Model Predictive Control

Chapter 5: Invariance

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Outline

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1. Objectives of Constrained Control

2. Invariance

3. Controlled Invariance

4. Summary

5. Practical Computation of Invariant Sets

Constrained Control

$$x(k+1) = g(x(k), u(k)) \qquad\qquad x, u \in \mathcal{X}, \mathcal{U}$$

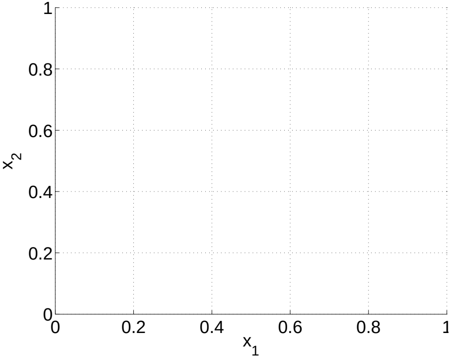
Design control law $u(k) = \kappa(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$
2. Is stable: $\lim_{k\to\infty} x(k) = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

This lecture is about how to ensure #1

(Remaining lectures cover 2-4)

Limitations of Linear Controllers



System:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

Constraints:

$$\mathcal{X} := \{x \mid \|x\|_\infty \le 5\}$$
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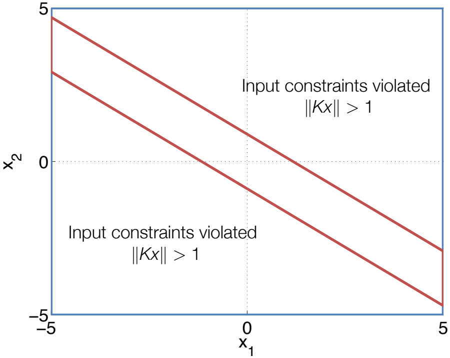
Consider an LQR controller, with $Q = I$, $R = 1$.

Does linear control work?

Limitations of Linear Controllers



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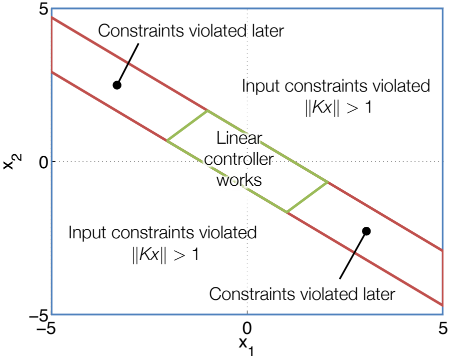

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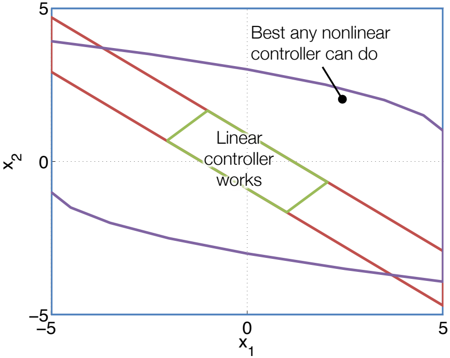

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Does linear control work?

Yes, but the region where it works is very small

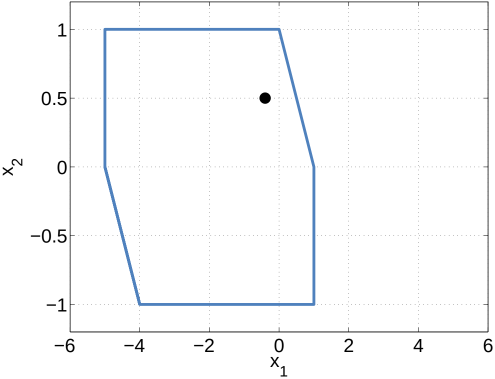
Use nonlinear control (MPC) to increase the region of attraction

Learning Objectives - Constrained Systems

- Learn definition and meaning of **invariance**
  - Region in which an **autonomous** system satisfies the constraints **for all time**
- Learn definition and meaning of **controlled invariance**
  - Region for which there **exists a controller** so that the system satisfies the constraints **for all time**
- Learn how to (conceptually) compute these sets
- Learn how to compute an ellipsoidal invariant set

Invariance: Which states are "good"?

The initial state is in the constraints. Is the next one?



System:

$$\dot{x} = \begin{bmatrix} -2\zeta\omega & -\omega^2 \\ 1 & 0 \end{bmatrix} x$$

where $\omega = 10$, $\zeta = 0.01$, sampled at 10Hz.

Constraints:

$$\mathcal{X} := \left\{ x \,\middle|\, \begin{array}{l} -5 \leq x_1 \leq 1 \\ -1 \leq x_2 \leq 1 \\ -5 \leq x_1 + x_2 \leq 1 \end{array} \right\}$$

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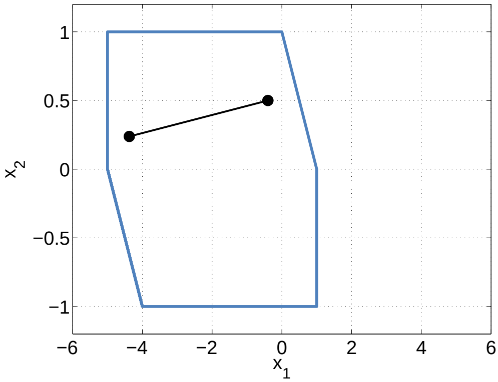

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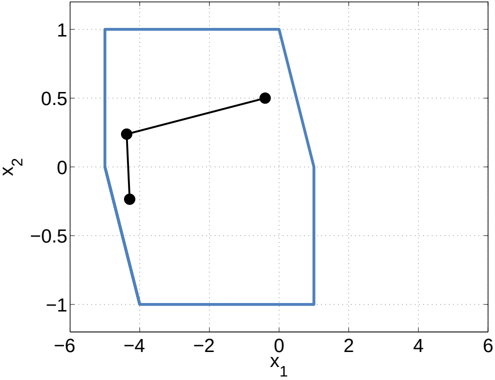

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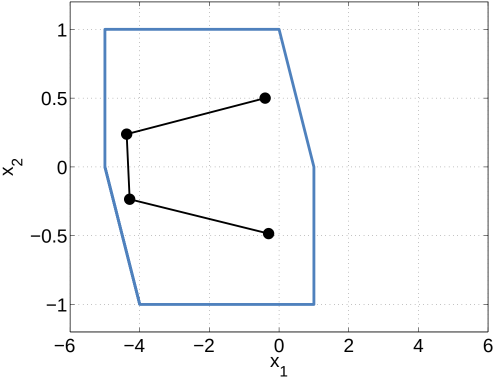

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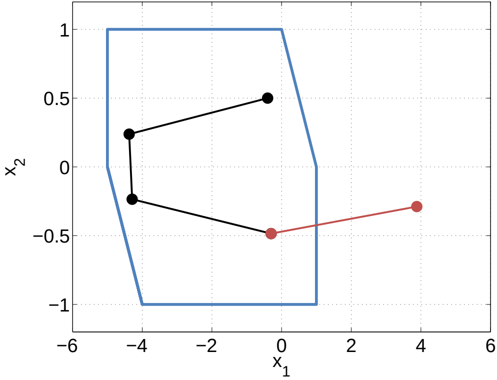
Yup, next one? Yup... Yup... Uh oh



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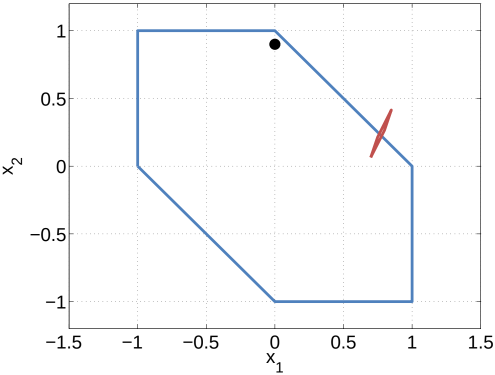
Look an infinite distance into the future to determine if the trajectory beginning at the current state always remains in the constraints.

Controlled Invariance: Does a good input exist?

The initial state is in the constraints. Can we choose the next one to be?



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We can choose from a set of inputs $\Rightarrow$ Set of possible next states

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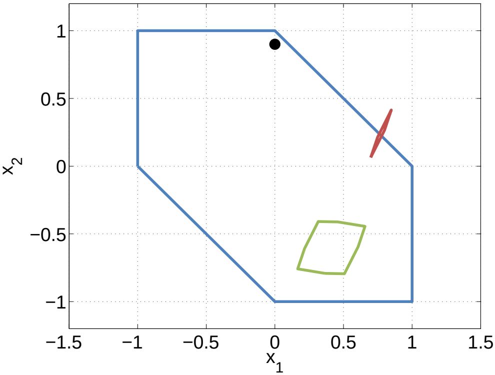

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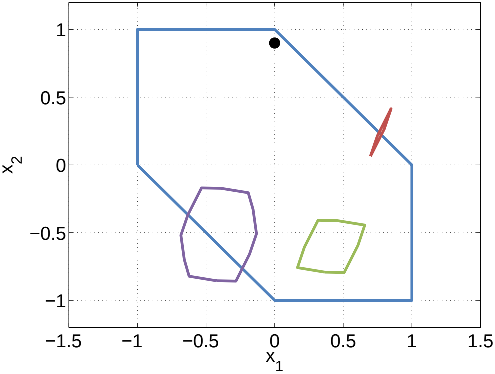

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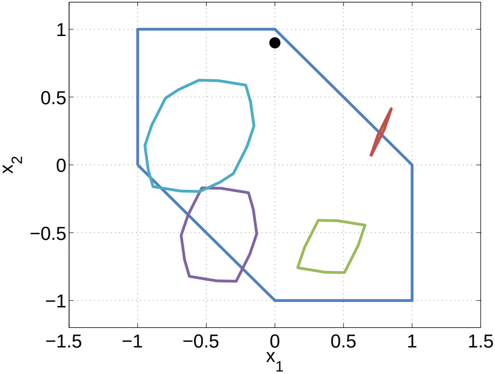

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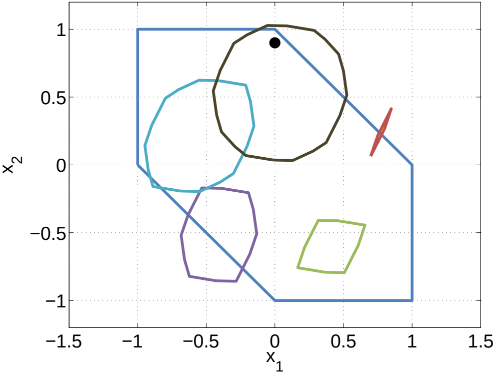

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We can choose from a set of inputs $\Rightarrow$ Set of possible next states

Controlled invariance: Will there always exist a valid input that will maintain constraints?

Outline

Invariance

Constraint satisfaction, for an **autonomous** system $x(k+1) = g(x(k))$, or **closed-loop** system $x(k+1) = g(x(k), \kappa(x(k)))$ for a **given** controller κ.

Positive Invariant set
A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system $x(k+1) = g(x(k))$ if $$x(k) \in \mathcal{O} \;\; \Rightarrow \;\; x(k+1) \in \mathcal{O} \;\; , \;\; \forall k \in \{0, 1, \ldots\}$$

If the invariant set is within the constraints, it provides a set of initial states from which the trajectory will never violate the system constraints.

Maximal Positive Invariant Set $\mathcal{O}_\infty$
The set $\mathcal{O}_\infty \subset \mathcal{X}$ is the maximal invariant set with respect to $\mathcal{X}$ if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets that contain the origin.

The maximal invariant set is the set of all states for which the system will remain feasible if it starts in $\mathcal{O}_\infty$.

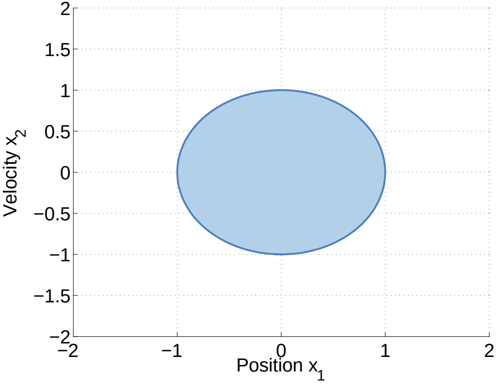
Pre-Sets

Pre Set

Given a set S and the dynamic system $x(k+1) = g(x(k))$, the **pre-set** of S is the set of states that evolve into the target set S in one time step:

$$\text{pre}(S) := \{x \mid g(x) \in S\}$$

Pre-Set Example : Pendulum



Pendulum:

$$x(k+1) = x(k) + \begin{bmatrix} x_2(k) \\ -9.8\sin x_1(k) - x_2(k) \end{bmatrix}$$

(Discretized with forward Euler at 1Hz)

Target set:

$$T := \{x \mid \|x\|_2 \le 1\}$$

Which states will be in the target set at the next point in time?

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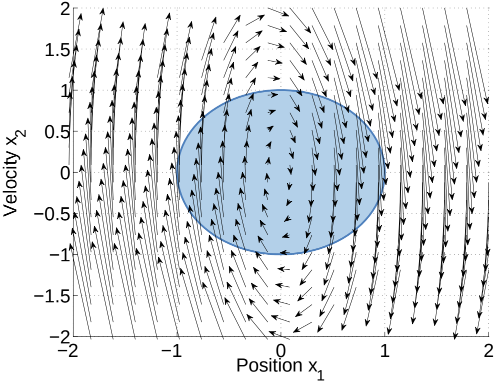

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Consider the phase diagram.

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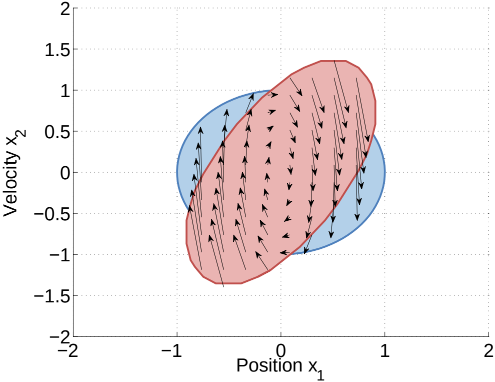

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$$T := \{x \mid \|x\|_2 \leq 1\}$$

Which states will be in the target set at the next point in time?

Consider the phase diagram.

Pre-set is those states that will be in T in one time-step

Extremely difficult to compute, except in special cases.

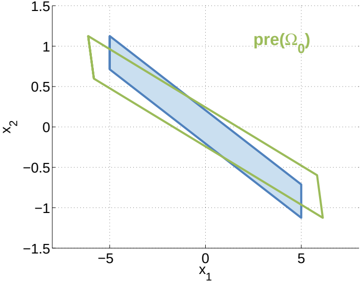
Pre-Set Computation: Linear Autonomous System

Pre Set

Given a set S and the dynamic system $x(k+1) = Ax(k)$, the **pre-set** of S is the set of states that evolve into the target set S in one time step:

$$\mathrm{pre}(S) := \{x \,|\, Ax \in S\}$$

If $S := \{x \,|\, Fx \leq f\}$, then $\mathrm{pre}(S) = \{x \,|\, FAx \leq f\}$



$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

$$\begin{bmatrix} -5 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \|u(k)\|_\infty \leq 0.1$$

Where $u(k) = Kx(k)$, with K the optimal LQR controller for $Q = I$, $R = 90$.

Invariant Set Conditions

Theorem: Geometric condition for invariance

A set $\mathcal{O}$ is a positive invariant set if and only if

$$\mathcal{O} \subseteq \mathsf{pre}(\mathcal{O})$$

We prove the contrapositive for both the necessary and sufficient conditions.

Necessary If $\mathcal{O} \nsubseteq \mathsf{pre}(\mathcal{O})$, then $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \mathsf{pre}(\mathcal{O})$. From the definition of $\mathsf{pre}(\mathcal{O})$, $g(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is not a positive invariant set.

Sufficient If $\mathcal{O}$ is not a positive invariant set, then $\exists \bar{x} \in \mathcal{O}$ such that $g(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \mathsf{pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \mathsf{pre}(\mathcal{O})$.

□

Note that $\mathcal{O} \subseteq \mathsf{pre}(\mathcal{O}) \Leftrightarrow \mathsf{pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

Computing Invariant Sets

Conceptual Algorithm to Compute Invariant Set

Input: g, $\mathcal{X}$
Output: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathcal{X}$
loop
  $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{O}_\infty = \Omega_i$
  end if
end loop

The algorithm generates the set sequence $\{\Omega_i\}$ satisfying $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$ and it terminates when $\Omega_{i+1} = \Omega_i$ so that Ω_i is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(k+1) = g(x(k))$.

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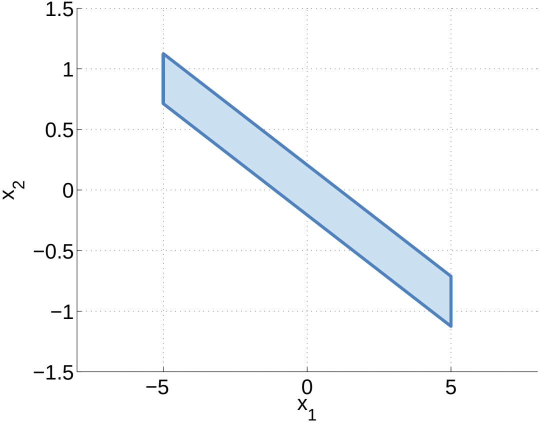
Requirements:

- Represent set Ω_i (Often as polytopes)
- Pre-set computation
- Intersection
- Equality test (bi-directional subset)

Computing Invariant Sets



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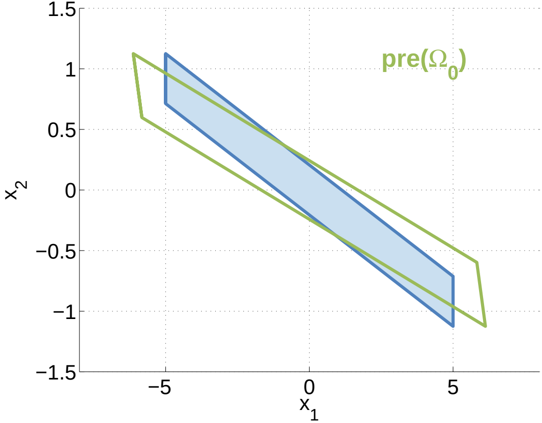

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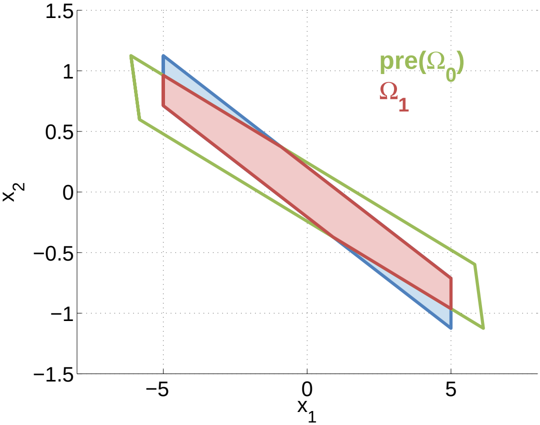

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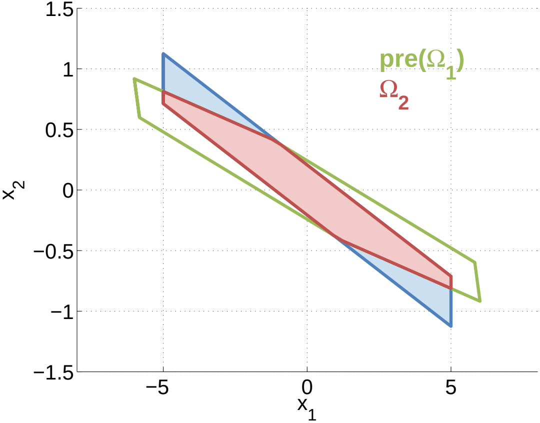

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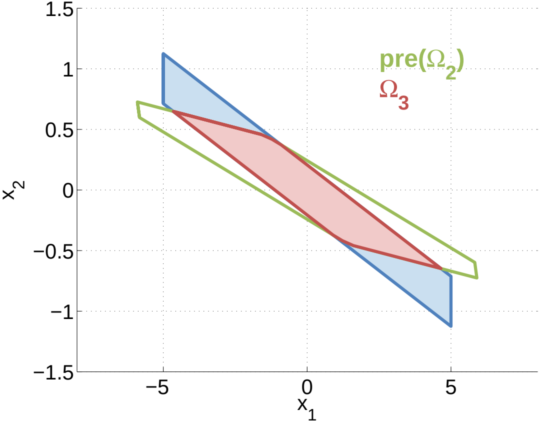

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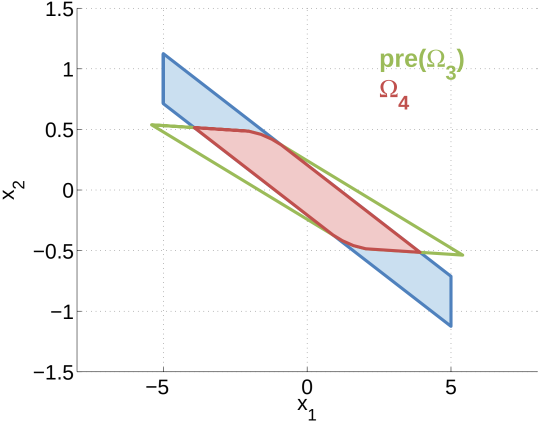
$-0.1 \leq u(k) \leq 0.1$

Where $u(k) = Kx(k)$, with K the optimal LQR controller for $Q = I$, $R = 90$.

Computing Invariant Sets



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Output: $\mathcal{O}_\infty$

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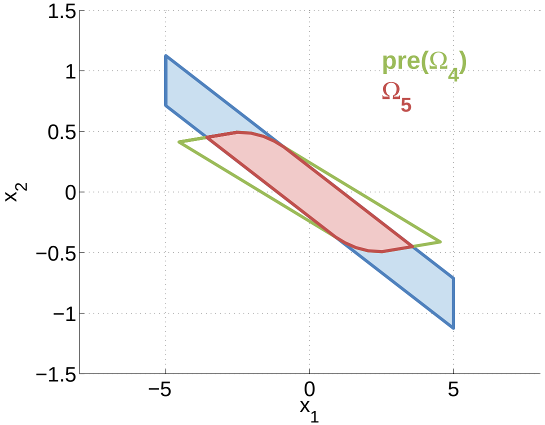

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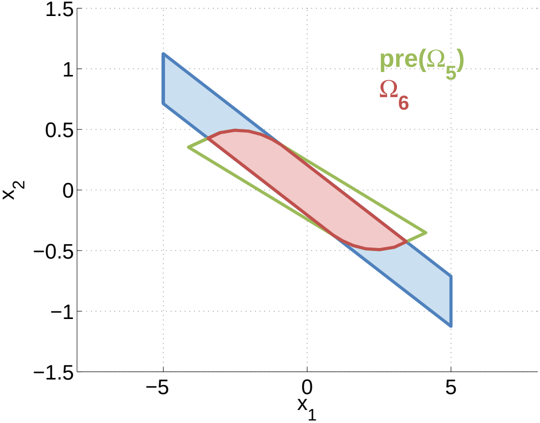

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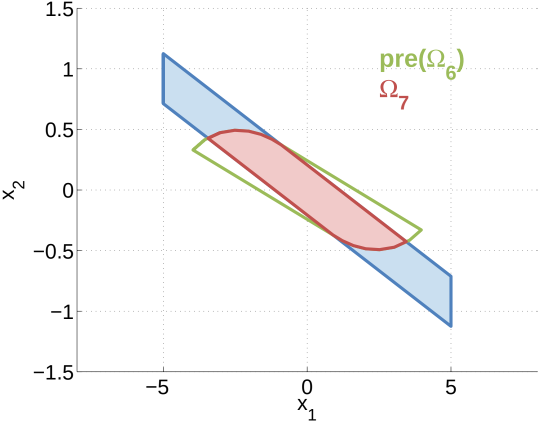

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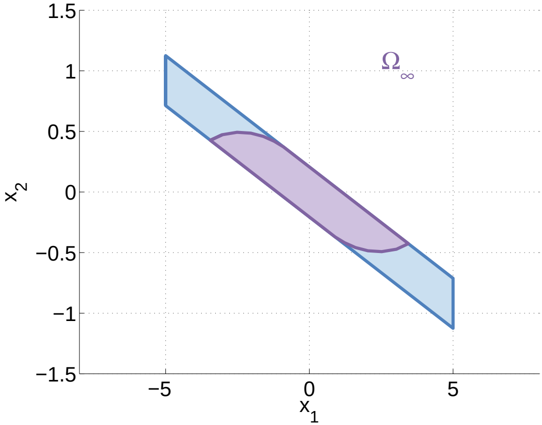
Computing Invariant Sets



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Where $u(k) = Kx(k)$, with K the optimal LQR controller for $Q = I$, $R = 90$.

Outline

Controlled Invariance

Control Invariant Set

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(k) \in \mathcal{C} \quad \Rightarrow \quad \exists u(k) \in \mathcal{U} \text{ such that } g(x(k), u(k)) \in \mathcal{C} \quad \text{for all } k \in \mathbb{N}^+$$

Defines the states for which there exists a **controller** that will satisfy constraints for **all time**.

Maximal Control Invariant Set $\mathcal{C}_\infty$

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(k+1) = g(x(k), u(k))$ subject to the constraints $(x, u) \in \mathcal{X} \times \mathcal{U}$ if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated.

Conceptual Calculation of Control Invariant Sets

Concept of a pre-set extends to systems with exogenous inputs

$$\mathrm{pre}(S) := \{x \,|\, \exists u \in \mathcal{U} \text{ s.t. } g(x,u) \in S\}$$

The same geometric condition holds for control invariant sets

A set $\mathcal{C}$ is a control invariant set if and only if $\mathcal{C} \subseteq \mathrm{pre}(\mathcal{C})$

As a result, the same conceptual algorithm can be used:

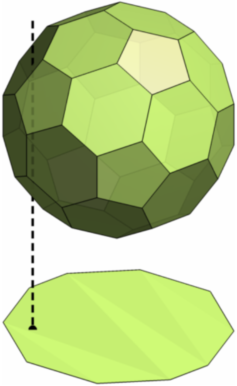
$\Omega_0 \leftarrow \mathcal{X}$
loop
  $\Omega_{i+1} \leftarrow \mathrm{pre}(\Omega_i) \cap \Omega_i$
  if $\Omega_{i+1} = \Omega_i$ **then**
    return $\mathcal{C}_\infty = \Omega_i$
  end if
end loop

However, it is now much harder to compute the pre-set!

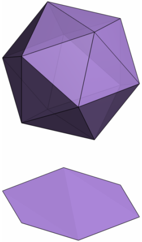
Pre-Set Computation: Controlled System

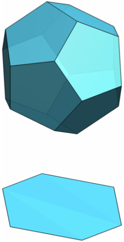
Consider the system $x(k+1) = Ax(k) + Bu(k)$ under the constraints $u(k) \in \mathcal{U} := \{u \mid Gu \leq g\}$ and the set $S := \{x \mid Fx \leq f\}$.

$$
\begin{aligned}
\text{pre}(S) &= \{x \mid \exists u \in \mathcal{U},\ Ax + Bu \in S\} \\
&= \{x \mid \exists u \in \mathcal{U},\ FAx + FBu \leq f\} \\
&= \left\{ x \;\middle|\; \exists u, \begin{bmatrix} FA & FB \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} f \\ g \end{bmatrix} \right\}
\end{aligned}
$$

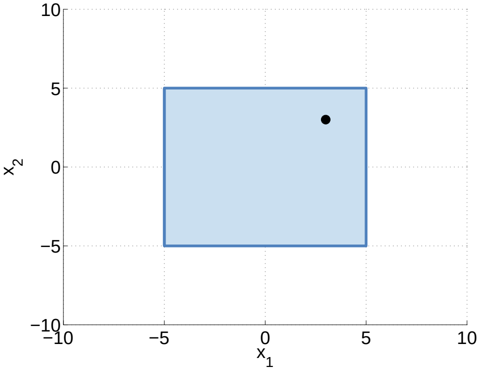


This is a **projection** operation.





Computing Control Invariant Sets



System:

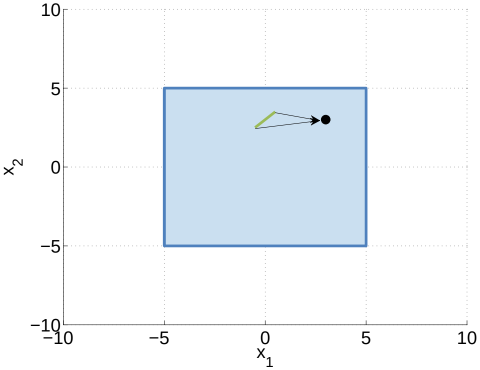
$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

Constraints:

$$\|x(k)\|_\infty \leq 5$$
$$\|u(k)\|_\infty \leq 1$$

An entire set of states can map into each point

Computing Control Invariant Sets



Algorithm:

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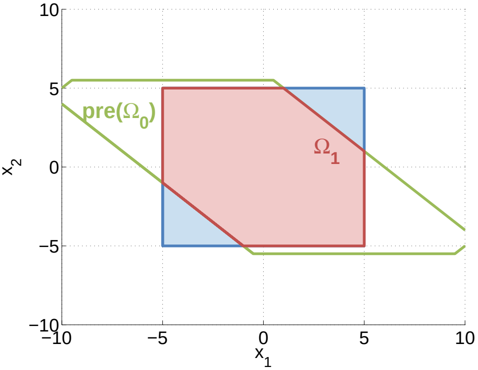
Computing Control Invariant Sets



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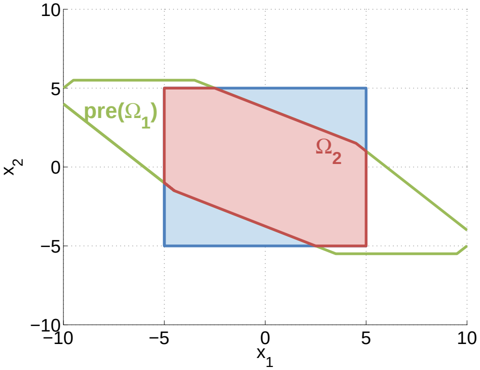
An entire set of states can map into each point

The pre-set is a lot larger, but much more difficult to compute

Computing Control Invariant Sets



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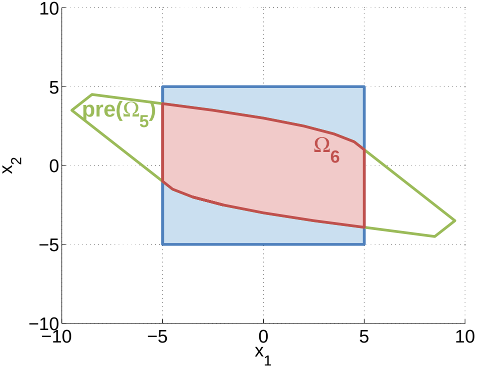

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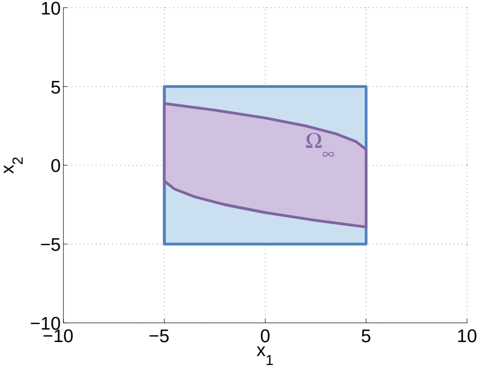

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An entire set of states can map into each point

The pre-set is a lot larger, but much more difficult to compute

The maximum control invariant set is the best any controller can do

Control Invariant Set $\Rightarrow$ Control Law

Let C be a control invariant set for the system $x(k+1) = g(x(k), u(k))$.

A control law $\kappa(x(k))$ will guarantee that the system $x(k+1) = g(x(k), \kappa(x(k)))$ will satisfy the constraints **for all time** if:

$$g(x, \kappa(x)) \in C \qquad \text{for all } x \in C$$

We can use this fact to **synthesize** a control law from a control invariant set by solving an optimization problem:

$$\kappa(x) := \operatorname{argmin} \{ f(x, u) \mid g(x, u) \in C \}$$

where f is any function (including $f(x, u) = 0$).

This doesn't ensure that the system will converge, but it will satisfy constraints.

Relation to MPC

- A control invariant set is a powerful object
- If one can compute one, it provides a direct method for synthesizing a control law that will satisfy constraints
- The maximal control invariant set is the best any controller can do!!!

So why don't we always compute them?

Relation to MPC

- A control invariant set is a powerful object
- If one can compute one, it provides a direct method for synthesizing a control law that will satisfy constraints
- The maximal control invariant set is the best any controller can do!!!

So why don't we always compute them?

We can't...

- Constrained linear systems : Often too complex
- (Constrained) nonlinear system : (Almost) always too complex

$\Rightarrow$ MPC **implicitly** describes a control invariant set such that it's easy to represent and compute.

Outline

Summary: Invariant Sets

- Core component of MPC problem
- Special case: Linear System / Polyhedral Constraints
  - Polyhedral invariant set
    - Can represent the maximum invariant set
    - Can be complex (many inequalities) for more than $\sim 5 - 10$ states
    - Resulting MPC optimization will be a quadratic program

  - Ellipsoidal invariant set
    - Smaller than polyhedral (not the maximal invariant set)
    - Easy to compute for large dimensions
    - Fixed complexity
    - Resulting MPC optimization will be a quadratically constrained quadratic program

Summary: Control Invariant Sets

Special case: Linear system, polyhedral constraints.

- Very difficult to compute
- Very complex
- Very useful

Next week:
Turn an invariant set into a control invariant set with tractable computation
(This is what MPC does)

Outline

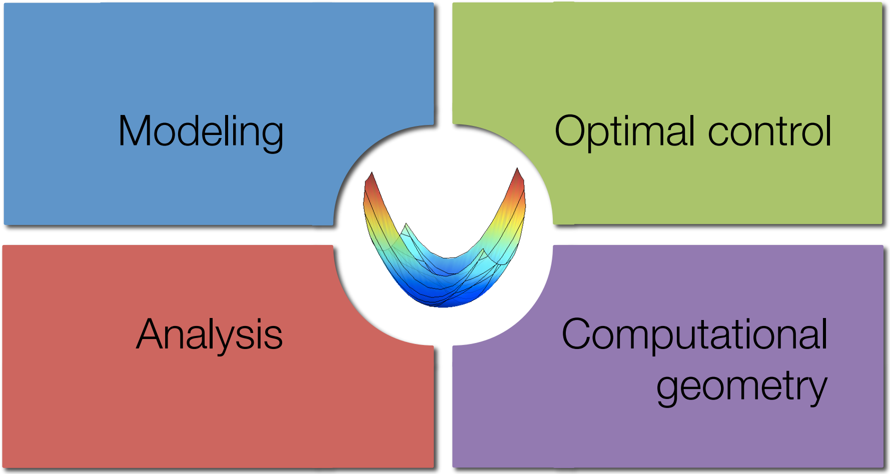
Outline

5. Practical Computation of Invariant Sets

    Polytopes

    Ellipsoids

Polytopes: MultiParametric Toolbox (MPT) (see supplementary material)



http://control.ee.ethz.ch/~mpt/

Outline

5. Practical Computation of Invariant Sets

    Polytopes

    Ellipsoids

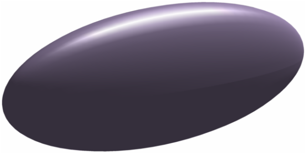
Ellipsoids

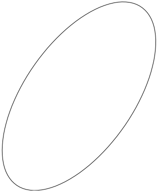
Ellipse

Let $P \succeq 0$ by a symmetric and positive-definite matrix in $\mathbb{R}^{n\times n}$ and $x_c \in \mathbb{R}^n$. The set

$$E := \left\{x \,\middle|\, (x - x_c)^T P(x - x_c) \leq 1\right\}$$

is an **ellipse**.





Ellipsoids are useful because the complexity of evaluating containment is always quadratic in the dimension, whereas polyhedra can be arbitrarily complex.

Invariant Sets from Lyapunov Functions

Lemma: Invariant set from Lyapunov function

If $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the system $x(k+1) = g(x(k))$, then

$$Y := \{x \,|\, V(x) \leq \alpha\}$$

is an invariant set for all $\alpha \geq 0$.

"Proof": We have the basic properties:

- $V(x) \geq 0$ for all x
- $V(g(x)) - V(x) < 0$

The second property implies that once $V(x(k)) \leq \alpha$, $V(x(j))$ will be less than α for all $j \geq k \to$ Invariance □

We often want the largest invariant set contained in our constraints.

If V is a Lyapunov function for the system $x(k+1) = g(x(k))$, and our constraints are given by the set $\mathcal{X}$, then we maximize α such that

$$Y_\alpha := \{x \,|\, V(x) \leq \alpha\} \subseteq \mathcal{X}$$

Invariant Sets from Lyapunov Functions

Consider the system $x(k+1) = Ax(k)$, and assume $P \succ 0$ satisfies the condition

$$A^T PA - P \prec 0$$

Then the function $V(x(k)) = x(k)^T P x(k)$ is a Lyapunov function.

Our goal is to find the largest α such that the invariant set Y_α is contained in the system constraints $\mathcal{X}$:

$$Y_\alpha := \left\{x \,\middle|\, x^T P x \leq \alpha\right\} \subset \mathcal{X} := \{x \,|\, Fx \leq f\}$$

Equivalently, we want to solve the problem:

$$\begin{aligned} &\max_{\alpha}\ \alpha \\ &\text{subj. to } h_{Y_\alpha}(F_i) \leq f_i \text{ for all } i \in \{1, \ldots, n\} \end{aligned} \tag{1}$$

Maximum Ellipsoidal Invariant Sets

Support of an ellipse:

$$\begin{aligned} h_{Y_\alpha}(\gamma) = &\max_x \gamma^T x \\ &\text{subj. to } x^T P x \le \alpha \end{aligned} \tag{2}$$

Change of variables $y := P^{1/2}x$

$$\begin{aligned} h_{Y_\alpha}(\gamma) = &\max_y \gamma^T P^{-1/2} y \\ &\text{subj. to } y^T y \le \sqrt{\alpha}^2 \end{aligned} \tag{3}$$

which can be solved by inspection:

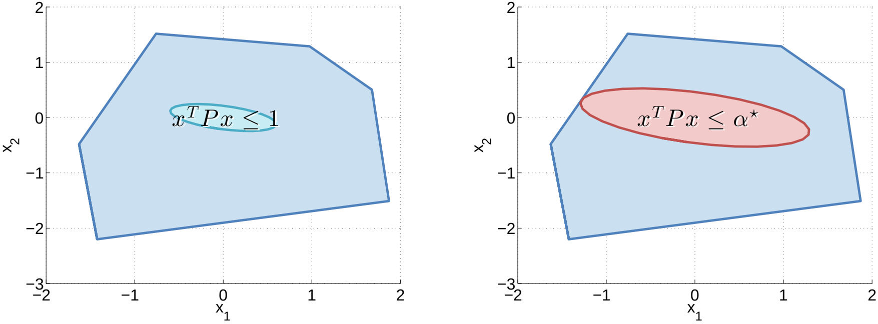
$$h_{Y_\alpha}(\gamma) = \gamma^T P^{-1/2} \frac{P^{-1/2}\gamma}{\|P^{-1/2}\gamma\|}\sqrt{\alpha} = \|P^{-1/2}\gamma\|\sqrt{\alpha}$$

Maximum Ellipsoidal Invariant Sets

Largest ellipse in a polytope is now a one-dimensional optimization problem:

$$\alpha^\star = \max_{\alpha} \ \alpha \quad \text{s.t.} \ \ \|P^{-1/2}F_i^T\|^2\alpha \le f_i^2 \text{ for all } i \in \{1,\ldots,n\}$$

$$= \min_{i\in\{1,\ldots,n\}} \frac{f_i^2}{F_i P^{-1} F_i^T}$$



It is possible to optimize over P, maximizing the volume of the ellipse, subject to stability and containment constraints (convex semi-definite program)

Largest Invariant Ellipsoid within Constraints (supplementary)

Convex optimization problem, which returns the largest invariant ellipoid within a polytope $\mathbb{X} = \{x \,|\, Fx \leq f\}$ (where we define $\tilde{P} := P^{-1}$)

$$\begin{aligned} \min_{\tilde{P}} \;& -\log\det \tilde{P} \\ \text{subj. to } & \begin{bmatrix} \tilde{P} & \tilde{P}A^T \\ A\tilde{P} & \tilde{P} \end{bmatrix} \succeq 0 \\ & F_i \tilde{P} F_i^T \leq f_i^2, \quad \text{for all } i = 1 \dots n \end{aligned}$$

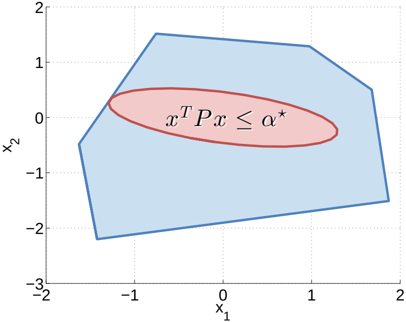
Notes:

- The volume of an ellipse is proportional to $\log\det P^{-1}$
- $\|P^{-1/2} F_i^T\|^2 = F_i P^{-1} F_i^T$
- $A^T P A - P \preceq 0 \Rightarrow \tilde{P} A^T P A \tilde{P} - \tilde{P} \preceq 0$ (pre- and postmultiply by $\tilde{P}$)
  $\Rightarrow$ Apply Schur complement

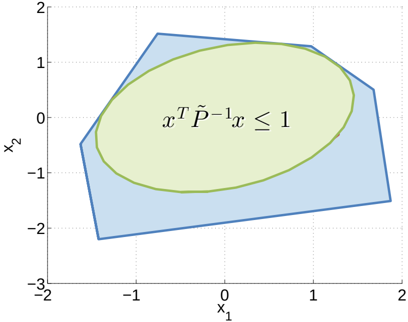
The largest volume ellipse centered at zero within the polytope $\mathbb{X}$ is then

$$\mathcal{E} = \left\{ x \,\middle|\, x^T \tilde{P}^{-1} x \leq 1 \right\} \subset \mathbb{X}$$

Example Revisited



Maximum volume ellipse using some matrix P.



Maximum volume ellipse resulting from any quadratic Lyapunov function.