e\}}$ is vertex of spanning tree polytope of $(V, E \setminus \{e\})$.

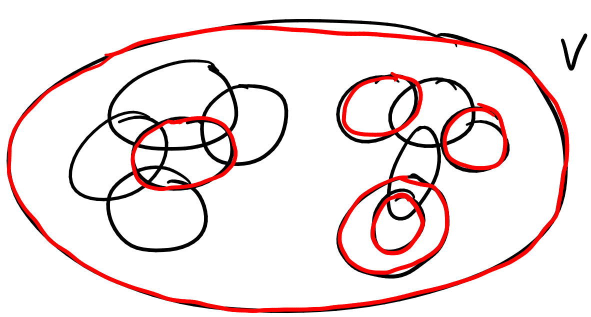
Let $\mathcal{F} = \{ S \subseteq V : y(E[S]) = |S| - 1 \}$

y-tight spanning tree constraints

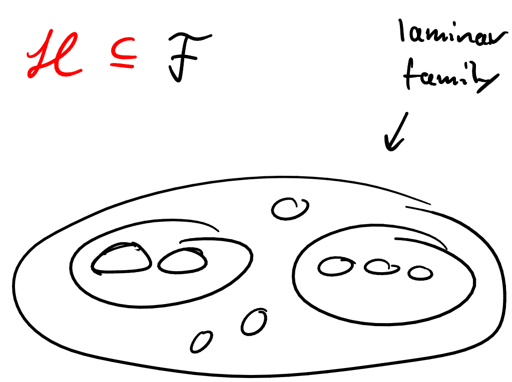
$y \in \text{vertices}(P) \implies y$ is unique sol. to

$$\circledast \qquad x(E[S]) = |S| - 1 \qquad \forall\, S \in \mathcal{F}$$

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$.



$\mathcal{H} \subseteq \mathcal{F}$



Lemma 5.23

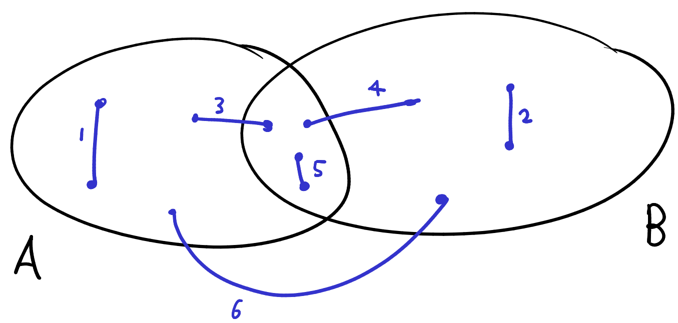
For any sets $A, B \subseteq V$, we have

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A\setminus B, B\setminus A)} = \chi^{E[A\cup B]} + \chi^{E[A\cap B]} \ ,$$

which implies

$$\chi^{E[A]} + \chi^{E[B]} \leq \chi^{E[A\cup B]} + \chi^{E[A\cap B]} \ .$$

Proof



→ 6 "edge types" in $E[A \cup B]$.

For each edge type, contribution to lhs and rhs is same.

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A\setminus B, B\setminus A)} \quad = \quad \chi^{E[A\cup B]} + \chi^{E[A\cap B]}$$

	$\chi^{E[A]}$	+	$\chi^{E[B]}$	+	$\chi^{E(A\setminus B, B\setminus A)}$	=	$\chi^{E[A\cup B]}$	+	$\chi^{E[A\cap B]}$
1	1		0		0		1		0
2	0		1		0		1		0
3	1		0		0		1		0
4	0		1		0		1		0
5	1		1		0		1		1
6	0		0		1		1		0

#

Lemma 5.24

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{F}$ and $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$. In particular, this implies by Lemma 5.23

$$\chi^{E[S_1]} + \chi^{E[S_2]} = \chi^{E[S_1 \cup S_2]} + \chi^{E[S_1 \cap S_2]} \ .$$

Proof

Back to : Each equality in $*$ is implied by $\square$.