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Mathematical Optimization – Problem set 14

`https://moodle-app2.let.ethz.ch/course/view.php?id=4844`

Problem 1: The polar of a polytope

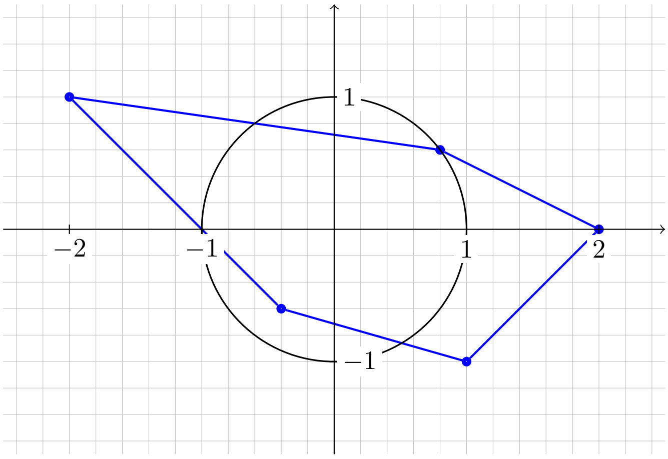
For a set $X \subseteq \mathbb{R}^n$, the polar X° of X is defined as

$$X^\circ := \{y \in \mathbb{R}^n : y^T x \leq 1 \text{ for all } x \in X\} \ .$$

Let $P \subseteq \mathbb{R}^2$ be the polytope defined by

$$P := \operatorname{conv}\left(\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0.8\\0.6\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}-0.4\\-0.6\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}\right) \ ,$$

as depicted in blue below.



Draw the polar P° of P. Use the space given above!

Problem 2: Properties of the polar

In this problem, we want to further investigate the properties of the polar.

(a) Provide an example of a full-dimensional polytope P that satisfies $(P^\circ)^\circ \neq P$. For your polytope P, compute both P° and $(P^\circ)^\circ$.

(b) Let $X \subseteq \mathbb{R}^n$ be a closed convex set that contains the origin. Show that $(X^\circ)^\circ = X$.

Hint: You can use a separation theorem, as for example Theorem 1.47 from the script.

Problem 3: Exponentially many facets and polynomially many vertices

Let $n \in \mathbb{Z}_{\geq 1}$. Provide an inequality description of a full-dimensional polytope $P \subseteq \mathbb{R}^n$ such that P has exponentially (in n) many facets, but only polynomially (in n) many vertices. Prove that your polytope P indeed has the required number of facets and vertices.

Hint: Try to first construct P°.

Problem 4: NP-completeness of integer programming feasibility

The satisfiability problem (SAT) is one of the first problems that was proved to be NP-complete, a result known as the Cook-Levin Theorem. In SAT, we deal with a finite set of variables $X =$

$\{x_1, \ldots, x_n\}$ that can each take values in $\{\texttt{true}, \texttt{false}\}$. Boolean expressions are built from such variables, the operators $\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction), as well as parenthesis. For any assignment $f\colon X \to \{\texttt{true}, \texttt{false}\}$, a boolean expression can be evaluated following the rules given in the table below.

x_1	x_2	$\neg x_1$	$x_1 \wedge x_2$	$x_1 \vee x_2$
true	true	false	true	true
true	false	false	false	true
false	true	true	false	true
false	false	true	false	false

Table 1: Evaluation of boolean operators.

SAT is the problem of deciding whether for a given boolean expression, there exists an assignment to its variables that evaluates to `true`. For example, we can see that the formula

$$(x_1 \vee x_2) \wedge \neg\big((x_1 \vee x_3) \wedge (\neg x_2 \vee x_3 \vee x_4)\big)$$

has a satisfying assignment $(x_1, x_2, x_3, x_4) = (\texttt{true}, \texttt{true}, \texttt{false}, \texttt{false})$, while the simple formula $x_1 \wedge \neg x_1$ does not have a satisfying assignment.

We consider a special case of the SAT problem that turns out to be equally hard, namely CNF-SAT, where the boolean formulas given as the problem input are in conjunctive normal form (CNF), i.e., they are of the form

$$\bigwedge_{i=1}^{p} \bigvee_{j=1}^{q_p} \ell_{ij} = (\ell_{11} \vee \ldots \vee \ell_{1q_1}) \wedge \ldots \wedge (\ell_{p1} \vee \ldots \vee \ell_{pq_p}) \ ,$$

where $\ell_{ij} \in \{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}$ are literals, i.e., variables or negations of variables. Note that expressions of the form $\bigvee_{j=1}^{q_p} \ell_{ij}$ for $i \in [p]$ in the above formula are called clauses of the formula.

(a) Given a boolean formula in conjunctive normal form as introduced above, write an integer linear program such that the boolean formula is feasible if and only if the integer linear program is.

Hint: Introduce variables in the integer program that correspond to the variables of the given formula. Try to write one constraint per clause such that the following holds: In an assignment corresponding to an IP solution that satisfies a constraint, the clause corresponding to this constraint evaluates to `true`.

(b) Conclude from (a) and NP-completeness of CNF-SAT that deciding feasibility of an integer linear program is NP-complete, as well.

Programming exercises

Complete the notebook `14_sudokuSolver.ipynb`, where you use integer programs to implement a Sudoku solver and to test whether a given Sudoku has a unique solution.