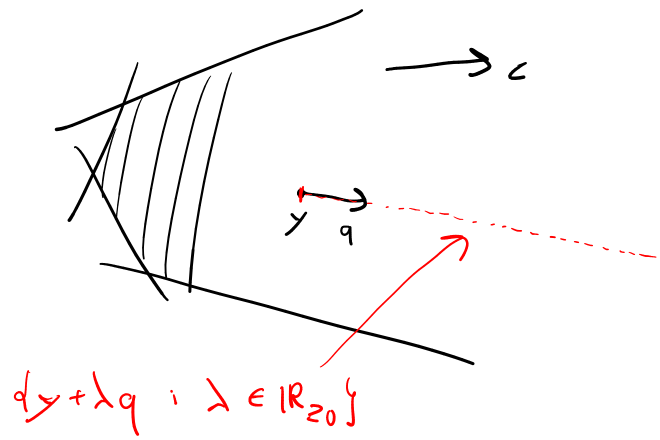
What do we expect from an LP algorithm?

- Detect type of LP (finite opt, unbounded, infeasible).
- If LP is finite → return optimal solution.
- If LP is unbounded → return certificate of unboundedness

$$\max \ c^T x \quad Ax \le b$$

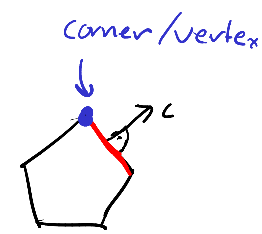


with

(i) $Ay \le b$ (ii) $Aq \le 0$ (iii) $c^T q > 0$

Further nice-to-haves of an LP algorithm

- In case of finite LP:
  - If possible, return optimal solution that is a corner.
  - Certificate of optimality.

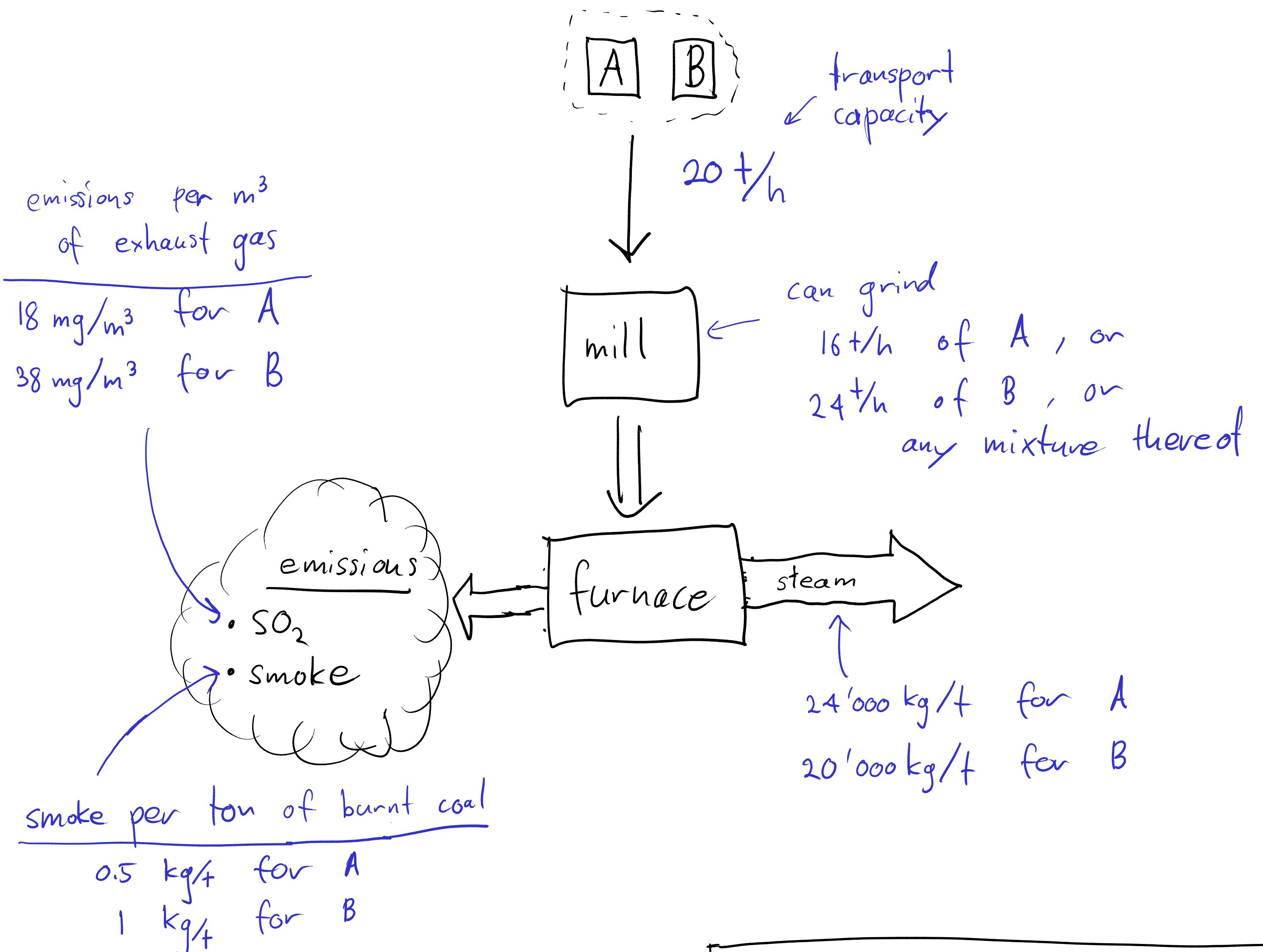


- In case of infeasible LP:
  - Certificate of infeasibility.

1.1.2 Example applications

Example 1: production planning

Coal power plant with 2 types of coal: A & B.



Emission limits

Maximum amount of sulfur dioxide (SO_2)	30 mg/m^3 air
Maximum amount of smoke	12 kg/h

Goal

Maximize tons of steam per hour under given constraints.

Modeling as an LP

Decision variables

$x_1 \in \mathbb{R}_{\geq 0}$: tons of coal A burnt per hour

$x_2 \in \mathbb{R}_{\geq 0}$: " " " B " " "

Objective function

$$\max \quad 24\, x_1 + 20\, x_2$$

Constraints

smoke emission : $\underbrace{0.5\, x_1 + 1 \cdot x_2}_{\text{kg of smoke emitted per hour}} \leq \underbrace{12}_{\text{emission limit for smoke}}$

SO_2 : $18 \cdot \dfrac{x_1}{x_1 + x_2} + 38 \dfrac{x_2}{x_1 + x_2} \leq 30$

$\left.\begin{array}{l}\text{multiply by} \\ x_1 + x_2\end{array}\right\} \longrightarrow \quad 12\, x_1 - 8 x_2 \geq 0$

(linearization)

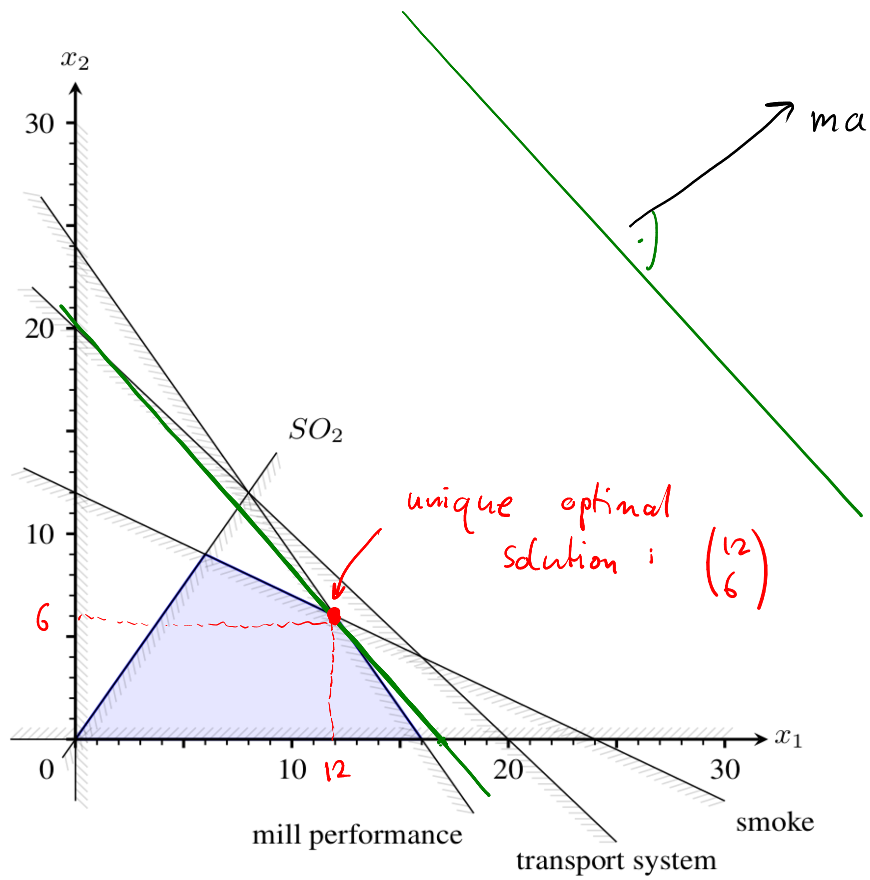
transport constraint : $x_1 + x_2 \leq 20$

mill performance : $\frac{1}{16} x_1 + \frac{1}{24} x_2 \leq 1$

→ Mathematical formulation :

$$
\begin{array}{lrcrcl}
\max & 24x_1 & + & 20x_2 & & \\
 & \frac{1}{2}x_1 & + & x_2 & \leq 12 & \text{(smoke emission)} \\
 & x_1 & + & x_2 & \leq 20 & \text{(transport capacity)} \\
 & \frac{1}{16}x_1 & + & \frac{1}{24}x_2 & \leq 1 & \text{(mill performance)} \\
 & 12x_1 & - & 8x_2 & \geq 0 & \text{(}SO_2\text{ emission)} \\
 & x_1 & & & \geq 0 & \multirow{2}{*}{\Big\} \text{(non-negativity)}} \\
 & & & x_2 & \geq 0 &
\end{array}
$$

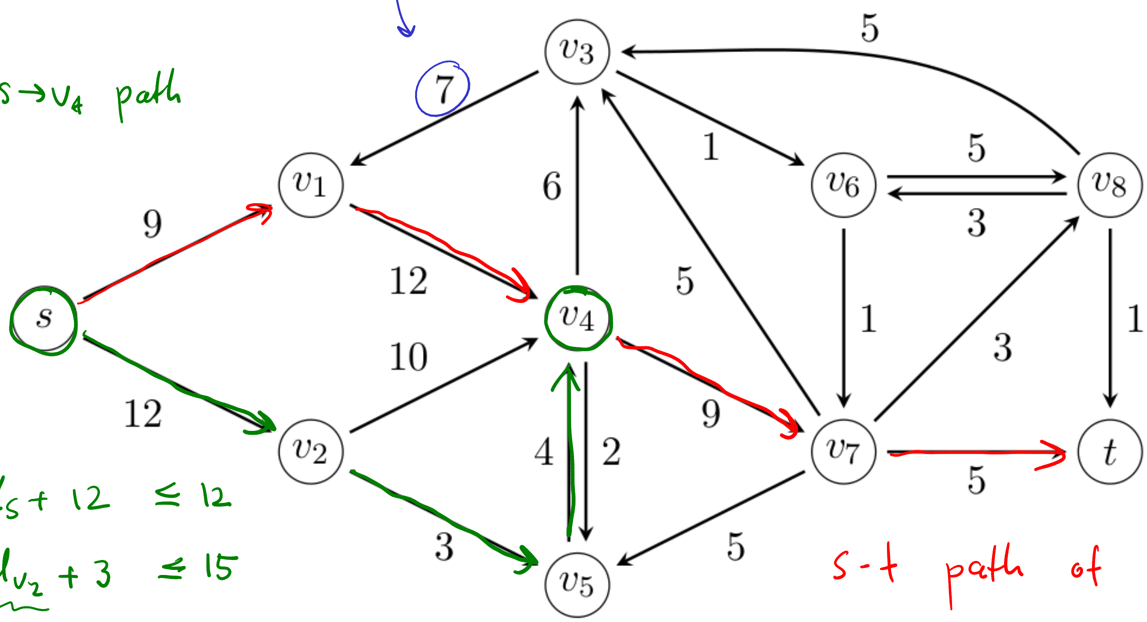
"Graphical solution method"



Example 2 : shortest s-t path

$\ell((v_3,v_1))$: length of arc $v_3 \to v_1$

shortest $s \to v_4$ path



$d_s = 0$

$d_{v_2} \le d_s + 12 \le 12$

$d_{v_5} \le \underbrace{d_{v_2}}_{\le 12} + 3 \le 15$

$d_{v_4} \le d_{v_5} + 4 \le 19$

s-t path of length:

$9 + 12 + 9 + 5 = 35$

Variables : $d_s, d_t, d_{v_1}, d_{v_2}, \ldots, d_{v_8}$

(think of d_u as distance from s to u)

$$\max d_t$$

$$d_s = 0$$

$$d_w \le d_u + \ell((u,w)) \quad \text{for every } u \to w \text{ arc}$$

$$d_u \ge 0 \quad \text{for every vertex } u$$

s u w t

These constraints make sure that d_u is no larger than the actual $s \to u$ distance.

An optimal LP solution: