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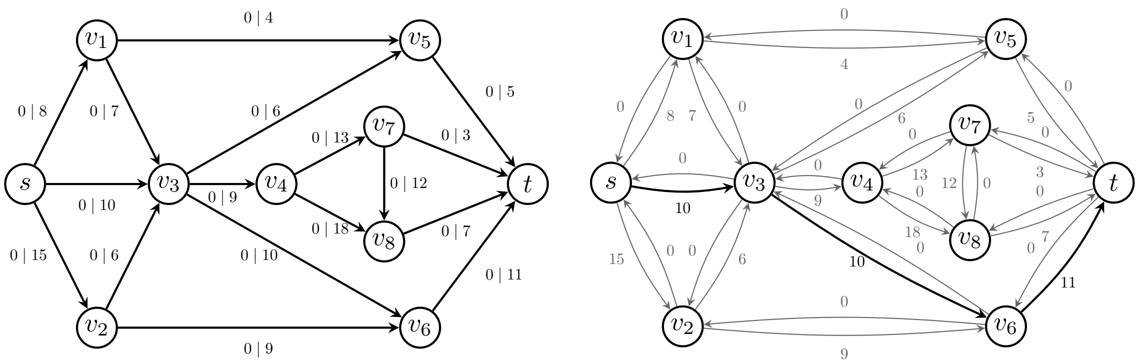
Mathematical Optimization – Solutions to problem set 7

`https://moodle-app2.let.ethz.ch/course/view.php?id=4844`

Problem 1: The algorithm of Ford-Fulkerson and the value of a flow

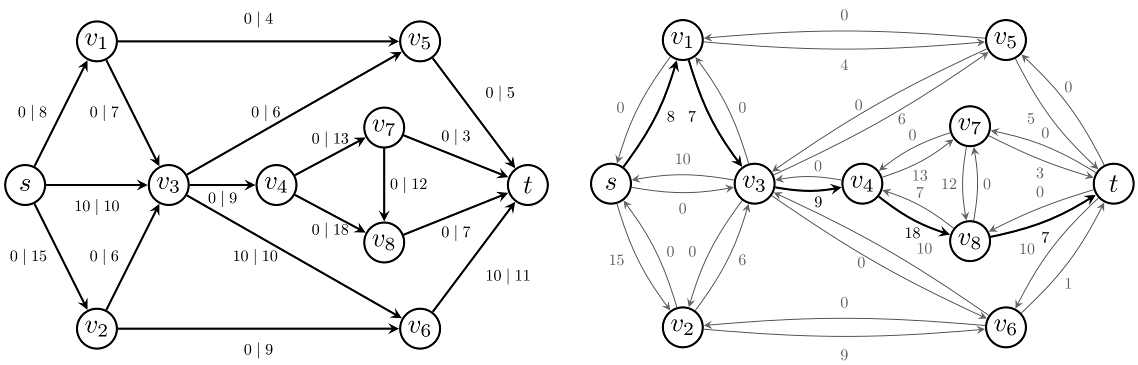
(a) The steps in the algorithm of Ford and Fulkerson heavily rely on the choice of f-augmenting paths in the process of the algorithm. The actual maximum flow f^* that the algorithm computes may not be unique, but for a fixed input it will always return the same maximum flow value $\nu(f^*)$ no matter which f-augmenting paths are chosen in the process of the algorithm.

Each of the subsequent Figures 1 to 7 corresponds to one iteration of the algorithm of Ford and Fulkerson: The left subfigure presents the current flow f, while the right subfigure shows the corresponding residual graph G_f with resudiual capacities u_f, and with our choice of an f-augmenting path highlighted. While in the intermediate iterations, we only have to find any f-augmenting path in G_f, the last iteration (i.e., when we reached an optimal flow f^*) needs a formal argument that certifies inexistence of an f^*-augmenting path in G_{f^*}.



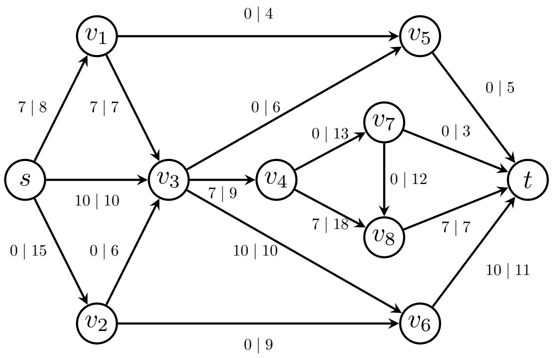
(a) The given G with zero flow $f = 0$. (b) An f-augmenting path with $\gamma = 10$.

Figure 1: First iteration of Ford-Fulkerson in G.

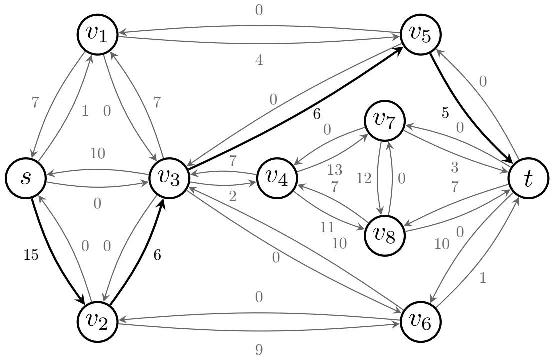


(a) Intermediate flow f of value 10. (b) An f-augmenting path with $\gamma = 7$.

Figure 2: Second iteration of Ford-Fulkerson in G.

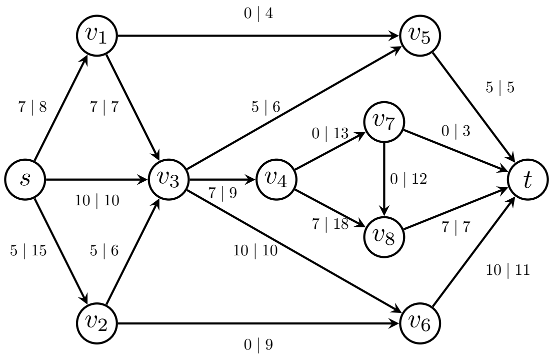


(a) Intermediate flow f of value 17.

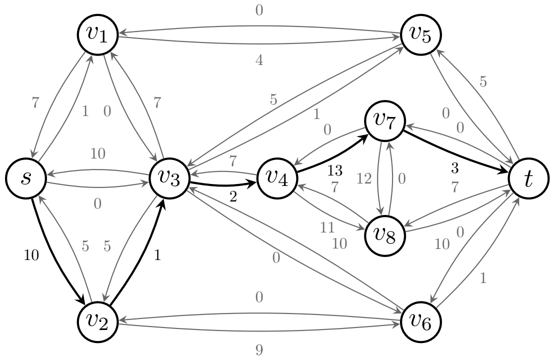


(b) An f-augmenting path with $\gamma = 5$.

Figure 3: Third iteration of Ford-Fulkerson in G.

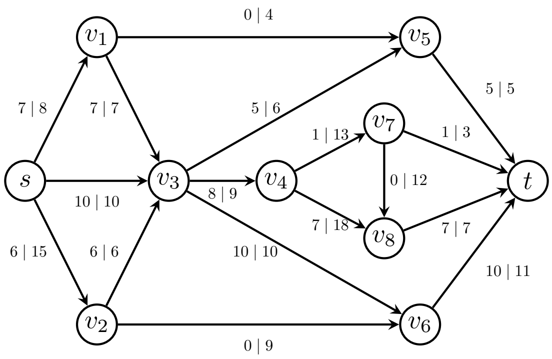


(a) Intermediate flow f of value 22.

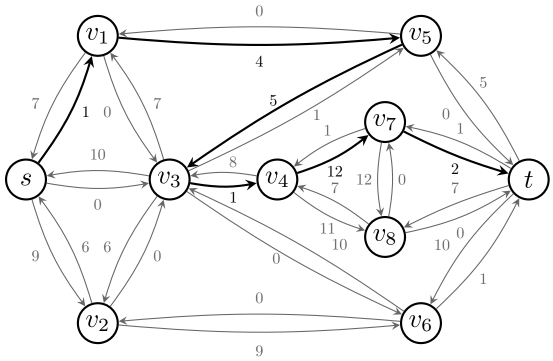


(b) An f-augmenting path with $\gamma = 1$.

Figure 4: Fourth iteration of Ford-Fulkerson in G.

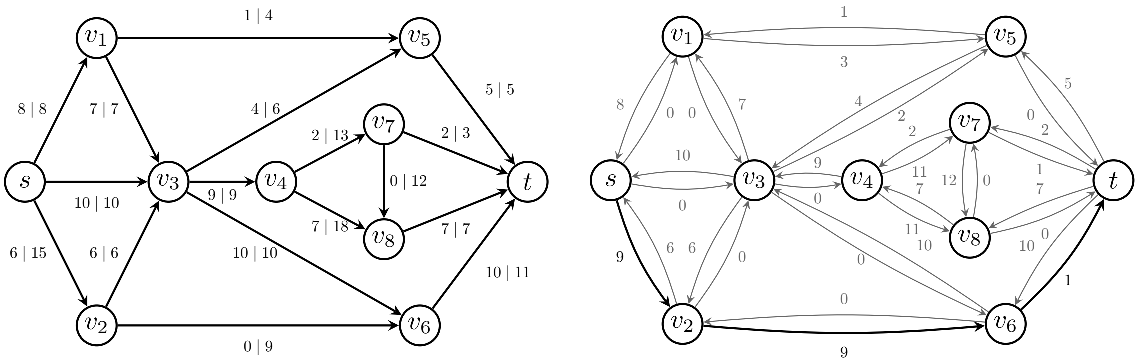


(a) Intermediate flow f of value 23.



(b) An f-augmenting path with $\gamma = 1$.

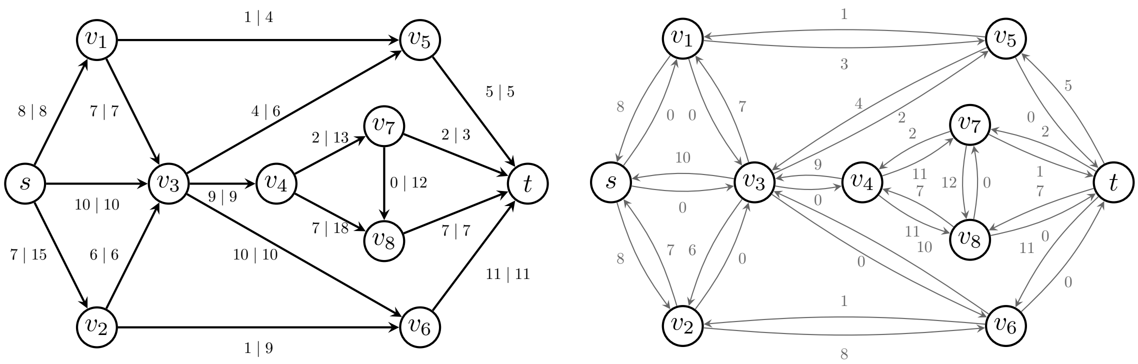
Figure 5: Fifth iteration of Ford-Fulkerson in G.



(a) Intermediate flow f of value 24.

(b) An f-augmenting path with $\gamma = 1$.

Figure 6: Sixth iteration of Ford-Fulkerson in G.



(a) Final flow f^* of value 25.

(b) No f^*-augmenting path exists.

Figure 7: Seventh and last iteration of Ford-Fulkerson in G. There does not exist an augmenting path, thus the algorithm terminates and the flow is optimal.

In the last residual graph in Figure 7b, we can see that $C \coloneqq \{s, v_1, v_2, v_3, v_5, v_6\}$ is a cut with $u_{f^*}(\delta^+(C)) = 0$, which proves that there is no f^*-augmenting path with strictly positive augmentation volume. Thus, we indeed found a maximum s-t flow in G. Its value is

$$\nu(f^*) = f(\delta^+(s)) - f(\delta^-(s)) = (8 + 10 + 7) - 0 = 25 \ .$$

Alternatively, we can calculate the value of f^* by summing up the increments of all iterations, which gives $\nu(f^*) = 10 + 7 + 5 + 1 + 1 + 1 = 25$.

(b) Consider the s-t cut $S = V \setminus \{t\}$, and note that $\delta^+(S) = \delta^-(t)$ and $\delta^-(S) = \delta^+(t)$. By Lemma 4.3 from the script, we thus have

$$\nu(f) = f(\delta^+(S)) - f(\delta^-(S)) = f(\delta^-(t)) - f(\delta^+(t)) \ .$$

Problem 2: Flow through intermediate vertices

By the max-flow min-cut theorem, we know that there exists an s_1-s_ℓ flow fo value at least k if an only if the value of a minimum s_1-s_ℓ cut is at least k. To see this, let C be a minimum s_1-s_ℓ cut. It is sufficient to prove that C has value at least k.

Observe that for some $i \in \{1, \ldots, \ell - 1\}$, C is also a s_i-s_{i+1} cut, as $s_1 \in C$ and $s_\ell \notin C$. By assumption, we know that there exists an s_i-s_{i+1} flow f_i of value at least k. Using the max-flow min-cut theorem once again, this implies that the minimum s_i-s_{i+1} cut has value at least k, and hence in particular, the cut C has value at least k. This is what we wanted to show.

Problem 3: Max-flow min-cut via duality I

(a) First, let us show that the optimal value of the linear program is no smaller than the value of any maximum s-t flow in G. For each arc $a \in A$, set the variable f_a to be the flow through the arc a, and set the variable ν to the value of the s-t flow. By the definition of an s-t flow, all the constraints of the linear program are satisfied, hence the optimal value of the linear program is at least as big as the value of the flow.

Now let us show that the optimal value of the linear program is no larger than the value of a maximum s-t flow. Let (f, ν) be an optimal solution of the linear program. Consider an s-t flow in G such that the flow through arc $a \in A$ is equal to f_a. Since (f, ν) satisfies the constraints of the linear program, the constructed flow is valid and of value ν. Therefore, the optimal value of the linear program is less or equal to the value of a maximum s-t flow.

Together, this proves that the optimal value of the linear program equals the value of a maximum s-t flow in G.

(b) Following the standard procedure, we arrive at the dual linear program

$$\begin{array}{lcccccll} \min & \sum_{a\in A} u_a z_a & & & & & & \\ & y_t & - & y_s & & \geq & 1 & \\ & y_v & - & y_w & + \; z_a & \geq & 0 & \forall a = (v,w) \in A \\ & & & & y_v & \in & \mathbb{R} & \forall v \in V \\ & & & & z_a & \in & \mathbb{R}_{\geq 0} & \forall a \in A \;. \end{array} \tag{D}$$

(c) To show that the constraint family $y_v - y_w + z_a \geq 0 \; \forall a = (v, w) \in A$ in the dual linear program (D) is satisfied, we take an arbitrary arc $a \in A$ and consider the following three cases.

(i) If the arc a points from $v \in C$ to $w \notin C$, then we have $y_v = 0$, $y_w = 1$, and $z_a = 1$. Thus, $y_v - y_w + z_a = 0$ and the corresponding constraint is satisfied (equality holds).

(ii) If the arc a points from $v \notin C$ to $w \in C$, then we have $y_v = 1$, $y_w = 0$, and $z_a = 0$. Thus, $y_v - y_w + z_a = 1$ and the corresponding constraint is satisfied (strict inequality holds).

(iii) If the arc a points from $v \in C$ to $w \in C$ or from $v \notin C$ to $w \notin C$, then we have $y_v = y_w$ and $z_a = 0$. Thus, $y_v - y_w + z_a = 0$ and the corresponding constraint is satisfied (equality holds).

In any case, the constraint corresponding to the arc a is satisfied. Next, the constraint $y_t - y_s \geq 1$ holds because $y_s = 0$ and $y_t = 1$, by definition. Thus, (y, z) is a feasible solution of the dual linear program (D). The value of this solution is indeed

$$\sum_{a\in A} u_a z_a = \sum_{a\in\delta^+(C)} u_a = u(\delta^+(C)) \;.$$

(d) Let $C \subseteq V$ be an s-t cut in G. Then by (c), there exists a feasible solution (y, z) of the dual linear program (D) corresponding to the cut C with value $u(\delta^+(C))$. Weak linear programming duality (Theorem 1.84) implies that the value of the dual feasible solution is greater or equal to the value of any primal-feasible solution. Finally, by (a), the optimal value of the primal linear program equals the value of a maximum s-t flow in G. Therefore, the value of any s-t cut is at least as big as the value of any s-t flow.

Note that we can not simply replace weak duality with strong duality to prove the strong max-flow min-cut theorem here, since we have not shown that there always exists an s-t cut in G with value equal to the optimal value of the dual linear program.

(e) (i) Let (y, z) be an arbitrary optimal solution of the dual linear program (D). For every $v \in V$, define $\tilde{y}_v := y_v - y_s$. Note that the constraints of (D) depend only on differences $y_v - y_w$ for $v, w \in V$ and not on the values of the variables themselves, while the value of the solution depends only on the z variables and not on y. Therefore, the constraints and the value of the solution remain unchanged after replacing y with $\tilde{y}$, as $\tilde{y}_v - \tilde{y}_w = y_v - y_w$ for all $v, w \in V$. Thus, $(\tilde{y}, z)$ is a feasible solution of (D) with the same value as (y, z), so $(\tilde{y}, z)$ is also optimal. Therefore, without loss of generality, we may consider optimal solutions of the dual linear program (D) such that $y_s = 0$.

(ii) To prove that (Y, Z) is feasible for (D), we need to check that all the constraints are fulfilled. For any arc $a = (v, w) \in A$, we have $Z_a \geq Y_w - Y_v$, hence

$$Y_v - Y_w + Z_a \geq Y_v - Y_w + Y_w - Y_v = 0 \ ,$$

thus the first set of constraints is satisfied. Since (y, z) is feasible for (D), we have $y_t - y_s \geq 1$, and by assumption, $y_s = 0$, hence $y_t \geq 1$. Thus, $Y_s = 0$ and $Y_t = 1$, so the constraint $Y_t - Y_s \geq 1$ holds with equality. By construction, we have $Y_v \in \mathbb{R}$ for every $v \in V$ and $Z_a \in \mathbb{R}_{\geq 0}$ for every $a \in A$. Therefore, (Y, Z) is a feasible solution of (D).

The bound on its expected value can be obtained as follows. First, we use linearity of expectation to rewrite the expected value as

$$\mathbb{E}\left[\sum_{a \in A} u_a Z_a\right] = \sum_{a \in A} u_a \mathbb{E}[Z_a] \ .$$

Next, we substitute the definition of Z_a into $\mathbb{E}[Z_a]$, which gives

$$\begin{aligned} \mathbb{E}[Z_a] &= \mathbb{E}[\max\{0, Y_w - Y_v\}] \\ &= 0 \cdot \Pr[Y_w - Y_v \leq 0] + \mathbb{E}[Y_w - Y_v \colon Y_w - Y_v > 0] \cdot \Pr[Y_w - Y_v > 0] \ . \end{aligned}$$

The first term in the latter sum gives a zero contribution to the expected value. Note that if $Y_w - Y_v > 0$, then in fact $Y_w - Y_v = 1$, and $Y_w - Y_v > 0$ holds precisely if $\theta \in (y_v, y_w]$. Thus,

$$\mathbb{E}[Z_a] = \Pr[\theta \in (y_v, y_w]] \leq \max\{0, y_w - y_v\} = z_a \ .$$

Altogether, we obtain the desired bound

$$\mathbb{E}\left[\sum_{a \in A} u_a Z_a\right] \leq \sum_{a \in A} u_a z_a \ .$$

(iii) As was mentioned in (ii), we have $Y_s = 0$ and $Y_t = 1$, so $s \in C$ and $t \notin C$, i.e., C is an s-t cut. To calculate its value $u(\delta^+(C))$, note that C contains precisely all vertices $v \in V$ with $Y_v = 0$, while all vertices in the complement of C satisfy $Y_v = 1$. Consequently, for every edge $a = (v, w) \in A$, we have

$$Z_a = \max\{0, Y_w - Y_v\} = \begin{cases} 1 & \text{if } a \in \delta^+(C) \\ 0 & \text{else} \end{cases} \ ,$$

hence we obtain the desired equality

$$u(\delta^+(C)) = \sum_{a \in \delta^+(C)} u_a = \sum_{a \in A} u_a Z_a \ .$$

By part (ii), this implies that the expected value of C is at most the dual linear program value. Consequently, there exists a cut C with at most this value. On the other hand, by part (c), the dual linear program value is at most the value of a minimum s-t cut in G. Thus, the two values are equal.

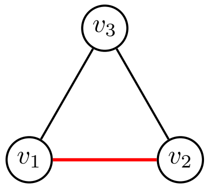
(iv) Note that both the primal and the dual linear programs are feasible (the zero flow is feasible for the primal, and the cut $C = \{s\}$ leads to a feasible solution for the dual following the construction in part (c)). Thus, by strong duality, their values agree.

By part (a), the primal optimal value is equal to the value of a maximum s-t flow, and by point (iii), the dual linear program value equals the value of a minimum s-t cut. By the above, these two values are the same, which is precisely what the strong max-flow min-cut theorem states.

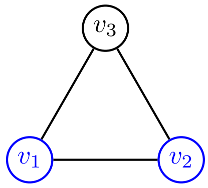
Problem 4: Matchings and vertex covers in bipartite graphs

(a) Let $M \subseteq E$ be a maximum matching in G. Suppose that there exists a vertex cover $C \subseteq V$ in G such that $|C| < |M|$. Since M is a matching, we have that $\left|\bigcup_{e \in M} e\right| = 2 \cdot |M|$. Thus, by the pigeonhole principle, there exists an edge $e = \{v, w\} \in M$ such that $v, w \notin C$. This contradicts the fact that C is a vertex cover, since neither of the two endpoints of the edge e lies in C. Therefore, our initial assumption is wrong, so $|C| \geq |M|$ holds for any vertex cover $C \subseteq V$ in G.

(b) For arbitrary graphs, the cardinality of a maximum matching is not necessarily equal to the cardinality of a minimum vertex cover. The simplest counterexample showing this is a triangle (see Figure 8). Every two edges share a vertex, so the cardinality of a maximum matching is 1. On the other hand, for any vertex there exists an edge not incident to it. Any vertex cover must therefore have size at least 2. Since it is easy to see that any two vertices form a vertex cover, the cardinality of a minimum vertex cover is 2.



(a) Maximum matching.



(b) Minimum vertex cover.

Figure 8: In triangles, the cardinality of a maximum matching is strictly smaller than the cardinality of a minimum vertex cover.

(c) Let a bipartite graph $G = (V, E)$ with bipartition $V = L \,\dot{\cup}\, R$ be given. In order to prove the desired result, we consider an auxiliary digraph $D = (W, A)$ with arc capacities u, to which we will later apply the max-flow min-cut theorem (Corollary 4.14 in the script) to get a vertex cover with the desired size. The digraph D and the capacities u are constructed as follows:

- The vertices of the digraph D are defined as $W := V \,\dot{\cup}\, \{s, t\} = L \,\dot{\cup}\, R \,\dot{\cup}\, \{s, t\}$, i.e., we add two new vertices s (the source) and t (the sink) to the vertices of G.
- The arcs of the digraph D are given by a disjoint union $A := A_1 \cup A_2 \cup A_3$ of arc sets defined by

$$\begin{aligned} A_1 &:= \{(s, v) \colon v \in L\} \quad , \\ A_2 &:= \{(v, w) \colon v \in L,\ w \in R,\ \{v, w\} \in E\} \quad , \\ A_3 &:= \{(w, t) \colon w \in R\} \quad . \end{aligned}$$

- For every arc $a \in A$, we define its capacity by

$$u(a) := \begin{cases} 1 & \text{if } a \in A_1 \cup A_3 \\ \infty & \text{if } a \in A_2 \end{cases} \quad .$$

An illustration of the thereby constructed digraph D is given in Figure 9.

The proof consists of two steps. We show that (i) the value of a maximum s-t flow in D equals the cardinality of a maximum matching in G, and (ii) the value of a minimum s-t cut in D equals the cardinality of a minimum vertex cover for G. If we know this, then by the max-flow min-cut theorem (Corollary 4.14 in the script), the value of a maximum s-t flow equals the value of a minimum s-t cut in D, thus implying that the cardinality of a maximum matching is the same as the cardinality of a minimum vertex cover for G. Consequently, it is sufficient to prove (i) and (ii).

(i) To prove that the value of a maximum s-t flow in D equals the cardinality of a maximum matching in G, we show that every integral s-t flow in D corresponds to a matching in G whose cardinality equals the value of the flow, and every matching in G corresponds to a feasible s-t flow in D whose value equals the cardinality of the matching. Recall that since our capacities are integral, there exists a maximum s-t flow that is integral.

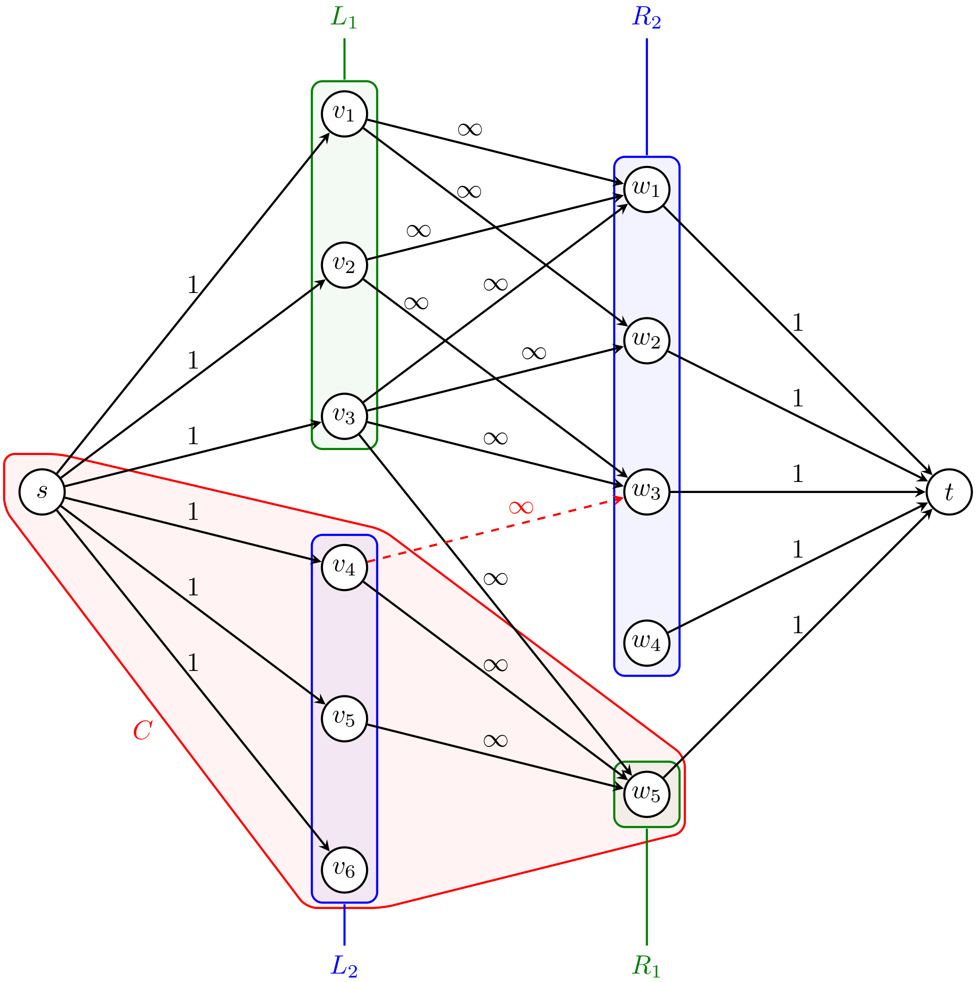


Figure 9: An example auxiliary digraph D. In a minimum s-t cut C, there cannot be arcs like the one highlighted in red.

- Let M be a matching in G. Consider an s-t flow f in D that sends flow 1 through arcs (s, v), (v, w), and (w, t) for every $\{v, w\} \in M$ (with $v \in L$ and $w \in R$) and sends flow 0 through every other arc. Since M is a matching, f is a feasible s-t flow. Moreover, by construction of f, its value is equal to the cardinality of the matching M.
- Let f be an integral s-t flow in D. Denote by $M \subseteq E$ the subset of edges of G corresponding to arcs in A_2 (i.e, arcs in D with infinite capacity) whose flow in f is non-zero. Since f is integral and every vertex in $L \subseteq W$ has exactly one incoming are with capacity 1, the flow on these arcs is actually exactly 1. Note that no two arcs in A_2 with flow 1 and capacity ∞ are adjacent, since for every $v \in L$ and $w \in R$ we have

$$u\left((s, v)\right) = u\left((w, t)\right) = 1 \ .$$

Therefore, no two edges in M share a vertex, so M is a matching. Moreover, we can express the value of the s-t flow f with the help of the s-t cut $C := \{s\} \cup L$, namely

$$\nu(f) = f(\delta^+(C)) - f(\delta^-(C)) \ .$$

Since there are only arcs in D that go from s to L, from L to R, or from R to t, we have that

$$\nu(f) = f(\delta^+(C)) = \sum_{a \in A_2} f(a) = |A_2| = |M| \ ,$$

so the cardinality of M is equal to the value of f.

This concludes the proof of (i).

(ii) To prove that the value of a minimum s-t cut in D equals the cardinality of a minimum vertex cover for G, we show that every minimum s-t cut in D corresponds to a vertex cover for G whose cardinality equals the value of the cut, and every minimum vertex cover for G corresponds to an s-t cut in D whose value equals the cardinality of the vertex cover.

- Let C be a minimum s-t cut in D. As illustrated in Figure 9, let $L_1 := L \setminus C$, $R_1 := R \cap C$, $L_2 := L \cap C$, and $R_2 := R \setminus C$. We claim that the desired vertex cover for G is given by $S := L_1 \cup R_1 \subseteq V$. Note that there are no arcs with infinite capacity pointing out of the minimum s-t cut C (since the s-t cut $\{s\}$ has finite value). Thus, there are no arcs in D pointing from L_2 to R_2. Therefore, every arc in A_2 (and thus also every edge in E) is incident to a vertex in S, so S is a vertex cover for G. Moreover, the cardinality of the vertex cover S equals the value of the s-t cut C, since for every vertex in $S = L_1 \cup R_1$, there is precisely one arc (with capacity 1) pointing out of C, and there are no other arcs exiting C.
- Let S be a minimum vertex cover for G, and set $L_1 := S \cap L$, $R_1 := S \cap R$, $L_2 := L \setminus L_1$, and $R_2 := R \setminus R_1$. We consider the s-t cut $C := \{s\} \cup L_2 \cup R_1 \subseteq W$ and claim that its value equals the cardinality of S. Note that there is precisely one arc (with capacity 1) going out of C for every vertex in $S = L_1 \cup R_1$. Moreover, the fact that S is a vertex cover for G ensures that there exist no arcs going from L_2 to R_2, and thus there are no other arcs exiting C. This shows that $u(\delta^+(C)) = |S|$, as desired.

Problem 5: Vertex connectivity

(a) Let $G = (V, A)$ be a directed graph and let $s, t \in V$ with $s \neq t$ such that there is no arc from s to t in G. In order to prove the vertex connectivity version of Menger's Theorem, we construct an auxiliary digraph $D = (W, B)$ with arc capacities u, to which we will later apply the max-flow min-cut theorem (Corollary 4.14 in the script). The digraph D and the capacities u are constructed as follows:

- For each vertex of the original graph $v \in V$, W contains the two vertices v^- and v^+.
- We separate the arcs of the digraph D into two sets $B := B_1 \cup B_2$, where

$$B_1 := \left\{(v^-, v^+) \colon v \in V\right\} \quad , \quad \text{and} \quad B_2 := \left\{(v^+, w^-) \colon (v, w) \in A\right\} .$$

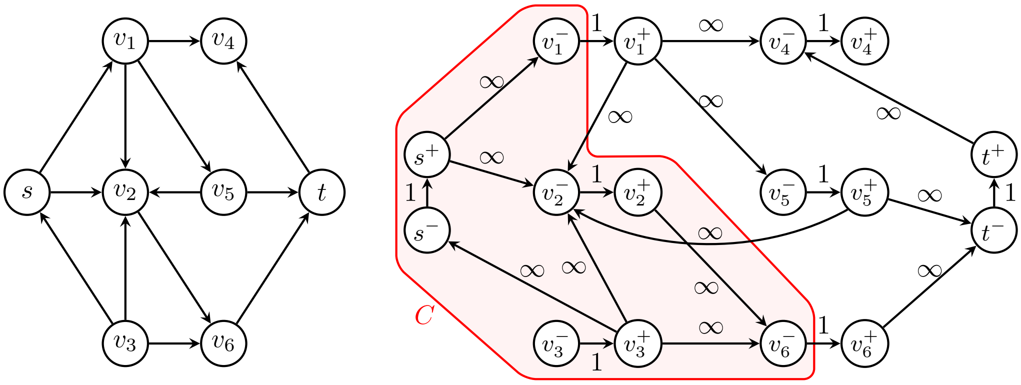
  The arcs in B_2 correspond to the arcs in the original graph G, whereas the arcs B_1 connect the two copies v^- and v^+ of a vertex $v \in V$ in D.
- For every arc $b \in B$, we define its capacity by

$$u\,(b) := \begin{cases} 1 & \text{if } b \in B_1 \\ \infty & \text{if } b \in B_2 \end{cases} .$$

An example for the above construction is given in Figure 10.

Note that the construction of D ensures the following: Any s-t path P in G can be transformed into an s^+-t^- path P' in D by replacing every arc (v, w) in P by (v^+, w^-) and any vertex $v \notin \{s, t\}$ in P by the sequence $v^-, (v^-, v^+), v^+$ (and s by s^+, and t by t^-). Moreover, since the only arc exiting a vertex v^- in D is (v^-, v^+), the reverse transformation works as well, i.e., any s^+-t^- path P' in D corresponds to an s-t path P in G. Additionally, two s-t paths P_1 and P_2 in G are internally vertex-disjoint if and only if the corresponding s^+-t^- paths P_1' and P_2' in D are arc-disjoint.

Our proof of the vertex connectivity version of Menger's Theorem consists of two steps: We show that (i) the maximum number of internally vertex-disjoint s-t paths in G equals the value of a maximum s^+-t^- flow in D, and (ii) the cardinality of a minimum s-t-disconnecting set in G



(a) Original graph G.

(b) Auxiliary graph D.

Figure 10: An example graph $G = (V, A)$ and the corresponding auxiliary graph $D = (W, B)$.

equals the value of a minimum s^+-t^- cut in D. By the max-flow min-cut theorem (Corollary 4.14 in the script), the value of a maximum s^+-t^- flow equals the value of a minimum s^+-t^- cut in D. It thus suffices to show (i) and (ii).

(i) We prove that the maximum number of internally vertex-disjoint s-t paths in G equals the value of a maximum s_+-t_- flow in D by showing that every set of internally vertex-disjoint s-t paths in G corresponds to an (integral) s^+-t^- flow in D whose value equals the number of paths, and that every integral s^+-t^- flow in D corresponds to a set of internally vertex-disjoint s-t paths in G whose cardinality equals the value of the flow. Note that this indeed proves the desired statement (i) since all capacities are integral (or infinite) and thus there is a maximum s^+-t^- flow in D which is integral.

   – Let $P_1, \dots, P_k$ be a set of internally vertex-disjoint s-t paths in G. Consider a flow f in D that sends flow 1 along all s^+-t^- paths $P'_1, \dots, P'_k$ in D which correspond to $P_1, \dots, P_k$. Since the paths $P_1, \dots, P_k$ are internally vertex-disjoint, the paths $P'_1, \dots, P'_k$ in D are arc-disjoint, implying that f is a feasible s^+-t^- flow in D. Moreover, the value of the flow f equals k, since there are precisely k units of flow sent through arcs exiting s^+ (namely the arcs corresponding to the first arcs in each of the k paths).

   – Let f be an integral s^+-t^- flow in D with $\nu(f) = k$. Since there is no direct arc from s^+ to t^-, every arc in D can carry flow at most 1 (since all arcs of the form (v^-, v^+) have capacity 1). Thus, the flow f would still be feasible if all arcs had unit capacity. A result from the lecture (Theorem 4.19 in the script) therefore implies that there are k arc-disjoint s^+-t^- paths $P'_1, \dots, P'_k$ in D, which, as mentioned above, can be transformed into k internally vertex-disjoint s-t paths $P_1, \dots, P_k$ in G.

(ii) To prove that the cardinality of a minimum s-t-disconnecting set in G equals the value of a minimum s^+-t^- cut in D, we show that an s-t-disconnecting set in G corresponds to an s^+-t^- cut in D whose value equals the cardinality of the disconnecting set, and that a minimum s^+-t^- cut in D corresponds to an s-t-disconnecting set in G whose cardinality equals the value of the cut.

   – Let $S \subseteq V \setminus \{s, t\}$ be a set disconnecting s and t in G. For every $v \in S$, delete the arc (v^-, v^+) from D and denote by C the set of vertices in D which remain reachable from s^+. We claim that $t^- \notin C$. Indeed, let P' be an s^+-t^- path in D and let P be the corresponding s-t path in G. Since S disconnects s and t in G, P contains a vertex $v \in S$. This means that the s^+-t^- path P' in D contains the arc (v^-, v^+) and is therefore disconnected after this arc is deleted. Thus, $s^+ \in C$ and $t^- \notin C$, so C is an s^+-t^- cut in D. Note that no arc (v^+, w^-) leaves the cut C, because otherwise we would have $v^+ \in C$ and $w^- \notin C$, contradicting the fact that w^- can be reached from s^+ by first going to v^+ (which is possible since $v^+ \in C$) and then going along the arc (v^+, w^-) (arcs of that type did not get removed when the cut C was constructed). Thus, every arc leaving C

is a unit-capacity arc of the form (v^-, v^+) and hence corresponds to a vertex in S (since $v^- \in C$ and $v^+ \notin C$, this arc was deleted when we constructed C). Therefore, the value of the cut C equals the cardinality of the set S.

– Let $C \subseteq W$ be a minimum s^+-t^- cut in D. We first observe that the set $\{s^+\} \cup \{v^- \mid v \in V \setminus \{t\}\}$ is an s^+-t^- cut in D of finite value (since $(s^+, t^-) \notin B$, only arcs with capacity 1 are leaving the cut). This implies that the minimum s^+-t^- cut C must also have finite value and therefore no arc leaving C can have capacity ∞. We now define

$$S \coloneqq \left\{v \in V \colon \left(v^-, v^+\right) \in \delta^+(C)\right\} \ .$$

Since only unit-capacity arcs of the form (v^-, v^+) are leaving the cut C, the cardinality of the set S equals the value of the s^+-t^- cut C. We claim that S is also a set that disconnects s and t in G. Indeed, consider an arbitrary s-t path P in G. Note that the corresponding path P' in D starts with $s^+ \in C$ and ends with $t^- \notin C$, so there must exists an arc $(v, w) \in B$ on the path P' such that $v \in C$ and $w \notin C$. Therefore, $(v, w) \in \delta^+(C)$, and thus, as noted previously, there must exist a vertex $u \in V$ such that $v = u^-$ and $w = u^+$. Since the arc $(u^-, u^+) = (v, w)$ lies on the path P', the path P uses the vertex u. Moreover, by construction of S, we also have $u \in S$. Thus, every s-t path P in G goes through some vertex $u \in S$, meaning that S is a set disconnecting s and t in G.

(b) Note that arcs from s to t are s-t paths with no internal vertex. Thus, adding these to a set of s-t paths which were pairwise internally vertex-disjoint does not destroy this property. Therefore, if the original graph G contains such arcs, then we can delete them from G, apply the previous result, and then add the arcs back to the graph and include them in the set of pairwise internally vertex-disjoint s-t paths obtained previously.