5.4 Bipartite matching polytope

$G = (V, E)$

$\mathcal{M} \subseteq 2^E$: all matchings in G

Theorem 5.12

The bipartite matching polytope $P_{\mathcal{M}}$ is given by

$$P_{\mathcal{M}} = \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \; \forall v \in V\} \; . \tag{5.7}$$

We prove the statement by showing (ii) and (iii) of the "recipe".

<u>Proof of point (ii)</u> ← $P_{\mathcal{M}}$ contains correct set of integral points.

5.4.1 Integrality through TU-ness

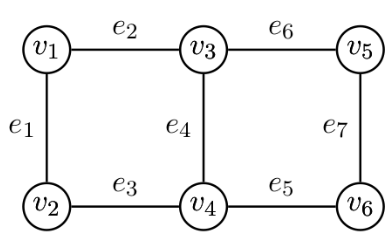
$$P = \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \quad \forall v \in V\} = \{x \in \mathbb{R}^E : Ax \leq b, \ x \geq 0\}$$

$\rightsquigarrow$ We will show that A is TU and then invoke:

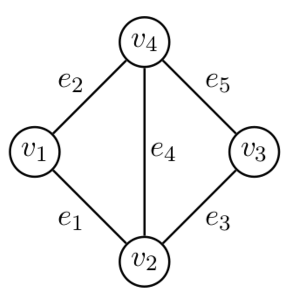
Theorem 5.8

Let $A \in \mathbb{Z}^{m \times n}$. Then,

$$A \text{ is TU} \quad \Leftrightarrow \quad P = \{x \in \mathbb{R}^n \colon Ax \leq b, x \geq 0\} \text{ is integral } \forall\, b \in \mathbb{Z}^m.$$



$$A = \begin{array}{cc} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$



$$A = \begin{array}{cc} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Theorem 5.13

Let $G = (V, E)$ be an undirected graph with vertex-edge incidence matrix A. Then,

$$G \text{ is bipartite } \Leftrightarrow A \text{ is TU.}$$

5.4.3 Some implications coming from inequality description of P_M

Perfect bipartite matching polytope

Theorem 5.14

The perfect matching polytope of a bipartite graph $G = (V, E)$ is given by

$$P = \left\{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) = 1 \ \forall v \in V\right\} \ .$$

Corollary 1.14

Let P be a polyhedron. Then a face of a face of P is itself a face of P.

Perfect matchings in bipartite d-regular graphs

Theorem 5.15

Let $d \in \mathbb{Z}_{\geq 1}$. Every d-regular bipartite graph admits a perfect matching.

5.5 Polyhedral description of short s-t paths

Consider directed graph $G=(V,A)$ and $s,t \in V$, $s \neq t$.

Consider :

$$P = \left\{ x \in [0,1]^A \;\middle|\; x(\delta^+(v)) - x(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \end{cases} \quad \forall v \in V \right\}.$$

This is s-t flow polytope of unit flow.

$$P = \left\{ x \in [0,1]^A \;\middle|\; x(\delta^+(v)) - x(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \end{cases} \quad \forall v \in V \right\}$$

Theorem 5.16

The vertex-arc incidence matrix $D \in \{-1,0,1\}^{V \times A}$ of any directed (loopless) graph $G = (V, A)$ is TU.

5.6 Spanning trees and r-arborescences

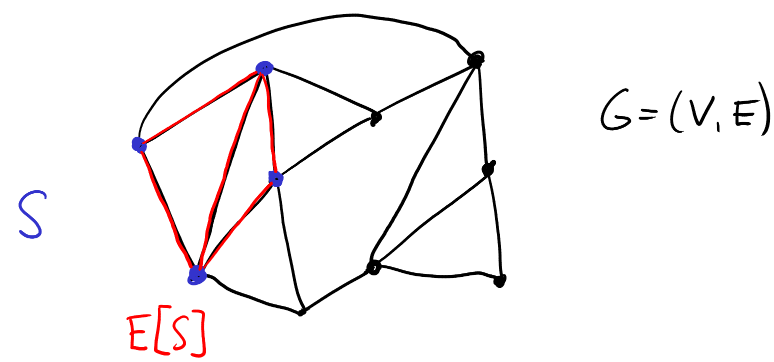
5.6.1 Spanning tree polytope

Theorem 5.17

The spanning tree polytope of an undirected loopless graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

All edges with both endpoints in S.



→ Exponentially many constraints.

→ Problem sets: All constraints can be facet defining (depending on input graph G).

5.6.2 The r-arborescence polytope

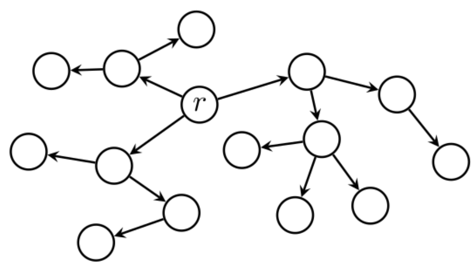
Definition 5.18: Arborescence, r-arborescence

Let $G = (V, A)$ be a directed graph. An *arborescence* in G is an arc set $T \subseteq A$ such that

(i) T is a spanning tree (when disregarding the arc directions), and

(ii) there is one vertex r from which all arcs are directed away, i.e., every vertex $v \in V$ can be reached from r using a directed path in T.

The vertex r in condition (ii) is called the *root* of the arborescence, and T is called an *r-arborescence*.



Theorem 5.19

The arborescence polytope of a directed loopless graph $G = (V, A)$ is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \;\middle|\; \begin{array}{ll} x(A) = |V| - 1 & \\ x(A[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \\ x(\delta^-(v)) \leq 1 & \forall v \in V \end{array} \right\} ,$$

where $A[S] \subseteq A$ for $S \subseteq V$ denotes all arcs with both endpoints in S.

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \colon x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\} .$$