5.8.3 Upper bounds on edges of minimally k-edge-connected graphs

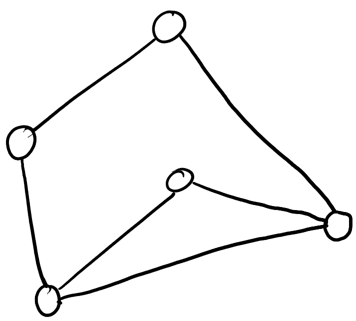
Let $G = (V, E)$ be an undirected graph.

Definition

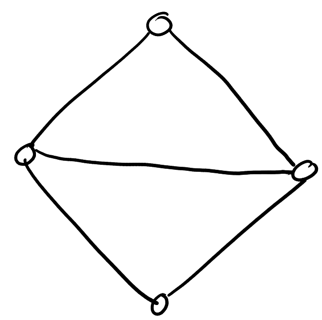
G is minimally k-edge-connected if

(i) G is k-edge-connected, and

(ii) for any $e \in E$, the graph $(V, E \setminus \{k\})$ is not k-edge-connected.



minimally 2-edge-connected



2-edge connected but not minimally 2-edge connected

Due to Menger's Theorem, a graph $G = (V, E)$ is minimally k-edge-connected if

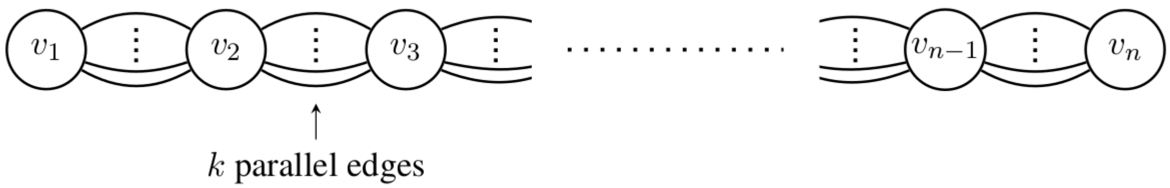
(i) $|\delta(S)| \geq k \quad \forall\, S \subseteq V$, and

(ii) $\forall\, e \in E \;\; \exists\, S \subseteq V$ s.t. $e \in \delta(S)$ and $|\delta(S)| = k$.

Theorem 5.30

Let $G = (V, E)$ be a minimally k-edge-connected graph. Then $|E| \leq k \cdot (|V| - 1)$.

This bound is tight:



Proof of Theorem 5.30

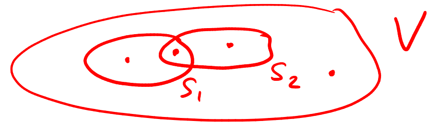
Lemma

For any $S_1, S_2 \subseteq V$:

$$x^{\delta(S_1)} + x^{\delta(S_2)} = x^{\delta(S_1 \cup S_2)} + x^{\delta(S_1 \cap S_2)} + 2x^{E(S_1 \setminus S_2,\, S_2 \setminus S_1)}$$

↳ Proof is analogous to proof of Lemma 5.27.

$S_1 \setminus S_2,\ S_2 \setminus S_1,\ S_1 \cap S_2,\ V \setminus (S_1 \cup S_2) \neq \emptyset$



$|\delta(S_1)| = |\delta(S_2)| = k$

Lemma

Let $S_1, S_2 \subseteq V$ be two crossing sets that are minimum cuts.
Then $|\delta(S_1 \cup S_2)| = |\delta(S_1 \cap S_2)| = k$ and $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$.

$\hookrightarrow$ Proof is analogous to proof of Lemma 5.28. (exercise)