Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 7-2

Recall:

$$\int_{\Omega}(u\Delta v - v\Delta u)dV = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \tag{1}$$

$$u(M_0) = -\frac{1}{4\pi}\iint_{\Gamma}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) - \frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right]dS. \tag{2}$$

0.1 Basic Properties of Harmonic Functions

Theorem 0.1 (Flux Properties). *Let the function $u(x, y, z)$ be a harmonic function in the region Ω, and it has first order continuous partial derivatives on $\overline{\Omega}$. Then*

$$\iint_{\Gamma}\frac{\partial u}{\partial n}dS = 0, \tag{3}$$

where Γ is the boundary of the region Ω, and $\mathbf{n}$ is the direction of the outer normal of Γ.

Proof. Just take u as a harmonic function and $v = 1$ in equation (1), then formula (3) can be obtained. □

Alternative Proof Using Gauss's Theorem.

$$\int_{\Gamma}\frac{\partial u}{\partial n}\,dS = \int_{\Gamma}\mathbf{n}\cdot\nabla u\,dS = \int_{\Omega}\Delta u\,dV = 0.$$

□

Formula (3) shows that the integral of the normal derivative of a harmonic function along the boundary of the region is 0. For a stable temperature field, this means that the amount of heat flowing into and out of the object through the object interface is equal. Otherwise, the thermal dynamic equilibrium cannot be maintained, and the temperature will be unstable.

Physical Interpretation via Electrostatics

- The equation $\Delta u = 0$ describes a **steady-state** condition.
- In electrostatics, u can be interpreted as an **electric potential** in a charge-free region.
- Gauss' flux theorem in electrostatics states:

$$\oint_{\Gamma}\mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \leftarrow \boxed{\text{For Green function in next section, one positive charge!}}$$

  where $Q = 0$ in a charge-free domain.

- Hence, due to $\mathbf{E} = -\nabla u$, from physical perspective, using the Gauss' flux theorem, the net flux of $\int_\Gamma \frac{\partial u}{\partial n} dS = \int_\Gamma \mathbf{n} \cdot \nabla u dS = -\int_\Gamma \mathbf{n} \cdot \mathbf{E} dS = 0$ through the boundary is zero.

- "The electric field lines entering are equal to the electric field lines exiting" (see Fig. 1).

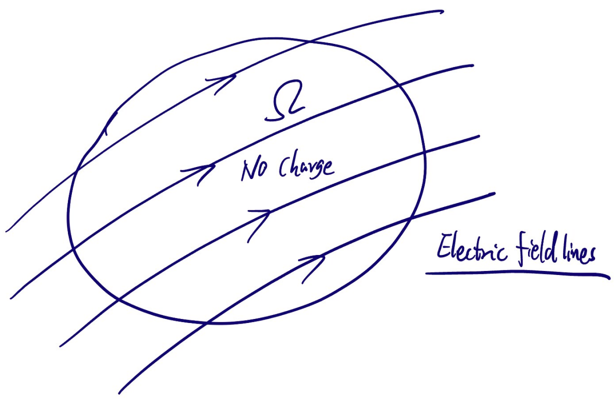

Figure 1: Gauss' flux theorem

From formula (3), it can be deduced that a **necessary condition** for the Neumann problem

$$\begin{cases} \Delta u(x,y,z) = 0, & (x,y,z) \in \Omega \\ \frac{\partial u}{\partial n}\big|_\Gamma = f(x,y,z) \end{cases}$$

to have a solution is $\iint_\Gamma f(x,y,z) dS = 0$.

Application to Boundary Value Problems

- If $\int_\Gamma f \, dS \neq 0$, the boundary value problem has **no solution**.

Theorem 0.2 (Mean Value Theorem). *Let the function $u(M)$ be* ***harmonic*** *in the region Ω, and M_0 be any point in the region Ω. If Γ_a is a* ***spherical surface*** *centered at M_0 with radius a, and this sphere is completely inside the region Ω, then*

$$u(M_0) = \frac{1}{4\pi a^2} \iint\limits_{\Gamma_a} u dS. \tag{4}$$

Conditions: $\begin{cases} \text{(1) } u \text{ is } \textbf{harmonic} \\ \text{(2) } \Gamma_a \text{ a } \textbf{spherical surface} \text{ (not every surface possesses such a good property)} \end{cases}$

Comparison with the General Case

- For a general function (may not be harmonic), the spherical mean approximates the function's value at the center only in the limit as the radius tends to zero.

- For harmonic functions, this equality holds for any radius, making the condition much stronger.

- This demonstrates that harmonic functions are uniquely determined once boundary values are given.

Key Conditions for Applying the Theorem

- The function must be harmonic, meaning it satisfies Laplace's equation.
- The averaging surface must be a sphere.
- A common mistake is applying the mean value property to non-spherical regions, where it does not hold.

Proof. Apply formula (2) to the spherical surface Γ_a, we get

$$u(M_0) = -\frac{1}{4\pi} \iint\limits_{\Gamma_a} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

Since

$$\iint\limits_{\Gamma_a} \frac{1}{r} \frac{\partial u}{\partial n} dS = \frac{1}{a} \iint\limits_{\Gamma_a} \frac{\partial u}{\partial n} dS = 0 \quad \text{(using Theorem 0.1)}$$

and

$$\iint\limits_{\Gamma_a} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = \iint\limits_{\Gamma_a} u \frac{\partial}{\partial r} \left(\frac{1}{r} \right) dS = \iint\limits_{\Gamma_a} u \left(-\frac{1}{r^2} \right) dS = -\frac{1}{a^2} \iint\limits_{\Gamma_a} u dS,$$

then equation (4) is proved. □

Theorem 0.3 (Extremum Principle)**.** *Let the function $u(x, y, z)$ be a* ***harmonic function*** *in the region Ω, and it is* ***continuous on*** *$\Omega + \Gamma$ and* ***not a constant*** *function. Then its* ***maximum and minimum*** *values can* ***only*** *be achieved* ***on the boundary*** *Γ.*

Continuity Condition

- A function u is harmonic in a domain Ω if it satisfies the Laplace equation $\Delta u = 0$ in Ω and $u \in C^2(\Omega)$.
- Additionally, u must be continuous up to the boundary Γ of Ω.
- If u is not continuous up to the boundary Γ, then it may have jumps or discontinuities at the boundary.
- In this case, boundary conditions such as $u|_\Gamma = f$ lose their meaning, as the function inside Ω is not constrained by the boundary values.
- For boundary conditions to meaningfully affect the interior solution, u must be continuously extendable to Γ.

Implicit Assumption of Continuity

- From now on, whenever we refer to a harmonic function in Ω, we assume it is continuous up to the boundary.
- This implicit assumption ensures that boundary values play a role in determining the solution inside Ω.

Maximum and Minimum Value Principle

- If u is harmonic in Ω and not a constant function, then u
  - **can attain** its maximum and minimum values;
  - **only** on the **boundary** Γ.
- This is a **stronger condition** than the general case of **continuous** functions, which are **only guaranteed to attain extreme values** in bounded domains.
- The principle remains valid even if Ω is unbounded or partially unbounded.

Strong Maximum Principle

- The maximum and minimum values of u cannot be attained in the interior of Ω, unless u is a constant function.
- If u is a harmonic function in a domain Ω, then only two possibilities exist:
  1. u is a **constant** function.
  2. If u is **not constant,** then its maximum and minimum values must occur on the **boundary** Γ.

Significance of the Principle

- The principle highlights the restrictive nature of harmonic functions compared to general continuous functions.
- It ensures that boundary values fully determine the behavior of a harmonic function in the interior.

Proof. The extremum principle can be easily proved by the mean value theorem. We only need to prove the case of the maximum value (by replacing u with $-u$, the case of the minimum value can be reduced to the case of the maximum value).

We use the method of proof by contradiction. Suppose the function u achieves its maximum value at a certain point $M_1 \in \Omega$. Then it can be deduced that u must be **identically equal to a constant**, and $u = u(M_1)$. This contradicts the condition that u is **not** a constant function.

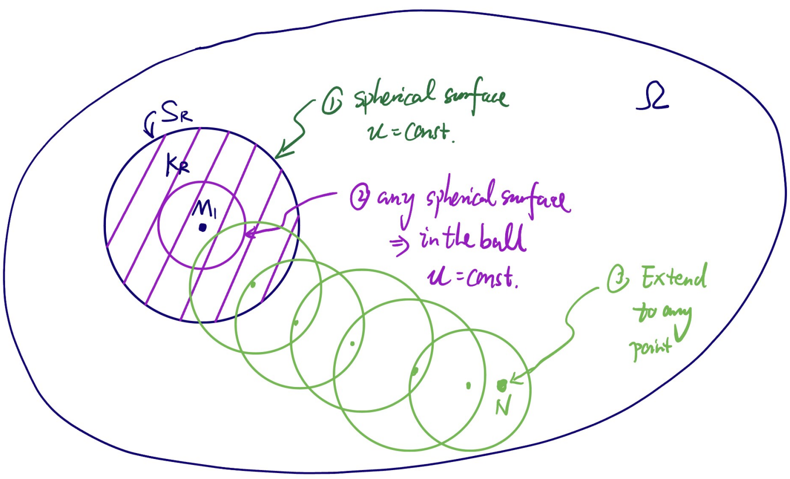

Figure 2: Extrmum Theorem

Take M_1 as the center and make a sphere K_R with an arbitrary radius R such that it is completely contained in the region Ω. Denote the spherical surface of K_R as S_R. Then on S_R, we have $u(M) = u(M_1)$.

(1) On S_R: In fact, if there exists a point M such that $u(M) < u(M_1)$, then by the **continuity** of the function, we can surely find a neighborhood of this point on the spherical surface S_R where also $u(M) < u(M_1)$. Therefore, even if $u(M) = u(M_1)$ on the remaining part of the spherical surface S_R, we still have

$$\iint\limits_{S_R} u(M)dS < \iint\limits_{S_R} u(M_1)dS$$

In fact,

$$\frac{1}{4\pi R^2}\iint\limits_{S_R} u(M)dS < \frac{1}{4\pi R^2}\iint\limits_{S_R} u(M_1)dS = u(M_1)$$

But by the **mean value formula** (4), we have

$$\frac{1}{4\pi R^2}\iint\limits_{S_R} u(M)dS = u(M_1)$$

This is a contradiction. Then on the spherical surface S_R, we have $u(M) \equiv u(M_1)$.

(2) On K_R: Similarly, on the spherical surface with M_1 as the center and an arbitrary radius $r(r \leq R)$, we also have $u(M) \equiv u(M_1)$. Therefore, in the entire sphere K_R, we always have $u(M) \equiv u(M_1)$.

(3) Any point $N \in \Omega$: Now we prove that $u(M) \equiv u(M_1)$ holds for all points in Ω. Arbitrarily take a point $N \in \Omega$. In the region Ω, make a polyline l connecting the two points M_1 and N. Denote the minimum distance from the polyline l to the boundary Γ of the region Ω as d. Due to the arbitrariness of the point N, we obtain that $u(M) \equiv u(M_1)$ holds in the entire region Ω. This contradicts the problem assumption. Then the extremum principle is proved. □

Corollary 0.1 (Principle of Comparison)**.** *Let u and v be both harmonic functions in the region Ω, and they are continuous on $\Omega + \Gamma$. If the inequality $u \leq v$ holds on the boundary Γ of Ω, then the inequality also holds in Ω, and the equality holds in Ω if and only if $u \equiv v$.*

Statement of the Theorem

- Suppose u and v are harmonic functions in a domain Ω.
- If $u \leq v$ (i.e., $u - v \leq 0$) on the boundary Γ, then $u \leq v$ (i.e., $u - v \leq 0$) in the entire domain Ω. Therefore, inspire us to define $w = u - v$ (one function can apply Extreme Theorem).

Proof Outline

- Define $w = u - v$.
- Since u and v are harmonic, we have:

$$\Delta u = 0, \quad \Delta v = 0.$$

- Subtracting the two equations gives:

$$\Delta w = \Delta u - \Delta v = 0.$$

  Thus, w is also harmonic in Ω.
- By assumption, $w \leq 0$ on Γ.
- Apply the maximum principle:
  - The maximum and minimum of a harmonic function in Ω must occur on the boundary unless the function is constant.
  - If $w > 0$ at some interior point, this contradicts the maximum principle, since the boundary values are non-positive.
- Therefore, $w \leq 0$ in Ω, which implies $u \leq v$ in Ω.

Remarks

- This result is a direct consequence of the maximum principle.
- The standard method of proof involves contradiction: assuming $w > 0$ somewhere inside leads to a contradiction with the maximum principle.
- The comparison principle is frequently used in solving boundary value problems for harmonic functions.

Corollary 0.2 (Application: Uniqueness Theorem)**.** *Proving the **uniqueness** of solutions to the Dirichlet problem using the extremum principle*

$$\begin{cases} \Delta u(x,y,z) = 0, \quad (x,y,z) \in \Omega \\ u|_\Gamma = f(x,y,z) \end{cases} \tag{5}$$

Well-posedness (Recall)

To ensure **well-posedness** of (5), we require the solution to satisfy three properties:

- **Existence:** A solution must exist.
- **Uniqueness:** The solution must be unique.
- **Stability:** Small changes in the boundary data should result in small changes in the solution.

In this section, we focus on proving the uniqueness of the solution using the maximum principle.

Summary of Uniqueness Proof in Differential Equations

The common approach to proving the uniqueness of solutions in differential equations is as follows:

- We use a contradiction argument.
- Assume there are two solutions u_1 and u_2.
- Define $w = u_1 - u_2$.
- Prove that $w = 0$, which implies $u_1 = u_2$.

Thus, the proof of uniqueness is reduced to demonstrating that the associated homogeneous problem for w has only the trivial solution. Various tools can be employed for this purpose, including:

- Maximum and Minimum Value Principle,
- Energy Methods (e.g. wave equations, there is no Extreme Theorems), and
- Other analytical tools.

Uniqueness Proof

Analysis of w

- Since both u_1 and u_2 satisfy the Laplace equation:

  $$\Delta u_1 = 0, \quad \Delta u_2 = 0,$$

  subtracting these equations gives:

  $$\Delta w = \Delta(u_1 - u_2) = 0.$$

  This implies that w is a harmonic function in Ω.

- On the boundary Γ, we have:
$$w = u_1 - u_2 = f - f = 0.$$
That is, $w = 0$ on Γ.

Application of the Maximum Principle

- The strong maximum principle states that a non-constant harmonic function attains its maximum and minimum on the boundary.
- Since w is harmonic and vanishes on Γ, the only possibility is $w \equiv 0$ in Ω.
- Thus, $u_1 = u_2$, proving uniqueness.

Proof. Let u_1 and u_2 be two solutions of problem (5). Then $u = u_1 - u_2$ is a harmonic function in Ω, that is, $\Delta u = 0$, and $u|_\Gamma = 0$. By the **extremum principle**, u can neither be greater than 0 nor less than 0 in Ω. So on $\Omega + \Gamma$, we have $u \equiv 0$, that is, $u_1 \equiv u_2$. This proves the uniqueness of solutions to the Dirichlet problem. $\square$

1 Green's Function

- In Section 4.1, we solved the Laplace equation **without** considering **boundary** conditions. The solution was a **harmonic function**, expressed in an **integral** form.
- In Section 4.2, we **aim** to **incorporate** boundary conditions and investigate the form of the solution.
- However, we **cannot solve** the equation with boundary conditions in this section. Instead, we aim to **find a transformation** (**the main objective in this section**) between u and a function v.
- Through this transformation, we convert the boundary value problem for u into a problem for v, i.e.,
$$\Delta v = 0, \quad \text{and} \quad v = \frac{1}{4\pi r}.$$

For a function u that is a harmonic function in the region Ω and has first order continuous partial derivatives on $\Omega + \Gamma$, we have the equality

$$u(M_0) = -\frac{1}{4\pi} \iint\limits_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

This integral expression shows that the value of the function u **inside the region** Ω can be expressed in terms of the **values of the function** u and its **normal derivative** $\frac{\partial u}{\partial n}$ on the **boundary** Γ. However, the solutions to the Dirichlet problem or the Neumann problem **cannot** be directly obtained from formula (2).

For example, for the Dirichlet problem, the value of u on Γ is given, while the value of $\frac{\partial u}{\partial n}$ on the boundary Γ is unknown. Since the solution to the Dirichlet problem is unique, the value of $\frac{\partial u}{\partial n}$ on the boundary Γ **cannot** be **arbitrarily specified**.

So, in order to solve the Dirichlet problem, we naturally first think of eliminating $\frac{\partial u}{\partial n}$ from formula (2). For this purpose, we need to introduce the concept of Green's function. We also need to rely on Green's second formula

$$\iiint\limits_{\Omega} (u\Delta v - v\Delta u) d\Omega = \iint\limits_{\Gamma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS. \tag{6}$$

- The integral expression of a harmonic function has a geometric meaning: it describes the value of the harmonic function u at any point inside a region Ω, depended on the boundary values.
- Specifically, for a harmonic function u inside Ω, its value at any point depends on:

$$u = \text{Boundary value of } u \quad \text{and} \quad \text{the normal derivative of } u \text{ on the boundary.}$$

- This property suggests that the boundary values of u and its normal derivative determine all internal values. In other words, the boundary conditions determine the solution inside the region.
- If we consider a system as follows,

$$\begin{cases} \Delta u(x,y,z) = 0, \quad (x,y,z) \in \Omega \\ u|_\Gamma = f(x,y,z) \\ \frac{\partial u}{\partial n}\big|_\Gamma = g(x,y,z) \leftarrow \boxed{\text{Extra bdry. may break the wellposedness, leading to no solution}} \end{cases}$$

- Since (5) leads to the unique solution u and this u yields a specific $\frac{\partial u}{\partial n}\big|_\Gamma$. If it is not equal to g, then there is no solution.
- **Idea:** Add one term A and subtract its variant B. Consider the form $A = \int_{\partial\Omega} kv\frac{\partial u}{\partial n}$ since we want to use A to **eliminate** $\frac{\partial u}{\partial n}$ in (2). The requirement is $A = B$ on the boundary, but actually the functions $A \neq B$. On careful consideration: (1) $A \neq B$ is only in the forms of expression. (2) As long as they are **equal on the boundary**. Green's second formula helps us find the answer (detailed idea can be found in Fig. 3).

Summary:

For a harmonic function $u \in C^1(\Omega \cup \Gamma)$ in the region Ω, there is the integral representation of u.

$$u(M_0) = -\frac{1}{4\pi} \iint\limits_\Gamma \left[u(M) \frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS$$

The value of u inside Ω is expressed in terms of the value of u and the normal derivative $\frac{\partial u}{\partial n}$ of u on the boundary Γ.

However, the solutions to the Dirichlet or Neumann problems cannot be directly obtained from (2). Because for the value of u and the normal derivative $\frac{\partial u}{\partial n}$ of u on the boundary Γ, only one of them can be prescribed.

For example, for the Dirichlet problem, the value of u on the boundary Γ is given, and $\frac{\partial u}{\partial n}\big|_\Gamma$ is unknown. Since the solution to the Dirichlet problem is unique, $\frac{\partial u}{\partial n}\big|_\Gamma$ cannot be arbitrarily specified (otherwise, there may be no solution, redundant information).

To solve the Dirichlet problem, naturally, the first idea is to try to eliminate $\frac{\partial u}{\partial n}$ from the formula (2). For this purpose, the concept of the Green's function needs to be introduced.

In Green's second formula (1), if we take both u and v as harmonic functions in the region Ω and they have continuous first order partial derivatives on $\Omega + \Gamma$, then we get

$$0 = \iint\limits_{\Gamma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Adding the above formula to formula (2), we obtain

$$u(M_0) = \iint\limits_{\Gamma} \left\{ u \cdot \frac{\partial}{\partial n} \left(v - \frac{1}{4\pi r_{MM_0}} \right) + \left(\frac{1}{4\pi r_{MM_0}} - v \right) \frac{\partial u}{\partial n} \right\} dS. \tag{6}$$

If we choose a harmonic function v such that $v|_\Gamma = \left.\frac{1}{4\pi r_{MM_0}}\right|_\Gamma$, then the $\frac{\partial u}{\partial n}$ term in formula (6) disappears, and we have

$$u(M_0) = -\iint\limits_{\Gamma} u\frac{\partial}{\partial n} \left(\frac{1}{4\pi r_{MM_0}} - v \right) dS. \tag{7}$$

Let

$$G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v, \tag{8}$$

Then formula (7) can be expressed as

$$u(M_0) = -\iint\limits_{\Gamma} u\frac{\partial G}{\partial n} dS.$$

where $G(M, M_0)$ is called the **Green's function** for Laplace's equation (or the source function for the Dirichlet problem). And $G(M, M_0)$ is identically equal to 0 on the boundary Γ.

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $\Omega + \Gamma$, for the Dirichlet problem of Laplace's equation

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_\Gamma = f(x, y, z) \end{cases} \tag{9}$$

If a solution with first order continuous partial derivatives exists on $\Omega + \Gamma$, then the solution to problem (9) can be expressed as

$$u(M_0) = -\iint\limits_{\Gamma} f(x, y, z)\frac{\partial G}{\partial n} dS. \tag{10}$$

Since

$$G = \frac{1}{4\pi r} - v,$$

$$\Delta \left(\frac{1}{r} \right) = -4\pi\delta(r - r_0) \Rightarrow \Delta \left(\frac{1}{4\pi r} \right) = -\delta(r - r_0).$$

Since $\Delta u = 0$

$$\Rightarrow \Delta G = \Delta \left(\frac{1}{4\pi r} - v \right) = -\delta(r - r_0).$$

$$\Rightarrow \begin{cases} \Delta G = -\delta(r - r_0) \\ G|_\Gamma = 0 \end{cases}.$$

G is the second kind Green's function

- In fact, the role of $u(M_0) = -\iint_\Gamma f\frac{\partial G}{\partial n} ds$ is to provide a variable transformation between u and v. Through this variable transformation, the problem of

$$\begin{cases} \Delta u = 0 \\ u|_\Gamma = f \end{cases}$$

can be transformed into the problem of

$$\left\{\begin{array}{l}\Delta v = 0\\ v|_\Gamma = \frac{1}{4\pi r}\end{array}\right. \leftarrow \boxed{v \text{ harmonic}} \quad \text{or} \quad \left\{\begin{array}{l}\Delta G = -\delta(r - r_0)\\ G|_\Gamma = 0\end{array}\right. \leftarrow \boxed{G \sim \frac{1}{r_{MM_0}} \text{ singular at } M_0}$$

$\uparrow$ $\boxed{\text{Do not means we have solved } u \text{ since } v \text{ and } G \text{ have to be solved in next section}}$

The second problem is simpler and independent of f.

- **Simplification of the Problem:**
  - For different f, only the same v problem needs to be solved if the domain is the same.
  - This **simplifies** or **categorizes** the problem (8).
- Recall to convert a difficult problem into a familiar or simpler one:
  - Function Transformation
  - Variable Transformation

  Then transformations $u(M_0) = -\iint_\Gamma f \frac{\partial G}{\partial n}\, ds$ plays a role of such transformation.

Mathematical Approach solving the Green functions

- When solving a problem involving Green functions G or v, we can use mathematical methods such as the separation of variables.
  - Separation of Variables.
  - Eigenfunction Expansion.

Semi-Physical Approach (introduced in Section 4.3)

- Instead of purely mathematical methods, we will use a semi-physical approach.
- This method provides a deeper understanding of the physical significance of G.
- The key idea is to interpret the differential equation in a physical context.

Given the Problems:

$$\left\{\begin{array}{l}\Delta u = 0\\ u|_\Gamma = f\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\Delta G = -\delta(r - r_0)\\ G|_\Gamma = 0\end{array}\right. \qquad (\text{where } \Gamma = \partial\Omega)$$

By integrating $G\Delta u - u\Delta G$ over Ω:

$$\int_\Omega \underbrace{G\Delta u}_{=0} - \underbrace{u\Delta G}_{-u\delta(r-r_0)}\, dV = \int_{\partial\Omega} \underbrace{G\frac{\partial u}{\partial n}}_{=0} - \underbrace{u\frac{\partial G}{\partial n}}_{f\frac{\partial G}{\partial n}}\, dS.$$

Then,

$$\int_\Omega u\delta(r - r_0)\, dV = -\int_{\partial\Omega} f\frac{\partial G}{\partial n}\, dS \quad \Rightarrow \quad u(r_0) = -\int_{\partial\Omega} f\frac{\partial G}{\partial n}\, dS.$$

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $\Omega + \Gamma$, for the Dirichlet problem of Poisson's equation

$$\begin{cases}\Delta u(x, y, z) = F, & (x, y, z) \in \Omega\\ u|_\Gamma = f(x, y, z)\end{cases}$$

If a solution with first order continuous partial derivatives exists on $\Omega + \Gamma$, then this solution must be expressible as

$$u(M_0) = -\iint\limits_{\Gamma} f \frac{\partial G}{\partial n} dS - \iiint\limits_{\Omega} F G d\Omega.$$

This is because, similar to before, according to Green's second identity, we have $\Delta V = 0$, $\Delta u = F$.

$$-\iiint\limits_{\Omega} vF d\Omega = \iint\limits_{\Omega} (u\Delta v - v\Delta u) d\Omega = \iint\limits_{\Gamma} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds \quad ①$$

$$u(M_0) = -\frac{1}{4\pi} \iint\limits_{\Gamma} \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) dS - \frac{1}{4\pi} \iiint\limits_{\Omega} \frac{F(M)}{r} d\Omega \quad ②$$

Adding ① and ② gives

$$u(M_0) = \iint\limits_{\Gamma} \left[u\frac{\partial}{\partial n}\left(v - \frac{1}{4\pi r}\right) - \left(v - \frac{1}{4\pi r}\right)\frac{\partial u}{\partial n} \right] dS + \iiint\limits_{\Omega} F\left(v - \frac{1}{4\pi r}\right) d\Omega$$

Let $(v - \frac{1}{4\pi r})\big|_{\Gamma} = 0$, then

$$u(M_0) = -\iint\limits_{\Gamma} u \frac{\partial G}{\partial n} dS - \iiint\limits_{\Omega} F G d\Omega.$$

Application of formula (10) to solve the Dirichlet problem for Laplace's equation When applying formula (10) to solve the Dirichlet problem for Laplace's equation, the key lies in finding the Green's function $G(M, M_0)$ in formula (8), where v is the solution to the following special Dirichlet problem:

$$\begin{cases} \Delta v = 0, & (x, y, z) \in \Omega \\ v|_{\Gamma} = \frac{1}{4\pi r_{MM_0}}\Big|_{\Gamma} \end{cases} \tag{21}$$

The Green's function determined by this function v is called the Green's function for the first boundary value problem. (For some special regions, such as spherical regions and half-spaces, the Green's function can be obtained.)

1.0.1 Physical meaning of Green's function in electrostatics

Suppose **a unit positive charge** is placed at point M_0. Then in free space, the **electric potential** it generates is $\frac{1}{4\pi r_{MM_0}}$. If the point charge at M_0 is enclosed within a **closed conducting surface** that is **grounded**, then the **electric potential** inside the conducting surface can be represented by the function

$$G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v,$$

This function is identically equal to 0 on the conducting surface. The function $-v$ exactly represents the electric potential generated by the induced charges on the conducting surface (see Fig. 4).

Physical Interpretation of Green's Function

To understand the meaning of G, we consider translating the equation into a physical problem.

Electrostatic Analogy

In electrostatics, Laplace's equation with a delta function source describes the potential due to a point charge:

$$\nabla^2 G = -\delta(r - r_0)$$

This equation represents:

- A unit point charge located at a specific position.
- The resulting electrostatic potential field in a given domain Ω.
- Conducting boundaries modifying the potential due to induced charges.

Boundary Conditions and Physical Setup

The boundary condition $G = 0$ on the boundary translates to:

- A grounded conductor surrounding the domain.
- Induced charges appearing on the conductor.
- The resulting electrostatic field being different from the free-space solution.

The problem consists of three key elements

- A conducting shell.
- A unit positive charge.
- A grounded connection.

Supplementary 3: Define the Green's function for the first boundary value problem in the plane and derive the integral expression for the solution of this problem

Differences

- In 3D: Green's function is solved in a given region Ω.
- In 2D: The boundary becomes a line or a curve, reducing the problem dimension.
- $4\pi\ (3D) \to 2\pi\ (2D)$.
- The fundamental solution $\frac{1}{r}\ (3D) \to \ln\left(\frac{1}{r}\right)\ (2D)$.

The solution involves finding relationships between functions u and G and their derivatives.

For this purpose, we need to rely on the formula

$$u(M_0) = -\frac{1}{2\pi}\int_C \left[u(M)\frac{\partial}{\partial n}\left(\ln\frac{1}{r_{MM_0}}\right) - \ln\frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right] dS. \tag{11}$$

and the second Green's formula in the plane

$$\iint_D (u\Delta v - v\Delta u) d\sigma = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS. \tag{12}$$

In Green's formula (12), if we take both u and v as harmonic functions in the region D and they have continuous first order partial derivatives on $D + C$, then we get

$$0 = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.$$

Adding the above formula to formula (11) gives

$$u(M_0) = \int_C \left\{u\frac{\partial}{\partial n}\left(v - \frac{1}{2\pi}\ln\frac{1}{r_{MM_0}}\right) + \left(\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}} - v\right)\frac{\partial u}{\partial n}\right\} dS. \tag{13}$$

If we choose a harmonic function v such that $v|_C = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}\Big|_C$, then the $\frac{\partial u}{\partial n}$ term in formula (13) disappears, and we have

$$u(M_0) = -\int_C u \frac{\partial}{\partial n} \left(\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v \right) dS. \tag{14}$$

Let

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v, \tag{15}$$

Then formula (14) can be expressed as

$$u(M_0) = -\int_C u \frac{\partial G}{\partial n} dS. \tag{16}$$

where $G(M, M_0)$ is called the Green's function for the two dimensional Laplace's equation (or the source function for the Dirichlet problem). And $G(M, M_0)$ is identically equal to 0 on the boundary C.

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $D + C$, for the Dirichlet problem of the two dimensional Laplace's equation

$$\begin{cases} \Delta u(x, y) = 0, & (x, y) \in D \\ u|_C = f(x, y) \end{cases} \tag{17}$$

If a solution with first - order continuous partial derivatives exists on $D+C$, then the solution to problem (17) can be expressed as

$$u(M_0) = -\int_C f(x, y) \frac{\partial G}{\partial n} dS.$$

When applying formula (16) to solve the Dirichlet problem for Laplace's equation, the key lies in finding the Green's function $G(M, M_0)$ in formula (15), where v is the solution to the following special Dirichlet problem:

$$\begin{cases} \Delta v = 0, & (x, y) \in D \\ v|_C = \left(\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} \right)\Big|_C \end{cases}$$

The Green's function determined by this function v is called the Green's function for the first boundary value problem. (For some special regions, such as circular regions and half-planes, the Green's function can be obtained.)

1.1 Properties of Green's function

Theorem 1.1. *The Green's function $G(M, M_0)$ satisfies Laplace's equation everywhere except at the point $M = M_0$. When $M \to M_0$, $G(M, M_0)$ tends to infinity, and its order is the same as that of $\frac{1}{r_{MM_0}}$.*

Theorem 1.2. *The Green's function $G(M, M_0)$ is identically equal to 0 on the boundary Γ.*

These two Theorems precisely describe

$$\begin{cases} \Delta G = -\delta(r - r_0) \\ G|_\Gamma = 0 \end{cases}$$

IDEAS: We $\boxed{\text{hope}}$

$$-\frac{1}{4\pi}\int_{\partial\Omega}(-B+A)\,dS=0, \qquad (*)$$

where two functions A, B to be determined.

Then

$$u(M_0)=-\frac{1}{4\pi}\int_{\partial\Omega}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-B+\boxed{A-\frac{1}{r}\frac{\partial u}{\partial n}}\right]dS \qquad (**)$$

$\boxed{\text{have common term } \frac{\partial u}{\partial n}}$ $\to$ $\underbrace{\text{take } A=4\pi v\frac{\partial u}{\partial n} \text{ to eliminate } \frac{1}{r}\frac{\partial u}{\partial n}}_{\text{This requires}}$

$$\boxed{v|_{\Gamma}=\frac{1}{4\pi r_{MM_0}}} \qquad (1)$$

Since $A=4\pi v\frac{\partial u}{\partial n}$, $(*)\Rightarrow \int_{\partial\Omega}-B+4\pi v\frac{\partial u}{\partial n}\,dS=0$

In view of symmetries

$\boxed{\text{Aesthetic criterion}}$

take $B=4\pi u\frac{\partial v}{\partial n}$

$$4\pi\int_{\partial\Omega}-u\frac{\partial v}{\partial n}+v\frac{\partial u}{\partial n}\,dS=0$$

$\|$ Green's second formula

$$4\pi\int_{\Omega}-u\Delta v+v\cancel{\Delta u}^{\,0}\,dS=0$$

We need v is also harmonic, i.e. $\boxed{\Delta v=0}$! (2)

Green 1st function

Therefore, we need take v as below

$$\begin{cases}\Delta v=0\\ v|_{\Gamma}=\dfrac{1}{4\pi r_{MM_0}}\end{cases} \xRightarrow{A,B\to(**)} \boxed{u(M_0)=-\int_{\partial\Omega}u(M)\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}-v\right)dS}$$

bdry condi. f

denoted as G Green 2nd function.

Figure 3: Ideas of Green functions

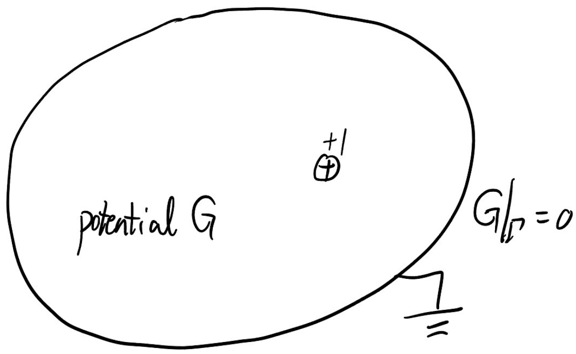

Figure 4: Physical meaning of Green functions G