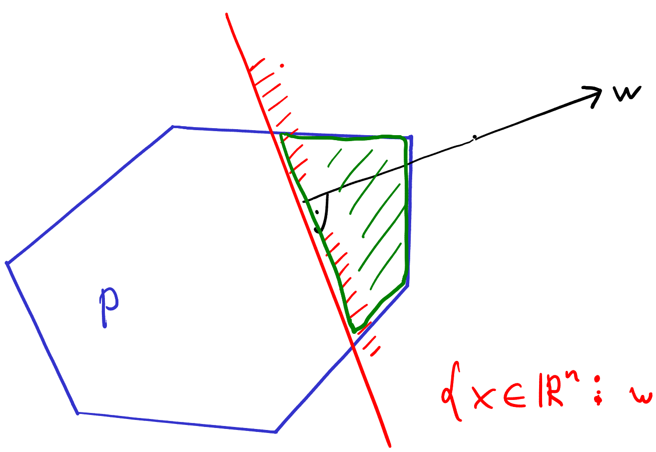
6.4 Ellipsoid Method for finding point in full-dimensional $\{0,1\}$-polytope

We start with simpler question (checking feasibility):

Given a separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, find a point $x \in P$.

Checking feasibility is closely related to optimization



$$\max \; w^T x$$
$$x \in P$$

$H_\nu \cap P$

$\{x \in \mathbb{R}^n : w^T x \geq \nu\} = H_\nu$

Basics on ellipsoids

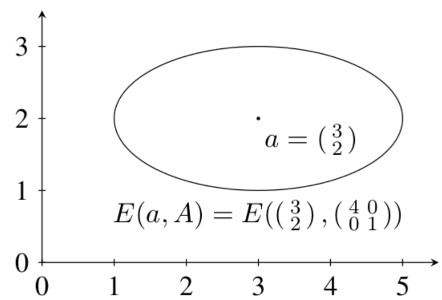
Definition 6.3: Ellipsoid

An ellipsoid in $\mathbb{R}^n$ is a set

$$E(a, A) := \{x \in \mathbb{R}^n : (x-a)^\top A^{-1}(x-a) \le 1\} \ ,$$

where $a \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. The point a is called the *center* of the ellipsoid $E(a, A)$.

this implies that A is symmetric



$x^T A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$

Equivalently, an ellipsoid is the image of the unit ball under an affine bijection:

$A \in \mathbb{R}^{n\times n}$ positive definite $\iff$ $A = QQ^T$ for some full-rank matrix $Q \in \mathbb{R}^{n\times n}$.

$$\begin{aligned}
E(a,A) &= \{x \in \mathbb{R}^n : (x-a)^T A^{-1}(x-a) \le 1\} \\
&= \{x \in \mathbb{R}^n : (x-a)^T (Q^{-1})^T Q^{-1}(x-a) \le 1\} \\
&= \{x \in \mathbb{R}^n : \|Q^{-1}(x-a)\|_2^2 \le 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n,\ \|y\|_2^2 \le 1\} \\
&= \{Qy + a : y \in \mathbb{R}^n,\ \|y\|_2 \le 1\}
\end{aligned}$$

$y = Q^{-1}(x-a)$

$x = Qy + a$

$$\begin{aligned} A^{-1} &= (QQ^T)^{-1} \\ &= (Q^T)^{-1} Q^{-1} \end{aligned}$$

6.4.1 (High-level) description of Ellipsoid Method

Algorithm 8: Ellipsoid Method

Input : Separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, and an ellipsoid $E_0 = E(a_0, A_0)$ with $P \subseteq E_0$.

Output: A point $y \in P$.

$i = 0$.

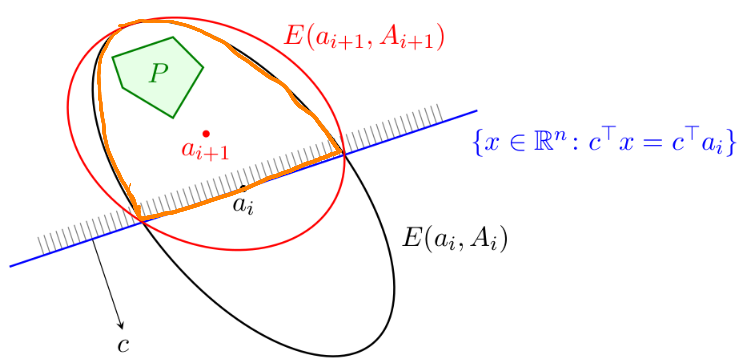
while $a_i \notin P$ *(checked with separation oracle)* **do**

  Get $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top a_i\}$, using separation oracle.

  Find min. volume ellipsoid $E_{i+1} = E(a_{i+1}, A_{i+1})$ containing $E_i \cap \{x \in \mathbb{R}^n : c^\top x \leq c^\top a_i\}$.

  $i = i + 1$.

return a_i.



Two key questions:

- (How quickly) does the Ellipsoid Method terminate?
- How to compute $E_{i+1} = E(a_{i+1}, A_{i+1})$?

6.4.2 Getting a bound on the number of iterations

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} \ .$$

Before proving Lemma 6.4, we show that it implies following bound on number of iterations.

Lemma 6.5

The Ellipsoid Method will stop after at most $2(n+1)\ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right)$ iterations.

Proof

Let $L \in \mathbb{Z}_{\geq 0}$ be last iteration of Ellipsoid Method, i.e., value of i when it terminates.

$$P \subseteq E_L \implies \mathrm{vol}(P) \leq \mathrm{vol}(E_L) \overset{\text{Lemma 6.4}}{\leq} \mathrm{vol}(E_0) \cdot e^{-\frac{L}{2(n+1)}}$$

$$\implies L \leq 2(n+1) \ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P)}\right) .$$

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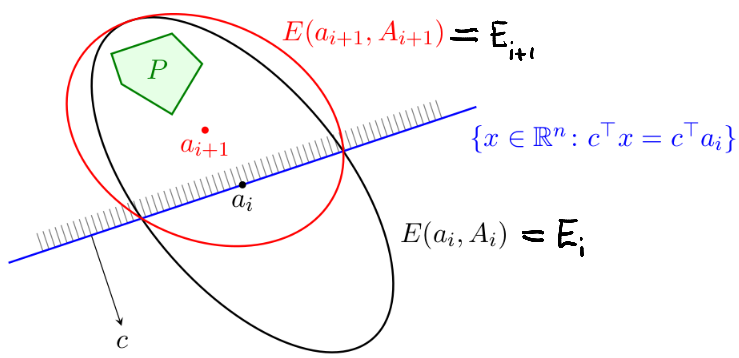
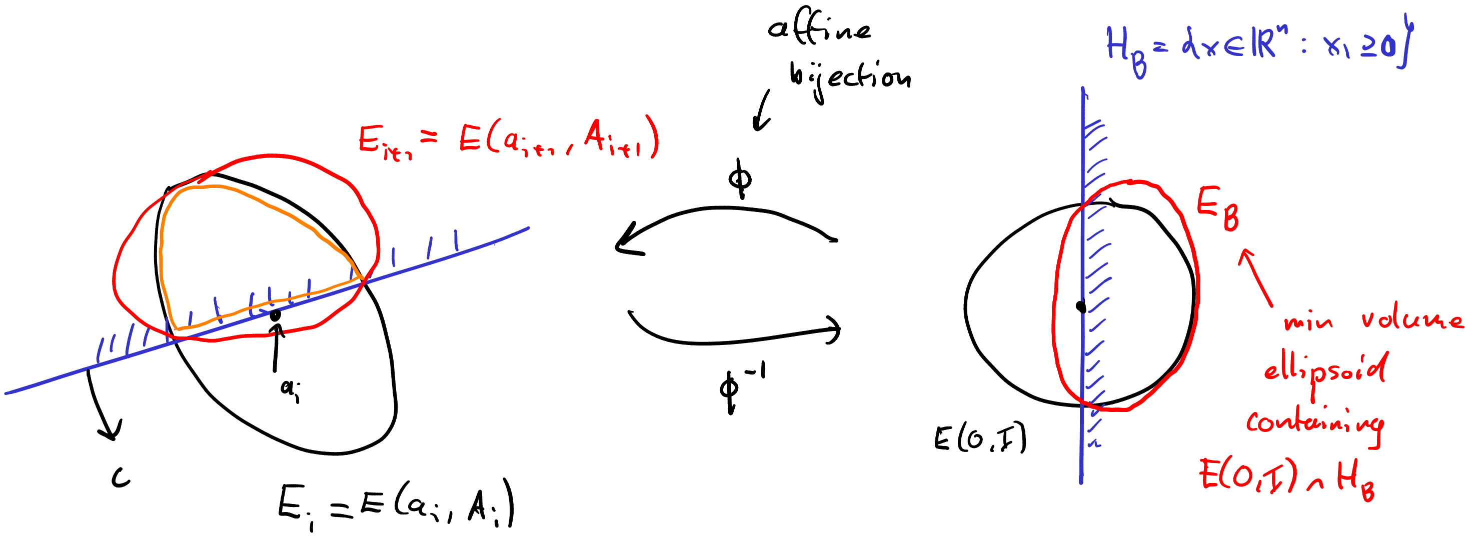
Proof of Lemma 6.4 and explicit description for E_{i+1}



$H_i = \{x \in \mathbb{R}^n : c^\top x \le c^\top a_i\}$

What is ratio between $\mathrm{vol}(E_{i+1})$ and $\mathrm{vol}(E_i)$?

This question can be reduced to unit ball case



$H_i = \{x \in \mathbb{R}^n : c^\top x \le c^\top a_i\}$

$\exists$ affine bijection $\phi : \mathbb{R}^n \to \mathbb{R}^n$ s.t. (i) $\phi(E(0,I)) = E_i$

(ii) $\phi(H_B) = H_i$

Claim

$\phi(E_B)$ is minimum volume ellipsoid containing $E_i \cap H_i$.

Proof

ϕ is injection

$$E(0,I) \cap H_B \subseteq E_B \implies \phi(E_B) \supseteq \phi\left(E(0,I) \cap H_B\right) = \underbrace{\phi(E(0,I))}_{=E_i} \cap \underbrace{\phi(H_B)}_{=H_i}$$

Analogous reasoning implies $\phi^{-1}(E_{i+1}) \supseteq E(0,I) \cap H_B$

min volume ellipsoid containing $E_i \cap H_i$.

$$\xRightarrow[\text{ellipsoid containing } E(0,I) \cap H_B]{E_B \text{ is smallest volume}} \quad \mathrm{vol}(E_B) \leq \mathrm{vol}\left(\phi^{-1}(E_{i+1})\right)$$

$$\xRightarrow[\text{all volumes by same non-negative factor.}]{\text{An affine transformation scales}} \quad \mathrm{vol}\left(\phi(E_B)\right) \leq \mathrm{vol}\left(E_{i+1}\right)$$

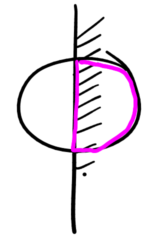
$\#_{\text{claim}}$

Lemma 6.7

Let $H_B = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Then the ellipsoid

$$E_B = \left\{ x \in \mathbb{R}^n \;\middle|\; \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{j=2}^n x_j^2 \leq 1 \right\} \tag{6.7}$$

contains $E(0, I) \cap H_B$.



Proof

Let $x \in E(0,I) \cap H_B$.

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{j=2}^n x_j^2$$

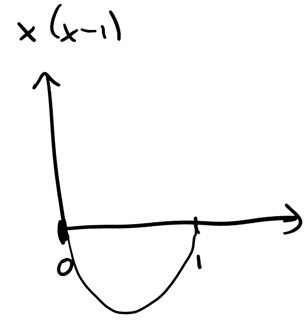
$$= \frac{1}{n^2} x_1^2 \left((n+1)^2 - n^2 + 1\right) - \frac{2}{n^2} x_1 (n+1) + \underbrace{\frac{1}{n^2} + \frac{n^2-1}{n^2} \underbrace{\sum_{j=1}^n x_j^2}_{\leq 1 \;\uparrow\; x \in E(0,I)}}_{\leq 1}$$

$$\leq \frac{1}{n^2} x_1^2 (2n+2) - \frac{2}{n^2} x_1 (n+1) + 1$$

$$= \frac{2}{n^2}(n+1) \underbrace{x_1(x_1 - 1)}_{\leq 0} + 1 \leq 1$$

$$\uparrow$$

$$\underset{\substack{\uparrow \\ x \in H_B}}{0 \leq} x_1 \underset{\substack{\uparrow \\ x \in E(0,I)}}{\leq} 1$$



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E_B is actually min volume ellipsoid containing $E(0,I) \cap H_B$.
(see problem sets)

Proof of Lemma 6.4

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}} .$$

By above results: $\dfrac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} \overset{\leq}{=} \dfrac{\mathrm{vol}(E_B)}{\mathrm{vol}(E(0,I))}$, where

$$E_B = \left\{ x \in \mathbb{R}^n : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{j=2}^{n} x_j^2 \leq 1 \right\}$$

(defined as in Lemma 6.7)

$$E_B = E(a, A) := \{ x \in \mathbb{R}^n : (x-a)^T A^{-1} (x-a) \leq 1 \}$$

$\psi(x) = Qx + a$ where $Q \in \mathbb{R}^{n \times n}$ s.t. $A = QQ^T$

$$\Rightarrow \quad E(a, A) = \psi(E(0, I))$$

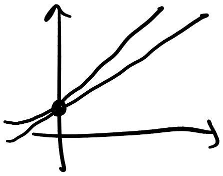
$$a = \left(\frac{1}{n+1}, 0, 0, \dots, 0 \right)$$

$$A^{-1} = \begin{pmatrix} \left(\frac{n+1}{n}\right)^2 & & & 0 \\ & \frac{n^2-1}{n^2} & & \\ & & \ddots & \\ 0 & & & \frac{n^2-1}{n^2} \end{pmatrix} \quad \Rightarrow \quad A = \begin{pmatrix} \left(\frac{n}{n+1}\right)^2 & & & 0 \\ & \frac{n^2}{n^2-1} & & \\ & & \ddots & \\ 0 & & & \frac{n^2}{n^2-1} \end{pmatrix}$$

$$\Rightarrow \quad \frac{\mathrm{vol}(E_B)}{\mathrm{vol}(E(0,I))} = \frac{\mathrm{vol}(\psi(E(0,I)))}{\mathrm{vol}(E(0,I))} = |\det Q|$$

$$= \sqrt{\det A} = \frac{n}{n+1} \cdot \left(\frac{n^2}{n^2-1} \right)^{\frac{n-1}{2}}$$

$A = QQ^T$



$e^x > 1 + x$

$x \in \mathbb{R} \setminus \{0\}$

$$= \left(1 - \frac{1}{n+1}\right) \cdot \left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}}$$

$$< e^{-\frac{1}{n+1}} \cdot e^{\frac{n-1}{2(n^2-1)}}$$

$$= e^{-\frac{1}{n+1}} \cdot e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}$$

#

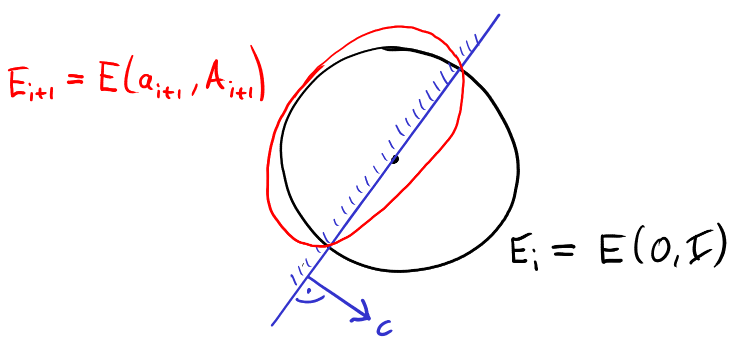
6.4.3 From the unit ball to the general case

We obtained explicit description of E_{i+1} if

- $E_i = E(0, I)$, and
- $H_i = \{x \in \mathbb{R}^n : x_1 \geq 0\}$

From this we obtain explicit description of E_{i+1} for the general case by transforming description of this special case through an approviate affine bijection.

General half-space cutting $E(0,I)$



$E_i = E(0, I)$

$H_i = \{x \in \mathbb{R}^n : c^T x \leq 0\}$,

with $\|c\|_2 = 1$

$\longrightarrow$ see problem sets

$E_{i+1} = E(a_{i+1}, A_{i+1})$, where

$$a_{i+1} = -\frac{1}{n+1} c$$

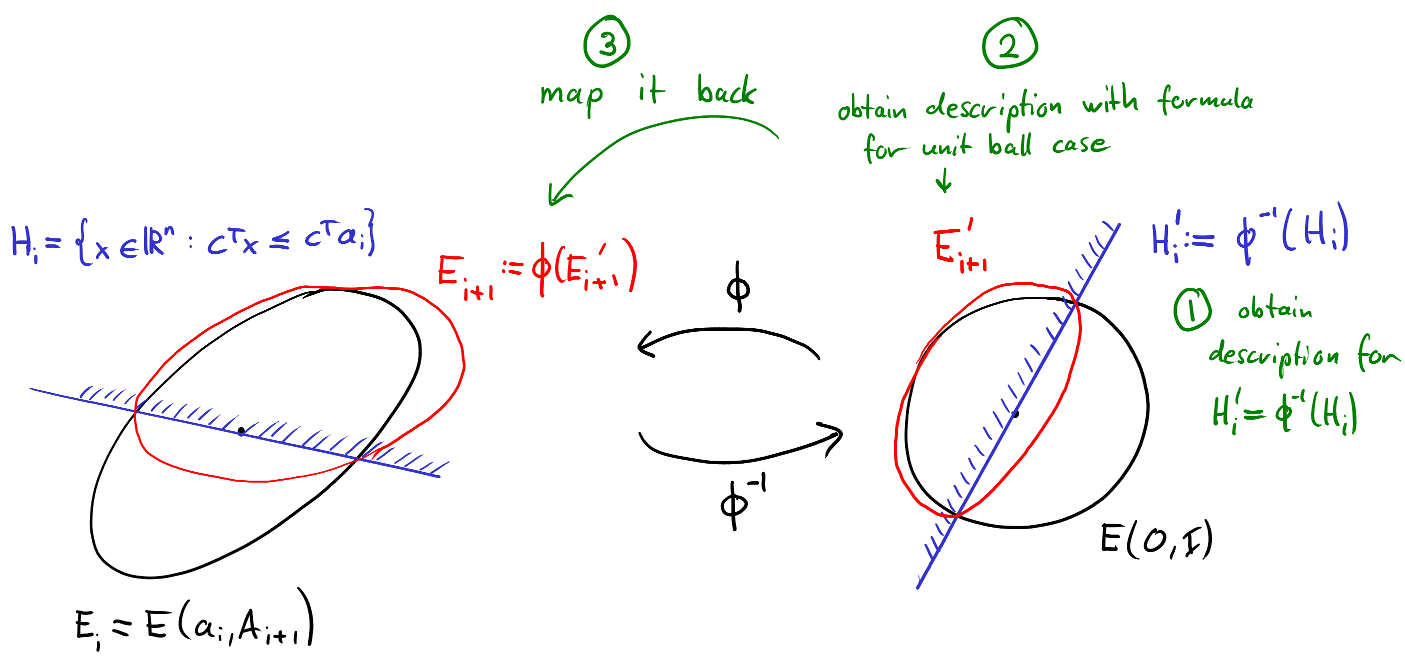
$$A_{i+1} = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} c c^T\right)$$

General case

Let $E_i = E(a_i, A_i)$ be a general ellipsoid, and let
$H_i = \{x \in \mathbb{R}^n : c^T x \leq c^T a_i\}$ be a general halfspace going through a_i.

$$\left.\begin{array}{l} \text{Let } Q_i \in \mathbb{R}^{n\times n} \text{ s.t. } A_i = Q_i Q_i^T. \\ \phi(x) := Q_i x + a_i . \end{array}\right\} \Rightarrow \phi(E(0,I)) = E_i .$$

Plan:



$$\phi(x) = Q_i x + a_i$$
$$\phi^{-1}(x) = Q_i^{-1}(x - a_i)$$

① $H_i' := \phi^{-1}(H_i) = \{ Q_i^{-1}(x-a) : x \in \mathbb{R}^n,\ c^T x \le c^T a_i \}$

$y = Q_i^{-1}(x-a)$

$$= \{ y \in \mathbb{R}^n : c^T(Q_i y + a) \le c^T a_i \}$$

$$= \{ y \in \mathbb{R}^n : (Q_i^T c)^T y \le 0 \}$$

$\Rightarrow H_i' = \{ x \in \mathbb{R}^n : d^T x \le 0 \}$ with $d := \dfrac{Q_i^T c}{\sqrt{c^T Q_i Q_i^T c}}$

② By unit ball case:

$$E_{i+1}' = E(a_{i+1}', A_{i+1}'), \text{ where}$$

$$a_{i+1}' = -\frac{1}{n+1} d$$

$$A_{i+1}' = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} dd^T \right)$$

③ $E_{i+1} = \phi(E_{i+1}') = \{ \phi(x) : x \in \mathbb{R}^n,\ (x - a_{i+1}')^T (A_{i+1}')^{-1} (x - a_{i+1}') \le 1 \}$

$$= \{ y \in \mathbb{R}^n : (Q_i^{-1}(y-a_i) - a_{i+1}')^T (A_{i+1}')^{-1} (Q_i^{-1}(y-a_i) - a_{i+1}') \le 1 \}$$

$\phi(x) = Q_i x + a_i$

$\phi^{-1}(x) = Q_i^{-1}(x - a_i)$

$$= \{ y \in \mathbb{R}^n : (y - a_i - Q_i a_{i+1}')^T (Q_i^{-1})^T (A_{i+1}')^{-1} Q_i^{-1} (y - a_i - Q_i a_{i+1}') \le 1 \}$$

$$= \{ y \in \mathbb{R}^n : (y - a_i - Q_i a_{i+1}')^T (Q_i A_{i+1}' Q_i^T)^{-1} (y - a_i - Q_i a_{i+1}') \le 1 \}$$

Recall: $a'_{i+1} = -\frac{1}{n+1} d$

$$a_{i+1} = a_i + Q_i a'_{i+1} = a_i - \frac{1}{n+1} Q_i d = a_i - \frac{1}{n+1} b,$$

where $b = Q_i d = Q_i \frac{Q_i^T c}{\sqrt{c^T Q_i Q_i^T c}} = \frac{A_i c}{\sqrt{c^T A_i c}}$

$$\left(A_{i+1}^{-1} = (Q_i^{-1})^T (A'_{i+1})^{-1} Q_i^{-1} \right) \qquad A'_{i+1} = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} dd^T\right)$$

$$\Rightarrow A_{i+1} = Q_i A'_{i+1} Q_i^T = \frac{n^2}{n^2-1}\left(A_i - \frac{2}{n+1} Q_i d d^T Q_i^T\right)$$

$$= \frac{n^2}{n^2-1}\left(A_i - \frac{2}{n+1} bb^T\right)$$

Hence, we can now write more explicitely how an iteration of the Ellipsoid Method looks.

Algorithm 9: Ellipsoid Method

Input : Separation oracle for a polytope $P \subseteq \mathbb{R}^n$ with $\dim(P) = n$, and an ellipsoid $E_0 = E(a_0, A_0)$ with $P \subseteq E_0$.

Output: A point $y \in P$.

$i = 0$.

while $a_i \notin P$ *(checked with separation oracle)* **do**

- Get $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top a_i\}$, using separation oracle.
- Let $b = \frac{A_i c}{\sqrt{c^\top A_i c}}$.
- Let $a_{i+1} = a_i - \frac{1}{n+1} b$.
- Let $A_{i+1} = \frac{n^2}{n^2-1}(A_i - \frac{2}{n+1} bb^\top)$.
- $i = i + 1$.

return a_i.

6.4.4 From checking feasibility to optimization over $\{0,1\}$-polytopes

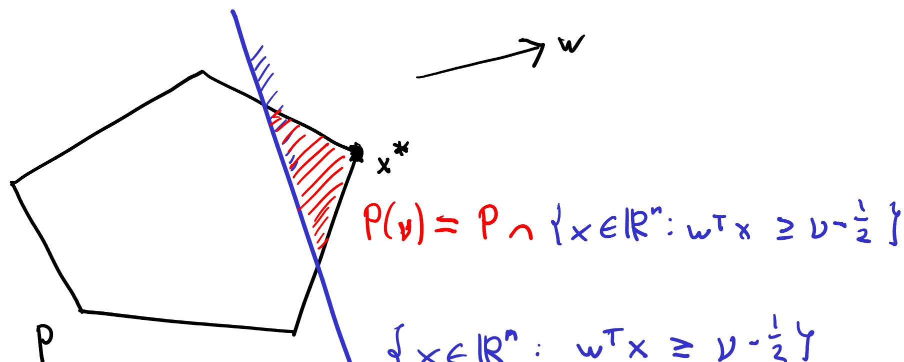
Let $P \subseteq \mathbb{R}^n$ be a full-dimensional $\{0,1\}$-polytope.

We want to solve:

$$\max_{x \in P} w^T x$$

for some $w \in \mathbb{Z}^n$.

Getting optimal LP value $\nu^* = \max\{w^T x : x \in P\}$



$w_{max} = \max\{|w_k| : k \in [n]\}$

P is $\{0,1\}$-polytope $\Rightarrow \exists\, x^* \in P \cap \{0,1\}^n$ with $w^T x^* = \nu^*$.

$$\Rightarrow \quad \nu^* \in [-n w_{max}, n w_{max}] \cap \mathbb{Z}$$

For $\nu \in [-n w_{max}, n w_{max}] \cap \mathbb{Z}$ let

$$P(\nu) = P \cap \{x \in \mathbb{R}^n : w^T x \geq \nu - \tfrac{1}{2}\}$$

Observe:

- ν^* is largest value of $\nu \in [-nw_{max}, nw_{max}] \cap \mathbb{Z}$ s.t. $P(\nu) \neq \emptyset$.
- For any $\nu \in \mathbb{Z}$, either $P(\nu) = \emptyset$ or $\dim(P(\nu)) = n$.

$\rightarrow$ We do binary search over $[-nw_{max}, nw_{max}] \cap \mathbb{Z}$ to find largest $\nu \in [-nw_{max}, nw_{max}] \cap \mathbb{Z}$ s.t. $\underline{P(\nu) \neq \emptyset}$.

we check this with Ellipsoid Method

<u>Plan</u>

We run Ellipsoid on $P(\nu)$ for sufficiently many iterations (enough to find a point in $P(\nu^*)$) to find a point in $P(\nu)$ if $P(\nu) \neq \emptyset$. If Ellipsoid fails to find a point in $P(\nu)$ after that many iterations, we conclude $P(\nu) = \emptyset$.

Starting ellipsoid

Choose $E_0 = E(a_0, A_0)$ to be smallest ellipsoid containing $[0,1]^n$.

$\rightarrow$ E_0 has center $a_0 = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)^T$ and radius $\frac{1}{2}\sqrt{n}$.

$$\mathrm{vol}(E_0) = \left(\tfrac{1}{2}\sqrt{n}\right)^n \underbrace{\mathrm{vol}(E(0,I))}_{\leq 2^n} \leq n^{n/2}$$

$E(0,I) \subseteq [-1,1]^n$

$$\Rightarrow \log(\mathrm{vol}(E_0)) = O(n \log n)$$

Bounding number of iterations

Recall:

Lemma 6.4

$$\frac{\mathrm{vol}(E_{i+1})}{\mathrm{vol}(E_i)} < e^{-\frac{1}{2(n+1)}}\ .$$

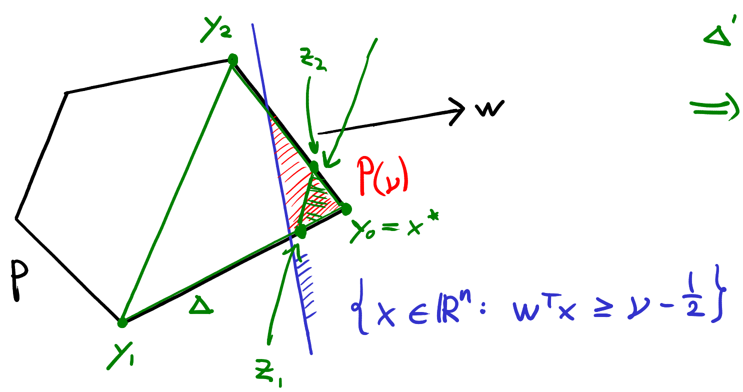
$\Rightarrow$ # of iterations $\leq 2(n+1) \ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P(\nu))}\right)$

Assuming $P(\nu) \neq \emptyset$, we need lower bound on $\mathrm{vol}(P(\nu))$.

Plan

$\Delta' \leftarrow$ scaled-down version of Δ s.t.
$\Delta' \subseteq P(\nu)$

$\Rightarrow \mathrm{vol}(\Delta') \leq \mathrm{vol}(P(\nu))$



Note that $P(\nu) \neq \emptyset$ implies $x^* \in P(\nu)$.

Let $y_0 = x^* \in \text{vertices}(P)$, and let $y_1, y_2, \dots, y_n \in \text{vertices}(P)$ s.t. $\Delta = \text{conv}(\{y_0, \dots, y_n\})$ is full-dimensional.

Let Δ' be simplex obtained by scaling Δ towards y_0 with scaling factor $\alpha = \frac{1}{2n w_{max}}$.

$\Rightarrow$ vertices of Δ' are

- $z_0 = y_0$
- $z_i = y_0 + \alpha(y_i - y_0) \quad \forall\, i \in [n]$

To show : $\Delta' \subseteq P(\nu)$ or equivalently, $z_i \in P(\nu) \ \forall\, i \in [n]$.

Recall : $P(\nu) = P \cap \{x \in \mathbb{R}^n : w^T x \geq \nu - \frac{1}{2}\}$.

Cleary, $z_i \in P$. Moreover

$$w^T z_i = w^T(y_0 + \alpha(y_i - y_0)) = \underbrace{w^T y_0}_{= \nu^* \geq \nu} + \alpha \underbrace{w^T(y_i - y_0)}_{\geq -n w_{max}} \geq \nu - \frac{1}{2}.$$

$\Rightarrow \Delta' \subseteq P(\nu). \Rightarrow \text{vol}(P(\nu)) \geq \text{vol}(\Delta')$

$$= \alpha^n \cdot \underbrace{\text{vol}(\Delta)}_{\geq \frac{1}{n!}}$$

$$\geq \left(\frac{1}{2n w_{max}}\right)^n \cdot \frac{1}{n!}$$

$$\Rightarrow -\log(P(\nu)) \leq O(n(\log n + \log w_{max}))$$

$\longrightarrow$ # of Ellipsoid iterations can be bounded

$$2(n+1)\ln\left(\frac{\mathrm{vol}(E_0)}{\mathrm{vol}(P(y))}\right) = O\left(n^2\left(\log n + \log w_{max}\right)\right).$$

Determining an optimal $\{0,1\}$-solution x^*

One can determine x^* coordinate-wise from x_1^*, x_2^* to x_n^*, by repeatedly solving LPs over P with slightly modified objectives.

To check whether there exists optimal solution x^* to $\max\{w^T x : x \in P\}$ with $x_1^* = 1$, ... solve

$$\bar{\nu} := \max\{\bar{w}^T x : x \in P\}, \text{ where}$$

$$\bar{w}_i = \begin{cases} w_1 + 1 & \text{if } i = 1 \\ w_i & \text{if } i \in [n] \setminus \{1\} \end{cases}$$

Observation: $\exists$ opt. sol. x^* with $x_1^* = 1 \iff \bar{\nu} = \nu^* + 1$.

Assume $\exists$ opt. sol x^* with $x_1^* = 1$.

↳ set $S = \{1,2\}$ and optimize $\nu_S = \max\{(w^S)^T x : x \in P\}$, where

$$w_i^S = \begin{cases} w_i + 1 & \text{if } i \in S, \\ w_i & \text{if } i \in [n] \setminus S. \end{cases}$$

Observation: $\exists$ opt. sol. x^* s.t. $x_i^* = 1 \ \forall i \in S \iff \nu_S = \nu^* + |S|$.

... (see problem sets)

$\Rightarrow$

Theorem 6.9

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x \colon x \in P\}$ using a polynomial number of elementary operations and calls to the separation oracle for P.

6.5 Comments on the non-full-dimensional case

Theorem 6.2

Let $P \subseteq \mathbb{R}^n$ be a $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x \colon x \in P\}$ using a polynomial number (in n) of elementary operations and calls to the separation oracle for P.