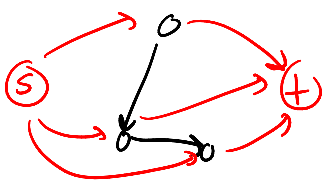
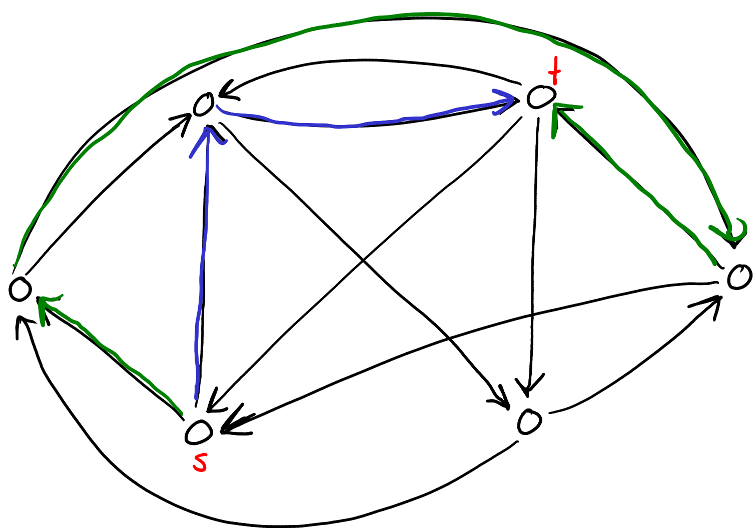
4.4 Applications of s-t flows

4.4.1 Arc-connectivity

Definition 4.17: k-arc-connectivity

A directed graph $G = (V, A)$ is *k-arc-connected* if for any two vertices $s, t \in V, s \neq t$, there are at least k arc-disjoint s-t paths in G.





Algorithm 4: Determining maximum number of arc-disjoint s-t paths.

Input: A directed graph $G = (V, A)$ and vertices $s, t \in V$, $s \neq t$.
Output: Maximum number of arc-disjoint s-t paths in G.

1. Define unit capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, i.e., $u(a) = 1\ \forall a \in A$.
2. Compute a maximum s-t flow f in G with capacities given by u.
3. **return** $\nu(f)$, the value of f.

Algorithm 4 is efficient:

(i) Step 1 takes $O(m)$ time.

(ii) Step 2 takes $O(\alpha(m+n))$ time, where α is the value of a maximum s-t flow.

By weak max-flow min-cut theorem

$$\alpha \leq u(\delta^+(s)) \leq m$$

$\Rightarrow$ Running time is $O(m(m+n))$.

Correctness of Algorithm 4

We will show:

> Let f be a maximum s-t flow in G with unit capacities and let $k \in \mathbb{Z}_{\geq 0}$ be the largest number of arc-disjoint s-t paths. Then
> $$\nu(f) = k$$

Will show $\nu(f) \geq k$ and $\nu(f) \leq k$.

Proof of $\nu(f) \geq k$

Observation 4.18: From arc-disjoint paths to flows

Let $G = (V, A)$ be a directed graph and $s, t \in V$ with $s \neq t$. Let $u\colon A \to \mathbb{Z}_{\geq 0}$ be unit capacities, i.e., $u(a) = 1 \ \forall a \in A$. If there are k arc-disjoint s-t paths $P_1, \ldots, P_k \subseteq A$ in G, then an s-t flow f of value k is obtained by setting

$$f(a) := \begin{cases} 1 & \text{if } a \in \bigcup_{i=1}^k P_i \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof of $\nu(f) \leq k$

Wlog, assume $\nu(f) \geq 1$ (If $\nu(f) = 0$, then we clearly have $\nu(f) \leq k$)

Moreover, we assume that f is integral, i.e., $f \in \mathbb{Z}_{\geq 0}^{A}$. (see Thm 4.16)

Let $U = \{a \in A : f(a) = 1\}$

Find s-t path $P \subseteq U$ in (V, U) (this could be done with BFS)

<u>Claim</u>: Because $\nu(f) \geq 1$, such a path exists.

Let $S = \{v \in V : \exists \text{ s-v path in } (V, U)\}$

Assume by sake of contradiction that $t \notin S$

$$\overset{\text{Lemma 4.3}}{\Longrightarrow} \quad \nu(f) = \underbrace{f(\delta^{+}(S))}_{=0} - \underbrace{f(\delta^{-}(S))}_{\geq 0} \leq 0 \quad \lightning$$

Now consider the s-t flow $f' : A \to \{0,1\}$ defined by

$$f'(a) = 1 \quad \forall a \in U \setminus P$$
$$f'(a) = 0 \quad \text{otherwise}$$

f' is an s-t flow, and $\nu(f') = \nu(f) - 1$.

→ Repeat above procedure $\nu(f)$ many times

→ we get $\nu(f)$ arc-disjoint s-t paths in G.

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