Algorithm 7: Edmonds-Karp algorithm

Input : Directed graph $G = (V, A)$ with arc capacities $u\colon A \to \mathbb{Z}_{\geq 0}$ and $s, t \in V$, $s \neq t$.
Output: A maximum s-t flow f.

$f(a) = 0 \ \ \forall a \in A$.
while $\exists f$*-augmenting path in* G_f **do**
  Find an f-augmenting path P in G_f minimizing $|P|$.
  Augment f along P and set f to augmented flow.
return f

Lemma 4.33

Let $G = (V, A)$ be a directed graph with arc capacities $u\colon A \to \mathbb{Z}_{\geq 0}$, and let $s, t \in V$ with $s \neq t$. Moreover, let f_1 be an s-t flow in G, and let f_2 be an s-t flow obtained by augmenting f_1 along a shortest augmenting path P in G_{f_1}. Then,

$$\begin{aligned} d_{f_1}(s, v) &\leq d_{f_2}(s, v) \quad \forall v \in V \ \ , \text{ and} \\ d_{f_1}(v, t) &\leq d_{f_2}(v, t) \quad \forall v \in V \ \ , \end{aligned}$$

where $d_f(v, w)$ denotes, for $v, w \in V$ and an s-t flow f, the length (in terms of number of arcs) of a shortest v-w path in G_f that only uses arcs with strictly positive f-residual capacity.

Theorem 4.34

Algorithm 7 runs in $O(nm^2)$ time.

Proof

Lemma 4.33 implies that augmenting paths have non-decreasing lengths throughout algo.

We can divide Edmonds-Karp algo into phases.

phase k: all augmentations on augmenting paths of length k

phases : $O(n)$.

We finish proof by showing that each phase performs $O(m)$ augmentations.

This proves statement because each augmentation takes $O(m)$ time.

Consider phase $k \in [n-1]$.

For an s-t flow f and $v, w \in V$, let

$$d_f(v,w) := \min\{|P| : \underset{\substack{\uparrow \\ G_f = (V,B)}}{P \subseteq B} \text{ is } v\text{-}w \text{ path with } u_f(b) > 0 \ \forall b \in P\}.$$

Claim

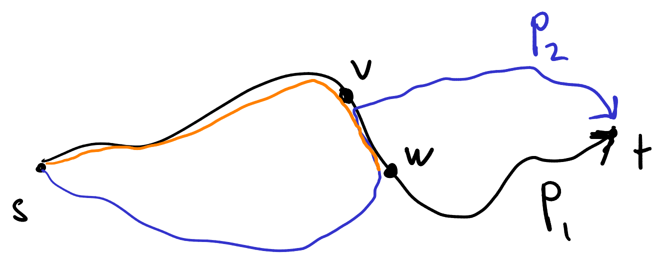
If an arc is used in an augmenting path in phase k, then its reverse arc is not used in any augmentation in phase k.

Proof of claim

Assume by sake of contradiction that $\exists\ (v,w) \in B$ s.t.

(i) (v,w) is used in a phase k augmenting path P_1 to augment the flow f_1.

(ii) (w,v) is part of a later augmenting path P_2 to augment some flow f_2.



$$|P_2| = \underbrace{d_{f_2}(s,w)}_{\geq d_{f_1}(s,w)} + 1 + \underbrace{d_{f_2}(v,t)}_{\geq d_{f_1}(v,t)} \geq \underbrace{d_{f_1}(s,w)}_{= d_{f_1}(s,v)+1} + 1 + d_{f_1}(v,t)$$

$$= d_{f_1}(s,v) + 2 + d_{f_1}(v,t) = |P_1| + 2$$

↯ Because $|P_1| = |P_2| = k$, as both augmentations happen in phase k. $\#_{\text{claim}}$

Claim implies:

In each phase, for every arc $a \in A$, augmentations either never use a or never use a^R.

Hence, once an arc becomes saturated, neither the arc nor its reverse version is used in same phase.

$\Longrightarrow$ # of times an arc gets saturated $\leq m$.

Each augmentation saturates at least one arc, by the way we define the augmentation volume.

$\Longrightarrow$ # augmentations in phase k $\leq m$.

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