5.4 Bipartite matching polytope

$G = (V, E)$

$\mathcal{M} \subseteq 2^E$: all matchings in G

Theorem 5.12

The bipartite matching polytope $P_{\mathcal{M}}$ is given by

$$P_{\mathcal{M}} = \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \ \forall v \in V\} \ . \tag{5.7}$$

We prove the statement by showing (ii) and (iii) of the "recipe".

<u>Proof of point (ii)</u> ← $P_{\mathcal{M}}$ contains correct set of integral points.

$[P_{\mathcal{M}} \cap \{0,1\}^E = \{\chi^M : M \in \mathcal{M}\}$

Let $F \subseteq E$.

$$\begin{aligned} F \text{ is a matching} &\iff |F \cap \delta(v)| \leq 1 \quad \forall v \in V \\ &\iff \chi^F(\delta(v)) \leq 1 \quad \forall v \in V \\ &\iff \chi^F \in P \ . \end{aligned}$$

To show (iii), we see 2 approaches:

integrality of $\{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \ \forall v \in V\} =: P$

- one uses TU-ness;
- one shows that P has no fractional extreme point. (see script)

5.4.1 Integrality through TU-ness

$$P = \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \quad \forall v \in V\} = \{x \in \mathbb{R}^E : Ax \leq b,\ x \geq 0\}$$

$\rightsquigarrow$ We will show that A is TU and then invoke:

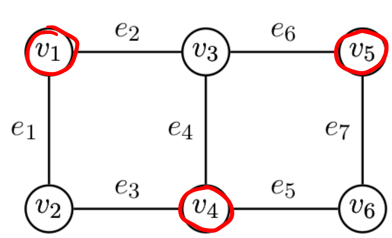
Theorem 5.8

Let $A \in \mathbb{Z}^{m \times n}$. Then,

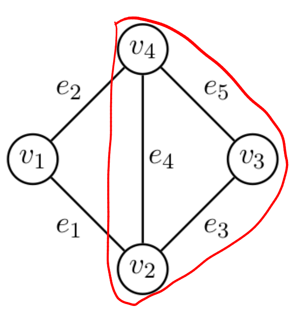
$$A \text{ is TU} \quad \Leftrightarrow \quad P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \text{ is integral } \forall\, b \in \mathbb{Z}^m.$$

$A \in \mathbb{R}^{V \times E}$ is vertex-edge incidence matrix of G:

$$A(v,e) = \begin{cases} 1 & \text{if } v \in e \quad (e \in \delta(v)) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, v \in V, e \in E$$



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x(e_1) \\ x(e_2) \\ \vdots \\ \vdots \\ x(e_7) \end{pmatrix} \leq 1$$



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

det $= -2$

Theorem 5.13

Let $G = (V, E)$ be an undirected graph with vertex-edge incidence matrix A. Then,

$$G \text{ is bipartite} \Leftrightarrow A \text{ is TU}.$$

Proof

$\Leftarrow$) Left as an excercise.

$\Rightarrow$) We use characterization of Ghouila-Houri wrt rows:

Let $R \subseteq V$ (subset of rows of A).

Let $V = X \,\dot\cup\, Y$ be bipartition s.t. every edge goes between X and Y.

We set $R_1 = R \cap X$,

$R_2 = R \cap Y$.

Let $e \in E$.

$$\underbrace{\overbrace{\sum_{v \in R_1} A(v,e)}^{\in \{0,1\}}}_{\leq \sum_{v \in X} A(v,e) = \sum_{v \in X} 1_{\{v \in e\}} = 1} - \underbrace{\overbrace{\sum_{v \in R_2} A(v,e)}^{\in \{0,1\}}}_{\leq \sum_{v \in Y} A(v,e) = \sum_{v \in Y} 1_{\{v \in e\}} = 1} \in \{-1, 0, 1\}$$

#

Recall: vertex cover in bipartite graphs:

$$P = \{x \in [0,1]^V : x(u) + x(v) \geq 1 \quad \forall \{u,v\} \in E\} = \{x \in [0,1]^V : A^T x \geq 1\}$$

vertex-edge inc. matrix

$$= \left\{x \in \mathbb{R}^V : x \geq 0, \ \underbrace{\begin{pmatrix} -A^T \\ I \end{pmatrix}}_{} x \leq \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$

TU matrix because A is TU

5.4.3 Some implications coming from inequality description of $P_{\mathcal{M}}$

Perfect bipartite matching polytope

Theorem 5.14

The perfect matching polytope of a bipartite graph $G = (V, E)$ is given by

$$P = \left\{x \in \mathbb{R}^E_{\geq 0} \colon x(\delta(v)) = 1 \ \forall v \in V\right\} \ .$$

Proof

P contains correct set of integral points. ✓

P is either empty or a face of $P_{\mathcal{M}} = \{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) \leq 1 \ \forall v \in V\}$.

Proposition 1.13

$P_{\mathcal{M}}$ is integral $\Longrightarrow$ P is integral.

Corollary 1.14

Let P be a polyhedron. Then a face of a face of P is itself a face of P.

#

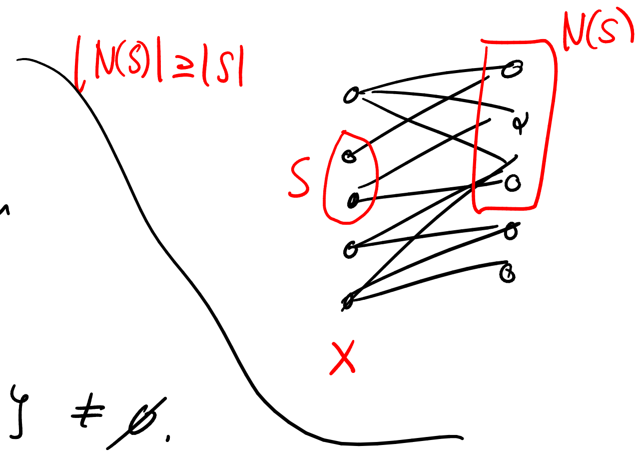
Perfect matchings in bipartite d-regular graphs

Theorem 5.15

Let $d \in \mathbb{Z}_{\geq 1}$. Every d-regular bipartite graph admits a perfect matching.

Proof

Existence of perfect matching in G is equivalent to



$$P := \{ x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) = 1 \quad \forall v \in V \} \neq \emptyset.$$

$\rightsquigarrow$ This is the case because:

$$\frac{1}{d} \cdot \chi^E \in P.$$

#

5.5 Polyhedral description of short s-t paths

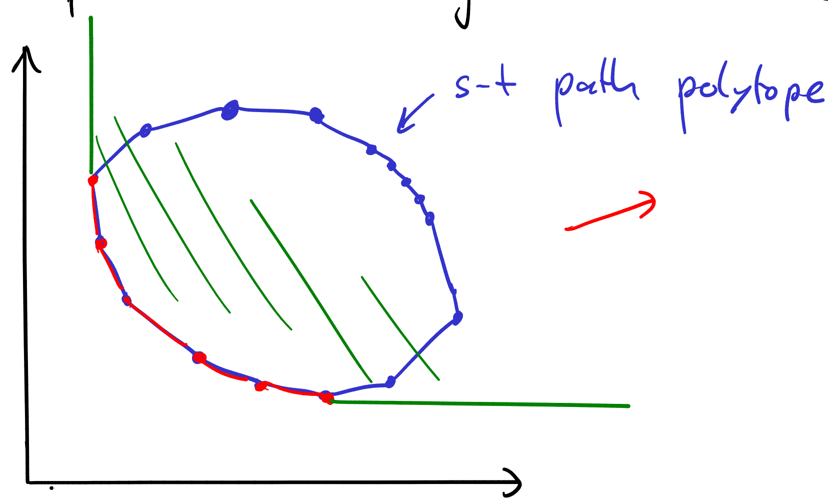
Consider directed graph $G=(V,A)$ and $s,t \in V$, $s \neq t$.

There is little hope to find useful description of s-t path polytope.

↳ Ability to solve LPs over that polytope allows for finding longest s-t paths.

→ This is NP-hard.

However, in shortest path problems, we typically have non-negative (or even positive) arc length $\ell : A \to \mathbb{R}_{\geq 0}$.

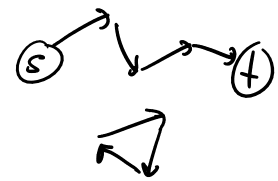


Consider :

$$P = \left\{ x \in [0,1]^A \;\middle|\; x(\delta^+(v)) - x(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \end{cases} \quad \forall v \in V \right\}.$$

This is s-t flow polytope of unit flow.

$P \cap \{0,1\}^A$ corresponds to disjoint union of :



- One s-t path, and
- any number of cycles.

P describes combinatorial polytope of these sets.

Assume we minimize some function $\ell : A \to \mathbb{R}_{>0}$ over P.

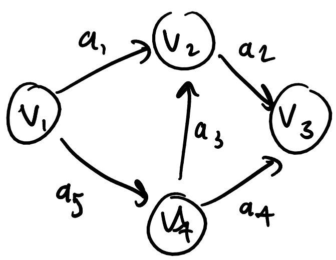
$$P = \left\{ x \in [0,1]^A \;\middle|\; x(\delta^+(v)) - x(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \end{cases} \quad \forall v \in V \right\}$$

$$P = \{ x \in \mathbb{R}^A : Dx = b,\ 0 \le x \le \mathbb{1} \}, \text{ where}$$

$$b \in \{-1,0,1\}^V \qquad b(v) = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

$$D \in \{-1,0,1\}^{V \times A} \quad \text{with} \quad D(v,a) = \begin{cases} 1 & \text{if } a \in \delta^+(v) \\ -1 & \text{if } a \in \delta^-(v) \\ 0 & \text{otherwise} \end{cases}$$

↑ vertex-arc incidence matrix



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notice $\quad P = \left\{ x \in \mathbb{R}^A : \begin{pmatrix} D \\ -D \\ I \end{pmatrix} x \le \begin{pmatrix} b \\ -b \\ \mathbb{1} \end{pmatrix},\ x \ge 0 \right\}$

$$D \text{ is TU} \implies \begin{pmatrix} D \\ -D \\ I \end{pmatrix} \text{ is TU}$$

Theorem 5.16

The vertex-arc incidence matrix $D \in \{-1, 0, 1\}^{V \times A}$ of any directed (loopless) graph $G = (V, A)$ is TU.

5.6 Spanning trees and r-arborescences

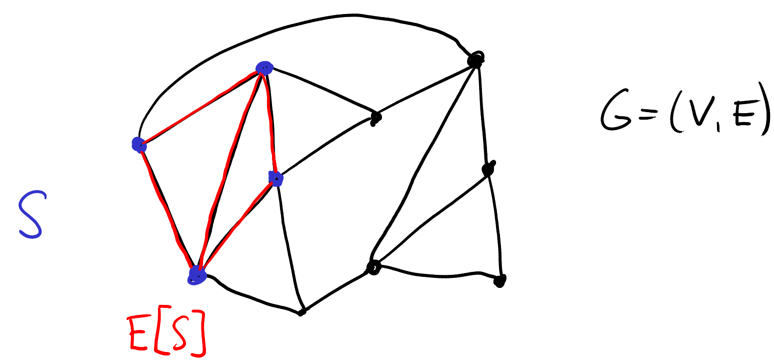
5.6.1 Spanning tree polytope

Theorem 5.17

The spanning tree polytope of an undirected loopless graph $G = (V, E)$ is given by

$$P = \left\{ x \in \mathbb{R}^E_{\geq 0} \;\middle|\; \begin{array}{ll} x(E) = |V| - 1 & \\ x(E[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \end{array} \right\} .$$

All edges with both endpoints in S.



→ Exponentially many constraints.

→ Problem sets : All constraints can be facet defining (depending on input graph G).

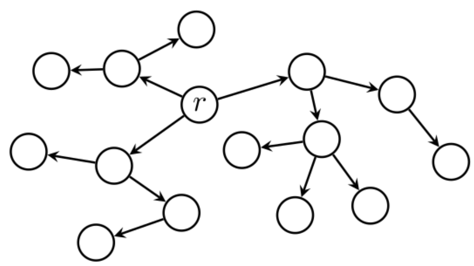
5.6.2 The r-arborescence polytope

Definition 5.18: Arborescence, r-arborescence

Let $G = (V, A)$ be a directed graph. An *arborescence* in G is an arc set $T \subseteq A$ such that

(i) T is a spanning tree (when disregarding the arc directions), and
(ii) there is one vertex r from which all arcs are directed away, i.e., every vertex $v \in V$ can be reached from r using a directed path in T.

The vertex r in condition (ii) is called the *root* of the arborescence, and T is called an r*-arborescence*.



Theorem 5.19

The arborescence polytope of a directed loopless graph $G = (V, A)$ is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} \;\middle|\; \begin{array}{ll} x(A) = |V| - 1 & \\ x(A[S]) \leq |S| - 1 & \forall S \subsetneq V, |S| \geq 2 \\ x(\delta^-(v)) \leq 1 & \forall v \in V \end{array} \right\} ,$$

where $A[S] \subseteq A$ for $S \subseteq V$ denotes all arcs with both endpoints in S.

Theorem 5.20

The dominant of the r-arborescence polytope is given by

$$P = \left\{ x \in \mathbb{R}^A_{\geq 0} : x(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \right\} .$$