

# Quasi-elliptic cohomology

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## Plan.

- Motivation and Definition
- Loop space construction: Witten's conjecture
- Geometric structure on Tate curve
- Global homotopy theory for elliptic cohomology theories
- Further problems
  - equivariant homotopy theory
  - chromatic homotopy theory
  - motivic homotopy theory

generalized cohomology theories  $\xrightleftharpoons{\text{1st Chern class}}$  formal groups

even periodic, multiplicative

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

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commutative 1-dimensional formal groups

- The additive formal group  $\mathbb{G}_a$ : periodic Eilenberg-MacLane spectrum.

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- Elliptic curves: elliptic cohomology?

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## Elliptic cohomology

[AHS][Lurie]

$R$ : commutative ring;  $C/R$ : elliptic curve over  $R$ .

$E$  is an elliptic cohomology theory if  $E^0(\text{pt}) \cong R$  and  $\text{Spf} E^0(\mathbb{C}P^\infty) \cong \widehat{C}$ .

## Moduli stack of elliptic curves

$$\{\text{elliptic curves } E \longrightarrow \operatorname{Spec} R\} = \operatorname{Hom}(\operatorname{Spec} R, M_{1,1}).$$

$\{\text{et} \acute{\text{a}}\text{l} \text{e elliptic curves over } R\} \longrightarrow \{\text{multiplicative cohomology theories}\}.$

## Tate K-theory

[AHS]

**Tate curve:** the generalized elliptic curve over  $\mathbb{Z}((q))$  classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

**The associated formal group:**  $\mathbb{G}_m$  over  $\mathbb{Z}((q))$ .

**Tate K-theory:** generalized elliptic cohomology associated to the Tate curve.

### Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.



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## Explicit Definition

$$QEII_G^*(X) \cong \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}^*(X^g)$$

- $G_{conj}^{tors}$ : a set of representatives of  $G$ -conjugacy classes in  $G^{tors}$ ;
- $\Lambda_G(g) = C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle$ ;
- $x \cdot [a, t] = x \cdot a$ , for all  $[a, t] \in \Lambda_G(g)$ ,  $x \in X^g$ .

## Relation with Tate K-theory

$$QEII_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K_{Tate}^*(X // G).$$



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## Representation theory

- Restriction map:  $RG \longrightarrow RH$ ;

## Equivariant K-theory

- Restriction map:  $K_G(X) \longrightarrow K_H(X)$ ;

## Quasi-elliptic cohomology

- Restriction map:  $QEll_G(X) \longrightarrow QEll_H(X)$ ;

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# Basic Properties of Quasi-elliptic cohomology

## Representation theory

- Restriction map:  $RG \longrightarrow RH$ ;
- Induced map:  $RH \longrightarrow RG$ .
- $RG \otimes RH \longrightarrow R(G \times H)$ .

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- Restriction map:  $K_G(X) \longrightarrow K_H(X)$ ;
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- $K_G^*(-)$  can be represented by an orthogonal  $G$ -spectrum;
- **Global K-theory exists.**

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Relevant Work

[Devoto][Ganter]

2007,  $G$ –equivariant Tate K-theory for finite groups  $G$  is modelled on the loop space of a global quotient orbifold.

# Relation between elliptic cohomology and Loop spaces

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Question

How can we construct elliptic cohomology theories from loop spaces?

## Review: Free Loop Space

$$LX = C^\infty(\mathbb{T}, X).$$

- $\mathbb{T}$ -action:  $\gamma \cdot t = (x \mapsto \gamma(s + t))$ .
- $LG$ -action:  $\gamma \cdot \delta = (s \mapsto \gamma(s) \cdot \delta(s))$ .
- $LG \rtimes \mathbb{T}$ -action:  $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s + t) \cdot \delta(s + t))$ .  
 $(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s + t_1), t_1 + t_2)$ .

Interpretation of the  $LG \rtimes \mathbb{T}$ -action

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$LG \rtimes \mathbb{T}$ : the extended gauge group

$$\begin{array}{ccc}
 G \times \mathbb{T} & \xrightarrow{(g,s) \mapsto (\gamma(s)g, s+t)} & G \times \mathbb{T} \\
 \downarrow & & \downarrow \\
 \mathbb{T} & \xrightarrow{s \mapsto s+t} & \mathbb{T}
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$LG \rtimes \mathbb{T}$ : act on loops  $G \times \mathbb{T} \longrightarrow G \times \mathbb{T} \xrightarrow{\tilde{\gamma}} X$

$$\begin{array}{ccc} & & \\ \downarrow & & \downarrow \\ \mathbb{T} & \longrightarrow & \mathbb{T} \end{array}$$

# The Answer: What is "Loop"?

## New Definition of Equivariant loops $Loop(X // G)$

[Rezk]

- Objects:

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

- $\pi$  : principal  $G$ -bundle over  $\mathbb{T}$

- $f$  :  $G$ -equivariant;

- Morphism  $(\alpha, t) : \{ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X \} \rightarrow \{ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X \}$ :

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## Relation with Bibundles

$$Bibun(\mathbb{T} // *, X // G)$$

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$\Lambda(X // G)$ : a subgroupoid of  $Loop(X // G)$  consisting of constant loops.

$$\Lambda(X // G) \cong \coprod_{g \in G_{conj}^{tors}} X^g // \Lambda_G(g)$$

$$QEII_G^*(X) = K_{orb}^*(\Lambda(X // G))$$

## Further Question

Can we use this new definition of loop spaces to construct elliptic cohomology theories?

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# Power operation of equivariant cohomology theories

## Power Operation of K-theory

[Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

## Power Operation of equivariant K-theory

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$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

## Wreath product $G \wr \Sigma_n$

$$(g_1, \dots, g_n, \sigma) \cdot (h_1, \dots, h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)}, \sigma \tau).$$

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## Definition of Equivariant Power Operation

[May][Ganter]

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# Quasi-elliptic cohomology has power operations

## Atiyah's Power Operation

[Ganter]

$V$ : a vector bundle over  $\Lambda(X//G)$ .

$P_n(V) := V^{\hat{\otimes}_{\mathbb{Z}[q^{\pm}]}^n}$  defines an operation

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## The Stringy Power Operation

[Huan]

$$\mathbb{P}_n = \prod_{(\underline{g}, \sigma) \in (G|\Sigma_n)_{conj}^{tors}} \mathbb{P}_{(\underline{g}, \sigma)}:$$

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## Theorem (Huan)

The Tate  $K$ -theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{\text{Tate}}(\text{pt} // \Sigma_N) / I_{\text{tr}}^{\text{Tate}} \cong \prod_{N=de} \mathbb{Z}((q)) [q_s'^{\pm}] / \langle q^d - q_s'^e \rangle,$$

where  $I_{\text{tr}}^{\text{Tate}}$  is the transfer ideal and  $q_s'$  is the image of  $q$  under the **stringy power operation**, the product goes over all the ordered pairs of positive integers  $(d, e)$  such that  $N = de$ .

The proof: (i) Apply representation theory, prove

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(ii) Apply the relation between  $QEII^*$  and  $K_{\text{Tate}}^*$ .

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- Key point of the construction: replace the restriction maps, which are identity maps in global homotopy theory, by equivariant weak equivalences.
- The category of almost global spectra is NOT equivalent to the category of global spectra.
- But they are Quillen equivalent, i.e. they are describing the same mathematical world.
- Global quasi-elliptic cohomology exists in almost global homotopy theory.
- What is better, infinitely many distinct cohomology theories can be globalized in almost global homotopy theory, **quasi-theories** etc.

## Quasi-theories

[Huan]

$$QE_{n,G}^*(X) := E^*(\wedge^n(X//G)) \cong \prod_{\sigma \in G_{\mathbb{Z}}^n} E_{n,\Lambda_G^n(\sigma)}^*(X^\sigma).$$

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## My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

More explicitly, if  $E^*$  can be globalized and  $A^*$  is a cohomology theory with divisible group  $\mathbb{G}_E \oplus (\mathbb{Q}/\mathbb{Z})^n$ , then  $A^*$  can also be globalized.

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# Future Problems in chromatic homotopy theory

Chromatic level:=height of the formal group.

| Chromatic level | complex oriented cohomology theory  |
|-----------------|---|
| 0               | ordinary cohomology<br>0th Morava K-theory $K(0)$                                     |
| 1               | complex K-theory $KU$<br>first Morava K-theory $K(1)$<br>first Morava E-theory $E(1)$ |
| 2               | elliptic cohomology<br>second Morava K-theory $K(2)$<br>second Morava E-theory $E(2)$ |
| n               | $n$ th Morava K-theory $K(n)$<br>$n$ th Morava E-theory $E(n)$                        |
| $\infty$        | complex bordism cohomology $MU$ .   |

Table: Chromatic homotopy theory



## Interesting point

The chromatic level of elliptic cohomology theories is **2**;  
the chromatic level of quasi-elliptic cohomology theory is **1**.

**Question:** How can we relate the two closed theories?

## Transchromatic character theory

- The character map:  $L \otimes R(G) \xrightarrow{\cong} CI(G; L)$ .
- Hopkins-Kuhn-Ravenel's generalized character map:

$$\chi_{n,p}^G : L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\cong} CI_{n,p}(G, X; L(E^*)).$$

- Stapleton extends the generalized character map for Morava  $E$ -theories:

$$C_t \otimes_{E_n^0} \Phi_G : C_t \otimes_{E_n^0} E_n^*(EG \times_G X) \xrightarrow{\cong} C_t^*(EG \times_G \text{Fix}_{n-t}(X)).$$

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**Question:** How can we relate the two closed theories?

## Transchromatic character theory

- The character map:  $L \otimes R(G) \xrightarrow{\cong} CI(G; L)$ .
- Hopkins-Kuhn-Ravenel's generalized character map:

$$\chi_{n,p}^G : L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\cong} CI_{n,p}(G, X; L(E^*)).$$

- Stapleton extends the generalized character map for Morava  $E$ -theories:

$$C_t \otimes_{E_n^0} \Phi_G : C_t \otimes_{E_n^0} E_n^*(EG \times_G X) \xrightarrow{\cong} C_t^*(EG \times_G \text{Fix}_{n-t}(X)).$$

## Conjecture

[Ganter][Huan]

Gronowski's elliptic cohomology theories and quasi-elliptic cohomology theory can be related by Stapleton's generalized character map.

## Conjecture

[Huan][Stapleton]

Generalized Morava  $E$ -theories have Strickland's theorem for  $A$ -level structures. Explicitly,

$$\mathrm{Spec}(E_n^0(L^h(BA))/I_{tr}^A) = \mathrm{Level}_A(\mathbb{G}_{E_n} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h)$$

- $A$ : any finite abelian group;
- $L(-)$ : the  $p$ -adic loop functor  $\mathrm{Hom}(*//\mathbb{Z}_p, X)$ .

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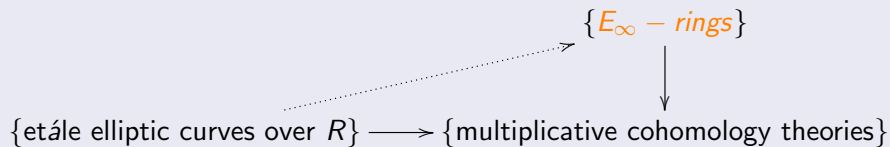
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## Goerss-Hopkins-Miller-Lurie Theorem



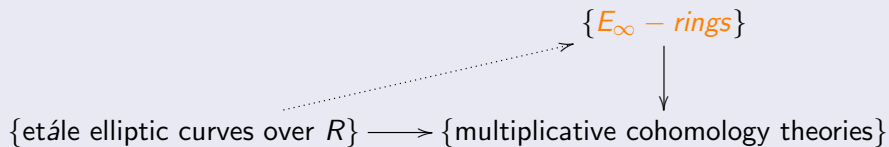
## Question

[Huan]

Do we have motivic Goerss-Hopkins-Miller-Lurie Theorem?

- 2009 Naumann, Spitzweck, Østvær: prove motivic Landweber exact functor theorem.  
Especially, motivic elliptic cohomology theories exist.
- 2009 Spitzweck: define motivic  $E_\infty$ -ring.

## motivic Goerss-Hopkins-Miller-Lurie Theorem



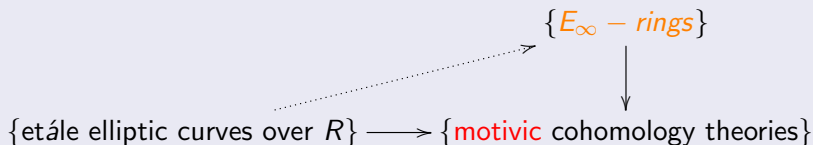
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*Thank you.*

<http://gagp.sysu.edu.cn/zhenhuan/Zhen-HUST-2018-Slides.pdf>

- Ando, "Isogenies of formal group laws and power operations in the cohomology theories  $E_n$ ", Duke J., 1995
- Ando, Hopkins, Strickland: "Elliptic spectra, the Witten genus and the theorem of the cube". Invent. Math. , 146(3):595–687, 2001.
- Ando, Hopkins, Strickland, "The sigma orientation is an  $H_\infty$  map", Amer. J. 2004;
- Atiyah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Baas, Dundas, Rognes: "Two-vector bundles and forms of elliptic cohomology", Topology, geometry and quantum field theory, 18–45, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Devoto, "Equivariant elliptic homology and finite groups", Michigan Math. J. , 43(1):3–32, 1996.
- Ganter, "Orbifold genera, product formulas, and power operations", Adv. Math, 2006;
- Ganter, "Stringy power operations in Tate K-theory", Homology, Homotopy, Appl., 2013;
- Gepner, "Homotopy Topoi and Equivariant Elliptic Cohomology", Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 1999.
- Ginzburg, Kapranov, Vasserot, "Elliptic algebras and equivariant elliptic cohomology I", available at arXiv:q-alg/9505012.
- Hopkins, Kuhn, Ravenel, "Generalized group characters and complex oriented cohomology theories", J. Am. Math. Soc. 13 (2000).
- Landweber, "Elliptic Curves and Modular Forms in Algebraic Topology: proceedings of a conference held at the Institute for Advanced Study", Princeton, September 1986, Lecture Notes in Mathematics.
- Lerman, "Orbifolds as stacks", Enseign. Math. (2) 56 (2010), no. 3–4, 315–363.
- Lurie, "A Survey of Elliptic Cohomology", in Algebraic Topology Abel Symposia Volume 4, 2009, pp 219–277.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schwede, "Global Homotopy Theory", global.pdf.
- Tomer M. Schlank, Nathaniel Stapleton, "A transchromatic proof of Strickland's theorem", Adv. Math. 285 (2015), 1415–1447.
- Stapleton, "Transchromatic generalized character maps", Algebr. Geom. Topol. 13 (2013), no. 1, 171–203.
- Strickland, "Morava E-theory of symmetric groups", Topology 37 (1998), no. 4.