Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Prof. Dr. Rico Zenklusen and Assistants
Contact: math.opt@ifor.math.ethz.ch

Institute for Operations Research
Department of Mathematics

Fall 2019

Mathematical Optimization – Solutions to problem set 2

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Finding a Chebychev center of a polyhedron

(a) If, as given in the hint, B is a ball with center y and radius r that is fully contained in a single halfspace $\{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$, then the point closest to the hyperplane $\{x \in \mathbb{R}^n : a_i^\top x = b_i\}$ is given by

$$z^{(i)} = y + \frac{a_i}{\|a_i\|_2} \cdot r \quad .$$

In fact, we can observe that the ball B is fully contained in the halfspace $\{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$ if and only if $z^{(i)}$ is. The latter can be written as $a_i^\top z^{(i)} \leq b_i$, which is equivalent to

$$a_i^\top y + \|a_i\|_2 \cdot r \leq b_i \quad . \tag{1}$$

To obtain a ball centered at y with radius r that is fully contained in the polyhedron P, we simply write $A = \begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix}^\top$ and $b = \begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix}^\top$ and require (1) for all $i \in [m]$. Maximizing the radius as an objective function, we get the linear program

$$\begin{array}{llll} \max & r & & \\ & a_i^\top y + \|a_i\|_2 \cdot r & \leq b_i & \forall i \in [m] \\ & y & \in \mathbb{R}^n & \\ & r & \in \mathbb{R}_{\geq 0} \ . & \end{array}$$

We argued above that any feasible solution of this linear program corresponds to a ball that is fully contained in P, thus the optimal value is at most the optimal radius.

On the other hand, if y is a Chebychev center and r is the corresponding maximum possible radius (note that we assume that such y and r exist in this part of the problem), then (y, r) is feasible for the linear program, and it has value r. Thus, an optimal solution of the linear program has value at least r.

Together, the previous paragraphs prove that the optimal solution value of the linear program is equal to the optimal radius, and every optimal solution y is a Chebychev center.

(b) We proved in the previous part that if the linear program is feasible and bounded, every optimal solution corresponds to a Chebychev center.

If the linear program is unbounded, this means that there exist feasible points (y, r) with arbitrarily large r. These correspond to balls inscribed into P with arbitrarily large radius. Thus, in this case there does not exist a Chebychev center.

If the linear program is infeasible, we know that not even a ball of radius 0 around any point y, i.e., the set $\{y\}$, is contained in P. Thus, P must be empty, in which case there does obviously not exist a Chebychev center, either.

To summarize, we can decide whether there exists a Chebychev center based on the linear program: There exists one if and only if the linear program is feasible and has finite optimal value.

Problem 2: Existence of vertices in full-rank polyhedra

(a) Assume for contradiction that there exist $v, w \in \mathbb{R}^n$ with $w \neq 0$ such that $L(v, w) \subseteq P$. This means that for all $\lambda \in \mathbb{R}$, we have $A(v + \lambda w) \leq b$, or equivalently,

$$\lambda \cdot Aw \leq b - Av \qquad \forall \lambda \in \mathbb{R} \quad .$$

We claim that this implies $Aw = 0$. Indeed, if for some coordinate $i \in [n]$, we have $(Aw)_i > 0$, choosing $\lambda > \frac{(b-Av)_i}{(Aw)_i}$ gives a contradiction to the above inequality. Analogously, if $(Aw)_i < 0$ for some $i \in [n]$, choosing $\lambda < \frac{(b-Av)_i}{(Aw)_i}$ gives a contradiction.

Thus, w is a non-zero vector that lies in the kernel of A—but this is impossible because A has full column rank.

(b) Suppose that the polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ is non-empty, i.e., there exists a point $y \in P$. If y is a vertex, we are done. Else, denote by $A^{(1)}$ the submatrix of A and by $b^{(1)}$ the subvector of b that correspond to all the constraints of P that are tight at y. Similarly, denote by $A^{(2)}$ the submatrix of A and by $b^{(2)}$ the subvector of b that correspond to the constraints that are not tight at y. This way, we separate the constraints that hold with equality and strict inequality at y.

Since y is not a vertex, there exist two points $y_1, y_2 \in P$ such that $y = \frac{y_1+y_2}{2}$. Note that

$$A^{(1)}y_1 = A^{(1)}y_2 = A^{(1)}y = b \ . \tag{2}$$

Indeed, if $\left(A^{(1)}y_1\right)_i > b_i$ for some $i \in [m]$, then $y_1 \in P$ violates one of the constraints of the polyhedron P, which is a contradiction; and if $\left(A^{(1)}y_1\right)_i < b_i$ for some $i \in [m]$, then

$$\left(A^{(1)}y_2\right)_i = \left(A^{(1)}\left(2y - y_1\right)\right)_i = 2\left(A^{(1)}y\right)_i - \left(A^{(1)}y_1\right)_i > 2b_i - b_i = b_i \ ,$$

so $y_2 \in P$ violates one of the constraints of the polyhedron P, which is a contradiction, as well.

Consider the line $L(y, w)$, where $w \coloneqq y_2 - y_1 \neq 0$. By (2), every point $z \in L(y, w)$ satisfies $A^{(1)}z = b^{(1)}$. Since A has full column rank, we know from part (a) that the polyhedron P cannot contain the full line $L(y, w)$. Consequently, there exists a point $z \in P \cap L(y, w)$ such that at least one tight constraint at z was not tight at y. By replacing y with z (and updating $A^{(1)}$, $A^{(2)}$, $b^{(1)}$, and $b^{(2)}$ accordingly), we thus increase the size of $A^{(1)}$ by at least 1. As A has full column rank, iteratively applying this procedure will lead to a point $z \in P$ for which $A^{(1)}$ has full column rank. Then, by Proposition 1.19, z is a vertex of the polyhedron P.

Problem 3: Finite linear programming optima are attained

(a) If $\dim(P) = 0$, then P contains precisely one point, and this point must attain the value γ.

(b) Let $w \in \ker(A) \setminus \{0\}$ and define the hyperplane $H = \{x \in \mathbb{R}^n \colon w^\top x = 0\}$. We claim that

(i) $P \cap H$ is non-empty,

(ii) $\gamma = \sup\{c^\top x \colon x \in P \cap H\}$, and

(iii) $\dim(P \cap H) = \dim(P) - 1$,

If these claims are true, it is indeed enough to find a point in the $(\dim(P) - 1)$-dimensional polyhedron $P \cap H$ that attains the supremum γ.

To prove the claims, let $y \in P$ and define $y_\lambda = y + \lambda w$ for $\lambda \in \mathbb{R}$. Observe that $y_\lambda \in P$ for all $\lambda \in \mathbb{R}$, as $Ay_\lambda = Ay + \lambda Aw = Ay \leqslant b$. The objective value of y_λ is

$$c^\top y_\lambda = c^\top y + \lambda c^\top w \ .$$

If $c^\top w \neq 0$, we see that for $\lambda \to \infty$ or $\lambda \to -\infty$ (depending on the sign of $c^\top w$), the objective value of the feasible point y_λ goes to ∞. This is impossible, as the supremum γ is bounded by assumption. Hence, we must have $c^\top w = 0$, and all points y_λ have the same objective value $c^\top y$.

It is easy to see that for $\lambda = -\frac{w^\top y}{w^\top w}$, we have $y_\lambda \in P \cap H$, which immediately proves (i). The fact that this transformation preserves objective values proves (ii). Finally, for any $y \in P \cap H$, the line $\{y_\lambda \colon \lambda \in \mathbb{R}\}$ is contained in P and orthogonal to H. Thus, intersecting P with H does indeed reduce the dimension by one, proving (iii).

(c) Observe that there are only finitely many non-singular subsystems $A'x \leq b'$ of $Ax \leq b$, and hence there are only finitely many options for finding a suitable point y', independently of the point y we start with. Thus, choosing one y' with maximum objective value $c^\top y'$ (among the finitely many options), we must have found a point y' such that $c^\top y' \geqslant c^\top y$ for all $y \in P$. Thus, $c^\top y' = \gamma$, as desired.

(d) To prove the claim, let $y \in P$, and let $A''x \leqslant b''$ be the subsystem of $Ax \leq b$ consisting of all y-tight constraints. If $\operatorname{rank}(A'') = n$, we are done, as we can then take $y' = y$ with a full-rank subsystem of $A''x \leq b''$ that uniquely defines y' when set to equality.

If $\operatorname{rank}(A'') < n$, let $w \in \ker(A'') \setminus \{0\}$ and consider $y_\lambda = y + \lambda w$ for $\lambda \in \mathbb{R}$. We show that there exists $\lambda \in \mathbb{R}$ such that $y_\lambda \in P$ and y_λ has at least one more tight constraint in $Ax \leq b$ compared to y, while additionally, $c^\top y_\lambda \geq c^\top y$. As A has full column rank, iterative applications of this argument eventually lead to a full-rank subsystem that uniquely defines a point of objective at least $c^\top y$, as desired.

To complete the proof, we assume without loss of generality that $c^\top w \geqslant 0$ (else, change the sign of w). We distinguish the following two cases:

- If $y_\lambda \in P$ for all $\lambda > 0$, we must have $c^\top w = 0$ because γ is finite. By problem 2(a) (note that we assume $\ker(A)$ to be trivial here, so A has full column rank), P does not contain the full line $L(y, w) = \{y_\lambda : \lambda \in \mathbb{R}\}$, so there exists a minimum value $\lambda < 0$ with $y_\lambda \in P$. This point y_λ has at least one extra tight constraint compared to y, as desired.
- On the other hand, if $y_\lambda \notin P$ for some $\lambda > 0$, then consider the maximum λ such that $y_\lambda \in P$. This point y_λ has at least one extra tight constraint compared to y, as desired.

Note that in both cases, $c^\top y_\lambda = c^\top y + \lambda \cdot c^\top w \geq c^\top y$: In the first case, $c^\top w = 0$; in the second, $c^\top w \geq 0$ and $\lambda \geq 0$. This completes the proof.

(e) We prove the initial statement by induction on $\dim(P)$. The base case $\dim(P) = 0$ is covered by (a). Thus, consider P with $\dim(P) = k > 0$ and assume we know the statement for dimensions at most $k - 1$. If $\ker(A) \neq \{0\}$, then by (b) and the inductive hypothesis, we are done. In the other case, namely $\ker(A) = \{0\}$, point (c) gives the conclusion, using the claim that is proved in point (d).

Problem 4: Polytopes and vertices

(a) Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$, be an inequality description of the given polytope. We claim that A has full column rank. If so, then by Problem 2(b), we are done: P is assumed to be non-empty, hence there must be a vertex.

To prove the claim, assume for contradiction that A does not have full column rank. Then, $\ker(A)$ is non-trivial, hence there exists $w \in \ker(A) \setminus \{0\}$. Let $y \in P$ and consider the line $L(y, w) = \{y + \lambda w : \lambda \in \mathbb{R}\}$. We have $L(y, w) \subseteq P$, as for any point $z = y + \lambda w \in L(y, w)$, we have $Az = Ay + \lambda Aw = Ay \leq b$. However, this is a contradiction: $L(y, w)$ is unbounded, while P is bounded, by assumption. This finishes the proof.

Remark: Note that the boundedness of P is important. The unbounded polyhedron $P = \mathbb{R}^n$ (with no constraints), for example, has no vertices.

(b) Denote $\nu = \max\left\{c^\top x : x \in P\right\}$. Since P is bounded, ν is finite. Thus, by Corollary 1.12, F is a face of P. By part (a) of this problem, F has a vertex y. Thus, $\{y\}$ is a face of P. By Corollary 1.13, a face of a face of P is a face of P, so $\{y\}$ is also a face of P. Equivalently, y is a vertex of P, and by construction, $c^\top y = \max\{c^\top x : x \in P\}$.

(c) Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$, be an inequality description of P. Denote $Q := \operatorname{conv}(\operatorname{vertices}(P))$ and let $\operatorname{vertices}(P) = \{x_1, x_2, \ldots, x_k\}$. We prove that $P = Q$ by showing that $Q \subseteq P$ and $P \subseteq Q$.

First, we prove that $Q \subseteq P$. To this end, consider a point $y \in Q$, which can by definition of Q be written as a convex combination of vertices of P, i.e., $y = \sum_{i=1}^k \lambda_i x_i$, where $\lambda \in [0,1]^k$ satisfies

$\sum_{i=1}^{k} \lambda_i = 1$. Then

$$Ay = A\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i A x_i \le \sum_{i=1}^{k} \lambda_i b = b \ ,$$

i.e., $y \in P$. Therefore, $Q \subseteq P$.

Now let us show that $P \subseteq Q$. By contradiction, assume there exists a point $y \in P \setminus Q$. Using the separation property given in the hint, we can find $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $a^\top y > \beta$ while $a^\top q < \beta$ for all $q \in Q$. By part (b), the optimization problem $\max\left\{a^\top x : x \in P\right\}$ has an optimal vertex solution $z \in P$. In particular, $a^\top z \ge a^\top y$. On the other hand, as $z \in Q$, we also have $a^\top z < \beta < a^\top y$, which is a contradiction to the previous inequality. Therefore, $P \setminus Q = \emptyset$, so $P \subseteq Q$.

Problem 5: Finite convex hull

(a) Every point in X satisfies all the constraint defining the polyhedron P, and P is convex, thus $\mathrm{conv}(X) \subseteq P$.

(b) To prove that $P \subseteq \mathrm{conv}(X)$, assume by contradiction that there exists a point $y \in P \setminus \mathrm{conv}(X)$ and consider the polyhedron

$$D := \{d \in \mathbb{R}^n : (y - x_i)^\top d \ge 1 \ \forall i \in [k]\} \ .$$

(i) To show that the polyhedron D has a vertex, we show that $D \neq \emptyset$, prove that the constraint matrix of the polyhedron D has full row rank, and use the result of Problem 2(b) of this problem set.

i. To show that the polyhedron D is non-empty, note that since $y \in P \setminus \mathrm{conv}(X)$ and $\mathrm{conv}(X)$ is a closed convex set, there exists a hyperplane separating y from $\mathrm{conv}(X)$, i.e., there exists a vector $a \in \mathbb{R}^n$ and a number $b \in \mathbb{R}$ such that $a^\top x \le b$ for every $x \in X$, and $a^\top y > b$. Thus $a^\top y - a^\top x > 0$ for every $x \in X$. Let $\overline{x} \in \arg\max\{a^\top x : x \in X\}$. Then setting $d := \frac{a^\top}{a^\top(y - \overline{x})}$, we obtain $d^\top x_i \le d^\top y - 1$ for all $i \in [k]$, which is equivalent to $(y - x_i)^\top d \ge 1$ for all $i \in [k]$. Thus $d \in D$, so $D \neq \emptyset$.

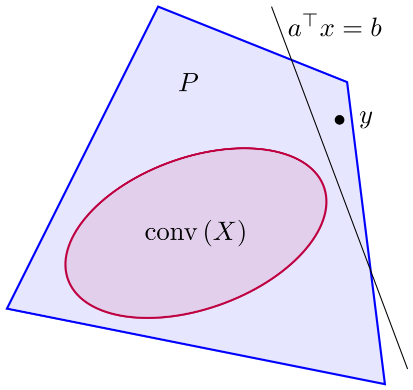

Figure 1: By the hyperplane separation theorem, there exists a hyperplane $a^\top x = b$ that separates the point $y \in P \setminus \mathrm{conv}(X)$ from the closed convex set $\mathrm{conv}(X)$. Our goal is to find such a hyperplane that is part of the inequality description of the polyhedron P.

ii. Now let us prove that the constraint matrix of the polyhedron D has full column rank. To achieve this, consider the rows of the constraint matrix: $(y - x_1)^\top, \ldots, (y - x_k)^\top$. Since the set X contains $n+1$ affinely independent points, there are $n+1$ affinely independent points among $x_1, \ldots, x_k$, say $x_1, \ldots, x_{n+1}$. Subtracting the first constraint from the next n constraints, we obtain the coefficient vectors $(x_1 - x_2)^\top, \ldots, (x_1 - x_{n+1})^\top$, which are linearly independent. This implies that the constraint matrix of the polyhedron D has n linearly independent rows, which means that the constraint matrix has full column rank.

(ii) Let d be a vertex of the polyhedron D. To see that the inequality $d^\top x \leq d^\top y - 1$ appears in the description of P, we have to check that there are n affinely independent points in X that satisfy the constraint with equality, and that all the other points in X satisfy the constraint, as well.

The latter is obvious, as $d \in D$ implies that $d^\top x_i \leq d^\top y - 1$ for all $i \in [k]$. To prove the former, we use that d is a vertex of D, hence there exist n linearly independent constraints that are tight at d. Denote by T the index set of these constraints. In particular, we have $(y - x_i)^\top d = 1$ for all $i \in T$, or equivalently, $d^\top x_i = d^\top y - 1$ for all $i \in T$, hence all $\{x_i\}_{i \in T}$ satisfy the constraint that we are considering with equality. Moreover, by choice of T, the rows $(y - x_i)^\top$ for $i \in T$ are linearly independent, so the points $\{x_i\}_{i \in T}$ and y are affinely independent, and thus also the n points $\{x_i\}_{i \in T}$ themselves.

Therefore, the hyperplane $d^\top x = d^\top y + 1$ contains n affinely independent points in X, so it was considered during the construction of the polyhedron P, and the inequality $d^\top x \leq d^\top y + 1$ must have been included in the inequality description of P. The point y does not satisfy this constraint, so $y \notin P$, contradicting the assumption.

(c) If X has $n + 1$ independent points, then by points (a) and (b), we have $\mathrm{conv}(X) = P$ where P is a polyhedron, so $\mathrm{conv}(X)$ is a polyhedron. Since $\mathrm{conv}(X)$ is bounded as X is finite, we get that $\mathrm{conv}(X)$ is actually a polytope.

(d) If $\mathrm{conv}(X)$ is not full-dimensional, all points in X lie in an affine subspace of dimension $k < n$. Choose k as small as possible. From linear algebra, we know that there exists an affine bijection Φ that transforms that subspace to $\mathbb{R}^k \times \{0\}^{n-k}$, and hence also transforms the points of X to that space. Consequently, $\mathrm{conv}(\Phi(X)) \subseteq \mathbb{R}^k \times \{0\}^{n-k}$. In particular, for the construction of $\mathrm{conv}(\Phi(X))$, we can consider only the first k coordinates of points in $\Phi(X)$ and construct their convex hull in $\mathbb{R}^k$, thus reducing the problem to the smaller dimension k. Note that the latter convex hull has dimension k due to the minimal choice of k, so indeed, we reduced to a full-dimensional case, obtaining a polytope in $\mathbb{R}^k$. Viewing this polytope as one in $\mathbb{R}^k \times \{0\}^{n-k}$, we get a polytope equal to $\mathrm{conv}(\Phi(X))$. Transforming this polytope back using Φ^{-1}, and employing the hint that the affine image of a polytope is a polytope, we get a polytope equal to $\mathrm{conv}(X)$, which finishes the proof.