5.3 Total unimodularity

One way to prove that $P = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ is integral, is by proving properties about constraint matrix A.

One strong and influential property: total unimodularity.

5.3.1 Definition and basic properties

Definition 5.3

A matrix is *totally unimodular* (TU) if the determinant of any square submatrix of it is either 0, 1, or -1.

example :
$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Remark 5.4

$A \in \mathbb{R}^{m \times n}$ is TU $\Rightarrow A \in \{-1, 0, 1\}^{m \times n}$.

↳ each entry of A is a square submatrix of A.

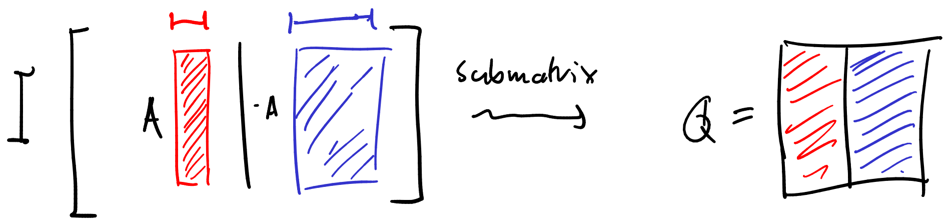
Remark 5.5

A is TU $\Leftrightarrow A^\top$ is TU.

$\hookrightarrow \det Q = \det Q^T$

Remark 5.6

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A \ -A]$, i.e., the $\mathbb{R}^{m\times 2n}$ matrix obtained by appending the columns of $-A$ to the columns of A.

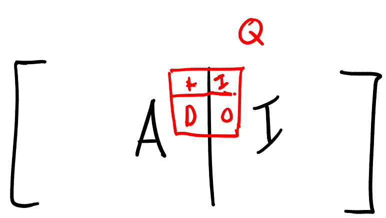


- Either $\vdash\!\dashv$ contains a negated column of $\vdash\!\dashv$ $\Rightarrow \det Q = 0$.
- If not $\Rightarrow$ Q is submatrix of A with some columns negated

$$\xRightarrow[A \text{ is TU}]{} \det(Q) \in \{-1, 0, 1\}.$$

Remark 5.7

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A \ I]$, i.e., the $\mathbb{R}^{m\times \overset{(n+m)}{\cancel{2n}}}$ matrix obtained by appending the columns of an $m \times m$ identity matrix I to the columns of A.



$\rightsquigarrow \det Q \in \{-\det D, \det D\}$

$\underset{\substack{\uparrow \\ A \text{ is TU}}}{\in} \{-1, 0, 1\}$

5.3.2 Integrality of polyhedra with TU constraint matrices

Theorem 5.8

Let $A \in \mathbb{Z}^{m \times n}$. Then,

$$A \text{ is TU} \quad \Leftrightarrow \quad P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \text{ is integral } \forall\, b \in \mathbb{Z}^m.$$

Proof

$\Rightarrow$) Let $y \in \text{vertices}(P)$.

$\Rightarrow$ y is unique sol. to square subsystem $Dx = d$ of

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \overset{\leq}{=} \begin{pmatrix} b \\ 0 \end{pmatrix}$$

constraints defining P

$$y = D^{-1} d$$

$Dx=d$ is full-rank system

$$A \text{ TU} \xRightarrow[\text{remarks}]{\text{prior}} \begin{pmatrix} A \\ -I \end{pmatrix} \text{ TU} \Rightarrow \det D \in \{-1, \cancel{0}, 1\}$$

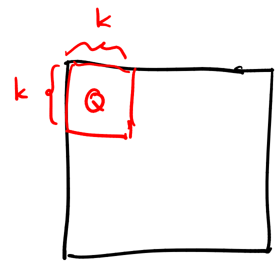
$$\xRightarrow{D \text{ integral}} D^{-1} \text{ integral}$$

$$\xRightarrow{d \text{ integral}} D^{-1} d = y \text{ is integral.}$$

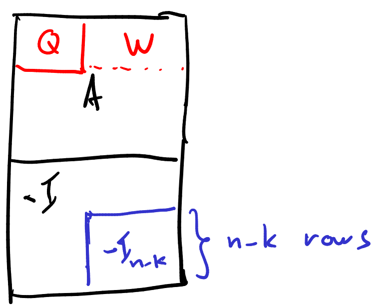
because d is subvector of $b \in \mathbb{Z}^m$

$\Leftarrow)$

Assume by sake of contradiction $\exists$ square submatrix $Q \in \mathbb{R}^{k\times k}$ of A with $\det Q \notin \{-1,0,1\}$.



$A \in \mathbb{R}^{m\times n}$



$$\xrightarrow{\text{submatrix}} \quad H = \begin{pmatrix} Q & W \\ 0 & -I \end{pmatrix}$$

$$\leadsto \quad H^{-1} = \begin{pmatrix} Q^{-1} & Q^{-1}W \\ 0 & -I \end{pmatrix}$$

Q is integer

$\boxed{\det Q} \notin \{-1,0,1\} \implies Q^{-1}$ is not integer.

$\det Q^{-1} = \frac{1}{\det Q} \notin \mathbb{Z}$

Let $j \in [n]$ s.t. j-th column of of Q^{-1} is not integral.

We now define a point $y \in \mathbb{R}^n$ with at least one fractional entry and a rhs $b \in \mathbb{Z}^m$ s.t. $y \in$ vertices(P), where "P_b"

$$P = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\}.$$

Definition of y

j-th canonical vector

all-ones vector

(i) First k entries of y : $Q^{-1}e_j + p \cdot \mathbb{1}$, where $p \in \mathbb{Z}_{\ge 0}$ is s.t. $Q^{-1}e_j + p \cdot \mathbb{1} \ge 0$

(ii) Remaining $n-k$ entries : 0

Due to fractionality of $Q^{-1}e_j \rightarrow y$ is fractional.

$$H = \begin{pmatrix} Q & W \\ 0 & -I \end{pmatrix}$$

$$H^{-1} = \begin{pmatrix} Q^{-1} & Q^{-1}W \\ 0 & -I \end{pmatrix}$$

First k entries of b : $e_j + pQ\mathbb{1} =: \tilde{b}$

$\Rightarrow$ y fulfills first k constraints of $Ax = b$ with equality.

$$[Q\ W] \cdot y = [Q\ W] \begin{pmatrix} Q^{-1}e_j + p\mathbb{1} \\ 0 \end{pmatrix} = e_j + pQ\mathbb{1} = \tilde{b}.$$

Choose last $m-k$ entries of b large enough s.t. $Ay \le b$.

$\Rightarrow$ $y \in$ vertices(P), because y is solution to subsystem

$$Hx = \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix} \quad \text{of} \quad \begin{pmatrix} A \\ -I \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

and $\text{rank}(H) = n$.

#

5.3.3 Characterization of Ghouila-Houri

Theorem 5.9: Characterization of Ghouila-Houri

A matrix $A \in \mathbb{R}^{m \times n}$ is TU if and only if for every subset of the rows $R \subseteq [m]$, there is a partition $R = R_1 \dot{\cup} R_2$ such that

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\} \quad \forall j \in [n] \ . \tag{5.6}$$

Proof

$\Rightarrow$) Assume A is TU and let $R \subseteq [m]$.

Let $d \in \mathbb{R}^m$:

$$d_i = \begin{cases} 1 & \text{if } i \in R, \\ 0 & \text{if } i \in [m] \setminus R \end{cases} \qquad (d = \chi^R)$$

$$Q := \left\{ x \in \mathbb{R}^m : \ A^T x \leq \lceil \tfrac{1}{2} A^T d \rceil, \ A^T x \geq \lfloor \tfrac{1}{2} A^T d \rfloor, \ x \leq d, \ x \geq 0 \right\}$$

Q is integral, because $\begin{pmatrix} A^T \\ -A^T \\ I \end{pmatrix}$ is TU.

$\frac{d}{2} \in Q \Rightarrow Q \neq \emptyset$

$\xRightarrow{Q \text{ is polytope}}$ Q has a vertex y, which is integer.

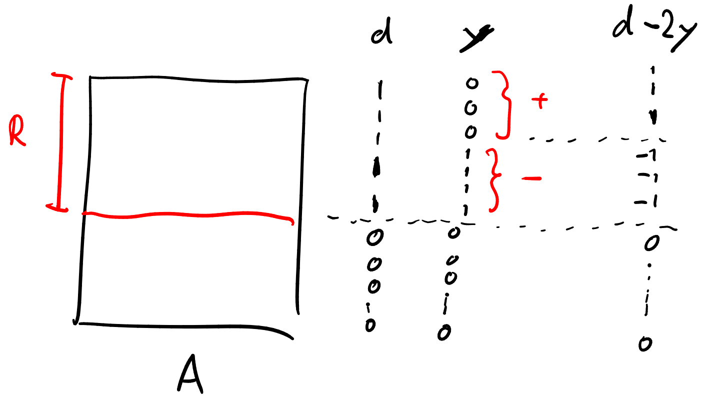
$y \in \{0, 1\}^m$

Let $R_1 = \{ i \in R : y_i = 0 \}$,

$R_2 = \{ i \in R : y_i = 1 \}$.

This partition fulfills the Ghouila-Houri criterion, because:

$$\sum_{i \in R_1} A_{i\cdot} - \sum_{i \in R_2} A_{i\cdot} = (d - 2y)^T A$$



$(d-2y)^T A$

$A^T(d-2y) \in \{-1,0,1\}^n$

to show

$A^T(d-2y)$

$$y \in Q \Rightarrow \lfloor \tfrac{1}{2} A^T d \rfloor \leq A^T y \leq \lceil \tfrac{1}{2} A^T d \rceil$$

$$\tfrac{1}{2} A^T d - \tfrac{1}{2} \cdot \mathbb{1} \leq A^T y \leq \tfrac{1}{2} A^T d + \tfrac{1}{2} \cdot \mathbb{1}$$

$$-\mathbb{1} \leq A^T(d-2y) \leq \mathbb{1}$$

$$\Rightarrow A^T(d-2y) \in \{-1,0,1\}^n$$

$\Leftarrow$)

$$A \in \mathbb{R}^{(k+1)\times(k+1)}$$

$$A = \begin{pmatrix} Q & c \\ r^T & 1 \end{pmatrix}$$

$$\xrightarrow{\text{row operations}} \quad \bar{A} = \begin{pmatrix} Q - c r^T & 0 \\ r^T & 1 \end{pmatrix} \qquad \det(\bar{A}) = \det(A)$$

$\rightarrow$ see script

Remark 5.10

Because A is TU if and only if $A^\top$ is TU, one can exchange the roles of rows and columns in Theorem 5.9.

Example 5.11

Consecutive-ones matrices are TU.

$$\begin{array}{cc}
\begin{matrix}+\\-\\ \\+\\-\end{matrix} & \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 1 & 0\\
\cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1}\\
0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}\\
 & \begin{matrix}0 & -1 & 0 & 0 & 0 & 1\end{matrix}
\end{array}
\qquad
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0\\
0 & 1 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}$$