1.3.8 Cycling and Bland's rule

Example 1.76 : Phase II of Simplex Method may cycle in case of degeneracy

$$\begin{array}{lrrrrrrrrrrrl}
\max\ z = & & & & 2x_3 & + & 2x_4 & - & 8x_5 & - & 2x_6 & & \\
& & x_2 & - & 7x_3 & - & 3x_4 & - & 7x_5 & + & 2x_6 & = & 0 \\
& x_1 & & + & 2x_3 & + & x_4 & - & 3x_5 & - & x_6 & = & 0 \\
& & & & & & & & & & x & \in & \mathbb{R}^6_{\geq 0}
\end{array}$$

We apply following rule the select pivot element:

(a) To select column (non-basic variable about to enter basis):
choose leftmost one among all possible options.

Columns with strictly negative objective entry.

(b) To select row (basic variable about to leave basis):
choose topmost one among all possible options.

Rows with smallest quotient among those leading to strictly positive pivot element.

(a)

	x_3	x_4	x_5	x_6	1
z	-2	-2	8	2	0
x_2	-7	-3	7	2	0
x_1	$\boxed{2}$	1	-3	-1	0

(b)

	x_1	x_4	x_5	x_6	1
z	1	-1	5	1	0
x_2	$\frac{7}{2}$	$\boxed{\frac{1}{2}}$	$-\frac{7}{2}$	$-\frac{3}{2}$	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0

(c)

	x_1	x_2	x_5	x_6	1
z	8	2	-2	-2	0
x_4	7	2	-7	-3	0
x_3	-3	-1	$\boxed{2}$	1	0

(d)

	x_1	x_2	x_3	x_6	1
z	5	1	1	-1	0
x_4	$-\frac{7}{2}$	$-\frac{3}{2}$	$\frac{7}{2}$	$\boxed{\frac{1}{2}}$	0
x_5	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

(e)

	x_1	x_2	x_3	x_4	1
z	-2	-2	8	2	0
x_6	-7	-3	7	2	0
x_5	$\boxed{2}$	1	-3	-1	0

(f)

	x_5	x_2	x_3	x_4	1
z	1	-1	5	1	0
x_6	$\frac{7}{2}$	$\boxed{\frac{1}{2}}$	$-\frac{7}{2}$	$-\frac{3}{2}$	0
x_1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0

(g)

	x_5	x_6	x_3	x_4	1
z	8	2	-2	-2	0
x_2	7	2	-7	-3	0
x_1	-3	-1	2	1	0

(end of example)

Definition 1.77: Bland's pivot rule

All variables are first ordered in an arbitrary way (e.g., according to increasing index). Whenever a pivot column or row is to be selected and there are several options, we choose the column or row corresponding to the variable appearing first in the fixed order.

Theorem 1.78

When applying Bland's rule for choosing the pivot, phase II of the Simplex Method does not cycle. More precisely, with Bland's rule, the Simplex Method will never encounter two tableaus with the same set of basic (and therefore also non-basic) variables.

Example 1.79

Using Bland's rule in Example 1.76 leads to a different pivot choice in tableau (f):

$$(f)\quad \begin{array}{c|cccc|c} & x_5 & x_2 & x_3 & x_4 & 1 \\ \hline z & 1 & -1 & 5 & 1 & 0 \\ \hline x_6 & \frac{7}{2} & \frac{1}{2} & -\frac{7}{2} & -\frac{3}{2} & 0 \\ x_1 & \frac{1}{2} & \boxed{\frac{1}{2}} & -\frac{3}{2} & -\frac{1}{2} & 0 \end{array}$$

This results in the following optimal tableau:

$$\begin{array}{c|cccc|c} & x_5 & x_1 & x_3 & x_4 & 1 \\ \hline z & 2 & 2 & 2 & 0 & 0 \\ \hline x_6 & 3 & -1 & -2 & -1 & 0 \\ x_2 & 1 & 2 & -3 & -1 & 0 \end{array}$$

Proof of Theorem 1.78

Proof by contradiction → Assume we cycle with Bland's rule.
Consider sequence of tableaus in cycle.

Assumptions wlog

- Each variable appears both as basic and non-basic in cycle.

  → If a variable is always basic → Delete corresponding row to obtain smaller example of cycle.

  → If a variable is always non-basic → Delete corresponding column.

- Because values of encountered tableaus are all the same, we can assume that it is zero.

Observations

When cycling, basic solution never changes.
Each variable is non-basic at some point.

$\Rightarrow$ Basic solution is all-zeros vector.

$\Rightarrow$ Rhs of any encountered tableau is all-zeros.

Let $x_1, x_2, \ldots, x_k$ be all variables in tableau.
(Numbered s.t. Bland's rule chooses lowest possible index.)

Let tableau (a) be a tableau in cycle, s.t. x_k is basic and will become non-basic in next tableau.

$$
(a) \qquad
\begin{array}{c|c|c}
 & x_i & \\
\hline
z & - & 0 \\
\hline
 & \ominus & 0 \\
 & \vdots & \vdots \\
 & \ominus & 0 \\
x_k & + & 0 \\
 & \ominus & 0 \\
 & \vdots & \vdots \\
 & \ominus & 0
\end{array}
$$

$\ominus$: entries that are ≤ 0

$-$: entries that are < 0

$+$: entries that are ≥ 0

Let tableau (b) be tableau in cycle s.t. x_k is non-basic and will become basic in next tableau.

$$
(b) \qquad
\begin{array}{c|ccccccc|c}
 & & & & x_k & & & & \\
\hline
z & \oplus & \cdots & \oplus & - & \oplus & \cdots & \oplus & 0 \\
\hline
 & & & & & & & & 0 \\
 & & & & & & & & \vdots \\
 & & & & & & & & 0
\end{array}
$$

$\oplus$: entries that are ≥ 0

$-$: entries that are < 0

By setting in tableau (a) all non-basic (i.e., free) variables to zero except for x_i, which we set to 1, a solution $d = (d_1, d_2, \dots, d_k)^T$ is obtained with:

$d_i = 1$

recall, this need not be feasible

(i) $d_k < 0$

(ii) $d_j \geq 0 \quad \forall j \in [k-1]$, and

(iii) objective value of d is strictly better than 0.

(a)

	x_i	
z	$-$	0
	$\ominus$	0
	$\vdots$	$\vdots$
	$\ominus$	0
x_k	$+$	0
	$\ominus$	0
	$\vdots$	$\vdots$
	$\ominus$	0

However, by plugging d into objective row of tableau (b), we get an objective value < 0. ↯

(b)

				x_k				
z	$\oplus$	$\cdots$	$\oplus$	$-$	$\oplus$	$\cdots$	$\oplus$	0
								0
								$\vdots$
								0

#