Almost Global Homotopy Theory

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Plan

- Preliminary: classical homotopy theory; stable homotopy theory.
- Model category;
- Equivariant homotopy theory;
- Global homotopy theory;
- Almost global homotopy theory;
- Examples: Quasi-theories.

Preliminary: classical homotopy theory

Top The category of topological spaces and continuous maps.
Top_* The category of based spaces and based maps.

a homotopy h between continuous maps f and g: $X \longrightarrow Y$

a map $h : X \times [0,1] \longrightarrow Y$ such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$.
f and g are **homotopic** $f \overset{h}{\simeq} g$ if there exists a homotopy between them.
Being homotopic is an equivalence relation.

$f : X \longrightarrow Y$ is a homotopy equivalence

there exists a map $g : Y \longrightarrow X$ and $g \circ f \backsimeq id_X$ and $f \circ g \backsimeq id_Y$.
$X \backsimeq Y$. X and Y are homotopy equivalent.

Ho(Top): the associated homotopy category

- objects: topological spaces.
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Homotopy groups: $\pi_n(X, x) := [S^n, X]_*$.

Example

$\pi_0(S^0) = S^0$. $\pi_1(S^1) \cong \mathbb{Z}$. $\pi_n(S^n) \cong \mathbb{Z}$.

$f : X \longrightarrow Y$ is a weak homotopy equivalence

$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

- Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.
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Fibration

Homotopy Lifting Property

$p: E \longrightarrow B$ is called a fibration if $p: E \longrightarrow B$ satisfies the Homotopy Lifting Property, i.e. given any map $f: X \longrightarrow E$ and homotopy $h: X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h}: X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dotted, "\overline{h}"] & B \end{tikzcd}$$

Example

• constant map; • covering space; • fiber bundle.

Serre fibration

A $p: E \longrightarrow B$ satisfying the homotopy lifting property with respect to $X = D^n$.

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Cofibrations: dual to fibration

Homotopy Extension Property

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The inclusion of a relative CW-complex is a cofibration.

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Spectrum

a sequence $\{X_n, \sigma_n\}_n$ where X_n are based spaces and $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$ are based maps.

Stable homotopy groups

$$\pi_n^S(X) = colim_{l\to\infty}\pi_{n+l}(S^l \wedge X).$$

Spectrum defines homology and cohomology

$$H_n(Y) = colim_{l\to\infty}\pi_{n+l}(Y_+ \wedge X_l);$$

$$H^n(Y) = colim_{l\to\infty}[S^l \wedge Y_+, X_{n+l}].$$

Brown Representation Theorem

(i) For each homology theory, there exists a spectrum representing it.
(ii) For each cohomology theory, there exists a spectrum representing it.

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$$H^n(Y) = colim_{l\to\infty}[S^l \wedge Y_+, X_{n+l}].$$

Brown Representation Theorem

(i) For each homology theory, there exists a spectrum representing it.
(ii) For each cohomology theory, there exists a spectrum representing it.

What is "homotopy theory"?

Motivating example: the category of topological spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Features

- **2-out-of-3**: If two of f, g, gf are weak homotopy equivalences, then so is the third.
- **Retracts**:

$$\begin{tikzcd} A \arrow[r, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & B \arrow[r, "r"'] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r, "i'"] \arrow[rr, bend right, "id"'] & B' \arrow[r, "r'"] & A' \end{tikzcd}$$

f is a retract of g.

If g is a weak homotopy equivalence/Serre fibration/retract of relative cell complex, then so is f.

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Quillen's perspective: Model Category

Definition: A model structure on a category $\mathcal{C}$

• Weak Equivalence • Fibration • Cofibration.

satisfying the axioms:

- **Retracts**;
- **2-out-of-3**;
- **Lifting**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

  The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

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Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
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$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w)\rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

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$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

$$\underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Elmendorf's theorem

We have the equivalence of the homotopy categories

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Model structures on equivariant stable homotopy theory

Right Philosophy for Stable Homotopy Theory (1990s) [MMSS]

$$\begin{tikzcd}
S-\text{Modules} & & \text{Symmetric Spectra} \\
& \text{Spectra} \arrow[ul, equal, "Q.E."'] \arrow[ur, equal, "Q.E."'] \arrow[dl, equal, "Q.E."] \arrow[dr, equal, "Q.E."] & \\
\text{Orthogonal Spectra} & & \text{Many Others}
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Which is the BEST model?

Orthogonal G−spectra.

Why BEST?

Combine the best feature of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

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Orthogonal $G-$spectrum [MMSS]

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
- a natural product $\mu : X \overline{\wedge} X \longrightarrow X \circ \bigoplus$

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The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG . K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
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- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

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$K^0_{\{e\}}(X) = K^0(X)$.
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- objects: inner product real spaces;
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An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

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- level model structure;
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Observation: restriction maps don't need to be identity maps.

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- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
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- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
    - linear isometric embedding: raising degree;
    - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
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The right subject: D_0^W−spectra [Huan]

The category D_0T^W

the full subcategory of D_0T consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in D_0T^W.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\leftrightarrows}} Sp_W^{D_0}$$

• The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
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Divisible group: a better algebraic object associated to an elliptic curve than formal group.

Example: Tate curve and Tate K-theory

$$0 \longrightarrow \mathbb{G}_m \longrightarrow Tate(q) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}_p} \mathbb{Q}_p)[p^k]) = K^0_{Tate}(B\mathbb{Z}_{p^k})$.

The general case

$$0 \longrightarrow F \longrightarrow \mathbb{G} \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

F: the formal group of a cohomology theory $\{E^*_{n,G}(-)\}_G$.
The corresponding cohomology theory $\{A^*_{n,G}(-)\}_G$.
The corresponding quasi-theory: $QE^*_{n,G}(X) = \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma)$.

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The globalization of Quasi-theories [Huan]

Theorem

If the theory $\{E^*_{n,G}(-)\}_G$ can be globalized, there is a D^W_0-spectrum representing the quasi-theory $\{QE^*_{n,G}(-)\}_G$.
In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

[Zhen Huan: *Quasi-elliptic cohomology*, PhD thesis]
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My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

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Some references

`https://huanzhen84.github.io/zhenhuan/Huan-HUST-2018.pdf`

- Ando, Hopkins, Strickland: "Elliptic spectra, the Witten genus and the theorem of the cube". Invent. Math. , 146(3):595–687, 2001.
- Ando, Hopkins, Strickland, "The sigma orientation is an H_∞ map", Amer. J. 2004;
- Atiyah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Atiyah,"Equivariant K-theory and completion", J. Differential Geometry 3 1969.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Bohmann, "Global orthogonal spectra", Homology, Homotopy and Applications. Vol 16 (2014), No 1, 313–332.
- Gepner, "Homotopy Topoi and Equivariant Elliptic Cohomology", Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 1999.
- Gepner, Henriques, "Homotopy Theory of Orbispaces", available at arXiv:math/0701916.
- Ginzburg, Kapranov, Vasserot, "Elliptic algebras and equivariant elliptic cohomology I", available at arXiv:q-alg/9505012.
- Greenlees, May, "Localization and completion theorems for MU- module spectra", Ann. of Math. (2), 146(3) (1997), 509–544.
- Katz, Mazur, "Arithmetic moduli of elliptic curves", Annals of Mathematics Studies, vol. 108, 1985.
- Lurie, "A Survey of Elliptic Cohomology", in Algebraic Topology Abel Symposia Volume 4, 2009, pp 219–277.
- Mandell, May, "Equivariant orthogonal spectra and S-modules", Mem.,Amer. Math. Soc. 159 (2002), no. 755, x+108 pp.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schwede, "Global Homotopy Theory", `global.pdf`.