Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 2-1

1 Fourier Series

Summary of Key Points on Fourier Series

- **Chapters Overview:** Chapters 2 and 5 of the book use **series solution methods** based on Fourier series.
- **New Perspective:** A new viewpoint is introduced that differs from previous calculus. It is recommended to compare this perspective with previous ones.
- **Representation of Fourier Series:**
  - Fourier series can be expressed using trigonometric functions.
  - Alternatively, they can be represented by exponential functions (using the imaginary unit i as in $e^{i\theta}$).
  - In this course, the trigonometric form is primarily used.
- **Applicability:** Fourier series are applicable only to **periodic functions**.
- **Conceptual Note:** The **idea** of **periodic extension** (due to above applicability) is critical for understanding subsequent material, even if it is not explicitly emphasized in the text.

Remember 4 things:

1. Analogous to linear expansion $y = \sum_{i=0}^{n} a_i e_i$
2. Odd and Even Functions
3. Orthogonality
4. Completeness and Inner Product Method for Finding Coefficients

1.1 Analogous to linear expansion

Suppose the function $f(x)$ with **period**[1] $2l$ can be expanded into a Fourier series, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

where the Fourier coefficients a_n, b_n satisfy

$$a_n = \frac{1}{l} \langle f(x), \cos \frac{n\pi x}{l} \rangle = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx \quad (n = 0, 1, 2, \cdots),$$

$$b_n = \frac{1}{l} \langle f(x), \sin \frac{n\pi x}{l} \rangle = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (n = 1, 2, 3, \cdots).$$

[1] The text does not explicitly state that the function is periodic; it is directly applied as such. However, this inherently involves an extension process, which introduces several important considerations. A clear understanding of this concept is essential for fully comprehending the subsequent material.

Analogous to linear expansion $\vec{\xi} = \sum_{i=0}^{n} a_i \vec{e_i}$

$$\vec{\xi} = a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n} = \sum_{i=1}^{n} a_i\vec{e_i}$$

where

$$\{\vec{e_i}\} \text{ is an } \textbf{orthonormal} \text{ basis.}$$

- Vector $\vec{\xi}$ $\leftrightarrow$Function $f(x)$;
- Basis $\{\vec{e_i}\}\leftrightarrow$ Basis $\{1, \sin, \cos, \cdots\}$.

To find the coefficients a_i, use the **inner product** as the **component**.

$$\langle\vec{\xi}, \vec{e_j}\rangle = \sum_{i=1}^{n} a_i\langle\vec{e_i}, \vec{e_j}\rangle = \sum_{i=1}^{n} a_i\delta_{ij}, \quad \text{(only orthogonality is needed!!!)}$$

$$\Rightarrow a_i = \langle\vec{\xi}, \vec{e_i}\rangle$$

Then

- Hope: A method for computing a_n.
- Question: To make such an analogy, further proof of **orthogonality** of the basis $\{1, \sin, \cos\}$ is also needed.
- **Orthogonality** need the **inner product**!
- Question: "**What is**" or "**how to define**" the inner product of functions?
  A: The **inner product** of two functions $f(x)$ and $g(x)$ over a period $[-l, l]$ is defined as:

$$\langle f, g\rangle = \int_{-l}^{l} f(x)g(x)\,dx$$

- Ideas for the above inner product: Compare $a_i = \langle\vec{\xi}, \vec{e_i}\rangle$ in linear algebra with $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$ in Fourier transformations. They are very similar. By comparing the two formulas, we find that the inner product is likely defined as the integral of the product of two functions. Based on this rough idea, we define it as above.
- Verify orthogonality of $\{1, \sin, \cos, \cdots\}$ according to this definitions.

1.2 Orthogonality

(1) Orthogonality: Trigonometric Function Series

The functions $\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \ldots, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, \ldots\}$ are orthogonal on the interval $[-l, l]$ (for any one period), that is:

$$\int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^{l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0,$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\,dx = \int_{-l}^{l} \cos\frac{n\pi x}{l}\,dx = 0.$$

Similarly,

$$\int_{-l}^{l} \sin\frac{(2m+1)\pi x}{2l}\sin\frac{(2n+1)\pi x}{2l}\,dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^{l} \cos\frac{(2m+1)\pi x}{2l}\cos\frac{(2n+1)\pi x}{2l}\,dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^{l} \sin\frac{(2m+1)\pi x}{2l}\cos\frac{(2n+1)\pi x}{2l}\,dx = 0,$$

(2) Trigonometric Product-to-Sum Formulas:

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Note

$$\int_{-\pi}^{\pi} \cos(\lambda x)\,dx = \frac{1}{\lambda}\int_{-\pi}^{\pi} \cos(\lambda x)\,dx = \frac{1}{\lambda}\left[\frac{\sin(\lambda x)}{\lambda}\right]_{-\pi}^{\pi} = \frac{2}{\lambda}\sin(\lambda\pi)$$

- For $m \neq n$:

$$\int_{-\pi}^{\pi} \cos(m+n)x\,dx = \frac{2}{m+n}\sin((m+n)\pi) = 0$$
$$\int_{-\pi}^{\pi} \cos(m-n)x\,dx = \frac{2}{m-n}\sin((m-n)\pi) = 0$$

- If $m = n$:

$$\int_{-\pi}^{\pi} \cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos(2mx))\,dx = \pi$$

$$\Rightarrow \|\cos(mx)\| = \sqrt{\pi}$$
$$\|\sin(mx)\| = \sqrt{\pi}$$

Remark 1.1. • *If we replace n with a half-integer (e.g., $n+\frac{1}{2}, n+\frac{3}{2}$, etc.), the resulting function set is still **orthogonal** (by similar calculations). That is, the functions*

$$\{1, \cos\frac{\frac{1}{2}\pi x}{l}, \sin\frac{\frac{1}{2}\pi x}{l}, \ldots, \cos\frac{(\frac{1}{2}+n)\pi x}{l}, \sin\frac{(\frac{1}{2}+n)\pi x}{l}, \ldots\}$$

*are **orthogonal** on the interval $[-l, l]$.*

- *In future discussions (e.g., in the heat equation), such function systems will appear naturally.*
- *One key difference:*
  - *When using integer indices, the period is $2l$.*
  - *When using half-integer indices, the period extends to $4l$.*
- *The verification method remains the same.*

1.3 Completeness and Inner Product Method for Finding Coefficients

- Analogous to linear algebra, linearly decompose a vector[2].
- The **difference** lies in that vector decomposition is **finite**, while function decomposition is **infinite**, involving issues of convergence, which should be strictly handled in mathematics.

Recall the vector decomposition:

$$\vec{\xi} = a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n} = \sum_{i=1}^{n} a_i\vec{e_i}$$

where

$$\{\vec{e_i}\} \text{ is an orthonormal basis.}$$

To find the coefficients a_i, use the **inner product** as the **component**.

$$\langle\vec{\xi}, \vec{e_j}\rangle = \sum_{i=1}^{n} a_i\langle\vec{e_i}, \vec{e_j}\rangle = \sum_{i=1}^{n} a_i\delta_{ij}, \quad \text{(only orthogonality is needed!!!)}$$

$$\Rightarrow a_i = \langle\vec{\xi}, \vec{e_i}\rangle$$

In summary,

- **Fourier series as a linear combination:** Similar to how a vector can be expressed as a linear combination of basis vectors, a function can be expanded using a set of basis functions.
- **Basis Functions:** In Fourier series, the sine and cosine functions serve as basis functions, similar to the basis vectors in linear algebra.
- **Coefficients Calculation:** The coefficients in a Fourier series expansion are obtained using an inner product, analogous to the method of projecting a vector onto a basis in linear algebra.

Function Decomposition (Analogous to Vector Decomposition):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

where

$$\left\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cdots\right\} \text{ is the orthonormal basis.}$$

To find the coefficients a_n, b_n, also use the **inner product** as the component of $f(x)$.

$$\langle f(x), \cos\frac{m\pi x}{l}\rangle = \langle a_m\cos\frac{m\pi x}{l}, \cos\frac{m\pi x}{l}\rangle = a_m l$$

$$\Rightarrow a_m = \frac{1}{l}\langle f(x), \cos\frac{m\pi x}{l}\rangle$$

Thus, above a_n, b_n, the inner product is defined as before.

Remark 1.2. *We only point out there is a significant difference between finite sums and infinite sums. Finite sums are true "=" decomposition. Infinite sums cannot generally be written as* $f(x) = \sum^{\infty}\cdots$, *and usually only a formal expansion can be obtained. The "=" implies* $\sum^{\infty} = \lim_{l\to\infty}\sum^{l}$ *has a meaning of convergence. Fourier series may diverge. To make this converge to* $f(x)$, $f(x)$ *must have certain restrictions. See references on Fourier analysis for more details.*

In the end, we summarize the above analogues in the following table 1:

[2] An orthonormal basis means

- If $m \neq n$, then $\langle e_m, e_n\rangle = 0$.
- If $m = n$, then $\langle e_n, e_n\rangle = \|e_n\|^2 = 1$.

Vector Decomposition (finite)	Function Decomposition (infinite)
$\vec{\xi} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \sum_{i=1}^{n} a_i\vec{e}_i$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$
$\{\vec{e}_i\}$ is an orthonormal basis.	$\{\frac{1}{\sqrt{2l}}, \frac{\cos\frac{\pi x}{l}}{\sqrt{l}}, \frac{\sin\frac{\pi x}{l}}{\sqrt{l}}, \cdots, \frac{\cos\frac{n\pi x}{l}}{\sqrt{l}}, \frac{\sin\frac{n\pi x}{l}}{\sqrt{l}}, \cdots\}$ orthogonormal basis.
To find the coefficients a_i, use the inner product as the component of ξ. $\langle\xi, \vec{e}_i\rangle = \sum_{i=1}^{n} a_i\langle\vec{e}_i, \vec{e}_j\rangle = \sum_{i=1}^{n} a_i\delta_{ij} \Rightarrow a_i = \langle\xi, \vec{e}_i\rangle$	To find coefficients a_n, b_n. Also use the inner product as the component of $f(x)$ in the basis. $\langle f(x), \cos\frac{n\pi x}{l}\rangle = \cdots$

Table 1: Comparison of Vector and Function Decomposition

1.4 Key Differences from Linear Algebra

- **Function vs. Vector:** The analogy replaces vectors with functions.
- **Infinite vs. Finite Dimensions:** Fourier series involves an infinite sum, extending linear algebra concepts to infinite-dimensional function spaces.
- **Functional Analysis Connection:** This extension leads to functional analysis, which generalizes linear algebra to infinite dimensions. We omit these in this lecture.

In summary,

- Fourier series can be understood **through the lens of linear algebra**.
  - Vector $\overrightarrow{\xi}$ $\leftrightarrow$Function $f(x)$;
  - Basis $\{\overrightarrow{e_i}\}\leftrightarrow$ Basis $\{1, \sin, \cos, \cdots\} \overset{\text{Chap 5}}{\leftrightarrow}$ Basis {Bessel functions}.
- The concept of an **inner product** is crucial for **computing Fourier coefficients**.
- The **transition** from **finite to infinite** dimensions introduces new challenges, such as **convergence issues**.

1.5 Odd and Even Functions

- When $f(x)$ is an odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad (n = 1, 2, 3, \cdots).$$

- When $f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 0, 1, 2, \cdots).$$

1.6 Generalizations

- **(Orthogonal basis and norms)** The set $\{\varphi_n(x)\}$ is called an **orthogonal basis** on the interval $[a, b]$ with respect to the weight function $q(x)$ ($q(x) = 1$ in the next chapter and $q(x) = x$ in Chapter 5) if

$$\langle\varphi_m(x), \varphi_n(x)\rangle := \int_a^b \varphi_m(x)\varphi_n(x)q(x)\,dx = 0 \quad (m \neq n).$$

If $m = n$, then

$$\|\varphi_n\| = \left[\int_a^b \varphi_n^2(x)q(x)\,dx\right]^{\frac{1}{2}} \quad \text{is called the } \textbf{norm} \text{ of } \varphi_n(x).$$

- **(Orthonormal basis)** For an orthogonal basis $\{\varphi_n(x)\}$, if it satisfies

$$\langle \varphi_m(x), \varphi_n(x)\rangle := \int_a^b \varphi_m(x)\varphi_n(x)q(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

  then we have the **orthonormal basis** on $[a, b]$. By normalizing the orthogonal basis $\{\varphi_n(x)\}$, we obtain the orthonormal basis $\left\{\frac{\varphi_n(x)}{\|\varphi_n\|}\right\}$.

- **(Example)** The set $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$ forms an orthogonal basis (the weight is 1) on the interval $[-\pi, \pi]$.

2 Method of Separation of Variables

Question 2.1. *Methods of Solving Equations: From elementary school to now, you have learned methods for solving equations.* ***What are the two basic approaches?***

Requirements: *Focus on the most* ***fundamental ideas****, without discussing specific methods like separation of variables or integrating factors. Just outline the two simplest and most direct concepts.*

A. ***Two Basic Approaches to Solving Equations***

1. ***Trial Method (Guessing)***

- *This method involves making educated guesses to arrive at a solution.*
- *It emphasizes discovering clues and refining guesses based on feedback from the results.*
- *This approach is a fundamental aspect of both the second and fifth chapters of the course.*
- *Also known as the "****method of trial and error,****" it serves not only as a solution technique but also as a research method.*
- *If you have limited information for your guess, you need to rely on reasoning. If you guess enough information, reasoning may not be necessary.*

2. ***Inverse Method***

- *For example, in linear algebra, $Ax = y$ if A is invertible, then $x = A^{-1}y$.*
- *It involves finding inverses (e.g., matrix inverses) to solve equations.*
- *This method can be generalized to include solving differential equations through integration, which is viewed as an inverse operation of differential. For example, by the separation of variables for ODEs, for a function $f(x)$, we can integrate $dy = f(x)dx$ to find y.*
- *Later chapters, such as the third and fourth, discuss methods like wave propagation, integral transform and Green functions, which also involve* ***inversion****.*

Application of Methods

- *The trial method resembles exploring an ancient tomb, where theorizing and practical experimentation guide the process.*
- *Encountering dead ends requires reevaluation and trying alternative paths to achieve a solution.*

Characteristics of the Separation of Variables Method

1. **Equation-Specific Methods**
   - There is no universal method for solving equations; different types require different approaches.
   - Equations must have specific characteristics for certain methods to apply.
2. **Conditions for Using the Separation of Variables Method**
   - **Applicability of Separation of Variables**
     - This method applies to problems involving linear wave, heat, or Laplace equations (in order to use the **principle of linear superposition**).
     - The key requirement is that both the equation and the boundary conditions must be **homogeneous** (Initial conditions can be either homogeneous or non-homogeneous).
   - **Essential Rule to Remember**

> – **Homogeneous equation** + **homogeneous boundary conditions** → Use separation of variables.

3. **Handling Non-homogeneous Cases**
   - **Non-homogeneous equations**: Solved using the **eigenfunction method** (Section 2.2).
   - **Non-homogeneous boundary conditions**: Solved using the **auxiliary function method** (Section 2.3).

> **Question 2.2.** *In the following solution process, we need to focus on two questions:*
>
> - *Where the **homogeneous equation** is used?*
> - *Where the **homogeneous boundary conditions** play a role?*

The Role of Intuition and Guessing

- Direct integration is often difficult due to the multi-variable derivatives (Chap. 3 gives some ideas).
- When traditional methods fail, educated guessing is necessary.
- Instead of blind guessing, one should use **physical intuition** to guide the search for solutions.
  - Since this course focuses on **Mathematical Physics Equations**, physical intuition is a valuable tool.
  - When mathematical approaches become challenging, **leveraging physical concepts** can provide new insights into the problem.
- For example, in wave equations, considering the physical behavior of vibrating strings helps identify possible solutions.

2.1 Free Vibration of a Bounded String

Consider the free vibration problem of a string fixed at both ends:

$$u_{tt} = a^2 u_{xx}, \quad (0 < x < l,\, t > 0), \tag{1}$$
$$u(0,t) = 0,\; u(l,t) = 0, \tag{2}$$
$$u(x,0) = \varphi(x),\; u_t(x,0) = \psi(x), \tag{3}$$

where $\varphi(x)$ and $\psi(x)$ are known functions.

The characteristic of this boundary value problem: Equation (1) is linear and homogeneous, hence the sum of various particular solutions is also a solution to this equation.

If enough particular solutions to equation (1) can be found, then their **linear combination** can be used to solve the mixed problem.

Physical Intuition:

To solve the boundary value problem (1)-(2), we first examine the physical model. From physics, we know that the sound emitted by musical instruments can be decomposed into various single tones of different frequencies.

Single tones are specific oscillation modes and serve as solutions to this equation.

Each single tone forms a sine curve when vibrating, and its amplitude depends on time. That is, each single tone can always be expressed as:

$$u(x,t) = c(t)\sin\lambda x, \quad \leftarrow \text{In general, an arbitrary function may not be decomposable in this way.} \tag{4}$$

The characteristic of this form is that $u(x,t)$ is the product of a function containing only the variable x and a function containing only the variable t, i.e., it has the form of separated variables.

A Priori Assumption in Equation Solving:

1. **Starting with an Assumption**
   - We begin solving the equation based on an **empirical observation**, e.g., (4).
   - The **empirical observation** is an assumption in following PDE solving. This initial assumption is known as an **a priori assumption**.
   - Beginners often find it **confusing** why the separation of variables method works and **why one can separate the variables like this at the beginning**. In fact, this is only an **(a priori) assumption**[3].
   - **Logical Progression** of a priori assumption:
     - Once the assumption is made, we proceed with mathematical reasoning.
     - If the approach works (i.e., verifiable solution, formal solution becomes the real solution), we obtain a solution.
     - If it fails, we **revise our assumption** and **try another** assumption or approach.
   - In nonlinear equations, similar assumptions are also applied.
2. **Separation of Variables as an Assumption**
   - The separation of variables method is fundamentally a **guessing approach**.
   - Instead of guessing the exact solution, we assume that the solution can be separated into independent functions (minimize guesses through more thorough derivations).
   - This assumption is based on physical laws and observed phenomena.

Now, let's try to find non-trivial solutions (i.e., not identically zero) of equation (1)

$$u_{tt} = a^2 u_{xx} \quad (0 < x < l,\ t > 0),$$

that satisfy the homogeneous boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0,$$

[3]The term "a priori assumption" is called "a priori" because it is a hypothesis made before analysis based on **existing experience, knowledge or theoretical frameworks**. This assumption does not rely on specific observations or experimental results but rather arises from established theoretical frameworks or known physical laws. Specifically:

1. **Definition**: An a priori assumption is a **hypothesis proposed before** conducting research or derivations, typically based on existing theories or concepts.
2. **Source**: These assumptions come from an understanding of certain **phenomena or intuition** about a problem, allowing researchers to derive conclusions without complete data.
3. **Importance**: A priori assumptions enable researchers to model and solve problems using existing knowledge in the absence of comprehensive information, thereby advancing theoretical development and application.

Thus, the term "a priori" emphasizes that these **assumptions exist prior to experience or observation**, reflecting a **theoretical preconception**.

Five Steps of the Method of Separation of Variables:

(1) Separation of Variables: Assume the solution can be expressed as the product

$$u(x,t) = X(x)T(t),$$

where $X(x)$ is a function of the variable x only, and $T(t)$ is a function of the variable t only.

Goal: Hope to obtain two ODEs to solve for X and T separately.

(2) PDE $\to$ ODEs: Substituting the assumed solution of the separated variable form into equation (1) gives

$$T''X = a^2X''T.$$

Transforming, we get:

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad \leftarrow \text{homogeneous equation} \tag{5}$$

Since the left and right sides of equation (5) remain constant when their independent variables change, let this constant be $-\lambda$, thus we obtain two ordinary differential equations:

$$T''(t) + \lambda a^2 T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0,$$

λ **is a constant:**

- **Language Description**: The left function is independent of x and is equal to the right function that is independent of t, then both functions are independent of t and x, then it is a constant.
- **Mathematical Approach**: Define a new function $G(t,x) = \frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)}$, then

$$\frac{\partial G}{\partial x} = 0 \quad \text{and} \quad \frac{\partial G}{\partial t} = 0 \quad \Rightarrow G \equiv \text{const}.$$

.

- A negative sign is added to the constant to maintain consistency with the established notation in subsequent sections.

1. **Series Solutions**: The method leads to solutions represented as series.
2. **Convergence Properties**:
   - **Different series** can converge to the **same** function.
   - This means that different students may arrive at different series forms, but the solution remains the same.
3. **Coefficient Variability**:
   - Constants (like a^2) can be placed in different parts of the equations, leading to differences in coefficients.
   - Although the resulting series may differ by a coefficient, the convergent function remains unchanged.

Thus, we normalize the steps to ensure consistency in the series solutions in this note.

We can solve these two ordinary differential equations to determine $T(t)$ and $X(x)$, thereby obtaining the particular solution of equation (1).

$$u(x,t) = X(x)T(t)$$

(3) Solving ODEs: To ensure that $u(x,t) = X(x)T(t)$ satisfies the homogeneous boundary conditions (2) $u(0,t) = 0$, $u(l,t) = 0$, it follows that:

$$T(t)X(0) = 0, \quad T(t)X(l) = 0. \quad \leftarrow \text{homogeneous boundary}$$

If $T(t) \equiv 0$, then $u(x,t) \equiv 0$, which is not a non-trivial solution. Therefore, it must be that:

$$X(0) = 0, \quad X(l) = 0.$$

Proof by Contradiction Assume that $X(0)$ is not identically equal to zero. Then $T(t)$ must be identically equal to zero. Substituting this back into the initial assumption leads us to conclude that u is identically equal to zero, resulting in the trivial solution.

Parameter Exploration

- The parameter λ is critical but initially unknown.
- Directly solving the equations will yield general solutions that still depend on λ.

Solution Approach

- Historically, mathematicians employed a trial-and-error method to explore potential solutions.
- The goal is to identify a viable path to the solution through systematic exploration.

Conclusion We will focus on the successful path identified through historical mathematical exploration.

(3a) $X(t)$-eq. $\xrightarrow{\text{by zero bdry condi.}}$ SL problem $\to$ Eigenvalues λ_n and eigenfunctions X_n:

To find the function $X(x)$, we only need to solve the following boundary value problem for the ordinary differential equation:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0. \tag{6}$$

If for certain values of λ, non-trivial solutions to problem (6) exist, then such λ values are called **eigenvalues**, and the corresponding non-trivial solutions $X(x)$ are called **eigenfunctions**, and we seek these. Such problems are commonly referred to as **Sturm-Liouville problems**.

Below we discuss λ under three scenarios:

$$\lambda < 0; \quad \lambda = 0; \quad \lambda > 0.$$

Preparatory Knowledge: General Solution of Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

The general solution formula for the differential equation $y'' + py' + qy = 0$ (*), where p and q are constants. The characteristic equation corresponding to equation (*) is $r^2 + pr + q = 0$.

1. If $\Delta \equiv p^2 - 4q > 0$, the general solution of equation (*) is

$$y = Ae^{r_1 x} + Be^{r_2 x}.$$

2. If $\Delta \equiv p^2 - 4q = 0$, the general solution of equation (*) is

$$y = (A + Bx)e^{rx}.$$

3. If $\Delta \equiv p^2 - 4q < 0$, $r_{1,2} = \alpha \pm i\beta$, the general solution of equation (*) is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Here, A and B are arbitrary constants.

Consider the differential equation $X''(x) + \lambda X(x) = 0$ with boundary conditions $X(0) = X(l) = 0$ (In fact, from Newtonian mechanics, one know λ determines the frequencies).

1. When $\lambda < 0$, problem (6) has no non-trivial solutions. In fact, the general solution of the equation is
$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$
from the boundary conditions, we get
$$A + B = 0,$$
$$Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} = 0.$$
Thus, $A = B = 0$, which implies $X(x) \equiv 0$.

2. When $\lambda = 0$, problem (6) has no non-trivial solutions. In fact, the general solution of the equation is
$$X(x) = (Ax + B)e^{\lambda x} = Ax + B.$$
From the boundary conditions, we get $A = B = 0$ (from geometric picture, two end points are 0, horizontal line, then $A = B = 0$), so the only solution is the trivial one where $X(x) \equiv 0$.

Consider the differential equation $X''(x) + \lambda X(x) = 0$ with boundary conditions $X(0) = X(l) = 0$.

3. When $\lambda > 0$, the general solution of the equation takes the form
$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$
From the boundary conditions, we get
$$X(l) = B\sin\sqrt{\lambda}l = 0, \quad X(0) = A = 0.$$
Assuming $X(x)$ is not identically zero, then $B \neq 0$, which implies $\sin\sqrt{\lambda}l = 0$ (Find zeros). Thus, we obtain
$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots). \quad \text{(eigenvalues)}$$
Hence, we find a set of non-zero solutions
$$X_n(x) = B_n \sin\frac{n\pi x}{l} \quad (n = 1, 2, \ldots). \quad \text{(eigenfunctions)}$$

Some remarks on the discrete values:

- To **ensure** the equation has a solution, the parameter λ must take on **specific discrete values**.
- These discrete values are referred to as **eigenvalues**, analogous to concepts in linear algebra, i.e., by (6) and let $\mathcal{A} := \partial_x^2$
$$X'' = -\lambda X \quad \Rightarrow \mathcal{A}X = -\lambda X$$
which formally consists with $Ax = \lambda x$ in linear algebra.
- Once λ is determined, substitute it back into the previously obtained solution form.
- The corresponding functions are called **eigenfunctions**, as they replace “vectors” in the context of functions, i.e., “Vector” $\rightarrow$ “function”.

- The **boundary conditions** help to **isolate** the values of λ; initially, λ could be any parameter, but the **boundary conditions constrain** it to specific **discrete** values.
- From a physical perspective, λ represents the **wave number**, and the boundary conditions constrain the **number** of **internal wave modes** allowed. The idea of discrete states is analogous to the behavior of waves, particularly in systems with fixed boundaries, such as standing waves.
- In quantum mechanics, atoms have discrete energy levels where electrons can exist, but not between these levels. These energy levels arise from solving certain equations and reflect the concept of discrete states.
- For standing waves, only specific frequencies (or wave numbers) can exist, determined by the boundary conditions.
- The wave number λ plays a crucial role in defining these specific frequencies, which are essential for the physical behavior of the system.
- Thus, the discrete values arise from boundary conditions, reflecting a fundamental nature of wave behavior in both quantum mechanics and classical wave mechanics.

(3b) $T(t)$-eq. $\xrightarrow{\text{substituting } \lambda_n}$ Find the general solution of T_n:
Now consider the differential equation

$$T''(t) + \lambda a^2 T(t) = 0,$$

with the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

Substituting these eigenvalues into equation (6), we get

$$T''(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0,$$

and the general solution is

$$T_n(t) = C_n \cos\frac{n\pi at}{l} + D_n \sin\frac{n\pi at}{l} \quad (n = 1, 2, \ldots).$$

Thus, we obtain the particular solution of equation (1) that satisfies the homogeneous boundary conditions (2) in the form of separated variables $u_n(x,t) = X_n(x)T_n(t)$.

The particular solution is given by

$$u_n(x,t) = \left(a_n \cos\frac{n\pi at}{l} + b_n \sin\frac{n\pi at}{l}\right) \sin\frac{n\pi x}{l} \quad (n = 1, 2, \ldots) \tag{7}$$

where $a_n = B_n C_n$, $b_n = B_n D_n$ are arbitrary constants.

Note the **initial conditions**

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

where $\varphi(x)$ and $\psi(x)$ are arbitrarily given. Generally speaking, any particular solution in (7) does **not** satisfy the given initial conditions (Note "any" function can be represented by the infinite sum of sin and cos, i.e., the Fourier series. This inspires us use superposition of u_n).

(4) Superposition of Series Solutions: Since equation (1) is linear and homogeneous, by the principle of linear superposition, the series

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi at}{l} + b_n \sin\frac{n\pi at}{l}\right) \sin\frac{n\pi x}{l} \tag{8}$$

is still a solution to equation (1) and satisfies the boundary conditions (2), i.e.,

$$u_{tt} = a^2 u_{xx}, \quad u(0,t) = 0, \quad u(l,t) = 0.$$

(5) Initial Conditions Determine Coefficients:

Question 2.3. *Under what conditions on a_n and b_n does equation (14) also satisfy the initial conditions*

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Thus, taking the derivative of (8) with respect to t, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} \left(-a_n \sin \frac{n\pi at}{l} + b_n \cos \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Key Question

- How do we compute the derivative of an infinite series?
- Can we differentiate term by term and then sum the results?

Potential Issue

- Given that a_n and b_n are undetermined coefficients, we cannot directly judge whether differentiation and summation can be interchanged.

Solution Approach

- Assume (also as a priori assunmption) term-by-term differentiation is valid (i.e., assume interchangeability).
- Verify the solution at the end.

Setting $t = 0$ in equation (8) and its corresponding derivative, and combining with the initial conditions, we have

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x)$$

This leads to

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = \varphi(x), \quad \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = \psi(x).$$

1. This is the Fourier expansion of the odd function $\varphi(x)$.
2. The function $\varphi(x)$ is defined on the interval $[0,l]$, but the Fourier series is applicable to **periodic** functions. This implies that $\varphi(x)$ can first be **extended oddly** and then **periodically** to become a **periodic function** before applying the Fourier expansion.

Since $\varphi(x)$ and $\psi(x)$ are defined on the interval $[0,l]$, a_n is the coefficient of the Fourier sine series expansion of $\varphi(x)$, and $b_n \frac{n\pi a}{l}$ is the coefficient of the Fourier sine series expansion of $\psi(x)$. That is,

$$a_n = \frac{2}{l} \int_0^l \underbrace{\varphi(x) \sin \frac{n\pi x}{l}}_{\text{even function} \to \frac{1}{l}\int_{-l}^{l} = \frac{2}{l}\int_0^l} dx,$$

$$b_n = \frac{2}{n\pi a}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx,$$

You can not write $a_n = \frac{1}{l}\int_{-l}^{l}\varphi(x)\sin\frac{n\pi x}{l}\,dx$ since the definition domain of φ is only $[0,l]$.

then the series (8) satisfies the initial conditions

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x).$$

Substituting the determined a_n and b_n from (9) into equation (8), we obtain the solution to the mixed problem (1)-(3).

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

where

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx, \\ b_n &= \frac{2}{n\pi a}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx. \end{aligned} \tag{9}$$

The method of solving described above is called the method of separation of variables.

The solution is given by

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

Potential Issues with the Solution

1. **Reliance on Assumptions**:
   - The derivation of the series solution involves multiple ac priori assumptions.
   - Examples include assuming the validity of variable separation and term-by-term differentiation.
   - These assumptions may affect the correctness of the solution.
2. **Verification of the Solution**:
   - To confirm the solution, it must be substituted back into the original equation and see if the derived series truly satisfies the equation.
   - It is necessary to verify whether the series converges to a valid function.
   - Without verifications, the solution is merely a **formal solution**.

Note: 1. The series solution (8) may **not necessarily converge**, hence it is sometimes referred to as a formal solution. However, under certain conditions in the existence theorem, (8) can indeed be guaranteed to be the classical solution to the boundary value problem (1)-(3).

$$\begin{cases} u_{tt}=a^2u_{xx} & (0<x<l,\, t>0), \\ u(0,t)=0,\, u(l,t)=0, \\ u(x,0)=\varphi(x),\, u_t(x,0)=\psi(x). \end{cases}$$

Theorem 2.1 (Existence Theorem)**.** *If $\varphi(x)\in C^4[0,l]$ (a function with continuous fourth derivatives), $\psi(x)\in C^3[0,l]$, and φ, φ'', ψ are zero at $x=0$ and $x=l$, then the classical solution to the initial boundary value problem (1)-(3) exists and can be expressed as series (8), with the coefficients determined by (9).*

$$
\begin{cases}
u_{tt} = a^2 u_{xx} & (0 < x < l,\, t > 0), \\
u(0,t) = 0,\, u(l,t) = 0, \\
u(x,0) = \varphi(x),\, u_t(x,0) = \psi(x),
\end{cases}
$$

$$
\begin{aligned}
u(x,t) = & \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \\
a_n = & \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l}\, dx, \\
b_n = & \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l}\, dx,
\end{aligned}
$$

Basic Steps of the Method of Separation of Variables:

1. **(Separation of Variables)** Assume separation of variables $u(x,t) = X(x)T(t)$
2. **(PDE → ODEs)** Substitute into the partial differential equation (PDE) to obtain two ordinary differential equations (ODEs) for $X(x)$ and $T(t)$ respectively
3. **(Solving ODEs)** Solve the ODEs:
   - $X(t)$-eq. $\overset{\text{by bdry condi.}}{\rightarrow}$ SL problem $\rightarrow$ Eigenvalues λ_n and eigenfunctions X_n (Standing waves appear here)
   - $T(t)$-eq. $\overset{\text{substituting } \lambda_n}{\rightarrow}$ Find the general solution of T_n
4. **(Superposition of Series Solutions)** Superpose the series solution $u = \sum^{\infty} u_n$
5. **(Initial Conditions Determine Coefficients)** Determine the Fourier coefficients using initial conditions

Some issues:

- The separation of variables method is initially based on physical observations suggesting that **variables can be separated**.
- However, as the process progresses, **the series solution obtained by superposition may not be variable separated.**
- The superposition of series disrupts the form of separable variables.