Almost global homotopy theory

Zhen Huan

Abstract. In this paper we develop the definition of a global orthogonal spectrum and its unitary version. It relates $G-$equivariant spectra by equivariant weak equivalence in a coherent way. This category of global spectra has a model structure Quillen equivalent to the $Fin-$global model structure on orthogonal spectra. We also show that there is a large family of equivariant cohomology theories, including quasi-elliptic cohomology, that can be globalized in the new context. Starting from one global ring spectrum, we can construct infinitely many distinct global ring spectra. Moreover, in light of the results in this paper, we ask whether we have the conjecture that the globalness of a cohomology theory is completely determined by the formal component of its divisible group and when the étale component of it varies the globalness does not change.

1. Introduction

Equivariant stable homotopy theory is the homotopy theory of topological spaces with group actions. Invented by G. B. Segal in the early 1970s [**20**], equivariant stable homotopy theory was motivated by Atiyah and Segal's work [**3**] on equivariant K-theory. The foundation of it was established systematically by tom Dieck, Segal and May afterwards. During the last decades this area has been very active. Equivariant stable homotopy theory has shown great power solving computational and conceptual problems in algebraic and geometric topology. In addition, even early in the history of equivariant stable homotopy theory, people noticed that many equivariant homotopy theories exist not only for one particular group, but for a family of groups in a uniform way and as equivariant K-theories, there are relations between equivariant homotopy theories for different groups, such as the connections embodied in the restriction maps, transfer maps, etc. These observations led to the birth of global homotopy theory. Globalness is a measure of the naturalness of a cohomology theory. Prominent examples of global homotopy theory include global K-theory, global bordism theory and global stable homotopy theory.

Orthogonal spectrum is a good model to interpret global spectrum, which reflects most symmetry. The idea of global orthogonal spectra was first inspired in Greenlees and May [**10**]. Several models of global homotopy theory have been established, which are equivalent to each other. Schwede develops a modern approach

2010 *Mathematics Subject Classification.* Primary 55.
The author was partially supported by NSF grant DMS-1406121.

to global homotopy theory in [**19**]. In [**5**] Bohmann uses the framework of enriched indexed categories to develop the definition of a global orthogonal spectrum. She shows that the Atiyah-Bott-Shapiro orientation can extend to this global context. Gepner and Henriques establish the basic theory of an unstable global homotopy theory in [**9**] via infinity categories. The underlying infinity category is equivalent to that of Schwede's orthogonal spaces. This theory is much easier to work with for elliptic cohomology.

The motivating example for almost global homotopy theory is quasi-elliptic cohomology [**17**][**13**][**14**]. This theory is a variant of elliptic cohomology theories. It is the orbifold K-theory of a space of constant loops. For global quotient orbifolds, it can be expressed in terms of equivariant K-theories and has the change-of-group isomorphisms. In a conversation, Ganter indicated that it has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede [**19**]. However, though many constructions on quasi-elliptic cohomology can be made in a neat way, there is not an effective way to prove or disprove this statement. We construct in [**12**] an orthogonal $G-$spectrum of quasi-elliptic cohomology for each compact Lie group G. As indicated in Remark 6.13 [**12**], these equivariant orthogonal spectra cannot arise from an orthogonal spectrum. Then it is even more difficult to see whether each elliptic cohomology theory, whose form is more intricate and mysterious than quasi-elliptic cohomology, can be globalized in the current setting.

In this paper we construct a new theory, almost global homotopy theory. We observe that if the equivariant cohomology theories $\{E^*_G(-)\}_G$ have the change-of-group isomorphisms and there exists an orthogonal $G-$spectrum $\{X_G(-), \sigma_{(-,-)}\}$ representing $E^*_G(-)$, then for any closed subgroup H of G and any $G-$representation V, we have a weak $H-$equivalence $X_G(V) \simeq_H X_H(V)$. In light of this observation, we combine the equivariant orthogonal spectra of $\{E^*_G(-)\}_G$ by equivariant weak equivalences to give the definition of a global spectrum. The category $Sp^W_{D_0}$ [Definition 3.6] of the resulting global spectra has a model structure on it Quillen equivalent to the $Fin-$global model structure on orthogonal spectra constructed in Theorem 4.3.17 [**19**].

Other than quasi-elliptic cohomology, there is a large family of equivariant cohomology theories that can extend to the new context of global spectra. In [**15**] we define quasi-theories, which is motivated by quasi-elliptic cohomology, whose divisible group is given by the Tate curve $Tate(q)$ [Section 2.6, [**1**]]. Starting from a cohomology theory E, we can construct a family of theories $QE^*_{n,G}(-)$ for any positive integer n and any compact Lie group G. Quasi-elliptic cohomology, the $n-$th generalized quasi-elliptic cohomology and the theories $QE^*_n(-)$ in [**12**] are all quasi-theories.

Quasi-theories have the change-of-group isomorphism, as shown in [**15**]. In [**16**] we construct an orthogonal $G-$spectrum for each quasi-theory. The construction generalizes that for quasi-elliptic cohomology in [**12**]. The equivariant orthogonal spectra for each quasi-theory cannot arise from an orthogonal spectrum, but do form a global spectra in the almost global homotopy theory.

In this paper we construct almost global homotopy theory and several model structures on the underlying category. In addition, we construct global spectra for quasi-theories in the new setting. We also define and construct unitary global spectra. In Section 2 we construct the new diagram category D_0 by adding restriction maps between group representations to the category $\mathbb{L}$ of inner product vector spaces. It is a Reedy category with the linear isometric embeddings raising the degree and the restriction maps lowering the degree. In Section 3 we introduce the category $Sp^W_{D_0}$ of global spectra, which is a full subcategory of the category of D_0-spectra. In Section 4 we construct unitary global spectra. In Section 5 we show the generalized Reedy model structure on $Sp^W_{D_0}$ is Quillen equivalent to the $Fin-$level model structure on orthogonal spectra in Proposition 4.3.7, [**19**]. In Section 6 we construct the global model structure on $Sp^W_{D_0}$ and show it is Quillen equivalent to the $Fin-$global model structure on orthogonal spectra in Theorem 4.3.17 [**19**]. In Section 7 we show quasi-theories have global version in this context. At the end of the paper, in Remark 7.11, we raise the question whether the *é*tale component in the divisible group associated to a cohomology theory plays no role in the globalization of the theory.

1.1. Acknowledgement. First I would like to thank my PhD advisor Charles Rezk. The main idea of this paper is motivated by him. I would also like to thank Matthew Ando for his support and encouragement all the time. I would like to thank Stefan Schwede for helpful conversation at Homotopy Theory Summer Berlin in June 2018 and a few email discussions. I would like to thank David Gepner and Nathaniel Stapleton for helpful conversations.

2. The categories D_0 and $D_0^{\mathbb{C}}$

In this section we introduce the new diagrams D_0 and $D_0^{\mathbb{C}}$.

2.1. The category D_0.

Definition 2.1. Let D be a category with objects (G, V, ρ) where V is an inner product vector space, G a compact group G and ρ a faithful group representations

$$\rho : G \longrightarrow O(V).$$

A morphism in D

$$\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$$

consists of a linear isometric embedding $\phi_2 : V \longrightarrow W$ and a group homomorphism $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$, which make the diagram commute.

$$\begin{tikzcd} G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(O(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & O(W) \end{tikzcd} \tag{2.1}$$

In other words, the group action of H on $\phi_2(V)$ is induced from that of G.

The composition of two morphisms

$$(G, V, \rho) \overset{(\phi_1,\phi_2)}{\longrightarrow} (H, W, \tau) \overset{(\psi_1,\psi_2)}{\longrightarrow} (K, U, \beta)$$

is defined to be

$$(\phi_1 \circ \psi_1|_{\beta^{-1}(O(\psi_2\circ\phi_2(V)))}, \psi_2 \circ \phi_2).$$

The composition is associative. The identity morphism in $D((G, V, \rho), (G, V, \rho))$ is (Id, Id).

All the maps in (2.1) are injective. Given a linear isometric embedding $\phi_2 : V \longrightarrow W$, $\tau^{-1}(\phi_{2*}(\rho(G)))$ is always nonempty since the identity element is in it. If ϕ_1 exists, it is unique.

LEMMA 2.2. *Two objects (G, V, ρ) and (H, W, τ) in D are isomorphic if and only if there is an isomorphism $G \longrightarrow H$ which makes V and W isomorphic as representations.*

The proof is left to the readers.

REMARK 2.3. This category D has the same objects as the category $\mathcal{I}_{\mathcal{G}}$ defined in Example 2.3, [**5**], from which Bohmann constructed a version of global spectra. However, the sets of morphisms in the two categories are very different. Roughly speaking, D has the same restriction maps as $\mathcal{I}_{\mathcal{G}}$ and less linear isometric embeddings than $\mathcal{I}_{\mathcal{G}}$. In addition, there is no obvious structure of enriched indexed category on D.

REMARK 2.4. Since the representation ρ in an object (G, V, ρ) of D is faithful, we may consider the group G as a closed subgroup of $O(V)$. Then the diagram (2.1) is equivalent to

$$\begin{tikzcd} G \arrow[r] & O(V) \arrow[d] \\ H \cap O(V) \arrow[u] \arrow[r] & O(W) \end{tikzcd} \tag{2.2}$$

where we consider $O(V)$ as a closed subgroup of $O(W)$ as well. All the maps in the diagram are inclusions.

In Section 6.1 [**13**], we discuss the topology on the space $D((G, V, \rho), (H, W, \tau))$ of morphisms. In addition, the space $D((G, V, \rho), (H, W, \tau))$ of morphisms inherits an $H-$action and $G-$action on it. For $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$ and $(g, h) \in G \times H$,

$$(g, h) \cdot (\phi_1, \phi_2) = (g\phi_1(h^{-1} - h)g^{-1}, h\phi_2(g^{-1} \cdot -)) \tag{2.3}$$

The $H-$action on ϕ_2 is left and the $G-$action on it is right, whereas the $H-$action on ϕ_1 is right and the $G-$action on ϕ_1 is left.

PROPOSITION 2.5. The category D is a symmetric monoidal category.

PROOF. The tensor product $+ : D \times D \longrightarrow D$ is defined by

$$((G, V, \rho), (H, W, \tau)) \mapsto (G \times H, V \oplus W, \rho \oplus \tau). \tag{2.4}$$

The unit object is $u = (e, 0, *)$ where e is the trivial group and $*$ is the unique map from e. From the property of product of representations, the tensor product is associative. And we have the isomorphism $(G \times H, V \oplus W, \rho \oplus \tau) \longrightarrow (H \times G, W \oplus V, \tau \oplus \rho)$.

It's straightforward to check it satisfies all the required diagrams.

$\square$

Note that each object (G, V, ρ) is isomorphic to $(\rho(G), V, i)$ in D with $i : \rho(G) \hookrightarrow O(V)$ the inclusion of closed subgroup. Let D' denote the full subcategory of D consisting of objects (G, V, i) with G a closed subgroup of $O(V)$ and i the inclusion. We will not lose any information if we study D' instead of D. We use (G, V) to denote (G, V, i) in the rest of the paper.

We are more interested in the category D_0 defined below.

Definition 2.6. D_0 is defined to be the full subcategory of D' consisting of those objects (G, V) with G a finite group.

D_0 is also a symmetric monoid category with the tensor product and unit defined in the proof of Proposition 2.5.

We can define a degree function on D_0 by

$$deg(G, V) = |G| \dim V,$$

the order of the group G times the dimension of the vector space V.

We show in Proposition 2.8 that D_0 is a generalized Reedy category in the sense of Definition 2.7, which is from [**4**]. Recall that a subcategory S of a category R is called wide if S has the same objects as R. Let $\mathrm{Iso}(R)$ denote the maximal subgroupoid of R.

Definition 2.7. A generalized Reedy structure on a small category R consists of wide subcategories R^+, R^-, and a degree-function $d : Ob(R) \longrightarrow \mathbb{N}$ satisfying the following four axioms:

(i) non-invertible morphisms in R^+ (resp. R^-) raise (resp. lower) the degree; isomorphisms in R preserve the degree;

(ii) $R^+ \cap R^- = \mathrm{Iso}(R)$;

(iii) every morphism f of R factors as $f = gh$ with $g \in R^+$ and $h \in R^-$, and this factorization is unique up to isomorphism;

(iv) If $\theta f = f$ for $\theta \in \mathrm{Iso}(R)$ and $f \in R^-$, then θ is an identity.

A generalized Reedy structure is dualizable if in addition the following axiom holds:

(iv)' If $f\theta = f$ for $\theta \in \mathrm{Iso}(R)$ and $f \in R^+$, then θ is an identity.

A (dualizable) generalized Reedy category is a small category equipped with a (dualizable) generalized Reedy structure.

A morphism of generalized Reedy categories $R \longrightarrow S$ is a functor which takes R^+ (resp. R^-) to S^+ (resp. S^-) and which preserves the degree.

Proposition 2.8. D_0 is a dualizable generalized Reedy category.

Proof. Let D_0^- be the subcategory of D_0 with the same objects as D_0 and morphisms $(G, V) \xrightarrow{(\alpha_1, \alpha_2)} (H, V')$ where $\alpha_1 : H \longrightarrow G$ a group homomorphism and α_2 an isometric isomorphism. In other words, the morphisms of D_0^- are all restrictions. We have the commutative diagrams

$$\begin{tikzcd} G \arrow[r] & O(V) \arrow[d, "\alpha_{2*}"] \\ H \arrow[u, "\alpha_1"] \arrow[r] & O(V') \end{tikzcd} \tag{2.5}$$

The left vertical map is injective. So the order of H is no larger than that of G. So

$$deg(G,V) \geq deg(H,V').$$

By Lemma 2.2 we know that if $(G,V) \overset{(\alpha_1,\alpha_2)}{\longrightarrow} (H,V')$ is not an isomorphism, $deg(G,V) > deg(H,V')$.

Let D_0^+ be the subcategory of D_0 with the same objects as D_0 and morphisms

$$(G,V) \overset{(\phi_1,\phi_2)}{\longrightarrow} (H,W)$$

where $\phi_1 : H \cap O(\phi_{2*}V) \longrightarrow G$ is a group isomorphism. In other words, the morphisms in D_0^+ are linear isometric embeddings. Since G is isomorphic to a subgroup of H and $\dim V \leq \dim W$, so $deg(G,V) \leq deg(H,W)$. $deg(G,V) = deg(H,W)$ if and only if (G,V) is isomorphic to (H,W).

Any morphism $(\phi_1,\phi_2) : (G,V) \longrightarrow (H,W)$ has the decomposition

$$(G,V) \overset{(i,Id)}{\longrightarrow} (\phi_1(H \cap \phi_{2*}(G)),V) \overset{(\phi_1,\phi_2)}{\longrightarrow} (H,W) \tag{2.6}$$

where $i : \phi_1(H \cap \phi_{2*}(G)) \longrightarrow G$ is the inclusion. Note that in the second morphism (ϕ_1,ϕ_2), $\phi_1 : H \cap \phi_{2*}(G) \longrightarrow \phi_1(H \cap \phi_{2*}(G))$ is a group isomorphism.

If $(\phi_1,\phi_2) = (f_1,f_2) \circ (\alpha_1,\alpha_2)$ with $(\alpha_1,\alpha_2) : (G,V) \longrightarrow (G',V')$ in D_0^- and $(f_1,f_2) : (G',V') \longrightarrow (H,W)$ in D_0^+, $\alpha_2 : V \longrightarrow V'$ is an isometric isomorphism and $f_1 : H \cap O(f_{2*}(V')) \longrightarrow G'$ is a group isomorphism. The group homomorphisms α_1 (resp. f_1) is uniquely determined by α_2 (resp. f_2). Note that $O(f_2(V')) \cap H = O(\phi_2(V)) \cap H$. We have the commutative diagram

$$\begin{tikzcd}
\phi_1(H \cap O(\phi_2(V))) \arrow[r] & O(V) \arrow[d, "\alpha_2"] \\
G' \arrow[u, "\phi_1 \circ f_1^{-1}"] \arrow[r] & O(V')
\end{tikzcd}$$

So we have the isomorphism in D_0

$$(\phi_1 \circ f_1^{-1}, \alpha_2) : (\phi_1(H \cap O(\phi_2(V))), V) \longrightarrow (G',V').$$

The morphism $(i, Id) : (G,V) \longrightarrow (\phi_1(H \cap O(\phi_2(V))), V)$ equals to the composition $(\phi_1 \circ f_1^{-1}, \alpha_2)^{-1} \circ (\alpha_1,\alpha_2)$ and $(\phi_1,\phi_2) : (\phi_1(H \cap O(\phi_2(V))), V) \longrightarrow (H,W,\tau)$ is the composition $(f_1,f_2) \circ (\phi_1 \circ f_1^{-1}, \alpha_2)$. So the decomposition (2.6) is unique up to isomorphism.

Let $(\beta_1,\beta_2) : (G,V) \longrightarrow (G',V')$ be a morphism in both D_0^+ and D_0^-. Then β_1 and β_2 are both isomorphism. So (β_1,β_2) is an isomorphism. By Lemma **??**, $D_0^+ \cap D_0^- = \mathrm{Iso}(D_0)$.

Let $\alpha = (\alpha_1,\alpha_2) : (G,V) \longrightarrow (G',V')$ be a morphism in D_0^- and $\theta' = (\theta'_1,\theta'_2) : (G',V') \longrightarrow (G',V')$ be a morphism in $\mathrm{Iso}(D_0)$. If $\theta' \circ \alpha = \alpha$, θ'_2 is the identity map. So θ'_1 is the identity.

Let $f = (f_1,f_2) : (G,V) \longrightarrow (H,W)$ be a morphism in D_0^+ and $\theta = (\theta_1,\theta_2) : (G,V) \longrightarrow (G,V)$ be a morphism in $\mathrm{Iso}(D_0)$. If $f \circ \theta = f$, θ_2 is the identity map. So θ_1 is the identity.

So D_0 is a dualizable generalized Reedy category in the sense of Definition 2.7. $\square$

REMARK 2.9. A skeleton of D_0 is the full subcategory of it consisting of elements of the form $(G, \mathbb{R}^n)$ with $n \geq 0$. It is more convenient to study with than D_0 itself. In the rest of the paper we use the same symbol D_0 to denote this skeleton.

2.2. The category $D_0^{\mathbb{C}}$. Let W be a complex inner product space, i.e. a finite dimensional $\mathbb{C}-$vector space equipped with a hermitian inner product $(-,-)$. Let rW denote the underlying real inner product spaces of W, i.e. the underlying finite dimensional $\mathbb{R}-$vector space equipped with the euclidean inner product

$$\langle v, w\rangle = \mathrm{Re}(v, w).$$

Let

$$C = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{Id}_{\mathbb{C}}, \tau\}$$

denote the Galois groups of $\mathbb{C}$ over $\mathbb{R}$ where $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex conjugation $\tau(\lambda) = \overline{\lambda}$.

DEFINITION 2.10. $L^{\mathbb{C}}$ is the complex isometries category whose objects are finite dimensional complex inner product spaces and morphism space between two objects V and W is the space of the pairs

$$(\psi, \sigma) \in L(rV, rW) \times C$$

that satisfy $\psi(\lambda \cdot v) = \sigma(\lambda) \cdot \psi(v)$ and $(\psi(v), \psi(v')) = \sigma((v, v'))$. The composition in $L^{\mathbb{C}}$ is defined by

$$(\psi, \sigma) \circ (\psi', \sigma') = (\psi\psi', \sigma\sigma')$$

and the identity morphism of V is $(\mathrm{Id}_V, \mathrm{Id}_{\mathbb{C}})$.

The extra piece of the unitary structure gives a richer global homotopy theory that is indexed not only on compact Lie groups, but on the larger class of augmented Lie groups.

DEFINITION 2.11. An augmented Lie group is a compact Lie group G equipped with a continuous homomorphism $\epsilon : G \longrightarrow C$, called the augmentation, to the Galois group of $\mathbb{C}$ over $\mathbb{R}$.

Let $G_{ev} = \epsilon^{-1}(Id_{\mathbb{C}})$ denote the even part of G and $G_{odd} = \epsilon^{-1}(\tau)$ the odd part of G. The product in the category of augmented Lie groups is defined to be the fiber product over C. Explicitly, the product of two augmented Lie groups G and K is the augmented Lie group $G \times_C K$ with $(G \times_C K)_{ev} = G_{ev} \times K_{ev}$ and $(G \times_C K)_{odd} = G_{odd} \times K_{odd}$.

EXAMPLE 2.12 (extended unitary group). The endomorphism group of a complex inner product space W in the category $L^{\mathbb{C}}$ is defined to be

$$\widetilde{U}(W) := L^{\mathbb{C}}(W, W).$$

The augmentation $\epsilon_W : \widetilde{U}(W) \longrightarrow C$ is defined to be $\epsilon_W(\varphi, c) = c$.

The extended unitary group is a closed subgroup of $O(rW)$.

The augmented Lie group contain compact Lie groups as the ones with trivial augmentation. It also contain the Real Lie groups of Atiyah and Segal in [**3**], which are defined as compact Lie groups equipped with an involution.

EXAMPLE 2.13 (Split augmented Lie groups). Let G be a compact Lie group equipped with an involution $\tau : G \longrightarrow G$ on it, namely a Real Lie group in the sense of [**2**]. We can construct an augmented Lie group from it. The semi-direct product $G \rtimes_\tau C$ is an augmented Lie group with the augmentation

$$G \rtimes_\tau C \longrightarrow C,\ (g, \sigma) \mapsto \sigma.$$

A real representation of $G \rtimes_\tau C$ amounts to a unitary representation V of G with a real structure $\tau : V \longrightarrow V$ such that

$$\tau(g \cdot v) = \tau(g) \cdot \tau(v),\ \forall g \in G, v \in V.$$

For the opposite direction, given an augmented Lie group, it is isomorphic to a $G \rtimes_\tau C$ for some Real Lie group G if and only if its augmentation has a multiplicative section, i.e., it has an odd element of order 2.

For any complex inner product space W, there is a canonical involution τ on $U(W)$ by complex conjugation. We have the isomorphism of augmented Lie groups

$$U(W) \rtimes_\tau C \longrightarrow \widetilde{U}(W),\ (\psi, \tau) \mapsto \psi \circ \tau_W. \tag{2.7}$$

DEFINITION 2.14. A real representation of an augmented Lie group G is a finite-dimensional complex inner product space V and a continuous homomorphism $\rho : G \longrightarrow \widetilde{U}(V)$, i.e., such that $\epsilon_V \circ \rho = \epsilon$.

DEFINITION 2.15. Let G be an augmented Lie group. An augmented right $G-$space is a right $G-$space A equipped with a continuous map $\epsilon : A \longrightarrow C$ such that

$$\epsilon(a \cdot g) = \epsilon(a) \cdot \epsilon(g)$$

for all $a \in A$ and all $g \in G$.

DEFINITION 2.16. A unitary space is a continuous functor from the complex isometries category $L^{\mathbb{C}}$ to the category of spaces, where $L^{\mathbb{C}}$ is the category in Definition 2.10. A morphism of unitary spaces is a natural transformation of functors. We denote the category of unitary spaces by spc^U.

DEFINITION 2.17. $D^{\mathbb{C}}$ is the category whose objects are (G, V, ρ) with G an augmented Lie group and (V, ρ) a faithful real representation of G, and the morphism space $D((G, V, \rho), (H, W, \tau))$ is the space of the pairs (ϕ_1, ϕ_2) with $\phi_2 \in L^{\mathbb{C}}(V, W)$ and $\phi_1 : \tau^{-1}(\widetilde{U}(\phi_2(V)))\rho(G) \longrightarrow G$ a group homomorphism, which make the diagram commute.

$$\begin{tikzcd} G \arrow[r, "\rho"] & \widetilde{U}(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(\widetilde{U}(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & \widetilde{U}(W) \end{tikzcd} \tag{2.8}$$

In other words, the action of the augmented Lie group H on $\phi_2(V)$ is induced from that of G.

PROPOSITION 2.18. The category $D^{\mathbb{C}}$ is a symmetric monoidal category.

PROOF. Let (G, V, ρ) and (H, W, τ) be two objects in $D^{\mathbb{C}}$. The tensor product of (G, V, ρ) and (H, W, τ) is defined to be $(G \times_C H, V \oplus W, \rho \times_C \tau)$. The product is obviously associative and commutative. The unit is $(1, , 1)$ where 1 is the trivial group equipped with the trivial augmentation.

□

Let $D_0^{\mathbb{C}}$ denote the full subcategory with objects (G,V,i) where G is a finite group and i is the inclusion of augmented Lie subgroup into $\widetilde{U}(V)$. We can omit i from the symbol.

3. The Category $Sp^W_{D_0}$

In this section we define the category $Sp^W_{D_0}$ of global spectra that we will work in. We implicitly add disjoint basepoints to the morphism spaces in D_0.

Definition 3.1 (The category D_0T). A D_0-space is a continuous functor $X: D_0 \longrightarrow T$ to the category of based compactly generated weak Hausdorff spaces. A morphism of D_0-spaces is a natural transformation. We use D_0T to denote the category of D_0-spaces.

Example 3.2. We can define the D_0-sphere. For each object (G,V) in D_0, define

$$S^{(G,V)} := S^V. \tag{3.1}$$

S^V inherits a $G-$action from that on V.

For any morphism $\phi = (\phi_1, \phi_2) : (G,V) \longrightarrow (H,W)$ in D_0. Define $S(\phi)$ to be

$$S^{\phi_2} : S^V \longrightarrow S^W. \tag{3.2}$$

In particular, $S(Id)$ is the identity map.

Definition 3.3 (The category Sp^{D_0}). A D_0-spectrum X consists of

- a based $G-$space $X(G,V)$;
- a based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G,V) \longrightarrow X(G\times H, V\oplus W)$ which is $G\times H-$equivariant

for any objects (G,V) and (H,W) of D_0.

In addition, $\sigma_{(G,V),(1,0)}$ is the identity map and

$$\sigma_{(G,V),(H_1\times H_2,W_1\oplus W_2)} = \sigma_{(G\times H_1,V\oplus W_1),(H_2,W_2)} \circ (S^{W_2} \wedge \sigma_{(G,V),(H_1,W_1)}).$$

A morphism of D_0-spectra $f: X \longrightarrow Y$ is a functor in D_0T compatible with the structure maps, i.e. for any objects (G,V) and (H,W) in D_0,

$$f(G\times H, V\oplus W) \circ \sigma^{(X)}_{(G,V),(H,W)} = \sigma^{(Y)}_{(G,V),(H,W)} \circ (f(G,V)\wedge S^W).$$

We use Sp^{D_0} to denote the category of D_0-spectra.

Remark 3.4. A D_0-spectrum can be interpreted as a diagram spectrum. First we give the structure maps a topology. Let (G,V) and (H,W) be two objects in D_0. Over the space $D_0((G,V),(H,W))$ of morphisms we define a vector bundle with total space

$$\xi((G,V),(H,W)) = \{(w,(\phi_1,\phi_2)) \in W \times D_0((G,V),(H,W)) | w \perp \phi_2(V)\}$$

and with the structure map $\xi((G,V),(H,W)) \longrightarrow D_0((G,V),(H,W))$ the projection to the second factor. Let $\mathbf{O}((G,V),(H,W))$ denote the Thom space of this bundle. We have

$$\mathbf{O}((G,V),(H,W)) \wedge V \cong W \wedge D_0((G,V),(H,W)). \tag{3.3}$$

A D_0-spectrum is a continuous based functor from $\mathbf{O}$ to the category $\mathbf{T}$ of based compactly generated weak Hausdorff spaces.

To study equivariant homotopy theory, a more interesting and reasonable category is defined below.

DEFINITION 3.5 (The category D_0T^W). Define D_0T^W to be the full category of D_0T consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an $H-$weak equivalence. We call the objects D_0^W-spaces.

DEFINITION 3.6 (The category $Sp_W^{D_0}$). A D_0^W-spectrum X is both a D_0-spectrum and a D_0-space in D_0T^W. The category $Sp_W^{D_0}$ is the full subcategory of Sp^{D_0} consisting of D_0^W-spectra.

We recall the definition of orthogonal spectra.

DEFINITION 3.7 (The category Sp^O of orthogonal spectra). An orthogonal spectrum consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$$

is $O(n) \times O(m)-$equivariant.

We use Sp^O to denote the category of orthogonal spectra.

D_0 is a topological category. So Sp^{D_0} is complete and cocomplete.

LEMMA 3.8. *$Sp_W^{D_0}$ is both complete and cocomplete.*

PROOF. We first show that $Sp_W^{D_0}$ is complete.

Let J be a small category and $F : J \longrightarrow Sp_W^{D_0}$ be a diagram. Since Sp^{D_0} is complete, the diagram $F : J \longrightarrow Sp_W^{D_0}$ has a limit X in Sp^{D_0}. Let $r : (G, V) \longrightarrow (H, V)$ be a restriction map. Let $ev_{(G,V)} : Sp_W^{D_0} \longrightarrow T$ be the evaluation map sending X to $X(G, V)$. Then we have diagrams

$$ev_{(G,V)} \circ F : J \longrightarrow T \text{ and } ev_{(H,V)} \circ F : J \longrightarrow T.$$

The limits of them are $X(G, V)$ and $X(H, V)$ respectively. For each integer n, we have the equivariant homotopy groups $\pi_n^H(-)$. The diagrams

$$\pi_n^H \circ ev_{(G,V)} \circ F \text{ and } \pi_n^H \circ ev_{(H,V)} \circ F$$

are isomorphic. So we have isomorphic limits

$$\pi_n^H(X(G, V)) \cong \pi_n^H(X(H, V)) \text{ for each } n.$$

Thus, $X(G, V)$ and $X(H, V)$ are $H-$weak equivalent. So X is an object in $Sp_W^{D_0}$.

We can prove that $Sp_W^{D_0}$ is cocomplete in a similar way.

□

We have a pair of adjoint functors between $Sp_W^{D_0}$ and the category Sp^O of orthogonal spaces. In Section 6.1 [**13**] we discuss a functor

$$l : L \longrightarrow D_0$$

sending $\mathbb{R}^n$ to $(\Sigma_n, \mathbb{R}^n)$. It gives a forgetful functor $U : Sp^{D_0} \longrightarrow Sp^O$. Let

$$Q : Sp_W^{D_0} \longrightarrow Sp^{D_0} \longrightarrow Sp^O$$

denote the composition of the inclusion and the forgetful functor U.

In addition, we can define a functor $P : Sp^O \longrightarrow Sp_W^{D_0}$ by sending an orthogonal spectrum Y to the D_0^W-spectrum

$$P(Y)(G, V) := Y(V); P(Y)(\phi_1, \phi_2) := Y(\phi_2) \tag{3.4}$$

for any object (G, V) and any morphism (ϕ_1, ϕ_2) in D_0.

Proposition 3.9.

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

is a pair of adjoint functors. In addition, the unit $\eta : Id_X \longrightarrow Q \circ P$ is an isomorphism.

Proof. Let X be an object in Sp^O and Y an object in $Sp_W^{D_0}$. We can define a natural isomorphism $F : Hom_{Sp^O}(-, Q(-)) \longrightarrow Hom_{Sp_W^{D_0}}(P(-), -)$.

For any functor $f : X \longrightarrow Q(Y)$, let $F(f)(G, V) : P(X)(G, V) \longrightarrow Y(G, V)$ be defined by the composition

$$\begin{array}{ccc} P(X)(\Sigma_n, V) & \xrightarrow{f(V)} & Y(\Sigma_n, V) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle Y(i, Id)} \\ P(X)(G, V) & \longrightarrow & Y(G, V) \end{array} \tag{3.5}$$

for any object (G, V) of $Sp_W^{D_0}$ and $n = dimV$.

In addition, we can define $\alpha : Hom_{Sp_W^{D_0}}(P(-), -) \longrightarrow Hom_{Sp^O}(-, Q(-))$. For any functor $\beta : P(X) \longrightarrow Y$, define $\alpha(\beta)(V) := f(\Sigma_n, V)$. α is the inverse of F.

The composition $Q \circ P$ is the identity. So η is an isomorphism. □

Example 3.10 (symmetric spectra). Each symmetric spectrum Y gives a D_0^W-spectrum QY.

We can also define the equivariant homotopy groups of D_0^W-spectrum.

Definition 3.11. For each D_0^W-spectrum Y,

$$\pi_k^G(Y) := \pi_k^G(Q(Y)), \tag{3.6}$$

for any finite group G and any positive integer k.

Remark 3.12. Note that for any D_0^W-spectrum, each restriction map in D_0 is mapped to an equivariant weak equivalence. Thus, the definition in (3.6) contains all the necessary information.

In addition, we can define the equivariant homotopy groups of D_0^W-spectrum in the canonical way as those of orthogonal spectra. We describe the construction explicitly. Let $s(\mathcal{U}_G)$ denote the poset, under inclusion of finite-dimensional $G-$subrepresentations of the chosen complete $G-$universe $\mathcal{U}_G$. Let X denote a D_0^W-spectrum Y. If $\psi = (Id, \psi_2) : (G, V) \longrightarrow (G, W)$ is a morphism in D_0 and $f : S^V \longrightarrow X(G, V)$ a continuous based map, we define $\psi_* f : S^W \longrightarrow X(G, W)$ as the composite

$$\psi_* f : S^W \cong S^V \wedge S^{W - \psi_2(V)} \xrightarrow{f \wedge S^{W - \psi_2(V)}} X(G, V) \wedge S^{W - \psi_2(V)} \xrightarrow{\psi_{2*}} X(G, W).$$

We obtain a functor from the poset $s(\mathcal{U}_G)$ to sets by sending $V \in s(\mathcal{U}_G)$ to

$$[S^V, X(G,V)]^G,$$

the set of $G-$equivariant homotopy classes of based $G-$maps from S^V to $X(G,V)$. For $V \subseteq W$ in $s(\mathcal{U}_G)$, the inclusion $i : V \longrightarrow W$ is sent to the map

$$[S^V, X(G,V)]^G \longrightarrow [S^W, X(G,W)]^G, \ [f] \mapsto [i_* f].$$

The 0-th equivariant homotopy group $\pi_0^G(X)$ is then defined as

$$\pi_0^G(X) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(G,V)]^G.$$

Let k be a positive integer. Define

$$\pi_k^G(X) := colim_{V \in s(\mathcal{U}_G)} [S^{V\oplus \mathbb{R}^k}, X(G,V)]^G \tag{3.7}$$

$$\pi_{-k}^G(X) := colim_{V \in s(\mathcal{U}_G)} [S^V, X(G,V \oplus \mathbb{R}^k)]^G. \tag{3.8}$$

4. Unitary D_0^W-spectra

In this section we define and study the unitary version of $Sp^W_{D_0}$.

Definition 4.1. A $D_0^{\mathbb{C}}-$space is a continuous functor from the category $D_0^{\mathbb{C}}$ to the category of spaces. A morphism of $D_0^{\mathbb{C}}-$spaces is a natural transformation of functors. We denote the category of $D_0^{\mathbb{C}}-$spaces by $D_0^{\mathbb{C}}T$.

Proposition 4.2. $D_0^{\mathbb{C}}$ is a generalized Reedy model category in the sense of Definition 2.7.

Example 4.3. We can define the $D_0^{\mathbb{C}}-$sphere. For each object (G,V) in $D_0^{\mathbb{C}}$, we can define

$$S^{(G,V)} := S^V. \tag{4.1}$$

S^V inherits a $G-$action from that on V.

Let $\phi = (\phi_1, \phi_2) : (G,V) \longrightarrow (H,W)$ be a morphism in $D_0^{\mathbb{C}}$.

$$S(\phi) := S^{\phi_2} : S^V \longrightarrow S^W. \tag{4.2}$$

Unitary $D_0^{\mathbb{C}}-$spectra are the stabilization of unitary $D_0^{\mathbb{C}}-$spaces.

Let $\mathbb{U}$ denote the category whose objects are those of $D_0^{\mathbb{C}}$ and the morphisms $\mathbb{U}((G,V),(H,W))$ between two objects (G,V) and (H,W) is the Thom space of the total space

$$\xi^{\mathbb{C}}((G,V),(H,W)) = \{(w,(\phi_1,\phi_2)) \in W \times D_0^{\mathbb{C}}((G,V),(H,W)) | w \perp \phi_2(V)\}$$

of the orthogonal complement vector bundle, whose structure map

$$\xi^{\mathbb{C}}((G,V),(H,W)) \longrightarrow D_0^{\mathbb{C}}((G,V),(H,W))$$

is the projection to the second factor.

Definition 4.4. A unitary $D_0^{\mathbb{C}}-$spectrum is a based continuous functor from $\mathbb{U}$ to the category $\mathbf{T}$ of based compactly generated weak Hausdorff spaces. A morphism is a natural transformation of functors. Let $Sp^{D_0,U}$ denote the category of unitary $D_0^{\mathbb{C}}-$spectra.

For any morphism $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ in $D_0^{\mathbb{C}}$, consider the one-point compactification of the inclusion of the fiber over ϕ of the bundle $\xi^{\mathbb{C}}((G, V), (H, W))$. We can define a continuous map

$$(-, \phi) : S^{W-\phi_2(V)} \longrightarrow \mathbb{U}((G, V), (H, W)),\ w \mapsto (w, \phi)$$

Let Y be a unitary $D_0^{\mathbb{C}}-$spectrum. The structure map of Y associated to ϕ is defined to be the composite

$$\phi_\star :\ Y(G, V) \wedge S^{W-\phi_2(V)} \xrightarrow{Y(G,V)\wedge(-,\phi)} Y(G, V) \wedge \mathbb{U}((G, V), (H, W)) \xrightarrow{Y} Y(H, W).$$

Definition 4.5 (The category $D_0^{\mathbb{C}}T^W$). Define $D_0^{\mathbb{C}}T^W$ to be the full category of $D_0^{\mathbb{C}}\mathbf{T}$ consisting of those objects $X : D_0^{\mathbb{C}} \longrightarrow \mathbf{T}$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an $H-$weak equivalence.

Definition 4.6 (The category $Sp_W^{D_0,U}$). A $D_0^{\mathbb{C},W}-$spectrum X is both a $D_0^{\mathbb{C}}-$spectrum and a $D_0^{\mathbb{C}}-$space in $D_0^{\mathbb{C}}T^W$. The category $Sp_{D_0}^{W,U}$ is the full subcategory of $Sp^{D_0,U}$ consisting of $D_0^{\mathbb{C},W}-$spectra.

The complexification functor c, underlying functor u and fixed point functors ψ between orthogonal and unitary spaces all have stable analogs.

$$\begin{tikzcd} Sp^U \arrow[r, bend left, "u"] \arrow[r, bend right, "\psi"'] & Sp \arrow[l, "c"'] \end{tikzcd}$$

Let X be an orthogonal spectrum and Y a unitary spectrum.

$$cX := X \circ r.$$

$$(uY)(V) := \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}}))$$

where V is a complex inner product space and $iV = i\mathbb{R} \otimes_{\mathbb{R}} V \subset V_{\mathbb{C}}$ is the imaginary part of V.

We can define a natural involution $\psi : uY \longrightarrow uY$. Define

$$\psi_V := \mathrm{Map}(S^\tau, Y(\tau_V)) : \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}})) \longrightarrow \mathrm{Map}(S^{iV}, Y(V_{\mathbb{C}}))$$

where $\tau : iV \longrightarrow iV$ is multiplication by -1.

$$(Y^\psi)(V) := \mathrm{Map}^C(S^{iV}, Y(V_{\mathbb{C}})).$$

5. Reedy Model Structure

In Section 6.4 [**13**] we show there is a Reedy model structure on the category D_0T, which is constructed in the canonical way. The subcategory D_0T^W inherits that model structure from D_0T, which is Quillen equivalent to the $Fin-$level model structure on orthogonal spaces (Definition 1.4.2 and Proposition 1.4.3, [**19**]). Analogously, we can also construct a Reedy model structure on the category Sp^{D_0} and $Sp_W^{D_0}$.

First we describe the construction of the Reedy model structure on $Sp_W^{D_0}$.

All the construction of the Reedy model structure presented on D_0-spaces in Section 6.4 [**13**] can be applied to D_0^W-spectra. Let X be an object in $Sp_W^{D_0}$. As in D_0T, we can also define the $r-$th latching object $L_r(X)$ and the $r-$th matching object $M_r(X)$ of it for each object r of D_0.

As in Definition 6.4.3, [**13**], we have the definition below.

DEFINITION 5.1. A map $f : X \longrightarrow Y$ in $Sp_W^{D_0}$ is called a

-Reedy cofibration if for each object r of D_0, the relative latching map $X_r \coprod_{L_r(X)} L_r(Y) \longrightarrow Y_r$ is an $Aut(r)-$cofibration.

-Reedy weak equivalence if for each r,

$$f(r)^H : X(r)^H \longrightarrow Y(r)^H$$

is a weak equivalence for each closed subgroup H of $Aut(r)$.

-Reedy fibration if for each r, the relative matching map

$$X_r^H \longrightarrow M_r(X)^H \times_{M_r(Y)^H} Y_r^H$$

is a Serre fibration for each closed subgroup H of $Aut(r)$.

The next result is the stable analog for the Reedy model structure on the category of D_0^W-spaces. The proof is completely parallel to that of Theorem 6.4.5 [**13**] , and we omit it.

THEOREM 5.2. *The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of* $Sp_W^{D_0}$.

THEOREM 5.3. *The Reedy model structure on* $Sp_W^{D_0}$ *is Quillen equivalent to the* $Fin-$*level model structure on orthogonal spectra in Proposition 4.3.7,* [**19**].

PROOF. We can check directly from definitions that Q maps Reedy fibrations to $Fin-$level fibrations, and maps Reedy weak equivalences to $Fin-$level equivalences. So Q preserves fibrations and acyclic fibrations. Thus, the pair of adjoint functors

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\leftrightarrows}} Sp_W^{D_0}$$

is a Quillen pair.

Then we show that $f : X \longrightarrow Y$ is a Reedy weak equivalence if $Q(f)$ is an $Fin-$level equivalence. For each n, $f(\Sigma_n, \mathbb{R}^n) : X(\Sigma_n, \mathbb{R}^n) \longrightarrow Y(\Sigma_n, \mathbb{R}^n)$ is a Σ_n-weak equivalence. Then since in the commutative diagram

$$\begin{tikzcd} X(\Sigma_n, \mathbb{R}^n) \arrow[r, "f(\Sigma_n, \mathbb{R}^n)"] \arrow[d, "X(i,Id)"'] & Y(\Sigma_n, \mathbb{R}^n) \arrow[d, "Y(i,Id)"'] \\ X(G, \mathbb{R}^n) \arrow[r, "f(G, \mathbb{R}^n)"] & Y(G, \mathbb{R}^n) \end{tikzcd} \tag{5.1}$$

both vertical maps are restriction maps and thus $G-$weak equivalence, so by 2-out-of-3 the bottom map is a $G-$weak equivalence. So f is a Reedy weak equivalence.

Thus, Q creates weak equivalence. Since the unit is an isomorphism, (P, Q) is a Quillen equivalence.

□

6. Global Model Structure

In Section 6.5 [**13**], we construct a global model structure on D_0-spaces. This model structure can also give a model structure on $Sp_W^{D_0}$. In this section we show it is Quillen equivalent to the $Fin-$global model structure on orthogonal spectra in Theorem 4.3.17 in [**19**].

First we describe the global model structure on $Sp_W^{D_0}$.

DEFINITION 6.1. A morphism $f : X \longrightarrow Y$ in $Sp_W^{D_0}$ is a global equivalence if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all the finite groups G and all integers k.

DEFINITION 6.2. A D_0^W-spectrum X is a global $\Omega-$spectrum if for any objects (G, V) and (G, W) of D_0, the adjoint structure map $\widetilde{\sigma}_{(G,V),(G,W)} : X(G, V) \longrightarrow \mathrm{map}_*(S^W, X(G \times G, V \oplus W)) \xrightarrow{\Delta_G^*} \mathrm{map}_*(S^W, X(G, V \oplus W))$ is a $G-$weak equivalence. Her $\Delta_G^* : G \longrightarrow G \times G$ is the diagonal map.

PROPOSITION 6.3. A D_0^W-spectrum X is a global $\Omega-$spectrum if and only if the unique morphism from X to a terminal D_0-space is a global fibration.

PROOF. Straightforward from the definitions. $\square$

DEFINITION 6.4. A morphism $f : X \longrightarrow Y$ in $Sp_W^{D_0}$ is a global fibration if it is a Reedy fibration and for each object $d = (G, V)$, $b = (G, W)$ of D_0, any subgroup H of G the square

$$\begin{array}{ccc} X(d)^H & \xrightarrow{\widetilde{\sigma}_{d,b}} & \mathrm{map}^H(S^W, X(G, V \oplus W)) \\ \Big\downarrow{\scriptstyle f(d)^H} & & \Big\downarrow{\scriptstyle \mathrm{map}^H(S^W, f(G, V \oplus W))} \\ Y(d)^H & \xrightarrow[\widetilde{\sigma}_{d,b}]{} & \mathrm{map}^H(S^W, Y(G, V \oplus W)) \end{array} \tag{6.1}$$

is a homotopy cartesian.

We proved the unstable version of Proposition 6.5 in Proposition 6.5.9 [**13**]. The proof is parallel to that.

PROPOSITION 6.5. (i) Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

be a pushout square in $Sp_W^{D_0}$ such that f is a global equivalence. If in addition f or g is an $h-$cofibration, then the morphism k is a global equivalence.

(ii) Let

$$\begin{array}{ccccc} C & \xleftarrow{g} & A & \xrightarrow{f} & B \\ {\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ C' & \xleftarrow[g']{} & A' & \xrightarrow[f']{} & B' \end{array}$$

be a commutative diagram in $Sp_W^{D_0}$ such that g and g' are $h-$cofibartions. If the morphisms α, β and γ are global equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta : C \cup_A B \longrightarrow C' \cup_{A'} B'.$$

(iii) Let

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"] & Y \end{tikzcd}$$

be a pullback square in $Sp_W^{D_0}$ in which f is a global equivalence. If in addition one of the morphisms f or h is a Reedy fibration, then the morphism g is also a global equivalence.

(iv) Every global equivalence that is also a global fibration is a Reedy weak equivalence.

Let J denote the set of generating trivial cofibrations in the Reedy model structure. Let K denote the set of morphisms that detect the squares (6.1) are homotopy cartesian. Let (V,ρ) and (W,ρ') be $G-$representations such that $d = (\rho(G), V)$ and $b = ((\rho\oplus\rho')(G\times G), V\oplus W)$ be two objects in D_0 and let $\phi : d \longrightarrow b$ be the morphism (Id_G, i_V) with $i_V : V \longrightarrow V\oplus W$ the inclusion. Then we have the restriction morphism

$$\lambda_{G,V,W}\ F_{G,V\oplus W}S^W \longrightarrow F_{G,V}.$$

This morphism is a global equivalence. Let

$$K := \bigcup_{G,V,W} Z(\lambda_{G,V,W}), \tag{6.2}$$

the set of all pushout products of sphere inclusions $S^{k-1} \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\lambda_{G,V,W}$, where the union goes over a set of representatives of the isomorphism classes of triples (G,V,W) with (G,V) and (G,W) objects of D_0. The right lifting property with respect to the union $J\cup K$ characterizes the global fibrations.

Proposition 6.6. A morphism of D_0-spaces is a global fibration if and only if it has the right lifting property with respect to the set $J\cup K$.

In Theorem 6.5.8 [**13**], we show and prove the global model structure on the category of D_0-spaces. In this paper we prove the global model structure on D_0^W-spectra.

Theorem 6.7. *The global equivalences, global fibrations and Reedy cofibrations on $Sp_W^{D_0}$ form a model structure, the global model structure on the category of $Sp_W^{D_0}$. The fibrant objects in the global model structure are the global $\Omega-$spectra. The global structure is proper, topological and compactly generated.*

Proof. The numbering of the model category axioms is as that in Definition 3.3, [**7**].

The category $Sp_W^{D_0}$ is complete and cocomplete(MC1), the global equivalences satisfy the 2-out-of-3 property(MC2) and the classes of global equivalences, global fibration and Reedy cofibrations are all closed under retracts(MC3). The Reedy model structure implies that each morphism in $Sp_W^{D_0}$ can be factored as a Reedy cofibration followed by a Reedy weak equivalence. Since Reedy weak equivalences are all global equivalences, this provides one of the factorizations as required by MC5.

For the other half of MC5, apply the small object argument to the set $J \cup K$. All morphisms in J are Reedy cofibrations and Reedy weak equivalence. Since $F_{G,V\oplus W}S^W$ and $F_{G,V}$ are Reedy cofibrant, the morphisms in K are also Reedy cofibrant, and they are global equivalences because the morphisms $\lambda_{G,V,W}$ are. The small object argument provides a functorial factorization of every morphism $\phi : X \longrightarrow Y$ in $Sp_W^{D_0}$ as composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in K and q has the right lifting property with respect to $J \cup K$. Since all morphisms in K are flat cofibrations and global equivalences, the morphism i is a flat cofibration and a global fibration.

Now we show the lifting properties of MC4. By Proposition 6.5 (iv) a morphism that is both a global fibration and a global equivalence is a Reedy weak equivalence, and hence an acyclic fibration in the Reedy model structure. So every morphism that is simultaneously global fibrations and a global equivalence has the right lifting property with respect to Reedy cofibrations. Now let $j : A \longrightarrow B$ be a Reedy cofibration that is also a global equivalence. We show that it has the left lifting property with respect to all global fibrations. Factor $j = q \circ i$, via the small object argument for $J \cup K$, where $i : A \longrightarrow W$ is a $J \cup K-$cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and hence an acyclic fibration in the Reedy model structure by Proposition 6.5 (iv). Since j is a Reedy cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, "="] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to all global fibrations. But then j itself has the lifting property. Thus we verified all the model category axioms. Meanwhile We have also specified sets of generating Reedy cofibrations I and generating acyclic cofibrations $J \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of Reedy cofibrations. So the global model structure is compactly generated.

Left properness of the global model structure follows from Proposition 6.5 (i) and the fact that Reedy cofibrations are $h-$cofibrations. Right properness follows from Proposition 6.5 (iii) since global fibrations are Reedy fibrations.

The proof the the global model structure is topological is formal.

□

THEOREM 6.8. *The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the $Fin-$global model structure on orthogonal spectra in Theorem 4.3.17* [**19**].

PROOF. The functor Q sends global fibrations to $Fin-$global fibration and it sends global equivalences to $Fin-$equivalences. So it preserves fibrations and acyclic fibrations. So the pair (P, Q) in Proposition 3.9 is a Quillen pair.

For a morphism $f : X \longrightarrow Y$ in $Sp_W^{D_0}$, if $Q(f)$ is a $Fin-$equivalence, by Definition 3.11, f is a global equivalence. So Q creates weak equivalence, the pair (P, Q) is a Quillen equivalence.

□

7. Examples: Quasi-theories

Quasi-elliptic cohomology is a variant of Tate K-theory. It is not an elliptic cohomology but does reflect the geometric features of the Tate curve. It has simpler form than most elliptic cohomology theories and can be expressed explicitly by equivariant K-theories, which can be globalized. A thorough introduction of this theory can be found in [**14**] and Chapter 2 [**13**].

In [**12**] we construct a $\mathcal{I}_G-$FSP representing quasi-elliptic cohomology up to a weak equivalence, which, however, cannot arise from an orthogonal spectrum. In [**16**] we apply the idea to construct a $\mathcal{I}_G-$FSP weakly representing each quasi-theory $QE^*_{n,G}(-)$. In this section we show these $\mathcal{I}_G-$FSP together define a D^W_0-spectrum.

7.1. Quasi-theories. In this section we recall the quasi-theories. The main reference for that is [**15**].

Let G be a compact Lie group and n denote a positive integer. Let G^{tors}_{conj} denote a set of representatives of $G-$conjugacy classes in the set G^{tors} of torsion elements in G. Let G^n_z denote set

$$\{\sigma = (\sigma_1, \sigma_2, \cdots \sigma_n) | \sigma_i \in G^{tors}_{conj}, [\sigma_i, \sigma_j] \text{ is the identity element in } G\}.$$

Let $\sigma = (\sigma_1, \sigma_2, \cdots \sigma_n) \in G^n_z$. Define

$$C_G(\sigma) := \bigcap_{i=1}^{n} C_G(\sigma_i); \tag{7.1}$$

$$\Lambda_G(\sigma) := C_G(\sigma) \times \mathbb{R}^n / \langle (\sigma_1, -e_1), (\sigma_2, -e_2), \cdots (\sigma_n, -e_n) \rangle. \tag{7.2}$$

where $C_G(\sigma_i)$ is the centralizer of each σ_i in G and $\{e_1, e_2, \cdots e_n\}$ is a basis of $\mathbb{R}^n$.

Let $q : \mathbb{T} \longrightarrow U(1)$ denote the representation $t \mapsto e^{2\pi i t}$. Let $q_i = 1 \otimes \cdots \otimes q \otimes \cdots \otimes 1 : \mathbb{T}^n \longrightarrow U(1)$ denote the tensor product with q at the $i-$th position and trivial representations at other position. The representation ring

$$R(\mathbb{T}^n) \cong R(\mathbb{T})^{\otimes n} = \mathbb{Z}[q_1^{\pm}, \cdots q_n^{\pm}].$$

We have the exact sequence

$$1 \longrightarrow C_G(\sigma) \longrightarrow \Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 0 \tag{7.3}$$

where the first map is $g \mapsto [g, 0]$ and the second map is $\pi([g, t_1, \cdots t_n]) = (e^{2\pi i t_1}, \cdots e^{2\pi i t_n})$. Then the map $\pi^* : R(\mathbb{T}^n) \longrightarrow R\Lambda_G(\sigma)$ equips the representation ring $R\Lambda_G(\sigma)$ the structure as an $R(\mathbb{T}^n)-$module.

This is Lemma 3.1 [**15**] presenting the relation between $RC_G(\sigma)$ and $R\Lambda_G(\sigma)$.

Lemma 7.1. *$\pi^* : R(\mathbb{T}^n) \longrightarrow R\Lambda_G(\sigma)$ exhibits $R\Lambda_G(\sigma)$ as a free $R(\mathbb{T}^n)-$module.*

There is an $R(\mathbb{T}^n)-$basis of $R\Lambda_G(\sigma)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|_{C_G(\sigma)}$ to $C_G(\sigma)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(\sigma)$.

Definition 7.2. For equivariant cohomology theories $\{E^*_H\}_H$ and any $G-$space X, the corresponding quasi-theory $QE^*_{n,G}(X)$ is defined to be

$$\prod_{\sigma \in G^n_z} E^*_{\Lambda_G(\sigma)}(X^\sigma).$$

EXAMPLE 7.3 (Motivating example: Tate K-theory and quasi-elliptic cohomology). Tate $K-$theory is the generalized elliptic cohomology associated to the Tate curve. The elliptic cohomology theories form a sheaf of cohomology theories over the moduli stack of elliptic curves $\mathcal{M}_{ell}$. Tate K-theory over Spec$\mathbb{Z}((q))$ is obtained when we restrict it to a punctured completed neighborhood of the cusp at ∞, i.e. the Tate curve $Tate(q)$ over Spec$\mathbb{Z}((q))$ [Section 2.6, [**1**]]. The divisible group associated to Tate K-theory is $\mathbb{G}_m \oplus \mathbb{Q}/\mathbb{Z}$.

Other than the theory over Spec$\mathbb{Z}((q))$, we can define variants of Tate K-theory over Spec$\mathbb{Z}[q]$ and Spec$\mathbb{Z}[q^{\pm}]$ respectively. The theory over Spec$\mathbb{Z}[q^{\pm}]$ is of especial interest. Inverting q allows us to define a sufficiently non-naive equivariant cohomology theory and to interpret some constructions more easily in terms of extensions of groups over the circle. The resulting cohomology theory is called quasi-elliptic cohomology [**17**][**13**][**14**]. Its relation with Tate K-theory is

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X) \tag{7.4}$$

which also reflects the geometric nature of the Tate curve. The idea of quasi-elliptic cohomology is motivated by Ganter's construction of Tate K-theory [**6**]. It is not an elliptic cohomology but a more robust and algebraically simpler treatment of Tate K-theory. This new theory can be interpreted in a neat form by equivariant K-theories. Some formulations in it can be generalized to equivariant cohomology theories other than Tate K-theory.

Quasi-elliptic cohomology $QEll^*_G(-)$ is exactly the quasi-theory $QK^*_{1,G}(-)$ in Definition 7.2.

EXAMPLE 7.4 (Generalized Tate K-theory and generalized quasi-elliptic cohomology). In Section 2 [**8**] Ganter gave an interpretation of $G-$equivariant Tate K-theory for finite groups G by the loop space of a global quotient orbifold. Apply the loop construction n times, we can get the $n-$th generalized Tate K-theory. The divisible group associated to it is $\mathbb{G}_m \oplus (\mathbb{Q}/\mathbb{Z})^n$.

With quasi-theories, we can get a neat expression of it. Consider the quasi-theory

$$QK^*_{n,G}(X) = \prod_{\sigma \in G^n_z} K^*_{\Lambda_G(\sigma)}(X^\sigma).$$

$QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^{\pm}]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}$ is isomorphic to the $n-$th generalized Tate K-theory.

7.2. Construction of the unitary $D^{\mathbb{C}}_0-$spectrum. In this section we show the construction of a $D^{\mathbb{C}}_0-$spectrum representing the theory $QE^*_{n,G}(-)$. The details of the construction are in Section 5 [**16**] with the main conclusion Theorem 5.18. Another reference is Chapter 4 [**13**].

Let G be a compact Lie group. We consider equivariant cohomology theories E that have the same key features as equivariant complex K-theories. More explicitly,
• The theories $\{E^*_G\}_G$ have the change-of-group isomorphism, i.e. for any closed subgroup H of G and $H-$space X, the change-of-group map $\rho^G_H : E^*_G(G \times_H X) \longrightarrow E^*_H(X)$ defined by $E^*_G(G \times_H X) \xrightarrow{\phi^*} E^*_H(G \times_H X) \xrightarrow{i^*} E^*_H(X)$ is an isomorphism where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$..
• There exists an orthogonal spectrum E such that for any compact Lie group G and "large" real $G-$representation V and a compact $G-$space B we have a bijection $E_G(B) \longrightarrow [B_+, E(V)]^G$. And (E_G, η^E, μ^E) is the underlying orthogonal

$G-$spectrum of E.

• Let G be a compact Lie group and V an orthogonal $G-$representation. For every ample $G-$representation W, the adjoint structure map $\widetilde{\sigma}^E_{V,W} : E(V) \longrightarrow \mathrm{Map}(S^W, E(V\oplus W))$ is a $G-$weak equivalence.

In Theorem 5.18 [**16**] we introduce the conclusion below.

Theorem 7.5. *For each positive integer n and each compact Lie group G, there is a well-defined functor $\mathcal{Q}_{G,n}$ from the category of orthogonal ring spectra to the category of $\mathcal{I}_G-$FSP sending E to $(QE_n(G,-), \eta^{QE_n}, \mu^{QE_n})$ that weakly represents the quasi-theory $QE^*_{n,G}$.*

We sketch the construction of the $\mathcal{I}_G-$FSP $(QE_n(G,-), \eta^{QE_n}, \mu^{QE_n})$ below. For more details please see [**16**].

- Let V be a real $G-$representation.
- Let $Sym(V) := \bigoplus\limits_{n\geq 0} Sym^n(V)$ denote the total symmetric power. It inherits a $G-$action from that on V.
- Let $S(G,V)_\sigma := Sym(V)\setminus Sym(V)^\sigma = \bigcup\limits_{i=1}^{n} Sym(V)\setminus Sym(V)^{\sigma_i}$ for each $\sigma = (\sigma_1, \cdots \sigma_n)\in G^n_z$.
- Let E denote the unitary global spectrum representing the theory E^*.
- $(V)^{\mathbb{R}}_\sigma$ is a specific real $\Lambda_G(\sigma)-$representation introduced in (A.2) in Appendix [**16**]. We show the construction below.

Let ρ be a complex $G-$representation with underlying space V. Let $i : C_G(\sigma)\hookrightarrow G$ denote the inclusion. Let $\{\lambda\}$ denote all the irreducible complex representations of $C_G(\sigma)$. We have the decomposition of $C_G(\sigma)-$representation i^*V into its isotypic components $i^*V \cong \bigoplus\limits_\lambda V_\lambda$ where V_λ denotes the sum of all subspaces of V isomorphic to λ. Each $V_\lambda = Hom_{C_G(g)}(\lambda, V)\otimes_{\mathbb{C}} \lambda$ is unique as a subspace.

Each V_λ can be equipped with a $\Lambda_G(\sigma)-$action. Each $\lambda(\sigma_i)$ is of the form $e^{\frac{2\pi i m_{\lambda i}}{l_i}} I$ with l_i the order of σ_i, $0 < m_{\lambda i}\leq l_i$ and I the identity matrix. As shown in Lemma 7.1, we have the well-defined complex $\Lambda_G(\sigma)-$representations

$$(V_\lambda)_\sigma := V_\lambda \odot_{\mathbb{C}} (q^{\frac{m_{\lambda 1}}{l_1}}\otimes\cdots\otimes q^{\frac{m_{\lambda n}}{l_n}})$$

and

$$(V)_\sigma := \bigoplus_\lambda (V_\lambda)_\sigma. \tag{7.5}$$

In addition,

$$(V)^{\mathbb{R}}_\sigma := (V\otimes_{\mathbb{R}}\mathbb{C})_\sigma \oplus (V\otimes_{\mathbb{R}}\mathbb{C})^*_\sigma \tag{7.6}$$

is real $\Lambda_G(\sigma)-$representation. It is faithful when V is a faithful real $G-$representation.

From the ingredients above, we construct

$$F_\sigma(G,V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)^{\mathbb{R}}_\sigma}, E((V)^{\mathbb{R}}_\sigma\oplus V^\sigma)) \tag{7.7}$$

We have the unit map $\eta_\sigma(G,V) : S^{V^\sigma}\longrightarrow F_\sigma(G,V)$ and the multiplication

$$\mu^F_{(\sigma,\tau)}((G,V),(H,W)) : F_\sigma(G,V)\wedge F_\tau(H,W)\longrightarrow F_{(\sigma,\tau)}(G\times H, V\oplus W)$$

cited in Proposition 5.2(i) [**16**].

$$QE_{n,\sigma}(G,V) := \{t_1a+t_2b \in F_\sigma(G,V) * S(G,V)_\sigma \,|\, \|b\| \leqslant t_2\}/\{t_1c_0+t_2b\} \tag{7.8}$$

where $*$ denotes the join.

$$QE_n(G,V) := \prod_{\sigma\in G_z^n} \mathrm{Map}_{C_G(\sigma)}(G, QE_{n,\sigma}(G,V)) \tag{7.9}$$

We use x_σ to denote the basepoint of $QE_{n,\sigma}(G,V)$. For each $v \in S^V$, there are $v_1 \in S^{V^\sigma}$ and $v_2 \in S^{(V^\sigma)^\perp}$ such that $v = v_1 \wedge v_2$. Let $\eta_\sigma^{QE_n}(G,V) : S^V \longrightarrow QE_{n,\sigma}(G,V)$ be the map

$$\eta_\sigma^{QE_n}(G,V)(v) := \begin{cases} (1-\|v_2\|)\eta_\sigma(G,V)(v_1) + \|v_2\|v_2, & \text{if } \|v_2\| \leqslant 1; \\ x_\sigma, & \text{if } \|v_2\| \geqslant 1. \end{cases} \tag{7.10}$$

The unit map is defined by
(7.11)

$$\eta^{QE_n}(G,V) : S^V \longrightarrow \prod_{\sigma\in G_z^n} \mathrm{Map}_{C_G(\sigma)}(G, QE_{n,\sigma}(G,V)),\ v \mapsto \prod_{\sigma\in G_z^n} (\alpha \mapsto \eta_\sigma^{QE_n}(G,V)(\alpha\cdot v)),$$

Next, we construct the multiplication map μ^{QE_n}. First we define a map

$$\mu_{(\sigma,\tau)}^{QE_n}((G,V),(H,W)) : QE_{n,\sigma}(G,V) \wedge QE_{n,\tau}(H,W) \longrightarrow QE_{n,(\sigma,\tau)}(G\times H, V\oplus W)$$

by sending a point $[t_1a_1+t_2b_1] \wedge [u_1a_2+u_2b_2]$ to
(7.12)

$$\begin{cases} [(1-\sqrt{t_2^2+u_2^2})\mu_{(\sigma,\tau)}^F((G,V),(H,W))(a_1\wedge a_2) & \text{if } t_2^2+u_2^2 \leq 1 \text{ and } t_2u_2 \neq 0; \\ +\sqrt{t_2^2+u_2^2}(b_1+b_2)], & \\ [(1-t_2)\mu_{(\sigma,\tau)}^F((G,V),(H,W))(a_1\wedge a_2)+t_2b_1], & \text{if } u_2=0 \text{ and } 0<t_2<1; \\ [(1-u_2)\mu_{(\sigma,\tau)}^F((G,V),(H,W))(a_1\wedge a_2)+u_2b_2], & \text{if } t_2=0 \text{ and } 0<u_2<1; \\ [1\mu_{(\sigma,\tau)}^F((G,V),(H,W))(a_1\wedge a_2)+0], & \text{if } u_2=0 \text{ and } t_2=0; \\ x_{\sigma,\tau}, & \text{Otherwise.} \end{cases}$$

where $x_{\sigma,\tau}$ is the basepoint of $QE_{n,(\sigma,\tau)}(G\times H, V\oplus W)$.

The basepoint of $QE_n(G,V)$ is the product of the basepoint of each factor $\mathrm{Map}_{C_G(\sigma)}(G, QE_{n,\sigma}(G,V))$, i.e. the product of the constant map to the basepoint of each $QE_{n,\sigma}(G,V)$.

We can define the multiplication $\mu^{QE_n}((G,V),(H,W)) : QE_n(G,V)\wedge QE_n(H,W) \longrightarrow QE_n(G\times H, V\oplus W)$ by

$$\Big(\prod_{\sigma\in G_z^n}\alpha_\sigma\Big) \wedge \Big(\prod_{\tau\in H_z^n}\beta_\tau\Big) \mapsto \prod_{\substack{\sigma\in G_z^n\\ \tau\in H_z^n}} \Big((\sigma',\tau') \mapsto \mu_{(\sigma,\tau)}^{QE_n}((G,V),(H,W))\big(\alpha_\sigma(\sigma')\wedge\beta_\tau(\tau')\big)\Big).$$

However, the equivariant orthogonal spectra cannot arise from a global spectrum in the sense of Remark 4.1.2, [**19**].

7.3. QE_n **is a global spectrum in** $Sp^W_{D_0}$**.** Via a completely parallel proof to that of Theorem 7.1.5, [**13**], we have the conclusion below.

THEOREM 7.6. *The* $\mathcal{I}_G-$*FSP* $QE_n : \mathcal{I}_G \longrightarrow G\mathcal{T}$ *together with the unit map* η^{QE_n} *defined in (7.11) and the multiplication* $\mu^{QE_n}((G,-),(G,-))$ *that weakly represents the quasi-theory* $QE^*_{n,G}(-)$ *with* G *varying all the finite groups can fit together to give a* D_0^W-*spectrum.*

THEOREM 7.7. QE_n *is a* D_0^W-*space.*

PROOF. Let (G,V) and (H,W) be two objects and $\phi=(\phi_1,\phi_2):(G,V)\longrightarrow(H,W)$ be a morphism in D_0. Let H' denote $H\cap O(\phi_2(V))$. Recall both ϕ_1 and ϕ_2 are injective. Note that for $\tau=(\tau_1,\cdots\tau_n)\in H^n_z$, $\phi_1(\tau)=(\phi_1(\tau_1),\cdots\phi_1(\tau_n))$ can be in G^n_z. ϕ_2 gives the linear isometric embedding

$$\phi_{2*}:(V)^{\mathbb{R}}_\tau\oplus V^{\phi_1(\tau)}\longrightarrow(W)^{\mathbb{R}}_\tau\oplus W^\tau,\ (v_1,v_2)\mapsto(\phi_2(v_1),\phi_2(v_2)). \tag{7.13}$$

It is equivariant in the sense that

$$\phi_{2*}([\phi_1(a),t]\cdot x)=[a,t]\cdot\phi_{2*}(x)$$

for any $x\in(V)^{\mathbb{R}}_\tau\oplus V^{\phi_1(\tau)}$, $[a,t]\in\Lambda_{H'}(\tau)$, and $[\phi_1(a),t]\in\Lambda_G(\phi_1(\tau))$.

Let

$$\beta:S^{(V)^{\mathbb{R}}_\tau}\longrightarrow E((V)^{\mathbb{R}}_\tau\oplus V^{\phi_1(\tau)})$$

be an element in $F_{\phi_1(\tau)}(G,V)$. We can define $F_\sigma(\phi_{2*})(\beta):S^{(W)^{\mathbb{R}}_\tau}\longrightarrow E((W)^{\mathbb{R}}_\tau\oplus W^\tau)$ by the composition

$$S^{(W)^{\mathbb{R}}_\tau}=S^{(W)^{\mathbb{R}}_\tau-(V)^{\mathbb{R}}_{\phi_1(\tau)}}\wedge S^{(V)^{\mathbb{R}}_{\phi_1(\tau)}}\xrightarrow{Id\wedge\beta}S^{(W)^{\mathbb{R}}_\tau-(V)^{\mathbb{R}}_{\phi_1(\tau)}}\wedge E((V)^{\mathbb{R}}_{\phi_1(\tau)}\oplus V^{\phi_1(\tau)})\longrightarrow$$

$$E((W)^{\mathbb{R}}_\tau\oplus V^{\phi_1(\tau)})\xrightarrow{E(Id\oplus\phi_2)}E((W)^{\mathbb{R}}_\tau\oplus W^\tau)$$

where the third map is the structure map of E and $Id\oplus\phi_2$ is the evident linear isometric embedding

$$(W)^{\mathbb{R}}_\tau\oplus V^{\phi_1(\tau)}\longrightarrow(W)^{\mathbb{R}}_\tau\oplus W^\tau.$$

It is $\mathbb{R}-$linear.

It's straightforward to check for morphisms

$$(G,V)\xrightarrow{\phi=(\phi_1,\phi_2)}(H,W)\xrightarrow{\psi=(\psi_1,\psi_2)}(K,U)$$

So we have

$$F_\sigma(\psi_{2*})\circ F_\sigma(\phi_{2*})(\beta)=F_\sigma((\psi\circ\phi)_{2*})(\beta). \tag{7.14}$$

and if ϕ is the identity map, $F_\sigma(\phi_{2*})$ is identity.

ϕ_2 also gives embeddings

$$Sym(V)\longrightarrow Sym(W)\text{ and }Sym(V)^{\phi_1(\tau)}\longrightarrow Sym(W)^\tau,$$

So we have well-defined

$$S(\phi_2):S(G,V)_{\phi_1(\tau)}\longrightarrow S(H,W)_\tau, \tag{7.15}$$

which is equivariant in the sense:

$$S(\phi_2)(\phi_1(\tau')\cdot y)=\tau'\cdot S(\phi_2)(y),\text{ for any }\tau'\in C_{H'}(\tau).$$

Then we have the join $\phi_{2*}*S(\phi_2):F_{\phi_1(\tau)}(G,V)*S(G,V)_{\phi_1(\tau)}\longrightarrow F_\tau(H,W)*S(H,W)_\tau$.

Let

$$\phi_{2\star}:QE_{n,\phi_1(\tau)}(G,V)\longrightarrow QE_{n,\tau}(H,W)$$

denote the quotient of the restriction $\phi_{2*} * S(\phi_2)|_{QE'_{n,\phi_1(\tau)}(G,V)}$.

Let $\tau \in H^n_z$. Let $f \in \mathrm{Map}_{C_G(\phi_1(\tau))}(G, QE_{n,\phi_1(\tau)}(G,V))$. Define $\widetilde{\phi_\star f} : C_{O(W)}(\tau) \times_{C_{H'}(\tau)} H' \longrightarrow \mathrm{Map}_{C_H(\tau)}(G, F_\tau(H,W))$ by

$$\widetilde{\phi_\star f}([\alpha, \tau']) = \alpha(\phi_{2\star} \circ f \circ \phi_1)(\tau') \tag{7.16}$$

Define

$$QE_n(\phi)_\tau : \mathrm{Map}_{C_G(\phi_1(\tau))}(G, QE_{n,\phi_1(\tau)}(G,V)) \longrightarrow \mathrm{Map}_{C_H(\tau)}(H, QE_{n,\tau}(H,V))$$

by

$$QE_n(\phi)_\tau(f)(g) := \begin{cases} \widetilde{\phi_\star f}(g), & \text{if } g \in C_{O(W)}(\tau) \times_{C_{H'}(\tau)} H'; \\ c_0, & \text{otherwise.} \end{cases} \tag{7.17}$$

where c_0 denotes the basepoint of $F_\tau(H,V)$.

$$QE_n(\phi) := \prod_{\tau \in H^n_z} QE_n(\phi)_\tau. \tag{7.18}$$

If ϕ is the identity map, $QE_n(\phi)$ is the identity map. If $\phi : (G,V) \longrightarrow (H,W)$ and $\psi : (H,W) \longrightarrow (K,U)$ are two morphisms in D_0, we have

$$QE_n(\psi \circ \phi) = QE_n(\psi) \circ QE_n(\phi).$$

So QE_n defines a D^W_0-space.

□

THEOREM 7.8. *QE_n is a D_0-FSP weakly representing QE_n. Especially, QE_n is a global ring spectrum in $Sp^{D_0}_W$.*

The proof is analogous to the proof of Theorem 7.2.3 [**13**]. It is straightforward and left to the readers.

REMARK 7.9. Given a unitary global spectrum representing E, we can also construct a unitary D^W_0-spectrum representing QE^*_n in the same way described in this section. The complex $\Lambda_G(\sigma)-$representations needed in the construction is constructed in Section A.1 [**16**].

REMARK 7.10. Theorem 7.5 guarantees that generalized Morava E-theories [**18**] can be weakly represented by $\mathcal{I}_G-$FSP. By analogous argument, they can be globalized in the new setting as well.

REMARK 7.11. In this paper we show that from one global ring spectrum E, we can construct infinitely many global spectrum $\{QE_n\}_{n\in\mathbb{N}}$. So there is a large family of examples for global homotopy theory. In addition, I doubt whether the conjecture is true that the globalness of a cohomology theory is completely determined the formal component of its divisible group and when the étale component of it varies the globalness does not change.

References

[1] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland: *Elliptic spectra, the Witten genus and the theorem of the cube*. Invent. Math. , 146(3):595–687, 2001.

[2] Atiyah, M.F.: *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) 17 1966 367–386.

[3] Atiyah, M.F. and Segal, G.B.: *Equivariant K-theory and completion*, J. Differential Geometry 3 1969 1–18.

[4] Clemens Berger, Ieke Moerdijk: *On an extension of the notion of Reedy category*, Mathematische Zeitschrift, December 2011, Volume 269, Issue 3, pp 977–1004.

[5] Bohmann, Anna Marie: *Global orthogonal spectra*, Homology, Homotopy and Applications. Vol 16 (2014), No 1, 313–332.

[6] Jorge A. Devoto: *Equivariant elliptic homology and finite groups*, Michigan Math. J. , 43(1):3–32, 1996.

[7] W.G.Dwyer and J.Spalinski: *Homotopy theories and model categories*, In: Handbook of algebraic topology, ed.I.M.James, Elsevier (1995), 73–126.

[8] Nora Ganter: *Stringy power operations in Tate K-theory*, 2007, available at arXiv: math/0701565.

[9] David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, available at arXiv:math/0701916.

[10] J.P.C. Greenlees and J.P. May: *Localization and completion theorems for MU- module spectra*, Ann. of Math. (2), 146(3) (1997), 509–544.

[11] M. Hovey, B.E. Shipley, and J.H. Smith: *Symmetric spectra*, Journal of the American Mathematical Society, 13(1):149–208, 2000.

[12] Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, available at arXiv:1703.06562.

[13] Zhen Huan: *Quasi-elliptic cohomology*, Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 2017. 290 pp. `http://hdl.handle.net/2142/97268`.

[14] Zhen Huan: *Quasi-elliptic cohomology I*. Advances in Mathematics. Volume 337, 15 October 2018, Pages 107-138.

[15] Zhen Huan: *Quasi-theories*. available at arXiv:1809.06651.

[16] Zhen Huan: *Quasi-theories and their equivariant orthogonal spectra*. available at arXiv:1809.07622.

[17] Charles Rezk: *Quasi-elliptic cohomology*, unpublished manuscript, 2011.

[18] Tomer M. Schlank, Nathaniel Stapleton: *A transchromatic proof of Strickland's theorem*, Adv. Math. 285 (2015), 1415–1447.

[19] Stefan Schwede: *Global Homotopy Theory*, v0.23/April 30, 2015, Preliminary and incomplete version, `http://www.math.uni-bonn.de/people/schwede/global.pdf`.

[20] G.B.Segal: *Equivariant stable homotopy theory*, Proceedings of I.C.M., Nice. (1970)

Zhen Huan, Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275 China

E-mail address: `huanzhen84@yahoo.com`