5 Polyhedral Approaches in Combinatorial Optimization

Combinatorial optimization problems can often be described by:

- A finite set N, called *ground set*,
- a family $\mathcal{F} \subseteq 2^N$ of *feasible sets*, also called *solutions*, and
- an *objective function* $w \colon N \to \mathbb{Z}$ to maximize/minimize.

The corresponding combinatorial optimization problem is then given by:

$$\boxed{\begin{array}{rl} \max / \min & w(F) := \sum_{e \in F} w(e) \\ & F \in \mathcal{F} \end{array}} \tag{5.1}$$

Some examples:

(i) **Maximum weight matchings**. Given is an undirected graph $G = (V, E)$ with non-negative edge weights $w \colon E \to \mathbb{Z}_{\geq 0}$.
   - Ground set: $N = E$.
   - Feasible sets: $\mathcal{F}$ is the family of all loopless sets $M \subseteq E$ such that no two edges of M have a common endpoint. Such a set M is called a *matching*.
   - Objective: maximize w.

   Two important special cases of the problem:
   - *Maximum cardinality matchings*: w is given by $w(U) = |U|$ for $U \subseteq E$, i.e., each edge has weight equal to one.
   - *Maximum weight/cardinality bipartite matchings*: $G = (V, E)$ is a *bipartite graph*, i.e., there is a bipartition $V = X \,\dot{\cup}\, Y$ such that each $e \in E$ has one endpoint in X and one in Y.

(ii) **Shortest s-t path problem**. Given is an undirected or directed graph $G = (V, E)$, two vertices $s, t \in V$, and a length function $\ell \colon E \to \mathbb{Z}_{\geq 0}$.
   - $N = E$,
   - $\mathcal{F}$ are all subsets of edges corresponding to s-t paths.
   - Objective: minimize $w = \ell$.

(iii) **Minimum weight spanning tree**. Given is an undirected graph $G = (V, E)$ and non-negative edge weights $w \colon E \to \mathbb{Z}_{\geq 0}$.
   - $N = E$,
   - $\mathcal{F}$ are all sets $F \subseteq E$ that correspond to spanning trees.
   - Objective: minimize w.

5.1 Polyhedral descriptions of combinatorial optimization problems

Consider the family $\mathcal{F} \subseteq 2^N$ of feasible sets of a combinatorial optimization problem (5.1). We can describe each solution $F \in \mathcal{F}$ by its *incidence/characteristic vector* $\chi^F \in \{0,1\}^N$, where

$$\chi^F(e) = \begin{cases} 1 & \text{if } e \in F \ , \\ 0 & \text{if } e \in N \setminus F \ . \end{cases}$$

The polytope that *corresponds* to $\mathcal{F}$ is the polytope $P_{\mathcal{F}} \subseteq [0,1]^N$ whose vertices are precisely incidence vectors of solutions, i.e.,

$$P_{\mathcal{F}} = \operatorname{conv}(\{\chi^F \colon F \in \mathcal{F}\}) \ , \tag{5.2}$$

where we recall that for any set $X \subseteq \mathbb{R}^n$, $\operatorname{conv}(X) \subseteq \mathbb{R}^n$ is the *convex hull* of X, i.e., the unique smallest convex set containing X. Also, remember that for a set $X \subseteq \mathbb{R}^n$, the *convex hull* $\operatorname{conv}(X)$ of X can be described by all *convex combinations* of points in X, i.e.,

$$\operatorname{conv}(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; k \in \mathbb{Z}_{>0}, x_i \in X \text{ and } \lambda_i \in \mathbb{R}_{\geq 0} \ \forall i \in [k] \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Furthermore, by Carathéodory's Theorem, one can fix k to be equal to $n+1$ in the above description of $\operatorname{conv}(X)$, still obtaining the same set.

Hence, $P_{\mathcal{F}}$ is a $\{0,1\}$-polytope, i.e., all its vertices only have coordinates within $\{0,1\}$. Notice that solving the original combinatorial optimization problem (5.1) is equivalent to finding a vertex solution—also called *basic feasible solution*—to the following LP, where we interpret $w \colon N \to \mathbb{Z}$ as a vector $w \in \mathbb{Z}^N$:

$$\boxed{\begin{array}{ll} \max / \min & w^\top x \\ & x \in P_{\mathcal{F}} \end{array}} \tag{5.3}$$

The main challenge with this approach is that the description of $P_{\mathcal{F}}$ given by (5.2) refers to all sets in $\mathcal{F}$, which are typically exponentially (in $n := |N|$) many. Problems where $|\mathcal{F}|$ is small are often not that interesting. In particular, if $|\mathcal{F}| = O(\operatorname{poly}(n))$, then one could try to solve the combinatorial optimization problem efficiently by simply checking all feasible solutions. Of course, such an approach would still require the availability of a method to enumerate all feasible solutions.

Our goal is to get an inequality-description of $P_{\mathcal{F}}$, i.e., write $P_{\mathcal{F}}$ as $P_{\mathcal{F}} = \{x \in \mathbb{R}^N \colon Ax \leq b\}$. This has many advantages:

- Often, $P_{\mathcal{F}}$ only has $O(\operatorname{poly}(n))$ facets, even though it has $2^{\Omega(n)}$ vertices, i.e., feasible solutions. In this case, $P_{\mathcal{F}}$ can be described compactly by its facets.

→ Think about $P = [0,1]^n$. P has 2^n vertices but only $2n$ facets:

$$P = \{x \in \mathbb{R}^n \colon 0 \leq x_i \leq 1 \ \forall i \in [n]\} \ ,$$

where $[n] \coloneqq \{1, \dots, n\}$.

- One good inequality description of $P_{\mathcal{F}}$ allows for optimizing any linear objective function.
  → Combinatorial algorithms are often less versatile. E.g., Edmonds' blossom shrinking algorithm is a very elegant combinatorial algorithm for finding a maximum cardinality matching. However, it is hard to generalize this approach to the weighted case.
- Even when $P_{\mathcal{F}}$ has exponentially many facets, a description of them can often be obtained. If so, we can often still solve the LP efficiently.
  → For example, this might be possible by employing the Ellipsoid Method. We will discuss the Ellipsoid Method later in class, when we talk about the "equivalence" between separation and optimization.
- An inequality-description of $P_{\mathcal{F}}$ is often helpful when trying to solve a related combinatorial optimization problem, e.g., the original one with some additional linear constraints.
- The LP dual of (5.3) can often be interpreted combinatorially. Possible implications:
  - Optimality certificates through strong duality. E.g., many classical certificates like max-flow min-cut can be derived this way.
  - Fast algorithms based on dual such as primal-dual algorithms.
- Get a better understanding of the original combinatorial optimization problem. In particular, one can use elegant polyhedral proof techniques to derive results.
- Leverage general polyhedral techniques to design strong algorithms. E.g., network simplex algorithms are a way to interpret the Simplex Method combinatorially.

5.2 Meta-recipe for finding inequality-descriptions

There is no one-fits-all method. However, one typical high-level approach is to "guess" the polytope and show that it is the right one, following the recipe below:

(i) Determine (guess) a candidate polytope $P \subseteq [0,1]^N$ defined by some linear inequality description $P = \{x \in \mathbb{R}^N \colon Ax \leq b\}$, which we want to prove to be equal to $P_{\mathcal{F}}$.

(ii) Prove that P contains the correct set of integral points, i.e.,

$$P \cap \{0,1\}^N = \{\chi^F \colon F \in \mathcal{F}\}.$$

This is normally quite easy. Notice that $\{\chi^F \colon F \in \mathcal{F}\} = P_{\mathcal{F}} \cap \{0,1\}^N$.

(iii) Show that P is a $\{0,1\}$-polytope. Because $P \subseteq [0,1]^N$, this is equivalent to showing that P is integral.

If one can show (ii) and (iii), then $P = P_{\mathcal{F}}$, because P only contains $\{0,1\}$-vertices by (iii), and those vertices are the same as the ones of $P_{\mathcal{F}}$ due to (ii).

Typically, the most difficult step of the above recipe is point (iii), i.e., to show that P is integral. Also, we remark that the above recipe can be used iteratively to correct a potentially wrong guess of the inequality description. More precisely, if we realize that the statement we have to prove in step (ii) or (iii) is incorrect, then we know that our candidate inequality description that we established in point (i) is erroneous. Moreover, the insights we gain when thinking about why point (ii) or (iii) is not correct often help to improve the inequality description coming from step (i). Iterating this procedure is one way to come up with the correct description.

5.2.1 Example: bipartite vertex cover

To exemplify the above recipe, we consider the bipartite vertex cover problem. We first define the notion of a vertex cover in a general graph.

Definition 5.1: Vertex cover

Let $G = (V, E)$ be an undirected graph. A vertex cover of G is a subset $S \subseteq V$ such that for every edge $e \in E$, at least one of its endpoints is in S.

The minimum vertex cover problem, where each vertex has a non-negative cost and the goal is to find a minimum cost vertex cover, is a classical combinatorial optimization problem. Our goal here is to describe the vertex cover polytope for a bipartite graph. Hence, let $G = (V, E)$ be a bipartite graph and let $\mathcal{F} \subseteq 2^V$ be the family of all vertex covers in G. Our goal is to obtain an inequality description of the vertex cover polytope $P_{\mathcal{F}}$, i.e.,

$$\mathcal{F} = \operatorname{conv}\left(\{\chi^F : F \in \mathcal{F}\}\right) \quad .$$

We follow the recipe and start with the following natural candidate:

$$P := \left\{x \in [0,1]^V : x(u) + x(v) \geq 1 \ \forall \{u,v\} \in E\right\} \quad .$$

Notice that this candidate is essentially a literal translation of the property of being a vertex cover into linear constraints. This completes point (i) of the recipe. Through step (ii) and (iii) of the recipe, we now prove that this is a correct inequality description for the bipartite vertex cover polytope.

Theorem 5.2

The vertex cover polytope of a bipartite graph $G = (V, E)$ can be described by

$$P = \left\{x \in [0,1]^V : x(u) + x(v) \geq 1 \ \forall \{u,v\} \in E\right\} \quad .$$

Proof. We now show point (ii) of the recipe, i.e., that P contains the correct set of integral points, which means $P \cap \{0,1\}^V = \{\chi^F : F \in \mathcal{F}\}$. Clearly, if $S \subseteq V$ is a vertex cover, then

$\chi^F \in P$. Moreover, consider a point $y \in P \cap \{0,1\}^V$. Because y is a $\{0,1\}$-point, it can be written as $y = \chi^S$, where $S \coloneqq \{v \in V : y(v) = 1\}$. Due to the constraints of P, we have

$$1 \leq y(u) + y(v) = |S \cap \{u,v\}| \quad \forall \{u,v\} \in E \ .$$

Hence, at least one of the endpoints of any edge $\{u,v\} \in E$ is in S. Thus, S is a vertex cover and hence $y = \chi^S \in \{\chi^F : F \in \mathcal{F}\}$.

It remains to show point (iii) of the recipe, i.e., that P is integral. For the sake of deriving a contradiction, assume that P is not integral. Hence, P contains a fractional vertex y, i.e., $y \in P \setminus \{0,1\}^V$. Let $V = A \dot{\cup} B$ be a bipartition of the vertices such that every edge has one endpoint in A and one in B. We define

$$\begin{aligned} W_A &\coloneqq \{u \in A : y(u) \in (0,1)\} \quad \text{and} \\ W_B &\coloneqq \{u \in B : y(u) \in (0,1)\} \end{aligned}$$

to be the vertices in A and B, respectively, with fractional y-value. Because y is not integral by assumption, we have $W_A \cup W_B \neq \emptyset$. We will derive a contradiction by showing that y is not an extreme point of P. Let

$$\epsilon \coloneqq \min\left\{\min\{y(u), 1 - y(u)\} : u \in W_A \cup W_B\right\} \ .$$

Note that $\epsilon > 0$. Now define for any $\delta \in \mathbb{R}$:

$$y^\delta \coloneqq y + \delta \cdot (\chi^{W_A} - \chi^{W_B}) \ .$$

We claim that $y^\epsilon \in P$ and $y^{-\epsilon} \in P$. Notice that this will lead to the desired contradiction because $y = \frac{1}{2}(y^\epsilon + y^{-\epsilon})$ and $y^\epsilon \neq y^{-\epsilon}$ because $\chi^{W_A} - \chi^{W_B}$ is not the all-zeros vector as $W_A \cup W_B \neq \emptyset$. Hence, this implies that y is not an extreme point of P and therefore also not a vertex of P due to Proposition 1.19. Hence, it remains to show $y^\epsilon, y^{-\epsilon} \in P$. In fact, it suffices to show $y^\epsilon \in P$, which implies $y^{-\epsilon} \in P$ by exchanging the roles of A and B.

By our choice of ϵ, we clearly have $y^\epsilon \in [0,1]^V$. Hence, it remains to check the constraints of P induced by the edges. Let $\{u,v\} \in E$. If either $y(u) = 1$ or $y(v) = 1$, then $y^\epsilon(u) + y^\epsilon(v) \geq 1$ because if a y-value is 1 at a certain entry, then so is the value of y^ϵ at the same entry, because y^ϵ only modifies fractional entries of y. Hence, we only have to check whether $y^\epsilon(u) + y^\epsilon(v) \geq 1$ if both $y(u)$ and $y(v)$ are fractional. However, in this case one of these two y-values will increase by ϵ and the other one decrease by ϵ, and thus

$$y^\epsilon(u) + y^\epsilon(v) = y(u) + y(v) \geq 1 \ ,$$

where the inequality follows from $y \in P$. □

5.3 Total unimodularity

There are various techniques to show integrality of a polyhedron, and we cover many different approaches here. In the example of the bipartite vertex cover problem, we have already seen one method, where we disproved the existence of fractional extreme points (see proof of Theorem 5.2). One elegant way to prove that certain polyhedra are integral is based on showing that the constraint matrix A used in the linear inequality description has a very strong structural property, known as *total unimodularity*. Even though most constraint matrices of integral polyhedra are not totally unimodular, it is still important to understand when we deal with matrices exhibiting this strong property, as this leads to short and elegant proofs, and allows for deriving many additional interesting properties.

5.3.1 Definition and basic observations

Definition 5.3

A matrix is *totally unimodular* (TU) if the determinant of any square submatrix of it is either 0, 1, or -1.

Remark 5.4

$A \in \mathbb{R}^{m\times n}$ is TU $\Rightarrow A \in \{-1,0,1\}^{m\times n}$.

This follows by observing that each single entry of the matrix A is itself a square submatrix and its determinant is the entry itself.

Remark 5.5

A is TU $\Leftrightarrow A^\top$ is TU.

This follows by the definition of TU matrices and the fact that the determinant of any square matrix is equal to the determinant of its transpose.

Remark 5.6

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A\ -A]$, i.e., the $\mathbb{R}^{m\times 2n}$ matrix obtained by appending the columns of $-A$ to the columns of A.

Indeed, $[A\ -A]$ is TU because any square submatrix of $[A\ -A]$ either contains a column of A and its negation in $-A$, in which case the determinant is zero, or it is a square submatrix of A with some columns multiplied by -1, in which case we have a determinant within $\{-1,0,1\}$ due to TU-ness of A.

Remark 5.7

If $A \in \mathbb{R}^{m\times n}$ is TU, then so is $[A\ I]$, i.e., the $\mathbb{R}^{m\times 2n}$ matrix obtained by appending the columns of an $m\times m$ identity matrix I to the columns of A.

Indeed, any determinant of a submatrix of $[A\ I]$ is a determinant of a submatrix of A, which is TU by assumption.

5.3.2 Integrality of polyhedra with TU constraint matrices

The main algorithmic motivation for studying totally unimodular matrices comes from the following theorem.

Theorem 5.8

Let $A \in \mathbb{Z}^{m\times n}$. Then,

$$A \text{ is TU} \quad \Leftrightarrow \quad P = \{x \in \mathbb{R}^n \colon Ax \leq b, x \geq 0\} \text{ is integral } \forall\, b \in \mathbb{Z}^m .$$

Proof. $\Rightarrow$) Assume that A is TU and $b \in \mathbb{Z}^m$, and let $P = \{x \in \mathbb{R}^n \colon Ax \leq b, x \geq 0\}$. To show that P is integral consider a vertex $y \in \text{vertices}(P)$. By Proposition 1.19, y is the unique solution to some subsystem $Dx = d$ of the system

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix} \quad , \tag{5.4}$$

which defines P. Moreover, we can choose D to be a square matrix because any linear system with a unique solution contains a square subsystem with the same unique solution. Hence,

$$y = D^{-1} d \quad .$$

Now observe that D^{-1} is an integral matrix, because D is a sub-matrix of the matrix in (5.4), which is TU by Remark 5.6 and 5.7; therefore, because D is full-rank, we have $\det(D) \in \{-1, 1\}$ which, together with $D \in \mathbb{Z}^{n\times n}$, implies integrality of D^{-1}. Moreover, also $d \in \mathbb{Z}^n$ is an integral vector because its entries are a subset of the entries of the right-hand side of (5.4). Hence, $y = D^{-1}d$ is integral, as desired.

$\Leftarrow$) We show this direction by proving the contraposition. Hence, assume that A is not TU, which implies that there is some square submatrix Q of A with $\det(Q) \notin \{-1, 0, 1\}$. Because the statement we want to prove is invariant with respect to row and column permutations, we assume without loss of generality that Q consists of the first k rows and k columns of A. Notice that this implies $k \leq \min\{m, n\}$ because $A \in \mathbb{R}^{m\times n}$. Consequently, when defining an $n \times n$ submatrix of $\left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right)$ consisting of the k first rows and $n-k$ last rows, a square matrix $H \in \mathbb{R}^{n\times n}$ is obtained with $\det(H) \notin \{-1, 0, 1\}$, namely

$$H = \begin{pmatrix} Q & W \\ 0 & -I \end{pmatrix} \quad ,$$

where I in the matrix above is an $(n-k) \times (n-k)$ identity matrix and $W \in \mathbb{Z}^{k\times(n-k)}$. Hence, we have

$$H^{-1} = \begin{pmatrix} Q^{-1} & Q^{-1}W \\ 0 & -I \end{pmatrix} \quad .$$

Notice that because $\det(Q) \notin \{-1, 0, 1\}$ we have that Q^{-1} is not an integral matrix. One easy way to see this is by observing that $\det(Q^{-1}) = 1/\det(Q) \notin \mathbb{Z}$ and using that a matrix with a non-integral determinant must have at least one non-integral entry. Let $j \in [k]$ be the index of a column of Q^{-1} that is not integral. We now define a vector $b \in \mathbb{Z}^m$ such that $P = \{x \in \mathbb{R}^n \colon Ax \leq b, x \geq 0\}$ is not integral, which will finish the proof. For this we start by defining the first k coordinates of b, i.e., those that correspond to the rows of Q. Let $e_j \in \{0,1\}^k$ be the j-th canonical unit vector in dimension k, and let $\mathbf{1} \in \{0,1\}^k$ be the all-ones vector. Moreover, let $p \in \mathbb{Z}$ be an integer that is big enough such that

$$Q^{-1}e_j + p \cdot \mathbf{1} \geq 0 \ . \tag{5.5}$$

We set the first k entries of b to $e_j + p \cdot Q\mathbf{1}$. We will set the remaining coordinates of b such that the following vector $y \in \mathbb{R}^n$ is a vertex of P: The first k coordinates of y are $Q^{-1}e_j + p \cdot \mathbf{1}$, all other coordinates of y are 0. First observe that y is fractional because the j-th column of Q^{-1} is fractional. We also have $y \geq 0$ due to (5.5). Moreover, y is by construction the unique solution to the subsystem of $Ax \leq b, x \geq 0$ that was used to define H. This subsystem is of the form $Hx = d$, where the first k rows of d are $e_j + p \cdot Q\mathbf{1}$, and all the other entries of d are zero. Hence, by choosing the last $m-k$ entries of b large enough, to make sure that y fulfills $Ay \leq b$, we have by Proposition 1.19 that y is a vertex of P. □

Hence, if we have an inequality description of a polyhedron as in Theorem 5.8 with A being TU and b integral, then the above theorem immediately implies that the polyhedron is integral. However, we still need techniques to check whether a given matrix is totally unimodular.

5.3.3 The characterization of Ghouila-Houri

The definition of total unimodularity as well as Theorem 5.8 do not provide very useful means to check that some matrix is totally unimodular. One of the most useful characterization of total unimodularity is the following theorem by Ghouila and Houri.

Theorem 5.9: Characterization of Ghouila-Houri

A matrix $A \in \mathbb{R}^{m \times n}$ is TU if and only if for every subset of the rows $R \subseteq [m]$, there is a partition $R = R_1 \,\dot\cup\, R_2$ such that

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\} \quad \forall j \in [n] \ . \tag{5.6}$$

Proof. $\Rightarrow$) We start by assuming that A is TU and show that the stated row-partitioning property holds. Hence, let $R \subseteq [m]$ be a subset of the rows of A. Let $d \in \mathbb{R}^m$ be defined by

$$d_i = \begin{cases} 1 & \text{if } i \in R \ , \\ 0 & \text{if } i \in [m] \setminus R \ , \end{cases}$$

and we define the polytope

$$Q \coloneqq \left\{ x \in \mathbb{R}^n \colon A^\top x \leq \left\lceil \frac{1}{2} A^\top d \right\rceil, \ A^\top x \geq \left\lfloor \frac{1}{2} A^\top d \right\rfloor, \ x \leq d, \ x \geq 0 \right\} \ .$$

By Theorem 5.8, Q is an integral polytope, because its constraint matrix (without including the non-negativity constraints) is

$$\begin{pmatrix} A^\top \\ -A^\top \\ I \end{pmatrix} \quad ,$$

which is TU due to Remark 5.5, 5.6, and 5.7. Furthermore, $Q \neq \emptyset$ because $\frac{d}{2} \in Q$. Therefore Q has a vertex, say y, which must be integral. Due to the upper and lower bounds in the definition of Q, we have $y \in \{0,1\}^n$. We now define the following partition of R into R_1 and R_2:

$$\begin{aligned} R_1 &:= \{i \in R \colon y_i = 0\} \quad , \\ R_2 &:= \{i \in R \colon y_i = 1\} \quad . \end{aligned}$$

This partition indeed fulfills (5.6) because

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} = (d - 2y)^\top A_{\cdot j} = (A^\top)_{j \cdot}(d - 2y) \in \{-1, 0, 1\} \qquad \forall j \in [n] \quad ,$$

where the fact that the expression $(A^\top)_{j\cdot}(d - 2y)$ is within $\{-1,0,1\}$ follows from integrality of $d - 2y$ together with $y \in Q$.

$\Leftarrow$) We now show the converse by showing that for any square matrix A, if A fulfills the row-partition property (5.6), then $\det(A) \in \{-1,0,1\}$. Notice that it suffices to consider square matrices. Indeed, for a general matrix B satisfying the row-partitioning property of Ghouila-Houri, we need to show that each square submatrix A of B fulfills $\det(A) \in \{-1,0,1\}$. If B satisfies the row-partitioning property of Ghuila-Houri, then so does A. If the implication we aim at proving holds for square matrices, then this implies $\det(A) \in \{-1,0,1\}$ as desired.

We show this statement by induction on the size k of the square matrix $A \in \mathbb{R}^{k \times k}$. For $k = 1$, the statement clearly holds. Hence, we consider a matrix $A \in \mathbb{R}^{(k+1)\times(k+1)}$, for $k \in \mathbb{Z}_{\geq 1}$, that fulfills the row-partition property (5.6), and we assume that the row-partition property on any square matrix of size at most k implies that its determinant is within $\{-1,0,1\}$.

By letting R be a single row of A, we obtain that each entry of A is within $\{-1,0,1\}$. If A is the all-zeros matrix, the statement trivially holds; hence, assume that A has some non-zero entry. Because the statement is invariant with respect to row and column permutations and multiplying rows and columns by -1, we can assume that $A_{k+1,k+1} = 1$. We denote by $\begin{pmatrix} r^\top & 1 \end{pmatrix}$ the last row of A, by $\binom{c}{1}$ the last column of A, and by $Q \in \mathbb{R}^{k \times k}$ the submatrix of A consisting of the first k rows and columns, i.e.,

$$A = \begin{pmatrix} Q & c \\ r^\top & 1 \end{pmatrix} \quad .$$

Through elementary row operations—more precisely, by adding or subtracting the last row to some of the other rows—we obtain the following matrix:

$$\overline{A} = \begin{pmatrix} Q - c \cdot r^\top & 0 \\ r^\top & 1 \end{pmatrix} \quad .$$

Because these row operations do not change the determinant of a matrix, we have $\det(A) = \det(\overline{A}) = \det(Q - c \cdot r^\top)$. Hence, it suffices to show $\det(Q - c \cdot r^\top) \in \{-1,0,1\}$. To this end, we show that $Q - c \cdot r^\top$ fulfills the row-partition condition (5.6), which then implies the statement by the inductive hypothesis. Thus, let $R \subseteq [k]$ be a subset of the rows of $Q - c \cdot r^\top$. We distinguish two cases depending on the parity of $|R \cap \operatorname{supp}(c)| = |R \cap \{i \in [k] \colon c_i \neq 0\}|$.

Case $|R \cap \operatorname{supp}(c)|$ even: By assumption, the Ghuila-Houri criterion holds for the matrix $\begin{pmatrix} Q & c \end{pmatrix}$, because it is a submatrix of A. Hence, there is a vector $d \in \{-1,0,1\}^k$ with

(i) $\operatorname{supp}(d) = R$,
(ii) $d^\top Q \in \{-1,0,1\}^k$, and
(iii) $d^\top c \in \{-1,0,1\}$.

More precisely, d encodes a partition $R = R_1 \mathbin{\dot\cup} R_2$, where the 1-entries of d correspond to the rows in R_1 and the -1-entries to the rows in R_2. We show that this same partition of R can be used for $Q - c \cdot r^\top$. Note that because $|R \cap \operatorname{supp}(c)|$ is even and $\operatorname{supp}(d) = R$, we must have that $d^\top c$ is even. Thus, because $d^\top c \in \{-1,0,1\}$, we have $d^\top c = 0$. However, this, together with (ii), implies

$$d^\top (Q - c \cdot r^\top) = d^\top Q \in \{-1,0,1\}^k \ ,$$

showing that the partition of R defined by d fulfills (5.6), as desired.

Case $|R \cap \operatorname{supp}(c)|$ odd: In this case we use the fact that the matrix A fulfills the row-partition condition for the rows $R \cup \{k+1\}$. Again, we can represent the partition of $R \cup \{k+1\}$ by a vector in $\{-1,0,1\}^{k+1}$. We can assume that row $k+1$ is in the second part of the partition, because the two parts are exchangeable, i.e., condition (5.6) is invariant with respect to exchanging R_1 and R_2. Hence, there is a partition of the rows $R \cup \{k+1\}$ of A fulfilling (5.6) that can be described by a vector $\left(\begin{smallmatrix} d \\ -1 \end{smallmatrix}\right) \in \{-1,0,1\}^{k+1}$, where $d \in \{-1,0,1\}^k$. Thus, we have

(i) $\operatorname{supp}(d) = R$,
(ii) $d^\top Q - r^\top \in \{-1,0,1\}^k$, and
(iii) $d^\top c - 1 \in \{-1,0,1\}$.

We claim that d describes a desired partition of the rows R of the matrix $Q - c \cdot r^\top$. First observe that because $|R \cap \operatorname{supp}(c)|$ is odd and $\operatorname{supp}(d) = R$, we have that $d^\top c$ is odd. Due to (iii), this implies $d^\top c = 1$. Hence, together with (ii), this leads to

$$d^\top (Q - c \cdot r^\top) = d^\top Q - r^\top \in \{-1,0,1\}^k \ ,$$

showing that the partition of R defined by d fulfills (5.6), as desired. □

Remark 5.10

Because A is TU if and only if $A^\top$ is TU, one can exchange the roles of rows and columns in Theorem 5.9.

Example 5.11: Consecutive-ones matrices

To exemplify how the characterization of Ghuila-Houri can be used, we show that consecutive-ones matrices are TU. A *consecutive-ones* matrix is a $\{0,1\}$-matrix $A \in \{0,1\}^{m \times n}$ such that either in each column all the 1s appear consecutively, or in each row all the 1s appear consecutively. Below are two examples of consecutive-ones matrices.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

We will show that any consecutive-ones matrix is TU.

Consider a consecutive-ones matrix $A \in \mathbb{R}^{m\times n}$, where the 1s are consecutive in each column. (This covers the general case because a consecutive-ones matrix where the 1s are consecutive in each row is just the transpose of one where they are consecutive in each column.) Now consider any subset $R \subseteq [m]$ of the rows of A. We have to find a partitioning of them satisfying (5.6). Let $B \in \mathbb{R}^{r\times n}$ be the submatrix of A consisting of the rows in R; hence, $r := |R|$. Notice that B is also a consecutive-ones matrix where the 1s appear consecutively in each column. We partition its rows by alternatingly putting them into R_1 and R_2. More precisely, all rows with an odd row index in B are assigned to R_1 and the other ones to R_2. One can easily check that this partition fulfills (5.6) because in each column the difference between the number of 1s assigned to R_1 and the number of 1s assigned to R_2 is at most one.

5.4 Bipartite matching polytope

Let $G = (V, E)$ be an undirected, bipartite graph with corresponding bipartition $V = X \mathbin{\dot\cup} Y$, and let $\mathcal{M} \subseteq 2^E$ be all matchings of G. Hence, using the notation introduced previously, we have $N = E$ and $\mathcal{F} = \mathcal{M}$.

Theorem 5.12

The bipartite matching polytope $P_{\mathcal{M}}$ is given by

$$P_{\mathcal{M}} = \{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) \leq 1 \ \forall v \in V\} \ . \tag{5.7}$$

We will prove Theorem 5.12 by first showing (ii), and then presenting two general techniques that prove (iii). Let $P = \{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) \leq 1 \ \forall v \in V\}$. Hence, we have to show $P = P_{\mathcal{M}}$.

Proof of point (ii). We prove (ii) by showing that for any $F \subseteq E$ we have

$$\chi^F \in P \cap \{0,1\}^N \Leftrightarrow F \in \mathcal{M} \ .$$

We distinguish between $F \in \mathcal{M}$ and $F \in 2^E \setminus \mathcal{M}$.

If $F \in \mathcal{M}$, then, as F is a matching, we have $|F \cap \delta(v)| \leq 1 \ \forall v \in V$. Notice that $|F\cap\delta(v)| = \chi^F(\delta(v))$. Hence, $\chi^F(\delta(v)) \leq 1 \ \forall v \in V$, and thus $\chi^F \in P$, as desired.

If $F \in 2^E \setminus \mathcal{M}$, then, as F is not a matching, there exists a vertex $v \in V$ such that $|F\cap\delta(v)| \geq 2$, which can be rephrased as $\chi^F(\delta(v)) \geq 2$. Hence, $\chi^F \notin P$. $\square$

It remains to prove (iii).

We will see two proofs of (iii):
- one showing that the constraint matrix is totally unimodular,
- one showing that P has no fractional extreme points.

5.4.1 Integrality through TU-ness

We can write P in the following form:

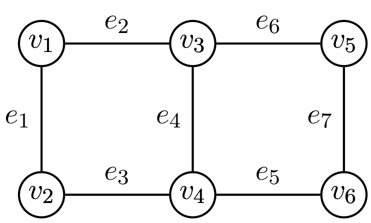
$$P = \{x \in \mathbb{R}^E \colon Ax \leq b, x \geq 0\} \enspace ,$$

where A and b are defined as follows. First observe that because P contains one constraint for each vertex, and one variable for each edges, we have $A \in \mathbb{R}^{V \times E}$ and $b \in \mathbb{R}^V$. The right-hand side of each constraint of P, that is not a non-negativity constraint, is 1, i.e., $b = \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^V$ is the all-ones vector.

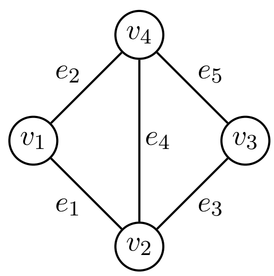
The matrix A is the *vertex-edge incidence matrix* of G, i.e., for every $v \in V$ and $e \in E$,

$$A(v,e) = \begin{cases} 1 & \text{if } v \in e \enspace , \\ 0 & \text{if } v \notin e \enspace . \end{cases}$$

Figure 5.1 shows two example graphs—one bipartite and the other one non-bipartite—with their corresponding vertex-edge incidence matrices, denoted by A.



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 5.1: Example graphs with corresponding vertex-edge incidence matrices.

Theorem 5.13

Let $G = (V, E)$ be an undirected graph with vertex-edge incidence matrix A. Then,

$$G \text{ is bipartite } \Leftrightarrow A \text{ is TU.}$$

Proof. $\Leftarrow$) Left as an exercise.

$\Rightarrow$) We use the characterization of Ghouila-Houri to show that A is totally unimodular if $G = (V, E)$ is bipartite with bipartition $V = X \dot{\cup} Y$.

Consider a subset of the rows of A. This subset corresponds to some subset of the vertices $R \subseteq V$. We partition R into $R_1 = R \cap X$ and $R_2 = R \cap Y$. For any column of A, which corresponds to some edge $e \in E$, we have

$$\underbrace{\sum_{v \in R_1} A_{v,e}}_{=|e \cap R_1| \leq |e \cap X| = 1} - \underbrace{\sum_{v \in R_2} A_{v,e}}_{=|e \cap R_2| \leq |e \cap Y| = 1} \in \{-1, 0, 1\}.$$

Notice that the above statement follows by the fact that each of the two sums in the above expression is either 0 or 1. Thus, their difference is either -1, 0, or 1. This shows that the criterion of Ghouila and Houri holds for the matrix A, implying that A is totally unimodular. $\square$

Combining Theorem 5.13 and Theorem 5.8, we obtain that $P = \{x \in \mathbb{R}^E : Ax \leq \mathbf{1}, x \geq 0\}$ is integral, and thus $P = P_{\mathcal{M}}$ as desired.

5.4.2 Integrality by disproving existence of fractional extreme points

Let $x \in P$ be a vertex of P. For sake of contradiction assume $x \notin \{0, 1\}^E$. Let $U \subseteq E$ be all edges with fractional values, i.e.,

$$U = \{e \in E : 0 < x(e) < 1\} \ .$$

Since x is not integral, $U \neq \emptyset$. We distinguish two cases, depending on whether U contains a cycle.

First case: U contains a cycle $C \subseteq U$.

Hence, C must be even since G is bipartite. Let $C = \{e_1, \ldots, e_k\} \subseteq U$ with k even be the edges along C. Let

$$\begin{aligned} W_1 &= \{e_i : i \in [k], i \text{ odd}\} \ , \\ W_2 &= \{e_i : i \in [k], i \text{ even}\} \ . \end{aligned}$$

For $\epsilon \in \mathbb{R}$, let $x^\epsilon \in \mathbb{R}^E$ be defined by

$$x^\epsilon = x + \epsilon \cdot \chi^{W_1} - \epsilon \cdot \chi^{W_2} \ .$$

We will show that there is some $\rho > 0$ such that $x^\rho, x^{-\rho} \in P$. This contradicts x being a vertex, or equivalently, an extreme point; indeed, x would be the midpoint of the two distinct points

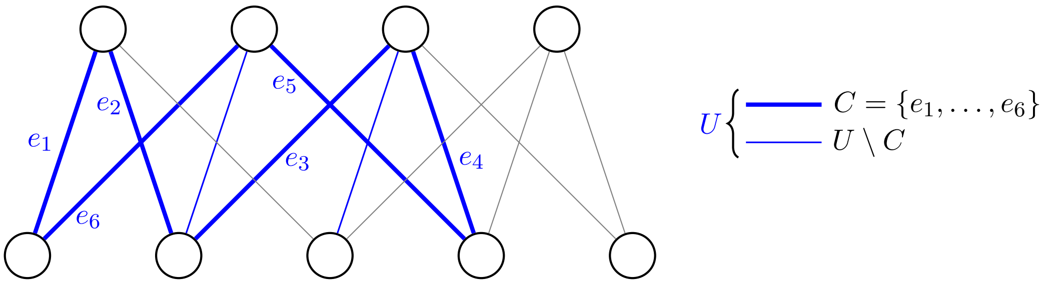


Figure 5.2: Example for first case of proof where U contains a cycle C.

$x^\rho \in P$ and $x^{-\rho} \in P$. Without loss of generality we only prove $x^\rho \in P$; the statement $x^{-\rho} \in P$ reduces to this case by exchanging the roles of W_1 and W_2, which can be done by renumbering the edges on C.

Let

$$\rho = \min_{e \in C}\{x(e)\} \ .$$

We clearly have $\rho > 0$ since $x(e) > 0 \ \forall e \in U$ and $C \subseteq U$. For any $v \in V$, we have

$$x^\rho(\delta(v)) = x(\delta(v)) \leq 1 \ ,$$

which is true even for any $\rho \in \mathbb{R}$. Furthermore, for $e \in E \setminus C$, we have $x^\rho(e) = x(e) \geq 0$, and for $e \in C$ we have

$$x^\rho(e) \geq x(e) - \underbrace{\rho}_{\leq x(e)} \geq 0 \ .$$

Thus, $x^\rho, x^{-\rho} \in P$, leading to a contradiction of x being a vertex.

Second case: U does not contain cycles.

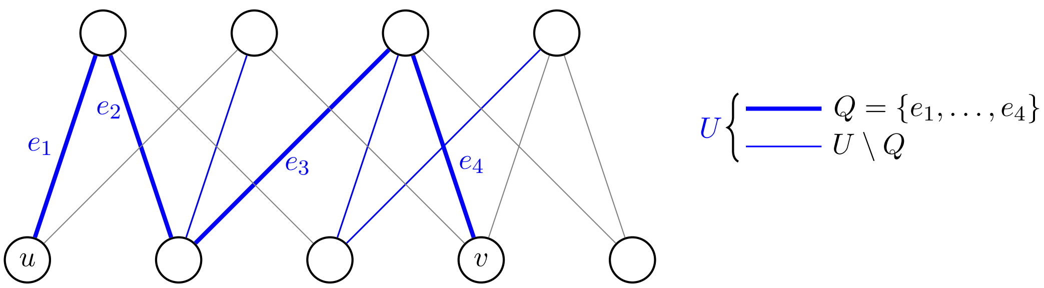


Figure 5.3: Example for second case of proof where U is a forest.

Hence U are the edges of a forest. Let $Q \subseteq U$ be any maximal path in U, i.e., Q is the unique path between some pair of leaf vertices u, v of U. We number the edges in $Q = \{e_1, \dots, e_k\}$ in the way they are encountered when traversing Q from u to v. Let

$$W_1 = \{e_i \colon i \in [k], i \text{ odd}\} \ ,$$
$$W_2 = \{e_i \colon i \in [k], i \text{ even}\} \ .$$

Again, we define for $\epsilon \in \mathbb{R}$

$$x^\epsilon = x + \epsilon \cdot \chi^{W_1} - \epsilon \cdot \chi^{W_2} \ ,$$

and this time we set

$$\rho = \min_{e \in Q}\{\min\{x(e), 1 - x(e)\}\} \ .$$

By definition of ρ, we clearly have $\rho > 0$ and $x^\rho, x^{-\rho} \in [0,1]^E$. Consider x^ρ (reasoning for $x^{-\rho}$ is identical). For all $w \in V \setminus \{u, v\}$ we have $x^\rho(\delta(w)) = x(\delta(w)) \leq 1$. It remains to show $x^\rho(\delta(u)) \leq 1$ and $x^\rho(\delta(v)) \leq 1$. W.l.o.g. we only show $x^\rho(\delta(u)) \leq 1$, since the roles of u and v can be exchanged by reversing the path P.

Notice that only one edge of U is adjacent to u since u was chosen to be a leaf vertex of the forest U. Thus all edges $e \in \delta(u) \setminus U$ satisfy $x(e) \in \{0, 1\}$. However, no edge $e \in \delta(u) \setminus U$ can satisfy $x(e) = 1$ since this would imply $x \notin P$ as $x(\delta(u)) \geq x(e) + x(e_1) = 1 + x(e_1) > 1$. Hence, of all edges $e \in \delta(u)$, e_1 is the only edge with $x(e) > 0$. Thus $x^\rho(\delta(u)) = x^\rho(e_1) \leq 1$, since $x^\rho \in [0,1]^E$. Hence $x^\rho \in P$.

Thus x can again be expressed as the midpoint of $x^\rho, x^{-\rho} \in P$ with $x^\rho \neq x^{-\rho}$ and therefore cannot be a vertex of P. Hence, P is integral, and therefore $P = P_{\mathcal{M}}$.

5.4.3 Some implications coming from the inequality description of $P_{\mathcal{M}}$

Often, one can derive interesting results from an inequality description of a polytope. We provide two examples below linked to the bipartite matching polytope. More precisely, we first show an example how an inequality description of one polytope can sometimes be used to derive an inequality description of a related one. The second example shows how the polyhedral description that we derived for the bipartite matching polytope can be used to obtain a very short and elegant proof of a classical theorem in Graph Theory.

Perfect bipartite matching polytope A matching $M \in \mathcal{M}$ is called *perfect*, if it touches all vertices, i.e., $|M| = |V|/2$. Of course, a perfect matching can only exist if $|V|$ is even. Furthermore, for a bipartite graph $G = (V, E)$ with corresponding bipartition $V = X \dot\cup Y$, one needs $|X| = |Y|$ for a perfect matching to exist.

Theorem 5.14

The perfect matching polytope of a bipartite graph $G = (V, E)$ is given by

$$P = \left\{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) = 1 \ \forall v \in V\right\} \ .$$

Proof. Again, one can easily check that P contains the correct set of integral points. Its integrality follows by observing that P is a face of the matching polytope $P_{\mathcal{M}}$ or the empty set, and hence an integral polytope.

A key property we exploit here is that a F face of an integral polyhedron Q is itself integral. This is a consequence of Corollary 1.14, which states that a face of a face is a face. Indeed, a vertex y of F is a 0-dimensional face of F. By Corollary 1.14, y is also a face of Q. Because y is 0-dimensional, it must be a vertex of Q. We now can exploit that Q is integral, which implies integrality of y. □

Perfect matchings in bipartite d-regular graphs A graph is called *d-regular* for some $d \in \mathbb{Z}_{\geq 0}$ if every vertex has degree d.

Theorem 5.15

Let $d \in \mathbb{Z}_{\geq 1}$. Every d-regular bipartite graph admits a perfect matching.

Proof. Using Theorem 5.14, let $P = \{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) = 1 \ \forall v \in V\}$ be the perfect bipartite matching polytope of a d-regular bipartite graph. Notice that the point $x \in \mathbb{R}^E$ given by $x(e) = \frac{1}{d}$ for every $e \in E$ satisfies $x \in P$. Hence, $P \neq \emptyset$, and P therefore contains a vertex, which corresponds to a perfect matching. □

5.5 Polyhedral description of shortest s-t paths

Let $G = (V, A)$ be a directed graph, and let $s, t \in V$ with $s \neq t$. Recall that a path is by definition vertex-disjoint, i.e., the same vertex is encountered at most once when traversing the path.

No "good" polyhedral description is known of the s-t path polytope, i.e., one over which we could optimize any linear function in polynomial time. This is not surprising, because such a description would imply $\mathcal{P} = \mathcal{NP}$ since one could solve the longest s-t path problem, which is well-known to be $\mathcal{NP}$-hard. In particular, one can solve the Hamiltonian path problem if one can find longest s-t paths: It suffices to find a longest s-t path with unit edge lengths for all $O(n^2)$ pairs of vertices $s, t \in V$, $s \neq t$, and check whether one of them is Hamiltonian.

Still, when trying to find shortest paths with respect to any positive length function, it is not hard to find a polyhedral approach. Consider the polytope

$$P = \left\{ x \in [0,1]^A \;\middle|\; x(\delta^+(v)) - x(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \end{cases} \quad \forall v \in V \right\} .$$

Notice that P is simply the flow polytope of an s-t flow problem where a unit flow has to be sent from s to t in a directed graph with all arc capacities being 1. Consider first the integral points in P. Notice that integral points in P do not correspond one-to-one to s-t paths. More precisely, each s-t path is an integral point in P, however, some integral points in P are not s-t paths. More precisely, one can easily prove (and we leave this as an exercise) that integral points in P correspond precisely to the disjoint union of an s-t path and possibly some additional cycles.

Hence, if we can show integrality of P, then for any positive weights (or lengths) $w \colon A \to \mathbb{Z}_{>0}$, a basic solution to $\min\{w^\top x \colon x \in P\}$ will correspond to a w-shortest s-t path. Thus, P

allows us to efficiently find shortest s-t paths with linear programming techniques. Notice that even if the weights w are non-negative instead of positive, i.e., $w\colon E \to \mathbb{Z}_{\geq 0}$, we can still find a shortest path via linear programming on P. An optimal vertex solution will correspond to a minimum weight set $U \subseteq E$ that is a disjoint union of an s-t path and cycles, where all cycles must have zero length by optimality of U. One can easily show that any s-t path $P \subseteq U$ is a shortest s-t path in G. The same approach works even if the weights are allowed to have negative values, but have the property of being *conservative*, which means that no cycle has strictly negative weight. Conservative weights cover the classical shortest path settings considered by specialized shortest path algorithms.

To show integrality of P, we will show that its corresponding constraint matrix is TU. First observe that P can be rewritten as

$$P = \left\{x \in \mathbb{R}^A \colon Dx = b, \mathbf{1} \geq x \geq 0\right\} \quad ,$$

where $\mathbf{1} = \chi^A$ is the all-ones vector, $b \in \{-1,0,1\}^V$ is defined by

$$b(v) = \begin{cases} 1 & \text{if } v = s \ , \\ -1 & \text{if } v = t \ , \\ 0 & \text{if } v \in V \setminus \{s,t\} \ , \end{cases}$$

and $D \in \{-1,0,1\}^{V \times A}$ is the *vertex-arc incidence matrix* of the *directed* (loopless) graph G, which is defined as follows: for $v \in V$, $a \in A$:

$$D(v,a) = \begin{cases} 1 & \text{if } a \in \delta^+(v) \ , \\ -1 & \text{if } a \in \delta^-(v) \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

It thus suffices to show that D is TU to obtain integrality of P.

Theorem 5.16

The vertex-arc incidence matrix $D \in \{-1,0,1\}^{V \times A}$ of any directed (loopless) graph $G = (V, A)$ is TU.

Proof. We apply the Ghouila-Houri characterization to the rows of D. For any subset $R \subseteq V$ of the rows, we choose the partition $R_1 = R$ and $R_2 = \emptyset$. Since each column of D has only zeros except for precisely one 1 and one -1, summing any subsets of the elements of any column will lead to a total sum of either -1, 0, or 1. Hence,

$$\sum_{v \in R_1} D_{v,a} \in \{-1,0,1\} \quad \forall a \in A \ ,$$

as desired. Or, more formally, for $a = (u,v) \in A$, we have

$$\sum_{w \in R_1} D_{w,a} = \underbrace{\mathbf{1}_{u \in R_1}}_{\in\{0,1\}} - \underbrace{\mathbf{1}_{v \in R_1}}_{\in\{0,1\}} \in \{-1,0,1\} \ ,$$

where, for $w \in V$, $\mathbf{1}_{w \in R_1} = 1$ if $w \in R_1$ and $\mathbf{1}_{w \in R_1} = 0$ otherwise. $\square$

5.6 Spanning trees and r-arborescences

Spanning trees and their directed counterparts, known as arborescences, are among the most basic combinatorial structures in Graph Theory. Contrary to the polytopes we have seen so far, the combinatorial polytopes corresponding to spanning trees and arborescences have an exponential number of facets. In this section, we only provide a brief discussion of their corresponding polytopes. We prove the correctness of the presented linear inequality descriptions later on when we introduce a powerful proof technique known as combinatorial uncrossing.

5.6.1 Spanning tree polytope

Let $G = (V, E)$ be an undirected graph. A *spanning tree* in G is an edge set $T \subseteq E$ that connects all vertices and does not contain a cycle.

For any set $S \subseteq V$, we denote by $E[S] \subseteq E$ all edges with both endpoints in S, i.e.,

$$E[S] \coloneqq \{e \in E \colon e \subseteq S\} \ .$$

Theorem 5.17

The spanning tree polytope of an undirected loopless graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} \ .$$

Again, one can easily check that P contains the correct set of integral points, i.e., the $\{0,1\}$-points in P are precisely the incidence vectors of spanning trees. To check this, the following equivalent definition of spanning trees is useful: A set $T \subseteq E$ is a spanning tree if and only if $|T| = |V| - 1$ and T does no contain any cycle.

The constraints of the spanning tree polytope are often divided into two groups, namely the *non-negativity constraints* $x \geq 0$, and all the other constraints which are called *spanning tree constraints*.

5.6.2 The r-arborescence polytope

Definition 5.18: Arborescence, r-arborescence

Let $G = (V, A)$ be a directed graph. An *arborescence* in G is an arc set $T \subseteq A$ such that

(i) T is a spanning tree (when disregarding the arc directions), and
(ii) there is one vertex r from which all arcs are directed away, i.e., every vertex $v \in V$ can be reached from r using a directed path in T.

The vertex r in condition (ii) is called the *root* of the arborescence, and T is called an *r-arborescence*.

Figure 5.4 shows an example of an r-arborescence.

Notice that condition (ii) can equivalently be replaced by

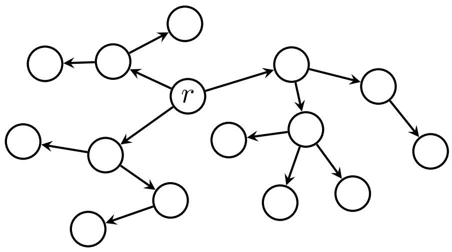


Figure 5.4: Example of an r-arborescence.

(ii') Every vertex has at most one incoming arc.

Theorem 5.19

The arborescence polytope of a directed loopless graph $G = (V, A)$ is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \;\middle|\; \begin{array}{ll} x(A) = |V| - 1 & \\ x(A[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \\ x(\delta^-(v)) \leq 1 & \forall v \in V \end{array} \right\} ,$$

where $A[S] \subseteq A$ for $S \subseteq V$ denotes all arcs with both endpoints in S.

A polyhedron that is closely related to the arborescence polytope and has a very elegant description is the *dominant of the arborescence polytope*. The dominant of the arborescence polytope will also provide an excellent example to show how integrality of a polyhedron can be proven using a technique called *combinatorial uncrossing*.

Apart from sometimes having a simpler description, the dominant of a polytope can also often be used for optimization. For example, consider the problem of finding a minimum weight r-arborescence with respect to some positive arc weights $w \in \mathbb{Z}^A_{>0}$. Let P be the r-arborescence polytope. Then this problem corresponds to minimizing $w^\top x$ over all $x \in P$. However, this is equivalent to minimizing $w^\top x$ over $x \in \mathrm{dom}(P)$. Indeed, any $x \in \mathrm{dom}(P)$ can be written as $x = y + z$, where $y \in P$ and $z \in \mathbb{R}^A_{\geq 0}$. Therefore for $x \in \mathrm{dom}(P)$ to be a minimizer of $w^\top x$, we must have $z = 0$; for otherwise, $w^\top y < w^\top x$ and $y \in P \subseteq \mathrm{dom}(P)$, violating that $x \in \mathrm{dom}(P)$ minimizes $w^\top x$.

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \colon x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\} .$$

We will prove integrality of this polyhedron later, when talking about combinatorial uncrossing.

5.7 Non-bipartite matchings

We will start by introducing the perfect matching polytope and then derive therefrom the description of the matching polytope.

5.7.1 Perfect matching polytope

Theorem 5.21

The perfect matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(S)) \geq 1 & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\} .$$

Proof. It is easy to check that P contains the correct set of integral points. Thus, it remains to show integrality of P.

By sake of contradiction assume that there are graphs $G = (V, E)$ for which P is not integral. Among all such graphs let $G = (V, E)$ be a one that minimizes $|V| + |E|$, i.e., we look at a smallest bad example, and let P be the corresponding polytope as defined in Theorem 5.21. Notice that we must have that $|V|$ is even. For otherwise the polytope P is indeed the perfect matching polytope because it is empty, which follows from the constraint $x(\delta(V)) \geq 1$, which is impossible to satisfy since $\delta(V) = \emptyset$.

Let $y \in P$ be a vertex of P that is fractional. We start by observing some basic properties following from the fact that we chose a smallest bad example. In particular, G is connected. For otherwise, a smaller bad example is obtained by only considering one of its connected components containing a y-fractional edge, which violates minimality of G. Moreover, there is no 0-edge, i.e., an edge $e \in E$ such that $y(e) = 0$, because such an edge could be deleted leading to a smaller bad example. Similarly, there is no 1-edge $e = \{u, v\} \in E$, because in this case there can be no other edge f except for e that is incident with either u or v, since we would have $y(f) = 0$ due to $y(\delta(u)) = 1$ and $y(\delta(v)) = 1$, and we already know that there is no 0-edge. Finally, $\delta(\{u, v\}) = \emptyset$ implies that G is not connected, because G must contain at least one edge besides e because there is an edge with fractional y-value. Hence, y is fractional on all edges, i.e., $y(e) \in (0, 1)$ $\forall e \in E$.

Since y is a vertex of P it can be defined as the unique solution to $|E|$ linearly independent constraints of P that are *y-tight*, i.e., tight for the point y. If it is clear from the context, we also often just talk about *tight* constraints. P has three types of constraints: *degree constraints* ($x(\delta(v)) = 1$ for $v \in V$), *cut constraints* ($x(\delta(S)) \geq 1$ for $S \subseteq V, |S|$ odd), and *non-negativity constraints* ($x(e) \geq 0$ for $e \in E$). Notice that none of the non-negativity constraints are tight since $y > 0$. Hence, y is the unique solution to a full-rank linear system of the following type:

$$\begin{aligned} x(\delta(v)) &= 1 \quad \forall v \in W \ , \\ x(\delta(S)) &= 1 \quad \forall S \in \mathcal{F} \ , \end{aligned} \tag{5.8}$$

where $W \subseteq V$, $\mathcal{F} \subseteq \{S \subseteq V : |S| \text{ odd}\}$, and $|W| + |\mathcal{F}| = |E|$ because the system is full-rank. Without loss of generality, we assume that $\mathcal{F}$ only contains sets $S \subseteq V$ such that $|S| \neq 1$ and

$|S| \neq |V| - 1$. Indeed, if $|S| = 1$, then the constraint $x(\delta(S)) = 1$ can be handled as a tight degree constraints. The same holds for $|S| = |V| - 1$, in which case the constraint $x(\delta(S)) = 1$ is identical to $x(\delta(V \setminus S)) = 1$, which is a degree constraint.

We now distinguish between two cases depending on whether $\mathcal{F} = \emptyset$ or not.

Case $\mathcal{F} = \emptyset$: We start with an observation that is independent of this case. Namely, each vertex must have degree at least two, since a degree-one vertex v would need to be adjacent to a 1-edge to satisfy the degree constraint $y(\delta(v)) = 1$. This implies $|E| \geq |V|$. Furthermore, since (5.8) is full-rank, we have $|W| = |E|$. Finally, $W \subseteq V$ implies $|W| \leq |V|$, and by combining all above inequalities we get $|V| \leq |E| = |W| \leq |V|$. Hence, $|E| = |W| = |V|$, i.e., the system (5.8) contains all degree constraints, and G is a graph where each vertex has degree precisely 2. Since G is connected, it must be a single cycle containing all vertices. Moreover, because $|V|$ is even, it is an even cycle. Let $V = \{v_1, \dots, v_n\}$ be a consecutive numbering of the vertices when going along the cycle. We now get a contradiction by showing that the system (5.8) is not full-rank. Indeed, we have

$$\sum_{i \in [n], i \text{ even}} \chi^{\delta(v_i)} = \sum_{i \in [n], i \text{ odd}} \chi^{\delta(v_i)} \ ,$$

which shows that there is a linear dependence within the degree constraints.

Case $\mathcal{F} \neq \emptyset$: Let $S \in \mathcal{F}$. As discussed, we have $|S| > 1$ and $|V \setminus S| > 1$. We consider two graphs, $G_1 = G/S$ and $G_2 = G/(V \setminus S)$, which are obtained from G by contracting S and $V \setminus S$, respectively. Contracting S means that we replace all vertices in S by a single new vertex v_S. Furthermore, all edges with both endpoints in S are deleted, and any edge with one endpoint in S and the other one $v \in V \setminus S$ outside of S is replaced by an edge between v_S and v. Let y_1 and y_2 be the restriction of y to all non-deleted edges in G_1 and G_2, respectively. One can easily observe that $y_i \in P_{G_i}$ for $i \in \{1, 2\}$, where P_{G_i} is the polytope as defined by Theorem 5.21 for the graph G_i. Notice that both graphs G_1 and G_2 are strictly smaller than G in terms of the sum of their number of vertices and edges. Since G was a smallest bad example, the two polytopes P_{G_1} and P_{G_2} are therefore the perfect matching polytopes of G_1 and G_2. Hence, we can write y_i for $i \in \{1, 2\}$ as a convex combination of perfect matchings in G_i. In particular, there is some $N \in \mathbb{Z}_{>0}$ such that for $i \in \{1, 2\}$

$$y_i = \frac{1}{N} \sum_{j=1}^{N} \chi^{M_i^j} \ , \tag{5.9}$$

where M_i^j is a perfect matching in G_i for $i \in \{1, 2\}$ and $j \in [N]$. Notice that both graphs G_1 and G_2 contain the edges $\delta(S)$. More precisely, for both graphs G_1, G_2, the set $\delta(S)$ consists of all edges incident with the vertex representing the contracted set S or $V \setminus S$. Hence, each perfect matching M_i^j contains precisely one edge of $\delta(S)$. Furthermore, for $i \in \{1, 2\}$, each edge $e \in \delta(S)$ must be contained in precisely $N \cdot y_i(e)$ matchings of the family $\{M_i^j\}_{j \in [N]}$ for (5.9) to be true. Additionally, since both y_1 and y_2 are just restrictions of y, they coincide on the edges $\delta(S)$. Hence, $y(e) = y_1(e) = y_2(e)$ for each $e \in \delta(S)$. Thus, for every $e \in \delta(S)$, there is the same number of perfect matchings in $\{M_1^j\}_{j \in [N]}$ that contain e as there are perfect matchings in

$\{M_2^j\}_{j\in[N]}$ containing e. We can therefore choose the numberings of those matchings, i.e., the indices j, such that

$$M_1^j \cap \delta(S) = M_2^j \cap \delta(S) \quad \forall j \in [N] \ .$$

This implies that $M_1^j \cup M_2^j$ is a perfect matching in G. Hence, we have

$$y = \frac{1}{N}\sum_{j=1}^{N} \chi^{M_1^j \cup M_2^j} \ ,$$

which implies that y is a convex combination of perfect matchings in G. This violates the fact that y is a fractional vertex of P, because we were able to write y as a convex combination of $\{0,1\}$-points of P, thus finding a non-trivial convex combination that expresses y. □

5.7.2 Matching polytope

As we show in the proof of the theorem below, the description of the perfect matching polytope can now be used to prove integrality of the matching poytope.

Theorem 5.22

The matching polytope of an undirected graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(\delta(v)) \leq 1 & \forall v \in V \\ x(E[S]) \leq \frac{|S|-1}{2} & \forall S \subseteq V, |S| \text{ odd} \end{array} \right\} \ .$$

Proof. As usual, it is easy to check that P contains the correct set of integral points, i.e., $P \cap \{0,1\}^E$ are all incidence vectors of matchings in G. Let $x \in P$. We have to show that x is a convex combination of matchings. We use the following proof plan. We will define an auxiliary graph $H = (W, F)$ that extends G, and we will also extend x to a point in $y \in [0,1]^F$, which we can show to be in the perfect matching polytope of H using Theorem 5.21. Hence, y can be written as a convex combination of perfect matchings of H. From this convex decomposition we then derive that x is a convex combination of matchings of G.

We start by constructing the auxiliary graph $H = (W, F)$. Let $G' = (V', E')$ be a copy of $G = (V, E)$, i.e., for every vertex $v \in V$ there is a vertex $v' \in V'$ and for every edge $e = \{u, v\} \in E$ there is an edge $e' = \{u', v'\} \in E'$. We define $H = (W, F)$ to be the disjoint union of G and G', where we add all edges of the type $\{v, v'\}$ for $v \in V$. More formally,

$$\begin{aligned} W &= V \cup V' \ , \\ F &= E \cup E' \cup \{\{v, v'\} \colon v \in V\} \ . \end{aligned}$$

Furthermore, we define $y \in [0,1]^F$ by

$$\begin{aligned} y(e) &= x(e) & \forall e \in E \ , \\ y(e') &= x(e) & \forall e \in E \ , \\ y(\{v, v'\}) &= 1 - x(\delta(v)) & \forall v \in V \ . \end{aligned}$$

We now show that y is in the perfect matching polytope of H, which is described by Theorem 5.21. First, we clearly have $y \geq 0$ and $y(\delta_H(w)) = 1$ for $w \in W$. Let $Q \subseteq W$ be a set with $|Q|$ odd. It remains to show $y(\delta_H(Q)) \geq 1$ for y to be in the perfect matching polytope of H. Let $A = Q \cap V$, and $B' = Q \cap V'$. Furthermore, we define $B = \{v \in V : v' \in B'\}$. We have

$$y(\delta_H(Q)) = \underbrace{y(\delta_H(Q) \cap E)}_{=\text{I}} + \underbrace{y(\delta_H(Q) \cap E')}_{=\text{II}} + \underbrace{y(\delta_H(Q) \cap \{\{v,v'\} : v \in V\})}_{=\text{III}} \ .$$

Now observe that

$$\begin{aligned}
\text{I} &= x(\delta(A)) \ , \\
\text{II} &= x(\delta(B)) \ , \\
\text{III} &= \sum_{v \in A \setminus B} (1 - x(\delta(v))) + \sum_{v \in B \setminus A} (1 - x(\delta(v))) \\
&= |A \setminus B| - 2x(E[A \setminus B]) - x(\delta(A \setminus B)) + |B \setminus A| - 2x(E[B \setminus A]) - x(\delta(B \setminus A)) \ .
\end{aligned}$$

Furthermore,

$$\begin{aligned}
x(\delta(A)) + x(\delta(B)) &= x(\delta(A \setminus B)) + x(\delta(B \setminus A)) + 2x(E(A \cap B, V \setminus (A \cup B))) \\
&\geq x(\delta(A \setminus B)) + x(\delta(B \setminus A)) \ ,
\end{aligned}$$

where for $S_1, S_2 \subseteq V$, the set $E(S_1, S_2) \subseteq E$ consists of all edges in G with one endpoint in S_1 and the other in S_2. Combining the above, we obtain

$$y(\delta_H(Q)) \geq |A \setminus B| - 2x(E[A \setminus B]) + |B \setminus A| - 2x(E[B \setminus A]) \ . \tag{5.10}$$

Now, notice that for any set $S \subseteq V$, we have

$$2x(E[S]) \leq \sum_{v \in S} x(\delta(v)) \leq |S| \ , \tag{5.11}$$

where the first inequality follows by observing that every edge $\{u, v\} \in E[S]$ appears twice in the middle term: once in $\delta(u)$ and once in $\delta(v)$. Furthermore, the second inequality follows by $x(\delta(v)) \leq 1 \ \forall v \in V$, which holds since $x \in P$.

Notice that

$$|Q| = |A \setminus B| + |B \setminus A| + 2|A \cap B| \ .$$

Hence, since $|Q|$ is odd, either $|A \setminus B|$ or $|B \setminus A|$ must be odd. Without loss of generality assume that $|A \setminus B|$ is odd. As $x \in P$, we have

$$x(E[A \setminus B]) \leq \frac{|A \setminus B| - 1}{2} \ . \tag{5.12}$$

We finally obtain

$$\begin{aligned}
y(\delta_H(Q)) &\geq |A \setminus B| - 2x(E[A \setminus B]) + |B \setminus A| - 2x(E[B \setminus A]) && \text{(by (5.10))} \\
&\geq |A \setminus B| - 2x(E[A \setminus B]) && \text{(by (5.11))} \\
&\geq 1 && \text{(by (5.12))},
\end{aligned}$$

thus implying that y is in the perfect matching polytope of H. Hence, y can be written as a convex combination of perfect matchings in H, i.e.,

$$y = \sum_{i=1}^{k} \lambda_i \underbrace{\chi^{M_i}}_{\in\{0,1\}^F} ,$$

where $k \in \mathbb{Z}_{>0}$, $\lambda_i \geq 0$ for $i \in [k]$, $\sum_{i=1}^{k} \lambda_i = 1$, and $M_i \subseteq F$ for $i \in [k]$ is a perfect matching in $H = (W, F)$. Since x is the restriction of y to the edges in E, we get

$$x = y|_E = \sum_{i=1}^{k} \lambda_i \underbrace{\chi^{M_i \cap E}}_{\in\{0,1\}^E} ,$$

where we use the characteristic vector notation χ in the equation above with respect to the ground set E instead of F, i.e., $\chi^{M_i \cap E} \in \{0,1\}^E$. Thus, x is a convex combination of the matchings $M_1 \cap E, \ldots, M_k \cap E$ in the graph G. This proves that P is contained in the matching polytope. Furthermore, since P contains the correct integral points, it contains the matching polytope. Hence, P is the matching polytope. □

5.8 Combinatorial uncrossing

Combinatorial uncrossing is a technique to extract a well-structured set system out of a large family of sets. It can be used in various contexts, one of which is to show the integrality of polyhedra. Here, the main goal of combinatorial uncrossing can be summarized as follows:

> Given a heavily overdetermined linear system that uniquely defines a point, find a well-structured full-rank subsystem.

5.8.1 Integrality of spanning tree polytope

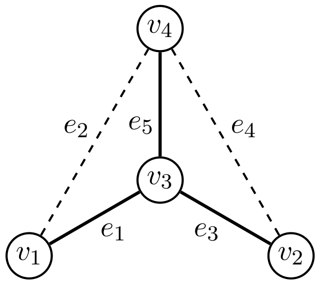
We recall the description of the spanning tree polytope claimed by Theorem 5.17.

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

As it is common for many combinatorial optimization problems, the spanning tree polytope is highly degenerate. Figure 5.5 shows a degenerate vertex of the spanning tree polytope together with the tight constraints.

Notice that the constraint matrix corresponding to the spanning tree polytope is not TU. Moreover, even if we only consider tight constraints, the resulting linear subsystem may not be TU. A submatrix of the tight constraints having a determinant not within $\{-1, 0, 1\}$ is highlighted in blue in Figure 5.5.

However, it turns out that for any vertex y of the spanning tree polytope, there is always a full-rank linear subsystem among the y-tight constraints that is TU and thus implies integrality of y. We will prove the existence of such a subsystem using combinatorial uncrossing.



— $y = (1,0,1,0,1)$

spanning tree constraints:

$$\begin{array}{rc}
 & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{array}{r} \{v_1,v_2\} \\ \{v_1,v_3\} \\ \{v_1,v_4\} \\ \{v_2,v_3\} \\ \{v_2,v_4\} \\ \{v_3,v_4\} \\ \{v_1,v_2,v_3\} \\ \{v_1,v_2,v_4\} \\ \{v_1,v_3,v_4\} \\ \{v_2,v_3,v_4\} \\ \{v_1,v_2,v_3,v_4\} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{array}
y \overset{=}{\leq}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$

non-negativity constraints:

$$\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
y \overset{=}{\geq}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Figure 5.5: The vector y is the incidence vector of the spanning tree $\{e_1, e_3, e_5\}$. Tight spanning tree constraints (with respect to y) and tight non-negativity constraints are highlighted in red. There are 7 tight spanning tree constraints and 2 tight non-negativity constraints. Hence, the vertex y of the spanning tree polytope is degenerate because the total number of tight constraints, which is 9, is strictly larger then the dimension in which y lives, i.e., 5. Furthermore, the constraint matrix that corresponds to only the tight constraints is not TU. This can be verified by considering the subsystem of the tight spanning tree constraints that correspond to the columns e_1, e_3, e_5 and the rows $\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$, as highlighted in blue in the above figure. The determinant of this 3×3 subsystem is -2.

Proof of integrality of P Let $y \in P$ be a vertex of P. Without loss of generality, we delete from $G = (V, E)$ all edges $e \in E$ with $y(e) = 0$, to obtain a smaller graph such that y restricted to $\{e \in E \mid y(e) > 0\}$ is still a vertex of the polytope P in the reduced graph. Hence, we assume from now on that G is this reduced graph and therefore $\mathrm{supp}(y) = E$.

Consider all y-tight, or simply tight, *spanning tree (ST) constraints*. Each tight ST constraint

$x(E[S]) = |S|-1$ corresponds to some set $S \subseteq V, |S| \geq 2$. We represent all tight ST constraints by their corresponding set S, thus obtaining the *family of tight sets* $\mathcal{F}$ given by

$$\mathcal{F} = \{S \subseteq V : |S| \geq 1,\ y(E[S]) = |S| - 1\} \ .$$

For technical reasons that will become clear later, we also include sets of size 1 in $\mathcal{F}$. Notice that they correspond to constraints that are trivially fulfilled.

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$, and consider the system:

$$x(E[S]) = |S| - 1 \quad \forall S \in \mathcal{H} \ . \tag{5.13}$$

We will show that y is the unique solution to (5.13). Notice that this will imply integrality of y since (5.13) is a TU system with integral right-hand side, as it has the consecutive-ones property with respect to the rows.

As the set of all tight spanning tree constraints uniquely defines y, it suffices to show that any tight spanning tree constraint is implied by (5.13), i.e.,

$$\chi^{E[S]} \text{ is a linear combination of } \{\chi^{E[H]} \mid H \in \mathcal{H}\} \qquad \forall S \in \mathcal{F} \ . \tag{5.14}$$

We showed in the problem sets in a very general context that (5.14) is indeed equivalent to showing that every tight spanning tree constraint is implied by (5.13).

To this end, we need a better understanding of the structure of y-tight constraints, because we need to show that $\mathcal{H}$ contains a rich enough family of y-tight constraints such that (5.14) holds. The lemma below states an elementary yet crucial structural property that will allow us to make a concrete statement about the laminar family $\mathcal{H}$.

Lemma 5.23

For any sets $A, B \subseteq V$, we have

$$\chi^{E[A]} + \chi^{E[B]} + \chi^{E(A\setminus B, B\setminus A)} = \chi^{E[A\cup B]} + \chi^{E[A\cap B]} \ ,$$

which implies

$$\chi^{E[A]} + \chi^{E[B]} \leq \chi^{E[A\cup B]} + \chi^{E[A\cap B]} \ .$$

Proof. In words, the equality in the lemma states that any edge $e \in E$ appears the same number of times in the edge sets $E[A], E[B], E(A\setminus B, B\setminus A)$ as it appears in the sets $E[A\cup B], E[A\cap B]$. One can easily check that Lemma 5.23 holds by verifying the equality for each coordinate, i.e., each edge, where the edges can be grouped into the following *edge types*. The type of an edge is determined by where its endpoints lie among the four sets $A \setminus B, B \setminus A, A \cap B, E \setminus (A \cup B)$. Figure 5.6 shows the 10 different edge types. Clearly, two edges of the same edge type have the same contribution to the left-hand side and right-hand side of the equality in Lemma 5.23. Hence, it suffices to check whether each edge type has the same contribution on each side of the equality, which is easy to observe. □

Lemma 5.23 allows for deriving relations about tight sets. In particular, if two sets with non-empty intersection are tight, then so are their union and intersection.

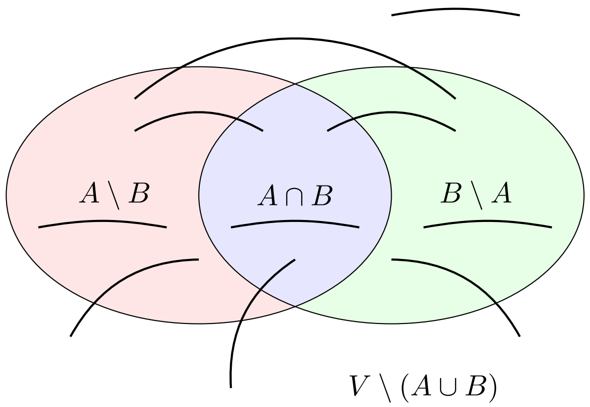


Figure 5.6: There are 10 types of edges in terms of where their endpoints lie with respect to the sets A and B, as shown above. One can easily check that each type appears precisely the same number of times in $E[A \cup B]$ and $E[A \cap B]$ as it appears in $E[A]$, $E[B]$, and $E(A \setminus B, B \setminus A)$.

Lemma 5.24

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2, S_1 \cup S_2 \in \mathcal{F}$ and $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$. In particular, this implies by Lemma 5.23

$$\chi^{E[S_1]} + \chi^{E[S_2]} = \chi^{E[S_1 \cup S_2]} + \chi^{E[S_1 \cap S_2]} \quad .$$

Proof. By Lemma 5.23 we have

$$y(E[S_1]) + y(E[S_2]) + y(E(S_1 \setminus S_2, S_2 \setminus S_1)) = y(E[S_1 \cup S_2]) + y(E[S_1 \cap S_2]) \quad . \tag{5.15}$$

Using (5.15), we obtain

$$\begin{aligned}
|S_1| + |S_2| - 2 &= \underbrace{|S_1 \cup S_2| - 1}_{\geq y(E[S_1 \cup S_2])} + \underbrace{|S_1 \cap S_2| - 1}_{\geq y(E[S_1 \cap S_2])} \\
&\geq y(E[S_1 \cup S_2]) + y(E[S_1 \cap S_2]) && \text{(using } y \in P) \\
&= \underbrace{y(E[S_1])}_{=|S_1|-1} + \underbrace{y(E[S_2])}_{=|S_2|-1} + y(E(S_1 \setminus S_2, S_2 \setminus S_1)) && \text{(by (5.15))} \\
&= |S_1| + |S_2| - 2 + \underbrace{y(E(S_1 \setminus S_2, S_2 \setminus S_1))}_{\geq 0} && \text{(because } S_1, S_2 \in \mathcal{F}) \\
&\geq |S_1| + |S_2| - 2 \quad .
\end{aligned}$$

Hence, all inequalities above must be satisfied with equality, thus implying $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $y(E(S_1 \setminus S_2, S_2 \setminus S_1)) = 0$. Finally, since $y > 0$, we have that $y(E(S_1 \setminus S_2, S_2 \setminus S_1)) = 0$ implies $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, as desired.

□

Notice that for Lemma 5.24 we need $\mathcal{F}$ to contain singleton sets to cover the case $S_1, S_2 \in \mathcal{F}$ with $|S_1 \cap S_2| = 1$.

Definition 5.25: Intersecting sets

Two sets $S_1, S_2 \subseteq V$ are called *intersecting* if $S_1 \cap S_2 \neq \emptyset$, $S_1 \setminus S_2 \neq \emptyset$, and $S_2 \setminus S_1 \neq \emptyset$.

Notice that a family of sets is laminar if and only if no two of its sets are intersecting.

Lemma 5.26

The statement (5.14) holds and y is thus uniquely defined by the TU system (5.13). Therefore, y is integral.

Proof. Assume by the sake of contradiction that (5.14) does not hold. Let

$$Q = \mathrm{span}(\{\chi^{E[H]} : H \in \mathcal{H}\}) \subseteq \mathbb{R}^E \quad .$$

Assuming (5.14) not to hold is thus equivalent to the existence of $S \in \mathcal{F}$ with $\chi^{E[S]} \notin Q$. Among all tight spanning tree constraints that violate (5.14), let $S \in \mathcal{F}$ be one such that

$$\mathcal{H}_S = \{H \in \mathcal{H} : S \text{ and } H \text{ are intersecting}\}$$

has smallest size. Notice that $\mathcal{H}_S \neq \emptyset$, for otherwise we could have added S to $\mathcal{H}$ without destroying laminarity of $\mathcal{H}$, thus contradicting maximality of $\mathcal{H}$. Let $H \in \mathcal{H}_S$. By Lemma 5.24 we have $S \cap H, S \cup H \in \mathcal{F}$ and

$$\chi^{E[S]} + \chi^{E[H]} = \chi^{E[S\cap H]} + \chi^{E[S\cup H]} \quad .$$

Notice that $\chi^{E[S]} \notin Q$ and $\chi^{E[H]} \in Q$. Hence, at least one of $\chi^{E[S\cap H]}, \chi^{E[S\cup H]}$ is not in Q. However, we have

$$\begin{aligned} |\mathcal{H}_{S\cap H}| &< |\mathcal{H}_S| \quad \text{and} \\ |\mathcal{H}_{S\cup H}| &< |\mathcal{H}_S| \quad , \end{aligned}$$

because there is no set in $\mathcal{H}$ that is intersecting with any of the two sets $S \cap H$ or $S \cup H$ but not S. Furthermore, the set H is intersecting with S but not with $S \cap H$ or $S \cup H$. (See problem sets.) This contradicts the choice of S. $\square$

5.8.2 Integrality of dominant of r-arborescence polytope

We recall the description of the dominant of the r-arborescence polytope:

$$P = \left\{x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \ \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\} \quad .$$

To show its integrality, we provide an analogous proof to the case of the spanning tree polytope.

Proof of integrality of P Let $y \in P$ be a vertex of P. As for the spanning tree case, we can assume that the underlying graph $G = (V, A)$ has no arcs $a \in A$ with $y(a) = 0$, since those arcs can be removed and the restriction of y to its support is still a vertex in the polyhedron P that corresponds to the reduced graph. We denote by $\mathcal{F} \subseteq 2^V$ the set of all y-tight constraints:

$$\mathcal{F} = \{S \subseteq V \setminus \{r\} : y(\delta^-(S)) = 1\} \ .$$

Hence, y is the unique solution to the linear system

$$x(\delta^-(S)) = 1 \quad \forall S \in \mathcal{F} \ . \tag{5.16}$$

Let $\mathcal{H} \subseteq \mathcal{F}$ be a maximal laminar subfamily of $\mathcal{F}$. As in the spanning tree case, we show that the following system implies all constraints of (5.16):

$$x(\delta^-(S)) = 1 \quad \forall S \in \mathcal{H} \ . \tag{5.17}$$

Again, the linear system (5.17) is TU, which can be seen by applying the Ghouila-Houri criterion with respect to the rows as follows. Consider any subset of the rows, which can be represented by a laminar subfamily $\mathcal{F}' \subseteq \mathcal{H}$. We partition this subfamily $\mathcal{F}'$ into a '$+$' and '$-$' group as follows. The topmost sets of $\mathcal{F}'$, i.e., the ones not contained in any other set of $\mathcal{F}'$ are in the '$+$' group, their children are in the '$-$' group, and we continue alternating that way (see Figure 5.7). One can easily verify that this partition leads to a vector with entries $\{-1, 0, 1\}$ as desired. TU-ness of (5.17) also follows from results shown in the problem sets, where we considered a system $x(\delta^+(S))$ for S being part of an arbitrary laminar family. The system (5.17) easily reduces to this case by reversing the directions of the arcs.

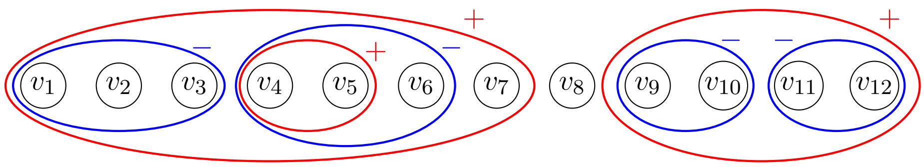


Figure 5.7: Illustration of how we apply the Ghouila-Houri criterion to the subfamily $\mathcal{F}'$, which is the laminar family depicted in the figure.

Hence, it remains to show that every constraint of (5.16) is implied by (5.17). Similar to the spanning tree case, we need to get a better understanding of directed cuts to deduce that $\mathcal{H}$ is a family rich enough to ensure that (5.17) implies (5.16). For this, we start with a basic property on directed cuts.

Lemma 5.27

For any two sets $S_1, S_2 \subseteq V$, we have

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} + \chi^{A(S_1 \setminus S_2, S_2 \setminus S_1)} + \chi^{A(S_2 \setminus S_1, S_1 \setminus S_2)} \ ,$$

which implies in particular

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} \geq \chi^{\delta^-(S_1 \cap S_2)} + \chi^{\delta^-(S_1 \cup S_2)} \ .$$

Proof. This proof can be done analogously to the proof of Lemma 5.23, by considering all different arc types. □

From Lemma 5.27, we can derive properties on tight cuts.

Lemma 5.28

If $S_1, S_2 \in \mathcal{F}$ with $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $A(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, $A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset$. In particular, this implies by Lemma 5.27

$$\chi^{\delta^-(S_1)} + \chi^{\delta^-(S_2)} = \chi^{\delta^-(S_1 \cup S_2)} + \chi^{\delta^-(S_1 \cap S_2)} \ .$$

Proof. By Lemma 5.27 and non-negativity of y we have

$$\begin{aligned} &y(\delta^-(S_1)) + y(\delta^-(S_2)) = \\ &\quad y(\delta^-(S_1 \cup S_2)) + y(\delta^-(S_1 \cap S_2)) + y(A(S_1 \setminus S_2, S_2 \setminus S_1)) + y(A(S_2 \setminus S_1, S_1 \setminus S_2)) \ . \end{aligned} \tag{5.18}$$

Using (5.18), we obtain

$$\begin{aligned} 2 &= y(\delta^-(S_1)) + y(\delta^-(S_2)) \\ &= y(\delta^-(S_1 \cup S_2)) + y(\delta^-(S_1 \cap S_2)) && \text{(by (5.18))} \\ &\qquad + \underbrace{y(A(S_1 \setminus S_2, S_2 \setminus S_1))}_{\geq 0} + \underbrace{y(A(S_2 \setminus S_1, S_1 \setminus S_2))}_{\geq 0} \\ &\geq \underbrace{y(\delta^-(S_1 \cup S_2))}_{\geq 1} + \underbrace{y(\delta^-(S_1 \cap S_2))}_{\geq 1} \\ &\geq 2 \ . && \text{(because } y \in P\text{)} \end{aligned}$$

Hence, all inequalities above must be satisfied with equality, thus implying $S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{F}$ and $y(A(S_1 \setminus S_2, S_2 \setminus S_1)) = y(A(S_2 \setminus S_1, S_1 \setminus S_2)) = 0$. Finally, because $y > 0$, we have $A(S_1 \setminus S_2, S_2 \setminus S_1) = A(S_2 \setminus S_1, S_1 \setminus S_2) = \emptyset$, as desired. □

We are now ready to show that every constraint of (5.16) is implied by (5.17). Again, by sake of contradiction assume that this is not the case. Let $S \in \mathcal{F}$ be a set such that the constraint

$x(\delta^-(S)) = 1$ is not implied by (5.17), and among all such sets we choose one such that the number of sets in $\mathcal{H}$ that are intersecting with S is minimum. We define

$$Q = \text{span}(\{\chi^{\delta^-(H)} : H \in \mathcal{H}\}) \ .$$

Hence, $S \in \mathcal{F}$ satisfies $\chi^{\delta^-(S)} \notin Q$. Let $H \in \mathcal{H}$ be a set such that S and H are intersecting. By Lemma 5.28 we have

$$\underbrace{\chi^{\delta^-(S)}}_{\notin Q} + \underbrace{\chi^{\delta^-(H)}}_{\in Q} = \chi^{\delta^-(S\cup H)} + \chi^{\delta^-(S\cap H)} \ .$$

Hence, at least one of $\chi^{\delta^-(S\cup H)}$ and $\chi^{\delta^-(S\cap H)}$ is not in Q. Since both $S \cup H$ and $S \cap H$ are intersecting with a strictly smaller number of sets in $\mathcal{H}$ than S, this contradicts the choice of S and finishes the proof, implying that each constraint of (5.16) is implied by (5.17).

5.8.3 Upper bound on number of edges of minimally k-edge-connected graphs

So far, we used combinatorial uncrossing to prove that certain polyhedra are integral. More precisely, we had inequality descriptions of the polyhedra where constraints where defined by sets, and we considered a maximal laminar family of sets that corresponded to tight constraints. Laminarity of the set system allowed us to show that the corresponding constraint matrix is TU, and combinatorial uncrossing arguments were employed to argue that they are a full-rank system. We now consider a quite different setting, not immediately related to polyhedra, where combinatorial uncrossing can be applied. This highlights nicely the versatility of combinatorial uncrossing, which goes far beyond the study of polyhedra.

Let $G = (V, E)$ be an undirected graph on n vertices. We recall that G is k-edge-connected for $k \in \mathbb{Z}_{\geq 0}$ if there are k edge-disjoint paths between any pair of vertices. By Menger's Theorem—a special case of the max-flow min-cut theorem—this is equivalent to the property that every cut $S \subsetneq V, S \neq \emptyset$ has size at least k, i.e., $|\delta(S)| \geq k$. A k-edge-connected graph G is called *minimally* k-edge-connected if removing any edge from G leads to a graph that is not k-edge-connected anymore. Hence, a graph is minimally k-edge-connected if and only if each edge is in a cut of size k, which is the size of a minimum cut because G is k-edge-connected. We are interested in determining the maximum number of edges that a minimally k-edge-connected graph on n vertices can have.

Figure 5.8 shows an example of a minimally k-edge-connected graph with $k \cdot (n-1)$ edges. We will show that this number is tight using combinatorial uncrossing. The notion of *crossing sets* is key in the proof.

Definition 5.29: Crossing sets

Two sets $S_1, S_2 \subseteq V$ are called *crossing* if none of the following sets is empty: $S_1 \cap S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$, and $V \setminus (S_1 \cup S_2)$.

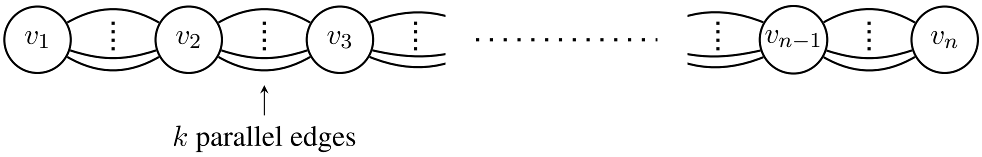


Figure 5.8: Example of a minimally k-edge-connected graph with n vertices and $k \cdot (n-1)$ edges.

In other words, two sets $S_1, S_2 \subseteq V$ are crossing if they are intersecting and $S_1 \cup S_2 \neq V$. One key property of crossing sets is that their intersection and union can be interpreted as cuts. (We recall that $S \subseteq V$ is a cut if $\emptyset \neq S \neq V$.)

Using combinatorial uncrossing, we now show a tight upper bound on the number of edges of a minimally k-edge-connected graph.

Theorem 5.30

Let $G = (V, E)$ be a minimally k-edge-connected graph. Then $|E| \leq k \cdot (|V| - 1)$.

Proof. We call a family $\mathcal{H} \subseteq 2^V$ of minimum cuts in G a *certifying family* if each edge is in at least one of them, i.e., for every $e \in E$ there is a cut $S \in \mathcal{H}$ such that $e \in \delta(S)$ and for each $S \in \mathcal{H}$ we have $|\delta(S)| = k$. We choose the name certifying family because $\mathcal{H}$ certifies that the graph is minimally k-edge-connected. Our goal is to construct a small certifying family $\mathcal{H}$ of size $|\mathcal{H}| \leq n - 1$, where $n := |V|$. Notice that the existence of such a small certifying family proves the lemma since $E \subseteq \bigcup_{S \in \mathcal{H}} \delta(S)$, and thus $|E| \leq (n-1) \cdot k$, because all cuts in $\mathcal{H}$ have size k since they are minimum.

Notice that for each cut $S \subseteq V$, also the set $V \setminus S$ is a cut containing the same edges. We can therefore choose an arbitrary vertex $r \in V$ and focus only on minimum cuts not containing r. An advantage of doing so is that whenever two cuts not containing r are intersecting, then they are also crossing.

To obtain $\mathcal{H}$, we start with an arbitrary certifying family $\mathcal{F} \subseteq 2^V$, which can be obtained as follows. For each $e \in E$, let

$$\mathcal{C}_e = \{S \subseteq V \setminus \{r\} : e \in \delta(S), |\delta(S)| = k\} \ .$$

By minimality of G we have $\mathcal{C}_e \neq \emptyset$ for every $e \in E$; for otherwise, e could be removed without destroying k-edge-connectivity. For each $e \in E$, let $S_e \in \mathcal{C}_e$ be an arbitrary cut in $\mathcal{C}_e$. We define

$$\mathcal{F} = \{S_e : e \in E\} \ .$$

Hence, $\mathcal{F}$ may contain up to $|E|$ sets, but possibly fewer, since we may have $S_e = S_{e'}$ for two different edges $e, e' \in E$.

In the same way we proved Lemma 5.28 one can show that for any two minimum cuts S_1, S_2 in G that are crossing, also $S_1 \cup S_2$ and $S_1 \cap S_2$ are minimum cuts, and furthermore $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$. Since all cuts in $\mathcal{F}$ are minimum, we can apply this reasoning to any two cuts in $\mathcal{F}$. We first want to turn $\mathcal{F}$ into a laminar certifying family. Assume that $\mathcal{F}$ is not laminar. Hence, there are two intersecting cuts $S_1, S_2 \in \mathcal{F}$, which are also crossing because $r \notin S_1 \cup S_2$.

Let
$$\mathcal{F}' = (\mathcal{F} \setminus \{S_1, S_2\}) \cup \{S_1 \cup S_2, S_1 \cap S_2\} \ .$$
Thus, $\mathcal{F}'$ is still a family of minimum cuts. Furthermore, since $E(S_1 \setminus S_2, S_2 \setminus S_1) = \emptyset$, we have
$$\delta(S_1) \cup \delta(S_2) = \delta(S_1 \cup S_2) \cup \delta(S_1 \cap S_2) \ .$$
Hence, $\mathcal{F}'$ remains a certifying family. We claim that if we repeat this uncrossing procedure as long as there are at least two intersecting cuts in our certifying family, we will eventually end up with a laminar family of cuts. To see that this uncrossing procedure will stop, we introduce a potential function ϕ for certifying families:
$$\phi(\mathcal{F}) = \sum_{S \in \mathcal{F}} |S| \cdot |\overline{S}| \ ,$$
where $\overline{S} = V \setminus S$. Notice that for any two intersecting sets $S_1, S_2 \in \mathcal{F}$, we have
$$|S_1||\overline{S_1}| + |S_2||\overline{S_2}| > |S_1 \cup S_2||\overline{S_1 \cup S_2}| + |S_1 \cap S_2||\overline{S_1 \cap S_2}| \ . \tag{5.19}$$
Hence, whenever we do uncrossing to go from a certifying family $\mathcal{F}$ to another one $\mathcal{F}'$, we have $\phi(\mathcal{F}) > \phi(\mathcal{F}')$. Notice that the potential may even decrease more than by the slack in the inequality (5.19), because the uncrossing may lead to sets that already exist in the family, in which case also the size of the family decreases. In any case, the potential strictly decreases after each uncrossing step. Since the potential is by definition non-negative for any certifying family and decreases by at least one unit at each step, the uncrossing procedure must stop. Hence, there are no two intersecting sets in the final certifying family $\mathcal{L}$, and thus, $\mathcal{L}$ is laminar. Recall from the problem sets that a laminar family (not containing the empty set) on a ground set of size n may still have up to $2n-1$ sets. Notice, however, that since $\mathcal{L}$ is a laminar family over the set $V \setminus \{r\}$, which has size $n-1$, we actually have $|\mathcal{L}| \leq 2n-3$. We further purge the family $\mathcal{L}$ to obtain $\mathcal{H}$ as follows. As long as there is a set $L \in \mathcal{L}$ such that the children $L_1, \dots, L_k$ of L in the family $\mathcal{L}$ form a partition of L, then we remove the set L from $\mathcal{L}$. We call this constellation of sets $L, L_1, \dots, L_k$ an *obstruction*. The resulting family will still be a certifying family since any edge contained in $\delta(L)$ is also contained in precisely one of the cuts $L_1, \dots, L_k$, i.e., $\delta(L) \subseteq \bigcup_{i=1}^k \delta(L_i)$. Let $\mathcal{H}$ be the resulting family, which does not contain any obstructions anymore. Notice that $\mathcal{H}$ is a laminar family over $V \setminus \{r\}$. Every laminar family $\mathcal{H}'$ on $n-1$ elements without obstructions and not containing the empty set satisfies $|\mathcal{H}'| \leq n-1$, since for each set $H' \in \mathcal{H}'$, there is an element of the ground set such that H' is the smallest set containing this element. Thus, $\mathcal{H}$ can have at most as many sets as there are elements in $V \setminus \{r\}$, which implies $|\mathcal{H}| \leq n-1$ and completes the proof.

□