1.3.4 Exchange step / pivoting

Goal: Getting from one tableau to another one.

↑ an equation system in tableau form

Example of exchange step / pivoting

y_1	y_2	y_3	x_1	x_2	1
$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
$-\frac{3}{4}$	0	1	$\boxed{\frac{3}{4}}$	0	6

↑ pick non-zero element in non-basic column

row operations ⟶

y_1	y_2	y_3	x_1	x_2	1
$\frac{1}{2}$	0	$-\frac{1}{3}$	0	1	8
$\frac{6}{4}$	1	$-\frac{7}{3}$	0	0	18
-1	0	$\frac{4}{3}$	1	0	8

- Add $-\dfrac{\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3}$ of 3rd row to 1st one.
- Add $-\dfrac{\frac{7}{4}}{\frac{3}{4}} = -\frac{7}{3}$ of 3rd row to 2nd one.
- Multiply 3rd row by $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$.

Formal description of exchange step/pivoting

$k \in [m+n]$

	y_1	y_2	y_3	x_1	x_2	1
	$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
	$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
$i \in [m]$	$-\frac{3}{4}$	0	1	$\boxed{\frac{3}{4}}$	0	6

Let's call this matrix $A \in \mathbb{R}^m$

$b \in \mathbb{R}^m$

$\rightsquigarrow$

y_1	y_2	y_3	x_1	x_2	1
$\frac{1}{2}$	0	$-\frac{1}{3}$	0	1	8
$\frac{6}{4}$	1	$-\frac{7}{3}$	0	0	18
-1	0	$\frac{4}{3}$	1	0	8

$A' \in \mathbb{R}^{m \times (m+n)}$

$b' \in \mathbb{R}^m$

Condition : $\underline{i} \in [m]$, $\underline{k} \in [m+n]$ such that $\underline{A_{ik}} \neq 0$

pivot row

pivot column

pivot element

Pivoting formulas

(i) $A'_{i\ell} = \dfrac{A_{i\ell}}{A_{ik}}$ $\qquad \ell \in [m+n]$

(ii) $A'_{j\ell} = A_{j\ell} - \dfrac{A_{jk} \cdot A_{i\ell}}{A_{ik}}$ $\qquad \ell \in [m+n], \qquad j \neq i, j \in [m]$

(iii) $b'_i = \dfrac{b_i}{A_{ik}}$

(iv) $b'_j = b_j - \dfrac{A_{jk} \cdot b_i}{A_{ik}}$ $\qquad j \in [m]$ with $j \neq i$,

1.3.5 Short tableau

long tableau

y_1	y_2	y_3	x_1	x_2	1
$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
$-\frac{3}{4}$	0	1	$\frac{3}{4}$	0	6

$\longleftrightarrow$

short tableau

	y_1	x_1	1
x_2	$\frac{1}{4}$	$\frac{1}{4}$	10
y_2	$-\frac{1}{4}$	$\frac{7}{4}$	32
y_3	$-\frac{3}{4}$	$\frac{3}{4}$	6

Some advantages of short tableau:

- Short tableau makes basis explicit.
  - → Removes ambiguities regarding which basis we want to focus on with a tableau in case basis is not unique.
- Short tableau has more natural dual interpretation (to be discussed later).
- Short tableau is more compact.

1.3.6 Exchange step in short tableau

$$
\begin{array}{c|cc|c}
 & y_1 & \boxed{x_1} & 1 \\
\hline
x_2 & \frac{1}{4} & \frac{1}{4} & 10 \\
y_2 & -\frac{1}{4} & \frac{7}{4} & 32 \\
\boxed{y_3} & -\frac{3}{4} & \boxed{\frac{3}{4}} & 6
\end{array}
\quad \xrightarrow{\text{exchange step}} \quad
\begin{array}{c|cc|c}
 & y_1 & y_3 & 1 \\
\hline
x_2 & \frac{1}{2} & -\frac{1}{3} & 8 \\
y_2 & \frac{6}{4} & -\frac{7}{3} & 18 \\
x_1 & -1 & \frac{4}{3} & 8
\end{array}
$$

pivot element

A b A' b'

basis : $B = (x_2, y_2, y_3)$

basis : $B' = (x_2, y_2, x_1)$

Pivoting formulas for short tableau

$$
\begin{aligned}
A'_{ik} &= \frac{1}{A_{ik}} && \textit{(pivot element)} \\
A'_{jk} &= -\frac{A_{jk}}{A_{ik}} && \textit{(pivot column)} \quad j \in [m], \quad j \neq i \\
A'_{i\ell} &= \frac{A_{i\ell}}{A_{ik}} && \textit{(pivot row)} \quad \ell \in [n], \quad \ell \neq k \\
A'_{j\ell} &= A_{j\ell} - \frac{A_{jk} \cdot A_{i\ell}}{A_{ik}} && j \in [m], \quad j \neq i \\
& && \ell \in [n], \quad \ell \neq k \\
b'_i &= \frac{b_i}{A_{ik}} \\
b'_j &= b_j - \frac{A_{jk} \cdot b_i}{A_{ik}} && j \in [m], \quad j \neq i
\end{aligned}
$$

Example 1.60

(a):

$B = (x_1, x_2, x_3)$

	x_4	x_5	1
x_1	2	$\boxed{1}$	3
x_2	1	-1	0
x_3	1	0	2

with basic solution

$(x_1, x_2, x_3, x_4, x_5) = (3, 0, 2, 0, 0)$

(b):

	x_4	x_1	1
x_5	2	1	3
x_2	$\boxed{3}$	1	3
x_3	1	0	2

$(0, 3, 2, 0, 3)$

(c):

	x_2	x_1	1
x_5	$-\frac{2}{3}$	$\frac{1}{3}$	1
x_4	$\frac{1}{3}$	$\boxed{\frac{1}{3}}$	1
x_3	$-\frac{1}{3}$	$-\frac{1}{3}$	1

$(0, 0, 1, 1, 1)$

(d):

	x_2	x_4	
x_5	-1	-1	0
x_1	1	3	3
x_3	0	1	2

$(3, 0, 2, 0, 0)$

Inequality description coming from (a)

$$\begin{array}{rcrcl} 2x_4 & + & x_5 & \leq & 3 \\ x_4 & - & x_5 & \leq & 0 \\ x_4 & & & \leq & 2 \\ x_4 & & & \geq & 0 \\ & & x_5 & \geq & 0 \end{array}$$