6 Ellipsoid Method

How to solve linear programs over polyhedra with exponentially many constraints?

For example:

Theorem 5.17

The spanning tree polytope of an undirected loopless graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset\} .$$

Theorem 5.21

The perfect matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\} .$$

Ellipsoid method can solve LPs over above polyhedra.

Its main ingredient is:

A separation oracle for the polyhedron over which to optimize.

6.1 Separation problem

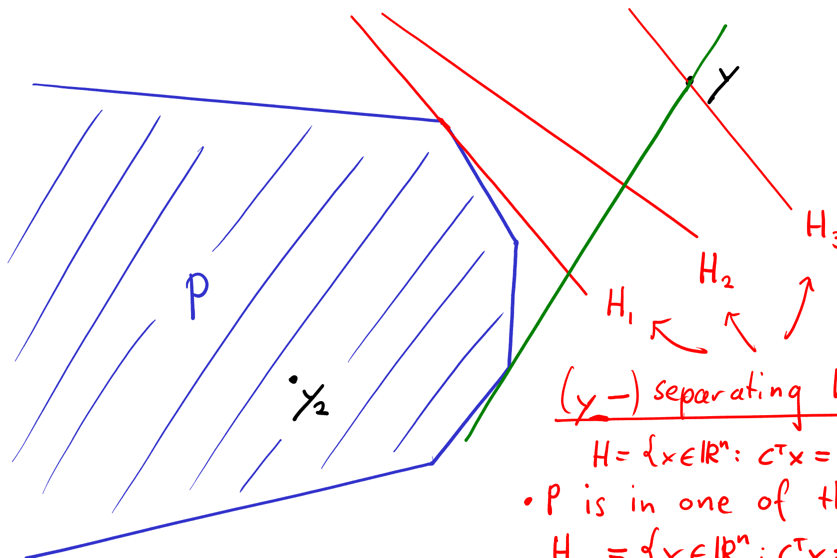
Separation problem for a polyhedron $P \subseteq \mathbb{R}^n$:

Definition 6.1: Separation problem & separation oracle

Given a point $y \in \mathbb{R}^n$:

- Decide whether $y \in P$, and if this is not the case,
- find $c \in \mathbb{R}^n$ such that $P \subseteq \{x \in \mathbb{R}^n : c^\top x < c^\top y\}$.

A procedure that solves the separation problem (for P) is often called a *separation oracle* (for P).



(y-) separating hyperplanes:

$H = \{x \in \mathbb{R}^n : c^\top x = \alpha\}$ s.t.

- P is in one of the closed halfspaces $H_{\leq} = \{x \in \mathbb{R}^n : c^\top x \leq \alpha\}$ & $H_{\geq} = \{x \in \mathbb{R}^n : c^\top x \geq \alpha\}$,
- y is in the other one, and
- either $P \cap H = \emptyset$ or $y \notin H$.

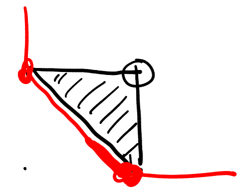
6.2 Optimization results based on Ellipsoid Method

Theorem 6.2

Let $P \subseteq \mathbb{R}^n$ be a $\{0,1\}$-polytope for which we are given a separation oracle. Furthermore, let $w \in \mathbb{Z}^n$. Then the Ellipsoid Method allows for finding an optimal vertex solution to the linear program $\max\{w^\top x : x \in P\}$ using a polynomial number (in n) of operations and calls to the separation oracle for P.

→ Even polyhedra with exponentially many facets can admit efficient separation oracles.

6.3 Example application



Minimum weight r-arborescence

Let $G=(V,A)$ be a directed graph with arc weights $w: A \to \mathbb{Z}_{\geq 0}$.

Our goal: Find r-arborescence $T \subseteq A$ minimizing $w(T)$.

→ Reduces to minimizing w over dominant P of r-arborescence polytope.

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset\} \ .$$

$$\min \ w^T x \quad x \in P$$

We cannot apply thm. 6.2 to above LP because P is (in general) not a $\{0,1\}$-polytope.

Let

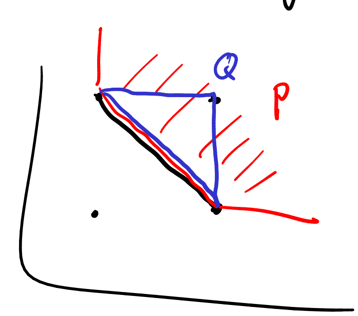
$$Q := P \cap [0,1]^A$$

One can show: P is dominant of $\{0,1\}$-polytope $\Rightarrow$ $P \cap [0,1]^A$ is integral (see problem sets)

consider →

$$\min \ w^T x \quad x \in Q$$

instead of above LP



By Theorem 6.2, to solve $\bullet$, it suffices to find separation oracle for

$$Q = \{ x \in [0,1]^A : x(\delta^-(S)) \geq 1 \quad \forall\, S \subseteq V \setminus \{r\},\ S \neq \emptyset \}$$

Separation oracle for Q

Let $y \in \mathbb{R}^A$.

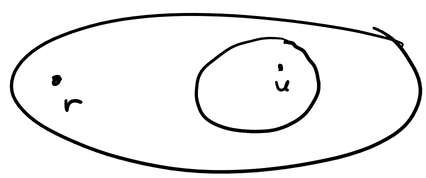
We first check whether $y \in [0,1]^A$. If not, say $y_a > 1$ for some $a \in A \Longrightarrow$ $x_a \leq 1$ is a violated constraint.

$\Longrightarrow$ $x_a = 1$ is y-separating hyperplane.

How to check $y(\delta^-(S)) \geq 1 \quad \forall\, S \subseteq V \setminus \{r\}$, $S \neq \emptyset$?

For each $u \in V \setminus \{r\}$, we will check whether

$$y(\delta^-(S)) \geq 1 \quad \forall\, S \subseteq V \text{ that is a } u\text{-}r \text{ cut}$$



This can be checked by finding minimizer $S_{u,r}$ of

$$\min \{ y(\delta^-(S)) : S \subseteq V \text{ is } u\text{-}r \text{ cut} \}$$

$$= \min \{ y(\delta^+(U)) : U \subseteq V \text{ is } r\text{-}u \text{ cut} \}$$

This is a minimum r-u cut problem in $G = (V, A)$ with capacities y.

$\hookrightarrow$ Can be solved efficiently

If $y(\delta^-(S_{u,r})) \geq 1 \quad \forall u \in V \setminus \{r\} \implies y \in Q$.

Otherwise, if $\exists u \in V \setminus \{r\}$, s.t. $y(\delta^-(S_{u,r})) < 1$.

$\implies x(\delta^-(S_{u,r})) \geq 1$ is a violated constraint.

$\implies x(\delta^-(S_{u,r})) = 1$ is a separating hyperplane.