Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 1-2

1 Heat Conduction Equation

- **Main Topic**: The heat equation is the second of the three key equations in the course, alongside the wave equation and the potential equation. The heat equation differs from the wave equation, and their solutions have fundamentally different properties, necessitating separate study.
- **Physical Meaning**: The heat equation models the transfer of heat. For example, in a classroom, when people enter, they radiate heat, and temperature differences create heat flow from high to low temperature, as described by the second law of thermodynamics (entropy increase).
- **Motivation**: The key question is how the temperature changes over time in response to heat transfer. This dynamic process requires an equation to model the change of temperature, which depends on both space (3D) and time.
- **Approach to Formulating the Equation**:
  - **Selection of Represention Variables**: Temperature $T(t, x)$ is chosen as the primary variable because it represents the physical phenomenon we are interested in (heat transfer).
  - **Choice of Local vs Global Perspective**: A global (overall) perspective is used, as we are interested in energy conservation across the entire system, not just at a point.
  - **Physical Law**: Energy conservation is the relevant physical law, as heat transfer involves the movement of energy.
  - **Ideal Assumption**: The empirical Fourier's Law of heat conduction (in the level of hooke's law, so as assumption instead of physical law) is assumed and use this empirical formula as an assumption. This law expresses that the heat flow is proportional to the temperature gradient.

Starting from the heat conduction problem in an object G, we derive the heat conduction equation. If the temperature within the object varies, heat flows from areas of higher temperature to areas of lower temperature. The temperature at any point in the object at a given time is represented as $u(t, x, y, z)$.

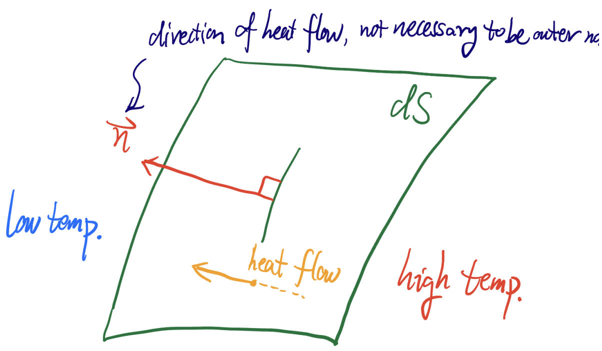
- **Fourier's Law**: The propagation of heat satisfies Fourier's experimental law (see Fig.1a): The heat dQ flowing through an infinitesimal area dS in an infinitesimal time dt is proportional to the directional derivative of the temperature u along the normal direction of the surface dS[1], i.e.,

$$dQ = -k(x, y, z)\frac{\partial u}{\partial n} dSdt = -k(x, y, z) n \cdot \nabla u dSdt = -k(x, y, z)\nabla u \cdot d\vec{S}dt \quad \Leftrightarrow \quad d\vec{q} = -k\nabla u$$

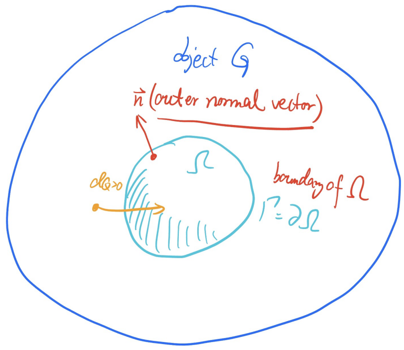
  where $k(x, y, z)$ is the thermal conductivity of the material at the point (x, y, z) and is positive. When the material is homogeneous and isotropic, k is a constant. The negative sign in the formula indicates that heat always flows in the direction opposite to the temperature gradient[2] (from higher temperature to lower temperature).

[1] Note $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma \xlongequal{\text{this language is more compact and concise}} n\cdot\nabla u$, $(\cos\alpha, \cos\beta, \cos\gamma)$ is direction cosine and it is a **unit** directional vector.

[2] The direction of the gradient ∇f is the direction in which the function $f(x, y, z, \dots)$ increases most rapidly. This is why the gradient is often referred to as the **direction of steepest ascent.**



(a) heatflow1



(b) heatflow2

- Fourier's law is typically introduced in thermodynamics or heat studies, but it is not often discussed in the context of equilibrium thermodynamics, which focuses on systems in a state of balance (e.g., Carnot engines).
- The current context involves **non-equilibrium thermodynamics**[3], which deals with systems that are not in thermal equilibrium[4].
- **Empirical Formula**: Fourier's law is an empirical formula that describes heat flow and temperature gradients.

1.1 Derivation of Heat Equations

Consider the heat flux entering a region Ω and see Fig.1b.

- Heat flux $d\vec{q}$ is considered to flow **inward** into a region.
- The direction of heat flow is **opposite** to the outward normal vector of the boundary.

1.1.1 Energy Conservation in Heat Transfer

- The heat flux entering a region Ω increases the temperature of the material inside.
- According to the **principle of energy conservation**, the heat energy entering the region must be equal to the increase in internal energy of the system.

[3] Non-equilibrium thermodynamics is essential for understanding processes like **thermoelectric effects**, where a temperature difference generates electrical energy.

[4] In non-equilibrium thermodynamics, a key principle is that:

* **Thermodynamic Flux is Proportional to Thermodynamic Force**.
* **Thermodynamic Flux**: Refers to the flow of a physical quantity such as heat, particles, or momentum.
* **Thermodynamic Force**: The driving force that causes the flux, such as a temperature **gradient** (for heat conduction), concentration **gradient** (for diffusion), or velocity **gradient** (for momentum transfer). For example, gravitational force and electric field intensity are both the negative gradient of a potential.

The relationship can be expressed as:

$$J = -L \cdot X$$

where:

* J is the thermodynamic flux (e.g., heat flow, mass flux).
* X is the thermodynamic force (e.g., temperature **gradient**, concentration **gradient**).
* L is the phenomenological coefficient (often called the Onsager coefficient), which determines the proportionality.

This principle is foundational in understanding non-equilibrium processes, such as:

* Heat conduction
* Diffusion
* Electrical currents in thermoelectric materials

- The **specific heat capacity** (denoted as c) is a physical property of a material that quantifies the amount of heat energy required to raise the temperature of a unit mass of the substance by one degree Celsius (or one Kelvin). It is defined as:

$$c = \frac{1}{m}\frac{dQ}{dT} \quad \Rightarrow \quad dQ = cmdT \quad \Rightarrow \quad \Delta Q = cm\Delta T.$$

  where:

  - c is the specific heat capacity (in units of $\mathrm{J} \cdot \mathrm{kg}^{-1} \cdot \mathrm{K}^{-1}$),
  - m is the mass of the substance (in kilograms),
  - dQ is the infinitesimal amount of heat energy added to the substance (in joules),
  - dT is the infinitesimal change in temperature (in Kelvin or Celsius).

- Mathematically, the total heat entering the region is given by:

$$\int_{t_1}^{t_2} \int_{\Gamma} dQ = \int_{t_1}^{t_2} \int_{\Gamma} k\mathbf{n} \cdot \nabla u \, dSdt = \int_{\Omega} c(x,y,z)\rho(x,y,z)(u(t_2,x,y,z) - u(t_1,x,y,x))\, dV \qquad (1)$$

Let us explore possible approaches to simplify (1).

1. Objective

- We aim to derive a **differential equation** from a given **integral equation**.
- The key approach is to **eliminate the integral sign** by transforming the equation into a local form.

2. Strategy for Simplification

- Simplify both terms on the left-hand and right-hand sides separately.
- HOPE[5]: Attempt to merge the integrals into a single integral equation.
- If successful, apply the fundamental theorem of calculus to remove the integral.

3. Handling Different Types of Integrals

- The left-hand side consists of a **double spatial integral** and a **single time integral**.
- The right-hand side consists of a **triple spatial integral** with no time integral.
- To unify these, convert the double integral into a triple integral, then introduce a time integral where necessary.

4. Applying Integral Theorems

- Use mathematical transformations to convert between integral forms.
- The most fundamental theorems for this process are:
  - **Gauss's theorem** (Divergence theorem)

    Theorem 1.1. *Let Ω be a compact region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and let $\mathbf{F}$ be a continuously differentiable vector field on Ω. Then:*

$$\int_{\Omega} \nabla \cdot \mathbf{F}\, dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS \left(= \int_{\partial\Omega} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\, dS \right),$$

    where:

    * $\nabla \cdot \mathbf{F}$ *is the divergence of* $\mathbf{F} = (P,Q,R)$,

[5] "Hope" drives mathematical thinking, as it formulates conjectures from clues that we can explore.

* dV *is the volume element in* $\mathbb{R}^n$*,*
* $\mathbf{n}$ *is the outward unit normal to the surface* $\partial\Omega$*,*
* dS *is the surface element on* $\partial\Omega$*.*

- **Green's theorem**
- **Stokes' theorem**

- These three theorems are actually special cases of a unified theorem in vector calculus, Stokes' theorem[6] (see [Lee09, §9.1]).

- **Steps to Derive the Heat Equation (for simplicity if k =constant)**[7]:

$$\begin{array}{ccc}
\underbrace{\int_{t_1}^{t_2}\int_{\partial\Omega} k\mathbf{n}\cdot\nabla u\,dSdt} & \xlongequal{\text{enery conservation}} & \underbrace{\int_{\Omega} c(x,y,z)\rho(x,y,z)(u(t_2,x,y,z)-u(t_1,x,y,x))\,dV} \\
\Downarrow\text{Gauss} & & \Downarrow\text{N-L} \\
\underbrace{\int_{t_1}^{t_2}\int_{\Omega} k\nabla\cdot\nabla u\,dVdt} & & \underbrace{\int_{\Omega} c(x,y,z)\rho(x,y,z)\int_{t_1}^{t_2}\frac{\partial u}{\partial t}\,dtdV} \\
\Downarrow & & \Downarrow \\
\int_{t_1}^{t_2}\int_{\Omega} k\Delta u\,dVdt & \xlongequal{\text{enery conservation}} & \int_{t_1}^{t_2}\int_{\Omega} c(x,y,z)\rho(x,y,z)\frac{\partial u}{\partial t}\,dVdt
\end{array}$$

Then

$$\int_{t_1}^{t_2}\left(\int_{\Omega} c(x,y,z)\rho(x,y,z)\frac{\partial u}{\partial t} - k\Delta u\right) dVdt = 0 \tag{2}$$

- We ask a question:

Problem 1.1. *Consider a case where the integral of a function equals zero, i.e.,*

$$\int_{\Omega} f(x)\,dx = 0$$

where Ω *is an* ***arbitrary*** *region. Can we conclude that* $f(x) = 0$ *for all* x*?*

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[6] That is,

Theorem (Stokes' Theorem)**.** *Let* M *be a smooth manifold with boundary* ∂M*, and let* ω *be a smooth differential form on* M*. Then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where $d\omega$ *is the exterior derivative of* ω*, and the integrals on both sides are taken with respect to the appropriate volume forms on* M *and* ∂M*, respectively.*

[7] Mathematical analysis often involves two fundamental perspectives:

1. **Integral (Global) Perspective**
   - Typically uses the **Newton-Leibniz formula**:
   $$\int_a^b f'(x)\,dx = f(b) - f(a).$$
   - This perspective is useful for analyzing overall changes and accumulated quantities.
2. **Differential (Local) Perspective**
   - Often relies on the **Mean Value Theorem (MVT)** and **Taylor Expansion**.
   - The Mean Value Theorem states that for a differentiable function $f(x)$, there exists a point c in (a, b) such that:
   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
   - Taylor expansion approximates functions locally as:
   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots.$$
3. **Summary**
   - **Integral perspective** is used for global analysis, leveraging the Newton-Leibniz formula.
   - **Differential perspective** focuses on local behavior, using the Mean Value Theorem and Taylor expansion.

- A simple **counterexample** exists: odd functions (e.g., sine and cosine) integrated over a symmetric interval can result in zero, yet the function itself is not zero.

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{for odd functions, but} \quad f(x) \neq 0 \,\text{for all}\, x$$

- The **key difference** here lies in the fact that in (3) and Problem 1.1, the integration is done over a region Ω which can be **arbitrarily chosen**.
- If a function's integral equals zero **over any region** Ω, we can conclude that the function must be **identically zero**. This is because (**a proof by contradiction**):
  * If $f(x)$ is non-zero at some point, say $f(x_0) > 0$, then in a small neighborhood around x_0, the function would remain positive.
  * Integrating over this small neighborhood would yield a positive result, leading to a contradiction with the integral being zero.
- Thus, the key insight is the **arbitrary selection** of the integration region Ω. Once we allow for arbitrary regions and time intervals, the result follows that the function must be zero everywhere in the chosen domain.

• Thus, (3) yields the heat equation (for simlicity, taking ρ, k, c to be constants for homogeneous matter):

$$c\rho\frac{\partial u}{\partial t} = k\Delta u \quad \Leftrightarrow \quad \frac{\partial u}{\partial t} = a^2\Delta u \quad \text{where } a^2 := \frac{k}{c\rho} \quad \text{(homogeneous)} \tag{3}$$

• Heat: $\frac{\partial u}{\partial t} = a^2\Delta u$; Recall wave $\frac{\partial^2 u}{\partial t^2} = a^2\Delta u$ and compare them:

- **Similarities and main difference:**
  * Both equations are similar in form, involving spatial derivatives.
  * The wave equation involves **second-order time derivatives**, while the heat equation involves a **first-order time derivative**.
- **Wave Equation:**

$$u_{tt} - a^2 u_{xx} = 0$$

- The wave equation models **oscillations or vibrations** and the solution is often represented by **trigonometric functions**.
- **Heat Equation:**

$$u_t - a^2 u_{xx} = 0$$

- The heat equation models diffusion and the solution involves **exponential decay**, showing how **heat dissipates over time**.

1.2 Heat Conduction Equation with Internal Heat Sources

• **Non-Homogeneous Term:**

- Consider a function $f(x, y, z, t)$, where f depends on the independent variables x, y, z, t. This is called a non-homogeneous term.
- Similar to ODEs, the non-homogeneous term f represents an external source or forcing function.

• **Heat Source:**

- In the case of a non-homogeneous heat equation, the right-hand side term $f(x, y, z)$ represents a heat source.
- When there is no heat source, the heat flow into a region balances the heat leaving it, with no additional heating from the inside.
- With a heat source, the heat flow into the region is augmented by the internal heat generation, so the total heat entering the region is the sum of the **external heat** and the **heat generated** within the domain.

- **Conclusion:**
  - The derivation for the non-homogeneous case follows the same steps as the homogeneous case, but with an **additional term** representing the **internal heat source**.

If the body contains **internal heat sources** (e.g., due to electric currents or chemical reactions), let $F(x, y, z, t)$ represent the heat source density (the amount of heat generated per unit volume per unit time). The heat generated within Ω from t_1 to t_2 is:

$$Q_3 = \int_{t_1}^{t_2} \int_{\Omega} F(x, y, z, t) dV dt$$

By conservation of energy, the total heat balance is:

$$Q_1 + Q_3 = Q_2$$

Combining the previous equations:

$$\int_{t_1}^{t_2} \int_{\Omega} [\nabla \cdot (k\nabla u) + F(x, y, z, t)] \, dV dt = \int_{\Omega} c\rho[u(x, y, z, t_2) - u(x, y, z, t_1)] dV$$

Rearranging and using the fundamental theorem of calculus:

$$\int_{t_1}^{t_2} \int_{\Omega} \left[c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - F(x, y, z, t) \right] dV dt = 0$$

Since Ω is **arbitrary** and the integrand is **continuous**, we obtain the **non-homogeneous** heat conduction equation:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (k\nabla u) + F(x, y, z, t)$$

For a homogeneous material with $a^2 = \frac{k}{c\rho}$, the equation becomes:

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(x, y, z, t)$$

where $f(x, y, z, t) = \frac{F(x,y,z,t)}{c\rho}$.

1.3 Initial and Boundary Conditions for the Heat Conduction Equation

1.3.1 Initial Conditions

The initial condition describes the temperature distribution within the body at the initial time $t = 0$:

$$u(x, y, z, 0) = \phi(x, y, z)$$

where $\phi(x, y, z)$ is a known function representing the initial temperature distribution.

- The number of required initial conditions depends on the order of the time derivative.
- Since the heat equation involves a first-order time derivative (u_t), only the function $u(x, t)$ itself needs to be specified at $t = 0$.

1.3.2 Boundary Conditions (Study: Physical meaning, math form and names)

The boundary conditions describe the physical conditions at the boundary of the body. There are three main types of boundary conditions:

- **First Type (Dirichlet Boundary Condition)**:
  - Specifies the function value u on the boundary.
  - **Physically**, this means the temperature (or other dependent variable) on the boundary is given.

- **Mathematically**: if the boundary surface S of the body G has a known temperature distribution $\mu_1(t)$:

$$u(x,y,z,t)\big|_S = \mu_1(t)$$

.

- If the boundary is kept at a constant temperature 0, this is called a homogeneous Dirichlet condition:

$$u(x,y,z,t)\big|_S = 0.$$

- **Second Type (Neumann Boundary Condition)**:
  - Specifies the normal derivative of u on the boundary.
  - In a 3D setting, this means providing the directional derivative of u along the outward normal $\mathbf{n}$.
  - **Mathematically:**

$$\frac{\partial u}{\partial n}|_S = \mu_2(x,y,z,t).$$

  - **Physically**[8], The heat flux q **across the boundary** S per unit area and per unit time is given. According to Fourier's law,

$$-k\frac{\partial u}{\partial n}\bigg|_S = q \quad \Rightarrow \frac{\partial u}{\partial n}|_S = \mu_2(x,y,z,t)$$

  where $\mu_2(x,y,z,t) = -q/k$ is a known function defined on S and $t \geq 0$.

  - If the boundary is **insulated** (no heat flux), this is called a homogeneous Neumann condition:

$$\frac{\partial u}{\partial n}\bigg|_S = 0.$$

- **Third Type Boundary Condition (Robin Boundary Condition)**
  - A linear combination of Dirichlet and Neumann conditions.
  - Mathematically:

$$\frac{\partial u}{\partial n}\bigg|_S + \alpha u\big|_S = \mu_3(t)$$

  where α is a heat transfer coefficient.

  - Consider a body in contact with a surrounding medium. Let u be the temperature of the body and u_1 be the temperature of the surrounding medium. If the temperatures are different, heat exchange occurs at the boundary S. According to **Newton's law of cooling**[9], the heat flux

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[8] To understand the physical meaning, one only needs to interpret what $\frac{\partial u}{\partial n}$ represents.

[9] **Heat Transfer in Continuous and Discontinuous Media**

Fourier's Law of Heat Conduction

* Describes heat conduction in a **continuous medium**.
* States that heat flux $\mathbf{q}$ is proportional to the temperature gradient:

$$d\mathbf{q} = -k\nabla T$$

  where k is the thermal conductivity.
* Applicable within the **same medium**, assuming a **smooth temperature variation**.

Newton's Law of Cooling

* Describes heat transfer between **two different media** (e.g., a solid surface and surrounding air).
* The temperature is generally **discontinuous** at the interface.
* Since temperature is **not continuous**, **derivatives cannot** be directly applied.
* Empirical law states:

$$\Delta\mathbf{q} = -h(u - u_1)$$

  .

Mathematical Basis

* Both Fourier's and Newton's laws can be understood as first-order **Taylor expansions**.
* Fourier's law corresponds to a spatial Taylor expansion within a continuous medium.
* Newton's law is based on an empirical relationship due to the discontinuity at the interface.

through the boundary is proportional to the temperature difference:

$$dQ = -h(u - u_1)dtdS$$

where h is the **heat transfer coefficient**.

Since heat cannot accumulate on the surface of an object, consider an infinitely close closed surface S_1 inside the object that is tangent to the object's surface S. The heat flux through surface S_1 should be equal to the heat flux through surface S (see Fig.2).

The heat flowing through surface S_1 is given by $dQ = -k\frac{\partial u}{\partial n}dSdt$, which leads to the relationship:

$$-k\frac{\partial u}{\partial n}dSdt = h(u - u_1)dSdt \implies k\frac{\partial u}{\partial n} + hu = hu_1$$

This can be rewritten as:

$$\left(\frac{\partial u}{\partial n} + \alpha u\right)\Bigg|_S = \mu_3(x, y, z, t)$$

where $\alpha = \frac{h}{k}$, and $f_3(x, y, z, t)$ is a known function defined on $(x, y, z) \in S, t \geq 0$.

This type of boundary condition is particularly useful when the body is in contact with a medium that can exchange heat.

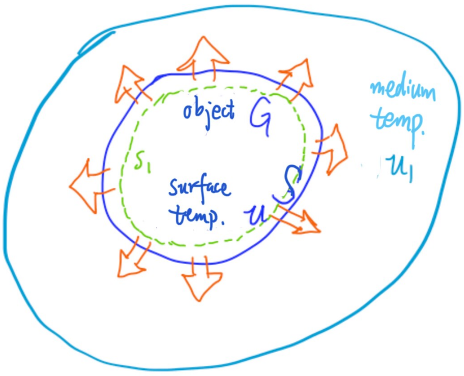

Figure 2: Robin condition

1.4 Laplace's Equation and Boundary Value Problems

1.4.1 Laplace's Equation

Laplace's equation (or **harmonic equation**) describes the steady-state distribution of a physical quantity, such as temperature or potential. In three dimensions, it is written as:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

A function $u \in C^2$ that satisfies Laplace's equation is called a **harmonic function**. That is

$$\begin{cases} (1)\ u \in C^2 \\ (2)\ \Delta u = 0 \end{cases} \tag{4}$$

Relationship with Other Equations:

- Laplace's equation can be derived by observing heat and wave equations.
- The **key difference** between Laplace's equation and the heat or wave equation is the **absence of time dependence** $\frac{\partial u}{\partial t} \equiv 0$.
- For example, in the heat equation, the temperature changes over time, while in Laplace's equation, the temperature is independent of t (steady-state).

Physical Interpretations:
1. Wave Interpretation

- Laplace's equation can be seen as describing the state of a damped wave or oscillation that reaches equilibrium.
- As time progresses, the oscillations decay, and eventually, they stop, reaching a balanced or steady state. This corresponds to the static nature of Laplace's equation.

2. Heat Interpretation

- Laplace's equation also describes heat conduction at equilibrium.
- This steady-state condition, where temperature does not change over time, results in the Laplace equation.

1.4.2 Poisson's Equation

Poisson's equation is a non-homogeneous version of Laplace's equation, where a source term $f(x, y, z)$ is present[10]:

$$\Delta u = -f(x, y, z)$$

1.4.3 Physical Interpretations:

The physical phenomenon of electrostatic field and potential distribution can also be described by the Laplace equation or Poisson's equation.

The electrostatic field is denoted as $\vec{E}$ and the charge density is $f(x, y, z)$, and by Gauss's law:

$$\underbrace{\iint_{\Gamma} \vec{E} \cdot \vec{n} dS}_{\Big\Downarrow \text{Gauss} \atop \iiint_{\Omega} \nabla \cdot \vec{E}\, dV} = \frac{1}{4\pi\varepsilon_0} \iiint_{\Omega} f\, dV$$

Assume $u(x, y, z)$ is the electric potential, then $\vec{E} = -\nabla u = -(u_x, u_y, u_z)$.

$$-\iiint_{\Omega} \nabla^2 u\, dV = \frac{1}{4\pi\varepsilon_0} \iiint_{\Omega} f\, dV \Rightarrow \Delta u = -\frac{f}{4\pi\varepsilon_0}, \quad \text{(by the arbitrariness of } \Omega)$$

If there is no charge distribution in G, $f = 0$, then u satisfies the Laplace equation:

$$\Delta u = 0.$$

Remark 1.1. *The Maxwell's equations can also derive the wave equation - electromagnetic wave.*

1.4.4 Boundary Value Problems

Boundary value problems for Laplace's equation involve finding a solution $u(x, y, z)$ that satisfies the equation within a domain Ω and meets certain boundary conditions on the boundary Γ:

- **First Boundary Value Problem (Dirichlet Problem)**:
  Given a continuous function f on the boundary Γ of a certain region Ω in space, it is required that the function $u(x, y, z)$ is continuous on the closed region $\Omega + \Gamma$ and harmonic within Ω, and coincides with the given function f on the boundary Γ.

$$\underbrace{u}_{\text{conti. in } \Omega+\Gamma;\text{harmonic in } \Omega} \Big|_{\Gamma} = \underbrace{f}_{\text{conti. on } \Gamma}$$

[10] The negative sign represents the eigenvalues of Δ operator is negative, recalling $\mathscr{A}u = \lambda u$ and letting $\mathscr{A} = \Delta$. See Chap. 2.

- **Second Boundary Value Problem (Neumann Problem)**:
  Given a continuous function f on the boundary Γ of a certain region Ω in space, it is required that the function $u(x, y, z)$ is continuous on the closed region $\Omega + \Gamma$, harmonic within Ω, and has a normal derivative $\frac{\partial u}{\partial n}$ existing on the boundary Γ.

$$\underbrace{\frac{\partial u}{\partial n}}_{\text{conti. in }\Omega+\Gamma;\text{harmonic in }\Omega;\partial_n u\text{ exists}} \Big|_{\Gamma} = \underbrace{f}_{\text{conti. on }\Gamma}$$

Remark 1.2.
- *The function u must be continuous not only inside the domain Ω but also on the boundary $\partial\Omega$.*
- *If u is discontinuous at the boundary, the boundary condition would be meaningless since it could be arbitrarily chosen. This would render the problem ill-posed.*
- *The boundary conditions must ensure both the continuity of u and the existence of its derivatives at the boundary, especially for problems involving second-type boundary conditions.*
- *This continuity condition is subtle but essential. Without it, the boundary conditions would not impose proper constraints on the problem, making the solution meaningless.*

1.5 Basic Concepts and Knowledge

- **Reason for Defining Concepts:**
  - When discussing equations, it was mentioned that equations with specific characteristics require particular methods for solving.
  - The concepts defined below describe these characteristics.
  - Understanding these concepts helps determine the appropriate methods to apply.

Definition 1.2 (PDE)**.** *An equation containing independent variables, unknown functions and their partial derivatives with respect to independent variables is called a partial differential equation (PDE), e.g.,*

$$F(\partial^\alpha u(x), \partial^\beta u(x), \dots, \partial u(x), u(x), x) = 0, \tag{5}$$

where α and β are multi-indices.

- Examples:
  - Second Order:
    $$U_{tt} = a^2 U_{xx}$$
  - Homogeneous:
    $$U_{xx} + U_{yy} = 0$$
  - Non-homogeneous:
    $$U_{xxy} + 2xU_{yy} + yU = xy$$
    $$(U_x)^2 + U_y = 8x^2$$

Definition 1.3 (Order)**.** *The highest order of the derivative of the unknown function in the PDE is called the "order of the PDE".*

Definition 1.4 (Linearity)**.** *If each term in the PDE is linear with respect to the unknown function and its partial derivatives (including higher-order derivatives), it is called a "linear PDE".*

$$F \textit{ is linear on } (\partial^k u, \partial^{k-1} u, \dots, \partial u, u)$$

i.e., it has the form:

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u = f(x),$$

where $\mathcal{A} = (a_\alpha(x), \dots, a(x), a_y)$.

$$\mathcal{A}u = f(x) \quad \textit{where} \quad u = (\partial^k u, \dots, \partial u, u).$$

Remark 1.3.
- *A **linear PDE** refers to an equation where the unknown function and its derivatives appear in a linear form.*
- *In this context, the function F is a linear function of the unknown function u and its derivatives.*
- *The linearity means that the equation involves linear combinations of u and its derivatives, with no higher powers or nonlinear terms of u.*
- *Specifically, F is a linear function of u and its partial derivatives, but not necessarily linear in the independent variables.*
- *The equation can often be expressed in matrix form, where the unknowns are vector functions (i.e., column vectors), and the coefficients depend on the independent variables.*

Definition 1.5 (Classical Solution)**.** *A function is called a **classical solution** of a partial differential equation if it **has all the continuous partial derivatives** required by the equation and **satisfies** the equation.*

Definition 1.6 (Free Term, Source Term[11], Nonhomogeneous Term)**.** *A term in a partial differential equation that does not contain the unknown function or its derivatives (i.e., only include function of independent variables). For example, in the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z)$$

$f(x, y, z)$ is the free term. If $f(x, y, z) = 0$, the equation is homogeneous; otherwise, it is non-homogeneous.

Remark 1.4.
- *The free term refers to a term in an equation that does not contain the unknown quantity.*
- *Specifically, it is a term that does not involve the unknown function or its derivatives.*
- *In other words, this term contains only the independent variables.*
- *Such a term is called the "free term" because it is independent of the unknown function and its derivatives.*

Remark 1.5.
- *We previously discussed boundary conditions as well.*
- *If a boundary condition is equal to zero, it is called a **homogeneous boundary condition**.*
- *If the right-hand side of the boundary condition contains a function of the independent variables, it is called a **non-homogeneous boundary condition**.*
- *These concepts of homogeneous and non-homogeneous may seem purely mathematical, but as we proceed to solve equations later in this class, you will see that they determine the method you should choose to solve the equation. Thus, these concepts are very important.*

Definition 1.7 (Well-Posedness of Boundary Value Problems)**.** *The well-posedness of a boundary value problem refers to the existence, uniqueness, and stability of the solution. A problem is well-posed if:*

1. *(Existence) A solution exists.*
2. *(Uniqueness) The solution is unique.*
3. *(Stability) The solution depends continuously on the initial and boundary data.*

Remark 1.6.
- *If the three conditions are satisfied, it means that the problem is given just the right conditions, neither too strong nor too weak.*
- *If the conditions are too strong, there may be no solution, violating the existence of the solution.*
- *If the conditions are too weak, there may be many solutions, so the solution is not unique.*

[11] This is because the free term often originates from external forces, heat sources.

- *Stability means that if you have found a solution to the problem and know a solution, and if the initial conditions deviate very slightly from the initial values of this solution, then the solution will always deviate very slightly from the value of the solution you obtained.*
- *In other words, if the input error is very small, the output error is also very small. This is what we commonly call stability.*
- *This is a very good condition because in practical work, you will often involve measurements, and you may calculate a solution through theoretical derivation.*
- *Then, you need to measure some things through experiments, and measurements will inevitably introduce errors.*
- *If there is no stability, then even if the input error is very small, the error may deviate significantly from the solution as time progresses.*
- *In that case, the actual measured solution may differ greatly from the theoretical value, making the theoretical value less reliable.*
- *Therefore, stability is also a very good requirement.*
- *However, in recent years, it has been found that instability is also very common in our real life.*
- *For example, if everything were stable, the universe would not have produced the Earth or humans.*
- *Theoretical calculations show that the universe would need much longer than its current age to produce the Earth and humans.*

1.6 Superposition Principle

- This principle of superposition runs through every solution method we will discuss later.
- I want to emphasize that it is called the **linear superposition principle**. The word "linear" is important because we are dealing with linear equations in this book.

Consider a second order linear partial differential equation of the form:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f_i, \quad (i = 1, 2, \cdots) \tag{6}$$

where A to F, and f_i are known functions of x and y in some region. If u_i is a solution corresponding to the source term f_i, if the series

$$u = \sum_{i=1}^{\infty} c_i u_i \tag{7}$$

converges, where $c_i(i = 1, 2, \cdots)$ are arbitrary constants and it can also be differentiated term by term twice, then the series (7) is a solution to the following equation

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = \sum_{i=1}^{\infty} c_i f_i$$

In particular, when the free term $f_i = 0$ in equation (1), the corresponding homogeneous equation is

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0. \tag{8}$$

If $u_i(i = 1, 2, \cdots)$ are solutions to equation (8), then the series (7) is also a solution to equation (8).

Ideas for the proof:

1. Let
$$\mathcal{A} = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F \quad \text{(linear operator)}$$
This is a linear operator.
2. From (6), we have $\mathcal{A}u_i = f_i$; thus, summing up gives $\sum \mathcal{A}u_i = \sum f_i$; hence, $\mathcal{A}(\sum u_i) = \sum f_i$.
3. Letting $u = \sum u_i$, $\mathcal{A}u = f$.

1.7 Classification of Second-Order PDEs

Question 1.1. *Why classify?*

A. *Phenomena vary greatly. By classifying, we capture commonalities. By studying one problem with commonalities, we can understand the results caused by these commonalities in all such problems.*

A general second-order linear partial differential equation has the following form:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = f, \tag{9}$$

where $a_{11}, a_{12}, a_{22}, b_1, b_2, c, f$ etc., are real functions of the independent variables x, y in the region Ω, and it is assumed that they are continuously differentiable.

Eq. (9) can be rewritten as

$$(\partial_x, \partial_y)\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=:M}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u + (b_1, b_2)\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u + cu = f \tag{10}$$

If at some point (x_0, y_0) in the region Ω,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M > 0,$$

then equation (9) is said to be **hyperbolic at the point** (x_0, y_0); if equation (9) is hyperbolic at every point in the region Ω, then equation (9) is said to be **hyperbolic** in this region.

If at some point (x_0, y_0) in the region Ω,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M = 0,$$

then equation (9) is said to be **parabolic at the point** (x_0, y_0); if equation (9) is parabolic at every point in the region Ω, then equation (9) is said to be **parabolic** in this region.

If at some point (x_0, y_0) in the region Ω,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M < 0,$$

then equation (9) is said to be **elliptic at the point** (x_0, y_0); if equation (4) is elliptic at every point in the region Ω, then equation (9) is said to be **elliptic** in this region.

- Hyperbolic equations correspond to hyperbolas, and their matrix form has a negative determinant (e.g., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola).
- Elliptic equations correspond to ellipses, and the matrix determinant is positive.
- Parabolic equations correspond to parabolas, where the determinant is zero.
- The classification depends primarily on the second-order terms of the PDE, specifically the determinant of the coefficient matrix.
- The matrix M discussed is known as a **metric matrix** in mathematics. It determines the structure of spacetime.
- The determinant of the matrix is negative for **hyperbolic** form, corresponding to **Lorentz geometry**.
- The classification into **hyperbolic, parabolic, and elliptical** corresponds to the conic sections and is related to **quadratic forms** (see (10)).
- **Quadratic form normalization** using congruent matrices is a process taught in linear algebra. This process can also be applied to **differential equations** to simplify a second order linear PDE.

An analogy:

The **hyperbola equation** is $a^2x^2 - b^2y^2 = 1$. Using the quadratic form:

$$(x, y) \quad \underbrace{\begin{pmatrix} a^2 & 0 \\ 0 & -b^2 \end{pmatrix}}_{=:M \Rightarrow \det M = -a^2b^2 < 0} \quad \begin{pmatrix} x \\ y \end{pmatrix}.$$

Generalize above hyperbola and take $x \to \partial_x$ and $y \to \partial_y$ (in linear algebra, x, y can be "any thing" which obeys certain linear relations, thus they can be differential operators). Equation classification depends on the highest degree:

$$(\partial_x, \partial_y) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

If $\det M < 0$, i.e., $\Delta = -\det M > 0$, then it is a hyperbola.

In addition, $\det M > 0$, it is an ellipse. If $\det M = 0$, it is a parabola, since the parabola has only one quadratic term.

$$\underbrace{ax^2 - by^2 = 1 \quad \text{a hyperbola}}$$

$$\Downarrow$$

$$\underbrace{ax^2 - by^2 = (x, y) \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{a hyperbolic polynomial}}$$

$$\Downarrow$$

$$\underbrace{x, y \text{ can be any math objects, indeterminate}}$$

$$\Downarrow$$

$$a\partial_x^2 - b\partial_x^2 = (\partial_x, \partial_y) \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

Details......To be continued......

1.7.1 Examples of PDE Classification

Wave Equation (Hyperbolic)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Here, $\Delta = a^2 > 0$, so it is hyperbolic.

Heat Equation (Parabolic)

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Here, $\Delta = 0$, so it is parabolic.

Laplace's Equation (Elliptic)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Here, $\Delta = -1 < 0$, so it is elliptic.

References

[Lee09] Jeffrey M. Lee. *Manifolds and differential geometry*. Number Volume 107 in Graduate Studies in Mathematics. American Mathematical Society, Providence, Rhode Island, 2009. Includes bibliographical references and index. Description based on print version record.