Model Predictive Control
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Exercise 1
System Theory Basics

Exercise 1 **Equilibrium Point and Linearization**

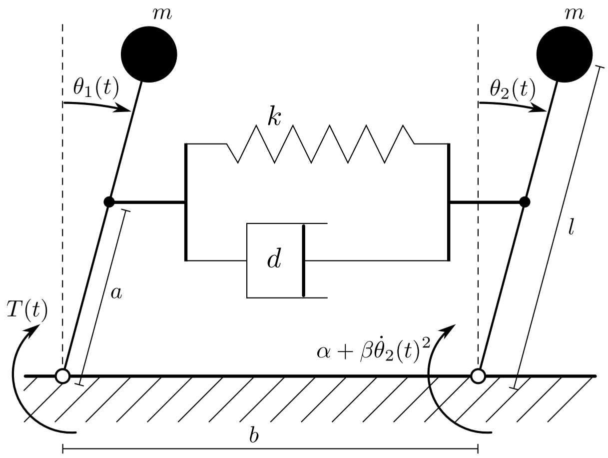

Figure 1: Inverted Double Pendulum

Consider the system in Fig. 1. This inverted double-pendulum consists of two point masses m on two rods of length l. The rods are supported in a pivot on a horizontal plane. At distance a from the pivot, they are connected by a spring-damper system. The spring has a spring constant k. Its neutral position is at $\theta_1(t) = \theta_2(t) = 0$. For simplicity, the vertical movement can be neglected. In the pivot of the left mass an external torque $T(t)$ can be applied. In the right pivot there is a disturbance torque of the form $\alpha + \beta\dot{\theta}_2(t)^2$.

The differential equations representing the dynamics of the described system are

$$
\begin{aligned}
ml^2\ddot{\theta}_1(t) &= mgl\sin\theta_1(t) + a\cos\theta_1(t)F(t) + T(t) \\
ml^2\ddot{\theta}_2(t) &= mgl\sin\theta_2(t) - a\cos\theta_2(t)F(t) + \alpha + \beta\dot{\theta}_2^2(t),
\end{aligned}
$$

where $F(t)$ is the force resulting from the spring-damper system.

1) Derive the force $F(t)$ dependent on $\theta_1(t), \dot{\theta}_1(t), \theta_2(t), \dot{\theta}_2(t)$.

2) Let $T = 0, \alpha = 0$, and $\frac{mgl}{2a^2k} \leq 1$. Calculate the equilibria of the system, denoted by a bar on the variables. Let $\bar{\theta}_2 = 0$ and $\bar{T} \neq 0, \alpha \neq 0$. Express $\bar{\theta}_1$ and $\bar{T}$ as a function of α such that the double pendulum is at an equilibrium.

3) Now let $\alpha = 0$. Linearize the system around the equilibrium point $\bar{\theta}_1 = \bar{\theta}_2 = \bar{T} = 0$ and derive the continuous-time state-space description

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
$$

where $u(t) = T(t) - \bar{T}$, and $y(t) = \theta_1(t) - \bar{\theta}_1$.

Hint: For small x the following approximations can be made: $\sin(x) \approx x$, and $\cos(x) \approx 1$.

Exercise 2 **Discretization of a LTI continuous-time state-space model**

Consider the following continuous-time dynamic system:

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2.7 \\ -3.1 & 1.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 & 2.1 \\ 1.1 & 3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

1) Discretize the system with $T_s = 0.5$ using Euler's first order approximation.

2) Discretize the system with $T_s = 0.5$ using exact discretization and check the result with the Matlab function `c2d`.

3) Compare the outputs of the continuous model and the discretized models in a dynamic simulation, starting from the same initial state and applying the same input. What would be a suitable discetization time T_s for the Euler discretization?

Hint: The following Matlab commands may be useful to solve the exercise: `expm` and `ode45`.

Exercise 3 **Sum of Lyapunov functions**

Let $V_i(x(k)) := x(k)^T P_i x(k)$ be a Lyapunov function for the system $x(k+1) = Ax(k)$ for $i = 1, 2$, with a rate of decrease of $x^T(k)\Gamma x(k)$, i.e.:

$$V_i(x(k+1)) - V_i(x(k)) \leq -x^T(k)\Gamma x(k) \ .$$

Show that $V(x(k)) = \alpha V_1(x(k)) + (1-\alpha)V_2(x(k))$ is also a Lyapunov function with a rate of decrease of $x^T(k)\Gamma x(k)$ for any $\alpha \in [0, 1]$.

Exercise 4 **Lyapunov Functions**

Consider the following discrete time LTI system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \end{aligned}$$

where

$$A = \begin{bmatrix} -0.4 & -1.1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

Further, consider a feedback controller for the above system of the form

$$u(k) = \begin{bmatrix} -4.4 & -4.45 & 0 \end{bmatrix} x(k)$$

and the function $V(x(k)) = x^T(k)Px(k)$, where $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Show that the equilibrium point $x(k) = 0$ of the closed-loop system is globally asymptotically stable using the Lyapunov function $V(x(k))$.