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Mathematical Optimization – Solutions to problem set 5

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Problem 1: Linear programm with unique optimal solution

The given short tableau corresponds to the linear program of the form

$$\begin{array}{llllllll} \max & z & & & & & & \\ & z & + & c^\top x & & & = & q \\ & & & Ax & + & y & = & b \\ & z & & & & & \in & \mathbb{R} \\ & & & x & & & \in & \mathbb{R}^n_{\geq 0} \\ & & & & & y & \in & \mathbb{R}^m_{\geq 0} \ . \end{array}$$

Let $(x, y, z) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}$ be a solution of this LP. The assumption $c > 0$ together with $x \geq 0$ implies $c^\top x \geq 0$, so the first constraint implies the upper bound $z \leq q$ for the value of the linear program, with equality if and only if $x = 0$.

Note that this upper bound is also achieved: The feasible short tableau gives a solution $(x, y, z) = (0, b, q)$. Thus, if (x^*, y^*, z^*) is an optimal solution, we have $z^* = q$. By the above, this implies $x^* = 0$. Employing the constraint $Ax + y = b$, we now immediately see that the unique way of complementing the solution is $y^* = b$. Thus, $(x^*, y^*, z^*) = (0, b, q)$ is the unique optimal solution.

Problem 2: Certifying infeasibility from phase I of the Simplex Method

(a) Take the vector $\mu \in \mathbb{R}^m_{\geq 0}$ with $\mu^\top A \geq 0$ and $\mu^\top b < 0$ and assume for the sake of deriving a contradiction that there exists a point $x \in \mathbb{R}^n_{\geq 0}$ such that $Ax \leq b$. We then have

$$0 \leq \mu^\top A \cdot x = \mu^\top \cdot Ax \leq \mu^\top b < 0 \ ,$$

which is a contradiction. Thus, there cannot exist an $x \in \mathbb{R}^n_{\geq 0}$ such that $Ax \leq b$.

(b) If the auxiliary problem has a strictly negative value, then $x_0 > 0$ in the corresponding optimal solution, thus x_0 must be a basic variable in the optimal tableau. Optimality implies that there is no improving direction, i.e., all tableau entries in the objective row are non-negative (except the objective value, which is negative). Thus, we transformed the initial short tableau

	x_1	$\dots$	x_n	x_0	1
$\tilde{z}$	0	$\dots$	0	1	0
x_{n+1}	A_{11}	$\dots$	A_{1n}	-1	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_{n+m}	A_{m1}	$\dots$	A_{mn}	-1	b_m

to an optimal short tableau of the form

	x_{f_1}	$\dots$	$x_{f_{n+1}}$	1
$\tilde{z}$	c_{f_1}	$\dots$	$c_{f_{n+1}}$	q
x_0		$\dots$		
$\vdots$	$\vdots$		$\vdots$	$\vdots$

with free variables having the indices $F := \{f_1, \dots, f_{n+1}\} \subseteq \{1, \dots, n+m\}$, $c_i \geq 0$ for all $i \in F$, and $q < 0$.

(c) Reading the tableaus obtained in (b) as systems of equations in tableau form and recalling the pivoting operations that were done to obtain one tableau from the other, we get that the equation corresponding to the objective, namely

$$\tilde{z} + \sum_{i \in F} c_i x_i = q \ ,$$

was obtained by starting from the equation corresponding to the objective in the initial tableau, which is

$$\tilde{z} + x_0 = 0 \ ,$$

and adding (combinations of) rows of the system

$$\left(\begin{array}{c|c|ccc} -1 & & 1 & & \\ \vdots & \text{A} & & \ddots & \\ -1 & & & & 1 \end{array}\right) \begin{pmatrix} x_0 \\ \vdots \\ x_{n+m} \end{pmatrix} = b \ .$$

In other words, there exists a vector $\lambda \in \mathbb{R}^m$ such that

$$\tilde{z} + x_0 + \lambda^\top \left(\begin{array}{c|c|ccc} -1 & & 1 & & \\ \vdots & \text{A} & & \ddots & \\ -1 & & & & 1 \end{array}\right) \begin{pmatrix} x_0 \\ \vdots \\ x_{n+m} \end{pmatrix} = \tilde{z} + \sum_{i \in F} c_i x_i \tag{1}$$

and

$$0 + \lambda^\top b = q \ . \tag{2}$$

Comparing coefficients of the variables $\{x_1, \dots, x_n\}$ in (1), we see that $\lambda^\top A \geq 0$, and the coefficients of the variables $\{x_{n+1}, \dots, x_{n+m}\}$ show that $\lambda \geq 0$ (note that both these arguments exploit that $c_i \geq 0$). Furthermore, (2) gives $\lambda^\top b = q < 0$. Thus, λ has the properties discussed in (a), and hence it is a certificate of infeasibility for the initial linear program.

Note that by comparing coefficients of the variables $\{x_{n+1}, \dots, x_{n+m}\}$ in (1), we can read off the components of λ, they are given by

$$\lambda_i = \begin{cases} 0 & \text{if } n+i \notin F \\ c_i & \text{if } n+i \in F \end{cases} \ ,$$

and can thus also be read directly from the optimal tableau.

Problem 3: The dual of a general linear program

(a) We introduce variables $y_+, y_- \in \mathbb{R}^{n_e}_{\geq 0}$ and $z' \in \mathbb{R}^{n_f}_{\geq 0}$ such that $y_+ - y_- = y$ (we interpret y_+ as the the positive and y_- as the negative part of y) and $-z' = z$ (by flipping the sign of the non-positive variable z we get the non-negative variable z'). This allows us to transform the given linear program into the following linear program (P') in canonical form.

$$\begin{array}{lrcrcrcrcl} \max & d^\top x & + & e^\top y_+ & - & e^\top y_- & - & f^\top z' & & \\ & Ax & + & By_+ & - & By_- & - & Cz' & \leq & a \\ & Dx & + & Ey_+ & - & Ey_- & - & Fz' & \leq & b \\ & -Dx & - & Ey_+ & + & Ey_- & + & Fz' & \leq & -b \\ & -Gx & - & Hy_+ & + & Hy_- & + & Kz' & \leq & -c \\ & & & & & & & x & \in & \mathbb{R}^{n_d}_{\geq 0} \\ & & & & & & & y_+ & \in & \mathbb{R}^{n_e}_{\geq 0} \\ & & & & & & & y_- & \in & \mathbb{R}^{n_e}_{\geq 0} \\ & & & & & & & z' & \in & \mathbb{R}^{n_f}_{\geq 0} \end{array} \tag{P'}$$

The dual of the linear program (P') is the following linear program (D').

$$
\begin{array}{lrcrcrcrcl}
\min & a^\top t_1 & + & b^\top t_2 & - & b^\top t_3 & - & c^\top t_4 & & \\
& A^\top t_1 & + & D^\top t_2 & - & D^\top t_3 & - & G^\top t_4 & \geq & d \\
& B^\top t_1 & + & E^\top t_2 & - & E^\top t_3 & - & H^\top t_4 & \geq & e \\
& -B^\top t_1 & - & E^\top t_2 & + & E^\top t_3 & + & H^\top t_4 & \geq & -e \\
& -C^\top t_1 & - & F^\top t_2 & + & F^\top t_3 & + & K^\top t_4 & \geq & -f \\
& & & & & & & t_1 & \in & \mathbb{R}^{n_a}_{\geq 0} \\
& & & & & & & t_2 & \in & \mathbb{R}^{\overline{n}_b}_{\geq 0} \\
& & & & & & & t_3 & \in & \mathbb{R}^{\overline{n}_b}_{\geq 0} \\
& & & & & & & t_4 & \in & \mathbb{R}^{\overline{n}_c}_{\geq 0}
\end{array} \qquad (D')
$$

(b) We first observe that the second and third inequality constraint of (D') actually imply an equality constraint. Additionally multiplying the fourth inequality constraint with -1 yields the following linear program (D'').

$$
\begin{array}{lrcrcrcrcl}
\min & a^\top t_1 & + & b^\top t_2 & - & b^\top t_3 & - & c^\top t_4 & & \\
& A^\top t_1 & + & D^\top t_2 & - & D^\top t_3 & - & G^\top t_4 & \geq & d \\
& B^\top t_1 & + & E^\top t_2 & - & E^\top t_3 & - & H^\top t_4 & = & e \\
& C^\top t_1 & + & F^\top t_2 & - & F^\top t_3 & - & K^\top t_4 & \leq & f \\
& & & & & & & t_1 & \in & \mathbb{R}^{n_a}_{\geq 0} \\
& & & & & & & t_2 & \in & \mathbb{R}^{\overline{n}_b}_{\geq 0} \\
& & & & & & & t_3 & \in & \mathbb{R}^{\overline{n}_b}_{\geq 0} \\
& & & & & & & t_4 & \in & \mathbb{R}^{\overline{n}_c}_{\geq 0}
\end{array} \qquad (D'')
$$

Next, we note that the variables t_2 and t_3 appear in pairs such that one could factor out $t_2 - t_3$ in the objective and all the constraints. Moreover, the variable t_4 always appears with a minus sign in front. Hence, we introduce variables $u \in \mathbb{R}^{m_a}_{\geq 0}$, $v \in \mathbb{R}^{m_b}$, and $w \in \mathbb{R}^{m_c}_{\leq 0}$ such that $u = t_1$, $v = t_2 - t_3$, and $w = -t_4$. This allows us to transform the dual linear program (D'') (respectively, (D')) into the following linear program (D).

$$
\begin{array}{lrcrcrcl}
\min & a^\top u & + & b^\top v & + & c^\top w & & \\
& A^\top u & + & D^\top v & + & G^\top w & \geq & d \\
& B^\top u & + & E^\top v & + & H^\top w & = & e \\
& C^\top u & + & F^\top v & + & K^\top w & \leq & f \\
& & & & & u & \in & \mathbb{R}^{m_a}_{\geq 0} \\
& & & & & v & \in & \mathbb{R}^{m_b} \\
& & & & & w & \in & \mathbb{R}^{m_c}_{\leq 0}
\end{array} \qquad (D)
$$

Our transformations ensure that (D') and (D) are equivalent. This can also be proved directly:

- Let (t_1, t_2, t_3, t_4) be a feasible solution of (D'). Then, $(u, v, w) = (t_1, t_2 - t_3, -t_4)$ is a feasible solution of (D) with the same objective value.
- Let (u, v, w) be a feasible solution of (D). Then,

$$(t_1, t_2, t_3, t_4) = (u, \max\{0, v\}, \max\{0, -v\}, -w)$$

  is a feasible solution of (D') with the same objective value.

Also note that (D) can again be interpreted as the problem of finding the best (i.e., smallest) upper bound on the optimal value of the original primal linear program given in the exercise.

Problem 4: Primal and dual pivot

(a) The short tableau corresponding to the linear program (P_1) has the form

$$\begin{array}{c|ccc|c} & x_1 & \dots & x_n & 1 \\ \hline z & -c_1 & \dots & -c_n & q \\ \hline y_1 & A_{11} & \dots & A_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ y_m & A_{m1} & \dots & A_{mn} & b_m \end{array}$$

with $q = 0$ and $z = c^\top x + q$. After performing a pivoting step on A_{ij}, we obtain the short tableau

$$\begin{array}{c|ccc|c} & x'_1 & \dots & x'_n & 1 \\ \hline z & -c'_1 & \dots & -c'_n & q' \\ \hline y'_1 & A'_{11} & \dots & A'_{1n} & b'_1 \\ \vdots & \vdots & & \vdots & \vdots \\ y'_m & A'_{m1} & \dots & A'_{mn} & b'_m \end{array} \tag{3}$$

where we denote

$$x'_k := \begin{cases} y_i & \text{if } k = j \\ x_k & \text{if } k \neq j \end{cases} \qquad \text{and} \qquad y'_\ell := \begin{cases} x_j & \text{if } \ell = i \\ y_\ell & \text{if } \ell \neq i \end{cases} ,$$

as well as

$$A'_{k\ell} = \begin{cases} \dfrac{1}{A_{ij}} & \text{if } k = i,\ \ell = j \text{ (pivot element)} \\ -\dfrac{A_{kj}}{A_{ij}} & \text{if } k \neq i,\ \ell = j \text{ (pivot column)} \\ \dfrac{A_{i\ell}}{A_{ij}} & \text{if } k = i,\ \ell \neq j \text{ (pivot row)} \\ A_{k\ell} - \dfrac{A_{i\ell} \cdot A_{kj}}{A_{ij}} & \text{if } k \neq i,\ \ell \neq j \end{cases} ,$$

and, finally,

$$b'_k = \begin{cases} \dfrac{b_i}{A_{ij}} & \text{if } k = i \\ b_k - \dfrac{A_{kj} \cdot b_i}{A_{ij}} & \text{if } k \neq i \end{cases} , \qquad c'_\ell = \begin{cases} -\dfrac{c_j}{A_{ij}} & \text{if } \ell = j \\ c_\ell - \dfrac{A_{i\ell} \cdot c_j}{A_{ij}} & \text{if } \ell \neq j \end{cases} , \quad \text{and} \quad q' = q + \frac{b_i \cdot c_j}{A_{ij}} .$$

(b) The short tableau in (3) corresponds to the canonical form linear program (P_2), and its dual is given by (D_2) below.

$$\begin{array}{rrcl} \max & (c')^\top x' & + & q' \\ & A'x' & \leq & b' \\ & x' & \in & \mathbb{R}^n_{\geq 0} \end{array} \quad (P_2) \qquad\qquad \begin{array}{rrcl} \min & (b')^\top y' & + & q' \\ & (A')^\top y' & \geq & c' \\ & y' & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad (D_2)$$

Transforming (D_2) into canonical form yields

$$\begin{array}{rrcl} \max & (-b')^\top y' & - & q' \\ & (-A')^\top y' & \leq & -c' \\ & y' & \in & \mathbb{R}^m_{\geq 0} \ , \end{array}$$

and the corresponding short tableau, using $w = (b')^\top y' + q'$, is

$$\begin{array}{c|ccc|c} & y'_1 & \dots & y'_m & 1 \\ \hline -w & b'_1 & \dots & b'_m & -q' \\ \hline x'_1 & -A'_{11} & \dots & -A'_{m1} & -c'_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x'_n & -A'_{1n} & \dots & -A'_{mn} & -c'_n \end{array} \tag{4}$$

(c) The original (primal) LP (P_1) and its dual (D_1) are the following.

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \le & b \\ & x & \in & \mathbb{R}^n_{\ge 0} \end{array} \quad (P_1) \qquad\qquad \begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & \ge & c \\ & y & \in & \mathbb{R}^m_{\ge 0}\ . \end{array} \quad (D_1)$$

After transforming (D_1) into canonical form

$$\begin{array}{rrcl} \max & -b^\top y & & \\ & -A^\top y & \le & -c \\ & y & \in & \mathbb{R}^m_{\ge 0}\ , \end{array}$$

we can write down the corresponding short simplex tableau

$$\begin{array}{c|ccc|c} & y_1 & \dots & y_m & 1 \\ \hline -w & b_1 & \dots & b_m & -q \\ \hline x_1 & -A_{11} & \dots & -A_{m1} & -c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & -A_{1n} & \dots & -A_{mn} & -c_n \end{array} \tag{5}$$

where again $q = 0$ and $w = b^\top y + q$.

(d) Starting from the tableau in (5) and pivoting on the element in the j^{th} row and i^{th} column of the matrix, i.e., the entry $-A_{ij}$, we get a tableau

$$\begin{array}{c|ccc|c} & y'_1 & \dots & y'_m & 1 \\ \hline -w & b''_1 & \dots & b''_m & -q'' \\ \hline x'_1 & -A''_{11} & \dots & -A''_{m1} & -c''_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x'_n & -A''_{1n} & \dots & -A''_{mn} & -c''_n \end{array} \tag{6}$$

Note that by the pivot choice, we can reuse the new variables x' and y', which were defined by

$$x'_k := \begin{cases} y_i & \text{if } k = j \\ x_k & \text{if } k \ne j \end{cases} \qquad \text{and} \qquad y'_\ell := \begin{cases} x_j & \text{if } \ell = i \\ y_\ell & \text{if } \ell \ne i \end{cases}\ .$$

Moreover, we can calculate the short tableau entries and get

$$-A''_{k\ell} = \begin{cases} -\dfrac{1}{A_{ij}}, & \text{if } k = i,\ \ell = j \text{ (pivot element)} \\ \dfrac{A_{kj}}{A_{ij}} & \text{if } k \ne i,\ \ell = j \text{ (pivot row)} \\ -\dfrac{A_{i\ell}}{A_{ij}} & \text{if } k = i,\ \ell \ne j \text{ (pivot column)} \\ -A_{k\ell} + \dfrac{A_{i\ell} \cdot A_{kj}}{A_{ij}} & \text{if } k \ne i,\ \ell \ne j \end{cases},$$

as well as

$$b''_k = \begin{cases} \dfrac{b_i}{A_{ij}} & \text{if } k = i \\ b_k - \dfrac{A_{kj} \cdot b_i}{A_{ij}} & \text{if } k \neq i \end{cases}, \qquad -c''_\ell = \begin{cases} \dfrac{c_j}{A_{ij}} & \text{if } \ell = j \\ -c_\ell + \dfrac{A_{i\ell} \cdot c_j}{A_{ij}} & \text{if } \ell \neq j \end{cases},$$

$$\text{and} \quad -q'' = -q - \frac{b_i \cdot c_j}{A_{ij}} .$$

(e) Comparing the definitions, we see that $A' = A''$, $b' = b''$, $c' = c''$, and $q' = q''$. Thus, the tableaus (4) and (6) are identical, which shows that indeed, the operations of pivoting and dualizing commute (given the right choice of the pivot element).

Problem 5: Strong LP duality and Farkas' Lemma

(a) First, assume that strong linear programming duality holds for primal/dual pairs with the primal in canonical form and consider a pair of primal/dual linear programs of the following form:

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \in & \mathbb{R}^n \end{array} \quad (P') \qquad \begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & = & c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad (D')$$

Introducing variables $u, v \in \mathbb{R}^n_{\geq 0}$ such that $u - v = x$ allows us to transform the linear program (P') into a linear program (P'_1) in canonical form and the linear program (D') into the corresponding dual program (D'_1):

$$\begin{array}{rrcrcl} \max & c^\top u & - & c^\top v & & \\ & Au & - & Av & \leq & b \\ & u & & & \in & \mathbb{R}^n_{\geq 0} \\ & & & v & \in & \mathbb{R}^n_{\geq 0} \end{array} \quad (P'_1) \qquad \begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & \geq & c \\ & -A^\top y & \geq & -c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad (D'_1)$$

By strong duality, if (P'_1) has a finite optimum attained by $(u^*, v^*) \in \mathbb{R}^{2n}_{\geq 0}$, then (D'_1) also has a finite optimum attained by $y^* \in \mathbb{R}^m_{\geq 0}$ and $c^\top u^* - c^\top v^* = b^\top y^*$. Then for $x^* := u^* - v^*$, which is feasible for the primal linear program (P'), we have $c^\top x^* = b^\top y^*$, where y^* is feasible for the dual (D'), i.e., strong duality holds for the primal/dual pair of the forms (P') and (D').

Now assume that strong duality holds for primal/dual pairs of the form (P') and (D') and consider a linear program (P) in canonical form together with its dual (D):

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & x & \in & \mathbb{R}^n_{\geq 0} \end{array} \quad (P) \qquad \begin{array}{rrcl} \min & b^\top y & & \\ & A^\top y & \geq & c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \quad (D)$$

Note that we can rewrite these in the form

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & b \\ & -I_n x & \leq & 0 \end{array} \quad (P_1) \qquad \begin{array}{rrcrcl} \min & b^\top y & & & & \\ & A^\top y & - & I_n z & = & c \\ & y & & & \in & \mathbb{R}^m_{\geq 0} \\ & & & z & \in & \mathbb{R}^n_{\geq 0} \end{array} \quad (D_1)$$

Here, I_n is the n-dimensional identity matrix. Moreover, we introduced a new vector of variables $z \in \mathbb{R}^n_{\geq 0}$ in the dual linear program (D). The linear programs (P_1) and (D_1) now have the form of the primal/dual pair (P') and (D'). By strong duality, if (P_1) has a finite optimum attained by $x^* \in \mathbb{R}^n$, then (P) also has a finite optimum attained by $(y^*, z^*) \in \mathbb{R}^{m+n}_{\geq 0}$ and $c^\top x^* = b^\top y^*$. Then x^* and y^* are also feasible for the primal linear program (P) and the dual linear program (D), respectively. Thus, strong duality holds for primal/dual pairs of linear programs with the primal linear program in canonical form.

(b) To prove Farkas' Lemma using strong linear programming duality, consider the following primal/dual pair of linear programs:

$$\begin{array}{rrcl} \max & c^\top x & & \\ & Ax & \leq & 0 \\ & x & \in & \mathbb{R}^n \end{array} \qquad (FP) \qquad\qquad \begin{array}{rrcl} \min & 0^\top y & & \\ & A^\top y & = & c \\ & y & \in & \mathbb{R}^m_{\geq 0} \end{array} \qquad (FD)$$

Observe the following:

- The primal linear program (FP) is always feasible, since $x = 0$ is a feasible solution. Therefore, (FP) either has a finite optimal value or it is unbounded.
- The feasible region of the linear program (FD) is precisely the set of points $y \in \mathbb{R}^m$ such that $A^\top y = c$ and $y \geq 0$, i.e., feasibility of the linear program (FD) is equivalent to alternative (ii) of Farkas' Lemma.
- Alternative (i) of Farkas' Lemma is equivalent to the primal linear program (FP) having solutions with strictly positive objective value. Since such a solution can be scaled by an arbitrary positive factor without loss of feasibility, we see that this is further equivalent to the primal linear program (FP) being unbounded.

To prove Farkas' Lemma, we distinguish the following two cases:

- Suppose that the primal (FP) is unbounded, in which case alternative (i) holds. By duality theory, an unbounded primal implies infeasibility of the dual (FD), hence alternative (ii) does not hold in this case.
- Suppose that the primal (FP) has a finite optimal value, in which case alternative (i) does not hold. By duality theory, the existence of a finite optimal value for the primal implies that the dual (FD) is feasible (and attains the same finite optimal value). This means that alternative (ii) holds in this case.

(c) Let x^* be an optimal solution of (P'), and let J be the index set of all the constraints (rows of A and b) that are tight at x^*. Setting $\overline{A} := A_{J\cdot}$ and $\overline{b} := b_J$, we consider the subsystem $\overline{A}x \leq \overline{b}$ of $Ax \leq b$ which comprises of all the constraints that are tight at x^*. Since x^* is an optimal solution, there is no improving direction from x^*. In other words, there can be no $v \in \mathbb{R}^n$ such that $c^\top v > 0$ and $\overline{A}v \leq 0$. Thus, by Farkas' Lemma, there exists $z \in \mathbb{R}^{|J|}_{\geq 0}$ such that $\overline{A}^\top z = c$. Following the hint, we define a vector $y^* \in \mathbb{R}^m_{\geq 0}$ by

$$y_i^* := \begin{cases} z_i & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases} .$$

By construction, y^* is a feasible solution of the linear program (D') which has a 0 in every entry corresponding to a constraint in (P') that is not tight at x^*. Clearly, it holds that

$$c^\top x^* = (y^*)^\top A x^* \leq (y^*)^\top b \ , \tag{7}$$

since $A^\top y^* = \overline{A}^\top z = c$, $y^* \geq 0$, and $Ax^* \leq b$. However, we actually have equality in (7). Indeed, by construction of y^*, we have

$$(y^*)^\top A = z^\top \overline{A} \qquad \text{and} \qquad (y^*)^\top b = z^\top \overline{b} \ .$$

Moreover, since $\overline{A}x \leq \overline{b}$ is the subsystem of $Ax \leq b$ which consists of all the constraints that are tight at x^* (i.e., we have $\overline{A}x^* = \overline{b}$), we conclude that

$$c^\top x^* = (y^*)^\top A x^* = z^\top \overline{A} x^* = z^\top \overline{b} = (y^*)^\top b \ .$$

Thus, strong linear programming duality holds.