Quasi-elliptic cohomology

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Overview

Plan.

- Motivation and Definition
- Loop space construction: Witten's conjecture
- Geometric structure on Tate curve
- Global homotopy theory for elliptic cohomology theories
- Further problems
  - equivariant homotopy theory
  - chromatic homotopy theory
  - motivic homotopy theory

$$\begin{tikzcd} \text{generalized cohomology theories} \arrow[r, bend left, "\text{1st Chern class}"] & \text{formal groups} \arrow[l, bend left] \end{tikzcd}$$

even periodic, multiplicative

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

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- Elliptic curves: elliptic cohomology?

Elliptic cohomology [AHS][Lurie]

R: commutative ring; C/R: elliptic curve over R.
E is an elliptic cohomology theory if $E^0(\mathrm{pt}) \cong R$ and $\mathrm{Spf} E^0(\mathbb{C}P^\infty) \cong \widehat{C}$.

Elliptic cohomology

Moduli stack of elliptic curves

$$\{\text{elliptic curves } E \longrightarrow \mathrm{Spec}R\} = Hom(\mathrm{Spec}R, M_{1,1}).$$

$\{$etále elliptic curves over $R\} \longrightarrow \{$multiplicative cohomology theories$\}$.

Tate K-theory [AHS]

Tate curve: the generalized elliptic curve over $\mathbb{Z}((q))$ classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

The associated formal group: $\mathbb{G}_m$ over $\mathbb{Z}((q))$.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

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Explicit Definition

$$QEll_G^*(X) \cong \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$$

- G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};
- $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$;
- $x \cdot [a, t] = x \cdot a$, for all $[a, t] \in \Lambda_G(g)$, $x \in X^g$.

Relation with Tate K-theory

$$QEll_G^*(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

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Basic Properties of Quasi-elliptic cohomology

Representation theory

- Restriction map: $RG \longrightarrow RH$;

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;

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- $RG \otimes RH \longrightarrow R(G \times H)$.

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- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;
- $K^*_G(-)$ can be represented by an orthogonal $G-$spectrum;
- Global K-theory exists.

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Relevant Work [Devoto][Ganter]

2007, $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

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Question

How can we construct elliptic cohomology theories from loop spaces?

Review: Free Loop Space

$$LX = C^{\infty}(\mathbb{T}, X).$$

- $\mathbb{T}-$action: $\gamma \cdot t = (x \mapsto \gamma(s+t))$.
- $LG-$action: $\gamma \cdot \delta = (s \mapsto \gamma(s) \cdot \delta(s))$.
- $LG \rtimes \mathbb{T}-$action: $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s+t) \cdot \delta(s+t))$.
$$(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s+t_1), t_1 + t_2).$$

Interpretation of the $LG \rtimes \mathbb{T}-$action

LG: the gauge group of the trivial $G-$bundle over $\mathbb{T}$.

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$LG \rtimes \mathbb{T}$: the extended gauge group

$$\begin{tikzcd} G \times \mathbb{T} \arrow[r, "{(g,s)\mapsto(\gamma(s)g,s+t)}"] \arrow[d] & G \times \mathbb{T} \arrow[d] \\ \mathbb{T} \arrow[r, "s\mapsto s+t"] & \mathbb{T} \end{tikzcd}$$

What is "Loop"? [Rezk]

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$LG \rtimes \mathbb{T}$: act on loops

$$\begin{tikzcd} G \times \mathbb{T} \arrow[r] \arrow[d] & G \times \mathbb{T} \arrow[r, "\widetilde{\gamma}"] \arrow[d] & X \\ \mathbb{T} \arrow[r] & \mathbb{T} & \end{tikzcd}$$

The Answer: What is "Loop"?

New Definition of Equivariant loops $Loop(X/\!\!/G)$ [Rezk]

- Objects:

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

  - π : principal $G-$bundle over $\mathbb{T}$
  - f : $G-$equivariant;

- Morphism $(\alpha, t) : \{ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X \} \longrightarrow \{ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X \}$:

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Relation with Bibundles

$Bibun(\mathbb{T}/\!\!/*, X/\!\!/G)$

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$\Lambda(X /\!\!/ G)$: a subgroupoid of $Loop(X /\!\!/ G)$ consisting of constant loops.

$$\Lambda(X /\!\!/ G) \cong \coprod_{g \in G^{tors}_{conj}} X^g /\!\!/ \Lambda_G(g)$$

$$QEll^*_G(X) = K^*_{orb}(\Lambda(X /\!\!/ G))$$

Further Question

Can we use this new definition of loop spaces to construct elliptic cohomology theories?

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Motivating examples: Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: $\mathrm{Spec}(E^0(B\Sigma_k)/I_{tr}) \cong Sub_{p^k}(\mathbb{G}_E)$.
  The Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton:

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Via transchromatic character theory.

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Are the geometric structures on elliptic curves classified in the same way?

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Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.

Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G \wr \Sigma_n}(X^{\times n})$$

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Quasi-elliptic cohomology has power operations

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!/G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

The Stringy Power Operation [Huan]

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)}:$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

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Tate Curve and Representation Theory

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q_s' is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

The proof: (i) Apply representation theory, prove

$$QEll^0_{\Sigma_N}(\mathrm{pt})/I_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle.$$

(ii) Apply the relation between $QEll^*$ and K^*_{Tate}.

Theorem (Huan)

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- infinity categories;
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- add restriction maps to the diagram;
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Almost global homotopy theory [Huan]

- Key point of the construction: replace the restriction maps, which are identity maps in global homotopy theory, by equivariant weak equivalences.
- The category of almost global spectra is NOT equivalent to the category of global spectra.
- But they are Quillen equivalent, i.e. they are describing the same mathematical world.
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Quasi-theories [Huan]

$$QE^*_{n,G}(X) := E^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma).$$

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My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the *é*tale component varies, the globalness does not change.

More explicitly, if E^* can be globalized and A^* is a cohomology theory with divisible group $\mathbb{G}_E \oplus (\mathbb{Q}/\mathbb{Z})^n$, then A^* can also be globalized.

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Future Problems in chromatic homotopy theory

Chromatic level:=height of the formal group.

Chromatic level	complex oriented cohomology theory
0	ordinary cohomology
	0th Morava K-theory $K(0)$
1	complex K-theory KU
	first Morava K-theory $K(1)$
	first Morava E-theory $E(1)$
2	elliptic cohomology
	second Morava K-theory $K(2)$
	second Morava E-theory $E(2)$
n	nth Morava K-theory $K(n)$
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∞	complex bordism cohomology MU.

Table: Chromatic homotopy theory

Future Problems in chromatic homotopy theory

Interesting point

The chromatic level of elliptic cohomology theories is 2;
the chromatic level of quasi-elliptic cohomology theory is 1.

Question: How can we relate the two closed theories?

Transchromatic character theory

- The character map: $L \otimes R(G) \xrightarrow{\cong} Cl(G; L)$.
- Hopkins-Kuhn-Ravenel's generalized character map:

$$\chi^G_{n,p} : L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\cong} Cl_{n,p}(G, X; L(E^*)).$$

- Stapleton extends the generalized character map for Morava E-theories:

$$C_t \otimes_{E^0_n} \Phi_G : C_t \otimes_{E^0_n} E^*_n(EG \times_G X) \xrightarrow{\cong} C^*_t(EG \times_G Fix_{n-t}(X)).$$

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Gronowski's elliptic cohomology theories and quasi-elliptic cohomology theory can be related by Stapleton's generalized character map.

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Generalized Morava $E-$ theories have Strickland's theorem for $A-$level structures. Explicitly,

$$\mathrm{Spec}(E_n^0(L^h(BA))/I_{tr}^A) = Level_A(\mathbb{G}_{E_n} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h)$$

- A: any finite abelian group;
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Goerss-Hopkins-Miller-Lurie Theorem

$$\begin{tikzcd} & \{E_\infty - rings\} \arrow[d] \\ \{\text{etále elliptic curves over } R\} \arrow[r] \arrow[ur, dotted] & \{\text{multiplicative cohomology theories}\} \end{tikzcd}$$

Question [Huan]

Do we have motivic Goerss-Hopkins-Miller-Lurie Theorem?

- 2009 Naumann, Spitzweck, Østvær: prove motivic Landweber exact funtor theorem.
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http://gagp.sysu.edu.cn/zhenhuan/Zhen-HUST-2018-Slides.pdf

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