1 Linear Programming and Polyhedra

1.1 Introduction to linear programming

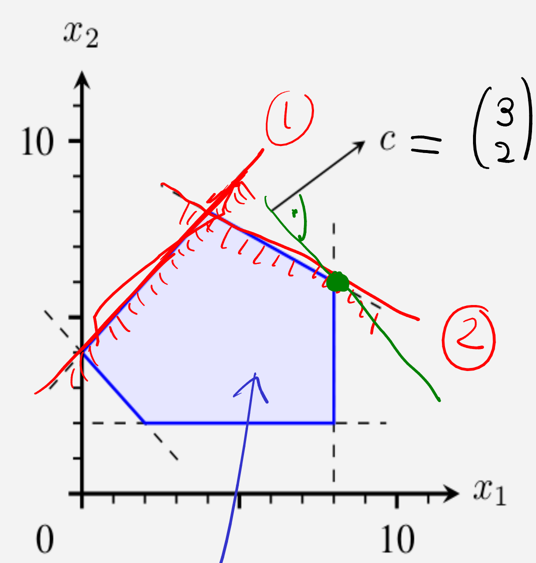
Getting some intuition : a 2D example

objective (function)

$$\boxed{\max \quad 3x_1 + 2x_2}$$

$$\begin{array}{rcrcl}
-x_1 & + & x_2 & \leq & 4 \quad (1) \\
x_1 & + & 2x_2 & \leq & 20 \quad (2) \\
x_1 & & & \leq & 8 \\
 & & x_2 & \geq & 2 \\
x_1 & + & x_2 & \geq & 4
\end{array}$$

constraints



feasible region

General and canonical form

$$
\begin{array}{lrcrcrcr}
\max & 6x_1 & + & 5x_2 & + & 5.5x_3 & & \\
\text{subject to} & 10x_1 & - & x_2 & - & 2.5x_3 & \geq & 11.5 \\
 & -21x_1 & + & x_2 & - & 6x_3 & \geq & -104 \\
 & 4.25x_1 & + & 2.75x_2 & - & x_3 & \leq & 24 \\
 & & & x_2 & & & \geq & 0 \\
 & 10x_1 & - & x_2 & + & 35x_3 & \geq & 49 \\
 & x_1 & & & + & 2x_3 & = & 12\ ,
\end{array}
$$

$$
\begin{array}{lrcl}
\max/\min & c^\top x & & \\
 & Ax & \leq & e \\
 & Bx & \geq & f \\
 & Cx & = & g
\end{array}
$$

(general LP)

$$
\begin{array}{lrcl}
\max & c^\top x & & \\
 & Ax & \leq & b \\
 & x & \geq & 0
\end{array}
$$

(LP in canonical form)

How to transform a general LP into canonical form?

- From min to max: $\min c^T x \rightarrow -\max (-c)^T x$
- Reduce to nonnegative variables:

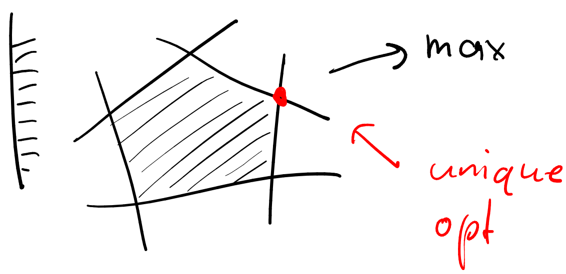
  Replace each variable $x_i \rightarrow x_i^+ - x_i^-$ with $x_i^+, x_i^- \geq 0$
- Reduce to $\leq$ constraints: $a^T x \geq \beta \rightarrow (-a)^T x \leq -\beta$

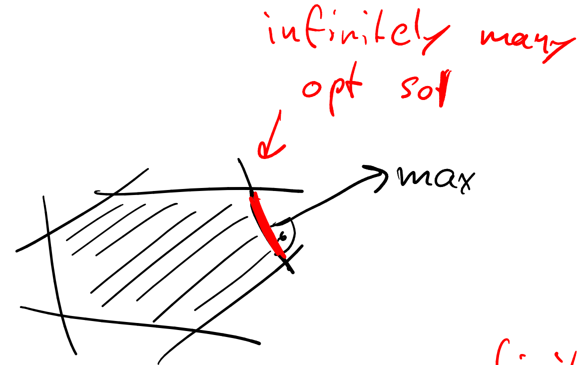
$$a^T x = \beta \rightarrow \begin{cases} a^T x \leq \beta \\ (-a)^T x \leq -\beta \end{cases}$$

1.1.1 Different types of LPs and goal of LP algorithms

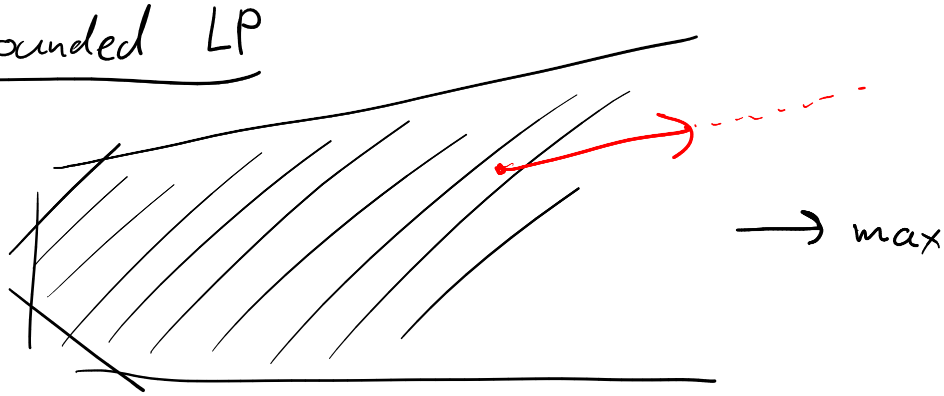
Each LP is one of 3 types:

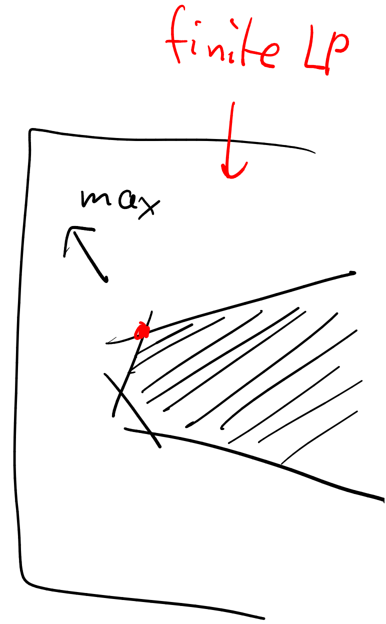
① LP with finite optimum



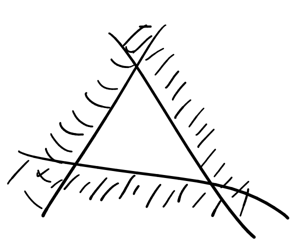


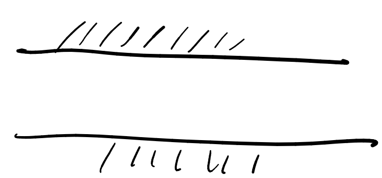
② Unbounded LP





③ Infeasible LP





$x_2 \geq 1$

$x_2 \leq 0$