Fall 2019

Mathematical Optimization – Problem set 12

`https://moodle-app2.let.ethz.ch/course/view.php?id=4844`

Problem 1: Laminar cut families and total unimodularity

Let $G = (V, A)$ be a directed graph and let $\mathcal{F} \subseteq 2^V$ be a laminar family. Show that the matrix $M \in \{0,1\}^{\mathcal{F} \times A}$, where the row corresponding to $S \in \mathcal{F}$ is given by $\chi^{\delta^+(S)}$, is totally unimodular.

Problem 2: Uncrossing directed cuts

Let $G = (V, A)$ be a directed graph, and assume that there exists a vertex $r \in V$ such that $(v, r) \in A$ for every $v \in V \setminus \{r\}$. We call a cut $S \subsetneq V$, $S \neq \emptyset$, *directed* if $\delta^-(S) = \emptyset$. Let $\mathcal{C}$ be the set of all directed cuts, i.e., $\mathcal{C} = \{S \subsetneq V : S \neq \emptyset,\ \delta^-(S) = \emptyset\}$. We are interested in subsets $U \subseteq A$ of arcs hitting all directed cuts. More precisely, we say that a subset of the arcs $U \subseteq A$ is *feasible* if $|U \cap \delta^+(C)| \geq 1$ for every directed cut $C \in \mathcal{C}$. We then denote the set of feasible subsets by $\mathcal{D}$, i.e., $\mathcal{D} = \{U \subseteq A : |U \cap \delta^+(C)| \geq 1\ \forall C \in \mathcal{C}\}$. We want to show that

$$P = \{x \in [0,1]^A : x(\delta^+(C)) \geq 1\ \forall C \in \mathcal{C}\}$$

is the polytope corresponding to $\mathcal{D}$, i.e., $P = \operatorname{conv}(\{\chi^D : D \in \mathcal{D}\})$.

(a) Show that P describes the correct set of integral points, i.e., $P \cap \{0,1\}^A = \{\chi^D : D \in \mathcal{D}\}$.

To prove that P is the correct description, it remains to show that P is integral. Let y be a vertex of P and define

$$\mathcal{F} = \{C \in \mathcal{C} : y(\delta^+(C)) = 1\}\ , \quad A_0 = \{a \in A : y(a) = 0\}\ , \quad \text{and} \quad A_1 = \{a \in A : y(a) = 1\}\ .$$

Thus, the entire system of y-tight constraints, to which y is the unique solution, is given by

$$\left\{\begin{array}{rcll} x(\delta^+(C)) & = & 1 & \forall C \in \mathcal{F} \\ x(a) & = & 0 & \forall a \in A_0 \\ x(a) & = & 1 & \forall a \in A_1 \end{array}\right. . \tag{S_0}$$

(b) Let $\mathcal{H}$ be a maximal laminar subfamily of $\mathcal{F}$. Show that the system

$$\left\{\begin{array}{rcll} x(\delta^+(H)) & = & 1 & \forall H \in \mathcal{H} \\ x(a) & = & 0 & \forall a \in A_0 \\ x(a) & = & 1 & \forall a \in A_1 \end{array}\right. \tag{S_1}$$

is equivalent to the system (S_0).

(c) Show that the system (S_1) is totally unimodular. Use this to conclude that P is indeed integral.

Problem 3: Cutting integer polyhedra

(a) Let $P \subseteq \mathbb{R}^n_{\geq 0}$ be an integral polyhedron with $P + \mathbb{R}^n_{\geq 0} = P$. Let $c \in \mathbb{Z}^n_{\geq 0}$ such that $x \leq c$ for each vertex of P. Prove that $P \cap \{x \in \mathbb{R}^n_{\geq 0} : x \leq c\}$ is an integral polyhedron.

(b) Use part (a) to show the following: If $P \subseteq \mathbb{R}^n$ is a $\{0,1\}$-polytope and $Q := \operatorname{dom}(P) \cap [0,1]^n$, then Q is a $\{0,1\}$-polytope.

Problem 4: Minimum-volume ellipsoid containing half-ball
Recall that an ellipsoid in $\mathbb{R}^n$ is a set of the form

$$E(a, A) = \{x \in \mathbb{R}^n : (x - a)^T A^{-1}(x - a) \leq 1\} \ ,$$

where $a \in \mathbb{R}^n$ is the center of the ellipsoid and $A \succ 0$ is a positive definite matrix in $\mathbb{R}^{n\times n}$. Let $H_B = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. In class (see Lemma 6.7), it is proved that the ellipsoid $E_B = E(\overline{a}, \overline{A})$ defined by

$$\overline{a} = \begin{pmatrix} \frac{1}{n+1} & 0 & 0 & \cdots & 0 \end{pmatrix}^T \qquad \text{and} \qquad \overline{A} = \begin{pmatrix} \left(\frac{n}{n+1}\right)^2 & 0 & \cdots & 0 \\ 0 & \frac{n^2}{n^2-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n^2}{n^2-1} \end{pmatrix}$$

contains the half-ball $E(0, I) \cap H_B$. In this problem, we prove that E_B is in fact the minimum minimum-volume ellipsoid containing $E(0, I) \cap H_B$ by showing the following claims.

(a) Let e_i denote the i-th unit vector in $\mathbb{R}^n$. Consider the set $S \subseteq \mathbb{R}^n$ defined by

$$S = \{e_1, e_2, -e_2, e_3, -e_3, \cdots, e_n, -e_n\} \ .$$

Note that S contains all the unit vectors and their negatives, except for $-e_1$. Observe that any ellipsoid enclosing $E(0, I) \cap H_B$ also encloses S.

(b) Write a semi-definite convex program to compute the minimum-volume ellipsoid enclosing S. For our purposes, it is convenient to consider a semi-definite convex program of the form

$$\begin{array}{rrcll} \min & f(x, X) & & & \\ & g_i(x, X) & \leq & b_i & \forall i \in [m] \\ & X & \succeq & 0 & \\ & x & \in & \mathbb{R}^n & \\ & X & \in & \mathbb{R}^{n\times n} & \end{array} \ , \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ for $i \in [m]$ are convex functions, and $b_i \in \mathbb{R}$ for $i \in [m]$.

Hint: Use the constraints of your convex program to enforce $(x - a)^T A^{-1}(x - a) \leq 1$ for all $x \in S$. Note, however, that this constraint is not convex in a and A. Show that it is enough to find a positive definite $D \in \mathbb{R}^{n\times n}$ and $p \in \mathbb{R}^n$ such that $\|Dx - p\|_2^2 \leq 1$ for all $x \in S$ (which is a convex constraint in D and p), and to then use $A = D^\top D$ and $a = D^{-1}p$. Moreover, for a suitable objective function, observe that the volume of the ellipsoid $E(a, A)$ is proportional to $\det(D)$, and use the convex function $X \to -\log\det X$.

(c) Show that if the convex program from part (b) has an optimal solution, then it is unique. This implies a minimum-volume ellipsoid containing S is uniquely defined, given that it exists.

(d) The Karush-Kuhn-Tucker (KKT) conditions are a sufficient optimality condition for a convex program of the form given in (1). These conditions state that (x^*, X^*) is an optimal solution of (1) if (x^*, X^*) is feasible and there exist non-negative multipliers $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla f(x^*, X^*) + \sum_{i\in[m]} \lambda_i \nabla g_i(x^*, X^*) = 0 \ ,$$
$$\text{and} \qquad \lambda_i(b_i - g_i(x^*, X^*)) = 0 \quad \forall i \in [m] \ .$$

Use the KKT conditions to prove that our candidate E_B is indeed the minimum volume ellipsoid enclosing S, and conclude that it is also the minimum volume ellipsoid containing $E(0, I) \cap H_B$.

Programming exercises
Complete the notebook `12_forestPolytopeSeparation.ipynb`, where you design and implement a separation oracle for the forest polytope.