5 Polyhedral Approaches in Combinatorial Optimization

Combinatorial optimization problems can often be described by:

(i) A finite set N, called <u>ground set</u>, $N = \{e_1, e_2, e_3\}$. $2^N = \{ \{e_1\}, \{e_2\}, \{e_3\} \; \{e_1, e_2, e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{\phi\} \}$.

(ii) a <u>family</u> $\mathcal{F} \subseteq 2^N$ of <u>feasible sets</u>, also called <u>solutions</u>, and
collection of sets.

(iii) an objective function $w: N \to \mathbb{R}$ to maximize or minimize.
<u>weight</u>

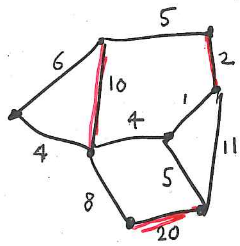
corresponding ⟿ problem

$$\boxed{\begin{array}{l} \max/\min \quad w(F) := \sum_{e \in F} w(e) \\ \qquad\qquad\quad F \in \mathcal{F} \end{array}}$$

<u>Examples</u>

given is undirected graph $G = (V, E)$ with non-negative edge weights $w: E \to \mathbb{R}_{\geq 0}$

<u>Maximum weight matchings</u>:



$G = (V, E)$

$w: E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$

(ii) Feasible sets: $\mathcal{F} = \{ M \subseteq E : M \text{ is a matching} \}$

(iii) Objective: maximize w

Well-known special cases:

- Maximum cardinality matching ⇝ $w(e) = 1 \;\; \forall e \in E$.
- Maximum cardinality/weight bipartite matchings ⇝ G is bipartite.

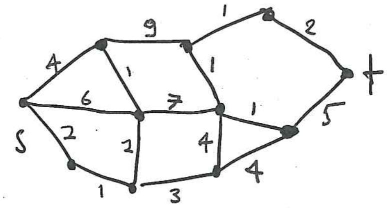
Shortest s-t path (in undir. graphs)

Given: • undir. graph $G=(V,E)$
• vertices $s,t \in V$
• non-neg. edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$.

(ii) Feasible sets: $\mathcal{F} = \{P \subseteq E : P \text{ is s-t path}\}$.

(iii) Objective: minimize $w = \ell$.



$\ell : E \to \mathbb{R}_{\geq 0}$

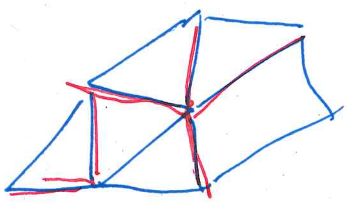
Minimum weight spanning tree ?

Given: • undir. graph $G=(V,E)$
• edge weights $w : E \to \mathbb{R}_{\geq 0}$

(i) Ground set: $N = E$.

(ii) Feasible sets: $\mathcal{F} = \left\{ F \subseteq E : \begin{array}{l} F \text{ is a spanning} \\ \text{tree in } G \end{array} \right\}$.

(iii) Objective: minimize w.



5.1 Polyhedral descriptions of combinatorial optimization problems

Let N be a finite (ground) set.

Definition

$\chi^U \in \mathbb{R}^N$

For $U \subseteq N$, we denote by χ^U its characteristic vector (also called incidence vector):

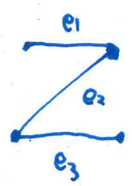
$$\chi^{U}(e) = \begin{cases} 1 & \text{if } e \in U. \\ 0 & \text{if } e \in N \setminus U. \end{cases}$$

Let $\mathcal{F} \subseteq 2^N$ be all feasible sets to a combinatorial optimization problem.

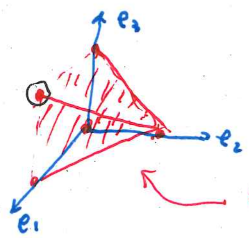
The (combinatorial) polytope that corresponds to $\mathcal{F}$ is the polytope $P_{\mathcal{F}} \subseteq [0,1]^N$ whose vertices are precisely $\{\chi^F : F \in \chi^F\}$, i.e.,

$$P_{\mathcal{F}} = \operatorname{conv}\left(\{\chi^F : F \in \mathcal{F}\}\right).$$

$G=(V,E)$.



$\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$



• All matchings

matching polytope. $P_{\mathcal{F}}$

$\mathcal{F} = \{M \subseteq E : M \text{ is a matching}\}$

$M = \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_3\}.$

$\max\ (1,1,1)^T x$
$x \in P_{\mathcal{F}}$

The combinatorial polytope allows for casting a combinatorial optimization problem into a linear program (and can be used for much more):

$$\boxed{\begin{array}{ll} \max/\min & w(F) \\ & F \in \mathcal{F} \end{array}} \rightsquigarrow \boxed{\begin{array}{ll} \max/\min & w^T x \\ & x \in P_{\mathcal{F}} \end{array}}$$

Optimal vertex solution to this LP is <u>characteristic vector of optimal solution of combinatorial optimization problem.</u>

<u>Key challenge</u>: Find explicit inequality description of $P_{\mathcal{F}}$.

$$\rightarrow \quad P_{\mathcal{F}} = \{x \in \mathbb{R}^N : Ax \leq b\}$$

<u>Some benefits of getting an inequality description:</u> (let $n := |N|$)

- Often, $\#$ facets of $P_{\mathcal{F}} = O(\text{poly } n)$.
  $\#$ vertices of $P_{\mathcal{F}} = 2^{\Omega(n)}$.

- If we can solve LPs over $P_{\mathcal{F}}$, then we can optimize <u>any linear objective.</u>

- Even when $P_{\mathcal{F}}$ has exponentially many facets, one can often get a description of them and even <u>solve LPs over $P_{\mathcal{F}}$.</u>

  for example, by using the Ellipsoid Method

- Being able to solve LPs over $P_{\mathcal{F}}$ often allows for <u>solving related problems</u>, for example by adding some extra constraints.

- The LP dual of $\max\{w^T x : x \in P_{\mathcal{I}}\}$ can often be interpreted combinatorially. Possible implications:
  - Natural optimality certificates through strong duality.
  - Fast algorithms based on dual such as primal-dual methods.

- Elegant polyhedral proof techniques.

⋮

5.2 Meta-recipe for finding inequality-descriptions

① Determine candidate description $P = \{x \in \mathbb{R}^N : Ax \leq b\} \subseteq [0,1]^N$.

this shows $P = P_{\mathcal{F}}$

② Prove
$$P \cap \{0,1\}^N = \{\chi^F : F \in \mathcal{F}\}.$$

③ Prove that P is integral.

↑ i.e., vertices $(P) \subseteq \mathbb{Z}^N$, which, because $P \subseteq [0,1]^N$, is same as vertices $(P) \subseteq \{0,1\}^N$

(i) P integral and $P \subseteq [0,1]^n$.

$\Rightarrow$ vertices$(P) \subseteq P \cap \{0,1\}^n$

(ii) Moreover if $v \in P \cap \{0,1\}^N \Rightarrow v \in$ vertices(P).

$v \in$ vertices $([0,1]^N)$.

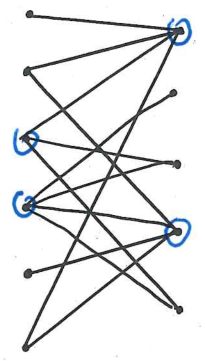
(i) & (ii) $\Rightarrow$ vertices$(P) = P \cap \{0,1\}^N \overset{②}{=} \{\chi^F : F \in \mathcal{F}\}$.

$\Rightarrow P = \text{conv}(\text{vertices}(P)) = \text{conv}(\{\chi^F : F \in \mathcal{F}\}) =: P_{\mathcal{F}}$

5.2.1 Example : bipartite vertex cover

Definition 5.1: Vertex cover

Let $G = (V, E)$ be an undirected graph. A vertex cover of G is a subset $S \subseteq V$ such that for every edge $e \in E$, at least one of its endpoints is in S.



$G = (V, E)$

$V = A \dot{\cup} B$

we follow the recipe.

① ~~follow the~~ Candidate description,

Theorem 5.2

The vertex cover polytope of a bipartite graph $G = (V, E)$ can be described by

$$P = \left\{ x \in [0,1]^V : x(u) + x(v) \geq 1 \ \forall \{u, v\} \in E \right\} .$$

1: pick the vertex

0: Not pick the vertex

Proof

②. Let $S \subseteq V$, S is a vertex cover $\iff$ $\{u,v\} \cap S \neq \emptyset$ $\forall \{u,v\} \in E$

at least one ~~vertex~~ vertex belongs to S. $\to$ $\iff \chi^S(u)+\chi^S(v) \geq 1$. $\forall \{u,v\} \in E$.

$\begin{cases} 1 & v \in S \\ 0 & v \in V/S \end{cases}$

$\iff \chi^S \in P$.

$P \cap \{0,1\}^N = \chi^S$

③. By sake of contradiction. assume has fraction vertex $y \in P \setminus \{0,1\}^V$

let $V = A \cup B$ be bipartion of the bipartite graph.

$W_A := \{u \in A : y(u) \in (0,1)\}$

$W_B := \{u \in B : y(u) \in (0,1)\}$.

$P := \{x \in [0,1]^V : x(u)+x(v) \geq 1, \forall \{u,v\} \in E\}$.

① vertex

② value of vertex $[0,1]$.

y is not integral $\Rightarrow W_A \cup W_B \neq \emptyset$

a ~~vector~~

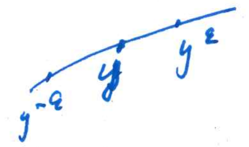
For $\delta \in \mathbb{R}$. $y^\delta := y + \delta(\chi^{W_A} - \chi^{W_B})$.

$y^\varepsilon := y + \varepsilon(\chi^{W_A} - \chi^{W_B})$

Non-all zero vector

Let $\varepsilon := \min\{\min\{y(u), 1-y(u)\} : u \in W_A \cup W_B\} > 0$.

$y^\varepsilon, y^{-\varepsilon} \in P$



to show $y^\varepsilon \in P$?

$y = \frac{1}{2}(y^\varepsilon + y^{-\varepsilon})$

Not a extreme point

$y^\varepsilon \in [0,1]^V$

$y^\varepsilon(u) + y^\varepsilon(v) = y(u) + y(v) \geq 1$