

# Almost Global Homotopy Theory

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- Preliminary: classical homotopy theory; stable homotopy theory.
- Model category;
- Equivariant homotopy theory;
- Global homotopy theory;
- Almost global homotopy theory;
- Examples: Quasi-theories.

# Preliminary: classical homotopy theory

*Top* The category of topological spaces and continuous maps.

*Top*<sub>\*</sub> The category of based spaces and based maps.

a **homotopy**  $h$  between continuous maps  $f$  and  $g: X \rightarrow Y$

a map  $h: X \times [0, 1] \rightarrow Y$  such that  $f(x) = h(x, 0)$  and  $g(x) = h(x, 1)$ .

$f$  and  $g$  are **homotopic**  $f \stackrel{h}{\simeq} g$  if there exists a homotopy between them.

Being homotopic is an **equivalence relation**.

$f: X \rightarrow Y$  is a **homotopy equivalence**

there exists a map  $g: Y \rightarrow X$  and  $g \circ f \simeq id_X$  and  $f \circ g \simeq id_Y$ .

$X \simeq Y$ .  $X$  and  $Y$  are homotopy equivalent.

**Ho(Top)**: the associated homotopy category

- objects: topological spaces.
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**Homotopy groups:**  $\pi_n(X, x) := [S^n, X]_*$ .

Example

$$\pi_0(S^0) = S^0. \quad \pi_1(S^1) \cong \mathbb{Z}. \quad \pi_n(S^n) \cong \mathbb{Z}.$$

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homotopy equivalence  $\Rightarrow$  weak homotopy equivalence.

**CW-complex:** nice spaces

$X^0$ : a discrete set.

$X^{n+1}$ : attach  $(n+1)$ -cells  $D^{n+1}$  to  $X^n$  along attaching maps  $S^n \longrightarrow X^n$ .

- Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.
- Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

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## Homotopy Lifting Property

$p : E \longrightarrow B$  is called a **fibration** if  $p : E \longrightarrow B$  satisfies the **Homotopy Lifting Property**, i.e. given any map  $f : X \longrightarrow E$  and homotopy  $h : X \times [0, 1] \longrightarrow B$  with  $h_0 = p \circ f$ , there exists an extension  $\bar{h} : X \times [0, 1] \longrightarrow E$  making the diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \downarrow i_0 & \nearrow \bar{h} & \downarrow p \\ X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

## Example

- constant map;
- covering space;
- fiber bundle.

## Serre fibration

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The inclusion of a relative CW-complex is a cofibration.

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# Stable Homotopy Theory

## Spectrum

a sequence  $\{X_n, \sigma_n\}_n$  where  $X_n$  are based spaces and  $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$  are based maps.

## Stable homotopy groups

$$\pi_n^S(X) = \operatorname{colim}_{l \rightarrow \infty} \pi_{n+l}(S^l \wedge X).$$

## Spectrum defines homology and cohomology

$$H_n(Y) = \operatorname{colim}_{l \rightarrow \infty} \pi_{n+l}(Y_+ \wedge X_l);$$

$$H^n(Y) = \operatorname{colim}_{l \rightarrow \infty} [S^l \wedge Y_+, X_{n+l}].$$

## Brown Representation Theorem

- (i) For each homology theory, there exists a spectrum representing it.
- (ii) For each cohomology theory, there exists a spectrum representing it.



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## Stable homotopy groups

$$\pi_n^S(X) = \operatorname{colim}_{l \rightarrow \infty} \pi_{n+l}(S^l \wedge X).$$

## Spectrum defines homology and cohomology

$$H_n(Y) = \operatorname{colim}_{l \rightarrow \infty} \pi_{n+l}(Y_+ \wedge X_l);$$

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- (i) For each homology theory, there exists a spectrum representing it.
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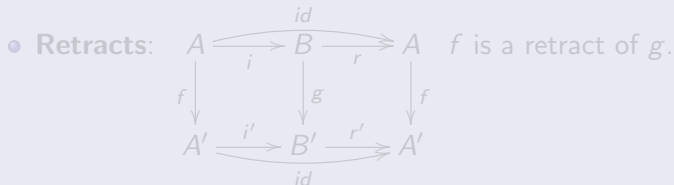
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## Motivating example: the category of topological spaces

- weak homotopy equivalence
- Serre fibration
- retract of relative cell complex

## Features

- **2-out-of-3:** If two of  $f$ ,  $g$ ,  $gf$  are weak homotopy equivalences, then so is the third.



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$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{r} & A \\ f \downarrow & & \downarrow g & & \downarrow f \\ A' & \xrightarrow{i'} & B' & \xrightarrow{r'} & A' \end{array}$$

$\text{---} id \text{---}$  (curved arrow from  $A$  to  $A$ )  
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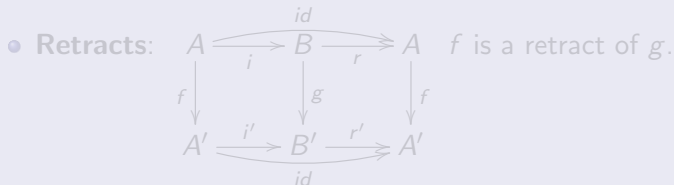
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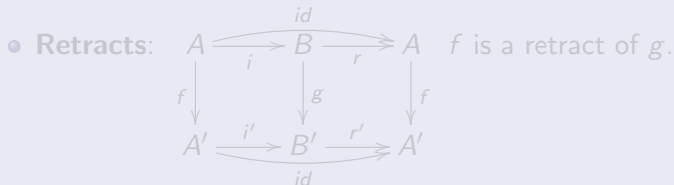
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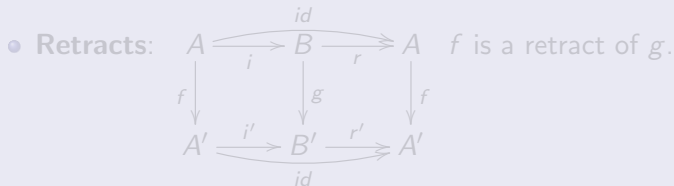
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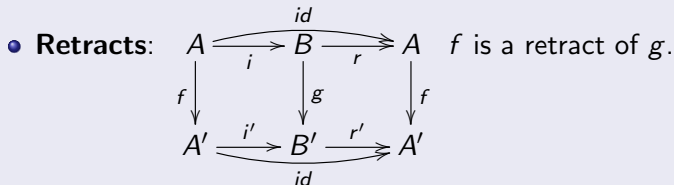
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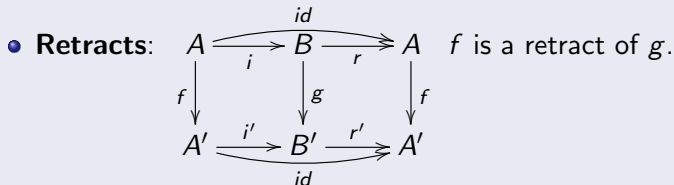
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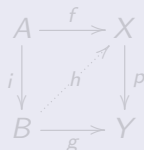
• Weak Equivalence • Fibration • Cofibration.

satisfying the axioms:

• Retracts;

• 2-out-of-3;

• **Lifting:**



The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

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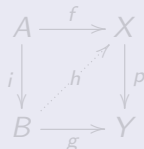
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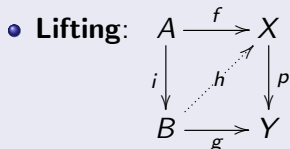
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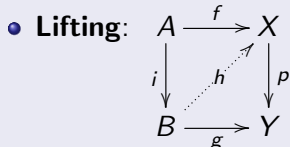
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# Morphisms in the Category of Model Categories

## Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$ : a category.  $\mathcal{W}$ : a subcategory of weak equivalences.

## The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as  $\mathcal{C}$ ;
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## Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction:  $(L \dashv R) : \mathcal{C} \xrightleftharpoons[L]{R} \mathcal{D}$ .

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$X^{n+1}$ : attach  $G$ -cells  $G/H \times D^{n+1}$  to  $X^n$  along attaching  $G$ -maps

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$$GTop \longrightarrow [Orb_G^{op}, Top]$$

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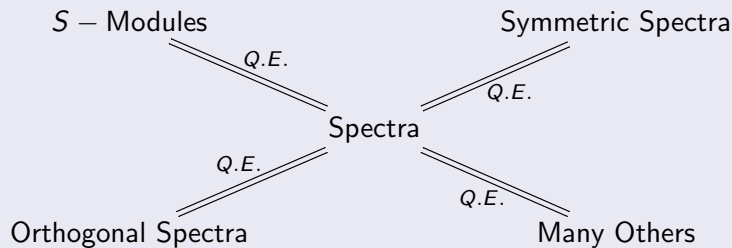
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## Right Philosophy for Stable Homotopy Theory (1990s)

[MMSS]



## Equivariant Stable Homotopy Theory (2000s)

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$S_G$  – Modules

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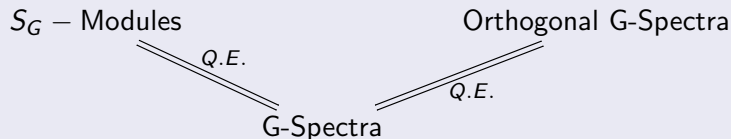
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# Model structures on equivariant stable homotopy theory

## Equivariant Stable Homotopy Theory (2000s)

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Combine the best feature of other models.

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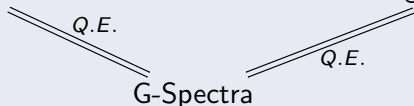
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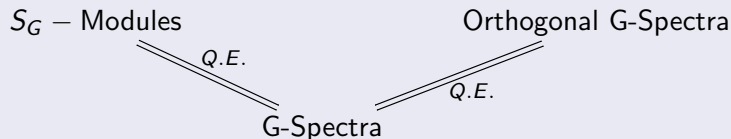
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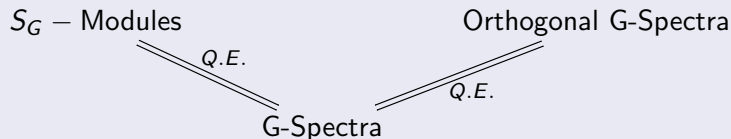
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$Top_G$ : the category of based  $G$ -spaces and continuous based maps.

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A  $G$ -continuous functor  $X : \mathcal{I}_G \rightarrow Top_G$ .

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An  $\mathcal{I}_G$ -space  $X$  with a natural transformation  $X(-) \wedge S^- \rightarrow X(- \oplus -)$  such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

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# The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

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Example (When  $G$  varies)

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- objects: inner product real spaces;
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# Motivating example: quasi-elliptic cohomology

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Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
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We constructed an orthogonal  $G$ -spectrum for  $QEII_G^*(-)$ , which cannot give a global spectrum in Schwede's sense.

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- $QEll_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K_{Tate}^*)_G(X)$ ;
- Change-of-group isomorphism:  $QEll_G^*(G \times_H X) \cong QEll_H^*(X)$ .

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie:** Elliptic cohomology theories can be globalized.
- **Nora Ganter:** Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal  $G$ -spectrum for  $QEll_G^*(-)$ , which cannot give a global spectrum in Schwede's sense.

[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

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Observation: restriction maps don't need to be identity maps.

- $\{E_G^n, \sigma_{G,n}\}_{n,G}$ : equivariant spectra representing  $\{E_G^*(-)\}_G$ .  
 $E_G^n \simeq_H E_H^n$  for  $H \xhookrightarrow{i} G$ .
- For an orthogonal spectrum  $X$ ,  $X(i^*(V)) = i^*X(V)$  for any  $G$ -representation  $V$ .

The new diagram  $D_0$ : add restriction maps to  $\mathbb{L}$

- objects:  $(G, V)$  with  $G \leq O(V)$  finite
- morphisms:  $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$  with  $\phi_2 : V \longrightarrow W$  a linear isometric embedding and  $\phi_1 : H \cap O(V) \longrightarrow G$  a group homomorphism.

$$\begin{array}{ccc}
 G & \longrightarrow & O(V) \\
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The feature of  $D_0$ 

- $D_0$  is a symmetric monoidal category.
- $D_0$  is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define  $D_0$ –space and  $D_0$ –spectrum.

A  $D_0$ –space is a continuous functor from  $D_0$  to the category of based compactly generated weak Hausdorff spaces.

A  $D_0$ –spectrum  $X$  consists of

- a based  $G$ –space  $X(G, V)$ ;
- a  $G \times H$ –equivariant based structure map
 
$$\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$$

A morphism of  $D_0$ –spectra: compatible with the structure maps.

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The category  $D_0 T^W$

the full subcategory of  $D_0 T$  consisting of those objects  $X : D_0 \rightarrow T$  that maps each restriction map  $(G, V) \rightarrow (H, V)$  to an  $H$ —weak equivalence.

$Sp_W^{D_0}$ : the category of  $D_0^W$ —spectra

A  $D_0^W$ —spectrum  $X$  is both a  $D_0$ —spectrum and a  $D_0$ —space in  $D_0 T^W$ .

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \xrightleftharpoons[P]{Q} Sp_W^{D_0}$$

- The Reedy model structure on  $Sp_W^{D_0}$  is Quillen equivalent to the *Fin*—level model structure on orthogonal spectra.
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Divisible group: a better algebraic object associated to an elliptic curve than formal group.

Example: Tate curve and Tate K-theory

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \text{Tate}(q) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$\mathbb{G}_m$ : formal group of Tate K-theory;  $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}_p} \mathbb{Q}_p)[p^k]) = K_{\text{Tate}}^0(B\mathbb{Z}_{p^k})$ .

The general case

$$0 \longrightarrow F \longrightarrow \mathbb{G} \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$F$ : the formal group of a cohomology theory  $\{E_{n,G}^*(-)\}_G$ .

The corresponding cohomology theory  $\{A_{n,G}^*(-)\}_G$ .

The corresponding quasi-theory:  $QE_{n,G}^*(X) = \prod_{\sigma \in G^n} E_{n,\Lambda_G^n(\sigma)}^*(X^\sigma)$ .

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$$0 \longrightarrow \mathbb{G}_m \longrightarrow \text{Tate}(q) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$\mathbb{G}_m$ : formal group of Tate K-theory;  $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}_p} \mathbb{Q}_p)[p^k]) = K_{\text{Tate}}^0(B\mathbb{Z}_{p^k})$ .

The general case

$$0 \longrightarrow F \longrightarrow \mathbb{G} \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$F$ : the formal group of a cohomology theory  $\{E_{n,G}^*(-)\}_G$ .

The corresponding cohomology theory  $\{A_{n,G}^*(-)\}_G$ .

The corresponding quasi-theory:  $QE_{n,G}^*(X) = \prod_{\sigma \in G_{\mathbb{Z}}^n} E_{n,\Lambda_G^n(\sigma)}^*(X^\sigma)$ .

$$A_{n,G}^*(X) \cong QE_{n,G}^*(X) \otimes_{\mathbb{Z}[q^\pm] \otimes^n \mathbb{Z}((q))} \mathbb{Z}((q))^{\otimes n}.$$

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## Theorem

If the theory  $\{E_{n,G}^*(-)\}_G$  can be globalized, there is a  $D_0^W$ -spectrum representing the quasi-theory  $\{QE_{n,G}^*(-)\}_G$ .

In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

[Zhen Huan: *Quasi-elliptic cohomology*, PhD thesis]

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## My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.



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*Thank you.*

<https://huanzhen84.github.io/zhenhuan/Huan-HUST-2018.pdf>

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