

Model Predictive Control

Chapter 4: Constrained Finite Time Optimal Control

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Learning Objectives

- Understand feasible set of constrained finite horizon optimal control (CFTOC) problem
- Write quadratic cost CFTOC as QP
- Write $1-\infty$ -norm cost CFTOC as LP
- Contrast properties of LP and QP solution

Constrained Control

$$x(k+1) = g(x(k), u(k)) \quad x, u \in \mathcal{X}, \mathcal{U}$$

Design control law $u(k) = \kappa(x(k))$ such that the system:

1. Satisfies constraints : $\{x(k)\} \subset \mathcal{X}, \{u(k)\} \subset \mathcal{U}$
2. Is stable: $\lim_{k \rightarrow \infty} x(k) = 0$
3. Optimizes “performance”
4. Maximizes the set $\{x(0) \mid \text{Conditions 1-3 are met}\}$

Constrained Infinite Time Optimal Control (what we would like to solve)

$$J_{\infty}^*(x(0)) = \min_{u(\cdot)} \sum_{i=0}^{\infty} l(x_i, u_i)$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i, \quad i = 0, \dots, \infty$$

$$x_i \in \mathcal{X}, u_i \in \mathcal{U}, i = 0, \dots, \infty$$

$$x_0 = x(0)$$

- **Stage cost** $l(x, u)$: “cost” of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
... but we can't compute it: there are an **infinite number of variables**

Constrained Finite Time Optimal Control (what we can sometimes solve)

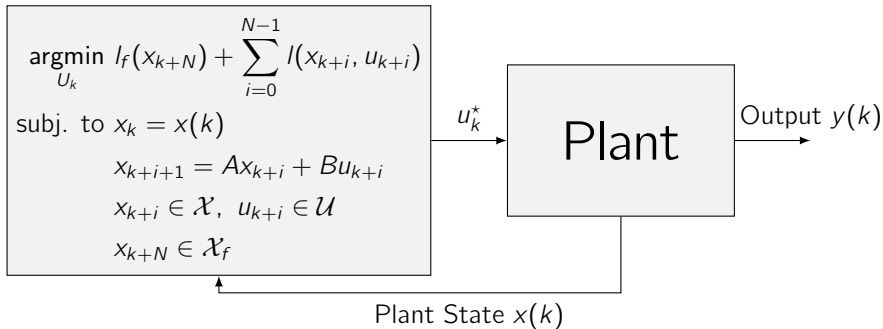
$$\begin{aligned} J_{k \rightarrow k+N|k}^*(x(k)) &= \min_{U_{k \rightarrow k+N|k}} \quad l_f(x_{k+N|k}) + \sum_{i=0}^{N-1} l(x_{k+i|k}, u_{k+i|k}) \\ \text{subj. to } &x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k}, \quad i = 0, \dots, N-1 \\ &x_{k+i|k} \in \mathcal{X}, u_{k+i|k} \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ &x_{k+N|k} \in \mathcal{X}_f \\ &x_{k|k} = x(k) \end{aligned} \quad (1)$$

where $U_{k \rightarrow k+N|k} = \{u_{k|k}, \dots, u_{k+N-1|k}\}$.

Truncate after a finite horizon:

- $l_f(x_{k+N|k})$: Approximates the 'tail' of the cost
- \mathcal{X}_f : Approximates the 'tail' of the constraints

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(k)$
- Find the optimal input sequence for the entire planning window N :
 $U_k^* = \{u_k^*, u_{k+1}^*, \dots, u_{k+N-1}^*\}$
- Implement only the first control action u_k^*

MPC (NMPC) Properties

Pros

- Any model
 - linear
 - nonlinear
 - single/multivariable
 - time delays
 - constraints
 - etc
- Any objective:
 - sum of squared errors
 - sum of absolute errors (i.e., integral)
 - worst error over time
 - economic objective
 - etc

Cons

- Very computationally demanding in the general case
- May or may not be stable
- May or may not be invariant

This lecture: Systems for which optimization is computationally tractable

Outline

1. Constrained Linear Optimal Control
2. Constrained Optimal Control: Quadratic cost
3. Constrained Optimal Control: 1-Norm and ∞ -Norm Cost
4. Receding Horizon Control Notation

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Constrained Linear Optimal Control

Cost function

$$J(x_0, U) = l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

- Squared Euclidian norm: $l_f(x_N) = x_N^\top P x_N$ and $l(x_i, u_i) = x_i^\top Q x_i + u_i^\top R u_i$, with $P \succeq 0$, $Q \succeq 0$, $R \succ 0$.
- $p = 1$ or $p = \infty$: $l_f(x_N) = \|P x_N\|_p$ and $l(x_i, u_i) = \|Q x_i\|_p + \|R u_i\|_p$, with P, Q, R full column rank matrices

Constrained Finite Time Optimal Control problem (CFTOC)

$$\begin{aligned} J^*(x(k)) &= \min_U J(x_0, U) \\ \text{subj. to } &x_{i+1} = A x_i + B u_i, \quad i = 0, \dots, N-1 \\ &x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ &x_N \in \mathcal{X}_f \\ &x_0 = x(k) \end{aligned} \tag{2}$$

N is the time horizon and $\mathcal{X}, \mathcal{U}, \mathcal{X}_f$ are polyhedral regions.

Feasible Sets

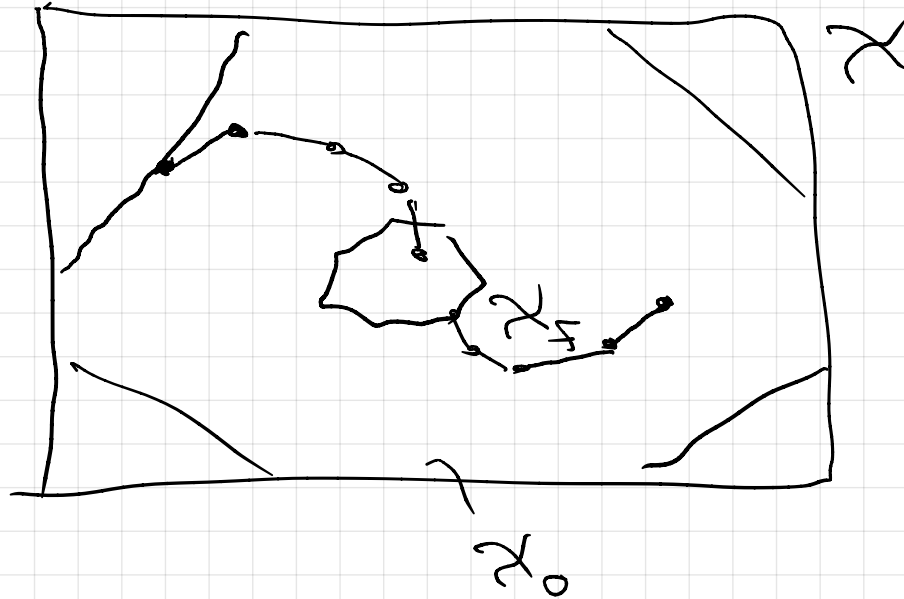
Set of initial states $x(0)$ for which the optimal control problem (2) is feasible:

$$\mathcal{X}_0 = \{x_0 \in \mathbb{R}^n \mid \exists(u_0, \dots, u_{N-1}) \text{ such that } x_i \in \mathcal{X}, u_i \in \mathcal{U}, \\ i = 0, \dots, N-1, x_N \in \mathcal{X}_f, \text{ where } x_{i+1} = Ax_i + Bu_i\}$$

In general \mathcal{X}_j is the set of states x_j at time j for which (2) is feasible:

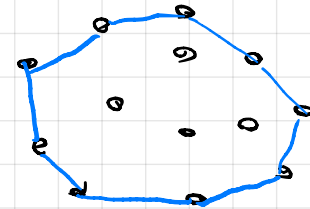
$$\mathcal{X}_j = \{x_j \in \mathbb{R}^n \mid \exists(u_j, \dots, u_{N-1}) \text{ such that } x_i \in \mathcal{X}, u_i \in \mathcal{U}, \\ i = j, \dots, N-1, x_N \in \mathcal{X}_f, \text{ where } x_{i+1} = Ax_i + Bu_i\},$$

The sets \mathcal{X}_j for $j = 0, \dots, N$ play an important role in the solution of the CFTOC problem. They are independent of the cost.



Note:

Approximate feasible set
convex hull of initial
feas. states



Reminder: Unconstrained Solution

For quadratic cost (squared Euclidian norm) and **no state and input constraints**:

$$\{x \in \mathcal{X}, u \in \mathcal{U}\} = \mathbb{R}^{n+m}, \mathcal{X}_f = \mathbb{R}^n$$

we have the **time-varying** linear control law

$$u_i^* = F_i x_i, \quad i = 0, \dots, N-1.$$

If $N \rightarrow \infty$, we have the **time-invariant** linear control law

$$u_i^* = F_\infty x_i, \quad i = 0, 1, \dots$$

Next we show how to compute finite time **constrained** optimal controllers.

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2. Constrained Optimal Control: Quadratic cost
3. Constrained Optimal Control: 1-Norm and ∞ -Norm Cost
4. Receding Horizon Control Notation

Transform Quadratic Cost CFTOC into QP

Quadratic Cost CFTOC

$$J^*(x(k)) = \min_U x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$$

$$\text{subj. to } x_{i+1} = A x_i + B u_i, \quad i = 0, \dots, N-1$$

$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad i = 0, \dots, N-1$$

$$x_N \in \mathcal{X}_f$$

$$x_0 = x(k)$$



QP Problem

$$\min_{z \in \mathbb{R}^n} \frac{1}{2} z^\top H z + q^\top z + r$$

$$\text{subj. to } Gz \leq h$$

$$Az = b$$

Outline

2. Constrained Optimal Control: Quadratic cost

- Construction of the QP with substitution

- Construction of the QP without substitution

- Quadratic Cost State Feedback Solution

Construction of the QP with substitution

Idea: Substitute the state equations (see lectures Week 2)

$$x_{i+1} = Ax_i + Bu_i, x_0 = x(k)$$

- **Step 1:** Rewrite the cost as

$$\begin{aligned} J(x(k), U) &= U^T H U + 2x(k)^T F U + x(k)^T Y x(k) \rightarrow \text{lecture 2} \\ &= [U^T \ x(k)^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} [U^T \ x(k)^T]^T \end{aligned}$$

$H = (S^y)^T \bar{Q} S^y + R > 0$

Note: $\begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} \succeq 0$ since $J(x(k), U) \geq 0$ by assumption.

- **Step 2:** Rewrite the constraints compactly as (details on the next slide)

$$GU \leq w + Ex(k)$$

- **Step 3:** Rewrite the constrained optimal control problem as

$$\begin{aligned} J^*(x(k)) &= \min_U [U^T \ x(k)^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} [U^T \ x(k)^T]^T \\ \text{subj. to} & \quad GU \leq w + Ex(k) \end{aligned}$$

QP construction

write problem in terms of sequences: $X = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$, $U = \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix}$

Cost: $\min X^T \bar{Q} X + U^T \bar{R} U$, $\bar{Q} = \begin{bmatrix} Q & & \\ & \ddots & \\ & & Q \end{bmatrix}$, $\bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$

Constraints:

$$A_x x_i \leq b_x$$

$$A_u u_i \leq b_u$$

$$A_f x_n \leq b_f$$

$$\Rightarrow \begin{bmatrix} \bar{A}_x X \\ \bar{A}_u U \end{bmatrix} \leq \begin{bmatrix} \bar{b}_x \\ \bar{b}_u \end{bmatrix} \text{ with}$$

$$\bar{A}_x = \begin{bmatrix} A_x & & \\ & \ddots & \\ & & A_x & \\ & & & A_f \end{bmatrix}$$

$$\bar{b}_x = \begin{bmatrix} b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

$$\bar{A}_u = \begin{bmatrix} A_u & & \\ & \ddots & \\ & & A_u \end{bmatrix}$$

$$\bar{b}_u = \begin{bmatrix} b_u \\ \vdots \\ b_u \end{bmatrix}$$

$$\Rightarrow \min X^T \bar{Q} X + U^T \bar{R} U$$

$$\text{s.t. } \bar{A}_x X = \bar{b}_x$$

$$\bar{A}_u U = \bar{b}_u$$

dynamics



difference between

substitution or no substitution

With substitution:

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{bmatrix} \begin{bmatrix} u \\ \vdots \\ u_{n-1} \end{bmatrix} \quad (\text{in slides})$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$X = S^x x(k) + S^u u \quad (*)$$

$$\Rightarrow \text{eliminate } X \text{ using } (*) \Rightarrow \begin{aligned} \bar{A}_u u &\leq \bar{b}_u \\ \bar{A}_x (S^x x(k) + S^u u) &\leq \bar{b}_x \\ \bar{A}_x S^u u &\leq \bar{b}_x - \bar{A}_x S^x x(k) \end{aligned}$$

$$\Rightarrow Gu \leq w + E x(k)$$

$$\Rightarrow \text{use } z = u \text{ to write QP}$$

Construction of QP constraints with substitution

Inequalities $GU \leq w + Ex(k)$ for \mathcal{X} , \mathcal{U} and \mathcal{X}_f given by:

$$\mathcal{X} = \{x \mid A_x x \leq b_x\} \quad \mathcal{U} = \{u \mid A_u u \leq b_u\} \quad \mathcal{X}_f = \{x \mid A_f x \leq b_f\}$$

Then G , E and w are defined as follows

$$G = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Solution of the QP with substitution

$$\begin{aligned} J^*(x(k)) = \min_U & \quad [U^\top \ x(k)^\top] \begin{bmatrix} H & F_Y^\top \\ F & Y \end{bmatrix} [U^\top \ x(k)^\top]^\top \\ \text{subj. to} & \quad GU \leq w + Ex(k) \end{aligned}$$

For a given $x(k)$, U^* can be found via a QP solver.

Outline

2. Constrained Optimal Control: Quadratic cost

Construction of the QP with substitution

Construction of the QP without substitution

Quadratic Cost State Feedback Solution

Construction of the QP without substitution (1/3)

Idea: Keep state equations as equality constraints (often more efficient)

We transform the CFTOC problem into the QP problem

$$J^*(x(k)) = \min_z \quad [z^\top \ x(k)^\top] \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} [z^\top \ x(k)^\top]^\top$$

$$\text{subj. to } G_{\text{in}} z \leq w_{\text{in}} + E_{\text{in}} x(k)$$

$$G_{\text{eq}} z = E_{\text{eq}} x(k)$$

- Define variable:

$$z = [x_1^\top \ \dots \ x_N^\top \ u_0^\top \ \dots \ u_{N-1}^\top]^\top$$

- Equalities from system dynamics $x_{i+1} = Ax_i + Bu_i$:

$$G_{\text{eq}} = \left[\begin{array}{cccc|cccc} I & & & & -B & & & \\ -A & I & & & & -B & & \\ & -A & I & & & & -B & \\ & & & \ddots & & & & \ddots \\ & & & & -A & I & & \\ & & & & & & -B & \end{array} \right], E_{\text{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

QP without substitution

Sparse formulation of dynamics:

$$x_0 = x(k)$$

$$x_1 = Ax_0 + Bu_0 \rightarrow -Ax_0 + x_1 = Bu_0$$

\vdots

$$x_{i+1} = Ax_i + Bu_i \rightarrow -Ax_i + x_{i+1} = Bu_i$$

$$\begin{bmatrix} I & 0 & \dots & 0 \\ -A & I & 0 & \dots & 0 \\ 0 & -A & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -A & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & B \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix} x(k)$$

$$G^x X = G^u U + E x(k)$$

$$\min X^T \bar{Q} X + U^T \bar{Q} U$$

$$\text{s.t. } \bar{A}_x X \leq \bar{b}_x$$

$$\bar{A}_u U \leq \bar{b}_u$$

$$G^x X = G^u U + \bar{F}^x(k)$$

$$z = \begin{bmatrix} X \\ U \end{bmatrix}$$

$$\min z^T \begin{bmatrix} \bar{Q} & 0 \\ 0 & \bar{Q} \end{bmatrix} z$$

$$\text{s.t. } \begin{bmatrix} \bar{A}_x & 0 \\ 0 & \bar{A}_u \end{bmatrix} z \leq \begin{bmatrix} \bar{b}_x \\ \bar{b}_u \end{bmatrix}$$

$$\begin{bmatrix} G^x & -G^u \end{bmatrix} z = \bar{F}^x(k)$$

Construction of the QP without substitution (2/3)

Inequalities $G_{\text{in}}z \leq w_{\text{in}} + E_{\text{in}}x(k)$ for \mathcal{X} , \mathcal{U} and \mathcal{X}_f given by:

$$\mathcal{X} = \{x \mid A_x x \leq b_x\} \quad \mathcal{U} = \{u \mid A_u u \leq b_u\} \quad \mathcal{X}_f = \{x \mid A_f x \leq b_f\}$$

Then matrices G_{in} , w_{in} and E_{in} are:

$$G_{\text{in}} = \begin{bmatrix} 0 & & & & 0 & & & & \\ & A_x & & & & & & & \\ & & \ddots & & & & & & \\ & & & A_x & & & & & \\ & & & & A_f & & & & \\ 0 & & & & & A_u & & & 0 \\ & 0 & & & & & A_u & & \\ & & \ddots & & & & & \ddots & \\ & & & 0 & & & & & A_u \\ & & & & 0 & & & & & A_u \end{bmatrix} \quad w_{\text{in}} = \begin{bmatrix} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{bmatrix}$$

$$E_{\text{in}} = \begin{bmatrix} -A_x^\top & 0 & \cdots & 0 \end{bmatrix}^\top$$

Construction of the QP without substitution (3/3)

Build cost function from MPC cost $x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$

$$\bar{H} = \begin{bmatrix} Q & & & & \\ & \ddots & & & \\ & & Q & & \\ & & & P & \\ \text{---} & & & & R & \text{---} \\ & & & & & \ddots & \\ & & & & & & R \end{bmatrix}$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```

Number of opt. variables

w/ sub: $N \cdot m$

"dense"

w/o sub: $N(n + m)$

sparse

Outline

2. Constrained Optimal Control: Quadratic cost

Construction of the QP with substitution

Construction of the QP without substitution

Quadratic Cost State Feedback Solution

Quadratic Cost State Feedback Solution

$$\begin{aligned} J^*(x(k)) = \min_u \quad & [U^\top \ x(k)^\top] \begin{bmatrix} H & F_Y^\top \\ F & Y \end{bmatrix} [U^\top \ x(k)^\top]^\top \\ \text{subj. to} \quad & GU \leq w + Ex(k) \end{aligned}$$

The CFTOC problem is a **multiparametric quadratic program (mp-QP)** with the following solution properties:

- The first component of the optimal solution has the form

$$u_0^* = \kappa(x(k)), \quad \forall x(k) \in \mathcal{X}_0,$$

$\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$

- The polyhedral sets $CR^j = \{x \in \mathbb{R}^n | H^j x \leq K^j\}$, $j = 1, \dots, N^r$ are a partition of the feasible polyhedron \mathcal{X}_0 .
- The value function $J^*(x(k))$ is convex and piecewise quadratic on polyhedra.

\Rightarrow Explicit MPC addresses how to compute this solution.

Example

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to constraints

$$-1 \leq u(k) \leq 1, \quad k = 0, \dots, 5$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad k = 0, \dots, 5$$

Compute the **state feedback** optimal controller $u^*(x(k))$ solving the CFTOC problem with $N = 6$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.1$, P the solution of the ARE, $\mathcal{X}_f = \mathbb{R}^2$.

Example

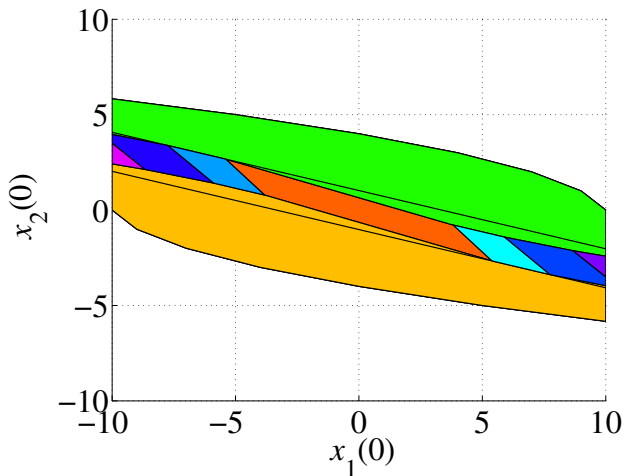


Figure: Partition of the state space for the affine control law $u^*(x)$ ($N_0^r = 13$)

Example

MPT Toolbox

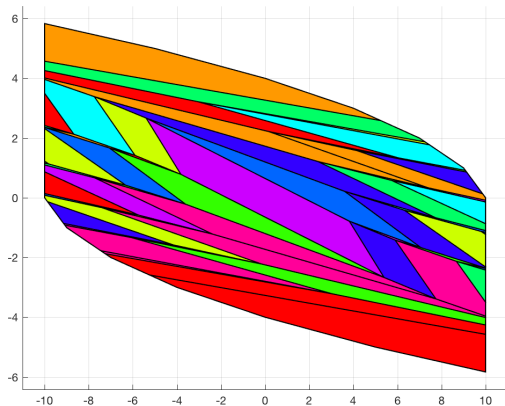


Figure: Partition of the state space for the affine control law $u^*(x)$ ($N_0^r = 61$)

Example

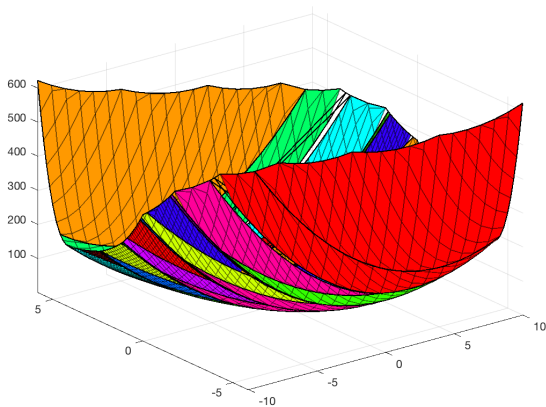


Figure: Value function for the affine control law $u^*(x)$ ($N_0^r = 61$)

Example

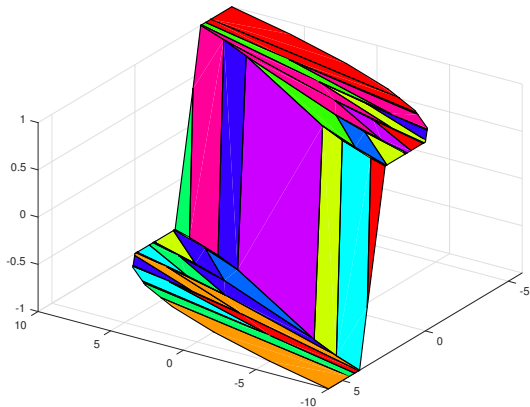


Figure: Optimal control input for the affine control law $u^*(0)$ ($N_0^r = 61$)

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4. Receding Horizon Control Notation

Transform 1- / ∞ -Norm Cost CFTOC into LP

1-Norm / ∞ -Norm Cost CFTOC

$$J^*(x(k)) = \min_U \|P x_N\|_p + \sum_{i=0}^{N-1} \|Q x_i\|_p + \|R u_i\|_p$$

$$\text{subj. to } x_{i+1} = A x_i + B u_i, \quad i = 0, \dots, N-1$$

$$x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad i = 0, \dots, N-1$$

$$x_N \in \mathcal{X}_f$$

$$x_0 = x(k)$$



LP Problem

$$\min_{z \in \mathbb{R}^n} c^\top z$$

$$\text{subj. to } Gz \leq h$$

$$Az = b$$

ℓ_∞ minimization

Constrained ℓ_∞ (Chebyshev) minimization:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \|x\|_\infty \\ \text{subj. to} \quad & Fx \leq g \end{aligned}$$

Write this as a max of linear functions.

Equivalent to:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & [\max \{x_1, \dots, x_n, -x_1, \dots, -x_n\}] \\ \text{subj. to} \quad & Fx \leq g \end{aligned}$$

ℓ_∞ minimization (cont'd)

Equivalent to:

$$\begin{array}{ll} \min_{x,t} t & \\ \text{subj. to} & x_i \leq t \quad i = 1, \dots, n \\ & -x_i \leq t \quad i = 1, \dots, n \\ & Fx \leq g \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min_{x,t} t & \\ \text{subj. to} & -\mathbf{1}t \leq x \leq \mathbf{1}t \\ & Fx \leq g \end{array}$$

-
- The notation ' $\mathbf{1}$ ' indicates a vector of ones.
 - The constraint $-\mathbf{1}t \leq x \leq \mathbf{1}t$ bounds the absolute value of every element of x with a common scalar variable t .

ℓ_1 minimization

Constrained ℓ_1 minimization:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} ||x||_1 \\ \text{subj. to } Fx \leq g \end{aligned}$$

Write this as a max of linear functions.

Equivalent to:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \left[\sum_{i=1}^m \max \{x_i, -x_i\} \right] \\ \text{subj. to } Fx \leq g \end{aligned}$$

ℓ_1 minimization (cont'd)

Equivalent to:

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n, t \in \mathbb{R}^m} & t_1 + \dots + \dots t_m \\ \text{subj. to} & x_i \leq t_i \quad i = 1, \dots, m \\ & -x_i \leq t_i \quad i = 1, \dots, m \\ & Fx \leq g \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min_{x \in \mathbb{R}^n, t \in \mathbb{R}^m} & \mathbf{1}^\top t \\ \text{subj. to} & -t \leq x \leq t \\ & Fx \leq g \end{array}$$

-
- The notation ' $\mathbf{1}$ ' indicates a vector of ones.
 - The constraint $-t \leq x \leq t$ bounds the absolute value of each component of x with a component of the vector variable t .

NB: trick was to add variables and write the problem in **epigraph** form.

Outline

3. Constrained Optimal Control: 1-Norm and ∞ -Norm Cost

Construction of the LP with substitution

1- / ∞ -Norm State Feedback Solution

$$\min \|Q x_i\|_\infty \Rightarrow \min \varepsilon_i^x$$

$$\text{s.t. } Q x_i \leq \varepsilon_i^x \mathbb{1}$$

$$-Q x_i \leq \varepsilon_i^x \mathbb{1}$$

$$\text{LP: } \min \|P x_n\|_\infty + \sum_{i=0}^{n-1} \|Q x_i\|_\infty + \|R u_i\|_\infty$$

s.t. constraints

$$\Rightarrow \min \varepsilon_n^x + \sum_{i=0}^{n-1} \varepsilon_i^x + \varepsilon_i^u$$

$$\text{s.t. } Q x_i \leq \varepsilon_i^x \mathbb{1}$$

$$-Q x_i \leq \varepsilon_i^x \mathbb{1}$$

$$P x_n \leq \varepsilon_n^x \mathbb{1}$$

$$-P x_n \leq \varepsilon_n^x \mathbb{1}$$

$$R u_i \leq \varepsilon_i^u \mathbb{1}$$

$$-R u_i \leq \varepsilon_i^u \mathbb{1}$$

constraints

\Rightarrow write in terms of X, U

and substitute dynamics

$$X = S^x x(k) + S^u U$$

Note: w/o sub keep X & U

$$\text{add } G^x X = G^u U + \bar{E} x(k)$$

\Rightarrow next slide: plug-in

$$x_i = S_i^x x(k) + S_i^u U$$

Construction of the LP with substitution

Recall that the ∞ -norm problem can be equivalently formulated as

$$\begin{aligned} \min_z \quad & \varepsilon_0^x + \dots + \varepsilon_N^x + \varepsilon_0^u + \dots + \varepsilon_{N-1}^u \\ \text{subj. to} \quad & -\mathbf{1}_n \varepsilon_i^x \leq \pm Q \left[A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \right], \\ & -\mathbf{1}_r \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\ & -\mathbf{1}_m \varepsilon_i^u \leq \pm R u_i, \\ & A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \in \mathcal{X}, \quad u_i \in \mathcal{U}, \\ & A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \\ & x_0 = x(k), \quad i = 0, \dots, N-1 \end{aligned}$$

Construction of the LP with substitution

The problem results in the following standard LP

$$\begin{array}{ll} \min_z & c^\top z \\ \text{subj. to} & \bar{G}z \leq \bar{w} + \bar{S}x(k) \end{array}$$

where $z := \{\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0^\top, \dots, u_{N-1}^\top\} \in \mathbb{R}^s$,
 $s := (m+1)N + N + 1$ and

$$\bar{G} = \begin{bmatrix} G_\varepsilon & G_u \\ 0 & G \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} S_\varepsilon \\ S \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_\varepsilon \\ w \end{bmatrix}$$

For a given $x(k)$, U^* can be obtained via an LP solver.

→ The 1-norm case works similarly.

Outline

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$$\begin{array}{ll} \min_z & c^\top z \\ \text{subj. to} & \bar{G}z \leq \bar{w} + \bar{S}x(k) \end{array}$$

The CFTOC problem is a **multiparametric linear program (mp-LP)** with the following solution properties:

- The first component of the multiparametric solution has the form

$$u_0^* = \kappa(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

$\kappa : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if } x \in CR^j, \quad j = 1, \dots, N^r$$

- The polyhedral sets $CR^j = \{x \in \mathbb{R}^n | H^j x \leq K^j\}$, $j = 1, \dots, N^r$ are a partition of the feasible polyhedron \mathcal{X}_0 .
- In case of multiple optimizers a PieceWise Affine control law exists.
- The value function $J^*(x(0))$ is convex and piecewise linear on polyhedra.

Solution properties: Quadratic vs. $1 - \|\cdot\|_\infty$ -norm cost

Let $n = \#$ optimization variables.

Quadratic cost (pos. def) : Solution is either

- unique and in the interior of feasible set \rightarrow no constraints active
- unique and on the boundary of feasible set \rightarrow at least 1 active constraint

Linear cost: Solution is either

- unbounded
- unique at a vertex of the feasible set \rightarrow at least n active constraints,
- a set of multiple optima \rightarrow at least 1 active constraint

Outline

1. Constrained Linear Optimal Control
2. Constrained Optimal Control: Quadratic cost
3. Constrained Optimal Control: 1-Norm and ∞ -Norm Cost
4. Receding Horizon Control Notation

Receding Horizon Control Notation (1/3)

Consider DT model

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\y(k) &= Cx(k) \\x(k) &\in \mathcal{X}, \quad u(k) \in \mathcal{U}, \quad \forall k \geq 0\end{aligned}$$

The Constrained Finite Time Optimal Control (CFTOC) Problem

$$\begin{aligned}J_{k \rightarrow k+N|k}^*(x(k)) &= \min_{u_{k \rightarrow k+N|k}} l_f(x_{k+N|k}) + \sum_{i=0}^{N-1} l(x_{k+i|k}, u_{k+i|k}) \\ \text{subj. to } &x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k}, \quad i = 0, \dots, N-1 \\ &x_{k+i|k} \in \mathcal{X}, \quad u_{k+i|k} \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ &x_{k+N|k} \in \mathcal{X}_f \\ &x_{k|k} = x(k)\end{aligned} \tag{4}$$

is solved at time k with $U_{k \rightarrow k+N|k} = \{u_{k|k}, \dots, u_{k+N-1|k}\}$.

Receding Horizon Control Notation (2/3)

- $x(k)$ is the state of the system at time k .
- $x_{i+k|k}$ is the state of the model at time $k+i$, predicted at time k obtained by starting from the current state $x_{k|k} = x(k)$ and applying to the system model

$$x_{k+1|k} = Ax_{k|k} + Bu_{k|k}$$

the input sequence $u_{k|k}, \dots, u_{k+i-1|k}$.

- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $k=1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $k=2$ starting from the current state $x(2)$.
- Similarly $u_{k+i|k}$ is read as "the input u at time $k+i$ computed at time k ".

Receding Horizon Control Notation (3/3)

- Let $U_{k \rightarrow k+N|k}^* = \{u_{k|k}^*, \dots, u_{k+N-1|k}^*\}$ be the optimal solution. The first element of $U_{k \rightarrow k+N|k}^*$ is applied to the system

$$u(k) = u_{k|k}^*(x(k)).$$

- The CFTOC problem is reformulated and solved at time $k+1$, based on the new state $x_{k+1|k+1} = x(k+1)$.

Receding horizon control law

$$\kappa_k(x(k)) = u_{k|k}^*(x(k))$$

Closed loop system

$$x(k+1) = Ax(k) + B\kappa_k(x(k)) := g_{cl}(x(k)), \quad k \geq 0$$

RHC: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $\kappa_k(x(k))$ becomes a time-invariant function of the initial state $x(k)$. Thus, we can simplify the notation as

$$\begin{aligned} J^*(x(k)) &= \min_U l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\ \text{subj. to } &x_{i+1} = Ax_i + Bu_i, \quad i = 0, \dots, N-1 \\ &x_i \in \mathcal{X}, \quad u_i \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ &x_N \in \mathcal{X}_f \\ &x_0 = x(k) \end{aligned} \tag{5}$$

where $U = \{u_0, \dots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $\kappa(x(k))$ for $\kappa_k(x(k))$.