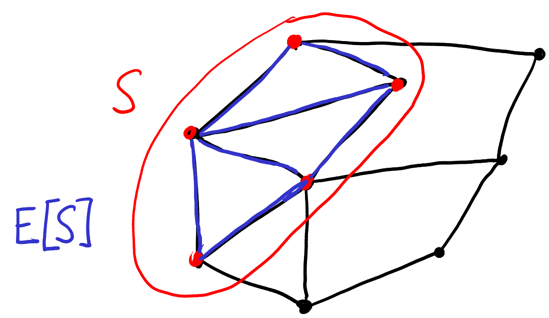
5.7.2 Matching polytope

$\left\{ x \in \mathbb{R}^E_{\geq 0} : \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\}$

Theorem 5.22

The matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) \leq 1 & \forall v \in V \\ x(E[S]) \leq \frac{|S|-1}{2} & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\}.$$



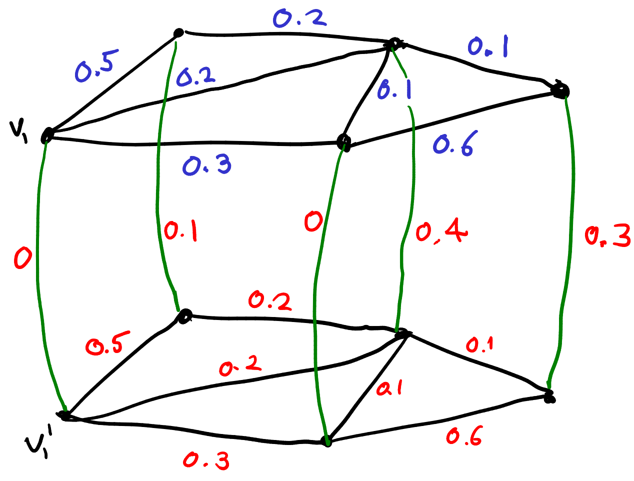
Proof of Theorem 5.22

- Easy to check that $P \cap \{0,1\}^E = \{ \chi^M : M \subseteq E \text{ is a matching in } G \}$

Let $x \in P$. We will show that x is convex combination of characteric vectors of matchings.

$\hookrightarrow$ Idea: Reduce to perfect matching case by constructing an auxiliary graph $H = (W, F)$.

$G=(V,E)$



$x \in P$

$\{\{v,v'\} : v \in V\}$

(V',E')

$\} \; y \in \mathbb{R}^F$

$H=(W,F)$

Formally,

$$V' = \{v' : v \in V\}$$
$$W = V \cup V'$$
$$E' = \{\{u',v'\} : \{u,v\} \in E\}$$
$$F = E \cup E' \cup \{\{v,v'\} : v \in V\}$$

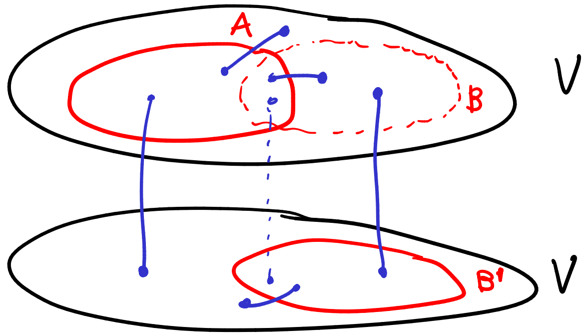
$$y(e) = x(e) \quad \forall\, e \in E$$
$$y(\{u',v'\}) = x(\{u,v\}) \quad \forall\, \{u,v\} \in E$$
$$y(\{v,v'\}) = 1 - x(\delta(v)) \quad \forall\, v \in V$$

We show that y is in the perfect matching polytope of $H=(W,F)$.

Clearly, $y \geq 0$ and $y(\delta_H(w)) = 1 \quad \forall\, w \in W$.

It remains to check, for $Q \subseteq W$ with $|Q|$ odd, that

$$y(\delta_H(Q)) \geq 1 \; .$$



$A = Q \cap V$

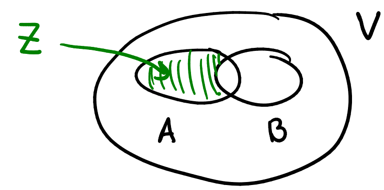
$B' = Q \cap V'$

$B = \{v \in V : v' \in B'\}$

$$y(\delta_H(Q)) = \underbrace{y(\delta_H(Q) \cap E)}_{x(\delta(A))} + \underbrace{y(\delta_H(Q) \cap E')}_{x(\delta(B))} + \underbrace{y(\delta_H(Q) \cap \{\{v,v'\} : v \in V\})}_{III}$$

$$III = \sum_{v \in A \setminus B} (1 - x(\delta(v))) + \sum_{v \in B \setminus A} (1 - x(\delta(v)))$$

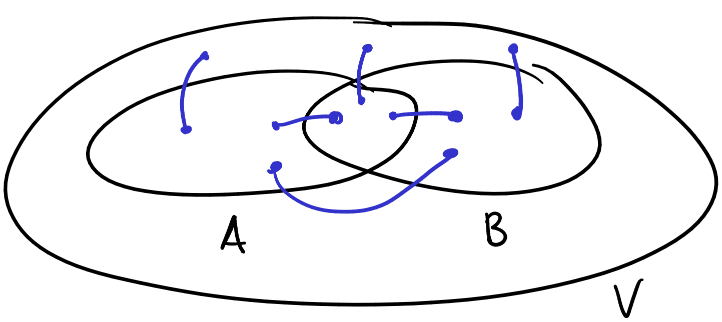
$$= |A \setminus B| - 2x(E[A \setminus B]) - x(\delta(A \setminus B))$$
$$+ |B \setminus A| - 2x(E[B \setminus A]) - x(\delta(B \setminus A))$$



$$\sum_{v \in Z} x(\delta(v)) = 2x(E[Z]) + x(\delta(Z))$$

$$x(\delta(A)) + x(\delta(B)) = x(\delta(A \setminus B)) + x(\delta(B \setminus A)) + \overbrace{\boxed{2x(E(A \cap B, V \setminus (A \cup B)))}}^{\geq 0}$$

$$\geq x(\delta(A \setminus B)) + x(\delta(B \setminus A))$$

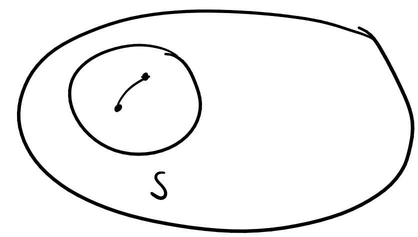


Hence,

$$y(\delta_H(Q)) \geq \underbrace{|A \setminus B| - 2x(E[A \setminus B])}_{(a)} + \underbrace{|B \setminus A| - 2x(E[B \setminus A])}_{(b)}$$

Observe that for any $S \subseteq V$:

$$|S| \geq \sum_{v \in S} \underbrace{x(\delta(v))}_{\leq 1} \geq 2x(E[S])$$

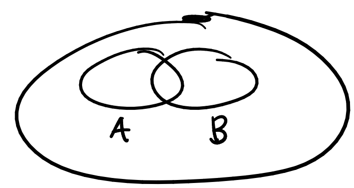


$\Rightarrow$ (a) ≥ 0, (b) ≥ 0

We show that, moreover, either (a) ≥ 1 or (b) ≥ 1.

$$|Q| = |A \cup B'| = |A| + |B| = |A \setminus B| + |B \setminus A| + 2|A \cap B|$$

is odd



$\Rightarrow$ Either $|A \setminus B|$ is odd or $|B \setminus A|$ is odd.

Wlog assume $|A \setminus B|$ is odd.

$$\Longrightarrow \quad x(E[A \setminus B]) \leq \frac{|A \setminus B| - 1}{2}$$

$$\Rightarrow |A \setminus B| - 2x(E[A \setminus B]) \geq 1 \Rightarrow (a) \geq 1 \Rightarrow y(\delta_H(Q)) \geq 1.$$

$\Rightarrow$ y is convex combination of perfect matching in H.

$$y = \sum_{i=1}^{k} \lambda_i \chi^{M_i}, \text{ where } k \in \mathbb{Z}_{\geq 1},$$

$$\lambda \in \mathbb{R}^k_{\geq 0}, \quad \sum_{i=1}^{k} \lambda_i = 1,$$

for $i \in [k]$: $M_i \subseteq F$ is perfect matching in H.

$$\Rightarrow \quad x = y\big|_E = \sum_{i=1}^{k} \lambda_i \underbrace{\chi^{M_i \cap E}}_{\in \{0,1\}^E}$$

$\rightarrow$ $M_i \cap E$ is matching in G $\forall\, i \in [k]$.

$\Rightarrow$ x is convex combination of matchings in G.

In other words:

$$P \subseteq \mathrm{conv}(\{\chi^M : M \subseteq E \text{ is matching in } G\})$$

P contains correct set of integral points:

$$P \cap \{0,1\}^E = \{\chi^M : M \subseteq E \text{ is matching in } G\}$$

$$P \supseteq \mathrm{conv}(P \cap \{0,1\}^E) = \mathrm{conv}(\{\chi^M : M \subseteq E \text{ is matching in } G\})$$

$$P = \mathrm{conv}(\{\chi^M : M \subseteq E \text{ is matching in } G\})$$

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