Model Predictive Control

Chapter 6: Feasibility and Stability

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Receding Horizon Control: The Motivation

$$x(k+1) = g(x(k), u(k)) \qquad\qquad x, u \in \mathcal{X}, \mathcal{U}$$

Design control law $u(k) = \kappa(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$
2. Is stable: $\lim_{k\to\infty} x(k) = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x(0) \mid \text{Conditions 1-3 are met}\}$

In this lecture, we will demonstrate that these objectives can be met in a predictive control framework.

Learning Objectives

- Contrast stability properties of LQR and MPC for constrained problems
- Understand why MPC by itself does not provide guarantees on stability and constraint satisfaction
- Pose sufficient conditions and prove guarantees on stability and constraint satisfaction

Outline

1. MPC: Key Points Illustrated

2. Loss of Feasibility and Stability in MPC

3. Feasibility and Stability Guarantees in MPC

4. Extension to Nonlinear MPC

Constrained Infinite Time Optimal Control (what we would like to solve)

$$
\begin{aligned}
J^\star_\infty(x(0)) = \min_{u(\cdot)} & \sum_{i=0}^{\infty} l(x_i, u_i) \\
\text{subj. to } & x_{i+1} = Ax_i + Bu_i, \ \ i = 0, \ldots, \infty \\
& x_i \in \mathcal{X}, u_i \in \mathcal{U}, i = 0, \ldots, \infty \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $l(x, u)$: "cost" of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ... but we can't compute it: there are an **infinite number of variables**

Constrained Finite Time Optimal Control (what we can sometimes solve)

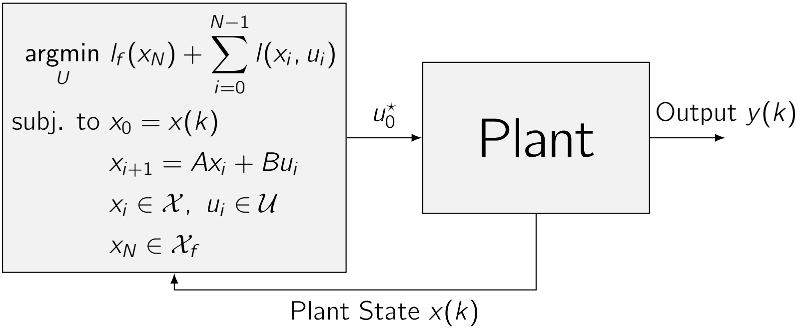
$$
\begin{aligned}
J^{\star}_{k\to k+N|k}(x(k)) = \min_{U_{k\to k+N|k}} \;& l_f(x_{k+N|k}) + \sum_{i=0}^{N-1} l(x_{k+i|k}, u_{k+i|k}) \\
\text{subj. to } & x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k},\;\; i = 0, \ldots, N-1 \\
& x_{k+i|k} \in \mathcal{X}, u_{k+i|k} \in \mathcal{U}, i = 0, \ldots, N-1 \\
& x_{k+N|k} \in \mathcal{X}_f \\
& x_{k|k} = x(k)
\end{aligned}
\tag{1}
$$

where $U_{k\to k+N|k} = \{u_{k|k}, \ldots, u_{k+N-1|k}\}$.

Truncate after a finite horizon:

- $l_f(x_{k+N|k})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(k)$
- Find the optimal input sequence for the entire planning window N: $U^\star = \{u_0^\star, u_1^\star, \ldots, u_{N-1}^\star\}$
- Implement only the first control action $u_0^\star$

Outline

1. MPC: Key Points Illustrated

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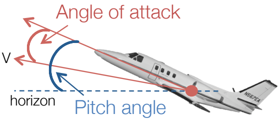
Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad/s ($\pm 60°$/s), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_\infty^\star(x(k)) = \min \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i$$
$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$
$$x_0 = x(k)$$

MPC

$$J^\star(x(k)) = \min_U \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{subj. to } x_{i+1} = Ax_i + Bu_i$$
$$x_i \in \mathcal{X},\ u_i \in \mathcal{U}$$
$$x_0 = x(k)$$

Assume: $Q = Q^T \succeq 0$, $R = R^T \succ 0$

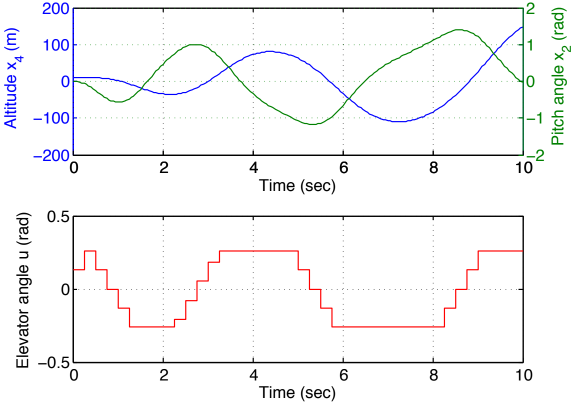
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$



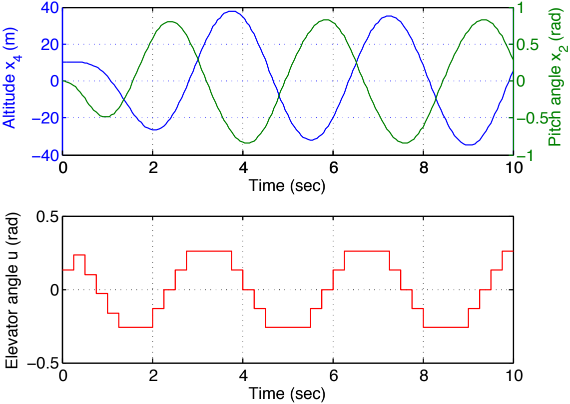
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_k| \leq 0.262$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

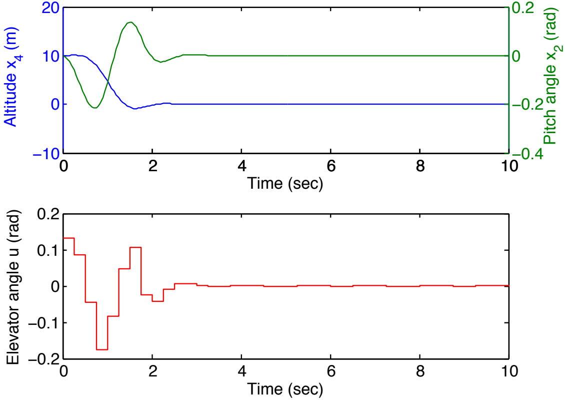
$\Rightarrow$ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_k| \leq 0.262$
and rate constraints $|\dot{u}_k| \leq 0.349$
approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I,\ R = 10,\ N = 10}$



The MPC controller considers all constraints on the actuator

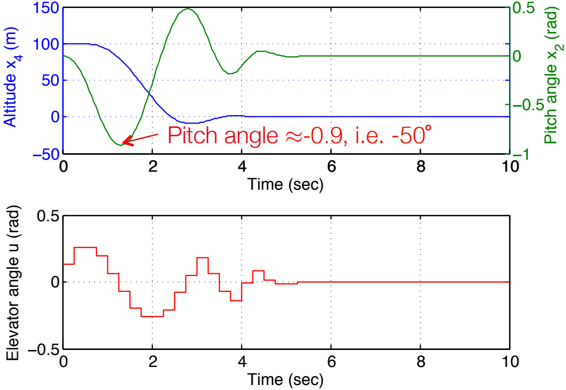
- Closed-loop system is stable
- Efficient use of the control authority

Example: Inclusion of state constraints

MPC controller with input constraints $|u_k| \leq 0.262$ and rate constraints $|\dot{u}_k| \leq 0.349$ approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec}, \\ Q = I,\ R = 10,\ N = 10\end{array}}$



Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e.
$x_0 = [0; 0; 0; 100]$

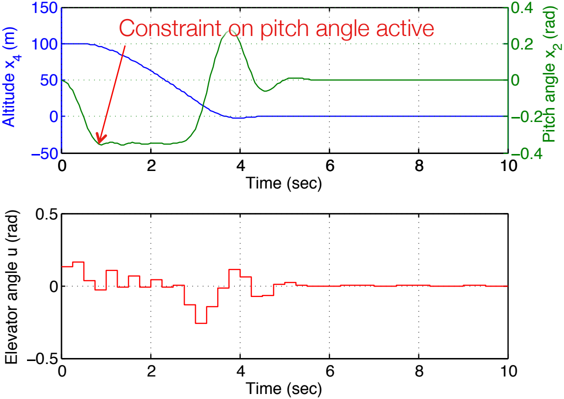
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_k| \leq 0.262$ and rate constraints $|\dot{u}_k| \leq 0.349$ approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



Add state constraints for passenger comfort:

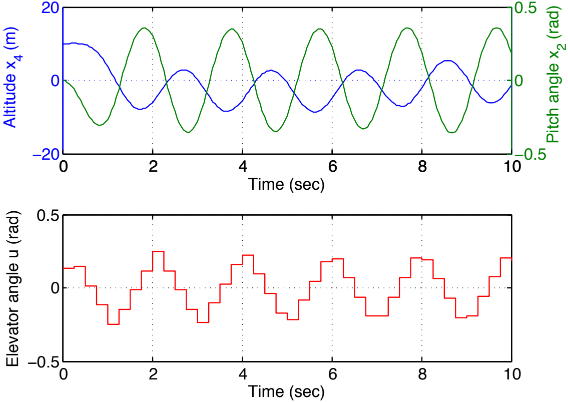
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_k| \leq 0.262$
and rate constraints $|\dot{u}_k| \leq 0.349$
approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I, \; R = 10, \; N = 4}$



Decrease in the prediction horizon causes loss of the stability properties

Outline

Loss of Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Example: Loss of feasibility - Double Integrator

Consider the double integrator

$$\begin{cases} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases}$$

subject to the input constraints

$$-0.5 \leq u(k) \leq 0.5$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Compute a receding horizon controller with quadratic objective with

$$N = 3,\ P = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 10.$$

Example: Loss of feasibility - Double Integrator

The QP problem associated with the RHC is

$$H = \begin{bmatrix} 13.50 & -10.00 & -0.50 \\ -10.00 & 22.00 & -10.00 \\ -0.50 & -10.00 & 31.50 \end{bmatrix}, \; F = \begin{bmatrix} -10.50 & 10.00 & -0.50 \\ -20.50 & 10.00 & 9.50 \end{bmatrix}, \; Y = \begin{bmatrix} 14.50 & 23.50 \\ 23.50 & 54.50 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.50 & -1.00 & 0.50 \\ -0.50 & 1.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ -0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.50 & 0.00 & 0.50 \\ -1.00 & 0.00 & 0.00 \\ 0.00 & -1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 \\ 0.00 & 0.00 & 0.00 \\ -0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.50 & 0.00 & -0.50 \\ -0.50 & 0.00 & 0.50 \\ 0.50 & 0.00 & -0.50 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}, \; E = \begin{bmatrix} 0.50 & 0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ -0.50 & -0.50 \\ -0.50 & -0.50 \\ 0.50 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 1.00 & 1.00 \\ -0.50 & -0.50 \\ -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.50 & -1.50 \\ 0.50 & 1.50 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.00 & -1.00 \end{bmatrix}, \; w = \begin{bmatrix} 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \\ 0.50 \\ 0.50 \\ 5.00 \\ 5.00 \\ 5.00 \\ 5.00 \end{bmatrix}$$

Example: Loss of feasibility - Double Integrator

1) MEASURE the state $x(k)$ at time instance k
2) OBTAIN $U^\star(x(k))$ by solving the CFTOC
3) IF $U^\star(x(k)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element $u_0^\star$ of $U^\star$ to the system
5) WAIT for the new sampling time $k+1$, GOTO 1)

Time step 1:

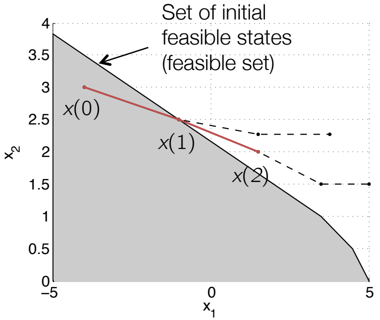
$$x_0 = [-4; 3], \quad u_0^\star(x(0)) = -0.5$$

Time step 2:

$$x_0 = [-1; 2.5], \quad u_0^\star(x(1)) = -0.5$$

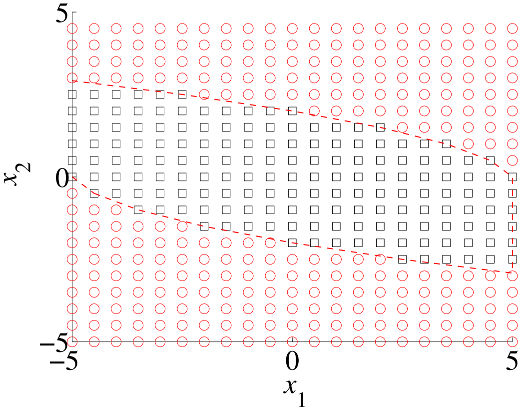
Time step 3:

$x_0 = [1.5; 2]$, Problem infeasible



Depending on initial condition, closed loop trajectory may lead to states for which optimization problem is infeasible.

Example: Loss of feasibility - Double Integrator



Boxes (**Circles**) are initial points leading (not leading) to feasible closed-loop trajectories

Example: Feasibility and stability are function of tuning

Unstable system $x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$

Input constraints $-1 \leq u(k) \leq 1$

State constraints $\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, Parameters: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Investigate the stability properties for different horizons N and weights R by solving the finite-horizon MPC problem in a receding horizon fashion...

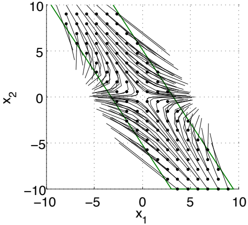
Example: Feasibility and stability are function of tuning

1. $R = 10$, $N = 2$: all trajectories unstable.
2. $R = 2$, $N = 3$: some trajectories stable.
3. $R = 1$, $N = 4$: more stable trajectories.

$*$ Initial points with convergent trajectories

$\circ$ Initial points that diverge

①



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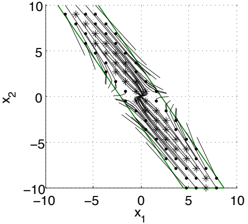

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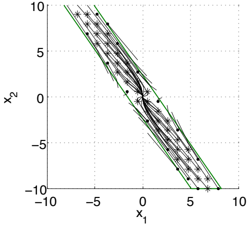
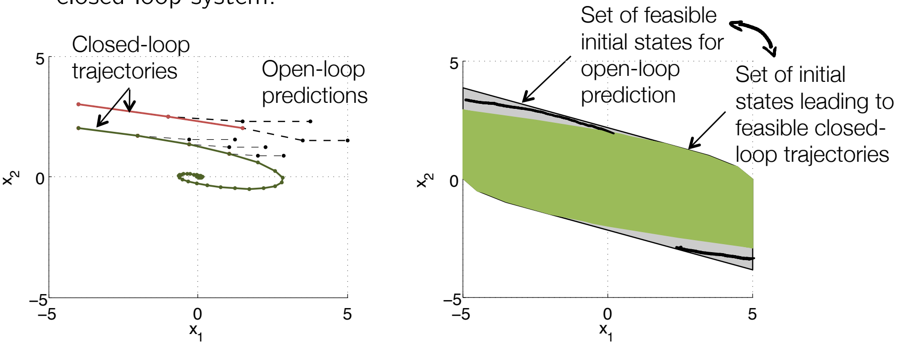


Green lines denote the set of all feasible initial points. They depend on the horizon N but not on the cost $R \Longrightarrow$ Parameters have complex effect and trajectories.

Summary: Feasibility and Stability

Problems originate from the use of a 'short sighted' strategy

⇒ Finite horizon causes deviation between the open-loop prediction and the closed-loop system:



Ideally we would solve the MPC problem with an infinite horizon, but that is computationally intractable

⇒ Design finite horizon problem such that it approximates the infinite horizon

Summary: Feasibility and Stability

- Infinite-Horizon
  If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence
  - If problem is feasible, the closed loop trajectories will be always feasible
  - If the cost is finite, then states and inputs will converge asymptotically to the origin

- Finite-Horizon
  RHC is "short-sighted" strategy approximating infinite horizon controller. But
  - **Feasibility**. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
  - **Stability**. The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J^\star(x(k)) = \quad & \min_{U} \quad l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) && \text{Terminal Cost} \\
& \text{subj. to} \\
& \quad x_{i+1} = Ax_i + Bu_i,\ \ i = 0, \ldots, N-1 \\
& \quad x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \ldots, N-1 \\
& \quad x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& \quad x_0 = x(k)
\end{aligned}
$$

$l_f(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

Outline

Feasibility and Stability of MPC: Proof

Main steps: feas. at $x(k)$ => feas. at $x(k+1)$

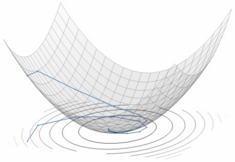
- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Reminder: Lyapunov Stability

Definition: Lyapunov function

Consider the equilibrium point $x = 0$. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. A function $V : \mathbb{R}^n \to \mathbb{R}$, continuous at the origin, finite for every $x \in \Omega$, and such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \in \Omega \setminus \{0\}$$
$$V(g(x)) - V(x) \leq -\alpha(x) \ \forall x \in \Omega \setminus \{0\}$$



where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite,

is called a **Lyapunov function**.

Theorem: Lyapunov stability (asymptotic stability)

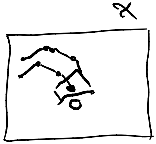
If a system admits a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** in Ω.

Feasibility and Stability of MPC: Proof

Main steps:

- Prove recursive feasibility by showing the existence of a feasible control sequence at all time instants when starting from a feasible initial point
- Prove stability by showing that the optimal cost function is a Lyapunov function

Two cases:



1. Terminal constraint at zero: $x_N = 0$
2. Terminal constraint in some (convex) set: $x_N \in \mathcal{X}_f$

General notation: Regulation to origin

$$J^\star(x(k)) = \min_U \quad \underbrace{l_f(x_N)}_{\text{terminal cost}} + \sum_{i=0}^{N-1} \underbrace{l(x_i, u_i)}_{\text{stage cost}}$$

$l(x,u) > 0$ for $x,u \neq 0$, $l(0,0) = 0$

$0 \in \text{int}\,\mathcal{X}$ $\quad 0 \in \text{int}\,\mathcal{U}$

Outline

3. Feasibility and Stability Guarantees in MPC

    Proof for $\mathcal{X}_f = 0$

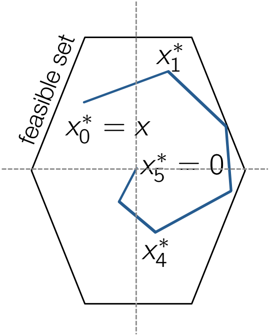
    General Terminal Sets

    Example

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

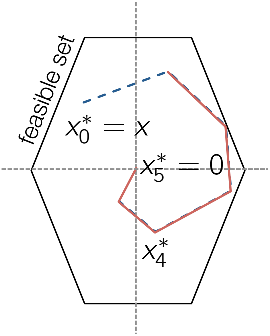
- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \ldots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and let $\{x(k),\ x_1^*,\ \ldots,\ x_N^*\}$ be the corresponding state trajectory $\quad x_N^* = 0$ by term constr.
- Apply $u(k) = u_0^\star$ and let system evolve to $x(k+1) = Ax(k) + Bu(k) = x^*_{1|k}$



Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and let $\{x(k),\ x_1^*,\ \dots,\ x_N^*\}$ be the corresponding state trajectory
- Apply $u(k) = u_0^\star$ and let system evolve to $x(k+1) = Ax(k) + Bu(k)$
- At $x(k+1) = x_1^\star$ the control sequence $\tilde{U} = \{\underbrace{u_1^\star,\ u_2^\star,\ \dots,\ u_{N-1}^\star}_{\in \mathcal{U}},\ \overset{\in \mathcal{U}}{0}\}$ is feasible (apply 0 control input $\Rightarrow A \underbrace{x_N^\star}_{=0} + B \cdot 0 = 0$)

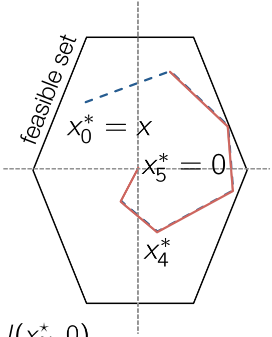


$\tilde{X} = \{\underbrace{x^*_{1|k}, \dots x^*_{N|k}}_{\in \mathcal{X}}, \underset{\in \mathcal{X}_f = 0}{0}\}$ -> not optimal, but feasible

$\Rightarrow$ **Recursive feasibility ✔**

Stability of MPC - Zero terminal state constraint

Terminal constraint: $x_N \in \mathcal{X}_f = 0$



Goal: Show $J^\star(x(k+1)) < J^\star(x(k)) \quad \forall x(k) \neq 0$

$J^*(x(k+1)) - J^*(x(k)) \leq -\alpha(x)$

$$J^\star(x(k)) = \underbrace{l_f(x_N^\star)}_{=0} + \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star)$$

$$J^\star(x(k+1)) \leq \tilde{J}(x(k+1)) = \sum_{i=1}^{N-1} l(x_i^\star, u_i^\star) + l(x_N^\star, 0)$$

$$= \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star) - l(x_0^\star, u_0^\star) + l(x_N^\star, 0)$$

$$= J^\star(x(k)) - \underbrace{l(x(k), u_0^\star)}_{\text{Subtract cost at stage } k} + \underbrace{l(0,0)}_{\text{Add cost for staying at 0=0}}$$

$\Rightarrow$ $J^\star(x)$ is a Lyapunov function $\rightarrow$ (Lyapunov) Stability ✔

Cost of "shifted" candidate solution

$$\tilde{J}(x(k+1)) = \sum_{i=0}^{N-1} l(x_i, \tilde{u}_i)$$

$$= \underbrace{\sum_{i=1}^{N-1} l(x^*_{i|k}, u^*_{i|k}) + l(x^*_{0|k}, u^*_{0|k})}_{} + \underbrace{l(0,0)}_{=0} - l(x^*_{0|k}, u^*_{0|k})$$

$$= J^*(x(k)) - \underbrace{l(x^*_{0|k}, u^*_{0|k})}_{>0}$$

$$J^*(x(k+1)) \underset{\substack{\uparrow \\ \text{optimality}}}{\leq} \tilde{J}(x(k+1)) < J^*(x(k))$$

$$J^*(x(k+1)) - J^*(x(k))$$

$$\leq \tilde{J}(x(k+1)) - J^*(x(k))$$

$$\leq -l(x(k), u^*(k))$$

Example: Impact of Horizon with Zero Terminal Constraint

System dynamics:

$$x(k+1) = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

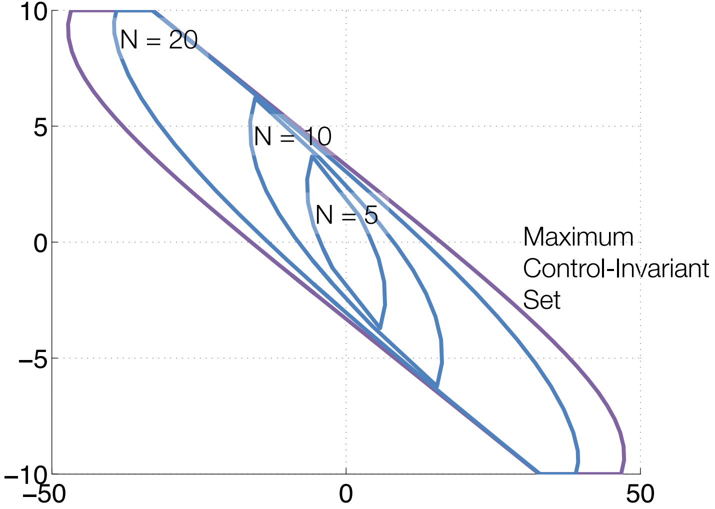
Constraints:

$$\mathcal{X} := \{x \,|\, -50 \leq x_1 \leq 50,\ -10 \leq x_2 \leq 10\} = \{x \,|\, A_x x \leq b_x\}$$
$$\mathcal{U} := \{u \,|\, \|u\|_\infty \leq 1\} = \{u \,|\, A_u u \leq b_u\}$$

Stage cost:

$$l(x_i, u_i) := x_i^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_i + u_i^\top u_i$$

Example: Impact of Horizon with Zero Terminal Constraint



The horizon can have a strong impact on the region of attraction.

Outline

3. Feasibility and Stability Guarantees in MPC

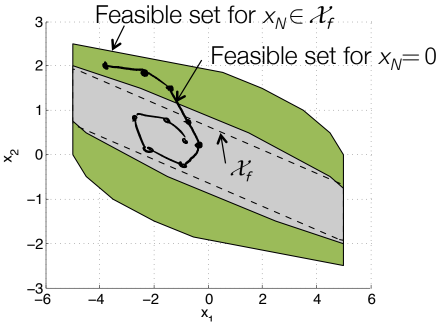
   Proof for $\mathcal{X}_f = 0$

   General Terminal Sets

   Example

Extension to More General Terminal Sets

Problem: The terminal constraint $x_N = 0$ reduces the size of the feasible set
Goal: Use convex set $\mathcal{X}_f$ to increase the region of attraction



Double integrator

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$-0.5 \leq u(k) \leq 0.5$$

$$N = 5, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 10$$

Goal: Generalize proof to the constraint $x_N \in \mathcal{X}_f$

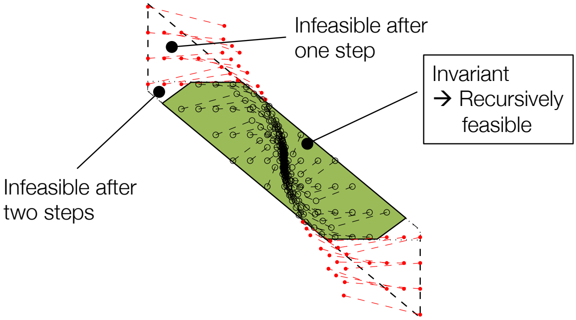
Invariant sets

Definition: Invariant set

A set $\mathcal{O}$ is called **positively invariant** for system $x(k+1) = g_{cl}(x(k))$, if

$$x(0) \in \mathcal{O} \Rightarrow x(k) \in \mathcal{O}, \quad \forall k \in \mathbb{N}_+$$

The positively invariant set that contains every closed positively invariant set is called the maximal positively invariant set $\mathcal{O}_\infty$.



Stability of MPC - Main Result

Assumptions

1. Stage cost is positive definite, i.e. it is strictly positive and only zero at the origin
2. Terminal set is **invariant** under the local control law $\kappa_f(x_i)$:

$$x_{i+1} = Ax_i + B\kappa_f(x_i) \in \mathcal{X}_f, \quad \text{for all } x_i \in \mathcal{X}_f$$

   All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ \kappa_f(x_i) \in \mathcal{U}, \quad \text{for all } x_i \in \mathcal{X}_f$$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$l_f(x_{i+1}) - l_f(x_i) \leq -l(x_i, \kappa_f(x_i)), \quad \text{for all } x_i \in \mathcal{X}_f$$

Under those 3 assumptions:

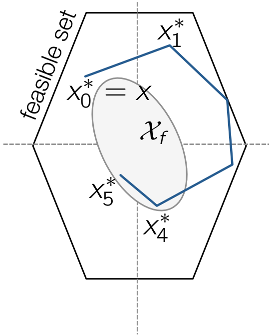
Theorem

The closed-loop system under the MPC control law $u_0^\star(x)$ is asymptotically stable and the set $\mathcal{X}_N$ is positive invariant for the system

$$x(k+1) = Ax(k) + Bu_0^\star(x(k)).$$

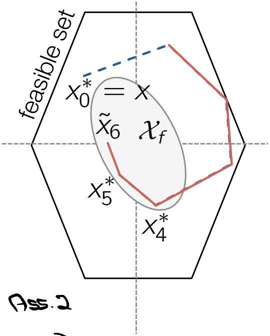
Stability of MPC - Outline of the Proof

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \dots,\ x_N^\star\}$ the corresponding state trajectory



Stability of MPC - Outline of the Proof

- Assume feasibility of $x(k)$ and let $\{u_0^\star,\ u_1^\star,\ \dots,\ u_{N-1}^\star\}$ be the optimal control sequence computed at $x(k)$ and $\{x(k),\ x_1^\star,\ \dots,\ x_N^\star\}$ the corresponding state trajectory



- At $x(k+1) = x_1^\star$, the control sequence $\tilde{U} = \{u_1^\star,\ u_2^\star,\ \dots,\ \kappa_f(x_N^\star)\}$ is feasible:

$x_N^\star$ is in $\mathcal{X}_f \rightarrow \kappa_f(x_N^\star)$ is feasible $\in \mathcal{U}$ by Ass.2

and $Ax_N^\star + B\kappa_f(x_N^\star)$ in $\mathcal{X}_f$ (by invariance) by Ass.2

$\tilde{X} = \{x^*_{1|k}, \dots\ x^*_{N|k}, Ax^*_N + B\kappa_f(x^*_N)\}$

$\Rightarrow$ **Terminal constraint provides recursive feasibility**

Asymptotic Stability of MPC - Outline of the Proof

$$J^\star(x(k)) = \sum_{i=0}^{N-1} l(x_i^\star, u_i^\star) + l_f(x_N^\star)$$

At $x(k+1) = x_1^\star,\ \tilde{U} = \{u_1^\star,\ u_2^\star,\ \ldots,\ \kappa_f(x_N^\star)\}$ is feasible & sub-optimal

$$\begin{aligned}
J^\star(x(k+1)) &\leq \sum_{i=1}^{N-1} l(x_i^\star, u_i^\star) + l(x_N^\star, \kappa_f(x_N^\star)) + l_f(Ax_N^\star + B\kappa_f(x_N^\star)) \\
&= \underbrace{\sum_{i=0}^{N-1} l(x_i^\star, u_i^\star)}_{J^\star(x(k)) - l_f(x_N^\star)} - l(x_0^\star, u_0^\star) + l(x_N^\star, \kappa_f(x_N^\star)) + l_f(Ax_N^\star + B\kappa_f(x_N^\star)) \\
&= J^\star(x(k)) - l(x(k), u_0^\star) + \underbrace{l_f(Ax_N^\star + B\kappa_f(x_N^\star)) - l_f(x_N^\star) + l(x_N^\star, \kappa_f(x_N^\star))}_{\leq 0 \text{ by Assumption 3}} \\
&\Longrightarrow J^\star(x(k+1)) - J^\star(x(k)) \leq -l(x(k), u_0^\star), \quad l(x,u) > 0 \text{ for } x, u \neq 0
\end{aligned}$$

$J^\star(x)$ is a Lyapunov function

$\Rightarrow$ The closed-loop system under the MPC control law is asymptotically stable

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

$$
\begin{aligned}
J^\star(x(k)) = \quad & \min_U \quad && x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i && \text{Terminal Cost} \\
& \text{subj. to} && && \\
& && x_{i+1} = Ax_i + Bu_i,\ \ i = 0, \ldots, N-1 && \\
& && x_i \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \ i = 0, \ldots, N-1 && \\
& && x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& && x_0 = x(k) &&
\end{aligned}
$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

- Design unconstrained LQR control law

$$F_\infty = -(B^\top P_\infty B + R)^{-1} B^\top P_\infty A$$

where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A^\top P_\infty A + Q - A^\top P_\infty B (B^\top P_\infty B + R)^{-1} B^\top P_\infty A$$

- Choose the terminal weight $P = P_\infty$
- Choose the terminal set $\mathcal{X}_f$ to be the maximum invariant set for the closed-loop system $x_{k+1} = (A + BF_\infty)x_k$:

$$x_{k+1} = Ax_k + BF_\infty(x_k) \in \mathcal{X}_f, \quad \text{for all } x_k \in \mathcal{X}_f$$

All state and input **constraints are satisfied** in $\mathcal{X}_f$:

$$\mathcal{X}_f \subseteq \mathcal{X},\ F_\infty x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_f$$

Choice of Terminal Sets and Cost - Linear System, Quadratic Cost

1. The stage cost is a positive definite function

2. By construction the terminal set is **invariant** under the local control law $\kappa_f(x) = F_\infty x$

3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_f$ and satisfies:

$$
\begin{aligned}
& x_{k+1}^\top P x_{k+1} - x_k^\top P x_k \\
& = x_k^\top(-P_\infty + A^\top P_\infty A + F_\infty^\top B^\top P_\infty A - F_\infty^\top R F_\infty)x_k \\
& = x_k^\top(-P_\infty + A^\top P_\infty A - A^\top P_\infty B(B^\top P_\infty B + R)^{-1} B^\top P_\infty A - F_\infty^\top R F_\infty)x_k \\
& = -x_k^\top(Q + F_\infty^\top R F_\infty)x_k
\end{aligned}
$$

All the Assumptions of the Feasibility and Stability Theorem are verified.

Example: Unstable Linear System

System dynamics:

$$x(k+1) = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$

Constraints:

$$\mathcal{X} := \{x \mid -50 \leq x_1 \leq 50,\ -10 \leq x_2 \leq 10\} = \{x \mid A_x x \leq b_x\}$$
$$\mathcal{U} := \{u \mid \|u\|_\infty \leq 1\} = \{u \mid A_u u \leq b_u\}$$

Stage cost:

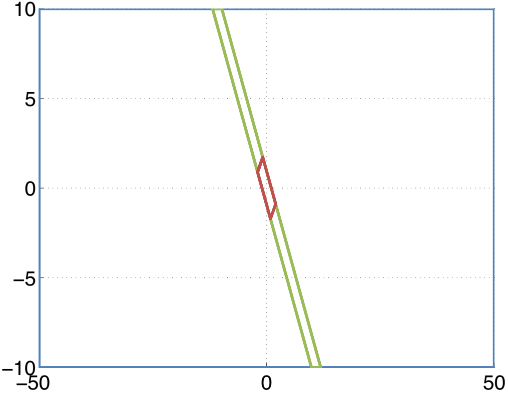
$$l(x_i, u_i) := x_i^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_i + u_i^\top u_i$$

Horizon: $N = 10$

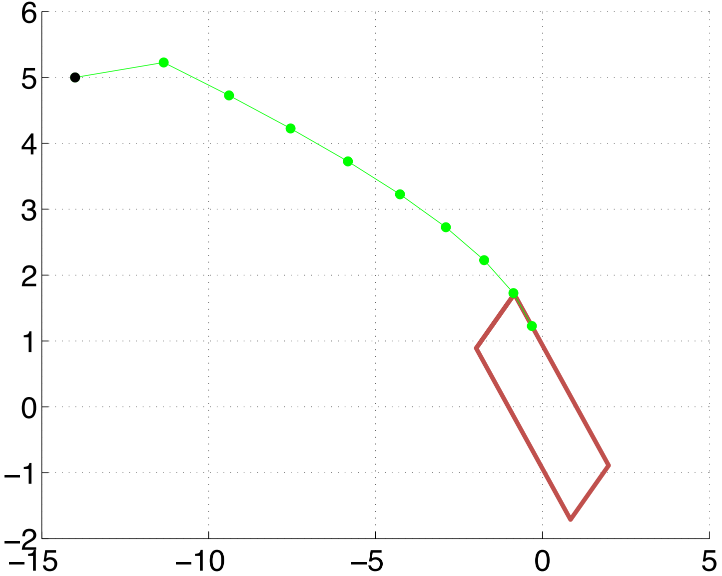
Example: Designing MPC Problem

1. Compute the optimal LQR controller and cost matrices: F_∞, P_∞
2. Compute the maximal invariant set $\mathcal{X}_f$ for the closed-loop linear system $x_{k+1} = (A + BF_\infty)x_k$ subject to the constraints

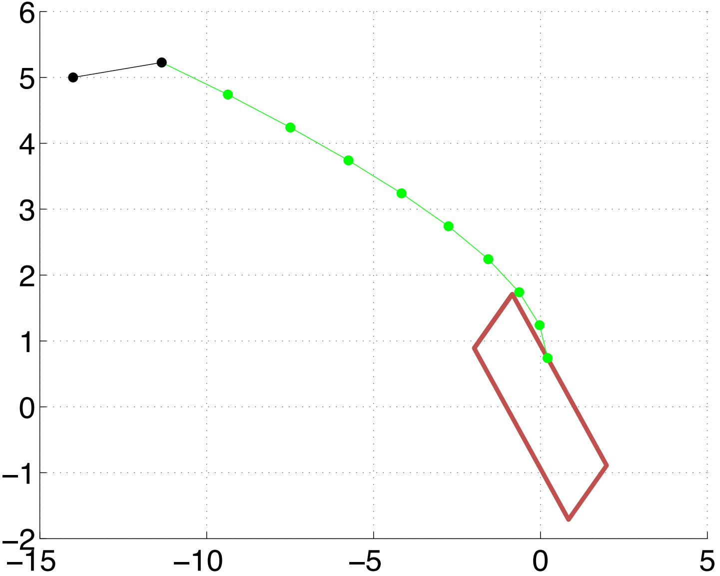
$$\mathcal{X}_{\mathrm{cl}} := \left\{ x \,\middle|\, \begin{bmatrix} A_x \\ A_u F_\infty \end{bmatrix} x \leq \begin{bmatrix} b_x \\ b_u \end{bmatrix} \right\}$$



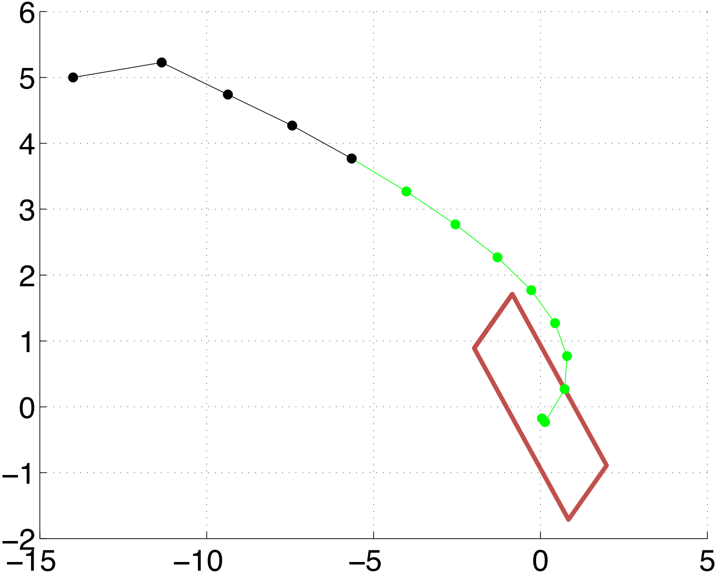
Example: Closed-loop behaviour



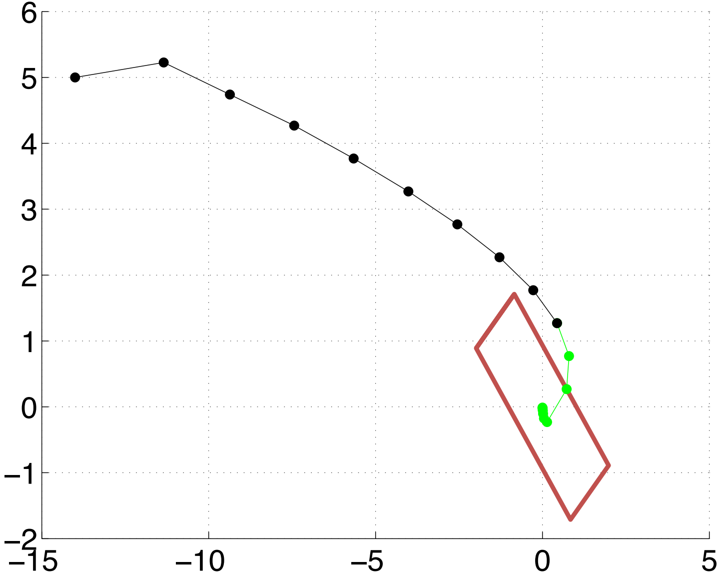
Example: Closed-loop behaviour



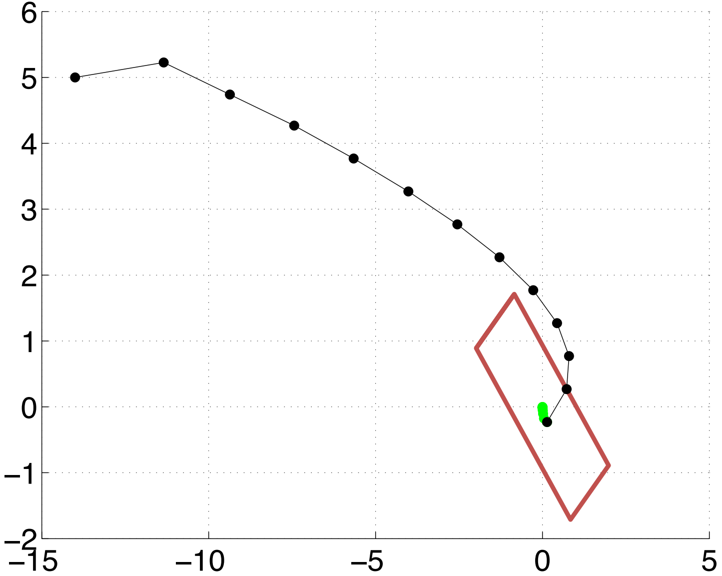
Example: Closed-loop behaviour



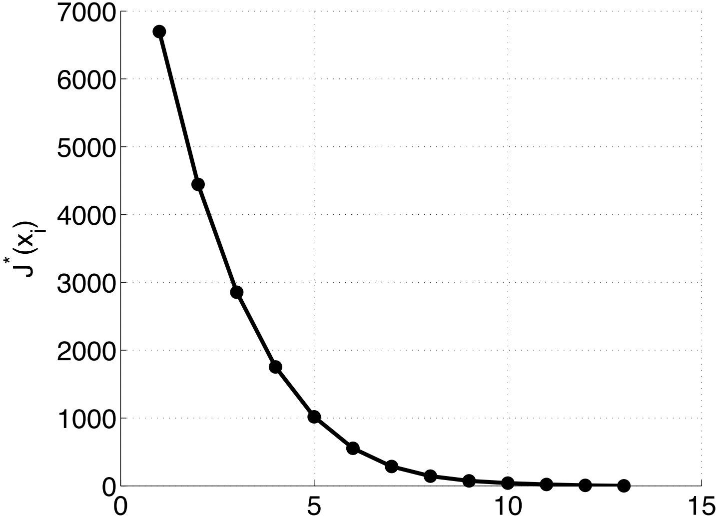
Example: Closed-loop behaviour



Example: Closed-loop behaviour



Example: Lyapunov Decrease of Optimal Cost



Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for feasibility and stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attraction for small N
- Solutions available for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

Outline

3. Feasibility and Stability Guarantees in MPC

   Proof for $\mathcal{X}_f = 0$

   General Terminal Sets

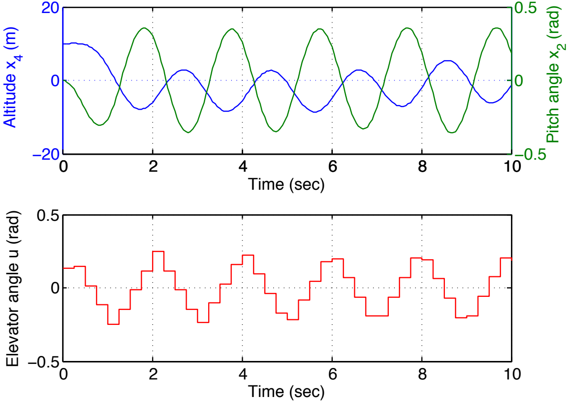
   Example

Example: Short horizon

MPC controller with input constraints $|u_k| \leq 0.262$ and rate constraints $|\dot{u}_k| \leq 0.349$ approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 4$



Decrease in the prediction horizon causes loss of the stability properties

Example: Short horizon

MPC controller with input constraints $|u_k| \leq 0.262$ and rate constraints $|\dot{u}_k| \leq 0.349$ approximated by $|u_i - u_{i-1}| \leq 0.349\, T_s$

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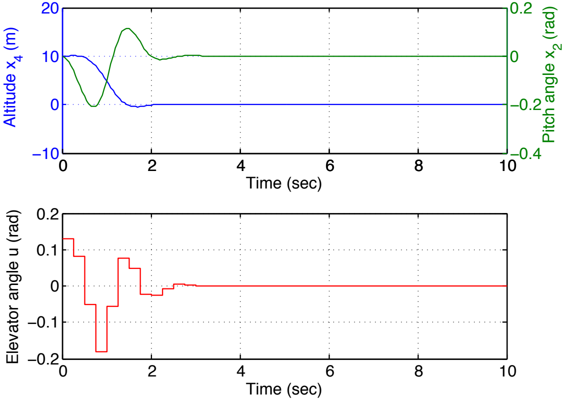

Inclusion of terminal cost and constraint provides stability

Outline

Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(k+1) = g(x(k), u(k))$

$$
\begin{aligned}
J^\star(x(k)) = \quad \min_{U} \quad & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{subj. to} \quad & x_{i+1} = g(x_i, u_i),\ \ i = 0, \dots, N-1 \\
& x_i \in \mathcal{X},\ \ u_i \in \mathcal{U},\ \ i = 0, \dots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

$\rightarrow$ Results can be directly extended to nonlinear systems.

However, computing the sets $\mathcal{X}_f$ and function l_f can be very difficult!

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We ‘fake’ infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.