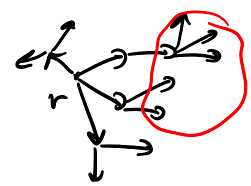
5.8.2 Integrality of dominant of r-arborescence polytope

Recall, for a directed graph $G=(V,A)$, we have:

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset\} .$$



Proof of integrality of P

$y \in$ vertices(P)

Wlog: $y(a) > 0 \quad \forall \; a \in A$ $\leadsto$ arcs $a \in A$ with $y(a) = 0$ can be removed.

$$\mathcal{F} = \{ S \subseteq V \setminus \{r\} : \; y(\delta^-(S)) = 1 \}$$

describes y-tight constraints

$\Rightarrow$ y is unique sol. to

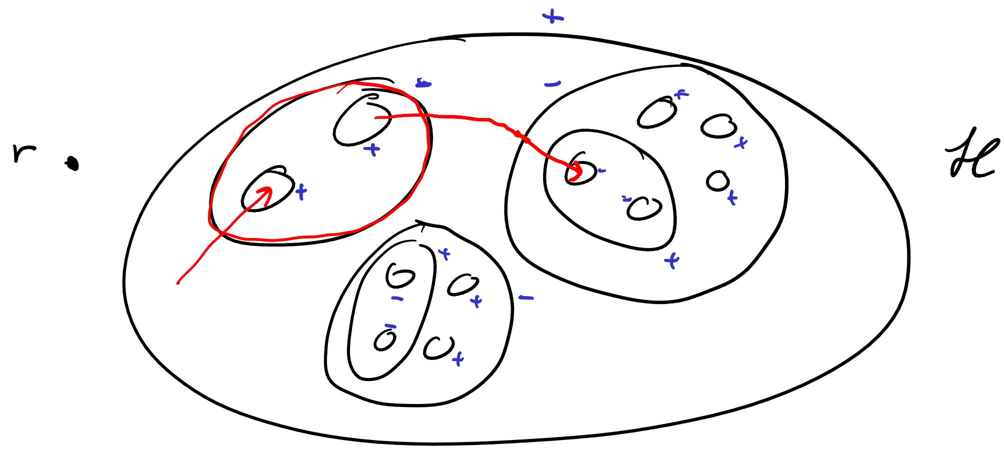
(*) $$x(\delta^-(S)) = 1 \quad \forall \; S \in \mathcal{F}$$

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$ and consider

$$\text{(☐)} \qquad x(\delta^-(H)) = 1 \qquad \forall\ H \in \mathcal{H}$$

Notice that (☐) is TU system:

We can use Ghouila-Houri:



Remains to show that every equation of (∗) is implied by (☐).

Lemma 5.27

For any two sets $S_1, S_2 \subseteq V$, we have

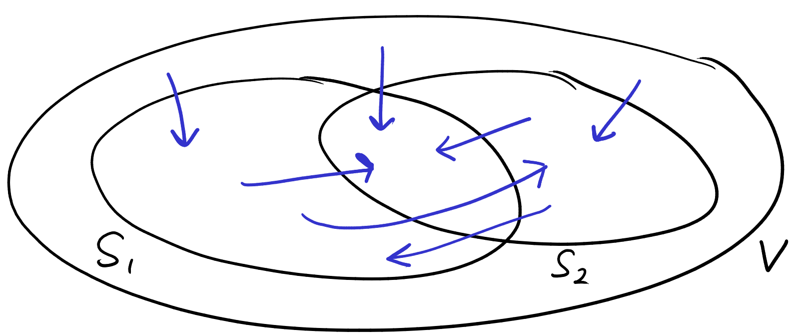
$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1\cap S_2)} + \chi^{\delta^-(S_1\cup S_2)} + \chi^{A(S_1\setminus S_2, S_2\setminus S_1)} + \chi^{A(S_2\setminus S_1, S_1\setminus S_2)} ,$$

which implies in particular

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} \geq \chi^{\delta^-(S_1\cap S_2)} + \chi^{\delta^-(S_1\cup S_2)} .$$

Proof

Idea: Check all different "arc types":



Lemma 5.28

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $A(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, $A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset$. In particular, this implies by Lemma 5.27

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} .$$

Proof

By lemma 5.27:

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} + \chi^{A(S_1\setminus S_2, S_2 \setminus S_1)} + \chi^{A(S_2\setminus S_1, S_1\setminus S_2)}$$

$$2 = \underbrace{y(\delta^-(S_1))}_{=1} + \underbrace{y(\delta^-(S_2))}_{=1} = \underbrace{y(\delta^-(S_1 \cup S_2))}_{\geq 1} + \underbrace{y(\delta^-(S_1 \cap S_2))}_{\geq 1} + \overbrace{y(A(S_1\setminus S_2, S_2\setminus S_1))}^{\geq 0} + \underbrace{y(A(S_2\setminus S_1, S_1\setminus S_2))}_{\geq 0} \geq 2$$

$S_1, S_2 \in \mathcal{F}$ $\qquad$ $y \in P$

$\Rightarrow$ Equality holds throughout.

$\Rightarrow$

- $y(\delta^-(S_1 \cup S_2)) = 1 \Rightarrow S_1 \cup S_2 \in \mathcal{F}$
- $y(\delta^-(S_1 \cap S_2)) = 1 \Rightarrow S_1 \cap S_2 \in \mathcal{F}$
- $\left.\begin{array}{l} y(A(S_1\setminus S_2, S_2\setminus S_1)) = 0 \\ y(A(S_2\setminus S_1, S_1\setminus S_2)) = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} A(S_1\setminus S_2, S_2, S_1) = \emptyset \\ A(S_2\setminus S_1, S_1\setminus S_2) = \emptyset \end{array}\right.$

#

Back to: $\square$ implies $\circledast$.

Assume by sake of contradiction that $\exists\ S \in \mathcal{F}$ s.t.

$$\chi^{\delta(S^-)} \notin Q := \operatorname{span}(\{\chi^{\delta^-(H)} : H \in \mathcal{H}\})$$

Let

$$\mathcal{H}_S := \{H \in \mathcal{H} : H \text{ and } S \text{ are intersecting}\}$$

$\Rightarrow S \notin \mathcal{H} \Rightarrow$ Adding S to $\mathcal{H}$ destroys laminarity, i.e.,

$\mathcal{H} \cup \{S\}$ is not laminar

$\Leftrightarrow \mathcal{H}_S \neq \emptyset$

Let S be a set with $\chi^{\delta^-(S)} \notin Q$ s.t. $|\mathcal{H}_S|$ is as small as possible

Let $H \in \mathcal{H}_S$.

By Lemma 5.28:

$$\underbrace{\chi^{\delta^-(S)}}_{\notin Q} + \underbrace{\chi^{\delta^-(H)}}_{\in Q} = \chi^{\delta^-(S \cup H)} + \chi^{\delta^-(S \cap H)}$$

$\Rightarrow$ At least one of $\chi^{\delta^-(S \cup H)}$ or $\chi^{\delta^-(S \cap H)}$ is not in Q.

However, $\mathcal{H}_{S \cup H} \subsetneq \mathcal{H}_S$ and $\mathcal{H}_{S \cap H} \subsetneq \mathcal{H}_S$.

$\rightsquigarrow$ This contradicts choice of S. $\#$