Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 6-1

3.3 Integral Transforms

3.3.1 Integral Transforms and Their Properties (Review)

If a function $f(x)$ is continuous and differentiable on $(-\infty, +\infty)$ and **absolutely integrable** (make sure the existence of the integral), then it has the **Fourier Transform**

$$\hat{f}(\lambda) = F(f) = \int_{-\infty}^{+\infty} f(x) e^{-ix\lambda} dx$$

and its **inverse Fourier Transform**

$$f(x) = F^{-1}(\hat{f}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\lambda) e^{ix\lambda} d\lambda.$$

If a function $f(t)$ does not grow faster than an exponential[1] on $(0, +\infty)$, then its **Laplace Transform** is defined as

$$F(s) = L(f) = \int_0^{+\infty} f(t) e^{-st} dt, \quad (\mathrm{Re} s > C)$$

Compared to the Fourier transform, the conditions for the existence of the Laplace transform are much weaker because the exponential factor $\exp(-\beta t)$ makes the integral converge more easily. However, this does not mean that any function has a Laplace transform without any conditions. In fact, the **sufficient conditions** for the existence of the **Laplace transform** can be described as follows:

1. The function $f(t)$ is piecewise continuous on the interval $[0, \infty)$;
2. There exist positive constants M and α such that for all $t \geq 0$, $|f(t)| \leq M \exp(Ct)$ holds, then the function $f(t)$ has a Laplace transform for all $\mathrm{Re} s > C$, i.e.,

$$\left| \int_0^{\infty} f(t) e^{-st}\, dt \right| < \infty$$

Hint: $|F(s)| \leq \int_0^{+\infty} |f(t)| e^{-\mathrm{Re}(s)t}\, dt \leq \int_0^{+\infty} M e^{(C-\mathrm{Re}(s))t}\, dt < +\infty$

Laplace Transform is denoted as $f(t) = L^{-1}(F(s))$. Laplace Transform can be viewed as the special case of Fourier Transform. Because, (let $s = \alpha + i\beta$) then $\mathrm{Re}(s) > 0$,

$$\begin{aligned} F(s) = L(f) &= \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-\alpha t - i\beta t} dt \\ &= \int_0^{\infty} \left[f(t) e^{-\alpha t} \right] e^{-i\beta t} dt = \int_{-\infty}^{+\infty} \tilde{f}(t) e^{-i\beta t} dt \end{aligned}$$

[1] There exist constants $M > 0$, $C > 0$ such that $|f(t)| \leq M e^{\alpha t}$ for all $t > 0$.

where $\tilde{f}(t) \equiv \begin{cases} f(t)e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$.

Using the inverse Fourier Transform, (when $t \geq 0$)

$$\tilde{f}(t) = f(t)e^{-\alpha t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha + i\beta)e^{i\beta t}d\beta \stackrel{\text{let } s=\alpha+i\beta}{=} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} F(s)e^{st}e^{-\alpha t}ds$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}ds$$

This form of calculation is usually more difficult, but when $F(s)$ satisfies certain conditions, it can be calculated using the residue theorem.

Laplace inverse transform is denoted as

$$f(t) = L^{-1}(F(s))$$

Theorem 0.1 (Using the residue theorem)**.** *Suppose $F(s)$ is analytic except only a finite number of isolated singular points $s_1, s_2, \cdots, s_n$ in the half-plane $Res \leq c$, and when $s \to \infty$, $F(s) \to 0$, then*

$$f(t) = \sum_{k=1}^{n} Res[F(s)e^{st}, s_k].$$

Basic idea: A differential equation (DE) (in a rough form): $Au = f$ where A is a **differential operator**.

Using Fourier (or Laplace) transform

$$\mathscr{F}(Au) = \mathscr{F}f \Rightarrow B\hat{u} = \mathscr{F}f \quad \text{(one equation can solve multiple unknowns)}$$

$$\Rightarrow \hat{u} = B^{-1}\mathscr{F}f \Rightarrow u = \mathscr{F}^{-1}(B^{-1}\mathscr{F}f) \text{ to find the solution.}$$

Key: Hope that the integral transform can **transform a differential equation** into an **algebraic equation**. Fourier and Laplace transforms have this effect.

Integral Transform Properties

(1) Linearity

Integral transforms exhibit linearity, which means:

$$F[af + bg] = aF[f] + bF[g],$$

$$L[af + bg] = aL[f] + bL[g],$$

where a and b are arbitrary constants.

(2) Differential Theorem 1

If f and f' can both have Fourier or Laplace transforms and are zero at infinity, then:

$$F[f'(x)] = i\lambda F[f(x)], \quad F[f''(x)] = (i\lambda)^2 F[f(x)],$$

$$F[f^{(n)}(x)] = (i\lambda)^n F[f(x)],$$

and

$$L[f'(t)] = sL[f(t)] - f(0),$$

$$L[f''(t)] = s^2 \underbrace{L[f(t)]}_{\text{viewed as } -1 \text{ order derivative}} - sf(0) - f'(0), \leftarrow \boxed{\text{order of polynomial} + \text{order of derivative} = 1}$$

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

- The proof involves using integration by parts to transfer the derivative to the exponential function.

(3) Differential Theorem 2

If $\hat{f}(\lambda) = F[f(x)], F(s) = L[f(t)]$, then:

$$\hat{f}'(\lambda) = F[-ixf] \quad \text{(Fourier transform)},$$
$$F'(s) = -L[tf(t)] \quad \text{(Laplace transform)}.$$

(4) Convolution Theorem

If the convolution of f and g is defined as:

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)dy,$$

and can undergo Fourier transform, then:

$$F[f * g] = F[f] \cdot F[g],$$

and thus:

$$F^{-1}[\hat{f} \cdot \hat{g}] = f * g.$$

For Laplace transforms, there is a similar convolution theorem:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau. \leftarrow \boxed{\text{According to Laplace transf. } \tau > 0 \text{ and } t - \tau > 0 \text{ imply } 0 < \tau < t}.$$

Definition and Origin

- Convolution is not originally defined in integral form but arises naturally in series multiplication.
- Given two infinite series:

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{m=0}^{\infty} b_m,$$

  their product can be computed in different ways.

Series Multiplication Methods

- Element-wise multiplication results in an infinite matrix:

$$\begin{bmatrix} a_0b_0 & a_0b_1 & a_0b_2 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Different summation methods yield different products:
  - Summing row by row.
  - Summing column by column.
  - Summing along diagonals (Cauchy product).

Cauchy Product

- Diagonal summation gives:
$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} a_i b_{n-i},$$
which is known as the Cauchy product formula.

Connection to Convolution

- Summation is a **discrete form** of integration.
- Replacing sums with integrals generalizes the concept to continuous convolution:
$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$
- The index transformation in series multiplication corresponds to variable substitution in convolution.
- Convolution originates as a redistribution method in multiplication but has many useful properties in mathematics.

(5) Frequency Shift Theorem (Translation Theorem)

For the variable of the transform, if $\hat{f}(\lambda) = F[f(x)], F(s) = L[f(t)]$, then:

$$F[f(x)e^{-i\lambda_0 x}] = \hat{f}(\lambda + \lambda_0) \quad \text{(Fourier transform)},$$
$$L[f(t)e^{-at}] = F(s+a) \quad \text{(Laplace transform)}.$$

- By definition, it can be proven directly.

(6) Delay Theorem

For the variable of the transform, if $\hat{f}(\lambda) = F[f(x)], F(s) = L[f(t)]$, then:

$$F[f(x-x_0)] = \hat{f}(\lambda)e^{-i\lambda x_0} \quad \text{(Fourier transform)},$$
$$L[f(t-t_0)u(t-t_0)] = F(s)e^{-st_0} \quad \text{(Laplace transform)},$$

where

$$u(t-t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases} \leftarrow \boxed{\text{Heaviside function}}$$

This can be simplified to:

$$L[f(t-t_0)] = F(s)e^{-st_0} \quad (t > t_0).$$

Proof of the Laplace Transform Delay Theorem. By the definition of the Laplace transform,

$$L[f(t-t_0)u(t-t_0)] = \int_0^{\infty} f(t-t_0)u(t-t_0)e^{-st}dt = \int_{t_0}^{\infty} f(t-t_0)e^{-st}dt$$

Let $y = t - t_0$, then the equation becomes

$$\text{Left side} = \int_0^{\infty} f(y)e^{-s(y+t_0)}dy = e^{-st_0}\int_0^{\infty} f(y)e^{-sy}dy = F(s)e^{-st_0} = \text{Right side}$$

□

Supplementary: Definition and Properties of the Dirac Delta Function

The Dirac delta function is a mathematical model abstracted from some physical phenomena, such as the impact force in mechanics, the explosion of a hydrogen bomb, etc. These physical phenomena have a common characteristic: the action time is extremely short, but the action intensity is extremely large. (Impulse function)

(1) Definition of the Dirac Delta Function: A function satisfying the following two conditions

1. $\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$
2. $\int_{-\infty}^{+\infty} \delta(x)dx = 1$

If the impulse action does not occur at $x = 0$, but occurs at $x = x_0$, then the function is denoted as $\delta(x - x_0)$, and satisfies

$$\delta(x - x_0) = \begin{cases} \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x - x_0)dx = 1$$

(2) Properties of the Dirac Delta Function:

1. Sampling Property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

Specifically,

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$$

- The sampling property considers the function F at the point where δ is infinite.

2. Symmetry: $\delta(x)$ is an even function, then

$$\delta(x - x_0) = \delta(x_0 - x)$$

Specifically,

$$\delta(x) = \delta(-x)$$

Naturally, there is also

$$\int_{-\infty}^{+\infty} f(x)\delta(x_0 - x)dx = f(x_0) = f * \delta(x_0)$$

Ex 0.1. *Find the Fourier Transform of $\delta(x + a)$, where a is a constant independent of the variable x.*

Solution. *By definition,*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx = \int_{-\infty}^{\infty} \delta(x + a)e^{-ix\lambda}dx$$

Using the properties of the δ function,

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

Thus,

$$F[\delta(x + a)] = e^{ia\lambda}$$

Similarly,

$$F[\delta(x - a)] = e^{-ia\lambda}$$

Using the linearity of the Fourier transform,

$$F\left[\frac{1}{2}[\delta(x+a)+\delta(x-a)]\right] = \frac{e^{ia\lambda}+e^{-ia\lambda}}{2} = \cos a\lambda$$

$$F\left[\frac{1}{2i}[\delta(x+a)-\delta(x-a)]\right] = \frac{e^{ia\lambda}-e^{-ia\lambda}}{2i} = \sin a\lambda$$

Thus, we have the formulas:

$$F^{-1}[\cos a\lambda] = \frac{1}{2}[\delta(x+a)+\delta(x-a)]$$

$$F^{-1}[\sin a\lambda] = \frac{1}{2i}[\delta(x+a)-\delta(x-a)]$$

Ex 0.2. *Find the Fourier Transform of*

$$f(x) = \begin{cases} 1, & |x| \le m \\ 0, & |x| > m \end{cases}$$

where $m > 0$.

Solution. *By definition and using* $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx = \int_{-m}^{m} e^{-ix\lambda}dx = \int_{-m}^{m}(\cos x\lambda - i\sin x\lambda)dx = 2\int_0^m \cos x\lambda dx = \frac{2\sin m\lambda}{\lambda}$$

From this, we can conclude:

$$F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \le m.$$

Ex 0.3. *Find the Inverse Fourier Transform of* $\hat{f}(\lambda) = e^{-\lambda^2 t}$*, where* $t > 0$.

Solution. *By definition,*

$$\begin{aligned} f(x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{ix\lambda}d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2 t}e^{ix\lambda}d\lambda \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2 t}(\cos x\lambda + i\sin x\lambda)d\lambda = \frac{1}{\pi}\int_0^{\infty}e^{-\lambda^2 t}\cos x\lambda d\lambda \end{aligned} \tag{1}$$

Differentiate $f(x)$ *and use integration by parts once to obtain:*

$$\frac{df(x)}{dx} + \frac{x}{2t}f(x) = 0 \Rightarrow f(x) = f(0)e^{-\frac{x^2}{4t}}$$

Note

$$\frac{df(x)}{dx} = -\frac{1}{\pi}\int_0^{\infty}\lambda e^{-\lambda^2 t}\sin(\lambda x)d\lambda$$

and

$$\begin{aligned} \frac{xf(x)}{2t} =& \frac{1}{\pi}\int_0^{\infty}\frac{x}{2t}e^{-\lambda^2 t}\cos(\lambda x)d\lambda = \frac{1}{\pi}\int_0^{\infty}\frac{1}{2t}e^{-\lambda^2 t}d(\sin(\lambda x)) \\ =& \frac{1}{\pi}\left[\frac{1}{2t}e^{-\lambda^2 t}\sin(\lambda x)\right]_0^{\infty} - \frac{1}{\pi}\int_0^{\infty}\frac{1}{2t}\sin(\lambda x)d(e^{-\lambda^2 t}) \\ =& -\frac{1}{\pi}\int_0^{\infty}\frac{1}{2t}e^{-\lambda^2 t}\cdot(-t)2\lambda\sin(\lambda x)d\lambda \\ =& \frac{1}{\pi}\int_0^{\infty}\lambda e^{-\lambda^2 t}\sin(\lambda x)d\lambda \end{aligned}$$

$$\Rightarrow \frac{df(x)}{dx} + \frac{x}{2t}f(x) = 0.$$

By (1),

$$f(0) = \frac{1}{\pi} \int_0^\infty e^{-\lambda^2 t} d\lambda$$

Using the Gaussian integral,

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

we conclude $f(0) = \frac{1}{2\sqrt{\pi t}}$*, and further*

$$f(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

From this example, we conclude:

$$F^{-1}[e^{-\lambda^2 t}] = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad (t > 0)$$

A novel approach is introduced to **solve integral problems** by transforming them into **differential equation problems**. This method, referred to as the "Spotlight Method," involves viewing a problem from a higher perspective to simplify the solution process.

Introduction

A problem-solving approach, inspired by a description of John Nash's thinking, is introduced. Instead of solving a problem directly, this method suggests gaining insight from a seemingly unrelated but broader perspective.

Core Idea

- Traditional approach: Solve the problem directly, like climbing a mountain to reach its peak.
- Nash's approach: Instead of climbing the given mountain, he first ascends a different, higher but gently sloping mountain (see Fig. 1).
- At the peak of the higher mountain, he uses a "searchlight" to illuminate the original mountain, gaining a clear view of its structure.
- This method suggests that solving a more general or advanced problem may provide clarity on a specific one.

Key Insights

- Reformulating a problem by looking at a broader or higher-level problem can simplify the original challenge.
- A shift in perspective often reveals hidden structures and deeper connections.
- Tackling an abstract or seemingly unrelated problem can offer new insights into the original one.

Key Ideas

- Instead of solving the integral directly, transform it into a differential equation.
- Exponential and trigonometric functions are useful due to their well-behaved derivatives:
  1. $\frac{d}{dx} e^x = e^x$
  2. $\frac{d^2}{dx^2} \cos x = -\cos x$

Figure 1: Mountain analogy

- The **goal** is to construct a function $f(x)$ satisfying a differential equation of the form:

$$\frac{d}{dx}f(x) = k(x)f(x)$$

  which simplifies the integration process.

Solution Strategy

1. Define $f(x)$ based on the given integral.
2. Compute its derivative to form a differential equation.
3. Utilize integration by parts to transform terms into a solvable form.
4. Ensure boundary conditions eliminate unnecessary terms.
5. Solve the differential equation, leading to an explicit solution.

Gaussian Integral Application

- The method confirms that the Gaussian integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

  is valid.

Ex 0.4. *Find the Inverse Fourier Transform of* $\hat{f}(\lambda) = e^{-|\lambda|y}$, *where* $y > 0$.

Solution. *By definition,*

$$
\begin{aligned}
f(x) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{ix\lambda}d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\lambda|y}e^{ix\lambda}d\lambda \\
=& \frac{1}{2\pi}\left(\int_{0}^{\infty} e^{\lambda(ix-y)}d\lambda + \int_{-\infty}^{0} e^{\lambda(ix+y)}d\lambda\right) \\
=& \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) = \frac{1}{\pi}\frac{y}{y^2+x^2}
\end{aligned}
$$

From this example, we conclude:

$$
F^{-1}[e^{-|\lambda|y}] = \frac{1}{\pi}\frac{y}{y^2+x^2} \quad (y>0)
$$

Common Fourier Transforms and Inverse Transforms

1. $F[\delta(x+a)] = e^{ia\lambda}$, $F[\delta(x-a)] = e^{-ia\lambda}$ and $F(\delta(x)) = 1$
2. $F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \le m$
3. $F^{-1}[e^{-\lambda^2 t}] = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} \quad (t>0)$
4. $F^{-1}[e^{-|\lambda|y}] = \frac{1}{\pi}\frac{y}{y^2+x^2} \quad (y>0)$
5. $F^{-1}[\cos a\lambda] = \frac{1}{2}[\delta(x+a)+\delta(x-a)]$ and $F^{-1}[\sin a\lambda] = \frac{1}{2i}[\delta(x+a)-\delta(x-a)]$.

Common Laplace Transforms and Inverse Transforms ($\mathrm{Re}\, s > 0$)

1. $L[\delta(t)] = 1$
2. $L[e^{-at}] = \frac{1}{s+a}$ (especially, $L[1] = \frac{1}{s}$)
3. $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$
4. $L[\sin at] = \frac{a}{s^2+a^2}$ and $L[\cos at] = \frac{s}{s^2+a^2}$
5. $L[e^{-at}\sin at] = \frac{a}{(s+a)^2+a^2}$ and $L[e^{-at}\cos at] = \frac{s+a}{(s+a)^2+a^2}$
6. $L^{-1}[F(s)e^{-sa}] = f(t-a) \quad (t>a)$ (Inverse Transform Form of the Delay Theorem)
7. $L^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{2}{\sqrt{\pi}}\int_a^{\infty}\frac{e^{-y^2}}{2\sqrt{t}}dy$ (Error Function)
8. $L^{-1}[e^{-a\sqrt{s}}] = \frac{a}{2\sqrt{\pi t^3}}e^{-\frac{a^2}{4t}}$. In fact, $L^{-1}[e^{-a\sqrt{s}}] = L^{-1}\left[s\cdot\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}\int_a^{\infty}\frac{e^{-y^2}}{2\sqrt{t}}dy\right] = \frac{a}{2\sqrt{\pi t^3}}e^{-\frac{a^2}{4t}}$

Ex 0.5. *Use Laplace Transform to solve*

$$
\begin{cases}
u''(t) + k^2 u(t) = f(t), \\
u(0) = 0, \quad u'(0) = 0.
\end{cases}
$$

Solution. *Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get:*

$$
s^2U(s) - su(0) - u'(0) + k^2U(s) = F(s)
$$

$$
\Rightarrow s^2U(s) + k^2U(s) = f(s)
$$

Thus,

$$
U(s) = \frac{1}{k}\frac{k}{s^2+k^2}\cdot F(s)
$$

$$
L[\sin at] = \frac{a}{s^2+a^2}
$$

Taking the inverse Laplace transform of the above equation gives:

$$
u(t) = \frac{1}{k}f(t) * \sin kt = \frac{1}{k}\int_0^t f(\tau)\sin k(t-\tau)d\tau
$$

3.3.2 Examples of Integral Transform Methods

The Integral transform is a powerful technique for solving differential equations, particularly linear ODEs and PDEs. It simplifies the problem by converting differential equations into algebraic equations, which are easier to solve.

Three-Step Method

The solution process consists of three clear steps:

1. **Transform to Frequency Domain:** Apply the Laplace transform to all functions in the equation. Use the differentiation property of the Laplace transform:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

   If the initial conditions are zero, the additional terms disappear.

2. **Solve the Algebraic Equation:** After transformation, the differential equation becomes an algebraic equation in terms of $F(s)$. Solve for $F(s)$.

3. **Inverse Laplace Transform:** Convert $F(s)$ back to the time domain using the inverse Laplace transform. This often involves convolution:

$$L^{-1}\{G(s)H(s)\} = (g * h)(t)$$

Key Insights

- **Dimensional Reduction:** Each application of the Integral transform reduces the number of the derivatives of independent variables:
  - An ODE becomes an algebraic equation.
  - A PDE with two variables becomes an ODE.
- **Analogy to a Thought Experiment:**
  - The process is compared to a classic riddle: "How do you put an elephant in a refrigerator?"
  - Step 1: Open the refrigerator door (*transform to frequency space*).
  - Step 2: Put the elephant inside (*solve the equation*).
  - Step 3: Close the door (*inverse transform back to time space*).
- **Hidden Complexity:** While the process seems straightforward, practical applications reveal underlying difficulties that require careful handling.

Conclusion

The Laplace transform provides a structured and efficient way to solve differential equations. Its strength lies in reducing complex differential operations into algebraic manipulations, making it a widely used tool in engineering and applied mathematics.

The advantage of integral transform methods is that they simplify the original equation into a simpler form, facilitating the solution.

In applications, for **initial value problems**, the **Fourier transform** (for **spatial variables**) is commonly used, while for problems with **boundary conditions**, the **Laplace transform** (for **time variables**) is used.

When solving partial differential equations (PDEs) using integral transforms, a key question arises:

- Should we use the **Fourier transform** or the **Laplace transform**?
- How do we choose the appropriate transform?

Understanding the selection criteria is essential, as real-world problems do not explicitly specify which transform to use.

Criteria for Choosing the Transform

Step 1: Determine Which Variable to Transform

- The **goal** of an integral transform is to **eliminate derivatives in one variable**, converting the PDE into an ordinary differential equation (ODE).
- The choice should align with the **given boundary or initial conditions**.
- If the problem provides **initial conditions in time** ($t = 0$), it is generally better to eliminate x, reducing the PDE to an ODE in t.
- If the problem provides **boundary conditions in space** (x), it is generally better to eliminate t.

Step 2: Decide Between Fourier and Laplace Transforms

- The choice depends on the domain of the variable:
  - The **Fourier transform** is defined for $x \in (-\infty, \infty)$.
  - The **Laplace transform** is typically used for functions defined on $x \geq 0$ or $t \geq 0$.
- If x extends over the entire real line, use the **Fourier transform**.
- If x is restricted to $x \geq 0$, use the **Laplace transform**.

Application to a Specific Problem

For a given PDE, follow these steps:

1. **Select the variable to transform** based on initial or boundary conditions.
2. **Choose the transform type** based on the domain of the variable.
3. **Apply the transform** to all relevant functions in the equation.
4. **Solve the transformed equation** in the frequency domain.
5. **Use the inverse transform** to return to the original domain.

Key Insight

- **Dimensional Reduction:** Each application of the Integral transform reduces the number of the derivatives of independent variables;
- In some cases, it is not necessary to explicitly compute certain transforms of the **free terms** and **data** since they will **transform back** in the final solution.

In application, we give a general principle (deviating from it may introduce additional complexity or difficulties, but alternative methods may still yield a solution but could be less efficient)

- Initial value problem $\rightarrow$ Use Fourier transform (for spatial variables $x \in (-\infty, +\infty)$) $\leftarrow$ Because initial conditions are given, so retain ∂_t derivative (solving ∂_t-eq needs initial data);
- Boundary value problem $\rightarrow$ Use Laplace transform (for temporal variables $t > 0$) $\leftarrow$ Because boundary conditions are given, so retain ∂_x derivative (solving ∂_x-eq needs boundary data).

Otherwise, additional **hidden conditions** need to be specified to determine the parameters.

Conclusion

By systematically selecting the appropriate transform and variable, PDEs can be simplified effectively. The choice of integral transform plays a crucial role in ensuring the problem remains solvable with minimal additional complexity.

Ex 0.6. *Solve the following problem:*

$$u_t = a^2 u_{xx} + f(x,t) \quad (-\infty < x < +\infty, t > 0), \tag{2}$$
$$u\big|_{t=0} = \varphi(x). \tag{3}$$

Key Considerations

- Two key questions must be addressed:
  1. Which variable should be transformed?
  2. Which integral transform should be used?
- The variable to be transformed should align with the given boundary or initial conditions to simplify solving the ODE.
- If initial conditions are provided, it is generally preferable to transform the spatial variable x.

Choosing the Right Transform

- The choice of transform depends on the domain of the variable:
  - **Fourier Transform**: Suitable for variables defined over the entire real line $(-\infty, \infty)$.
  - **Laplace Transform**: Suitable for variables defined for non-negative values $t \geq 0$.
- The definition domain of the variable helps determine whether to use the Laplace or Fourier transform.

Solution. *First, take the Fourier transform with respect to x, denote:*

$$F[u(x,t)] = U(\lambda, t), \quad F[f(x,t)] = G(\lambda, t), \quad F[\varphi(x)] = \Phi(\lambda)$$

- *Apply the Fourier transform to both sides of the given PDE.*

- *By linearity and using the differentiation property of the Fourier transform, the transform can be applied term by term*

$$\underbrace{F(\partial_t u)}_{=\partial_t U} = \underbrace{a^2 F\left(\partial_x^2 u\right)}_{=-a^2\lambda^2 U} + \underbrace{F(f)}_{=G}$$

where

$$F(\partial_t u) = \int_{-\infty}^{\infty} \partial_t u \cdot e^{-i\lambda x}\, dx = \int_{-\infty}^{\infty} \partial_t (u \cdot e^{-i\lambda x})\, dx = \partial_t \int_{-\infty}^{\infty} u \cdot e^{-i\lambda x}\, dx = \partial_t F(u) = \partial_t U$$

- *Using this property, the PDE is transformed into an ODE.*

Taking the Fourier transform of equation (2) *with respect to x, we get:*

$$\frac{dU(\lambda,t)}{dt} = -a^2\lambda^2 U(\lambda,t) + G(\lambda,t), \tag{4}$$

which satisfies the initial condition:

$$U(\lambda,t)\big|_{t=0} = \Phi(\lambda). \tag{5}$$

- *Various methods can be used to solve the ODE, such as:*
  - *Integrating factor method.*
  - *Method of variation of parameters.*
  - *Laplace transform.*

To solve the ordinary differential equation initial value problem (4)–(5), *denote:*

$$L[U(\lambda,t)] = \bar{U}(\lambda,s), \quad L[G(\lambda,t)] = \bar{G}(\lambda,s).$$

Taking the Laplace transform of equation (4) *with respect to t, and combining with condition* (5), *we get:*

$$s\bar{U}(\lambda,s) - \Phi(\lambda) = -a^2\lambda^2\bar{U}(\lambda,s) + \bar{G}(\lambda,s),$$

$$\Rightarrow \bar{U}(\lambda,s) = \frac{1}{s+\lambda^2 a^2}\Phi(\lambda) + \frac{1}{s+\lambda^2 a^2}\bar{G}(\lambda,s). \tag{6}$$

Applying the Inverse Transforms

- *After obtaining the solution in the frequency space, we must transform it back to the original space.*
- *This requires applying two inverse transforms:*
  - *Since two integral transforms have been applied, two* ***integral variables*** *appear in the solution.*
  - *It is* ***crucial to distinguish*** *which variable each inverse transform is applied to.*
  - *First, the inverse Laplace transform L^{-1} to revert from s-domain to t-domain.*
  - *Second, the inverse Fourier transform F^{-1} to revert from λ-domain to x-domain.*

Taking the inverse Laplace transform of both sides of equation (6), we get:

$$U(\lambda,t)=\Phi(\lambda)L^{-1}\left[\frac{1}{s+\lambda^2a^2}\right]+L^{-1}\left[\frac{1}{s+\lambda^2a^2}\bar{G}(\lambda,s)\right].$$

Taking the inverse Laplace transform of equation (6) and using $L[e^{-at}]=\frac{1}{s+a}$, *we get:*

$$\begin{aligned}U(\lambda,t)=&\Phi(\lambda)L^{-1}\left[\frac{1}{s+\lambda^2a^2}\right]+L^{-1}\left[\frac{1}{s+\lambda^2a^2}\bar{G}(\lambda,s)\right]\\=&\Phi(\lambda)e^{-a^2\lambda^2t}+G(\lambda,t)*e^{-a^2\lambda^2t}\\=&\Phi(\lambda)e^{-a^2\lambda^2t}+\int_0^t G(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}d\tau.\end{aligned}\tag{7}$$

Handling the Inverse Laplace Transform

- *The solution* $\tilde{U}$ *is expressed in terms of* s *and* λ.
- *Applying the inverse Laplace transform, we use the convolution property:*

$$L^{-1}\left[\frac{1}{s+\lambda^2a^2}\bar{G}(\lambda,s)\right]=L^{-1}\left[\frac{1}{s+\lambda^2a^2}\right]*G$$

  where $*$ *denotes convolution.*
- *If a term* ***does not depend on*** s, *it acts as a constant under the inverse Laplace transform.*
- *This results in:*

$$L^{-1}\left[\frac{1}{s+\lambda^2a^2}\Phi(\lambda)\right]=L^{-1}\left[constant(\lambda)\cdot\frac{1}{s+\lambda^2a^2}\right]=constant(\lambda)\cdot L^{-1}\left[\frac{1}{s+\lambda^2a^2}\right]$$

Final Step: Applying the Inverse Fourier Transform

- *After obtaining* $U(x,t)$, *apply the inverse Fourier transform:*

$$u(x,t)=F^{-1}[U(\lambda,t)]\tag{8}$$

- *This step restores the solution to its original spatial form.*

To find the solution to problem (2)–(3), we still need to take the inverse Fourier transform of $U(\lambda,t)$. *Taking the inverse Fourier transform of equation (7), we get:*

$$u(x,t)=F^{-1}\left[\Phi(\lambda)e^{-a^2\lambda^2t}\right]+\int_0^t F^{-1}\left[G(\lambda,\tau)e^{-a^2\lambda^2(t-\tau)}\right]d\tau.\leftarrow\boxed{\textit{Since } F^{-1} \textit{ is an integral for } x,\ \textit{commutable.}}$$

Using the convolution theorem, we get:

$$u(x,t)=\varphi(x)*F^{-1}\left[e^{-a^2\lambda^2t}\right]+\int_0^t f(x,\tau)*F^{-1}\left[e^{-a^2\lambda^2(t-\tau)}\right]d\tau.$$

Using the conclusion:

$$F^{-1}\left[e^{-\lambda^2t}\right]=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\quad(t>0)$$

we know

$$F^{-1}\left[e^{-a^2\lambda^2t}\right]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}.$$

Thus, we obtain:

$$u(x,t) = \varphi(x) * \left(\frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}\right) + \int_0^t f(x,\tau) * \left(\frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{x^2}{4a^2(t-\tau)}}\right) d\tau.$$

This is the solution to the original problem.

Ex 0.7. *Use Fourier Transform to Solve the Following Problem*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x). \leftarrow \boxed{\textit{initial data: tranf. } x \textit{ by Fourier}} \end{cases} \tag{9}$$

$$\uparrow \boxed{\textit{solved by the traveling wave method}}$$

Solution. *Assume*

$$\lim_{|x|\to\infty} u(x,t) = \lim_{|x|\to\infty} u_t(x,t) = 0.$$

(One might also try Laplace transform, but this hidden condition is needed.)

- *When applying the Laplace transform to t (since $t > 0$), derivatives transform as follows:*
  - *The second derivative u_{tt} transforms into $s^2 L[u]$ with initial condition terms.*
  - *Initial conditions introduce additional terms, but* ***boundary conditions*** *may be* ***lost****.*
- *The resulting equation is a second-order ODE in x, but* ***lacks explicit boundary conditions for*** *x.*
- *To solve for x, implicit conditions must be identified, such as:*
  - *A common assumption: u and u_x approach zero at infinity.*
  - *However, specific problems may require different assumptions.*
- *If Laplace transform is used instead of the standard approach, additional* ***hidden conditions*** *must be* ***extracted****.*
- *Therefore, in such cases, it is often* ***preferable*** *to use the Fourier transform instead.*

Taking the Fourier transform of both sides of equation (9) with respect to x, denote:

$$F[u(x,t)] = U(\lambda,t), \quad F[\varphi(x)] = \Phi(\lambda), \quad F[\psi(x)] = \Psi(\lambda).$$

Thus, we obtain:

$$\begin{cases} \frac{d^2U}{dt^2} = -a^2\lambda^2 U, \\ U(\lambda,t)\big|_{t=0} = \Phi(\lambda), \quad \frac{dU}{dt}(\lambda,t)\big|_{t=0} = \Psi(\lambda). \end{cases} \leftarrow \boxed{\partial_t \textit{ and } \int \textit{ (i.e., } F\textit{) are commutable.}} \tag{10}$$

Since $u_t\big|_{t=0} = \psi$, we obtain

$$F[\psi] = \int_{-\infty}^{+\infty} u_t(x,t)\Big|_{t=0} e^{-i\lambda x}\, dx = \left(\int_{-\infty}^{+\infty} u_t e^{-i\lambda x}\, dx\right)\Bigg|_{t=0} = \partial_t \left(\int_{-\infty}^{+\infty} u e^{-i\lambda x}\, dx\right)\Bigg|_{t=0}$$

Equation (10) is an ordinary differential equation with parameter λ. The solution to this initial value problem is:

$$U(\lambda,t) = \Phi(\lambda)\cos(a\lambda t) + \frac{\Psi(\lambda)}{a\lambda}\sin(a\lambda t). \tag{11}$$

Taking the inverse Fourier transform of equation (11), we get:

$$
\begin{aligned}
u(x,t) =& F^{-1}[\Phi(\lambda)\cos(a\lambda t)] + F^{-1}\left[\frac{\Psi(\lambda)}{a\lambda}\sin(a\lambda t)\right] \\
=& \varphi(x) * F^{-1}[\cos(a\lambda t)] + \frac{1}{a}\psi(x) * F^{-1}\left[\frac{\sin(a\lambda t)}{\lambda}\right].
\end{aligned} \tag{12}
$$

Using the conclusions:

$$
F^{-1}[\cos(a\lambda)] = \frac{1}{2}[\delta(x+a)+\delta(x-a)] \quad \textit{and} \quad F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \le m.
$$

Thus, we obtain:

$$
F^{-1}[\Phi(\lambda)\cos(a\lambda t)] = \varphi(x) * \frac{1}{2}[\delta(x+at)+\delta(x-at)] = \frac{1}{2}[\varphi(x+at)+\varphi(x-at)].
$$

$$
\begin{aligned}
& F^{-1}[\Phi(\lambda)] * F^{-1}[\cos(a\lambda t)] \\
=& \varphi(x) * \frac{1}{2}[\delta(x+at)+\delta(x-at)] \\
=& \frac{1}{2}\int_{-\infty}^{\infty} \varphi(x-\xi)[\delta(\xi+at)+\delta(\xi-at)]d\xi \leftarrow \boxed{\textit{use} \int_{-\infty}^{\infty} f(\xi)\delta(\xi-x_0)d\xi = f(x_0)} \\
=& \frac{1}{2}[\varphi(x+at)+\varphi(x-at)].
\end{aligned}
$$

Substituting the results into equation (12), the solution to the original problem (9) is:

$$
u(x,t) = \frac{1}{2}[\varphi(x+at)+\varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha.
$$

Ex 0.8. *Solve the Following Problem*

$$
\begin{cases}
u_{xx} + u_{yy} = 0 & (-\infty < x < +\infty, y > 0), \\
u\big|_{y=0} = g(x), & \lim_{x^2+y^2\to\infty} u(x,y) = 0.
\end{cases} \tag{13}
$$

Choice of Transformation

- Based on the first condition, since the given boundary condition involves y, we should transform with respect to x, a Fourier transform with respect to x, accordingly.
- However, the second condition complicates the choice, making it difficult to determine the proper action.
- Despite this, we prioritize the first condition, as transforming x ensures a boundary condition first, ensuring that **at least one** boundary condition is **properly handled**.
- Next, analyze the given second condition and determine **necessary modifications on the second condition** to facilitate solving the problem.

Solution. *Take the Fourier transform of equation (13) with respect to x, denote:*

$$
F[u(x,y)] = U(\lambda,y), \quad F[g(x)] = G(\lambda).
$$

Equation (13) becomes:

$$
\begin{cases}
\frac{d^2U}{dy^2} - \lambda^2 U = 0, \\
U(\lambda,0) = G(\lambda), \quad \lim_{y\to\infty} U(\lambda,y) = 0.
\end{cases} \tag{14}
$$

Handling the Limit Condition

- *The second condition is crucial. Since we have chosen to transform x, we need to ensure that we can transform (Fourier on x) the second boundary condition as well.*
- *To achieve this, x should not appear within the limit of the second condition; otherwise, the transformation cannot proceed.*

$$\begin{cases} \underbrace{F\left(\lim_{x^2+y^2\to\infty} u(x,y)\right)}_{\updownarrow} = 0 \\ \int_{-\infty}^{+\infty} \lim_{x^2+y^2\to\infty} u\cdot e^{-i\lambda x}\,dx \overset{weaken}{\Longrightarrow} \int \lim_{y\to\infty} u\cdot e^{-i\lambda x}\,dx = \lim_{y\to\infty} F(u) = 0 \end{cases}$$

- *This transformation allows us to* ***replace the original condition*** *with a* ***necessary*** *but* ***weaker*** *form.*
- *This original condition is stronger because it requires convergence to zero in all radial directions as $r\to\infty$, while the weaker condition only applies along the y-axis.*
- *The weaker condition does not guarantee that the function tends to zero along, for instance, the x-axis, making it insufficient in certain cases.*
- *Therefore, we replace the stronger condition with a weaker necessary condition to facilitate the transformation.*

Uniqueness of the Solution

- *A* ***condition*** *that is* ***too strong*** *may lead to* ***no solution,*** *while a* ***too-weak*** *condition may result in* ***non-uniqueness****.*
- *Using the* ***weakest necessary condition*** *does* ***not alter*** *the* ***uniqueness*** *of the solution.*
- *This can be understood via the "****bucket theory****" analogy:* ***weakening a non-determining condition*** *does* ***not change*** *the final solution (see Fig. 2).*
- *Since we can obtain a solution under the* ***weaker condition*** *which ensures the existence of the solution, the uniqueness of the solution are mathematically provable.*
- *Moreover, the existence and uniqueness of the original problem can also be established.*
- *Since the weaker condition still leads to a unique solution, the solution to the weaker condition problem is identical to that of the original problem.*

Solving equation (14) gives:

$$U(\lambda,y) = G(\lambda)e^{-|\lambda|y}.$$

Taking the inverse Fourier transform of the above equation, we get:

$$u(x,y) = F^{-1}[G(\lambda)e^{-|\lambda|y}] = g(x) * F^{-1}[e^{-|\lambda|y}]$$

Using the conclusion:

$$F^{-1}[e^{-|\lambda|y}] = \frac{y}{\pi(y^2+x^2)} \quad (y>0).$$

Thus, the solution to the original problem (13) is:

$$u(x,y) = \frac{y}{\pi}g(x) * \frac{1}{y^2+x^2} = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)}{y^2+(x-\xi)^2}d\xi.$$

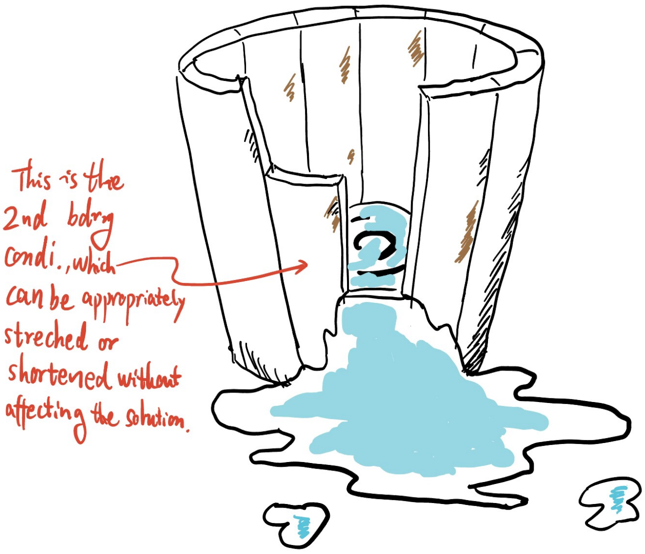

Figure 2: Bucket theory

Ex 0.9. *Solve the Following Problem*

$$u_t = a^2 u_{xx}, \quad (x > 0, t > 0), \tag{15}$$
$$u\big|_{t=0} = 0, \tag{16}$$
$$u\big|_{x=0} = f(t), \tag{17}$$
$$|u(x,t)| < M. \tag{18}$$

- This problem involves both initial and boundary conditions.
- The key question is: which transformation should be used in this case?
- However, only the Laplace transform can be applied because $x > 0$ and $t > 0$, which align with its definition.
- The best approach is to apply the Laplace transform with respect to t since its initial data is zero.
- Transforming with respect to t is preferable because it simplifies the equation.

Solution. *Take the Laplace transform of equations* (15), (17) *and* (18) *with respect to* t, *denote:*

$$L[u(x,t)] = U(x,s), \quad L[f(t)] = F(s).$$

Equations (15)–(18) *become:*

$$a^2 \frac{d^2U}{dx^2} - \underbrace{s}_{Note\ s \in \mathbb{C}} U = 0, \tag{19}$$
$$U(x,s)\big|_{x=0} = F(s), \tag{20}$$
$$|U(x,s)| < \overline{M} \leftarrow \boxed{\textit{different from } M, \textit{ bucket theory}}. \tag{21}$$

- *Condition (21) weakens Condition (18) like the previous example. That is, (21) is a necessary condition of (18).*
- *More specifically,*

$$|U(x,s)| = \left| \int_0^{+\infty} u(x,t)e^{-st}\, dt \right| \le M \int_0^{+\infty} e^{-Re(s)\cdot t}\, dt < \infty.$$

- ***Question:*** *Can the general solution be found using methods for second-order constant coefficient ODEs?*
- *The key issue: $s \in \mathbb{C}$ is a **complex number**, whereas standard methods are typically applied in the **real number**, i.e., $s \in \mathbb{R}$.*
- *In fact, the derivation of the general solution formula also applies to $s \in \mathbb{C}$. This derivation of the general solution formula holds identically in the complex domain, meaning the same process applies. That is, **assume a formal solution** $U = e^{\beta x}$ (trial method). Substituting it into (19) yields*

$$a^2\beta^2 e^{\beta x} - se^{\beta x} = 0 \Rightarrow a^2\beta^2 - s = 0 \Rightarrow \beta^2 = \frac{s}{a^2} \Rightarrow \beta = \pm\frac{\sqrt{s}}{a}.$$

- *Therefore, $s \in \mathbb{C}$ can be **treated as a real number**, and the general solution formula can be used as before.*

The general solution to equation (19) is:

$$U(x,s) = c_1 e^{-\frac{\sqrt{s}}{a}x} + c_2 e^{\frac{\sqrt{s}}{a}x}.$$

From condition (21), we know $c_2 = 0$. From condition (20), we know $c_1 = F(s)$. Thus, we have:

$$U(x,s) = F(s)e^{-\frac{\sqrt{s}}{a}x}. \tag{22}$$

Taking the inverse Laplace transform of equation (22), we get:

$$u(x,t) = L^{-1}[F(s)e^{-\frac{\sqrt{s}}{a}x}] = \underbrace{f(t)}_{\text{do not need transf. explicitly since it will be transformed back}} * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}].$$

Using the conclusion:

$$L^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{2}{\sqrt{\pi}}\int_a^{\infty} \frac{e^{-y^2}}{2\sqrt{t}}dy.$$

Thus, we have:

$$L^{-1}\left[\frac{1}{s}e^{-\frac{x\sqrt{s}}{a}}\right] = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-y^2}dy.$$

Using the first differentiation theorem of the Laplace transform again, we have

$$L^{-1}\left[e^{-\frac{x}{a}\sqrt{s}}\right] = L^{-1}\left[s\cdot\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}\right] = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}\int_t^{+\infty}\frac{1}{2a\sqrt{y}}e^{-y^2}dy\right] = \frac{x}{2a\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4a^2t}}.$$

Thus, the solution to the original problem (15)–(18) is:

$$u(x,t) = f(t) * \frac{x}{2a\sqrt{\pi}t^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2t}} = \frac{x}{2a\sqrt{\pi}}\int_0^t f(\tau)\frac{1}{(t-\tau)^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau.$$