5.8.2 Integrality of dominant of r-arborescence polytope

Recall, for a directed graph $G=(V,A)$, we have:

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\} .$$

Proof of integrality of P

Lemma 5.27

For any two sets $S_1, S_2 \subseteq V$, we have

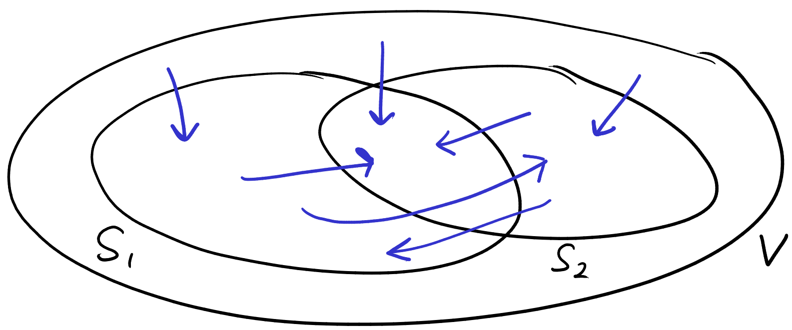
$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} + \chi^{A(S_1 \setminus S_2, S_2 \setminus S_1)} + \chi^{A(S_2 \setminus S_1, S_1 \setminus S_2)} \ ,$$

which implies in particular

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} \geq \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} \ .$$

Proof

Idea: Check all different "arc types":



Lemma 5.28

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $A(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, $A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset$. In particular, this implies by Lemma 5.27

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} \ .$$

Proof

Back to: $\square$ implies $*$.