# Quasi-elliptic cohomology

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#### Overview

#### Plan.

- Motivation and Definition
- Loop space construction: Witten's conjecture
- Geometric structure on Tate curve
- Global homotopy theory for elliptic cohomology theories
- Further problems
  - equivariant homotopy theory
  - chromatic homotopy theory
  - motivic homotopy theory

even periodic, multiplicative

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

generalized cohomology theories \_\_\_\_\_\_ formal groups

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#### commutative 1-dimensional formal groups

• The additive formal group  $\mathbb{G}_a$ : periodic Eilenberg-MacLane spectrum.

 $\underbrace{ \begin{array}{c} \text{1st Chern class} \\ \text{generalized cohomology theories} \end{array}}_{\text{1st Chern class}} \text{formal groups}$ 

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#### Elliptic cohomology

[AHS][Lurie]

R: commutative ring; C/R: elliptic curve over R.

E is an elliptic cohomology theory if  $E^0(\mathsf{pt}) \cong R$  and  $\mathsf{Spf}E^0(\mathbb{C}P^\infty) \cong \widehat{C}$ .

#### Moduli stack of elliptic curves

$$\{\text{elliptic curves } E \longrightarrow \text{Spec} R\} = Hom(\text{Spec} R, M_{1,1}).$$

 $\{$ etále elliptic curves over  $R\} \longrightarrow \{$ multiplicative cohomology theories $\}$ .

#### Tate K-theory

[AHS]

**Tate curve**: the generalized elliptic curve over  $\mathbb{Z}((q))$  classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

The associated formal group:  $\mathbb{G}_m$  over  $\mathbb{Z}((q))$ .

**Tate K-theory**: generalized elliptic cohomology associated to the Tate curve.

- relation with K-theory.
- relation with string theory;
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#### **Explicit Definition**

$$QEll_G^*(X) \cong \prod_{g \in G_{coni}^{tors}} K_{\Lambda_G(g)}^*(X^g)$$

- $G_{conj}^{tors}$ : a set of representatives of G-conjugacy classes in  $G^{tors}$ ;
- $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$ ;
- $x \cdot [a, t] = x \cdot a$ , for all  $[a, t] \in \Lambda_G(g)$ ,  $x \in X^g$ .

#### Relation with Tate K-theory

$$QEll_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K_{Tate}^*(X /\!\!/ G)$$

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#### Representation theory

• Restriction map:  $RG \longrightarrow RH$ ;

### Equivariant K-theory

• Restriction map:  $K_G(X) \longrightarrow K_H(X)$ ;

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### Representation theory

- Restriction map:  $RG \longrightarrow RH$ ;
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- $RG \otimes RH \longrightarrow R(G \times H)$ .

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- Induced map:  $K_H(X) \longrightarrow K_G(X)$ ;
- Künneth map:  $K_G^*(X) \otimes K_H^*(Y) \longrightarrow K_{G \times H}^*(X \times Y)$ ;

- Restriction map:  $QEII_G(X) \longrightarrow QEII_H(X)$ ;
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- Künneth map:  $QEll_G^*(X) \widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} QEll_H^*(Y) \longrightarrow QEll_{G\times H}^*(X\times Y)$ .

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- Change-of-group isomorphism:  $K_G(Y \times_H G) \stackrel{\cong}{\longrightarrow} K_H(Y)$ ;
- $K_G^*(-)$  can be represented by an orthogonal G-spectrum;
- Global K-theory exists.

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[Landweber]

$$LX = \mathbb{C}^{\infty}(\mathbb{T},X),$$

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#### Relevant Work

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#### Question

How can we construct elliptic cohomology theories from loop spaces?

What is "Loop"?

[Rezk]

#### Review: Free Loop Space

$$LX = C^{\infty}(\mathbb{T}, X).$$

- $\mathbb{T}$ -action:  $\gamma \cdot t = (x \mapsto \gamma(s+t))$ .
- LG-action:  $\gamma \cdot \delta = (s \mapsto \gamma(s) \cdot \delta(s))$ .
- $LG \times \mathbb{T}$ -action:  $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s+t) \cdot \delta(s+t)).$  $(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s+t_1), t_1 + t_2).$

#### Interpretation of the $LG \times \mathbb{T}$ -action

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What is "Loop"? [Rezk]

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#### Interpretation of the $LG \rtimes \mathbb{T}$ —action

*LG*: the gauge group of the trivial G-bundle over  $\mathbb{T}$ .

 $LG imes \mathbb{T}$ : the extended gauge group  $G imes \mathbb{T} \xrightarrow{(g,s) \mapsto (\gamma(s)g,s+t)} G imes \mathbb{T} \xrightarrow{g \mapsto s+t} \mathbb{T}$ 

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$$LG \rtimes \mathbb{T}$$
: act on loops  $G \times \mathbb{T} \longrightarrow G \times \mathbb{T} \xrightarrow{\widetilde{\gamma}} X$ 

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\mathbb{T} \longrightarrow \mathbb{T}$$

## The Answer: What is "Loop"?

## New Definition of Equivariant loops $Loop(X /\!\!/ G)$

[Rezk]

Objects:

$$\mathbb{T} \stackrel{\pi}{\longleftarrow} P \stackrel{f}{\longrightarrow} X$$

•  $\pi$  : principal G-bundle over  $\mathbb T$ 

- f : G—equivariant;
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#### Relation with Bibundles

 $Bibun(\mathbb{T}/\!\!/*, X/\!\!/G)$ 

- same objects;
- morphisms:  $(\alpha, Id)$ . No rotations.

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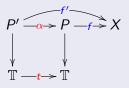
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 $\Lambda(X/\!\!/ G)$ : a subgroupoid of  $Loop(X/\!\!/ G)$  consisting of constant loops.

$$\Lambda(X/\!\!/G) \cong \coprod_{g \in G_{conj}^{tors}} X^g /\!\!/ \Lambda_G(g)$$

$$OFU^*(X) = K^* (\Lambda(X/\!\!/G))$$

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#### **Further Question**

### Motivating examples: Morava E-theories

- 1995, Matthew Ando: a classification of the level $-p^k$  structure of its formal group.
- 1998, Neil Strickland:  $\operatorname{Spec}(E^0(B\Sigma_{p^k})/I_{tr}) \cong \operatorname{Sub}_{p^k}(\mathbb{G}_E)$ . The Morava E—theory of the symmetric group  $\Sigma_n$  modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton:

$$\operatorname{Spec}(E^0(L^h(B\Sigma_{p^k}))/I_{tr}) \cong \operatorname{Sub}_{p^k}(\mathbb{G}_E \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h).$$

Via transchromatic character theory.

Power Operation the homotopy theory 

Power Operation its formal group.

#### Question

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### Power Operation of K-theory

[Atiyah]

$$P_n: K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory

Atiyah]

$$P_n: K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes}$$

Wreath product  $G \wr \Sigma_n$ 

$$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma \tau).$$
  
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# Quasi-elliptic cohomology has power operations

### Atiyah's Power Operation

[Ganter]

V: a vector bundle over  $\Lambda(X /\!\!/ G)$ .

 $P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^\pm]} n}$  defines an operation

$$P_n: QEII_G(X) \longrightarrow QEII_{G\wr \Sigma_n}(X^{\times n})$$

#### The Stringy Power Operation

Huan]

$$\begin{split} \mathbb{P}_n &= \prod_{(\underline{g},\sigma) \in (G \wr \Sigma_n)_{conj}^{tors}} \mathbb{P}_{(\underline{g},\sigma)} : \\ QEII_G(X) &\longrightarrow QEII_{G \wr \Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma) \in (G \wr \Sigma_n)_{conj}^{tors}} \mathsf{K}_{\mathsf{A}_{G \wr \Sigma_n}(\underline{g},\sigma)} ((X^{\times n})^{(\underline{g},\sigma)}) \end{split}$$

$$\mathbb{P}_{(\underline{g},\sigma)}: QEll_{G}(X) \xrightarrow{U^{*}} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{()_{k}^{\wedge}} K_{orb}(\Lambda_{(\underline{g},\sigma)}^{var}(X))$$

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### Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$\textit{K}_{\textit{Tate}}(\textit{pt}/\!\!/ \Sigma_{\textit{N}})/\textit{I}_{\textit{tr}}^{\textit{Tate}} \cong \prod_{\textit{N}=\textit{de}} \mathbb{Z}((\textit{q}))[\textit{\textbf{q}}_{\textit{s}}'^{\pm}]/\langle \textit{q}^{\textit{d}} - \textit{\textbf{q}}_{\textit{s}}'^{\textit{e}} \rangle,$$

where  $I_{tr}^{Tate}$  is the transfer ideal and  $q_s'$  is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that N = de.

The proof: (i) Apply representation theory, prove 
$$QEII^0_{\Sigma_N}(\mathrm{pt})/I^{QEII}_{tr}\cong\prod_{s}\mathbb{Z}[q^\pm][q'_s^\pm]/\langle q^d-q'_s{}^e\rangle.$$

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### My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

More explicitly, if  $E^*$  can be globalized and  $A^*$  is a cohomology theory with divisible group  $\mathbb{G}_E \oplus (\mathbb{Q}/\mathbb{Z})^n$ , then  $A^*$  can also be globalized.

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Chromatic level:=height of the formal group.

Chromatic level	complex oriented cohomology theory
0	ordinary cohomology
	0th Morava K-theory $K(0)$
1	complex K-theory <i>KU</i>
	first Morava K-theory $K(1)$
	first Morava E-theory $E(1)$
2	elliptic cohomology
	second Morava K-theory $K(2)$
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n	nth Morava K-theory $K(n)$
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$\infty$	complex bordism cohomology MU.

Table: Chromatic homotopy theory

## Interesting point

The chromatic level of elliptic cohomology theories is 2; the chromatic level of quasi-elliptic cohomology theory is 1

Question: How can we relate the two closed theories?

### Transchromatic character theory

- The character map:  $L \otimes R(G) \stackrel{=}{\longrightarrow} Cl(G; L)$ .
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$$\operatorname{Spec}(E_n^0(L^h(BA))/I_{tr}^A) = \operatorname{Level}_A(\mathbb{G}_{E_n} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h)$$

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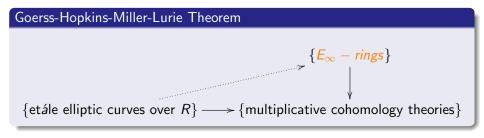
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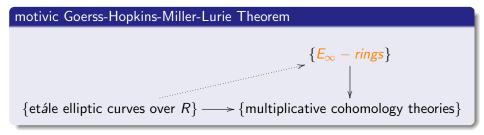
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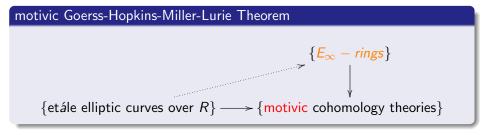
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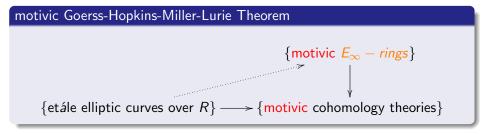
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### Some references

#### http://gagp.sysu.edu.cn/zhenhuan/Zhen-HUST-2018-Slides.pdf

- Ando, "Isogenies of formal group laws and power operations in the cohomology theories  $E_n$ ", Duke J., 1995
- Ando, Hopkins, Strickland: "Elliptic spectra, the Witten genus and the theorem of the cube". Invent. Math. 146(3):595-687, 2001.
- Ando, Hopkins, Strickland, "The sigma orientation is an H<sub>∞</sub> map", Amer. J. 2004;
- Ativah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Baas, Dundas, Rognes: "Two-vector bundles and forms of elliptic cohomology", Topology, geometry and quantum field theory, 18-45, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Devoto, "Equivariant elliptic homology and finite groups", Michigan Math. J., 43(1):3-32, 1996.
- Ganter, "Orbifold genera, product formulas, and power operations", Adv. Math. 2006;
- Ganter, "Stringy power operations in Tate K-theory", Homology, Homotopy, Appl., 2013;
- Gepner, "Homotopy Topoi and Equivariant Elliptic Cohomology", Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 1999.
- Ginzburg, Kapranov, Vasserot, "Elliptic algebras and equivariant elliptic cohomology I", available at arXiv:q-alg/9505012.
- Hopkins, Kuhn, Ravenel, "Generalized group characters and complex oriented cohomology theories", J. Am. Math. Soc. 13 (2000).
- Landweber, "Elliptic Curves and Modular Forms in Algebraic Topology: proceedings of a conference held at the Institute for Advanced Study". Princeton. September 1986. Lecture Notes in Mathematics.
- Lerman, "Orbifolds as stacks", Enseign. Math. (2) 56 (2010), no. 3-4, 315-363.
- Lurie, "A Survey of Elliptic Cohomology", in Algebraic Topology Abel Symposia Volume 4, 2009, pp 219-277.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schwede, "Global Homotopy Theory", global.pdf.
- Tomer M. Schlank, Nathaniel Stapleton, "A transchromatic proof of Strickland's theorem", Adv. Math. 285 (2015). 1415-1447.
- Stapleton, "Transchromatic generalized character maps", Algebr. Geom. Topol. 13 (2013), no. 1, 171–203.
- Strickland, "Morava E-theory of symmetric groups", Topology 37 (1998), no. 4.