7 Equivalence Between Optimization and Separation

Loosely speaking, the Ellipsoid Method shows that if one can separate (over a polyhedron) then one can also optimize (a linear function over it).

$$\begin{tikzcd} \text{separation} \arrow[r, "\text{Ellipsoid}", "\text{Method}"'] & \text{optimization} \arrow[l, bend left] \end{tikzcd}$$

It turns out that there is also a reverse connection, which is based on polarity.

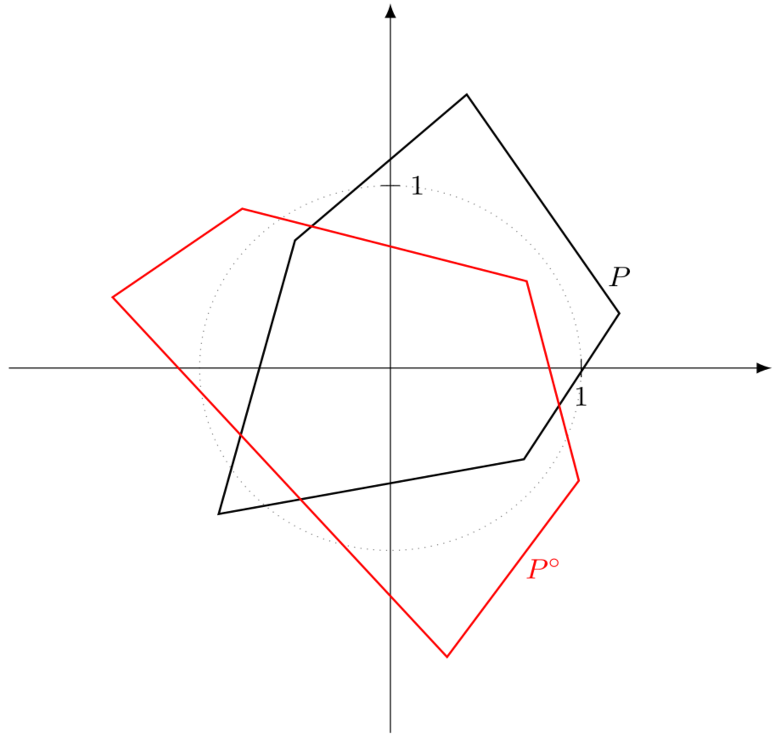
Definition

Let $X \subseteq \mathbb{R}^n$. The polar $X^\circ \subseteq \mathbb{R}^n$ of X is given by

$$X^\circ = \{ y \in \mathbb{R}^n : \; x^T y \leq 1 \quad \forall \, x \in X \}.$$

Clearly, if $A \subseteq B \subseteq \mathbb{R}^n$, then $B^\circ \subseteq A^\circ$.

Example 1



Example 2

Let $r \in \mathbb{R}_{>0}$ and consider $B(0,r) := \{x \in \mathbb{R}^n : \|x\|_2 \leq r\}$.

Then $\left(B(0,r)\right)^{\circ} = B\left(0, \frac{1}{r}\right)$.

Lemma 7.1

Let $X \subseteq \mathbb{R}^n$ be a compact (i.e., closed and bounded) convex set, containing the origin in its interior. Then

(a) X° is a compact convex set with the origin in its interior.
(b) $(X^\circ)^\circ = X$.

Proof

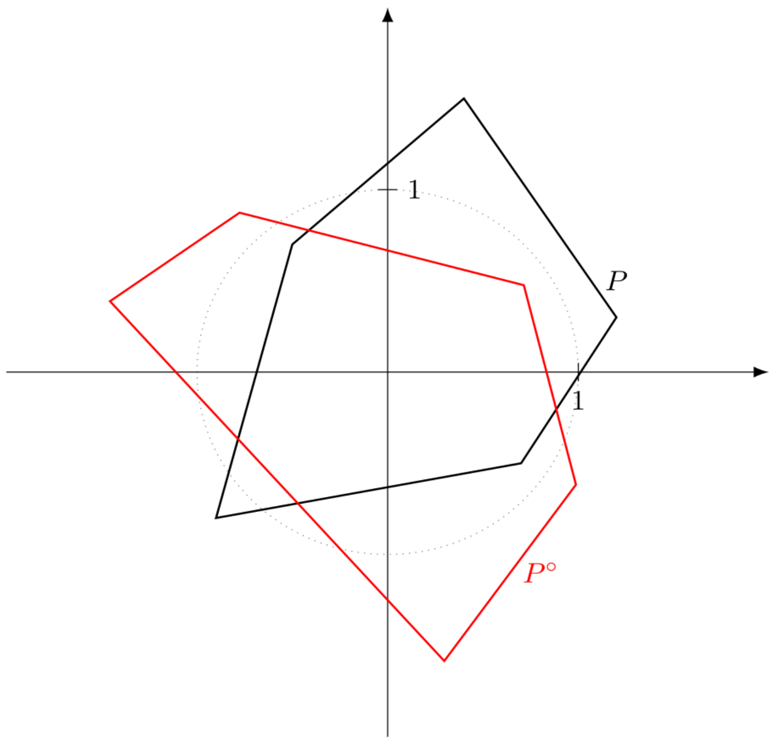
Theorem 1.47

Let $Y, Z \subseteq \mathbb{R}^n$ be two disjoint closed convex sets with at least one of them being compact, then there exists a strictly (Y, Z)-separating hyperplane.

Lemma 7.2

Let $P \subseteq \mathbb{R}^n$ be a polytope containing the origin in its interior. Then P° is a polytope. Moreover, for any $x \in \mathbb{R}^n$, we have

$$x \text{ is a vertex of } P \quad \Leftrightarrow \quad \{y \in \mathbb{R}^n \colon x^\top y \leq 1\} \text{ is facet-defining for } P^\circ.$$



Proof (of Lemma 7.2)

Lemma 1.20

Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a full-dimensional polyhedron, then each inequality $a^\top x \leq \beta$ of $Ax \leq b$ that is not facet-defining for P is redundant.

$\boxed{\text{Optimization over } P}$

$\boxed{\text{Separation over } P}$

$\boxed{\text{Separation over } P^\circ}$

$\boxed{\text{Optimization over } P^\circ}$