Mathematical Optimization. Problem Set 2. Chen Ni

problem 1

「Not complet yet」

(a). The closest distance from Ball to hyperplane is

$$d = \frac{|a_i^T y - b_i|}{\|a_i\|_2} - r.$$

Because $Ay \le b$.

$$\therefore \quad d = \frac{-a_i^T y + b_i}{\|a_i\|_2} - r.$$

The LP problem is then.

$$\max \quad r$$

$$\text{s.t.} \quad \frac{a_i^T y}{\|a_i\|_2} + r \le \frac{b_i}{\|a_i\|_2}.$$

$$Ay \le b$$

$$r \ge 0.$$

Let $\tilde{r} = (y^T, r)^T$, $\tilde{a_i} = (a_{i1}, \dots a_{in}, \|a_i\|_2)$, $C = (\underbrace{0, \dots, 0}_{n}, 1)^T$

we can rewrite LP problem:

$$\max \quad C^T \tilde{r}$$

$$\text{s.t.} \quad \tilde{A} \cdot \tilde{r} \le b.$$

~~Thi~~ This is a standard LP problem. There exist solution as assumped.

(b). No. if P is unbounded, and the c is in the unbounded direction there is possibilitties that solution is unbounded.

If we chose $r=0$. then $\tilde{A} \cdot \tilde{r} = A \cdot y$.

So the solution always exists if P itself has element inside $(P \neq \phi)$.

problem 2

(a) Because $A \in \mathbb{R}^{m \times n}$ have full column Rank. $A = [a_1, a_2, \dots a_n]$

i.e.. $a_1, \dots a_n$ are linearly independent

The distance from the point on ~~Line~~ $L(v,w)$ to a hyperplane $a_i^T x = b_i$ is.

$$d_{L\lambda} = \frac{a_i^T (v + \lambda w) - b_i}{\|a_i\|_2} = \frac{(a_i^T w) \cdot \lambda + a_i^T v - b_i}{\|a_i\|_2};$$

1) If $a_i^T w \neq 0$. We have some part ~~of~~ the line: $d_{L\lambda} > 0$ and the rest $d_{L\lambda} < 0$ and an intersection with $d_{L\lambda} = 0$, where $\lambda = \frac{-a_i^T v + b_i}{a_i^T w}$

in this case. $L(v,w) \nsubseteq P$. as there always some points (infinity) which doesn't satisfy $a_i^T x \le b_i$ ~~..~~ $i \in [k]$.

2) If $a_i^T w = 0$; the $d_{L\lambda}$ is fixed ~~..~~ for certain hyperplane. We show that at least one hyperplane will intersect with $L(\lambda, w)$. $(n \ge 2)$. Otherwise.

~~there exists two hyperplane whose distance are fixed say~~ We denote
~~these two hyperplane as~~ $a_j^T x = b_j$, $a_k^T x = b_k$.
~~they both have~~

$$a_j^T w = 0, \quad \cancel{a_k^T w = 0}.$$
$$\Rightarrow \cancel{(a_j - a_k)^T w = 0}$$

Continued,

for all $i = 1, 2, \dots n$. we have $a_i^T w = 0$.

Because a_i are linearly independent, so $\mathrm{span}\{a_1, a_2, \dots a_n\} = R^n$

$\forall x \in R^n$. we can find a combination $\{\beta_i\}$

$$x = \sum_{i=1}^{n} \beta_i a_i$$

Let $x = w \in R^n \neq 0$, assume $w = \sum_{i=1}^{n} \alpha_i a_i$

$$\|w\|_2 = w^T w = \sum_{i=1}^{n} \alpha_i a_i^T w = 0.$$

$$\Rightarrow w = 0. \quad \text{which is impossible}$$

So. we have at least one i, s.t. $a_i^T w \neq 0$. $\square$

(b). If P is empty, apparently it has no vertex.

If P is not empty, we show that P has at least one vertex.

Because $\{a_i\}$ are linearly independent; ~~so~~ there is no ~~parall~~ parallel hyperplanes

Assume two hyperplane $a_i^T x = b_i$, $a_j^T x = b_j$ has an intersection.

The intersection of two hyperplanes is a line. (*) ???

According to results from (a), there exists a hyperplane $a_k^T x = b_k$ which interacts

with this line, Then we get the intersection of this line

with the third hyperplane, and it's a point, y.

So y is the unique solution to $\bar{A}x = b$. where $\bar{A} = [a_i, a_j, a_k]$

So y is a vertex of P

Problem 3.

(a). If $\dim(P)=0$. ~~then p is a point~~.

then P contains only single points.

Because $\gamma := \sup\{c^T x : x \in P\}$ is finite, we can specify the exact $x^* \in P$

So that $c^T x = \gamma$. $\square$

(b) If $\ker(A) \neq \{0\}$. i.e. $\exists x \in \mathbb{R}^n, x \neq 0, Ax = 0$ $\quad (a_1, a_2, \dots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$

The we show that if a $(\dim(P)-1)$-dimensional polyhedra satisfied the ~~as~~ statement, then so does the dim p-dimensional polyhedra.

for $\forall x \in \mathbb{R}^n$, we can find a basis $\{e_i\}$

Because $\exists x^* \in \mathbb{R}^n, x^* \neq 0, Ax^* = 0$. take $e_1^* = \frac{x^*}{\|x^*\|}$, and ~~the expan~~

then extent to $\{e_1^*, e_2 \dots e_n\} = \mathcal{B}$

So for $\forall x \in \mathbb{R}^n$ the representation can be expressed.

$$x = \xi_1 e_1^* + \sum_{i=2}^{n} e_i \xi_i$$

Let $Q = \{\xi \mid x = \xi_1 e_1^* + \sum_{i=2}^{n} \xi_i e_i, x \in P\}$.

we have
$$Q = \{\xi \mid x = \xi_1 e_1^* + \sum_{i=2}^{n} \xi_i e_i, Ax \le b\}$$
$$= \{\xi \mid x = \xi_1 e_1^* + \sum_{i=2}^{n} \xi_i e_i, A(\xi_1 e_1^* + \sum_{i=2}^{n} \xi_i e_i) \le b\}$$
$$= \{\xi \mid x = \gamma e_1^* + \sum_{i=2}^{n} \xi_i e_i, \gamma \in \mathbb{R}, A(\sum_{i=2}^{n} \xi_i e_i) \le b\}$$

~~we can construct $B = [\,]$~~

~~s.t. $\forall x \in P$~~

~~$Bx = Bx$~~

So. ~~the first dimension of x is unstricted, which means~~.

~~P is actually~~ a

So the first axis is undecided. Because $c^T x$ has sup.

$\sup(c^T x : x \in P)$ exists. $\therefore$

$\therefore \sup\{c^T (\gamma e_1^* + \sum_{i=2}^{n} \xi_i e_i), x \in P\}$ exists.

and e_1^* won't make ~~any~~ difference. So we can ~~eliminate~~ neglect the e_1^*, and ~~to~~ to solve the (dim(P) − 1) problem $\square$

(c) Starting from $y^{(1)}$, according to claim, we can find a $y^{(2)}$ that $c^T y^{(2)} \geq c^T y^{(1)}$.

And so on, we can find a sequence $\{y^{(1)}, y^{(2)}, \ldots y^{(n)}\}$.

where $y^{(1)} \leq y^{(2)} \leq \ldots \leq y^{(n)}$ (Here for simplicity, we use $y^{(i)} \leq y^{(j)}$ to denote $c^T y^{(i)} \leq c^T y^{(j)}$)

~~and~~ ($c^T y$ is bounded) &

Because P is a closed set, so the sequence $\{y^{(i)}\}$ converges to some point x^*, and $x^* \in P$. $\square$

(d) Start with a point $y \in P$, we consider the y-tight constraints in the system $Ax \leq b$, $\tilde{A}x \leq \tilde{b}$.

If the subsystem $\tilde{A}x \leq \tilde{b}$ is full-rank, ~~then from (c)~~

then we can solve $\tilde{y} = \tilde{A}^{-1}\tilde{b} \in P$.

the unique solution exists. So it's vertex/extreme point