5.8.3 Upper bounds on edges of minimally k-edge-connected graphs

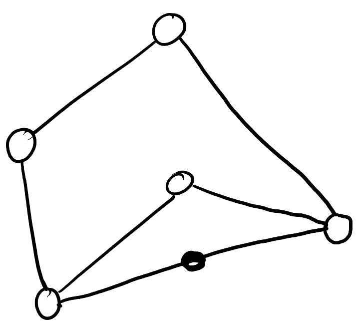
Let $G=(V,E)$ be an undirected graph.

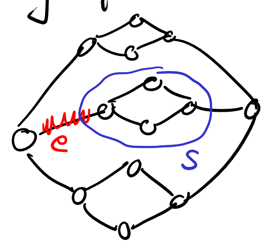
Definition

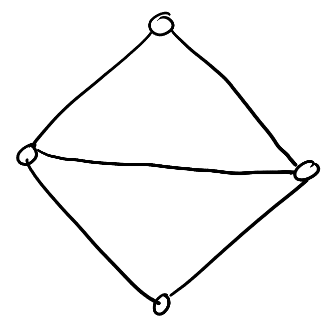
G is minimally k-edge-connected if

(i) G is k-edge-connected, and

(ii) for any $e \in E$, the graph $(V, E\setminus\{e\})$ is not k-edge-connected.







minimally 2-edge-connected

2-edge connected but not minimally 2-edge connected

Due to Menger's Theorem, a graph $G=(V,E)$ is minimally k-edge-connected if

(i) $|\delta(S)| \geq k \quad \forall\ \overset{S \neq \emptyset}{S \subsetneq V}$, and

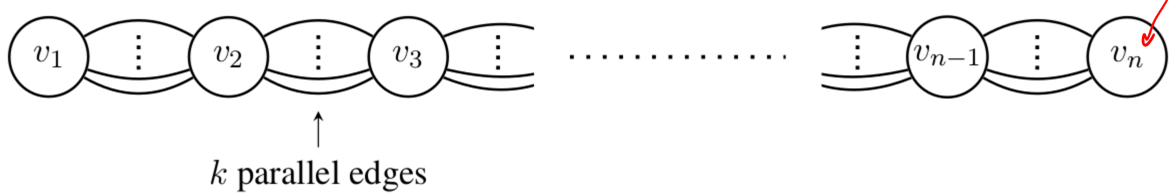
(ii) $\forall\, e \in E \ \exists\ S \subseteq V$ s.t. $e \in \delta(S)$ and $|\delta(S)| = k$.

Theorem 5.30

Let $G = (V, E)$ be a minimally k-edge-connected graph. Then $|E| \leq k \cdot (|V| - 1)$.

This bound is tight:

$n := |V|$



Proof of Theorem 5.30

Let $G=(V,E)$ be a minimally k-edge-connected graph.

We call $\mathcal{H} \subseteq 2^V$ a <u>certifying family</u> if:

(i) $|\delta(H)| = k \quad \forall \; H \in \mathcal{H}$, and

(ii) $\bigcup_{H \in \mathcal{H}} \delta(H) = E$.

→ Notice that a certifying family indeed certifies that some k-edge connected graph is minimally k-edge-connected.

Our goal: Show that there exists certifying family $\mathcal{H} \subseteq 2^V$ with $|\mathcal{H}| \leq |V| - 1$.

This implies the statement:

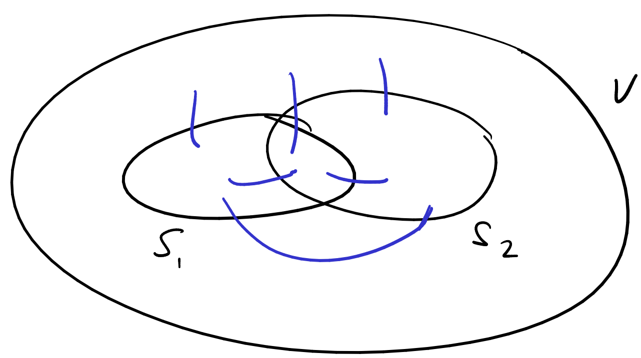
$$|E| = \left| \bigcup_{H \in \mathcal{H}} \delta(H) \right| \leq \sum_{H \in \mathcal{H}} \underbrace{|\delta(H)|}_{= k} = |\mathcal{H}| \cdot k = (|V|-1) \cdot k .$$

We start with arbitrary certifying family $\mathcal{F}$.

We fix an arbitrary vertex $r \in V$ and assume wlog:

$$r \notin H \quad \forall\, H \in \mathcal{H}.$$

→ For otherwise replace H by $V \setminus H$.



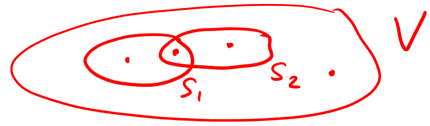
__Lemma__

For any $S_1, S_2 \subseteq V$:

$$\chi^{\delta(S_1)} + \chi^{\delta(S_2)} = \chi^{\delta(S_1 \cup S_2)} + \chi^{\delta(S_1 \cap S_2)} + 2\chi^{E(S_1 \setminus S_2,\, S_2 \setminus S_1)}$$

↳ Proof is analogous to proof of Lemma 5.27.

$S_1\setminus S_2,\ S_2\setminus S_1,\ S_1\cap S_2,\ V\setminus(S_1\cup S_2) \neq \emptyset$



$|\delta(S_1)| = |\delta(S_2)| = k$

Lemma

Let $S_1, S_2 \subseteq V$ be two $\overbrace{\text{crossing sets}}$ that are $\overbrace{\text{minimum cuts}}$.

Then $|\delta(S_1\cup S_2)| = |\delta(S_1\cap S_2)| = k$ and $E(S_1\setminus S_2, S_2\setminus S_1) = \emptyset$.

$$\chi^{\delta(S_1)} + \chi^{\delta(S_2)} = \chi^{\delta(S_1\cup S_2)} + \chi^{\delta(S_1\cap S_2)} + 2\chi^{E(S_1\setminus S_2, S_2\setminus S_1)}$$

$$2k = \underbrace{|\delta(S_1)|}_{=k} + \underbrace{|\delta(S_2)|}_{=k} = \underbrace{|\delta(S_1\cup S_2)|}_{\geq k} + \underbrace{|\delta(S_1\cap S_2)|}_{\geq k} + 2\underbrace{|E(S_1\setminus S_2, S_2\setminus S_1)|}_{\geq 0} \geq 2k$$

$\#$

We first obtain a laminar certifying family by repeating:

If $S_1, S_2 \in \mathcal{F}$ s.t. S_1 and S_2 are crossing, replace S_1, S_2 by $S_1\cup S_2$ and $S_1\cap S_2$:

$$\mathcal{F}' = (\mathcal{F}\setminus\{S_1,S_2\}) \cup \{S_1\cup S_2,\ S_1\cap S_2\}$$

By above lemma, $S_1\cup S_2$ and $S_1\cap S_2$ are min cuts.

$\mathcal{F}'$ is a certifying family because:

$$\chi^{\delta(S_1)} + \chi^{\delta(S_2)} = \chi^{\delta(S_1\cup S_2)} + \chi^{\delta(S_1\cap S_2)}$$

$$\Rightarrow \delta(S_1)\cup\delta(S_2) = \delta(S_1\cup S_2)\cup\delta(S_1\cap S_2)$$

Moreover, above procedure stops due to following potential function argument:

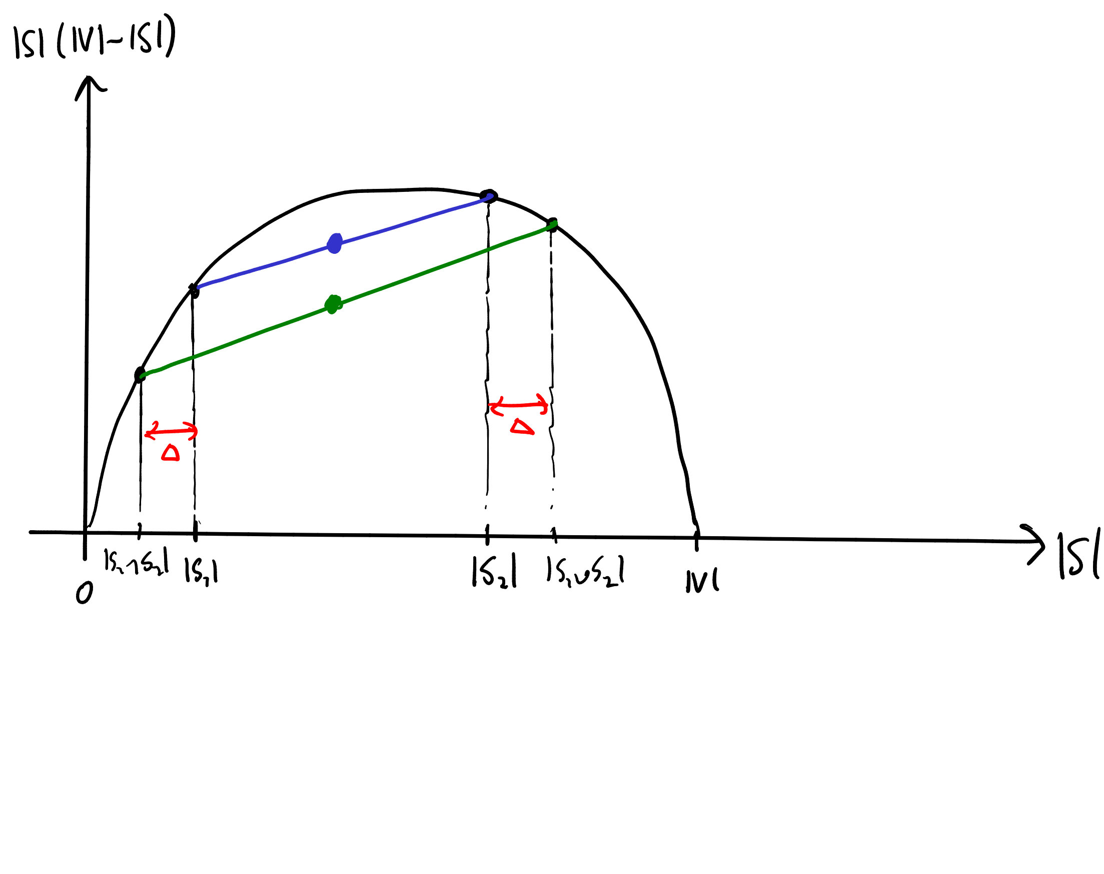
For any $\overline{\mathcal{F}} \subseteq 2^V$, let

$$\Phi(\overline{\mathcal{F}}) = \sum_{S \in \overline{\mathcal{F}}} |S| \cdot |V \setminus S|$$

Potential function strictly decreases at each uncrossing step.

For any $S_1, S_2 \subseteq V$ (crossing sets):

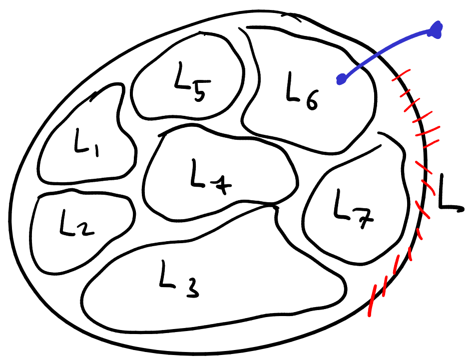
$$|S_1| \cdot |V \setminus S_1| + |S_2| |V \setminus S_2| > |S_1 \cup S_2| |V \setminus (S_1 \cup S_2)| + |S_1 \cap S_2| |V \setminus (S_1 \cap S_2)|$$



$\Longrightarrow$ We end up with a laminar certifying family $\mathcal{L} \subseteq 2^V$.

$$\Longrightarrow \quad |\mathcal{L}| \leq 2 \cdot |V \setminus \{r\}| - 1 = 2|V| - 3$$

(not small enough yet)



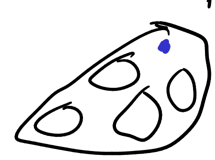
$L \in \mathcal{L}$

$L_1, \dots, L_q$: children of L in $\mathcal{L}$.

If $L_1, L_2, \dots, L_q$ partitions L, we call this an obstructions.

$\Longrightarrow$ We can delete L from $\mathcal{L}$.

$$\left(\text{Because} \quad \delta(L) \subseteq \bigcup_{i=1}^{q} \delta(L_i)\right)$$



$\rightsquigarrow$ New cert. family $\mathcal{H} \subseteq \mathcal{L}$.

$\rightarrow \quad |\mathcal{H}| \leq |V| - 1 \qquad (\exists \text{ injection from } \mathcal{H} \to V \setminus \{r\})$.

$\#$