Twisted Quasi-elliptic cohomology and twisted equivariant elliptic cohomology

Zhen Huan and Matthew Spong

Abstract. In this paper we construct a Chern character map from quasi-elliptic cohomology theory to Devoto's equivariant elliptic cohomology theory. Moreover, we define twisted quasi-elliptic cohomology. To provide a geometric interpretation of it, we define a version of twisted equivariant loop space via bibundles. In addition, we construct a twisted Chern character map from it to twisted equivariant elliptic cohomology theory.

1. Introduction

It is a classical result that the Chern character maps complex K-theory isomorphically onto complex cohomology. In the equivariant case, this is not always true. In [**22**] Rosu described $K^*_T(X)\otimes\mathbb{C}$ in terms of $H^*_T(X)\otimes\mathbb{C}$ via a globalised Chern character with T an abelian compact Lie group. In [**13**, Theorem 3.9] Freed, Hopkins and Teleman generalised the result to the twisted case. They described twisted equivariant K-theory via the twisted Chern character in terms of twisted equivariant cohomology of fixed-point sets with coefficients in certain equivariantly flat complex line bundles. Moreover, the construction of Chern character can be carried to higher chromatic level. In [**15**, Section 3] Ganter discussed elliptic Chern character map in the context of equivariant elliptic cohomology.

In our paper, based on the idea in [**22**], we construct a Chern character map from complex quasi-elliptic cohomology to Devoto's equivariant elliptic cohomology [**8**] when the group is finite. In this way we provide another Chern character map with the information of elliptic cohomology built in. Quasi-elliptic cohomology, introduced by the first author in [**17**], is a variant of Tate K-theory, which is the generalized elliptic cohomology associated to the Tate curve. The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity [Section 2.6, [**2**]]. The relation between quasi-elliptic cohomology and Tate K-theory can be expressed by

$$QEll^*(X/\!\!/G)\otimes_{\mathbb{Z}[q^{\pm}]}\mathbb{Z}((q))\cong K_{Tate}(X/\!\!/G).$$

As shown in Section 5, we connect complex quasi-elliptic cohomology with a Chern character map to Devoto's equivariant elliptic cohomology, which is constructed from fixed point spaces. The key role in the construction are the Atiyah-Segal map [**3**] and the Chern Character of complex K-theory.

In addition, we show the construction can be carried to twisted theories. In [**5**], to demonstrate the relation between physics and elliptic cohomology, Berwick-Evans constructed a twisted equivariant refinement of $TMF \otimes \mathbb{C}$ motivated by the geometry of $2|1-$dimensional supersymmetric sigma models and defined twisted equivariant elliptic cohomology. In Section 6 we construct twisted quasi-elliptic cohomology $QEll^*_\alpha(-)$. It has the relation with twisted equivariant Tate K-theory $\alpha K_{Tate}(X/\!\!/G)$

$$QEll^*_\alpha(X/\!\!/G)\otimes_{\mathbb{Z}[q^\pm]\mathbb{Z}((q))} \cong \alpha K_{Tate}(X/\!\!/G)$$

The first author gave a loop space construction of quasi-elliptic cohomology in Given an orbifold M, quasi-elliptic cohomology $QEll(M)$ is defined as the orbifold K-theory of a certain subgroupoid of the orbifold loop space $Loop(M)$, and takes values in modules over $\mathbb{Z}[q^\pm]$.

Quasi-elliptic cohomology is a variant of Tate K-theory, a form of elliptic cohomology at the Tate curve, which was introduced by the first author in [**18**]. Given an orbifold M, quasi-elliptic cohomology $QEll(M)$ is defined as the orbifold K-theory of a certain subgroupoid of the orbifold loop space $Loop(M)$, and takes values in modules over $\mathbb{Z}[q^\pm]$. In this paper, for a finite group G acting on a compact space X, we compare quasi-elliptic cohomology of the orbifold $X/\!\!/G$ to Devoto's G-equivariant elliptic cohomology of X [**8**]. We carry out the comparison by constructing a map from quasi-elliptic cohomology to Devoto's theory, the main ingredient of which is a version of the equivariant Chern character. The equivariant Chern character is a global (or delocalised) version of the ordinary Chern character, and has been constructed in various forms in the literature. For example, see [**10**] and [**7**], and also [**22**] and [**13**]. Notably, an interpretation of the equivariant Chern character was given in [**6**] as a version of super holonomy on constant super loops in $X/\!\!/G$.

After constructing the character map, we define twisted quasi-elliptic cohomology, and show that a twisted version of the character map compares this with Devoto's twisted equivariant elliptic cohomology. Along the way, we construct a twisted version of the orbifold loop space $Loop(X/\!\!/G)$.

In a future paper, we expect to be able to use these results to compare power operations in the two theories. Indeed, in her thesis [**18**], the first author defined power operations for quasi-elliptic cohomology, and we intend to explore operations in the twisted version also.

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2. Quasi-elliptic cohomology

2.1. Definition. In this section we recall the definition of quasi-elliptic cohomology in term of equivariant K-theory and state the conclusions that we need in this paper. For more details on quasi-elliptic cohomology, please refer [**17**].

Let X be a $G-$space. Let $G^{tors} \subseteq G$ be the set of torsion elements of G. Let $\sigma \in G^{tors}$. The fixed point space X^σ is a $C_G(\sigma)-$space. We can define a

$\Lambda_G(\sigma)-$action on X^σ by

$$[g,t]\cdot x := g\cdot x.$$

Then quasi-elliptic cohomology of the orbifold $X/\!\!/G$ is defined by

DEFINITION 2.1.

$$QEll^*(X/\!\!/G) := \prod_{g\in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g) = \Big(\prod_{g\in G^{tors}} K^*_{\Lambda_G(g)}(X^g)\Big)^G, \tag{2.1}$$

where G^{tors}_{conj} is a set of representatives of $G-$conjugacy classes in G^{tors}.

We have the ring homomorphism

$$\mathbb{Z}[q^\pm] = K^0_{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\mathrm{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection $[a,t]\mapsto e^{2\pi i t}$ and the second is via the collapsing map $X \longrightarrow \mathrm{pt}$. So $QEll^*_G(X)$ is naturally a $\mathbb{Z}[q^\pm]-$algebra.

PROPOSITION 2.2. The relation between quasi-elliptic cohomology and Tate K-theory is

$$QEll^*(X/\!\!/G)\otimes_{\mathbb{Z}[q^\pm]}\mathbb{Z}((q)) = K^*_{Tate}(X/\!\!/G). \tag{2.2}$$

2.2. Loop space. In [**17**, Section 2] Huan provides loop space construction for quasi-elliptic cohomology. We review that model in this section.

For any space X, we have the free loop space of X

$$LX := \mathbb{C}^\infty(S^1, X). \tag{2.3}$$

It comes with an evident action by the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by rotating the circle

$$t\cdot\gamma := (s\mapsto \gamma(s+t)),\ t\in S^1,\ \gamma\in LX. \tag{2.4}$$

Let G be a compact Lie group. Suppose X is a right G-space. The free loop space LX is equipped with an action by the loop group LG

$$\delta\cdot\gamma := (s\mapsto \delta(s)\cdot\gamma(s)),\ \text{for any } s\in S^1,\ \delta\in LX,\ \ \gamma\in LG. \tag{2.5}$$

Combining the action by group of automorphisms $Aut(S^1)$ on the circle and the action by LG, we get an action by the extended loop group ΛG on LX. $\Lambda G := LG\rtimes\mathbb{T}$ is a subgroup of

$$LG\rtimes Aut(S^1),\ (\gamma,\phi)\cdot(\gamma',\phi') := (s\mapsto \gamma(s)\gamma'(\phi^{-1}(s)), \phi\circ\phi') \tag{2.6}$$

with $\mathbb{T}$ identified with the group of rotations on S^1. ΛG acts on LX by

$$\delta\cdot(\gamma,\phi) := (t\mapsto \delta(\phi(t))\cdot\gamma(\phi(t))),\ \text{for any } (\gamma,\phi)\in\Lambda G,\ \text{and } \delta\in LX. \tag{2.7}$$

Let $\widetilde{\delta} : G\times\mathbb{T}\longrightarrow X$ denote the map $(g,t)\mapsto \delta(t)g$. The action on δ by (γ,t) can be interpreted as precomposing $\widetilde{\delta}$ with a $G-$bundle map covering the rotation ϕ.

$$\begin{tikzcd} G\times\mathbb{T} \arrow[r, "(g,t)\mapsto(\gamma(t)g,\phi(t))"] \arrow[d] & G\times\mathbb{T} \arrow[r, "\widetilde{\delta}"] \arrow[d] & X \\ \mathbb{T} \arrow[r, "\phi"] & \mathbb{T} & \end{tikzcd} \tag{2.8}$$

More generally, we have the definition of the equivariant loop space $Loop(X/\!\!/G)$ below.

Definition 2.3. We define the equivariant loop space $Loop(X/\!\!/G)$ as the category with objects

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

where π is a principal $G-$bundle over $\mathbb{T}$ and f is a $G-$map. A morphism

$$(\alpha, t) : \{ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X \} \longrightarrow \{ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X \}$$

consists of a $G-$bundle map α and a rotation t making the diagrams commute.

$$\begin{tikzcd} P' \arrow[r, "\alpha"] \arrow[d] \arrow[rr, bend left, "f'"] & P \arrow[r, "f"] \arrow[d] & X \\ \mathbb{T} \arrow[r, "t"] & \mathbb{T} & \end{tikzcd}$$

The groupoid $\Lambda(X/\!\!/G)$ is a subgroupoid of $Loop(X/\!\!/G)$ consisting of constant loops.

3. Devoto's equivariant elliptic cohomology over $\mathbb{C}$

Our reference for this and the next section is [**5**]. For G a finite group, in [**8**] Devoto defined an equivariant refinement of the elliptic cohomology of Landweber, Ravenel and Stong [**21**]. Let $\mathcal{C}(G)$ denote the set of pairs of commuting elements of G, and $L \subset \mathbb{C}^2$ the subspace of pairs (t_1, t_2) such that the imaginary part of t_1/t_2 is defined and positive. The group $SL_2(\mathbb{Z})$ acts on $L \times \mathcal{C}(G)$ from the right by

$$((t_1, t_2), (g, h)) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ((at_1 + ct_2, bt_1 + dt_2), (g^d h^{-b}, g^{-c} h^a))$$

and the group G acts on $L \times \mathcal{C}(G)$ from the right by

$$((t_1, t_2), (g, h)) \cdot k := ((t_1, t_2), (k^{-1}gk, k^{-1}hk)).$$

Since these actions commute, we have a right action of the group $G \times SL_2(\mathbb{Z})$ on $L \times \mathcal{C}(G)$.

Let $\mathcal{O}(L)$ denote the ring of holomorphic functions on L, and for an integer j. The $\mathbb{C}^\times$-action on L given by scaling both t_1 and t_2 induces a graded ring structure on $\mathcal{O}(L)$

$$\mathcal{O}(L) = \bigoplus_{j \in \mathbb{Z}} \mathcal{O}^j(L),$$

where $f \in \mathcal{O}^j(L)$ if and only if $f(\mu^2 t_1, \mu^2 t_2) = \mu^j f(t_1, t_2)$. The $SL_2(\mathbb{Z})$-invariant elements of $\mathcal{O}^j(L)$ are called the *weak modular forms of weight* $-j/2$, which we denote by MF^j_{weak}.

Let G act on a space X from the right, and denote by $X^{g,h} \subset X$ the subspace of points fixed by both g and h. The action of G on X induces isomorphisms

$$X^{g,h} \to X^{k^{-1}gk, k^{-1}hk} \tag{3.1}$$

sending $x \mapsto x \cdot k$ for all $k \in G$. Write $\mathcal{C}[G]$ for the orbit space of the action of G on $\mathcal{C}(G)$, and denote the image of a point (g, h) by $[g, h]$. The stabiliser of a pair (g, h) is the maximal subgroup $C_{g,h} \subset G$ which centralises both g and h. The action of G induces an action of $C_{g,h}$ on $X^{g,h}$. Thus, for each $k \in G$ we have an isomorphism

$$H^*_{C_{k^{-1}gk, k^{-1}hk}}(X^{k^{-1}gk, k^{-1}hk}) \to H^*_{C_{g,h}}(X^{g,h})$$

induced by (3.1).

DEFINITION 3.1. In degree $k \in \mathbb{Z}$, Devoto's G-equivariant elliptic cohomology of X is defined as the abelian group

$$\begin{aligned} Ell_G^k(X) &:= \bigoplus_{i+j=k} \left(\bigoplus_{(g,h)\in\mathcal{C}(G)} H^i_{C_{g,h}}(X^{g,h}) \otimes_{\mathbb{C}} \mathcal{O}^j(L) \right)^{G\times SL_2(\mathbb{Z})} \\ &\cong \bigoplus_{i+j=k} \left(\bigoplus_{[g,h]\in\mathcal{C}[G]} H^i_{C_{g,h}}(X^{g,h}) \otimes_{\mathbb{C}} \mathcal{O}^j(L) \right)^{SL_2(\mathbb{Z})} \end{aligned} \tag{3.2}$$

where the isomorphism follows by choosing a representative pair (g,h) for each conjugacy class in $\mathcal{C}(G)$. The equivalent definitions should be compared to the two definitions of quasi-elliptic cohomology in Definition 2.1.

REMARK 3.2. Note that the action of $SL_2(\mathbb{Z})$ is trivial on the cohomology, since $\{g,h\}$ and $\{g^d h^{-b}, g^{-c}h^a\}$ generate the same subgroup of G, and so

$$X^{g,h} = X^{g^d h^{-b}, g^{-c}h^a} \quad \text{and} \quad C_{g,h} = C_{g^d h^{-b}, g^{-c}h^a}.$$

REMARK 3.3. If $G = \{e\}$ is the trivial group, then

$$Ell_e^k(X) = \bigoplus_{i+j=k} H^i(X) \otimes_{\mathbb{C}} \mathcal{O}^j(L)^{SL_2(\mathbb{Z})} = H^*(X) \otimes_{\mathbb{C}} MF^*_{weak} \tag{3.3}$$

where the right hand side is the graded tensor product over $\mathbb{C}$ of the cohomology ring of X with the graded ring of weak modular forms.

REMARK 3.4. If $X = pt$, then

$$Ell_G^k(pt) = \left(\bigoplus_{(g,h)\in\mathcal{C}(G)} \mathcal{O}^k(L) \right)^{G\times SL_2(\mathbb{Z})} \cong \left(\bigoplus_{[g,h]\in\mathcal{C}[G]} \mathcal{O}^k(L) \right)^{SL_2(\mathbb{Z})}.$$

4. Twisted equivariant elliptic cohomology over $\mathbb{C}$

A 3-cocycle on G with values in $U(1)$ is a map

$$\alpha : G \times G \times G \to U(1)$$

satisfying

$$\frac{\alpha(g_1,g_2,g_3)\alpha(g_0,g_1g_2,g_3)\alpha(g_0,g_1,g_2)}{\alpha(g_0g_1,g_2,g_3)\alpha(g_0,g_1,g_2g_3)} = 1$$

for all $g_0, g_1, g_2, g_3 \in G$. Such a cocycle is called *normalised* if it evaluates to 1 on any triple containing the identity element $e \in G$. Recall the value of Ell_G^k on a point pt in (3.3). We may use α to twist the G-action on

$$\bigoplus_{(g,h)\in\mathcal{C}(G)} \mathcal{O}^k(L) \tag{4.1}$$

by defining it to be

$$h \cdot f_{g_1,g_2} = \frac{\alpha(g_2,h,g_1)\alpha(h,g_1,g_2)\alpha(g_1,g_2,h)}{\alpha(h,g_2,g_1)\alpha(g_1,h,g_2)\alpha(g_2,g_1,h)} f_{g_1,g_2}.$$

This is compatible with the $SL_2(\mathbb{Z})$-action (CHECK), which does not change. We denote the subgroup of holomorphic functions in (4.1) which satisfy this transformation property by

$$\bigoplus_{(g,h)\in\mathcal{C}(G)} \mathcal{O}^{k+\alpha}(L).$$

In [**5**], Berwick-Evans defined an α-twisted version of Devoto's equivariant elliptic cohomology for a G-space X, which we may write as follows (CHECK THAT THIS IS THE SAME)

$$Ell_G^{k+\alpha}(X) := \bigoplus_{i+j=k} \left(\bigoplus_{(g,h)\in\mathcal{C}(G)} H^i_{C_{g,h}}(X^{g,h}) \otimes \mathcal{O}^{j+\alpha}(L) \right)^{G\times SL_2(\mathbb{Z})}.$$

5. Chern character map

In [**17**] Huan formulated quasi-elliptic cohomology. It is a variant of Tate K-theory, the generalized elliptic cohomology associated to the Tate curve. In this section we construct a Chern character map from quasi-elliptic cohomology to Devoto's equivariant elliptic cohomology.

Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & C_G(\sigma) \arrow[r] \arrow[d, "="'] & S^1 \times C_G(\sigma) \arrow[r] \arrow[d, "c"] & S^1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & C_G(\sigma) \arrow[r] & \Lambda_G(\sigma) \arrow[r] & S^1 \arrow[r] & 1
\end{tikzcd}$$

where the middle vertical map sends (t,g) to $[Nt,g]$ and the right vertical map sends $e^{2\pi it}$ to $e^{2\pi iNt}$ with N the order of σ.

The Chern Character map (by Matthew Spong, Dileep Menon and me) is constructed as the composition

$$\begin{aligned}
QEll_G(X)\otimes\mathbb{C} = \prod_{[\sigma]\in G_{conj}} K_{\Lambda_G(\sigma)}(X^\sigma)\otimes\mathbb{C} &\xrightarrow{c^*} \prod_{[\sigma]\in G_{conj}} K_{S^1\times C_G(\sigma)}(X^\sigma)\otimes\mathbb{C} \\
&\xrightarrow{\cong} \prod_{[\sigma]\in G_{conj}} K_{C_G(\sigma)}(X^\sigma)\otimes\mathbb{Z}[q^\pm]\otimes\mathbb{C} \xrightarrow{AS} \prod_{[\sigma]\in G_{conj}} (\prod_{[\tau]\in C_G(\sigma)_{conj}} (K(X^{\sigma,\tau})\otimes\mathbb{C})^{C_G(\tau)}\otimes\mathbb{Z}[q^\pm]) \\
&\xrightarrow{chern} \prod_{[\sigma],[\tau]\in G_{conj},\sigma\tau=\tau\sigma} (H(X^{\sigma,\tau})\otimes\mathbb{C})^{C_G(\tau)}\otimes\mathbb{Z}[q^\pm] \\
&\xrightarrow{\cong} \prod_{[\sigma],[\tau]\in G_{conj},\sigma\tau=\tau\sigma} H_{C_G(\tau)}(X^{\sigma,\tau})\otimes\mathbb{C}\otimes\mathbb{Z}[q^\pm]
\end{aligned}$$

The first map is the restriction map. The third map is the Atiyah-Segal map in [**3**, Theorem 2]

$$K^*_G(X)\otimes\mathbb{C} \xrightarrow{\cong} \prod_{[\sigma]\in G_{conj}} (K^*(X^g)\otimes\mathbb{C})^{C_G(g)}.$$

The fourth map is the product of Chern character maps of K-theory.

6. Twisted Quasi-elliptic cohomology

Let G be a finite group, let $\alpha\in H^3(BG;\mathbb{R}/\mathbb{Z})$, and consider a G-space X. We could define twisted quasi-elliptic cohomology $QEll^\alpha(X/\!\!/G)$, as the orbifold K-theory of a twisted orbifold $\Lambda^\alpha(X/\!\!/G)$, which is defined as follows (based on the material in Section 3 of [**11**]).

First, we show that α determines an element in $H^2(BC_G(g); \mathbb{R}/\mathbb{Z})$ for each conjugacy class $[g]$. Let e be the evaluation map

$$e : S^1 \times \mathrm{Map}(S^1, BG) \to BG$$

and let π be the projection

$$\pi : S^1 \times \mathrm{Map}(S^1, BG) \twoheadrightarrow \mathrm{Map}(S^1, BG).$$

Define the class

$$\theta := \pi_* e^* \alpha \in H^2(\mathrm{Map}(S^1, BG); \mathbb{R}/\mathbb{Z}) \cong \bigoplus_{[g]} H^2(BC_G(g); \mathbb{R}/\mathbb{Z}),$$

which has degree two because the formalism of the Pontryagin-Thom construction means that the degree of $e^*\alpha$ drops by one when we push it forward along π_*. Note that we have also used the fact that the mapping space $\mathrm{Map}(S^1, BG)$ is homotopy equivalent to

$$\coprod_{[g]} BC_G(g).$$

In this way, θ determines an element θ_g in $H^2(BC_G(g); \mathbb{R}/\mathbb{Z})$ for each $[g]$.

We can now define the twisted orbifold. Recall that a 2-cocycle θ_g determines a central extension

$$1 \to \mathbb{T} \to C^{\alpha}_G(g) \to C_G(g) \to 1$$

with group multiplication given by

$$(a, h)(b, k) = (a + b + \theta_g(h, k), hk).$$

We have a well-defined $C^{\alpha}_G(g)-$action on X^g

$$(a, h) \cdot x := h \cdot x. \tag{6.1}$$

Example 6.1 (Twisted Inertia Groupoid $I^{\alpha}(X/\!\!/G)$). T. Dove constructed twisted Inertia groupoid in [**9**]. The twisted inertia groupoid $I^{tors}(X/\!\!/G)$ of the translation groupoid $X/\!\!/G$ is the groupoid with

objects: the space $\coprod\limits_{g\in G} X^g$

morphisms: the space $\coprod\limits_{g,g'\in G} C^{\alpha}_G(g, g') \times X^g$.

For $x \in X^g$ and $(\sigma, g) \in C^{\alpha}_G(g, g') \times X^g$, $(\sigma, g)(x) = \sigma x \in X^{g'}$.

Example 6.2 (Twisted orbifold loop space). In [**14**, Definition 2.3] Ganter defined orbifold loop space

$$\mathcal{L}(X/\!\!/G) := \coprod_{[g]} \mathcal{L}_g X/\!\!/C_G(g),$$

via which equivariant Tate K-theory can be constructed. The space $\mathcal{L}_g X$ is the space $\mathrm{Map}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}$,where l is the order of g. Dove formulated twisted orbifold loop space in [**9**]. There is a well-defined $C^{\alpha}_G(g)-$action on $\mathcal{L}_g X$ by

$$\gamma(a, h)(t) = \gamma(t + a)h$$

for $\gamma \in \mathcal{L}_g X$ and $(a, h) \in C^{\alpha}_G(g)$. It's straightforward to check that $\gamma(a, h)$ is indeed in $\mathcal{L}_g X$. The twisted orbifold loop space is defined as

$$\mathcal{L}^{\alpha}(X/\!\!/G) := \coprod_{[g]} \mathcal{L}_g X/\!\!/C^{\alpha}_G(g).$$

Note that on the space of constant loops X^g, the action by $C^\alpha_G(g)$ in (6.1) covers that by $C_G(g)$.

Let $\Lambda^\alpha_G(g)$ denote the quotient

$$\mathbb{R}\times C^\alpha_G(g)/\langle(-1,(0,g))\rangle.$$

We construct a twisted orbifold as follows

$$\Lambda^\alpha(X/\!\!/G):=\coprod_{g\in G^{tors}_{conj}} X^g/\!\!/\Lambda^\alpha_G(g).$$

We have the short exact sequence

$$1\longrightarrow\mathbb{R}/\mathbb{Z}\longrightarrow\Lambda^\alpha_G(g)\longrightarrow\Lambda_G(g)\longrightarrow 1.$$

The surjective map gives the map of orbifolds

$$\Lambda^\alpha(X/\!\!/G)\to\Lambda(X/\!\!/G)$$

given by sending a morphism $(x,r,(a,g))$ to (x,r,g) is an S^1-equivariant graded central extension in the sense of [**20**].

Definition 6.3 (Twisted Quasi-elliptic cohomology). $QEll^{\alpha *}_G(X):=K^*_{orb}(\Lambda^\alpha(X/\!\!/G))\cong\prod_{G^{tors}_{conj}}K^*_{\Lambda^\alpha_G(g)}(X^g)$.

Remark 6.4. In [**12**], for each $\tau\in H^3(BG;\mathbb{Z})\cong H^2(BG;U(1))$, Freed, Hopkins and Teleman constructed twisted K-groups

$$K^{\tau+*}_G(X)=K^*_{G^\tau}(X)$$

for $G-$space X where G^τ is the central extension

$$1\longrightarrow\mathbb{T}\longrightarrow G^\tau\longrightarrow G\longrightarrow 1.$$

In our case, $C^\alpha_G(g)$ defined above is the group $C^\theta_G(g)$ in terms of the symbol in [**12**]. We can see the twisted Inertia groupoid

$$K^*_{orb}(I^\alpha(X/\!\!/G))=\prod_{G^{tors}_{conj}}K^{\theta_g+*}_{C_G(g)}(X^g). \tag{6.2}$$

In addition,

$$K^{\theta_g+*}_{\Lambda_G(g)}(X^g)=K^*_{\Lambda^\alpha_G(g)}(X^g).$$

Thus

$$K^*_{orb}(\Lambda^\alpha(X/\!\!/G))=\prod_{g\in G^{tors}_{conj}}K^{\theta_g+*}_{\Lambda_G(g)}(X^g)=K^{\theta+*}_{orb}(\Lambda(X/\!\!/G)).$$

Example 6.5. When the space X is the single point space pt, each $K^0_{\Lambda^\alpha_G(g)}(X^g)$ in twisted quasi-elliptic cohomology $QEll^{\alpha 0}_G(X)$ is the Grothendieck group $R^{\theta_g}(G)$ of representations of $\Lambda^\alpha_G(g)$. By [**12**, Example 1.10], $K^0_{\Lambda^\alpha_G(g)}(X^g)$ is isomorphic to the twisted K-theory $K^{\theta_g+0}_{\Lambda_G(g)}(pt)$. Thus, $QEll^{\alpha 0}_G(pt)$ is isomorphic to the twisted orbifold K-theory $K^{\theta+0}_{orb}(\Lambda(pt/\!\!/G))$.

Example 6.6. When G is the trivial group and g is the identity element, $QEll_G(X)=K^*_{\mathbb{T}}(X)$. In this case, for any 2-cocycle α, $\Lambda^\alpha_G(g)=\Lambda_G(g)\cong\mathbb{T}\times\mathbb{T}$. Thus, $QEll^\alpha_G(X)=K_{\mathbb{T}\times\mathbb{T}}(X)$.

7. Twisted Loop space

Based on the loop space construction of quasi-elliptic cohomology, we can give twisted quasi-elliptic cohomology an interpretation via loop space.

EXAMPLE 7.1. Let $\mathbb{T}^2$ denote the trivial principal $\mathbb{T}-$bundle over S^1. $Bibun(\mathbb{T}^2/\!\!/\mathbb{T}, X/\!\!/G)$ is the category of bibundles from $\mathbb{T}^2/\!\!/\mathbb{T}$ to $X/\!\!/G$. Its relation with $Bibun(S^1/\!\!/*, X/\!\!/G)$ can be interpreted in this way.

For each object $\mathbb{T}^2/\!\!/\mathbb{T} \xleftarrow{p^\alpha} P^\alpha \xrightarrow{f^\alpha} X/\!\!/G$ of $Bibun(\mathbb{T}^2/\!\!/\mathbb{T}, X/\!\!/G)$, we have the diagram

$$\begin{tikzcd}
\mathbb{T}^2/\!\!/\mathbb{T} \arrow[d] & P^\alpha \arrow[l, "p^\alpha"'] \arrow[r, "f^\alpha"] \arrow[d] & X/\!\!/G \arrow[d, "="] \\
S^1/\!\!/* & P \arrow[l, "p"'] \arrow[r, "f"] & X/\!\!/G
\end{tikzcd}$$

where $S^1/\!\!/* \xleftarrow{p} P \xrightarrow{f} X/\!\!/G$ is an object in $Bibun(S^1/\!\!/*, X/\!\!/G)$, P^α is a principal $\mathbb{T}-$bundle over P, the left two vertical maps are both bundle projections, p^α is a bundle map covering p and f^α is the composition of the projection and f.

A morphism $m : P^\alpha \longrightarrow P^{'\alpha}$ in it is a $\mathbb{T}-$bundle map covering a bundle automorphism $\mathbb{T}^2/\!\!/\mathbb{T} \longrightarrow \mathbb{T}^2/\!\!/\mathbb{T}$ and making the diagrams commute

$$\begin{tikzcd}
\mathbb{T}^2/\!\!/\mathbb{T} \arrow[dd] & P^\alpha \arrow[l, "p^\alpha"'] \arrow[dd, "m"'] \arrow[dr, "f^\alpha"] & \\
& & X \\
\mathbb{T}^2/\!\!/\mathbb{T} & P^{'\alpha} \arrow[l, "p^{'\alpha}"'] \arrow[ur, "f^{'\alpha}"'] &
\end{tikzcd} .$$

DEFINITION 7.2 (Twisted equivariant loop space). $Loop^{twist}(X/\!\!/G)$ has objects the bibundles from $\mathbb{T}^2/\mathbb{T}$ to $X/\!\!/G$. A morphism $(m, t) : P^\alpha \longrightarrow P^{'\alpha}$ consists of a rotation t on S^1 and a $\mathbb{T}-$bundle map m covering a bundle isomorphism $\mathbb{T}^2/\!\!/\mathbb{T} \longrightarrow \mathbb{T}^2/\!\!/\mathbb{T}$ that covers t. and making the diagrams commute

$$\begin{tikzcd}
\mathbb{T}^2/\!\!/\mathbb{T} \arrow[dd] & P^\alpha \arrow[l, "p^\alpha"'] \arrow[dd, "m"'] \arrow[dr, "f^\alpha"] & \\
& & X \\
\mathbb{T}^2/\!\!/\mathbb{T} & P^{'\alpha} \arrow[l, "p^{'\alpha}"'] \arrow[ur, "f^{'\alpha}"'] &
\end{tikzcd} .$$

LEMMA 7.3. *The groupoid $\Lambda^\alpha(X/\!\!/G)$ is a subgroupoid of $Loop^{twist}(X/\!\!/G)$ with some constant loops* $\coprod_{g\in G^{tors}_{conj}} X^g$ *as objects.*

8. The twisted Chern character map

LEMMA 8.1. *Suppose that α_g has order n, and let l be the order of g. Then the order of $(0, g)$ in C^α_g divides nl.*

PROOF. We have

$$\begin{aligned} nl(0,g) &= (\alpha_g(g,g)+\alpha_g(g,g^2)+...+\alpha_g(g,g^{nl}),g^{nl}) \\ &= (n(\alpha_g(g,g)+\alpha_g(g,g^2)+...+\alpha_g(g,e)),e) \\ &= (0,e). \end{aligned}$$

The second equality holds since $g^{ml+k}=g^k$ for all integers m and k, and the third equality holds since α_g has order n. Therefore, since $nl(0,g)$ is the trivial element, the order of $(0,g)$ must divide nl. □

LEMMA 8.2. *Let $H\subset G$ be a normal subgroup, and let X be a compact G-space such that H acts trivially on X. A G-vector bundle E on X decomposes as*

$$E\cong \oplus_{\mu\in\hat{H}}E_\mu$$

where E_μ denotes the μ the direct sum is indexed by irreducible characters of H. For each $\mu\in\hat{H}$, choose an extension $\bar{\mu}$ of μ to G. Denote by $\mathbb{C}_\mu$ the trivial bundle corresponding to μ. Then

$$\begin{array}{ccc} K_G(X) & \longrightarrow & K_{G/H}(X)\otimes_{K_{G/H}}K_G(pt) \\ {[E_\mu]} & \longmapsto & [E_\mu\otimes\mathbb{C}_{-\bar{\mu}}]\otimes[\mathbb{C}_{\bar{\mu}}] \end{array}$$

is a well-defined isomorphism, natural in X.

PROOF. The map is well defined since, for a different choice of extension $\bar{\mu}'\in\hat{G}$, the difference $[\mathbb{C}_{\bar{\mu}-\bar{\mu}'}]$ lies in $K_{G/H}$, and cancels out. The map has an inverse given by pulling back the G/H-action on vector bundles to a G-action, and tensoring with an element of K_G. Naturality is clear. □

Consider the map

$$p_g: S^1\times C_g^\alpha\to\Lambda_g^\alpha$$

which sends $(t,(a,g))$ to $[Nt,(a,g)]$, where N is the order of $(0,g)$ in C_g^α.

We now study the following composite of maps.

$$\begin{aligned} QEll_G^\alpha(X)\otimes\mathbb{C} = \prod_{[g]}K_{\Lambda_G^\alpha(g)}(X^g)\otimes\mathbb{C} &\xrightarrow{p^*} \prod_{[g]}K_{S^1\times C_g^\alpha}(X^g)\otimes\mathbb{C} \\ &\xrightarrow{\cong} \prod_{[g]}K_{C_g}(X^g)\otimes\mathbb{C}\otimes R(S^1\times C_g^\alpha) \\ &\xrightarrow{AS} \prod_{[g]}(\prod_{[h]}(K(X^{g,h})\otimes\mathbb{C})^{C_{g,h}}\otimes R(S^1\times C_g^\alpha) \\ &\xrightarrow{Chern} \prod_{[g],[h],gh=hg}(H(X^{g,h})\otimes\mathbb{C})^{C_{g,h}}\otimes R(S^1\times C_g^\alpha) \end{aligned}$$

The map p^* is the change of groups map given on the gth factor by pulling back the $\Lambda_G^\alpha(g)$-action along p_g. Let N be the order of $(0,g)$ in C^α. Note that the kernel of p_g is equal to

$$\ker(p_g):=\{([-m/N],(\alpha_g(g,g)+...+\alpha_g(g,g^m),g^m))\in S^1\times C_g^\alpha : m\in\mathbb{Z}\},$$

which acts trivially on X^g. The image of p_g^* is generated by the $S^1\times C_g^\alpha$-vector bundles with trivial $\ker(p_g)$-action on fibers.

The second map is a special case of the isomorphism in Lemma 8.2, where we set $G = S^1 \times C_g^\alpha$, $H = S^1 \times \mathbb{R}/\mathbb{Z}$ and $G/H = C_g$. Note that $S^1 \times \mathbb{R}/\mathbb{Z}$ acts trivially on X^g.

It follows that the image of the composite of the first and second maps is generated by elements

$$[E_\mu \otimes \mathbb{C}_{-\bar{\mu}}] \otimes \mathbb{C}_{\bar{\mu}} \in K_{C_g}(X^g) \otimes R(S^1 \times C_g^\alpha) \tag{8.1}$$

for irreducible characters μ of $S^1 \times \mathbb{R}/\mathbb{Z}$ which are trivial on

$$\ker(p) \cap (S^1 \times \mathbb{R}/\mathbb{Z}) = \{([-m/N], \alpha_g(g,g) + ... + \alpha_g(g, g^m)) : m \in \mathbb{Z}\}.$$

The Atiyah-Segal map sends an element of the form (8.1) to

$$\left(\bigoplus_{h \in C_g} \sum_\nu \nu(h)[E_\mu \otimes \mathbb{C}_{-\bar{\mu}}]_\nu\right) \otimes \mathbb{C}_{\bar{\mu}} \tag{8.2}$$

where ν ranges over irreducible characters of C_g, and $[E_\mu \otimes \mathbb{C}_{-\bar{\mu}}]_\nu$ denotes the isomorphism class of the νth summand of the restriction of $E_\mu \otimes \mathbb{C}_{-\bar{\mu}}$ to X^h. Finally, the Chern character map sends the element (8.2) to

$$\left(\bigoplus_{h \in C_g} \sum_\nu \nu(h) ch([E_\mu \otimes \mathbb{C}_{-\bar{\mu}}]_\nu)\right) \otimes \mathbb{C}_{\bar{\mu}}.$$

References

[1] J.F.Adams: *Maps Between Classifying Spaces. II*, Inventiones mathematicae (1978), Volume: 49, page 1–66.

[2] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland: *Elliptic spectra, the Witten genus and the theorem of the cube*. Invent. Math. , 146(3):595–687, 2001.

[3] Michael Atiyah, Graeme Segal: *On equivariant Euler characteristics*, Journal of Geometry and Physics Volume 6, Issue 4, 1989, Pages 671–677.

[4] Tobias Barthel, Daniel Berwick-Evans, Nathaniel Stapleton: *Geometric Power Operations in Elliptic cohomology*.

[5] Daniel Berwick-Evans: *Twisted equivariant elliptic cohomology with complex coefficients from gauged sigma models*. Preprint. arXiv:1410.5500v3 [math.AT]

[6] D. Berwick-Evans and F. Han, *The equivariant Chern character as super holonomy on loop stacks*. Preprint. arXiv:1610.02362 [math.AT]

[7] J. Block and E. Getzler, *Equivariant cyclic homology and equivariant differential forms*, Ann. Sci. Ecole Normale Sup. 4 (1994).

[8] Jorge A. Devoto: *Equivariant elliptic homology and finite groups*, Michigan Math. J. , 43(1):3-32, 1996.

[9] T. Dove: *Title*, Master's Thesis (unpublished)

[10] M. Duflo and M. Vergne: *Cohomologie équivariante et descente*, Astérisque, 215, 5-108 (1993).

[11] D. Freed and F. Quinn: *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. 156 (1993).

[12] Daniel S. Freed, Michael J. Hopkins, Constantin Teleman: *Loop groups and Twisted K-theory I*, Journal of Topology, Volume 4, Issue 4, December 2011, Pages 737–798.

[13] Daniel S. Freed, Michael J. Hopkins, Constantin Teleman: *Twisted equivariant K-theory with complex coefficients*, Journal of Topology, Volume 1, Issue 1, January 2008, Pages 16–44.

[14] Nora Ganter: *Stringy power operations in Tate K-theory*, (2007), available at arXiv: math/0701565.

[15] Nora Ganter: *The elliptic Weyl character formula*, Compositio Math. 150 (2014) 1196–1234.

[16] Zhen Huan: *Quasi-Elliptic Cohomology and its Power Operations*, Journal of Homotopy and Related Structures, 13(4), 715–767.

[17] Zhen Huan: *Quasi-elliptic cohomology I*. Advances in Mathematics. Volume 337, 15 October 2018, Pages 107–138.

[18] Zhen Huan: *Quasi-elliptic cohomology*, Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 2017. 290 pp. `http://hdl.handle.net/2142/97268`.

[19] Michael J. Hopkins, Nicolas J. Kuhn, Douglas C. Ravenel: *Generalized group characters and complex oriented cohomology theories*, J. Am. Math. Soc. 13 (2000) 553–594.

[20] Kiran Luecke: *Completed K-theory and Equivariant Elliptic Cohomology*, 2019.

[21] P. Landwebder, D.Ravenel, R. Stong: *Periodic cohomology theories defined by elliptic curves*, The Čech Centennial 1993(M. Cenkl and H. Miller, eds.), Contemp. Math., vol. 181, Amer. Math. Soc. Boston, 1995.

[22] I. Rosu: *Equivariant K-theory and equivariant cohomology*, Mathematische Zeitschrift, Volume 243, Issue 3, (2003) 423–448, with an appendix by Allen Knutson and Ioanid Rosu.