Model Predictive Control

Chapter 4: Constrained Finite Time Optimal Control

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Learning Objectives

- Understand feasible set of constrained finite horizon optimal control (CFTOC) problem
- Write quadratic cost CFTOC as QP
- Write $1-/\ \infty-$norm cost CFTOC as LP
- Contrast properties of LP and QP solution

Constrained Control

$$x(k+1) = g(x(k), u(k)) \qquad\qquad x, u \in \mathcal{X}, \mathcal{U}$$

Design control law $u(k) = \kappa(x(k))$ such that the system:

1. Satifies constraints : $\{x(k)\} \subset \mathcal{X}$, $\{u(k)\} \subset \mathcal{U}$
2. Is stable: $\lim_{k\to\infty} x(k) = 0$
3. Optimizes "performance"
4. Maximizes the set $\{x(0) \,|\, \text{Conditions 1-3 are met}\}$

Constrained Infinite Time Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_\infty^\star(x(0)) = \min_{u(\cdot)} & \sum_{i=0}^{\infty} l(x_i, u_i) \\
\text{subj. to } & x_{i+1} = Ax_i + Bu_i,\ \ i = 0, \ldots, \infty \\
& x_i \in \mathcal{X}, u_i \in \mathcal{U}, i = 0, \ldots, \infty \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $l(x, u)$: "cost" of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ... but we can't compute it: there are an **infinite number of variables**

Constrained Finite Time Optimal Control (what we can sometimes solve)

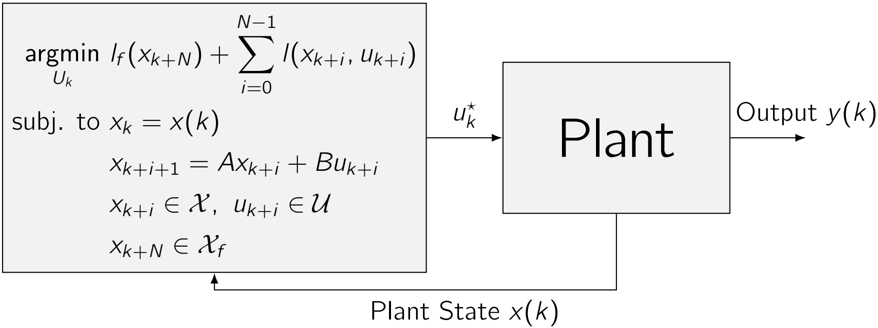
$$
\begin{aligned}
J^{\star}_{k\rightarrow k+N|k}(x(k)) = \min_{U_{k\rightarrow k+N|k}} \;& l_f(x_{k+N|k}) + \sum_{i=0}^{N-1} l(x_{k+i|k}, u_{k+i|k}) \\
\text{subj. to } & x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k},\;\; i = 0, \ldots, N-1 \\
& x_{k+i|k} \in \mathcal{X}, u_{k+i|k} \in \mathcal{U}, i = 0, \ldots, N-1 \\
& x_{k+N|k} \in \mathcal{X}_f \\
& x_{k|k} = x(k)
\end{aligned} \tag{1}
$$

where $U_{k\rightarrow k+N|k} = \{u_{k|k}, \ldots, u_{k+N-1|k}\}$.

Truncate after a finite horizon:

- $l_f(x_{k+N|k})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(k)$
- Find the optimal input sequence for the entire planning window N: $U_k^\star = \{u_k^\star, u_{k+1}^\star, \ldots, u_{k+N-1}^\star\}$
- Implement only the first control action $u_k^\star$

MPC (NMPC) Properties

Pros

- Any model
  - linear
  - nonlinear
  - single/multivariable
  - time delays
  - constraints
  - etc
- Any objective:
  - sum of squared errors
  - sum of absolute errors (i.e., integral)
  - worst error over time
  - economic objective
  - etc

Cons

- Very computationally demanding in the general case
- May or may not be stable
- May or may not be invariant

This lecture: Systems for which optimization is computationally tractable

Outline

1. Constrained Linear Optimal Control
2. Constrained Optimal Control: Quadratic cost
3. Constrained Optimal Control: 1-Norm and ∞-Norm Cost
4. Receding Horizon Control Notation

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1. Constrained Linear Optimal Control

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Constrained Linear Optimal Control

Cost function

$$J(x_0, U) = l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

- Squared Euclidian norm: $l_f(x_N) = x_N^\top P x_N$ and $l(x_i, u_i) = x_i^\top Q x_i + u_i^\top R u_i$, with $P \succeq 0$, $Q \succeq 0$, $R \succ 0$.
- $p = 1$ or $p = \infty$: $l_f(x_N) = \|Px_N\|_p$ and $l(x_i, u_i) = \|Qx_i\|_p + \|Ru_i\|_p$, with P, Q, R full column rank matrices

Constrained Finite Time Optimal Control problem (CFTOC)

$$\boxed{\begin{aligned} J^\star(x(k)) = \min_U \ & J(x_0, U) \\ \text{subj. to } & x_{i+1} = Ax_i + Bu_i, \ \ i = 0, \dots, N-1 \\ & x_i \in \mathcal{X}, \ \ u_i \in \mathcal{U}, \ \ i = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(k) \end{aligned} \qquad (2)}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Feasible Sets

Set of initial states $x(0)$ for which the optimal control problem (2) is feasible:

$$\begin{aligned}\mathcal{X}_0 = \quad & \{x_0 \in \mathbb{R}^n | \ \exists(u_0, \ldots, u_{N-1}) \text{ such that } x_i \in \mathcal{X}, \ u_i \in \mathcal{U}, \\ & i = 0, \ldots, N-1, \ x_N \in \mathcal{X}_f, \text{ where } x_{i+1} = Ax_i + Bu_i\}\end{aligned}$$

In general $\mathcal{X}_j$ is the set of states x_j at time j for which (2) is feasible:

$$\begin{aligned}\mathcal{X}_j = \quad & \{x_j \in \mathbb{R}^n | \ \exists(u_j, \ldots, u_{N-1}) \text{ such that } x_i \in \mathcal{X}, \ u_i \in \mathcal{U}, \\ & i = j, \ldots, N-1, \ x_N \in \mathcal{X}_f, \text{ where } x_{i+1} = Ax_i + Bu_i\},\end{aligned}$$

The sets $\mathcal{X}_j$ for $j = 0, \ldots, N$ play an important role in the solution of the CFTOC problem. They are independent of the cost.

Reminder: Unconstrained Solution

For quadratic cost (squared Euclidian norm) and **no state and input constraints**:

$$\{x \in \mathcal{X},\ u \in \mathcal{U}\} = \mathbb{R}^{n+m},\ \mathcal{X}_f = \mathbb{R}^n$$

we have the **time-varying** linear control law

$$u_i^\star = F_i x_i,\ i = 0, \dots, N-1.$$

If $N \to \infty$, we have the **time-invariant** linear control law

$$u_i^\star = F_\infty x_i,\ i = 0, 1, \dots$$

Next we show how to compute finite time **constrained** optimal controllers.

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4. Receding Horizon Control Notation

Transform Quadratic Cost CFTOC into QP

Quadratic Cost CFTOC

$$
\begin{aligned}
J^\star(x(k)) = \min_{U} \; & x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\
\text{subj. to } & x_{i+1} = Ax_i + Bu_i, \; i = 0, \dots, N-1 \\
& x_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; i = 0, \dots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

$$\Downarrow$$

QP Problem

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} \; & \frac{1}{2} z^\top H z + q^\top z + r \\
\text{subj. to } & Gz \leq h \\
& Az = b
\end{aligned}
$$

Outline

2. Constrained Optimal Control: Quadratic cost

    Construction of the QP with substitution

    Construction of the QP without substitution

    Quadratic Cost State Feedback Solution

Construction of the QP with substitution

Idea: Substitute the state equations (see lectures Week 2)

$$x_{i+1} = Ax_i + Bu_i, x_0 = x(k)$$

- **Step 1**: Rewrite the cost as

$$\begin{aligned} J(x(k), U) &= U^\top H U + 2x(k)^\top F U + x(k)^\top Y x(k) \\ &= [U^\top \ x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \ x(k)^\top]^\top \end{aligned}$$

Note: $\left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] \succeq 0$ since $J(x(k), U) \geq 0$ by assumption.

- **Step 2**: Rewrite the constraints compactly as (details on the next slide)

$$GU \leq w + Ex(k)$$

- **Step 3**: Rewrite the constrained optimal control problem as

$$\boxed{\begin{aligned} J^\star(x(k)) = \min_U \quad & [U^\top \ x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \ x(k)^\top]^\top \\ \text{subj. to} \quad & GU \leq w + Ex(k) \end{aligned}}$$

Construction of QP constraints with substitution

Inequalities $GU \leq w + Ex(k)$ for $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then G, E and w are defined as follows

$$G = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Solution of the QP with substitution

$$
\begin{aligned}
J^\star(x(k)) = \min_U \quad & [U^\top \ x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \ x(k)^\top]^\top \\
\text{subj. to} \quad & GU \leq w + Ex(k)
\end{aligned}
$$

For a given x(k), $U^\star$ can be found via a QP solver.

Outline

2. Constrained Optimal Control: Quadratic cost

    Construction of the QP with substitution

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Construction of the QP without substitution (1/3)

Idea: Keep state equations as equality constraints (often more efficient)

We transform the CFTOC problem into the QP problem

$$\begin{aligned} J^\star(x(k)) = \min_z \quad & [z^\top \; x(k)^\top] \left[\begin{smallmatrix} \bar{H} & 0 \\ 0 & Q \end{smallmatrix}\right] [z^\top \; x(k)^\top]^\top \\ \text{subj. to } \; & G_{\mathrm{in}} z \le w_{\mathrm{in}} + E_{\mathrm{in}} x(k) \\ & G_{\mathrm{eq}} z = E_{\mathrm{eq}} x(k) \end{aligned}$$

- Define variable:

$$z = \begin{bmatrix} x_1^\top & \dots & x_N^\top & u_0^\top & \dots & u_{N-1}^\top \end{bmatrix}^\top$$

- Equalities from system dynamics $x_{i+1} = Ax_i + Bu_i$:

$$G_{\mathrm{eq}} = \left[\begin{array}{ccccc:cccc} I & & & & & -B & & & \\ -A & I & & & & & -B & & \\ & -A & I & & & & & -B & \\ & & \ddots & \ddots & & & & & \ddots \\ & & & -A & I & & & & -B \end{array}\right], E_{\mathrm{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Construction of the QP without substitution (2/3)

Inequalities $G_{\text{in}} z \leq w_{\text{in}} + E_{\text{in}} x(k)$ for $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then matrices G_{in}, w_{in} and E_{in} are:

$$G_{\text{in}} = \left[\begin{array}{ccccc|ccccc}
0 & & & & & 0 & & & & \\
& A_x & & & & & 0 & & & \\
& & \ddots & & & & & \ddots & & \\
& & & A_x & & & & & 0 & \\
& & & & A_f & & & & & 0 \\
\hline
0 & & & & & A_u & & & & \\
& 0 & & & & & A_u & & & \\
& & \ddots & & & & & \ddots & & \\
& & & 0 & & & & & A_u & \\
& & & & 0 & & & & & A_u
\end{array}\right] \qquad w_{\text{in}} = \left[\begin{array}{c} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

$$E_{\text{in}} = \left[\, -A_x^\top \;\; 0 \;\; \cdots \;\; 0 \,\right]^\top$$

Construction of the QP without substitution (3/3)

Build cost function from MPC cost $x_N^\top P x_N + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i$

$$\bar{H} = \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```

Outline

2. Constrained Optimal Control: Quadratic cost

   Construction of the QP with substitution

   Construction of the QP without substitution

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Quadratic Cost State Feedback Solution

$$J^\star(x(k)) = \min_U \quad [U^\top \; x(k)^\top] \left[\begin{smallmatrix} H & F^\top \\ F & Y \end{smallmatrix}\right] [U^\top \; x(k)^\top]^\top$$
$$\text{subj. to} \qquad GU \le w + Ex(k)$$

The CFTOC problem is a **multiparametric quadratic program (mp-QP)** with the following solution properties:

- The first component of the optimal solution has the form

$$u_0^\star = \kappa(x(k)), \quad \forall x(k) \in \mathcal{X}_0,$$

  $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$

- The polyhedral sets $CR^j = \{x \in \mathbb{R}^n | H^j x \le K^j\}$, $j = 1, \dots, N^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
- The value function $J^\star(x(k))$ is convex and piecewise quadratic on polyhedra.

$\Rightarrow$ Explicit MPC addresses how to compute this solution.

Example

Consider the double integrator

$$\begin{cases} x(t+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

subject to constraints

$$-1 \leq u(k) \leq 1, \;\; k = 0, \ldots, 5$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \;\; k = 0, \ldots, 5$$

Compute the **state feedback** optimal controller $u^*(x(k))$ solving the CFTOC

problem with $N = 6$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.1$, P the solution of the ARE, $\mathcal{X}_f = \mathbb{R}^2$.

Example

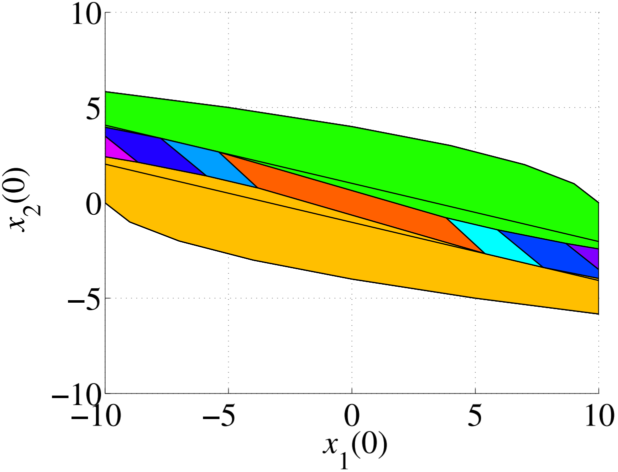

Figure: Partition of the state space for the affine control law $u^*(x)$ ($N_0^r = 13$)

Example

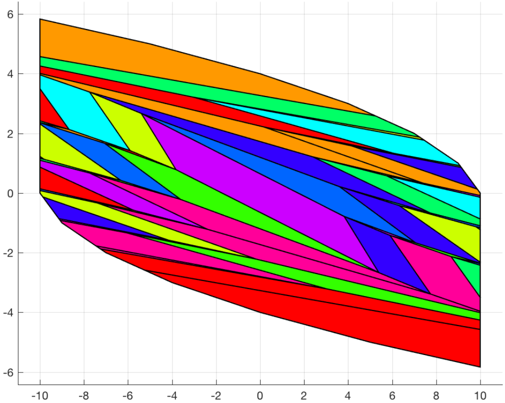

Figure: Partition of the state space for the affine control law $u^*(x)$ ($N_0^r = 61$)

Example

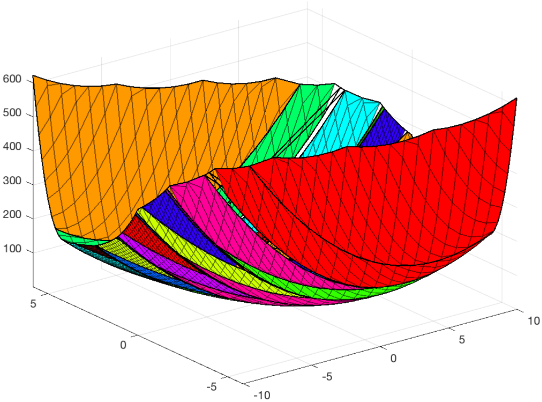

Figure: Value function for the affine control law $u^*(x)$ ($N_0^r = 61$)

Example

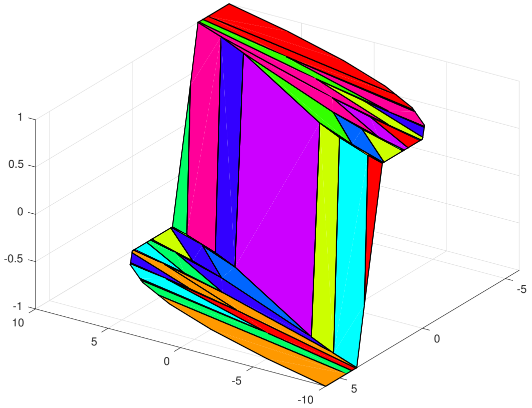

Figure: Optimal control input for the affine control law $u^*(0)$ ($N_0^r = 61$)

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Transform 1- / ∞-Norm Cost CFTOC into LP

1-Norm / ∞-Norm Cost CFTOC

$$
\begin{aligned}
J^\star(x(k)) = \min_U \ & \|Px_N\|_p + \sum_{i=0}^{N-1} \|Qx_i\|_p + \|Ru_i\|_p \\
\text{subj. to } & x_{i+1} = Ax_i + Bu_i,\ i = 0, \ldots, N-1 \\
& x_i \in \mathcal{X},\ u_i \in \mathcal{U},\ i = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(k)
\end{aligned}
$$

$$\Downarrow$$

LP Problem

$$
\begin{aligned}
\min_{z \in \mathbb{R}^n} \ & c^\top z \\
\text{subj. to } & Gz \leq h \\
& Az = b
\end{aligned}
$$

ℓ_∞ minimization

Constrained ℓ_∞ (Chebyshev) minimization:

$$\min_{x \in \mathbb{R}^n} \ ||x||_\infty$$
$$\text{subj. to } Fx \leq g$$

Write this as a max of linear functions.

Equivalent to:

$$\min_{x \in \mathbb{R}^n} \ [\max\{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}]$$
$$\text{subj. to } Fx \leq g$$

ℓ_∞ minimization (cont'd)

Equivalent to:

$$\begin{aligned} \min_{x,t}\ & t \\ \text{subj. to}\ & x_i \le t \quad i = 1, \dots, n \\ & -x_i \le t \quad i = 1, \dots, n \\ & Fx \le g \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \min_{x,t}\ & t \\ \text{subj. to}\ & -\mathbf{1}t \le x \le \mathbf{1}t \\ & Fx \le g \end{aligned}$$

- The notation '$\mathbf{1}$' indicates a vector of ones.
- The constraint $-\mathbf{1}t \le x \le \mathbf{1}t$ bounds the absolute value of every element of x with a common scalar variable t.

ℓ_1 minimization

Constrained ℓ_1 minimization:

$$\min_{x \in \mathbb{R}^n} \ ||x||_1$$
$$\text{subj. to } Fx \leq g$$

Write this as a max of linear functions.

Equivalent to:

$$\min_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \max\left\{x_i, -x_i\right\} \right]$$
$$\text{subj. to } Fx \leq g$$

ℓ_1 minimization (cont'd)

Equivalent to:

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}^m} \quad & t_1 + \ldots + \ldots t_m \\ \text{subj. to} \quad & x_i \le t_i \quad i = 1, \ldots, m \\ & -x_i \le t_i \quad i = 1, \ldots, m \\ & Fx \le g \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}^m} \quad & \mathbf{1}^\top t \\ \text{subj. to} \quad & -t \le x \le t \\ & Fx \le g \end{aligned}$$

- The notation '**1**' indicates a vector of ones.
- The constraint $-t \le x \le t$ bounds the absolute value of each component of x with a component of the vector variable t.

NB: trick was to add variables and write the problem in **epigraph** form.

Outline

3. Constrained Optimal Control: 1-Norm and ∞-Norm Cost

   Construction of the LP with substitution

   1- /∞-Norm State Feedback Solution

Construction of the LP with substitution

Recall that the $\infty-$norm problem can be equivalently formulated as

$$
\begin{aligned}
\min_{z} \quad & \varepsilon_0^x + \ldots + \varepsilon_N^x + \varepsilon_0^u + \ldots + \varepsilon_{N-1}^u \\
\text{subj. to} \quad & -\mathbf{1}_n \varepsilon_i^x \leq \pm Q \left[A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \right], \\
& -\mathbf{1}_r \varepsilon_N^x \leq \pm P \left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \right], \\
& -\mathbf{1}_m \varepsilon_i^u \leq \pm R u_i, \\
& A^i x_0 + \sum_{j=0}^{i-1} A^j B u_{i-1-j} \in \mathcal{X}, \;\; u_i \in \mathcal{U}, \\
& A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \\
& x_0 = x(k), \;\; i = 0, \ldots, N-1
\end{aligned}
$$

Construction of the LP with substitution

The problem results in the following standard LP

$$\begin{array}{ll} \min\limits_{z} & c^\top z \\ \text{subj. to} & \bar{G}z \leq \bar{w} + \bar{S}x(k) \end{array}$$

where $z := \{\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u, u_0^\top, \ldots, u_{N-1}^\top\} \in \mathbb{R}^s$,
$s := (m+1)N + N + 1$ and

$$\bar{G} = \begin{bmatrix} G_\varepsilon & G_u \\ 0 & G \end{bmatrix}, \; \bar{S} = \begin{bmatrix} S_\varepsilon \\ S \end{bmatrix}, \; \bar{w} = \begin{bmatrix} w_\varepsilon \\ w \end{bmatrix}$$

For a given $x(k)$, $U^\star$ can be obtained via an LP solver.

$\rightarrow$ The $1-$norm case works similarly.

Outline

3. Constrained Optimal Control: 1-Norm and ∞-Norm Cost

    Construction of the LP with substitution

    1- /∞-Norm State Feedback Solution

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$$\begin{array}{ll} \min\limits_{z} & c^\top z \\ \text{subj. to} & \bar{G}z \le \bar{w} + \bar{S}x(k) \end{array}$$

The CFTOC problem is a **multiparametric linear program (mp-LP)** with the following solution properties:

- The first component of the multiparametric solution has the form

$$u_0^\star = \kappa(x(0)), \quad \forall x(0) \in \mathcal{X}_0,$$

$\kappa : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PieceWise Affine on Polyhedra

$$\kappa(x) = F^j x + g^j \quad \text{if} \quad x \in CR^j, \quad j = 1, \dots, N^r$$

- The polyhedral sets $CR^j = \{x \in \mathbb{R}^n | H^j x \le K^j\}$, $j = 1, \dots, N^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.
- In case of multiple optimizers a PieceWise Affine control law exists.
- The value function $J^\star(x(0))$ is convex and piecewise linear on polyhedra.

Solution properties: Quadratic vs. $1-/\infty$-norm cost

Let n = # optimization variables.

Quadratic cost (pos. def) : Solution is either

- unique and in the interior of feasible set –> no constraints active
- unique and on the boundary of feasible set –> at least 1 active constraint

Linear cost: Solution is either

- unbounded
- unique at a vertex of the feasible set –> at least n active constraints,
- a set of multiple optima –> at least 1 active constraint

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Receding Horizon Control Notation (1/3)

Consider DT model
$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

$$x(k) \in \mathcal{X},\ u(k) \in \mathcal{U},\ \forall k \geq 0$$

The Constrained Finite Time Optimal Control (CFTOC) Problem

$$\begin{aligned} J^\star_{k\to k+N|k}(x(k)) = \min_{U_{k\to k+N|k}} \ & l_f(x_{k+N|k}) + \sum_{i=0}^{N-1} l(x_{k+i|k}, u_{k+i|k}) \\ \text{subj. to } & x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k},\ i = 0, \ldots, N-1 \\ & x_{k+i|k} \in \mathcal{X},\ u_{k+i|k} \in \mathcal{U},\ i = 0, \ldots, N-1 \\ & x_{k+N|k} \in \mathcal{X}_f \\ & x_{k|k} = x(k) \end{aligned} \tag{4}$$

is solved at time k with $U_{k\to k+N|k} = \left\{u_{k|k}, \ldots, u_{k+N-1|k}\right\}$.

Receding Horizon Control Notation (2/3)

- $x(k)$ is the state of the system at time k.
- $x_{i+k|k}$ is the state of the model at time $k+i$, predicted at time k obtained by starting from the current state $x_{k|k} = x(k)$ and applying to the system model

$$x_{k+1|k} = Ax_{k|k} + Bu_{k|k}$$

  the input sequence $u_{k|k}, \ldots, u_{k+i-1|k}$.
- For instance, $x_{3|1}$ represents the predicted state at time 3 when the prediction is done at time $k = 1$ starting from the current state $x(1)$. It is different, in general, from $x_{3|2}$ which is the predicted state at time 3 when the prediction is done at time $k = 2$ starting from the current state $x(2)$.
- Similarly $u_{k+i|k}$ is read as "the input u at time $k+i$ computed at time k".

Receding Horizon Control Notation (3/3)

- Let $U^\star_{k\to k+N|k} = \left\{u^\star_{k|k}, \dots, u^\star_{k+N-1|k}\right\}$ be the optimal solution. The first element of $U^\star_{k\to k+N|k}$ is applied to the system

$$u(k) = u^\star_{k|k}(x(k)).$$

- The CFTOC problem is reformulated and solved at time $k+1$, based on the new state $x_{k+1|k+1} = x(k+1)$.

Receding horizon control law

$$\kappa_k(x(k)) = u^\star_{k|k}(x(k))$$

Closed loop system

$$x(k+1) = Ax(k) + B\kappa_k(x(k)) := g_{cl}(x(k)),\ \ k \geq 0$$

RHC: Time-invariant Systems

As the system, the constraints and the cost function are time-invariant, the solution $\kappa_k(x(k))$ becomes a time-invariant function of the initial state $x(k)$. Thus, we can simplify the notation as

$$\begin{aligned} J^\star(x(k)) = \min_{U} \; & l_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \\ \text{subj. to } & x_{i+1} = Ax_i + Bu_i, \; i = 0, \dots, N-1 \\ & x_i \in \mathcal{X}, \; u_i \in \mathcal{U}, \; i = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(k) \end{aligned} \tag{5}$$

where $U = \{u_0, \dots, u_{N-1}\}$.

The control law and closed loop system are **time-invariant** as well, and we write $\kappa(x(k))$ for $\kappa_k(x(k))$.