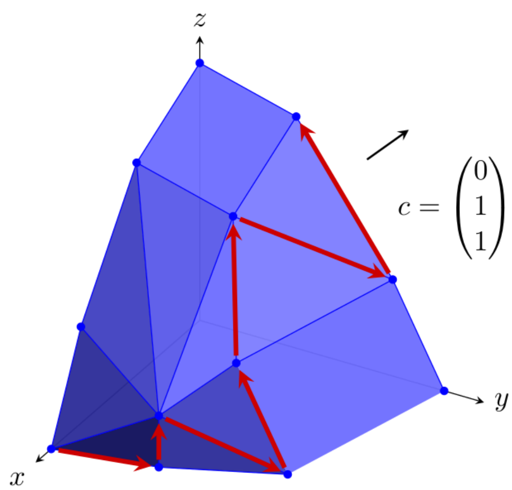
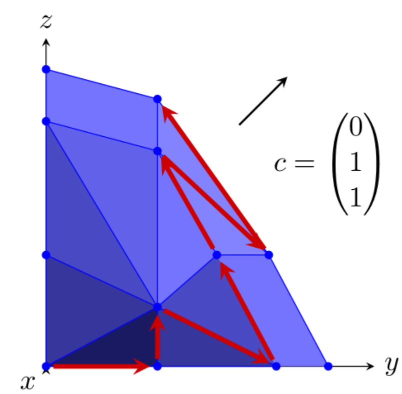
1.3 Simplex Method

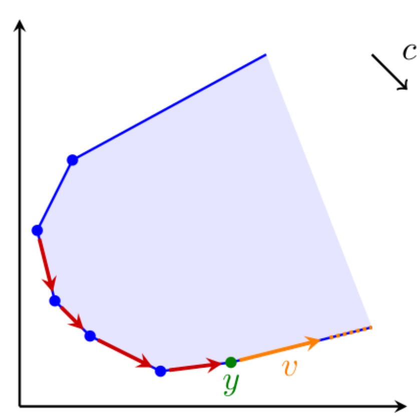
1.3.1 Geometric idea





Idea: Start at a vertex and iteratively walk to neighboring vertex with strictly improved objective value.

Plan for unbounded case:



1.3.2 From canonical to standard form

(LP in standard form)

$$\begin{aligned} \max\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

To introduce the Simplex Method, we start with canonical LP and transform it into standard form.

(original, canonical LP)

$$\begin{aligned} \max\ & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$(c \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m)$$

Let $y \in \mathbb{R}^m$

$\xrightarrow{\text{corresponding standard LP}}$

$$\begin{aligned} \max\ & c^T x \\ & Ax + y = b \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

Why start with canonical LP and then move to standard LP?

- The vertex-to-vertex walk of Simplex Method is wrt canonical LP.
- The algebraic realization of Simplex Method is done in standard LP form.

Remark

There is a one-to-one correspondance between

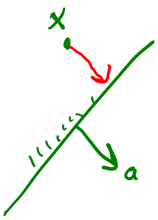
constraints of original canonical LP $\longleftrightarrow$ variables of corresponding standard LP

$$
\begin{array}{lrcrcl}
\max & -3x_1 & + & 2x_2 & & & \\
 & x_1 & - & x_2 & \leq & 1 & \leftrightarrow y_1 \\
 & -2x_1 & + & x_2 & \leq & 1 & \leftrightarrow y_2 \\
 & -x_1 & + & x_2 & \leq & 2 & \leftrightarrow y_3 \\
 & x_1 & & & \geq & 0 & \leftrightarrow x_1 \\
 & & & x_2 & \geq & 0 & \leftrightarrow x_2
\end{array}
$$

$$
\begin{array}{llccccrcrcl}
\max & z = & & & & - & 3x_1 & + & 2x_2 & & \\
 & & y_1 & & & + & x_1 & - & x_2 & = & 1 \\
 & & & y_2 & & - & 2x_1 & + & x_2 & = & 1 \\
 & & & & y_3 & - & x_1 & + & x_2 & = & 2 \\
 & & & & & & & & x & \in & \mathbb{R}^2_{\geq 0} \\
 & & & & & & & & y & \in & \mathbb{R}^3_{\geq 0}
\end{array}
$$

The variables measure the slack of the corresponding constraints.

$\rightarrow$ Due to this y_1, y_2, y_3 are often called slack variables.

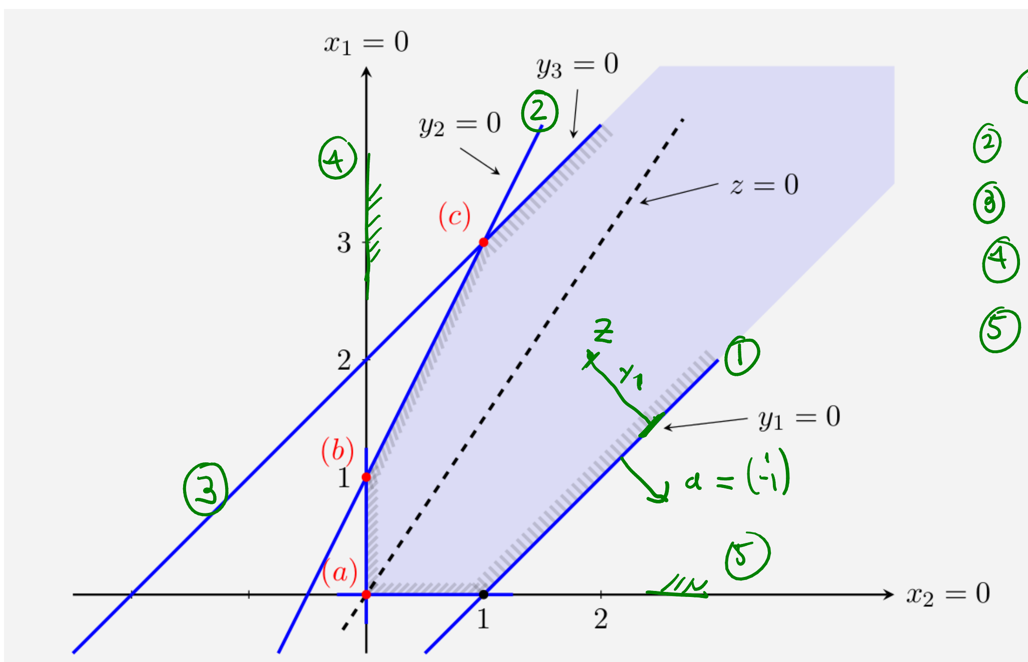


$$
\begin{array}{lrrrrrrrll}
\max & z = & & & - & 3x_1 & + & 2x_2 & & \\
 & & y_1 & & + & x_1 & - & x_2 & = & 1 \quad (1)\\
 & & & y_2 & - & 2x_1 & + & x_2 & = & 1\\
 & & & y_3 & - & x_1 & + & x_2 & = & 2\\
 & & & & & & & x & \in & \mathbb{R}^2_{\geq 0}\\
 & & & & & & & y & \in & \mathbb{R}^3_{\geq 0}
\end{array}
$$

$$a^T x \leq \beta$$

$$a^T\left(x + \lambda \frac{a}{\|a\|_2}\right) = \beta$$

$$\lambda \|a\|_2 = \beta - a^T x$$



(1) $x_1 - x_2 \leq 1$

(2) $-2x_1 + x_2 \leq 1$

(3) $-x_1 + x_2 \leq 2$

(4) $x_1 \geq 0$

(5) $x_2 \geq 0$

Some terminology

$$\begin{array}{ll} \max & c^T x \\ & Ax \le b \\ & x \ge 0 \end{array}$$

$$\begin{array}{ll} \max & c^T x \\ & Ax + y = b \\ & x \ge 0 \\ & y \ge 0 \end{array}$$

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

- A tuple $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ is a <u>solution of the LP</u> if $Ax + y = b$.
- A solution $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ is called <u>feasible</u> if $x \ge 0$, $y \ge 0$.

We first focus on studying solutions of the LP

Notice that $Ax + y = b$ always has solutions $\longleftarrow$ for example $\begin{array}{l} x = 0 \\ y = b \end{array}$

Tabular form of $Ax + y = b$

$$\begin{array}{rcrcll} \max z = & 400x_1 & + & 900x_2 & & \\ & x_1 & + & 4x_2 & \le & 40 \quad ① \\ & 2x_1 & + & x_2 & \le & 42 \quad ② \\ & 1.5x_1 & + & 3x_2 & \le & 36 \quad ③ \\ & x_1 & & & \ge & 0 \\ & & & x_2 & \ge & 0 \end{array}$$

$$\begin{array}{|ccccc|c} y_1 & y_2 & y_3 & x_1 & x_2 & 1 \\ \hline 1 & 0 & 0 & 1 & 4 & 40 \\ 0 & 1 & 0 & 2 & 1 & 42 \\ 0 & 0 & 1 & 1.5 & 3 & 36 \end{array}$$

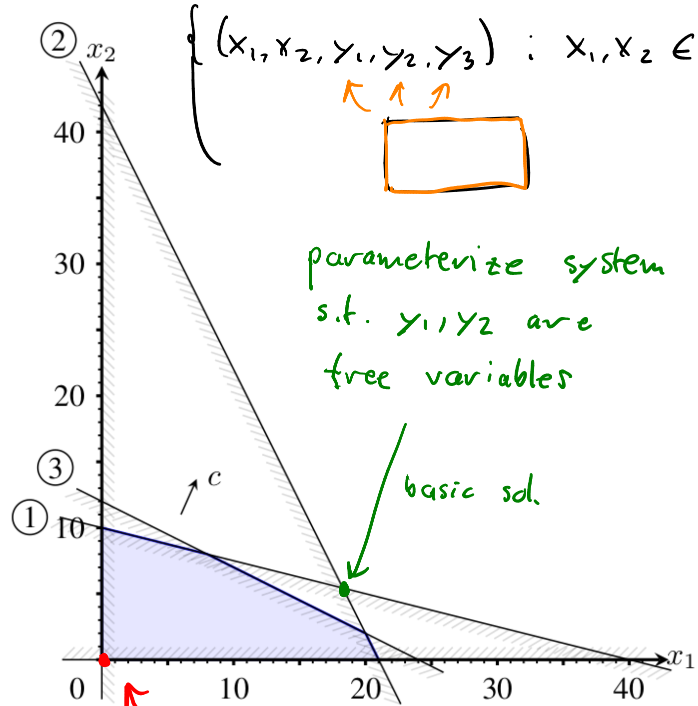
coefficient matrix — right-hand side

Parameterized form/system

$$\{(x_1, x_2, y_1, y_2, y_3) : x_1, x_2 \in \mathbb{R}\}$$

$$\boxed{\begin{aligned} y_1 &= 40 - x_1 - 4x_2 \\ y_2 &= 42 - 2x_1 - x_2 \\ y_3 &= 36 - 1.5x_1 - 3x_2 \end{aligned}}$$

Every choice of x_1, x_2 leads to unique solution by setting y_1, y_2, y_3 according to above equations.



(x_1, x_2)	are called *free variables*, and
(y_1, y_2, y_3)	are called *dependent variables*.

Basic solution to parameterized system

The basic solution to a parameterized system is the one that corresponds to setting all free variables to 0.

$$\begin{aligned} y_1 &= 40 - x_1 - 4x_2 \\ y_2 &= 42 - 2x_1 - x_2 \\ y_3 &= 36 - 1.5x_1 - 3x_2 \end{aligned} \quad \xrightarrow[\text{solution}]{\text{corresponding basic}} \quad \boxed{\begin{aligned} x_1 &= 0 \qquad y_1 = 40 \\ x_2 &= 0 \qquad y_2 = 42 \\ & \qquad\qquad y_3 = 36 \end{aligned}}$$

Parameterized forms are a way to represent a particular vertex solution of LP.

$$\begin{array}{c} \qquad\qquad Q \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{aligned} x_1 &\le 2 \qquad x_1, x_2 \ge 0 \\ x_1 &\le 3 \\ x_2 &\le 4 \end{aligned}$$

	y_1	y_2	y_3	x_1	x_2	1
	1	0	0	1	4	40
	0	1	0	2	1	42
	0	0	1	1.5	3	36

can be interpreted as being in parameterized form due to identity matrix.

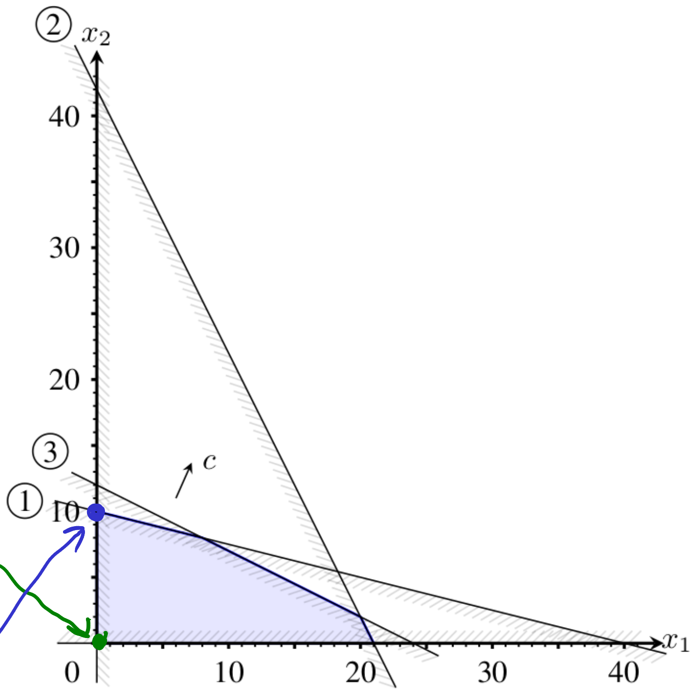
→ basic solution: $(x_1, x_2, y_1, y_2, y_3) = (0, 0, 40, 42, 36)$

equivalent equation system

	y_1	y_2	y_3	x_1	x_2	1
	$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
	$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
	$-\frac{3}{4}$	0	1	$\frac{3}{4}$	0	6

parameterized form

$$
\begin{aligned}
x_2 &= 10 - \tfrac{1}{4}y_1 - \tfrac{1}{4}x_1 \\
y_2 &= 32 + \tfrac{1}{4}y_1 - \tfrac{7}{4}x_1 \\
y_3 &= 6 + \tfrac{3}{4}y_1 - \tfrac{3}{4}x_1
\end{aligned}
$$



basic solution: $(x_1, x_2, y_1, y_2, y_3) = (0, 10, 0, 32, 6)$

→ To go from one vertex solution to another one, the Simplex Method goes from one parameterized form to an equivalent one.

1.3.3 Equivalent linear systems

Definition 1.50: Elementary row operations

(i) Change the order of the equations.
(ii) Multiply an equation with a non-zero real number.
(iii) Add a multiple of one equation to another one.

Example 1.51

	y_1	y_2	y_3	x_1	x_2		1
	1	0	0	1	4	1	40
$\boxed{\text{I}}$	0	1	0	2	1	0	42
	0	0	1	1.5	3	0	36

Multiply the first equation by $\frac{1}{4}$:

	y_1	y_2	y_3	x_1	x_2	1
	$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
$\rightarrow \boxed{\text{Ia}}$	0	1	0	2	1	42
	0	0	1	1.5	3	36

Add $(-1)\times$ first row to the second row, and add $(-3)\times$ first row to the third row:

	y_1	y_2	y_3	x_1	x_2	1
	$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
$\boxed{\text{II}}$	$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
	$-\frac{3}{4}$	0	1	$\frac{3}{4}$	0	6

Definition 1.52: Equivalent equation systems

Systems of linear equations, which can be transformed to each other using elementary row operations, are called *equivalent*.

Definition 1.53: Tableau form

An equation system $Ax = b$ with m equations in $n + m$ variables is in *tableau form* if its coefficient matrix contains an identity matrix. In this case:

- A tuple of variables whose corresponding columns—listed in the same order in which they appear in the tuple—form an identity matrix, are called a *basis*.
- The variables used in the basis are called *basic variables*.
- All other variables are called *non-basic variables*.
- The columns corresponding to basic variables are called *basic columns*.
- The columns corresponding to non-basic variables are called *non-basic columns*.
- The *basic solution* corresponding to a basis is the unique solution obtained by setting all non-basic variables to zero; hence, the basic variables will be set to b.

Equation system in tableau form

non-basic variables

y_1	y_2	y_3	x_1	x_2	1
$\frac{1}{4}$	0	0	$\frac{1}{4}$	1	10
$-\frac{1}{4}$	1	0	$\frac{7}{4}$	0	32
$-\frac{3}{4}$	0	1	$\frac{3}{4}$	0	6

basic variables

(x_2, y_2, y_3) is unique basis of this tableau

basic sol.: $(x_1, x_2, y_1, y_2, y_3)$

$= (0, 10, 0, 32, 6)$

Equation system in tableau form with more than one basis.

y_1	y_2	y_3	x_1	x_2	1
1	0	0	0	1	40
0	1	0	1	2	50
0	0	1	0	3	60

There are 2 bases:

(y_1, y_2, y_3) and

(y_1, x_1, y_3)