RESEARCH OUTLINE

ZHEN HUAN

An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. Most elliptic cohomology theories themselves are usually difficult to express explicitly and compute. Quasi-elliptic cohomology is constructed as an object both reflecting the geometric nature of elliptic curves and more practicable to study. It can be expressed by equivariant K-theories in a neat way. It was first introduced by Ganter inspired by Devoto's equivariant Tate K-theory [14].

It is an old idea of Witten [32] that the elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops. There have already been relevant conclusions. Tate K-theory is the generalized elliptic cohomology associated to the Tate curve. The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity [Section 2.6, [3]]. Tate K-theory is itself a distinctive subject to study. Its relation with string theory is better understood than most known elliptic cohomology theories. In addition, the definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold [Section 2, [17]]. We hope to explore the physical meaning of elliptic cohomology theories and unfold the relation between elliptic cohomology theories and the free loop space via equivariant Tate K-theory. We need a bridge connecting elliptic cohomology theories and Tate K-theory.

Morava E-theories have many properties that reflects other homotopy theories. They serve as motivating examples for the research on other cohomology theories. A classification of the level structure of its formal group is given in [1]. Strickland proved in [44] that the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group. Stapleton proved in [40] this result for generalized Morava E-theory via transchromatic character theory [42] [43]. In each case, the power operation serves as a bridge connecting the homotopy theory and its formal group. It is conjectured that we have classification theorems of the geometric structures of each elliptic curve in the same form.

Ginzburg, Kapranov and Vasserot have the conjecture [20] that any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A. In his thesis [19], Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms. We are interested in answering this question from a different perspective. We are trying to give an explicit construction of the orthogonal $G-$spectrum of each elliptic cohomology theory.

The idea of global orthogonal spectra was first inspired in Greenlees and May [21]. Many classical theories, equivariant stable homotopy theory, equivariant bordism, equivariant K-theory, etc, naturally exist not only for a single group but a specific family of groups in a uniform way. Several models of global homotopy theories have been established, including that by Bohmann [13] and Schwede [41]. The two model categories of global spectra are Quillen equivalent, as shown in [13]. But it is unclear whether global elliptic cohomology theory exists and how we should construct it.

Elliptic cohomology theories themselves cannot serve as the starting point to answer the questions above, which implies the necessity of an intermediate theory. It is supposed to be a theory that is more practicable to compute and study than elliptic cohomology theories and serves as a breakthrough point to answer the questions above.

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Below we give an outline of our current research and future plans. In Section 1, we introduce this intermediate theory, quasi-elliptic cohomology, and interpret quasi-elliptic cohomology by a loop space. In Section 2 we construct an elliptic power operation of the theory. Then via the power operation we use quasi-elliptic cohomology to study the geometric structure of the Tate curve in Section 3. In Section 4 and 5 we study the representing spectrum of the theory. In Section 4 we construct in a new category GwS of orthogonal $G-$spectra and construct in it the equivariant orthogonal spectrum of quasi-elliptic cohomology. This equivariant orthogonal spectrum, as shown in Section 5, cannot arise from a global spectrum. In Section 5 we establish a new global homotopy theory where quasi-elliptic cohomology can be globalized. This new theory is equivalent to Schwede's global homotopy theory.

1. The Construction of Quasi-elliptic cohomology

Quasi-elliptic cohomology is a variant of Tate K-theory. In this section we construct quasi-elliptic cohomology from a loop space via bibundles, which verifies the old idea of Witten indirectly.

If G is a Lie group and X is a manifold with a smooth $G-$action, the space of smooth unbased loops in the orbifold $X/\!\!/G$ in principle carries a lot of structure: on the one hand, it includes loops represented by continuous maps $\gamma : \mathbb{R} \longrightarrow X$ such that $\gamma(t+1) = \gamma(t)g$ for some $g \in G$; at the same time the circle acts on the loop space by rotation. We interpret a "loop" as a bibundle, i.e. a 1-morphism in the localization of the category of Lie groupoids with respect to the equivalence of Lie groupoids. The right loop space $Loop^{ext}(X/\!\!/G)$ is constructed from the category of bibundles from $S^1/\!\!/*$ to $X/\!\!/G$ with the circle rotations added as morphisms.

Quasi-elliptic cohomology $QEll^*_G(X)$ is defined to be the orbifold K-theory of a subgroupoid $\Lambda(X/\!\!/G)$ of $Loop^{ext}(X/\!\!/G)$ consisting of constant loops. More explicitly, $QEll^*_G(X)$ can be expressed in terms of the equivariant K-theory of X and its subspaces

$$QEll^*_G(X) := \prod_{\sigma\in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} K^*_{\Lambda_G(\sigma)}(X^\sigma)\Big)^G. \tag{1}$$

We can also construct quasi-elliptic cohomology for Lie groupoids. We study those bibundles from $S^1/\!\!/*$ to a Lie groupoid $\mathbb{G}$ and construct the constant loop space $\Lambda(\mathbb{G})$. $QEll^*(\mathbb{G})$ is defined to be $K^*_{orb}(\mathbb{G})$. When $\mathbb{G}$ is a global quotient $X/\!\!/G$, $\Lambda(\mathbb{G})$ is the groupoid $\Lambda(X/\!\!/G)$ and $QEll^*(\mathbb{G})$ is $QEll^*_G(X)$ defined by (1).

In [18], Ganter explains that Tate K-theory is really a cohomology theory for orbifolds, which is based on Devoto's definition of the equivariant theory. We have the relation

$$QEll^*(\mathbb{G}) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = K^*_{Tate}(\mathbb{G}). \tag{2}$$

Moreover, we define an involution for the theory which is compatible with its geometric interpretation. As equivariant K-theory, quasi-elliptic cohomology also has the Real and real version, which is shown in Chapter 5, [27].

The quasi-elliptic cohomology is presented originally in Rezk's unpublished manuscript [37] and in full detail in [27]. We will present it more thoroughly and deeply in a coming paper [25].

2. Power operation

Power operations in elliptic cohomology arise from isogenies of the underlying elliptic curve [2]. Moreover, Ando, Hopkins and Strickland studied the power operation for Morava E-theories in [4]. Ganter constructed the power operation for equivariant Tate K-theory in [17] and that for orbifold Tate K-theory in [18]. The questions arise whether there is a power operation of quasi-elliptic cohomology exhibiting the relation between the cohomology theory and Tate curve and what advantages it has over the power operation of Tate K-theory. The explicit relation between quasi-elliptic cohomology and equivariant K-theories guarantees the existence of a power operation analogous to the Atiyah's power operation on equivariant K-theories [7]. Ganter indicates in [17] and [18] that quasi-elliptic cohomology has a power operation reflecting the geometric nature of the Tate curve.

We construct a power operation $\{\mathbb{P}_n\}_{n\geq 0}$ for quasi-elliptic cohomology via explicit formulas that interwine the power operation in K-theory and natural symmetries of the free loop space. We have the theorem below.

THEOREM 2.1. *(Section 4.2, [24]) Quasi-elliptic cohomology has a power operation*

$$\mathbb{P}_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

that is elliptic in the sense: $\mathbb{P}_n$ can be uniquely extended to the stringy power operation

$$P_n^{string} : K_{Tate}(X/\!\!/G) \longrightarrow K_{Tate}(X^{\times n}/\!\!/(G\wr\Sigma_n))$$

of the Tate K-theory in [17], which is elliptic in the sense of [3].

The construction of the power operation $\{\mathbb{P}_n\}_{n\geq 0}$, which mixes power operation in K-theory with natural operation of dilating and rotating loops, can be generalized to other equivariant cohomology theories.

In addition, an elliptic power operation for orbifold quasi-elliptic cohomology exists. Ganter spelled out the axioms for orbifold theories with power operations in [18], and constructed the power operation for orbifold Tate K-theory in the same paper, which is closely related to the level structure and isogenies on Tate curve. We extend the power operation $\{\mathbb{P}_n\}_{n\geq 0}$ to the orbifold quasi-elliptic cohomology. The resulting power operation satisfies the axioms that Ganter concluded.

3. Classification of the finite subgroups of the Tate curve

As indicated in [4], power operation connects elliptic cohomology and elliptic curve. Via the power operation $\{\mathbb{P}_n\}_{n\geq 0}$ in Section 2, we explore the geometric nature of the Tate curve. In this process, quasi-elliptic cohomology provides a method that reduces facts such as the classification of geometric structures on the Tate curve into questions in representation theory.

It is already illuminating to study the power operation $\{\mathbb{P}_n\}_{n\geqslant 0}$ when X is a point pt with trivial group action. A big ingredient then is understanding $QEll(\mathrm{pt}/\!\!/\Sigma_N)$. Applying that we prove in [24] that the Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve, which is analogous to the principal result in Strickland [44] that the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of the formal group $\mathbb{G}_E$.

Our main conclusion is Theorem 3.1.

THEOREM 3.1. *(Theorem 6.4, [24]) The Tate K-theory of symmetric groups modulo a certain transfer ideal, $K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate}$, classifies finite subgroups of the Tate curve.*

Equivariant K-theory $K_G(\mathrm{pt})$ of a point is the representation ring of G. As shown in (1), quasi-elliptic cohomology can be expressed by equivariant K-theories. Thus, we can compute quasi-elliptic cohomology of a point by computing the representation ring of each $\Lambda_G(\sigma)$ whereas we cannot compute Tate K-theory directly with representation theory. To prove Theorem 3.1, we first compute the quasi-elliptic cohomology of symmetric groups modulo the transfer ideal, i.e. $QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{QEll}$. Then, applying the relation (2) between Tate K-theory and quasi-elliptic cohomology, we get the formula for $K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{Tate}$. Comparing it with the ring that classifies the finite subgroups of the Tate curve, we prove Theorem 3.1. The role of quasi-elliptic cohomology in the proof is crucial.

Applying the same idea, we proved that the Tate K-theory of any finite abelian group A modulo a certain transfer ideal classifies the $A-$Level structures of the Tate curve. The result will appear in a coming paper [30].

Moreover, we can define an operation $\{\overline{P}_N\}_{N\geq 0}$ of quasi-elliptic cohomology via the power operation $\{\mathbb{P}_N\}_{N\geq 0}$. It is a ring homomorphism and is analogous to the Adams operation of equivariant K-theories. It uniquely extends to an additive operation of the Tate K-theory

$$\overline{P^{string}}_n : K_{Tate}(X/\!\!/G) \longrightarrow K_{Tate}(X/\!\!/G) \otimes_{\mathbb{Z}((q))} K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I^{Tate}_{tr}$$

constructed in [17].

Applying the Strickland's theorem in [44], Ando, Hopkins and Strickland show in [4] that the additive power operation of Morava E-theories

$$E^0 \longrightarrow E(B\Sigma_{p^k})/I_{tr}$$

has a nice algebra-geometric interpretation in terms of the formal group and it takes the quotient by the universal subgroup. The additive operation $\{\overline{P}_N\}_{N\geq 0}$ also contains subtle geometric information. Via it, we construct the universal finite subgroup of the Tate curve in [28].

4. The orthogonal $G-$spectrum

One advantage of quasi-elliptic cohomology is that it is built using equivariant topological K-theory, each aspect of which has been studied thoroughly. Some constructions on quasi-elliptic cohomology can be made simpler than most elliptic cohomology theories, including the Tate K-theory.

Equivariant homotopy theory is homotopy theory of topological $G-$spaces. Mandell, May, Schewede and Shipley built several good model categories of equivariant spectra [36]. Equivariant orthogonal spectra, as shown in [35], is one of them. An orthogonal $G-$spectrum is defined from a $\mathcal{I}_G-$functor with $\mathcal{I}_G$ the category of orthogonal $G-$representations.

QUESTION 4.1. *Is there an orthogonal $G-$spectrum representing $QEll^*_G$?*

Applying equivariant homotopy theory, we construct an orthogonal $G-$spectrum representing quasi-elliptic cohomology in a new category GwS of orthogonal $G-$spectra.

DEFINITION 4.2. *The category GwS is the homotopy category of the category of orthogonal $G-$spectra with the weak equivalence defined by*

$$X \sim Y \text{ if } \pi_0(X(V)) = \pi_0(Y(V)), \tag{3}$$

for each faithful $G-$representation V.

An orthogonal $G-$spectrum X in GwS is said to represent a theory H^*_G if we have a natural map

$$\pi_0(X(V)) = H^V_G(S^0), \tag{4}$$

for each faithful $G-$representation V.

THEOREM 4.3. *([26]) For each compact Lie group G, there exists a commutative I_G-FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ representing $QEll^*_G$ in GwS.*

The construction of $(QE(G,-), \eta^{QE}, \mu^{QE})$ is explicit and can be applied to a family of theories, such as generalized Morava E-theories and equivariant Tate K-theory. Moreover, we show in [26] that for any equivariant cohomology theory E with the same key features as equivariant K-theory, the theory

$$QE^*_G(X) := \prod_{\sigma\in G^{tors}_{conj}} E^*_{\Lambda(\sigma)}(X^\sigma) = \Big(\prod_{\sigma\in G^{tors}} E^*_{\Lambda(\sigma)}(X^\sigma)\Big)^G \tag{5}$$

can be represented by an orthogonal $G-$spectrum in GwS. More explicitly, the theory E is supposed to have the features below.

• The theories $\{E^*_G\}_G$ have the change-of-group isomorphism, i.e. for any closed subgroup H of G and $H-$space X, the change-of-group map $\rho^G_H : E^*_G(G\times_H X) \longrightarrow E^*_H(X)$ defined by $E^*_G(G\times_H X) \xrightarrow{\phi^*} E^*_H(G\times_H X) \xrightarrow{i^*} E^*_H(X)$ is an isomorphism where ϕ^* is the restriction map and $i : X \longrightarrow G\times_H X$ is the $H-$equivariant map defined by $i(x) = [e, x]$.
• There exists an orthogonal spectrum E such that for any compact Lie group G and "large" real $G-$representation V and a compact $G-$space B we have a bijection $E_G(B) \longrightarrow [B_+, E(V)]^G$.
• Let G be a compact Lie group and V an orthogonal $G-$representation. For every ample $G-$representation W, the adjoint structure map $\widetilde{\sigma}^E_{V,W} : E(V) \longrightarrow \mathrm{Map}(S^W, E(V\oplus W))$ is a $G-$weak equivalence.

5. A global theory for elliptic cohomology theories

At the early beginning of equivariant homotopy theory people noticed that certain theories naturally exist not only for one particular group but for all groups in a specific class. This observation motivated the birth of global homotopy theory. In [41] the concept of orthogonal spectra is introduced, which is defined from $\mathbb{L}-$functors with $\mathbb{L}$ the category of inner product real spaces. Each global spectrum consists of compatible $G-$spectra with G across the entire category of groups and they reflect any symmetry. Globalness is a measure of the naturalness of a cohomology theory.

In Remark 4.1.6 [41], Schwede discussed the relation between orthogonal $G-$spectra and global spectra. We have the question associated to the underlying orthogonal $G-$spectrum of the I_G-FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ in Theorem 4.3.

QUESTION 5.1. *Can $\{(QE(G,-), \eta^{QE}, \mu^{QE})\}_G$ arise from an orthogonal spectrum?*

Ganter showed that $\{QEll^*_G\}_G$ have the change-of-group isomorphism, which is a good sign that quasi-elliptic cohomology may be globalized. By the discussion in Remark 4.1.6 [41], however, the answer to QUESTION 5.1 is no. Then it is even more difficult to see whether each elliptic cohomology theory, whose form is more intricate and mysterious than quasi-elliptic cohomology, can be globalized in the current setting.

Our solution is to establish a more flexible global homotopy theory where quasi-elliptic chomomology can fit into. We hope that it is easier to judge whether a cohomology theory, especially an elliptic cohomology theory, can be globalized in the new theory. In addition we want to show that the new global homotopy theory is equivalent to the current global homotopy theory.

We construct in [27] a category D_0 to replace $\mathbb{L}$ whose objects are (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations

$$\rho : G \longrightarrow O(V),$$

and whose morphism $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$ consists of a linear isometric embedding $\phi_2 : V \longrightarrow W$ and a group homomorphism $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$, which makes the diagram (6) commute.

(6)
$$\begin{tikzcd} G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(O(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & O(W) \end{tikzcd}$$

In other words, the group action of H on $\phi_2(V)$ is induced from that of G. Intuitively, the category D_0 is obtained by adding the restriction maps between representations into the category $\mathbb{L}$.

Instead of the category of orthogonal spaces, we study the category of D_0-spaces. The category of orthogonal spaces is a full subcategory of the category D_0T of D_0-spaces. Apply the idea of diagram spectra in [36], we can also define D_0-spectra and D_0-FSP.

Moreover, we notice that if the equivariant homotopy theories $\{E^*_G(-)\}_G$ is represented by a D_0-spectrum X, it has the property that $X(G,V) \simeq_H X(H,V)$ for any closed subgroup H of G. So what we really need to study are D_0^W-spectra.

DEFINITION 5.2 (The category $Sp_W^{D_0}$). *A D_0^W-spectrum X is both a D_0-spectrum and a D_0-space that maps each restriction map $(G,V) \longrightarrow (H,V)$ to an $H-$weak equivalence. The category $Sp_W^{D_0}$ is the category of D_0^W-spectra.*

Combining the orthogonal $G-$spectra of quasi-elliptic cohomology together, we get a well-defined unitary D_0^W-spectrum and unitary D_0^W-FSP. Thus, we can define global quasi-elliptic cohomology in the category of D_0-spectra.

THEOREM 5.3. *(Theorem 7.2.3* [27], [29]*)There is a unitary D_0^W-FSP weakly representing quasi-elliptic cohomology.*

Equipping a homotopy theory with a model structure is like interpreting the world via philosophy. Model category theory is an essential basis and tool to judge whether two homotopy theories describe the same world. We build several model structures on D_0T. First by the theory in [36], there is a level model structure on D_0T.

THEOREM 5.4. *(Theorem 6.3.4* [27], [29]*) $Sp_W^{D_0}$ is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.*

D_0 is a generalized Reedy category in the sense of [12]. We can construct a Reedy model structure on $Sp_W^{D_0}$.

THEOREM 5.5. *(Theorem 6.4.5* [27], [29]*)The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category $Sp_W^{D_0}$.*

In addition, this new global theory describes the same world of homotopy theories as that by Schwede.

THEOREM 5.6. *(*[29]*)There is a global model structure on $Sp_W^{D_0}$ Quillen equivalent to the $Fin-$global model structure on the orthogonal spectra constructed by Schwede in* [41].

6. Research Plan

During the coming two years, I will work on these families of research problems.

- **Generalize the conclusions/constructions on quasi-elliptic cohomology to elliptic cohomology theories.** I will work with Professor Thomas Schick and Professor Chenchang Zhu at Georg-August-Universität Gottingen on this project. Their fields are related to orbifolds and gouproids.
  - One problem we work on is related to groupoids and loop spaces. In Section 1 we used a loop space constructed from bibundles in the definition of quasi-elliptic cohomology. We want to know whether the relation between any elliptic cohomology theory and loop space can be interpreted by bibundles as well.
  - Via quasi-elliptic cohomology theory, we get the classification theorem of finite subgroups and that of level structures on Tate curve. The form of the conclusion is the same as those on Morava E-theories, and hopefully on generalized Morava E-theories. I am cooperating with Nathaniel Stapleton on this. We also want to show that the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve.
- **Explore the relation between quasi-elliptic cohomology and physics.** I will work on this project with Professor Thomas Schick and Professor Chenchang Zhu at Georg-August-Universität Gottingen. They are also interested in this project.

  - First I plan to express quasi-elliptic cohomology by group schemes since algebraic geometry is related to physics more closely.
  - Then I will construct some physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology. I am curious what distinct properties and advantages they may have.
- **Another representing object of quasi-elliptic cohomology** Other then equivariant spectra, another type of representing object of interest is two-vector bundles [9][10], which is constructed geometrically for elliptic cohomology. I will construct a geometric representing object via two-vector bundles.
- **Construct a global homotopy theory where global elliptic cohomology theories reside.** Quasi-elliptic cohomology serves as a breakthrough point of the construction.
  - Other than Schwede's global homotopy theory [41] which is built via the language of model categories, Gepner constructed a global homotopy theory via orbispaces in [19]. In addition, based on Gepner's theory, Rezk presented in his unpublished manuscript [38] a global homotopy theory with ideas from infinite category theory. The relation between the global theory by Schwede and that by Gepner and Rezk is intriguing and worth exploration.
  - I will study whether global elliptic cohomology theory resides in Gepner and Rezk' global homotopy theory and which theory is more suitable for constructing global elliptic cohomology theory.
  - In addition, I am studying the properties and features of the global homotopy theory in Section 5.
- **Quasi-elliptic cohomology's HKR character theory.**

  Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel (HKR) character theory [23]. Ganter expected the HKR theory for orbifold Tate K-theory to be established via Stapleton's framework of tanschromatic character map. Stapleton constructed in his paper [42] and [43], for each finite $G-$CW complex X and each positive integer n, a topological groupoid $Twist_n(X)$, whose construction is analogous to that of the groupoid $\Lambda(X/\!\!/G)$. With the topological groupoid $Fix_n(X)$ in [23], he constructed in [42] extensions of the generalized character map of Hopkins, Kuhn, and Ravenel [23] for Morava E-theory to every height between 0 and n. And with $Twist_n(X)$, in [43] he constructed the twisted character map which can canonically recover the transchromatic generalized character map in [42]. Apply this transchromatic character theory, he and Schlank provided a new proof of Strickland's theorem in [44] that the Morava E-theory of the symmetric group has an algebro-geometric interpretation after taking the quotient by a certain transfer ideal. Moreover, he showed in [11] this extended character theory can be used to compute the total power operation for the Morava E-theory of any finite group, up to torsion.

  I will construct the Hopkins-Kuhn-Ravenel (HKR) character theory for $QEll^*$ and that for Tate K-theory. It can probably be fit into Stapleton's transchromatic character theory.
- **A Generalization of quasi-elliptic cohomology.**

  We can generalize quasi-elliptic cohomology. Let n be any positive integer. Let G be a compact Lie group and X a $G-CW$ complex. Let $\sigma=(\sigma_1,\sigma_2,\cdots\sigma_n)$ with each $\sigma_i\in G^{tors}_{conj}$ and $[\sigma_i,\sigma_j]=e$. Let G^n_z be the set of of all such $(\sigma_1,\cdots\sigma_n)$s. Let $\Lambda^n(\sigma)$ be the quotient of $C_G(\sigma)\times\mathbb{R}^n$ by the normal subgroup generated by elements $(\sigma_i,-e_i)$ where $e_i=(0,\cdots 0,1,0,\cdots 0)\in\mathbb{R}^n$.

  Apply the idea of constructing $\Lambda(X/\!\!/G)$, we can construct a topological groupoid $\Lambda^n(X/\!\!/G)$. It is analogous to the topological groupoid $Fix_n(X)$ in Hopkins, Kuhn and Ravenel [23].

  Let E^* be any equivariant cohomology theory. We can define

$$\mathbf{E}^*(X):=\prod_{\sigma\in G^n_z}E_{\Lambda^n(\sigma)}(X^\sigma).$$

  Those results for quasi-elliptic cohomology on power operation and spectra can be constructed analogous to $\mathbf{E}^*$. I am also working on this.

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Zhen Huan, Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275 China

E-mail address: `huanzhen2016@gmail.com`