3 Basics on Graphs

$n^2 = O(n^3)$

$O(n^3) = n^2$

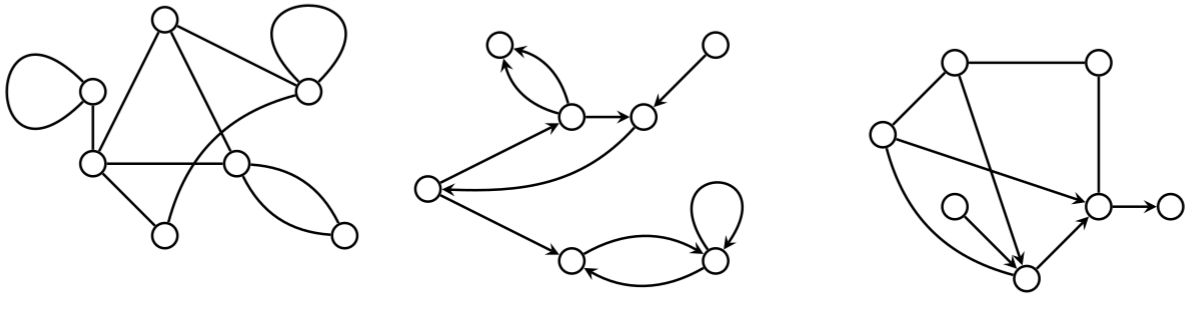
$O(n^2) = O(n^3)$

~~$O(n^3) = O(n^2)$~~

3.1 Some motivational examples

↪ See intro lecture and script.

3.2 Basic terminology and notation



(a) An undirected graph. (b) A directed graph. (c) A mixed graph.

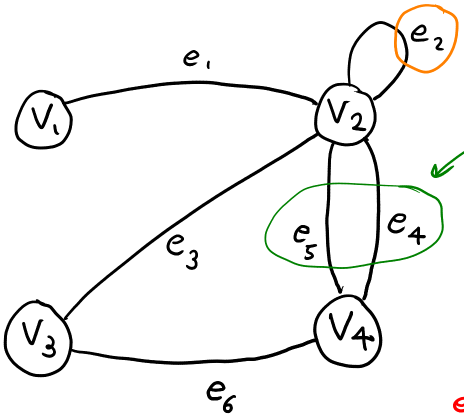
Figure 3.8: Different types of graphs.

Undirected graphs

$$G = (\overset{\text{vertices}}{V}, \overset{\text{edges}}{E})$$

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$



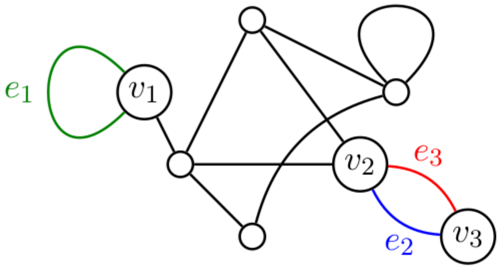
loop (both endpoints are same vertex)

parallel edges

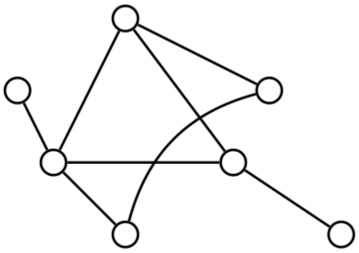
endpoints(e_4) = endpoints(e_5)

endpoints$(e_6) = \{v_3, v_4\}$

An undirected graph G is called <u>simple</u> if has neither loops nor parallel edges.

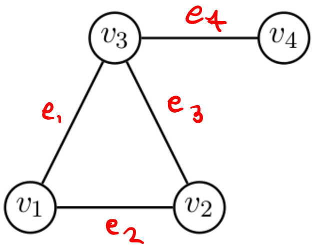


(a) A non-simple graph.



(b) A simple graph.

$e_1 = \{v_1, v_3\}$

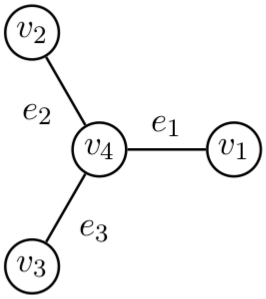


$$G = (V, E)$$

$$V = \{v_1, v_2, v_3, v_4\}$$

$$E = \{\underset{e_2}{\{v_1, v_2\}}, \underset{e_3}{\{v_2, v_3\}}, \underbrace{\{v_1, v_3\}}_{e_1}, \underset{e_4}{\{v_3, v_4\}}\}$$

Figure 3.10: Notation for simple graphs. Due to its convenience, this notation is also often used in the context of non-simple graphs.



v_1 is incident to e_1, but not to e_2 or e_3.

v_4 is adjacent to v_1, v_2, and v_3.

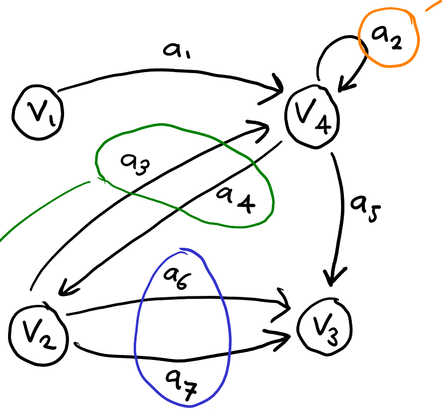
v_1 and v_2 are not adjacent.

Directed graphs

$$G = (\overset{\text{vertices}}{V}, \overset{\text{arcs}}{A})$$

$V = \{v_1, v_2, v_3, v_4\}$

$A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$



loop : $\text{head}(a_2) = \text{tail}(a_2)$

parallel arcs :

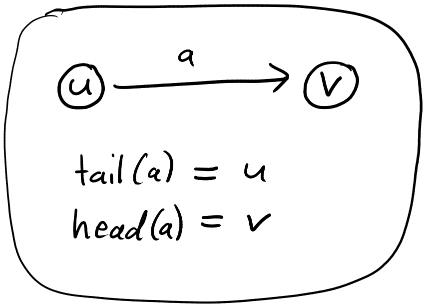
$\text{tail}(a_6) = \text{tail}(a_7)$

$\text{head}(a_6) = \text{head}(a_6)$

antiparallel:

$\text{tail}(a_3) = \text{head}(a_4)$

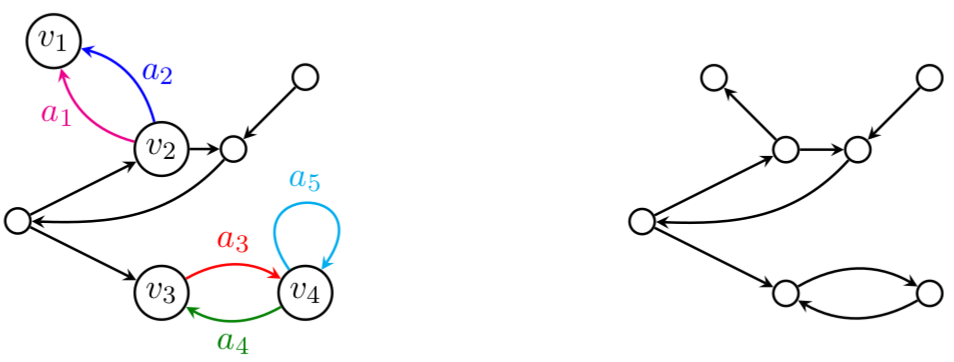
$\text{head}(a_3) = \text{tail}(a_4)$



$\text{tail}(a) = u$

$\text{head}(a) = v$

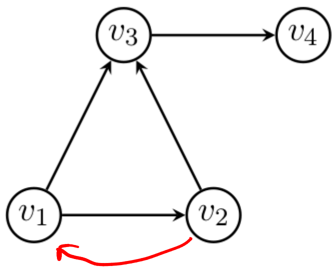
G is a <u>simple</u> directed graph if G has neither loops nor parallel arcs.



(a) A non-simple directed graph.

(b) A simple directed graph.

$a=(v_1,v_2)$



$$G=(V,A)$$

$$V=\{v_1,v_2,v_3,v_4\}$$

$$A=\{(v_1,v_2),(v_1,v_3),(v_2,v_3),(v_3,v_4)\}$$

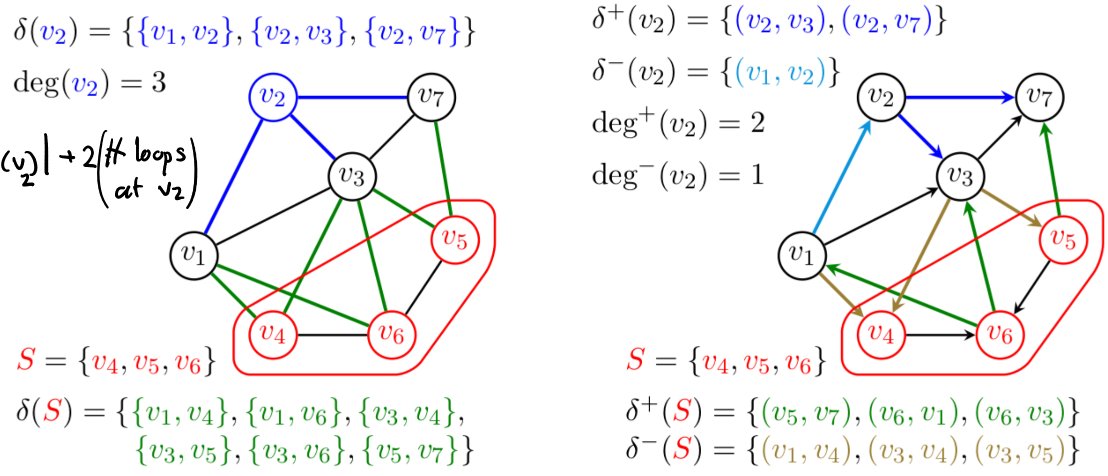
(v_2,v_1)

Figure 3.13: Notation for simple directed graphs. Due to its convenience, this notation is also used in the context of non-simple graphs.

Basic notation and terminology

$\delta(v) = \delta^+(v) \cup \delta^-(v)$

$e \notin \delta(v_2)$



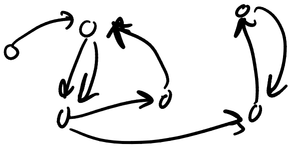
(a) Undirected graph.

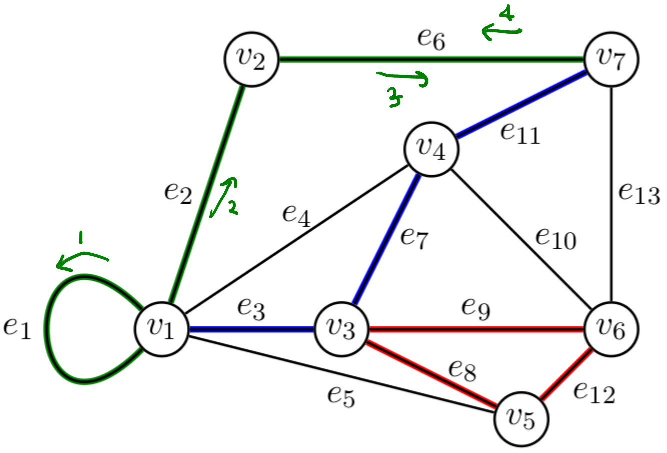
(b) Directed graph.

$\deg(v_2) = |\delta(v_2)| + 2(\text{\# loops at } v_2)$

$\delta(S) = \delta^+(S) \cup \delta^-(S)$

Walks, paths, and cycles

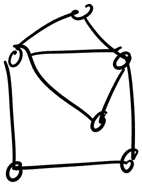


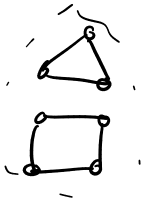


$v_1, e_1, v_1, e_2, v_2, e_6, v_7, e_6, v_2$ is a walk of length 4, but not a path, since v_1 and v_2 appear multiple times.

$v_1, e_3, v_3, e_7, v_4, e_{11}, v_7$ is a v_1-v_7-path of length 3.

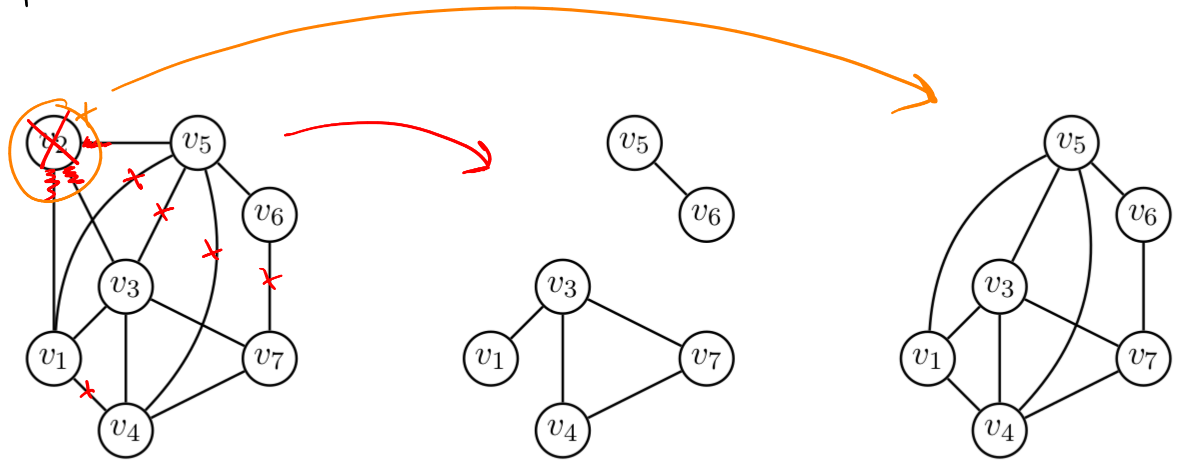
$v_3, e_8, v_5, e_{12}, v_6, e_9, v_3$ is a closed walk with pairwise distinct vertices (up to start and endpoint), hence it is a cycle of length 3.





A path is uniquely determined by its edge set: $\{e_3, e_7, e_{11}\}$.

Subgraphs



(a) A graph $G = (V, E)$.

(b) A subgraph of G.

(c) Induced subgraph $G[V \backslash \{v_2\}]$.

A graph $H=(W,F)$ is a subgraph of $G=(V,E)$ if:

(i) $W \subseteq V$ and

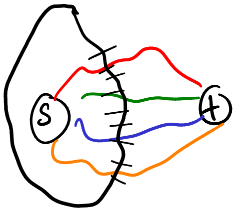
(ii) $F \boxed{\subseteq} \{e \in E : \text{endpoints}(e) \subseteq W\}$

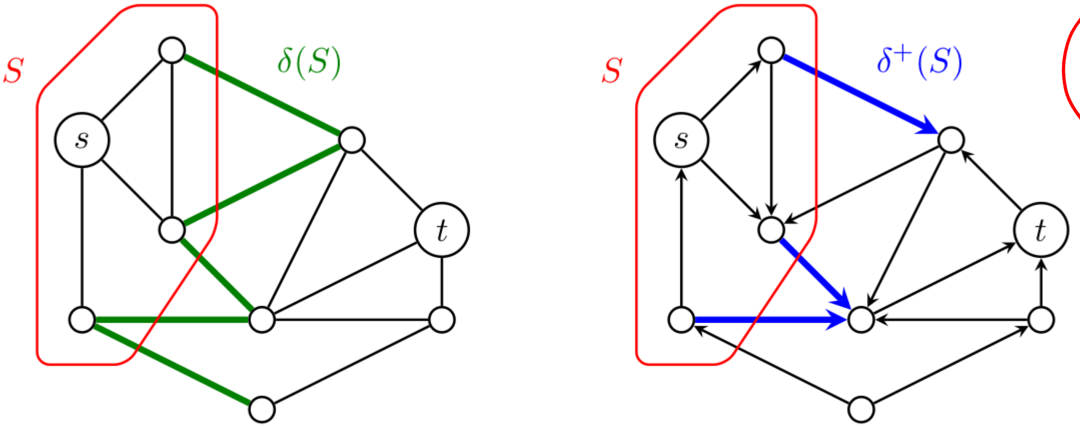
Let $G=(V,E)$ be a graph and $W \subseteq V$. The subgraph of G induced by W, is the graph $G[W] = (W,F)$ with

$$F \boxed{=} \{e \in E : \text{endpoints}(e) = W\}$$

($G[W]$ is called an *induced subgraph*.)

Cuts

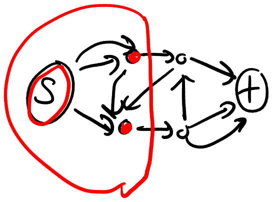




(a) Undirected graph.

(b) Directed graph.

Figure 3.18: An s-t cut S and the edges in the cut $\delta(S)$ (for the undirected case) and $\delta^+(S)$ (for the directed case).



A cut in $G=(V,E)$ is a vertex set $S \subseteq V$ with

(i) $S \neq \emptyset$, and

(ii) $S \neq V$.

For two vertices $s,t \in V$, an s-t cut is a vertex set $S \subseteq V$ with

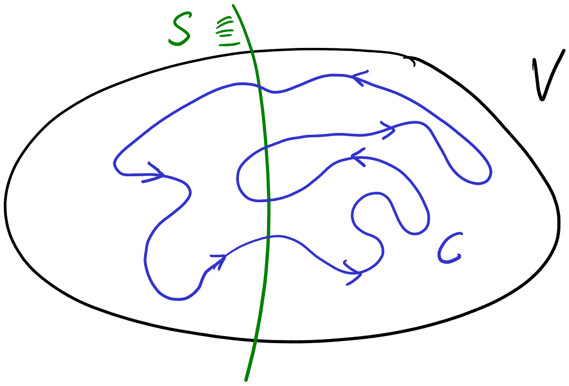
(i) $s \in S$, and

(ii) $t \notin S$.

Exercise 3.11

Let $G = (V, A)$ be a directed graph, S a cut in G, and $C \subseteq A$ a closed directed walk in G. Show that $|\delta^+(S) \cap C| = |\delta^-(S) \cap C|$.

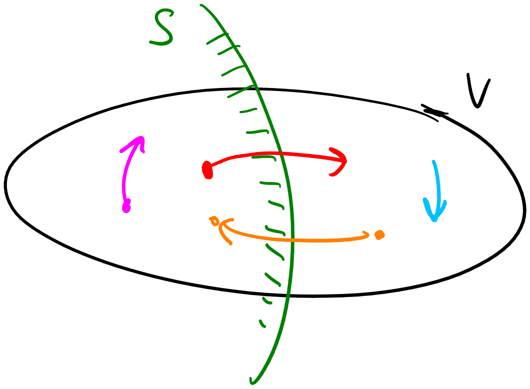
using every arc at most once



Proof

Let $H = (V, C)$ be the subgraph of G that contains all vertices of G and the arcs in C.

Our goal is to show : $$\underbrace{|\delta_H^+(S)|}_{= \delta_G^+(S) \cap C} = \underbrace{|\delta_H^-(S)|}_{= \delta_G^-(S) \cap C}$$



$$|\delta_H^+(S)| - |\delta_H^-(S)| = \sum_{v \in S} \underbrace{\left(\deg_H^+(v) - \deg_H^-(v)\right)}_{=0} = 0$$

↖ because C is closed walk

#

3.3 Data structures for graphs

We consider two datastructures to work with graphs:

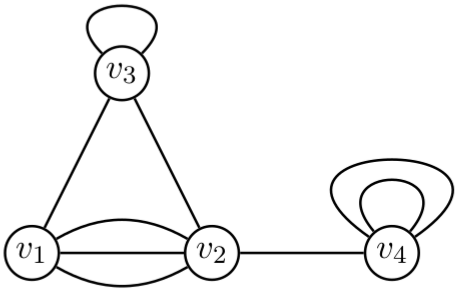
(1) adjacency matrix

(2) incidence list

Adjacency matrix

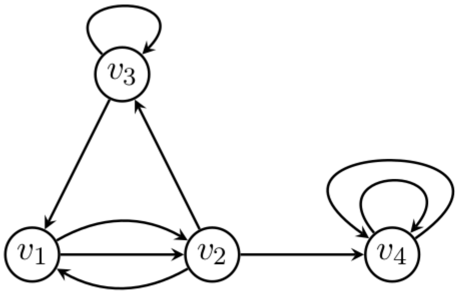
vertices: n

edges : m



$$M = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 4 \end{pmatrix} \end{array}$$

Figure 3.19: The adjacency matrix of an undirected graph.



$$M = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

Figure 3.20: The adjacency matrix of a directed graph.

Incidence list

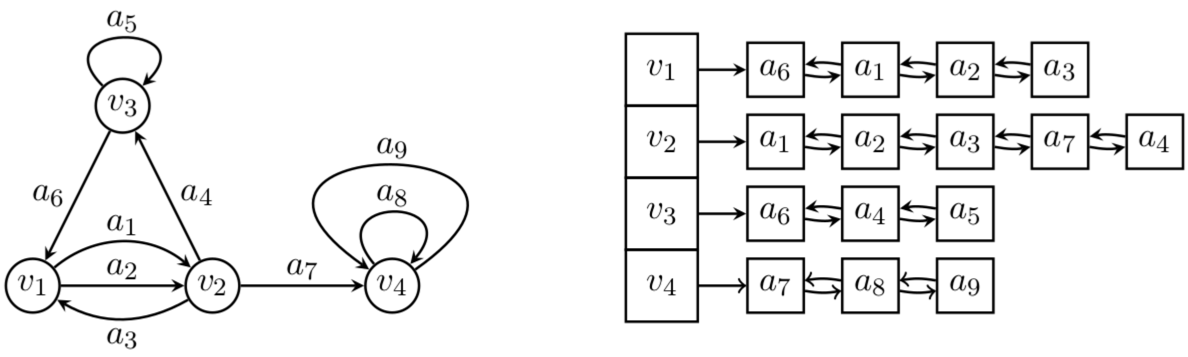


Figure 3.21: The incidence list of a directed graph.

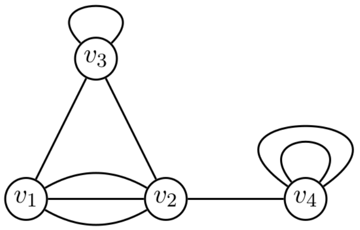
Brief comparison : adjacency matrix vs. incidence list

Space requirements

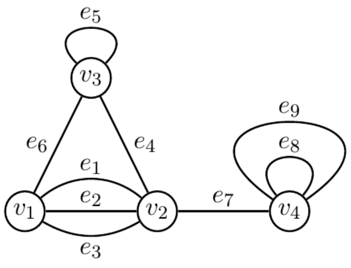
Let $G = (V, E)$ be a simple undirected graph, and let $n := |V|$, $m := |E|$.

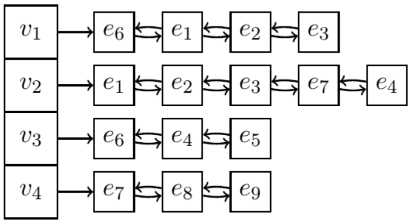
space needed for adjacency matrix : $\Theta(n^2)$
" " " incidence list : $\Theta(m+n)$

Algorithmic complexity of some basic operations



$$M = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 3 & 1 & 0 \\ v_2 & 3 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 2 & 0 \\ v_4 & 0 & 1 & 0 & 4 \end{array}$$





	$\deg(v)$ or $\delta(v)$	$\exists ?\{u, v\} \in E$	$\lvert E \rvert$
adjacency matrix	$O(n)$	$O(1)$	$O(n^2)$
incidence list	$O(\deg(v))$	$O(\min\{\deg(u), \deg(v)\})$	$O(m+n)$