Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 10-2

Recall

$$C_m = \frac{\int_0^R rf(r)J_n\left(\frac{\mu_m^{(n)}}{R}r\right)dr}{\frac{R^2}{2}J_{n+1}^2(\mu_m^{(n)})}. \tag{1}$$

1 5.4 Applications of Bessel Functions

Bessel functions have extremely wide applications. In this section, we only choose the simplest problems to illustrate the **key points and steps** of using Bessel functions to solve mathematical physics problems.

- Note: In higher-dimensional problems involving the **Laplace operator** on a **cylindrical domain** or **circular disk**, Bessel equations and functions arise.
- However, to **simplify calculations**, this section typically **assumes axisymmetry**, **reducing** the problem to a **two-dimensional** one.
- If axisymmetry is **not** present, the approach outlined in Section 5.1 can be followed.

Ex 1.1 (Heat Conduction Problem)**.** *Consider a uniform thin circular disk with a radius of 1. The temperature on the circumference is maintained at 0 degrees, and the initial temperature distribution inside the disk is $1-r^2$, where r is the polar radius of any point inside the disk. Try to find the temperature distribution inside the disk.*

Solution. *The temperature u to be found satisfies the two dimensional homogeneous heat conduction equation. Since the solution domain is a circular domain,* ***polar coordinates*** *are used. Because the definite solution conditions are* ***independent of*** *θ, so $u=u(r,t)$. Then the definite solution problem is as follows:*

$$u_t = a^2(u_{rr} + \frac{1}{r}u_r), \quad (0<r<1), \leftarrow \boxed{\textit{Homogeneous equation}} \tag{2}$$

$$u|_{r=1} = 0, \leftarrow \boxed{\textit{Homogeneous boundary}} \tag{3}$$

$$u|_{t=0} = 1-r^2. \tag{4}$$

Apply the ***method of separation of variables****. Let $u(r,t)=R(r)T(t)$ and substitute it into (2), we get*

$$RT' = a^2(R'' + \frac{1}{r}R')T.$$

$$\Longrightarrow \frac{T'}{a^2T} = \frac{R''+\frac{1}{r}R'}{R} = -\lambda.$$

From this, we obtain

$$T' + \lambda a^2 T = 0 \tag{5}$$

$$r^2R'' + rR' + \lambda r^2 R = 0 \tag{6}$$

From the physical meaning of the problem, we know that the temperature function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy the natural boundary condition

$$|R(0)| < +\infty \tag{7}$$

*And from the **homogeneous boundary condition** (3), we can get*

$$R(1) = 0 \tag{8}$$

Equations (6)-(8) form the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} r^2R'' + rR' + \lambda r^2 R = 0 \leftarrow \boxed{\textit{Identify } 0\textit{-order Bessel equation}} \\ |R(0)| < +\infty, \ R(1) = 0 \end{cases}$$

The general solution of the Bessel equation of order 0 (6) is

$$R(r) = CJ_0(\sqrt{\lambda}r) + DY_0(\sqrt{\lambda}r)$$

From the condition (7) $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (8) $R(1) = 0$, we get $J_0(\sqrt{\lambda}) = 0$, that is, $\sqrt{\lambda}$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (6) under the conditions (7) and (8) are

$$\begin{cases} \lambda_m^{(0)} = (\mu_m^{(0)})^2 & (m = 1, 2, \cdots) \\ R_m(r) = J_0(\mu_m^{(0)}r) \end{cases}$$

Now consider the equation

$$T' + \lambda a^2 T = 0$$

Substitute $\lambda_m^{(0)}$ into equation (5) to obtain its general solution

$$T_m(t) = C_m e^{-(\mu_m^{(0)}a)^2 t}$$

Then, using $u(r,t) = R(r)T(t)$, we can get

$$u_m(r,t) = C_m e^{-(\mu_m^{(0)}a)^2 t} J_0(\mu_m^{(0)}r)$$

According to the superposition principle, the solution of equation (2) that satisfies condition (3) is

$$u(r,t) = \sum_{m=1}^{\infty} C_m e^{-(\mu_m^{(0)}a)^2 t} J_0(\mu_m^{(0)}r) \tag{9}$$

Then, from the initial condition (5), we have

$$u(r,0) = \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(0)}r) = 1 - r^2$$

Using the Fourier-Bessel coefficient formula (1), we have

$$C_m = \frac{\int_0^1 r(1-r^2)J_0(\mu_m^{(0)}r)dr}{\frac{1}{2}J_1^2(\mu_m^{(0)})} \leftarrow \boxed{\textit{Don't forget the weight } r}$$

First, calculate the numerator. Let $\mu_m^{(0)}r = x$, then

$$\begin{aligned} \int_0^1 rJ_0(\mu_m^{(0)}r)dr &= \frac{1}{(\mu_m^{(0)})^2}\int_0^{\mu_m^{(0)}} xJ_0(x)dx \\ &= \frac{1}{(\mu_m^{(0)})^2}[xJ_1(x)]_0^{\mu_m^{(0)}} = \frac{1}{\mu_m^{(0)}}J_1(\mu_m^{(0)}) \end{aligned}$$

$$
\begin{tikzcd}
xJ_1 \arrow[r, bend left, "\frac{d}{dx}"] & x\boxed{J_0} \arrow[l, bend left, "\int"] \arrow[d, bend left, "\frac{d}{dx}"] \\
& \boxed{-J_1} \arrow[u, bend left, "\int"]
\end{tikzcd}
$$

Recall $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ *and*

$$
\begin{aligned}
\int_0^1 r^3 J_0(\mu_m^{(0)} r)dr &= \frac{1}{(\mu_m^{(0)})^4}\int_0^{\mu_m^{(0)}} x^3 J_0(x)dx \\
&= \frac{1}{(\mu_m^{(0)})^4}\int_0^{\mu_m^{(0)}} x^2 \cdot xJ_0(x)dx \\
&= \frac{1}{(\mu_m^{(0)})^4}\left[x^3 J_1(x)\Big|_0^{\mu_m^{(0)}} - 2\int_0^{\mu_m^{(0)}} x^2 J_1(x)dx\right]
\end{aligned}
$$

$\uparrow$ $\boxed{\textit{Another method: } \int_0^{\mu_m^{(0)}} x^2 J_1(x)dx = -\int_0^{\mu_m^{(0)}} x^2 dJ_0(x)}$

$$
\begin{aligned}
&= \frac{1}{(\mu_m^{(0)})^4}\left[(\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2x^2 J_2(x)\Big|_0^{\mu_m^{(0)}}\right] \\
&= \frac{1}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) - \frac{2}{(\mu_m^{(0)})^2} J_2(\mu_m^{(0)})
\end{aligned}
$$

- *The problem involves integrating the product of polynomials and Bessel functions.*
- *To integrate such products,* ***polynomials*** *are typically* ***decomposed****.*
- *The approach often involves* ***breaking down the polynomial into simpler terms*** *that can be more* ***easily integrated with Bessel functions****. That is, to generate* xJ_0, $x^n J_{n-1}$ *or* $x^{-n} J_{n+1}$ *terms which can be integrated.*

Substitute into C_m *to get*

$$
C_m = \frac{4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})}
$$

Substitute C_m *into* (9)*, and the solution of the problem* (2)-(4) *is*

$$
u(r,t) = \sum_{m=1}^{\infty} \frac{4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} J_0(\mu_m^{(0)} r) e^{-(\mu_m^{(0)} a)^2 t}
$$

Ex 1.2 (Electric Potential in a Cylindrical Domain)**.** *For an empty cylinder composed of conductor walls, with the height of the cylinder being* h *and the radius being* b*. Suppose the electric potential of the upper base is* U*, and the electric potentials of the side surface and the lower base are 0. Try to find the electric potential inside the cylinder.*

Solution. *Since the region is cylindrical, the cylindrical coordinate system is used. Because the boundary conditions are* ***independent of the angle*** φ*, the electric potential* u *to be found is only a function of* *two variables ρ and z, that is $u = u(\rho, z)$. Then the definite solution problem is as follows:*

$$u_{\rho\rho} + \frac{1}{\rho}u_\rho + u_{zz} = 0 \quad (0 < \rho < b, 0 < z < h), \leftarrow \boxed{\textit{Homogeneous equation}} \tag{10}$$

$$u|_{z=0} = 0,\, u|_{z=h} = U, \leftarrow \boxed{\textit{determine the coefficients}} \tag{11}$$

$$u|_{\rho=b} = 0, \leftarrow \boxed{\textit{Homogeneous boundary}} \tag{12}$$

where U is a constant.

*Apply the **method of separation of variables**. Let $u(\rho, z) = R(\rho)Z(z)$ and substitute it into (10), we get*

$$R''Z + \frac{1}{\rho}R'Z + RZ'' = 0$$

$$\frac{R'' + \frac{1}{\rho}R'}{R} = -\frac{Z''}{Z} = -\lambda$$

From this, we obtain

$$Z'' - \lambda Z = 0 \tag{13}$$

$$\rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \tag{14}$$

From the physical meaning of the problem, we know that the electric potential function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy

$$|R(0)| < +\infty. \tag{15}$$

And from the homogeneous boundary condition (12), we can get

$$R(b) = 0 \tag{16}$$

Equations (14)-(16) form the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} \rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \\ |R(0)| < +\infty, \;\; R(b) = 0 \end{cases}$$

The general solution of the Bessel equation of order 0 (14) is

$$R(\rho) = CJ_0(\sqrt{\lambda}\rho) + DY_0(\sqrt{\lambda}\rho)$$

From the condition (15) $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (16) $R(b) = 0$, we get $J_0(\sqrt{\lambda}b) = 0$, that is, $\sqrt{\lambda}b$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (14) under the conditions (15) and (16) are

$$\lambda_m^{(0)} = \left(\frac{\mu_m^{(0)}}{b}\right)^2, R_m(\rho) = J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) \quad (m = 1, 2, \cdots)$$

Now consider the equation

$$Z'' - \lambda Z = 0 \tag{17}$$

Substitute $\lambda_m^{(0)}$ into equation (17) to obtain its general solution

$$Z_m(z) = C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}.$$

Thus

$$u_m(\rho, z) = (C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right).$$

According to the superposition principle, the solution of equation (10) that satisfies condition (12) is

$$u(\rho, z) = \sum_{m=1}^{\infty} (C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) \tag{18}$$

From the first formula in condition (11), we have

$$u(\rho,0)=\sum_{m=1}^{\infty}(C_m+D_m)J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right)=0$$

So we get (comparing the coeficients–orthoganlity or linear independency)

$$C_m+D_m=0 \quad (m=1,2,\cdots) \tag{19}$$

From the second formula in condition (11), we have

$$u(\rho,h)=\sum_{m=1}^{\infty}(C_m e^{\frac{\mu_m^{(0)}}{b}h}+D_m e^{-\frac{\mu_m^{(0)}}{b}h})J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right)=U$$

Using the Fourier-Bessel coefficient formula (1), we have

$$C_m e^{\frac{\mu_m^{(0)}}{b}h}+D_m e^{-\frac{\mu_m^{(0)}}{b}h}=\frac{\int_0^b \rho U J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right)d\rho}{\frac{b^2}{2}J_1^2(\mu_m^{(0)})} \tag{20}$$

First, calculate the numerator. Let $\frac{\mu_m^{(0)}}{b}\rho=x$, *then*

$$\begin{aligned}\int_0^b \rho U J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right)d\rho &= \frac{b^2U}{(\mu_m^{(0)})^2}\int_0^{\mu_m^{(0)}} xJ_0(x)dx\\ &=\frac{b^2U}{(\mu_m^{(0)})^2}[xJ_1(x)]_0^{\mu_m^{(0)}}=\frac{b^2U}{\mu_m^{(0)}}J_1(\mu_m^{(0)})\end{aligned}$$

Then formula (21) simplifies to

$$C_m e^{\frac{\mu_m^{(0)}}{b}h}+D_m e^{-\frac{\mu_m^{(0)}}{b}h}=\frac{2U}{\mu_m^{(0)}J_1(\mu_m^{(0)})}$$

Solve equations (19) and (20) simultaneously, with $shx=\frac{e^x-e^{-x}}{2}$, *we get*

$$C_m=\frac{U}{\mu_m^{(0)}J_1(\mu_m^{(0)})sh\frac{\mu_m^{(0)}}{b}h}, D_m=-\frac{U}{\mu_m^{(0)}J_1(\mu_m^{(0)})sh\frac{\mu_m^{(0)}}{b}h}$$

Substitute C_m *and* D_m *into (18), and the solution of the problem (10)-(23) is*

$$u(\rho,z)=\sum_{m=1}^{\infty}\frac{2U}{\mu_m^{(0)}sh\frac{\mu_m^{(0)}}{b}hJ_1(\mu_m^{(0)})}sh\frac{\mu_m^{(0)}}{b}zJ_0\left(\frac{\mu_m^{(0)}}{b}\rho\right).$$

Ex 1.3 (Axisymmetric Vibration Problem of a Circular Membrane)**.** *Consider a circular membrane with a radius of B. The circumference is fixed. If a very small height $h>0$ is lifted at the center of the membrane and then held stationary, and suddenly released to let it vibrate, try to find the vibration law of the membrane.*

Solution. *Since the equation is homogeneous and the definite - solution conditions are independent of the angle θ, in the polar coordinate system, the displacement function u is only a function of two variables r and t, that is $u=u(r,t)$. Then the definite solution problem is as follows:*

$$u_{tt}=a^2(u_{rr}+\frac{1}{r}u_r) \quad (0<r<B), \tag{21}$$

$$u|_{r=B}=0, \tag{22}$$

$$u|_{t=0}=h(1-\frac{r}{B}),\ u_t|_{t=0}=0. \tag{23}$$

Apply the method of separation of variables. Let $u(r,t) = R(r)T(t)$ and substitute it into (21), we get

$$RT'' = a^2(R'' + \frac{1}{r}R')T$$

$$\Longrightarrow \frac{T''}{a^2T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda$$

From this, we obtain

$$T'' + a^2\lambda T = 0 \tag{24}$$

$$r^2R'' + rR' + \lambda r^2 R = 0 \tag{25}$$

From the physical meaning of the problem, we know that the displacement function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy

$$|R(0)| < +\infty$$

And from the homogeneous boundary condition (22), we can get

$$R(B) = 0 \tag{26}$$

This constitutes the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} r^2R'' + rR' + \lambda r^2 R = 0 \\ |R(0)| < +\infty, \ R(B) = 0 \end{cases}$$

The general solution of the Bessel equation of order 0 (25) *is*

$$R(r) = CJ_0(\sqrt{\lambda}r) + DY_0(\sqrt{\lambda}r)$$

From the boundedness condition $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (26) $R(B) = 0$, we get $J_0(\sqrt{\lambda}B) = 0$, that is, $\sqrt{\lambda}B$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (25) under the boundedness condition and (26) are

$$\lambda_m^{(0)} = \left(\frac{\mu_m^{(0)}}{B}\right)^2, R_m(r) = J_0\left(\frac{\mu_m^{(0)}}{B}r\right) \quad (m = 1, 2, \cdots)$$

Now consider the equation

$$T'' + a^2\lambda T = 0$$

Substitute $\lambda_m^{(0)}$ into equation (24) to obtain its general solution

$$T_m(t) = a_m \cos\frac{a\mu_m^{(0)}}{B}t + b_m \sin\frac{a\mu_m^{(0)}}{B}t$$

Thus

$$u_m(r,t) = (a_m \cos\frac{a\mu_m^{(0)}}{B}t + b_m \sin\frac{a\mu_m^{(0)}}{B}t)J_0\left(\frac{\mu_m^{(0)}}{B}r\right)$$

According to the superposition principle, the solution of equation (21) that satisfies condition (22) is

$$u(r,t) = \sum_{m=1}^{\infty}(a_m \cos\frac{a\mu_m^{(0)}}{B}t + b_m \sin\frac{a\mu_m^{(0)}}{B}t)J_0\left(\frac{\mu_m^{(0)}}{B}r\right) \tag{27}$$

From the second formula in the initial condition (23), we have

$$\sum_{m=1}^{\infty}\frac{a\mu_m^{(0)}}{B}b_mJ_0\left(\frac{\mu_m^{(0)}}{B}r\right) = 0$$

So we get

$$b_m = 0 \quad (m = 1, 2, \cdots)$$

From the first formula in the initial condition (23), we have

$$u(r,0)=\sum_{m=1}^{\infty}a_m J_0\left(\frac{\mu_m^{(0)}}{B}r\right)=h(1-\frac{r}{B})$$

Using the Fourier-Bessel coefficient formula (1), we have

$$a_m=\frac{\int_0^B rh(1-\frac{r}{B})J_0\left(\frac{\mu_m^{(0)}}{B}r\right)dr}{\frac{B^2}{2}J_1^2(\mu_m^{(0)})} \tag{28}$$

First, calculate the numerator. Let $\frac{\mu_m^{(0)}}{B}r=x$, *then*

$$\begin{aligned}h\int_0^B rJ_0\left(\frac{\mu_m^{(0)}}{B}r\right)dr=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^2}\int_0^{\mu_m^{(0)}}xJ_0(x)dx\\=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^2}\left[xJ_1(x)\right]_0^{\mu_m^{(0)}}=\frac{hB^2}{\mu_m^{(0)}}J_1\left(\mu_m^{(0)}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{h}{B}\int_0^B r^2J_0\left(\frac{\mu_m^{(0)}}{B}r\right)dr=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^3}\int_0^{\mu_m^{(0)}}x^2J_0(x)dx\\=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^3}\int_0^{\mu_m^{(0)}}x\cdot xJ_0(x)dx\\=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^3}\left[x^2J_1(x)\Big|_0^{\mu_m^{(0)}}-\int_0^{\mu_m^{(0)}}xJ_1(x)dx\right]\\=&\frac{hB^2}{\left(\mu_m^{(0)}\right)^3}\left[\left(\mu_m^{(0)}\right)^2J_1\left(\mu_m^{(0)}\right)+xJ_0(x)\Big|_0^{\mu_m^{(0)}}-\int_0^{\mu_m^{(0)}}J_0(x)dx\right]\\=&\frac{hB^2}{\mu_m^{(0)}}J_1\left(\mu_m^{(0)}\right)-\frac{hB^2}{\left(\mu_m^{(0)}\right)^3}\underbrace{\int_0^{\mu_m^{(0)}}J_0(x)dx}_{Generalized\ Hypergeometric\ Function}.\end{aligned}$$

Using the Fourier-Bessel coefficient formula (1), we have:

$$h\int_0^B rJ_0\left(\frac{\mu_m^{(0)}}{B}r\right)dr=\frac{hB^2}{\mu_m^{(0)}}J_1(\mu_m^{(0)})$$

$$\frac{h}{B}\int_0^B r^2J_0\left(\frac{\mu_m^{(0)}}{B}r\right)dr=\frac{hB^2}{\mu_m^{(0)}}J_1(\mu_m^{(0)})-\frac{hB^2}{(\mu_m^{(0)})^3}\int_0^{\mu_m^{(0)}}J_0(x)dx$$

Substitute the above results into formula (28):

$$a_m=\frac{2h}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}\int_0^{\mu_m^{(0)}}J_0(x)dx$$

$$a_m=\frac{2h}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}\int_0^{\mu_m^{(0)}}J_0(x)dx,b_m=0\quad(m=1,2,\cdots)$$

Substitute a_m *and* b_m *into (27), and the solution of the problem (21)-(23) is:*

$$u(r,t)=\sum_{m=1}^{\infty}\left[\frac{2h}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}\int_0^{\mu_m^{(0)}}J_0(x)dx\right]\cos\frac{a\mu_m^{(0)}t}{B}J_0\left(\frac{\mu_m^{(0)}}{B}r\right).$$

Ex 1.4. *Solve the following definite solution problem:*

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) \quad (0 < r < 1), \leftarrow \boxed{\text{leads to 1st oder Bessel equation}} \tag{29}$$

$$u|_{r=1} = 0, |u(r,t)| < +\infty, \tag{30}$$

$$u|_{t=0} = 1 - r. \tag{31}$$

- Replacing u_t with u_{tt} transforms the equation into a wave equation.
- Replacing u_t with $-u_{zz}$ transforms the equation into a potential equation in cylindrical coordinates.
- These substitutions allow for the creation of various new problems in PDEs.
- The corresponding solutions can be found by substituting the appropriate parts of the five steps of the method of separation of variables.

Solution. *Apply the method of separation of variables. Let $u(r,t) = R(r)T(t)$ and substitute it into (29), we get*

$$RT' = a^2 \left(R'' + \frac{1}{r} R' - \frac{1}{r^2} R \right) T$$

or

$$\frac{T'}{a^2 T} = \frac{R'' + \frac{1}{r} R' - \frac{1}{r^2} R}{R} = -\lambda$$

From this, we obtain

$$T' + \lambda a^2 T = 0 \tag{32}$$

$$r^2 R'' + r R' + (\lambda r^2 - 1) R = 0 \tag{33}$$

Note that this equation is a Bessel equation of order one. Using the definite solution conditions (30), we can get

$$R(1) = 0, |R(0)| < +\infty \tag{34}$$

(33) and (34) form the eigenvalue problem of the Bessel equation of order one. The corresponding eigenvalues and eigenfunctions are respectively

$$\lambda_m^{(1)} = (\mu_m^{(1)})^2, R_m(r) = J_1(\mu_m^{(1)} r) \quad (m = 1, 2, \cdots)$$

Substitute the eigenvalues into (32), we can get

$$T_m(t) = C_m e^{-(\mu_m^{(1)} a)^2 t}$$

Then, using $u(r,t) = R(r)T(t)$, we can get

$$u_m(r,t) = C_m e^{-(\mu_m^{(1)} a)^2 t} J_1(\mu_m^{(1)} r) \quad (m = 1, 2, \cdots)$$

According to the superposition principle, the general solution that satisfies equation (29) and condition (30) is

$$u(r,t) = \sum_{m=1}^{\infty} C_m e^{-(\mu_m^{(1)} a)^2 t} J_1(\mu_m^{(1)} r). \tag{35}$$

Finally, from the initial condition (31), we have

$$1 - r = \sum_{m=1}^{\infty} C_m J_1(\mu_m^{(1)} r)$$

Using the Fourier-Bessel coefficient formula (1), we have

$$C_m = \frac{\int_0^1 r(1-r) J_1(\mu_m^{(1)} r) dr}{\frac{1}{2} J_1^2(\mu_m^{(1)})} \tag{36}$$

First, calculate the numerator. Let $\mu_m^{(1)} r = x$, *then*

$$\int_0^1 r J_1(\mu_m^{(1)} r) dr = \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} x J_1(x) dx$$

$$= -\frac{1}{(\mu_m^{(1)})^2} [x J_0(x)]_0^{\mu_m^{(1)}} + \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} J_0(x) dx$$

$$= -\frac{J_0(\mu_m^{(1)})}{\mu_m^{(1)}} + \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} J_0(x) dx$$

and

$$\int_0^1 r^2 J_1(\mu_m^{(1)} r) dr = \frac{1}{(\mu_m^{(1)})^3} \int_0^{\mu_m^{(1)}} x^2 J_1(x) dx$$

$$= \frac{1}{(\mu_m^{(1)})^3} [x^2 J_2(x)]_0^{\mu_m^{(1)}} = \frac{J_2(\mu_m^{(1)})}{\mu_m^{(1)}}$$

Note that, from the recurrence formula, we can get

$$J_0(\mu_m^{(1)}) + J_2(\mu_m^{(1)}) = \frac{2}{\mu_m^{(1)}} J_1(\mu_m^{(1)}) = 0$$

Substitute the above results into (36) *and simplify to get*

$$C_m = \frac{2}{(\mu_m^{(1)})^2 J_1^2(\mu_m^{(1)})} \int_0^{\mu_m^{(1)}} J_0(x) dx$$

Substitute the value of C_m *into expression* (35) *to obtain the solution of the original problem* (29)-(31).

Ex 1.5. *Solve the following definite solution problem:*

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r \right) + A \quad (0 < r < 1), \leftarrow \boxed{\text{Non-homogeneous eq.}} \tag{37}$$

$$u|_{r=1} = 0, |u(r,t)| < +\infty, \leftarrow \boxed{\text{Homogeneous bdry.}} \tag{38}$$

$$u|_{t=0} = 0, \leftarrow \boxed{\text{Homogeneous initial data}} \tag{39}$$

where A *is a constant.*

Solution. *Apply the* ***method of eigenfunctions*** *(recall §2.4).*

Step 1: *First, for the non-homogeneous equation* (37), *the corresponding homogeneous equation is*

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r \right).$$

The system of eigenfunctions that simultaneously satisfies the homogeneous boundary condition (38) *is the Bessel function system* $\{J_0(\mu_m^{(0)} r)\}_{m=1}^{\infty}$.

Step 2: *Assume the solution is*

$$u(r,t) = \sum_{m=1}^{\infty} u_m(t) J_0(\mu_m^{(0)} r) \tag{40}$$

where $u_m(t)$ *is a function of* t *to be determined.*

Step 3: *Expand the free term* A *in the equation into a Fourier-Bessel series according to the corresponding Bessel function system:*

$$A = \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)} r) \tag{41}$$

where the coefficients are

$$f_m(t) = \frac{\int_0^1 r \cdot A \cdot J_0(\mu_m^{(0)} r) dr}{\frac{1}{2} J_1^2(\mu_m^{(0)})} = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})} \quad (m = 1, 2, \cdots).$$

Step 4: *Substitute (40) and (41) into (37) and simplify to get*

$$\sum_{m=1}^{\infty} u'_m(t) J_0(\mu_m^{(0)} r) - \sum_{m=1}^{\infty} a^2 u_m(t) \underbrace{\left\{ [J_0(\mu_m^{(0)} r)]'' + \frac{1}{r} [J_0(\mu_m^{(0)} r)]' \right\}}_{\text{Hope to replace it to } J_0(\mu_m^{(0)} r)\text{, then one can compare the coef.}}$$

$$= \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)} r). \tag{42}$$

- *This step features a significant innovation: the derivative terms are replaced by the Bessel equation.*
- *Only by making this substitution so that it satisfies the Bessel equation, does the first step you took earlier amount to solving the corresponding homogeneous Sturm-Liouville (S-L) problem.*
- *Similar situations will also occur in Chapter 2, for example,* $(\sin \frac{n\pi}{l} x)'' \propto \sin \frac{n\pi}{l} x$. *(Recall and compare the relevant content and calculations from Chapter 2 "How to invent the method of eigenfunctions?")*
- *This process can be substituted in because it is designed to use the S-L equation to handle (Recall Chapter 2 "How to invent the method of eigenfunctions?").*

From the Bessel equation of order zero, we know that:

$$r^2 [J_0(\mu_m^{(0)} r)]'' + r [J_0(\mu_m^{(0)} r)]' + (\mu_m^{(0)})^2 r^2 J_0(\mu_m^{(0)} r) = 0$$

Naturally, we have

$$[J_0(\mu_m^{(0)} r)]'' + \frac{1}{r} [J_0(\mu_m^{(0)} r)]' = -(\mu_m^{(0)})^2 J_0(\mu_m^{(0)} r)$$

Substitute the above formula into (42) and simplify to get

$$\sum_{m=1}^{\infty} [u'_m(t) + (\mu_m^{(0)} a)^2 u_m(t)] J_0(\mu_m^{(0)} r) = \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)} r).$$

By comparing the coefficients of the like terms on both sides, we can get

$$u'_m(t) + (\mu_m^{(0)} a)^2 u_m(t) = f_m(t) = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})}.$$

Step 5: *From the initial condition (39), we can get*

$$u_m(0) = 0.$$

Step 6: *Apply the general solution formula of the first order linear differential equation or the Laplace transform method to obtain*

$$u_m(t) = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})} \int_0^t e^{-(\mu_m^{(0)} a)^2 (t-\tau)} d\tau = \frac{2A}{(\mu_m^{(0)})^3 a^2 J_1(\mu_m^{(0)})} (1 - e^{-(\mu_m^{(0)} a)^2 t})$$

Finally, substitute the value of $u_m(t)$ into formula (40) to obtain the solution of the definite solution problem (37)-(39):

$$u(r, t) = \frac{2A}{a^2} \sum_{m=1}^{\infty} \frac{(1 - e^{-(\mu_m^{(0)} a)^2 t})}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} J_0(\mu_m^{(0)} r)$$

Notes on Solving Non-homogeneous Boundary Value Problems

- Students may ask how to use auxiliary functions to handle non-homogeneous boundary conditions when encountering non-homogeneous boundary.
- The approach involves constructing an auxiliary function.
- After constructing the auxiliary function, remove it from the variable to transform the problem into one that can be solved using eigenfunction methods and separation of variables.
- Although we haven't covered auxiliary functions explicitly, the method from Chapter 2 still applies.

Exercise Class

Ex 1.6. *Calculate definite integrals and indefinite integrals.*

1. $\int_0^1 x^3 J_0(\alpha x)dx$, *where* α *is a positive zero point of the Bessel function* $J_0(x)$ *of order zero, (that is* $J_0(\alpha) = 0 \Rightarrow J_1(\alpha) \neq 0$*).*
2. $\int x^4 J_1(x)dx$
3. $\int J_3(x)dx$ *(Hint: Express it in terms of* J_0*,* J_1 *and* J_2*, use the recurrence relations* $x^n J_n \xrightarrow{diff} x^n J_{n-1}$, $x^{-n} J_n \xrightarrow{diff} -x^{-n} J_{n+1}$*)*

Solution. *(1) We will obtain two different expressions of the same result by different recurrence relations. Let* $t = \alpha x$*, then* $dx = \frac{1}{\alpha}dt$*.*

$$\begin{aligned}
\int_0^1 x^3 J_0(\alpha x)dx &= \frac{1}{\alpha^4}\int_0^\alpha t^3 J_0(t)dt \\
&= \frac{1}{\alpha^4}\int_0^\alpha t^2 \cdot tJ_0(t)dt \\
&= \frac{1}{\alpha^4}\int_0^\alpha t^2 d(tJ_1(t)) \leftarrow \boxed{\textit{Integrate by parts}} \\
&= \frac{1}{\alpha^4}[\alpha^3 J_1(\alpha) - 2\int_0^\alpha t^2 J_1(t)dt]
\end{aligned}$$

Method 1: *Using* $x^n J_n(x) \xrightarrow{diff} x^n J_{n-1}(x)$*, we obtain*

$$\begin{aligned}
&= \frac{1}{\alpha^4}[\alpha^3 J_1(\alpha) - 2[t^2 J_2(t)]_0^\alpha] \\
&= \frac{1}{\alpha^4}[\alpha^3 J_1(\alpha) - 2\alpha^2 J_2(\alpha)] \\
&= \frac{1}{\alpha}J_1(\alpha) - \frac{2}{\alpha^2}J_2(\alpha)
\end{aligned}$$

Method 2:

$$= \frac{1}{\alpha^4}[\alpha^3 J_1(\alpha) + 2\int_0^\alpha t^2 J_0'(t)dt]$$

Since $J_0(\alpha) = 0$*, then* $\int_0^\alpha t^2 J_0'(t)dt = [t^2 J_0(t)]_0^\alpha - 2\int_0^\alpha tJ_0(t)dt$

$$\begin{aligned}
&= \frac{1}{\alpha^4}[\alpha^3 J_1(\alpha) + 2[t^2 J_0(t)]_0^\alpha - 4\int_0^\alpha tJ_0(t)dt] \\
&= \frac{1}{\alpha}J_1(\alpha) - \frac{4}{\alpha^3}J_1(\alpha)
\end{aligned}$$

Remark 1.1. • *Compare Method 1 and Method 2:* $\frac{2}{\alpha^2}J_2(\alpha) = \frac{4}{\alpha^3}J_1(\alpha) \Rightarrow J_2(\alpha) = \frac{2}{\alpha}J_1(\alpha)$.

- *This can be proven by the recurrence relations* $J_2(\alpha) + J_0(\alpha) = \frac{2}{\alpha}J_1(\alpha)$ *and* $J_0(\alpha) = 0$.

(2) ***Method 1:*** *Using* $x^n J_n(x) \xrightarrow{diff} x^n J_{n-1}(x)$,

$$\begin{aligned}
\int x^4 J_1(x)dx &= \int x^2 \cdot x^2 J_1(x)dx \\
&= \int x^2 d(x^2 J_2(x)) \\
&= x^2 \cdot x^2 J_2(x) - \int x^2 J_2(x)dx^2 \\
&= x^4 J_2(x) - 2\int x^3 J_2(x)dx \\
&= x^4 J_2(x) - 2\int d(x^3 J_3(x)) \\
&= x^4 J_2(x) - 2x^3 J_3(x) + C
\end{aligned}$$

Method 2: *By* $J_1(x) = -J_0'(x)$ *and* $x^n J_n(x) \xrightarrow{diff} x^n J_{n-1}(x)$,

$$\begin{aligned}
\int x^4 J_1(x)dx &= -\int x^4 dJ_0(x) \\
&= -x^4 J_0(x) + \int J_0(x)dx^4 \\
&= -x^4 J_0(x) + 4\int x^3 J_0(x)dx \\
&= -x^4 J_0(x) + 4\int x^2 d(xJ_1(x)) \\
&= -x^4 J_0(x) + 4x^3 J_1(x) - 4\int xJ_1(x)dx^2 \\
&= -x^4 J_0(x) + 4x^3 J_1(x) - 8\int x^2 J_1(x)dx \\
&= -x^4 J_0(x) + 4x^3 J_1(x) - 8x^2 J_2(x) + C.
\end{aligned}$$

(3) Idea: reduce the order of J_3 *to* J_0 *and* J_1*, use relations* $x^{-n}J_n(x) \xrightarrow{diff} -x^{-n}J_{n+1}(x)$*, then* $-x^{-1}J_1 \longrightarrow x^{-1}J_2$.

$$\begin{aligned}
\int J_3(x)dx &= \int x^2 \cdot x^{-2} J_3(x)dx \leftarrow \boxed{\textit{introduce } x^{-n} \textit{ to help}} \\
&= -\int x^2 d(x^{-2} J_2(x)) \\
&= -x^2 \cdot x^{-2} J_2(x) + \int x^{-2} J_2(x)dx^2 \\
&= -J_2(x) + 2\int x^{-1} J_2(x)dx \\
&= -J_2(x) - 2\int d(x^{-1} J_1(x)) \\
&= -J_2(x) - 2x^{-1} J_1(x) + C.
\end{aligned}$$

Ex 1.7. *1. Let* $\mu_m^{(0)}$ *be the m-th positive zero of the Bessel function* $J_0(x)$*. Try to expand the function* $f(x) = x^2 - 1$ *into a Fourier-Bessel series of* $J_0(\mu_m^{(0)}x)$ *on the interval* $(0,1)$*. (Hint: Use the recurrence formulas:* $\frac{d}{dx}[J_0(x)] = -J_1(x)$*,* $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$*)*

2. *Solve the following boundary value problem:*

$$\begin{cases} u_{tt} = a^2(u_{rr} + \frac{1}{r}u_r), \quad 0 \le r < 1, t > 0 \leftarrow \boxed{\text{change } u_{tt} \text{ to } u_t, -u_{zz} \text{ to create new exercise}} \\ u(1,t) = 0 \\ u(r,0) = f(r), \quad f(1) = 0 \leftarrow \boxed{\text{Compatibility condition–initial data is compatible with boundary}} \\ u_t(r,0) = 0 \end{cases}$$

Solution. *1.* $x^2 - 1 = \sum_{n=1}^{\infty} C_m J_0(\mu_m^{(0)} x)$, *where*

$$C_m = \frac{\int_0^1 x(x^2-1)J_0(\mu_m^{(0)}x)dx}{\frac{1}{2}J_1^2(\mu_m^{(0)})}$$

$$\begin{aligned} \int_0^{\mu_m^{(0)}} x^3 J_0(x)dx &= \int_0^{\mu_m^{(0)}} x^2(xJ_1(x))'dx \\ &= (\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2\int_0^{\mu_m^{(0)}} x^2 J_1(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) + 2\int_0^{\mu_m^{(0)}} x^2 J_0'(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 4\int_0^{\mu_m^{(0)}} xJ_0(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 4\mu_m^{(0)} J_1(\mu_m^{(0)}) \end{aligned}$$

or

$$\begin{aligned} \int_0^{\mu_m^{(0)}} x^3 J_0(x)dx &= \int_0^{\mu_m^{(0)}} x^2(xJ_1(x))'dx \\ &= (\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2\int_0^{\mu_m^{(0)}} x^2 J_1(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2(\mu_m^{(0)})^2 J_2(\mu_m^{(0)}) \end{aligned}$$

Therefore,

$$C_m = \frac{-8}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} \quad or \quad C_m = \frac{-4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})}.$$

2. Apply the ***method of separation of variables****. Let $u(x,t) = R(r)Z(z)$. Substitute it into the equation and separate variables to get*

$$T'' - \lambda a^2 T = 0$$

$$r^2 R'' + rR' + \lambda r^2 R = 0$$

From $|u(0,t)| < +\infty$ and $u(1,t) = 0$, we know $|R(0)| < +\infty$ and $R(1) = 0$. Solving the zero - order Bessel equation, we get the general solution

$$R(r) = CJ_0(\sqrt{\lambda}r) + DY_0(\sqrt{\lambda}r)$$

From the condition $|R(0)| < +\infty$, we know $D = 0$. Denote $\mu_m^{(0)}$ as the m-th positive zero of $J_0(x)$. Then from the condition $R(1) = 0$, we have $J_0(\sqrt{\lambda}) = 0$, so

$$\begin{cases} \lambda_m = (\mu_m^{(0)})^2 \\ R_m(r) = J_0(\mu_m^{(0)} r) \end{cases}$$

Substitute λ_m into the equation of T to get

$$T_m(t) = a_m \cos(\mu_m^{(0)} at) + b_m \sin(\mu_m^{(0)} at)$$

Then

$$u_m(r,t) = [a_m \cos(\mu_m^{(0)} at) + b_m \sin(\mu_m^{(0)} at)] J_0(\mu_m^{(0)} r)$$

According to the superposition principle

$$u(x,t) = \sum_{m=1}^{+\infty} [a_m \cos(\mu_m^{(0)} at) + b_m \sin(\mu_m^{(0)} at)] J_0(\mu_m^{(0)} r)$$

From the initial value $u(r,0) = \phi(r)$, *we get*

$$a_m = \frac{\int_0^1 r\phi(r) J_0(\mu_m^{(0)} r) dr}{\frac{1}{2} J_1^2(\mu_m^{(0)})}, \quad b_m = 0$$

Ex 1.8. 1. *Calculate the definite integral* $\int_0^1 x^3 J_0(\alpha x) dx$, *where* α *is a positive zero of the Bessel function* $J_0(x)$ *of order zero. (Hint: Use the recurrence formulas:* $J_0'(x) = -J_1(x)$, $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$*)*

2. *Solve the following boundary value problem:*

$$\begin{cases} u_t = u_{rr} + \frac{1}{r} u_r, & 0 \le r < 2, t > 0 \\ u(2,t) = 0, |u(0,t)| < +\infty, & t > 0 \\ u(r,0) = 4 - r^2, & 0 \le r \le 2 \end{cases}$$

Solution. *1.*

$$\begin{aligned} \int_0^1 x^3 J_0(\alpha x) dx &= \frac{1}{\alpha^4} \int_0^\alpha t^3 J_0(t) dt \\ &= \frac{1}{\alpha^4} \int_0^\alpha t^2 (tJ_1(t))' dt \\ &= \frac{1}{\alpha^4} (\alpha^3 J_1(\alpha) - 2 \int_0^\alpha t^2 J_1(t) dt) \\ &= \frac{J_1(\alpha)}{\alpha} + \frac{2}{\alpha^4} \int_0^\alpha t^2 J_0'(t) dt \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{4}{\alpha^4} \int_0^\alpha t J_0(t) dt \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{4J_1(\alpha)}{\alpha^3} \end{aligned}$$

or

$$\begin{aligned} \int_0^1 x^3 J_0(\alpha x) dx &= \frac{1}{\alpha^4} \int_0^\alpha t^3 J_0(t) dt \\ &= \frac{1}{\alpha^4} \int_0^\alpha t^2 (tJ_1(t))' dt \\ &= \frac{1}{\alpha^4} (\alpha^3 J_1(\alpha) - 2 \int_0^\alpha t^2 J_1(t) dt) \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{2}{\alpha^4} \int_0^\alpha (t^2 J_2)'(t) dt \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{2J_2(\alpha)}{\alpha^2} \end{aligned}$$

2. Apply the ***method of separation of variables****. Let* $u(x,t) = R(r)T(t)$. *Substitute it into the equation and separate variables to get*

$$T' + \lambda T = 0$$

$$r^2 R'' + rR' + \lambda r^2 R = 0$$

From $|u(0,t)| < +\infty$, *we have* $|R(0)| < +\infty$, *and from* $u(2,t) = 0$, *we have* $R(2) = 0$. *Solving the zero order Bessel equation, we get the general solution*

$$R(r) = CJ_0(\sqrt{\lambda} r) + DY_0(\sqrt{\lambda} r)$$

From the condition $|R(0)| < +\infty$, we know $D = 0$. Denote $\mu_m^{(0)}$ as the m-th positive zero of $J_0(x)$. Then from the condition $R(2) = 0$, we have $J_0(2\sqrt{\lambda}) = 0$, so

$$\begin{cases} \lambda_m = \frac{(\mu_m^{(0)})^2}{4} \\ R_m(r) = J_0(\frac{\mu_m^{(0)} r}{2}) \end{cases}$$

Substitute λ_m into the equation of T to get

$$T_m(t) = C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}}$$

Then

$$u_m(r,t) = C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0(\frac{\mu_m^{(0)} r}{2})$$

According to the superposition principle

$$u(x,t) = \sum_{m=1}^{+\infty} C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0(\frac{\mu_m^{(0)} r}{2})$$

From the initial condition, we have

$$\begin{aligned} C_m &= \frac{\int_0^2 r(4-r^2) J_0(\frac{\mu_m^{(0)} r}{2}) dr}{2J_1^2(\mu_m^{(0)})} \\ &= \frac{8\int_0^{\mu_m^{(0)}} x J_0(x) dx}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} - \frac{8\int_0^{\mu_m^{(0)}} x^3 J_0(x) dx}{(\mu_m^{(0)})^4 J_1^2(\mu_m^{(0)})} \\ &= \frac{8}{\mu_m^{(0)} J_1(\mu_m^{(0)})} - \frac{8}{\mu_m^{(0)} J_1(\mu_m^{(0)})} + \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} \\ &= \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{m=1}^{+\infty} \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0(\frac{\mu_m^{(0)} r}{2})$$

or

$$u(x,t) = \sum_{m=1}^{+\infty} \frac{16 J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0(\frac{\mu_m^{(0)} r}{2})$$

Ex 1.9. *Suppose there is a cylinder with radius a and height h, which is adiabatic (heat - insulated) from the outside. The initial temperature is $u_0(1 - \frac{\rho^2}{a^2})$. Find the temperature distribution and variation inside this cylinder.*

Solution. *Since the initial temperature is* ***independent of*** *θ and z, this problem is* ***independent*** *of θ and z. The problem can be translated as:*

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial u}{\partial \rho}) = 0 \quad \boxed{\Leftrightarrow \quad u_t = k(u_{\rho\rho} + \frac{1}{\rho} u_\rho)} \\ \frac{\partial u}{\partial \rho}\Big|_{\rho=a} = 0, \ u|_{\rho=0} \text{ is bounded} \leftarrow \boxed{\text{The second kind boundary}} \\ u\big|_{t=0} = u_0(1 - \frac{\rho^2}{a^2}) \end{cases}$$

Using the ***method of separation of variables.***

The separation of variables: *Let $u(\rho,t) = R(\rho)T(t)$. Then*

$$RT' - k\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) = RT' - k\frac{1}{\rho}(R' + \rho R'') = 0.$$

Dividing both sides by kRT, we get

$$\frac{T'}{kT} = \frac{R' + \rho R''}{\rho R} = -\lambda,$$

that is $R' + \rho R'' + \lambda\rho R = 0$ and $T' + k\lambda T = 0$.

Transform PDE to ODEs:

$$\begin{cases} T' + k\lambda T = 0 \\ \rho^2 R'' + \rho R' + \lambda\rho^2 R = 0 \end{cases}$$

The boundary conditions can be written as $R'(a)T(t) = 0$. To obtain a nontrivial solution, we have $R'(a) = 0$ and $|R(0)| < +\infty$.

Solve ODEs: *The equation $\rho^2 R'' + \rho R' + \lambda\rho^2 R = 0$ is a Bessel equation of order 0.*

Supplementary proof of $\lambda \geq 0$

The Bessel's equation is

$$\rho R'' + R' + \lambda\rho R = 0$$

Multiply both sides by R and integrate:

$$\int_0^a \rho R'' R \mathrm{d}\rho + \int_0^a R R' \mathrm{d}\rho + \lambda \int_0^a \rho R^2 \mathrm{d}\rho = 0.$$

Using integration by parts:

$$\int_0^a \rho R'' R \mathrm{d}\rho = \underbrace{[\rho R' R]_0^a}_{\textit{by bdry condition}=0} - \int_0^a R R' \mathrm{d}\rho - \int_0^a \rho (R')^2 \mathrm{d}\rho$$

It follows that: $\lambda \int_0^a \rho R^2 \mathrm{d}\rho = \int_0^a \rho (R')^2 \mathrm{d}\rho$, and then $\lambda \geq 0$.

- *Note: Compared with the first type boundary problem, λ can be 0 here.*
- *If $\lambda = 0 \Rightarrow R' = 0, \Rightarrow R = constant$. Unlike the first type problem, in the second type problem, $R(\rho) \not\equiv 0$, i.e., $R(\rho)$ could be a nonzero constant.*
- *Because in the first type problem, $\lambda = 0$ leads to $R(a) = 0$ and $R' = 0 \Rightarrow R \equiv 0$, while the second type boundary problem has no such issue.*

Case 1: when $\lambda > 0$. *The general solution of the Bessel equation of order 0 is $R(\rho) = CJ_0(\sqrt{\lambda}\rho) + DY_0(\sqrt{\lambda}\rho)$. From the boundedness $|R(0)| < +\infty$, we have $D = 0$. Then using $R'(a) = 0$, since $R'(\rho) = C\sqrt{\lambda}J_0'(\sqrt{\lambda}\rho) \implies R'(a) = C\sqrt{\lambda}J_0'(\sqrt{\lambda}a) = 0$, and $J_0'(\sqrt{\lambda}a) = -J_1(\sqrt{\lambda}a) = 0$ (recall in the first type boundary, here is $J_0(\sqrt{\lambda}a) = 0$. This is a main different!). Therefore, $R'(a) = 0 \implies J_1(\sqrt{\lambda}a) = 0$.*

Let $\mu_m^{(1)}$ be the positive zero of $J_1(x)$ $(m = 1, 2, \cdots)$. Then the eigenvalues are

$$\lambda_m = (\frac{\mu_m^{(1)}}{a})^2,$$

and the eigenfunctions are

$$R_m(\rho) = J_0\left(\frac{\mu_m^{(1)}}{a}\rho\right) \leftarrow \boxed{\text{It is not } J_0\left(\frac{\mu_m^{(0)}}{a}\rho\right)}, \quad (m = 1, 2, \cdots).$$

For the ODE of T, $T_m(t) = C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t}$. So $u_m(\rho, t) = C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0(\frac{\mu_m^{(1)}}{a}\rho)$.

Case 2 when $\lambda = 0$. *Solve $\rho^2 R'' + \rho R' = 0$, we get $R(\rho) = A + B\ln\rho$, then $R'(\rho) = \frac{B}{\rho}$. Using the boundary conditions $R'(a) = 0$ and $|R(0)| < +\infty$, we have $B = 0$, so $R_0(\rho) = A_0$. For the ODE of T, $T_0(t) = A_0^*$, so $u_0(\rho, t) = A_0^* A_0 \equiv C_0$.*

Superposition series solution

$$u(\rho, t) = C_0 + \sum_{m=1}^{\infty} C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0(\frac{\mu_m^{(1)}}{a}\rho)$$

Determine coefficients from initial values

$$u(\rho,0)=C_0+\sum_{m=1}^{\infty}C_mJ_0(\frac{\mu_m^{(1)}}{a}\rho)=u_0(1-\frac{\rho^2}{a^2})$$

- *We cannot use the Fourier-Bessel coefficients given previously because $\mu_m^{(1)}$ is a zero of J_1, not a zero of J_0.*

Proposition 1.1. *Prove that $\int_0^1 J_0(\mu_i^{(1)}x)xdx=0$, where $\mu_i^{(1)}$ is a zero of $J_1(x)$, that is $J_1(\mu_i^{(1)})=0$.*

Proof.

$$\int_0^1 J_0(\mu_i^{(1)}x)xdx=\frac{1}{(\mu_i^{(1)})^2}\int_0^{\mu_i^{(1)}}J_0(t)tdt=\frac{1}{(\mu_i^{(1)})^2}[tJ_1(t)]_0^{\mu_i^{(1)}}=\frac{1}{(\mu_i^{(1)})^2}\mu_i^{(1)}J_1(\mu_i^{(1)})=0$$

This shows that 1 and $J_0(\mu_i^{(1)}x)$ are **orthogonal**. □

Theorem 1.1. *Let μ_i be a positive zero of $J_n'(r)$. Then*

1. *Orthogonality: $\int_0^1 J_n(\mu_i r)J_n(\mu_k r)rdr=0$ $(i\neq k)$*
2. *And the norm: $\int_0^1 J_n^2(\mu_i r)rdr=\frac{1}{2}(1-\frac{n^2}{\mu_i^2})J_n^2(\mu_i)$*

Proof. 1. From the previous theorem, we know $\int_0^R rJ_n(\frac{\mu_m^{(n)}}{R}r)J_n(\frac{\mu_k^{(n)}}{R}r)dr=0$ $(m\neq k)$. Using the notation in the proof of the orthogonality theorem,

$$(\alpha_1^2-\alpha_2^2)\int_0^R rF_1(r)F_2(r)dr+\left[rF_2\frac{dF_1}{dr}-rF_1\frac{dF_2}{dr}\right]_0^R=0. \tag{43}$$

Let $R=1$, $\alpha_1=\mu_i$, $\alpha_2=\mu_k$, then $\frac{dF_1}{dr}\big|_{r=1}=\mu_iJ_n'(\mu_i)=0$, $\frac{dF_2}{dr}\big|_{r=1}=\mu_kJ_n'(\mu_k)=0$. So

$$(\alpha_1^2-\alpha_2^2)\int_0^1 rF_1(r)F_2(r)dr=0.$$

When $i\neq k$, $\int_0^1 rJ_n(\mu_ir)J_n(\mu_kr)dr=0$, and the orthogonality is proved.

2. Calculate $\int_0^1 J_n^2(\mu_ir)rdr$. Using (43),

$$\int_0^1 rF_1(r)F_2(r)dr=\frac{[rF_2\frac{dF_1}{dr}-rF_1\frac{dF_2}{dr}]_0^1}{\alpha_1^2-\alpha_2^2}.$$

Let $\alpha_1=\mu_i$, then $\frac{dF_1}{dr}\big|_{r=1}=0$. We further arrive at

$$\underbrace{\int_0^1 rJ_n(\mu_ir)J_n(\alpha_2r)dr}_{\text{a continuous function of }\alpha_2}=\frac{\alpha_2J_n(\mu_i)J_n'(\alpha_2)}{\mu_i^2-\alpha_2^2}$$

Then

$$\begin{aligned}\int_0^1 J_n^2(\mu_ir)rdr&=\lim_{\alpha_2\to\mu_i}\int_0^1 rJ_n(\mu_ir)J_n(\alpha_2r)dr\\&=\lim_{\alpha_2\to\mu_i}\frac{\alpha_2J_n(\mu_i)J_n'(\alpha_2)}{\mu_i^2-\alpha_2^2}\\&=\lim_{\alpha_2\to\mu_i}\frac{\alpha_2J_n(\mu_i)J_n''(\alpha_2)+J_n(\mu_i)J_n'(\alpha_2)}{-2\alpha_2}\\&=-\frac{\mu_iJ_n(\mu_i)J_n''(\mu_i)}{2\mu_i}=-\frac{1}{2}J_n(\mu_i)J_n''(\mu_i)\end{aligned}$$

Using the Bessel equation $J_n''(x) + \frac{1}{x}J_n'(x) + (1 - \frac{n^2}{x^2})J_n(x) = 0$ with $x = \mu_i$, $J_n''(\mu_i) = -(1 - \frac{n^2}{\mu_i^2})J_n(\mu_i)$. Then

$$\int_0^1 J_n^2(\mu_i r) r dr = \frac{1}{2}(1 - \frac{n^2}{\mu_i^2})J_n^2(\mu_i).$$

□

Let $a = 1,\ u(\rho, 0) = C_0 + \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(1)}\rho) = u_0(1 - \rho^2)$

$$C_0 = \frac{\int_0^1 u_0(1-\rho^2)\rho d\rho}{\int_0^1 \rho d\rho} = \frac{u_0 \int_0^1 (\rho - \rho^3) d\rho}{\frac{1}{2}} = \frac{1}{2}u_0,$$

$$C_m = \frac{\int_0^1 u_0(1-\rho^2)\rho J_0(\mu_m^{(1)}\rho) d\rho}{\int_0^1 J_0^2(\mu_m^{(1)}\rho)\rho d\rho} = \frac{\mu_0 \int_0^1 \left[\rho J_0(\mu_m^{(1)}\rho) - \rho^3 J_0(\mu_m^{(1)}\rho)\right] d\rho}{\frac{1}{2}J_0^2(\mu_m^{(1)})}.$$

Calculate

$$\int_0^1 \rho J_0(\mu_m^{(1)}\rho) d\rho = \frac{1}{(\mu_m^{(1)})^2}\int_0^{\mu_m^{(1)}} t J_0(t) dt = \frac{1}{(\mu_m^{(1)})^2}[t J_1(t)]_0^{\mu_m^{(1)}} = 0$$

and

$$\begin{aligned}
\int_0^1 \rho^3 J_0(\mu_m^{(1)}\rho) d\rho &= \frac{1}{(\mu_m^{(1)})^4}\int_0^{\mu_m^{(1)}} t^3 J_0(t) dt \\
&= \frac{1}{(\mu_m^{(1)})^4}\int_0^{\mu_m^{(1)}} t^2 d(t J_1(t)) \\
&= \frac{1}{(\mu_m^{(1)})^4}\left([t^3 J_1(t)]_0^{\mu_m^{(1)}} - 2\int_0^{\mu_m^{(1)}} t J_1(t) dt\right) \\
&= -\frac{2(\mu_m^{(1)})^2}{(\mu_m^{(1)})^4} J_2(\mu_m^{(1)}) = -\frac{2J_2(\mu_m^{(1)})}{(\mu_m^{(1)})^2}.
\end{aligned}$$

Substitute into C_m*, and using* $J_0(\mu_m^{(1)}) + J_2(\mu_m^{(1)}) = \frac{2}{\mu_m^{(1)}}J_1(\mu_m^{(1)}) = 0 \Rightarrow J_2(\mu_m^{(1)}) = -J_0(\mu_m^{(1)})$

$$C_m = -\frac{4u_0 J_0(\mu_m^{(1)})}{(\mu_m^{(1)})^2 J_0^2(\mu_m^{(1)})} = -\frac{4u_0}{(\mu_m^{(1)})^2 J_0(\mu_m^{(1)})}.$$

So

$$u(\rho, t) = \frac{1}{2}u_0 - \sum_{m=1}^{\infty}\frac{4u_0}{(\mu_m^{(1)})^2 J_0(\mu_m^{(1)})} e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0(\frac{\mu_m^{(1)}}{a}\rho).$$

References