

Methods and Ideas in Linear Partial Differential Equations

CHAO LIU¹

May 19, 2025

¹chao.liu.math@foxmail.com

To all my families, for supporting me to complete this book.

Contents

1	Introduction	2
1.1	String Vibration Equation and Definite Conditions	4
1.1.1	Derivation of the String Vibration Equation	4
1.1.2	Boundary and Initial Conditions	13
1.2	Heat Conduction Equation	16
1.2.1	Derivation of Heat Equations	18
1.2.2	Initial and Boundary Conditions for the Heat Conduction Equation	23
1.3	Laplace's Equation and Boundary Value Problems	25
1.4	Basic Concepts and Knowledge	27
1.4.1	Basic concepts	27
1.4.2	Well-Posedness	29
1.4.3	Superposition Principle	30
1.4.4	Classification of Second-Order PDEs	31
1.5	Fourier Series	33
1.5.1	Analogous to linear expansion	34
1.5.2	Orthogonality	35
1.5.3	Completeness and Inner Product Method for Finding Coefficients	36
1.5.4	Key Differences from Linear Algebra	38
1.5.5	Odd and Even Functions	38
1.5.6	Generalizations	39
2	Method of Separation of Variables	40
2.1	Free Vibration of a Bounded String	42
2.2	Heat Conduction in a Finite Rod	62
2.2.1	Consider the mixed problem of the homogeneous heat conduction equation (with boundary conditions of the first type)	62
2.2.2	Consider the Mixed Problem of the Homogeneous Heat Conduction Equation (Boundary Conditions are Both of the Second Type)	67
2.3	Boundary Value Problems of the Two-Dimensional Laplace Equation	72
2.3.1	Boundary Value Problem of the Laplace Equation in a Rectangular Domain	72
2.3.2	Boundary Value Problem of Laplace's Equation in a Circular Domain	77
2.4	Non-homogeneous Equations and Their Solving Problems	93
2.4.1	Forced Vibration Problem with Boundary Conditions	93
2.4.2	Heat Conduction in a Finite Rod with a Heat Source	110

2.4.3 Poisson Equation (Non-homogeneous Laplace Equation)	116
2.5 Problems with Non-homogeneous Boundary Conditions	121
2.6 Eigenvalues and Eigenfunctions (Introduction)	130
2.7 Exercise class	132
3 Method of Traveling Waves and Integral Transform Method	156
3.1 D'Alembert's Formula: Wave Propagation	157
3.1.1 D'Alembert's Solution for the String Vibration Equation	157
3.1.2 Solving Non-Homogeneous Wave Equations	165
3.1.3 Physical Meaning of D'Alembert's Solution	174
3.1.4 Dependency Region, Determination Region, and Influence Region	175
3.2 Initial Value Problems for Higher Dimensional Wave Equations	178
3.2.1 Kirchhoff's Formula for the Three-Dimensional Wave Equation	180
3.2.2 Dimension Reduction Method	190
3.2.3 Physical meaning of the Solution	194
3.3 Integral Transforms	199
3.3.1 Integral Transforms and Their Properties (Review)	199
3.3.2 Examples of Integral Transform Methods	211
3.4 Exercise class	227
4 Green's Function Method	245
4.1 Green's Formula and Its Applications	247
4.1.1 Spherically Symmetric Solutions (Raw Material)	248
4.1.2 Green's Formulas (Mathematical Tool)	253
4.1.3 Integral Expressions of Harmonic Functions	255
4.1.4 Basic Properties of Harmonic Functions	260
4.2 Green's Function	268
4.3 Applications of Green's Function—Solve Green functions G	283
4.3.1 Green's Function in the Half Space and the Dirichlet Problem	285
4.3.2 Green's Function in a Spherical Domain and the Dirichlet Problem	292
4.4 Trial and Error Method and Solution of Poisson's Equation	302
4.4.1 Trial and Error Method	302
4.4.2 Solution of Poisson's Equation	304
4.5 Exercise Class	306
5 Bessel Functions	318
5.1 Bessel Equation and Bessel Functions	322
5.1.1 Derivation of the Bessel Equation	322
5.1.2 Bessel Functions	327
5.2 Recurrence Formulas of Bessel Functions	338
5.3 Expansion into Series in Terms of Bessel Functions	343
5.3.1 Zeros of Bessel Functions	345
5.3.2 Orthogonality of the Bessel Function System	347
5.3.3 The Norm of Bessel Functions	350
5.3.4 Fourier-Bessel Series	351

5.4 Applications of Bessel Functions	353
5.5 Exercise Class	366

Preface

This book is based on the author's lecture notes for the course "Mathematical Physics Equations and Special Functions (MAT0701)" that the author taught at Huazhong University of Science and Technology from 2020 to 2025, and the same-titled textbook [4] published by Huazhong University of Science and Technology.

This book places more emphasis on guiding readers to explore the thinking context behind the knowledge. By adopting an original learning method, it deeply analyzes the origin and evolution of the methods for solving partial differential equations, helping readers understand where these methods come from and how they are conceived. It aims to enable readers not only to know what these methods are but also to understand why they work.

Acknowledgements

I am grateful for the help offered by my colleagues in the Mathematical Physics Equations Teaching Group.

Chao Liu

<https://chaoliu2019.github.io/chaoliu.github.io/>

1

Introduction

PDEs and ODEs

- Mathematical Physics Equations. A.K.A Partial Differential Eeuation → PDE (several independent variable, e.g. $\partial_x^2\Phi(x, y) + \partial_y^2\Phi(x, y) = 0$);
- Ordinary Differential Eeuation → ODE (one independent variable, e.g. $y'(t) = f(t)$)
- PDEs “approximate”, “translate” or even “precisely describe” the physical phenomenons.

Textbook

1. Chap 1: Derive the 3 types of PDEs (hyperbolic, parabolic and elliptic) and basic concepts;
2. Chap 2 – 4: Methods;
3. Chap 5: One special function–Bessel functions– tool for solving PDEs–methods can be applied to other special functions (see Chap 6).

Interesting Examples

The following two equations both describe the “gravity”:

- General relativity–Einstein Equation ($G = T$)–(Quasilinear) Wave Equation ($\partial_t^2g - \Delta g = T$)– gravitational wave (“like” water wave)!
- Newtonian gravity–Poisson Equation ($\Delta\Phi = \rho$)–Potential Equation–No wave (static distribution)!

Course Content Organization

- **By Equation Types:** Hyperbolic (wave equation), Elliptic (potential equation), Parabolic (heat equation).
- **By Methods:** Note the conditions under which each method applies.

- **Key feature of this Lecture:** Only **Linear** PDEs

Question 1.0.1. Why is “linearity” required?

A. The linearity is extremely important in this course:

- The **derivations of equations** need this concept. Otherwise, the derivations are very confusing;
- The solving **methods** strongly rely on the linear structures.
- In many circumstances, **not far from the linear cases**, nonlinear problems can be **approximated** by the **linear PDEs**.

Question 1.0.2. What is the difference between the objects studied in Linear Algebra and Calculus?

- A.
- **Linear Algebra:** **linear** mappings, **linear** transforms and **linear** functions (Represented by matrix...).
 - **Calculus:** **nonlinear** (C^k -differentiable or integrable) functions and mappings.

Question 1.0.3. What is the relationship between the **linear objects** in Linear Algebra and the **nonlinear objects** in Calculus?

- A.
- Core idea in calculus (**LEGO idea**): differentiation and integration;
 - Differentiation (**localization, disassemble**): nonlinear objects locally are **linear-local linearizations**;
 - Integrating (**assemble**) the local objects (differential elements) to return the nonlinear objects.

Claim 1.0.1. Once the problem is **locally linearized**, Linear Algebra steps in!

Question 1.0.4. What is the core theorem in Calculus?

- A.
- **Differentiation:** **Taylor expansions**—disassemble a function to all order of small parts including the linear part. $f \in C^k(\mathbb{R})$, then the **nonlinear function**

$$f(x) = f(x_0) + \underbrace{f'(x_0)\Delta x}_{\text{linear part (*)}} + \underbrace{\frac{1}{2}f''(x_0)\Delta x^2}_{\text{Hessian(**)}} + \dots$$

- The linear part (*) can use linear algebra, and if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $f'(x_0)$ becomes an $m \times n$ **gradient matrix**.
- The second order (**) is **Hessian** matrix–bilinear form and Quadratic form.
- Under local approximations, Δx small enough, getting rid of the higher order terms, $f(x) - f(x_0) \sim f'(x_0)\Delta x$.
- Integration: **Newton-Leibniz theorem**—assemble the linear parts of a function to this function.

- In the following derivation of the String Vibration Equation, we apply several **approximations up to the linear terms** (linearization);
- Alternatively, we could derive a **fully nonlinear equation first**, which is more complex, and then perform the **linearization at the end**.
- The derivation leads to some “ridiculous facts” due to the **linear level**. The **true behavior** is **hidden** in the **higher-order terms**.

1.1 String Vibration Equation and Definite Conditions

derive eqs, intro concepts, classify eqs and condi.—methods of solving eqs need this classifications

The string vibration equation was systematically studied by d'Alembert and others in the 18th century. It is a typical representative of a large class of partial differential equations.

1.1.1 Derivation of the String Vibration Equation

Question 1.1.1. How to build a PDE?

A 1.1.1. Build a PDE \Leftrightarrow Translate **Physical phenomenon** in terms of **Math. language**.

1. **Choice of Representation Quantities:** A representation quantity is a property you are interested in when describing a physical phenomenon. This could be a variable that captures how a certain property changes with respect to time and space.

Ex 1.1.1. In the case of a vibrating string, the property you're interested in is the displacement from the equilibrium position. Thus, you choose the displacement as your representation quantity.

Ex 1.1.2. In the case of heat conduction, the representation quantity would be temperature. For example, if you're considering a classroom where people emit heat, the temperature distribution in the room is what you want to study. In this case, the representation quantity is the temperature function, which describes how temperature varies spatially and temporally.

2. **Assumptions–Approximation and Simplification (Removing “Hair”):** In physics and mathematics, we often need to *simplify* models by *removing extraneous details*. This process of “removing the hair” refers to *eliminating unnecessary or irrelevant properties* to focus on the *core elements* of the system. The equations derived are thus *approximations of real-world phenomena*, rather than exact representations.

Ex 1.1.3. When deriving the equations for a vibrating string, assumptions are made to simplify the model, such as neglecting air resistance or assuming a perfectly elastic string. These simplifications are part of the *approximation process*.

3. **Translation of Physical Laws into Mathematical Form** (math.-phys. bridge). The physical laws governing the system must be *translated* into mathematical equations.

Ex 1.1.4. In the context of a vibrating string, the fundamental physical law we apply is **Newton’s second law** of motion. For a vibrating string, you would apply Newton’s second law to translate the physical phenomenon into a differential equation that describes the vibration.

Remark 1.1.1. When applying Newton’s second law in mechanics, remember it is primarily valid in the macroscopic world at low speeds. Furthermore, Newtonian mechanics applies to **point masses**, which is important to distinguish when studying different systems.

- In mechanical systems, when dealing with **point masses**, you apply **Newton’s second law**.
- However, in other systems like **fields** (e.g., electromagnetic or gravitational fields), different types of mechanics apply.

4. **Local vs Global Viewpoint** (calculus involves, compatible with the choice of the physical laws). Since we are dealing with a differential equation, we need to apply differential thinking. The localization in this context refers to differentiation, and the integration corresponds to the process of globalizing the problem. Therefore, we should break the problem into small elements and differentiate them.

- Localization (i.e., ideas of differentiation): differential equations reflect the **local properties**—require **localization**—take a local piece of the string, and study its properties.
- Integration (easy global analysis for some cases, e.g., Gauss’ flux theorem): since differential and integration are inverse operations, possible to use **global analysis** to derive **local equation** (i.e., PDE).

If you approach a problem from the perspective of fields, you’re considering it as a global problem. However, when dealing with point mass mechanics, you must localize the problem. This localization is closely tied to differentiation. The reason Newton developed both classical mechanics and calculus was because they stemmed from the same idea. Thus, the mechanics of point masses naturally lead to differentiation.

Ex 1.1.5. For a vibrating string, you consider small segments of the string as point masses, and apply Newton’s second law to each small piece to analyze the forces acting on it. This process involves localization.

Remark 1.1.2. Some further remarks on the “Local vs Global Viewpoint”:

- **Physical Laws of Point Mechanics and Fields:** Some physical laws are inherently related to the mechanics of point masses, while others apply to the system as a whole. For instance, some physical laws describe the behavior of systems as a collective whole, rather than at individual points.
- **Energy and Global Laws:** Energy-based laws are examples of global physical laws. When analyzing such systems, you don't localize, but rather treat the system as a whole. For example, energy conservation can be analyzed globally without the need to focus on individual points.

Ex 1.1.6. In the heat conduction problem, you consider the energy distribution across the entire system, which leads to the formulation of a global energy equation. The system is analyzed as a whole.

- **From Integral to Differential Equations:** When deriving equations such as the heat equation, the process starts with an integral formulation. Through integration, you obtain a global view. Then, by removing the integrals, you arrive at a differential equation. This process of going from integral to differential reflects a shift from a global to a local perspective.

Ex 1.1.7. In deriving the heat equation, the heat transfer is initially considered as an integral over the system's volume. By simplifying, you eventually obtain a differential equation that describes the local temperature change over time and space.

- **Integral vs Differential Approaches:** When deriving equations, both integral and differential viewpoints can be used. You may start with a global perspective (integral form), and eventually convert to a local (differential) form. For example, in analyzing a vibrating string, you can either use the differential approach with **Newton's second law**, or the integral approach with the **impulse theorem**.
- **Approximation Process and Ideal Assumptions:** The equations we derive are always approximations of real-world phenomena. The process involves making idealized assumptions to simplify the system, which helps in deriving solvable equations.

Consider a uniform, soft string of length L , fixed at both ends and stretched. Under external forces, the string undergoes small transverse vibrations near its equilibrium position. We **aim** to find the motion of each point on the string.

- **Choice of Representation Quantities:** The **displacement** from the equilibrium position.
- **Importance of Ideal Assumptions:** In order to simplify the equations and make them solvable, ideal assumptions are necessary. Without these assumptions, the number of variables and conditions will become too large, and the resulting equations would be highly complex, possibly unsolvable. Therefore, assumptions are made to reduce the complexity of the system, while still **maintaining its most important characteristics**.

- **Role of Simplifying Assumptions (ensure the linearity):** All assumptions made in the process are aimed at simplifying the problem into a linear equation with constant coefficients. While some assumptions may seem unreasonable or unrealistic, they help in achieving this simplified form, making the equations tractable.
- **Complex nonlinear equations are allowed:** If you don't simplify the problem and consider more factors, you can include more terms in the equation. However, as you reduce your assumptions, the resulting equation may become increasingly complex. Therefore, the equation we will derive here represents the simplest case. This does not mean it is the only possible formulation, but it is important to first understand this concept.

Assumptions

- The string is uniform, and its cross-sectional diameter is negligible compared to its length. The linear density is constant.
- The string is soft and does not resist bending¹. The tension at each point is along the tangent direction, and the elongation follows Hooke's law².
 - **Empirical formula:** Hooke's law is often treated as an empirical formula;
 - Instead of considering it purely as empirical, we can interpret it as a fact derived from the first-order approximation (linear approximation), based on Taylor expansion. If the tension, T , is continuous and has k -derivatives, it can be expanded in a Taylor series as:

$$\Delta T = \frac{dT}{dx}(x_0)\Delta x + \frac{1}{2} \frac{d^2T}{dx^2}(x_0)(\Delta x)^2 + \dots$$

In Hooke's law, we discard higher-order terms and consider only the first-order term, assuming Δx is small. This gives us the linear approximation, which is essentially Hooke's law.

Since we are aiming linear equations in this course, we limit the expansion to the linear term. This assumption simplifies the problem to the linear framework, which is the essence of Hooke's law.

- The string undergoes small transverse vibrations in a plane: That is, the position of the string always remains close to a straight line (the equilibrium position), and each point on the string performs small vibrations in a direction perpendicular to the straight line, all within the same plane. (“Small” refers to the amplitude of the string’s vibrations and the small tilt angle of the tangent at any point on the string). $u \ll 1$ and $u_x \ll 1$ (see Fig.1.3). The slope of the tangent line to the string with respect to x is equal to $\tan \alpha_i$, i.e., $u_x = \tan \alpha_i$. For small transverse vibrations, both u and u_x are very small. In other words, the angle α_i is also small.

¹The biggest difference between a teaching stick and yarn, for example, lies in their resistance to deformation – one exerts a force that resists change in shape.

²That is, within the elastic limit, the deformation of an object is directly proportional to the external force that causes the deformation $\Delta T = -k\Delta x$. In the case of a string, the direction of the applied force is opposite to the direction of the deformation. This is because the force tries to restore the object to its original position, pulling it back towards its equilibrium. The negative sign in Hooke's law indicates this opposition between the force and the deformation.

Remark 1.1.3. This is because if u_x is small, it remains small regardless of the scale at which the system is viewed, whereas the **vibration amplitude** u behaves differently. For example, in the “Ant-Man” scenario from the Marvel universe, an object appears small at a macroscopic scale, but when zoomed in, its size becomes more apparent. However, if the angle α remains small, the system will continue to appear small at any scale. Thus, we assume:

$$u \ll 1 \quad \text{and} \quad u_x \ll 1 \quad (\text{see Fig. 1.3}).$$

Additionally, since the small angle approximation holds, we have:

$$\alpha_1, \alpha_2 \text{ are small, and } u_x \approx \tan(\alpha_i).$$

No external forces

First, let’s discuss the case of string vibration without the influence of external forces during the vibration process. Keep in mind some key facts listed below before we proceed.

- The problem of string vibration ultimately involves **translating Newton’s Second Law** into a PDE.
- Since this is a **particle mechanics problem**, we need to **localize** the string by **analyzing a small segment**.
- Newton’s Second Law is applied to this small segment (viewed as a particle and perform a force analysis), where forces are balanced by acceleration.
- After **force analysis**, the equation of motion can be derived.
- **Momentum or impulse principles** can also be used, as they are nearly equivalent to Newton’s Second Law, though they are more general.
- We assume **no external forces** (e.g., gravity), so the only forces on both ends of the segment are the tension forces.
- The tension force is **tangential** to the string, and although the tangents at both ends are slightly different, we assume they are nearly the same due to the small size of the segment.

Coordinate System

We choose a coordinate system (see Fig.1.1) where:

- The equilibrium position of the string aligns with the x -axis, fixed at endpoints $x = 0$ and $x = L$.
- $u(x, t)$ denotes the transverse displacement of the point at position x along the string at time t .

Deriving the free transverse vibration equation (wave equation)

Let $T_1(t) := T(t, x)|_{x_1}$ and $T_2(t) := T(t, x)|_{x_2}$ denote tensions at points x_1 and $x_2 := x_1 + \Delta x$, respectively, directed along the local tangents (see Fig. 1.3).

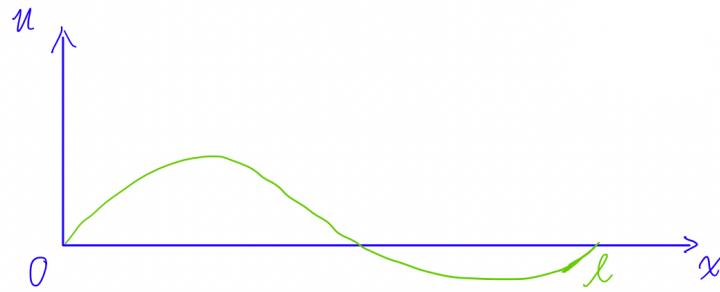


Figure 1.1: String 1

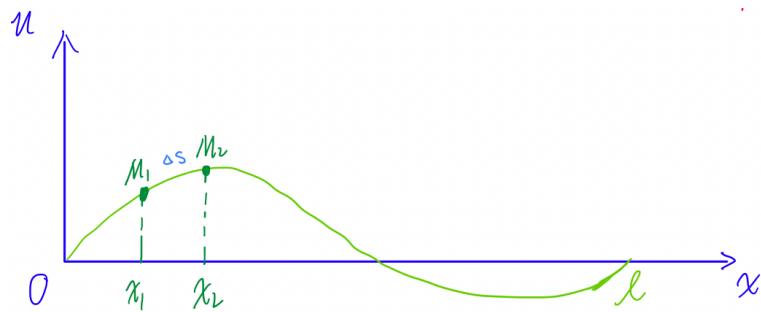


Figure 1.2: String 2

Performing a force analysis!

- **Horizontal components–balance:** For small vibrations, the horizontal components cancel: $T_1 \cos \alpha_1 = T_2 \cos \alpha_2$. Since
 - For transverse vibration, movement occurs only in the vertical direction.
 - In the horizontal direction, the two forces must balance each other.
 - This balance results in no movement in the horizontal direction.
- **Vertical components–Newton’s second law:** $T_2 \sin \alpha_2 - T_1 \sin \alpha_1 = \rho \Delta s \partial_t^2 u$. Since
 - The tension components in the vertical direction are given by: $T_2 \sin \alpha_2 - T_1 \sin \alpha_1$.
 - The net vertical force causes the string to move, so the forces must balance the mass times acceleration³.
 - The **mass** of the string segment is the linear density multiplied by the length of the segment $m = \rho \Delta s$.

³ Applying Newton’s Second Law: the net vertical force equals the mass times acceleration.

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

where mass is the linear density ρ times the string length L .

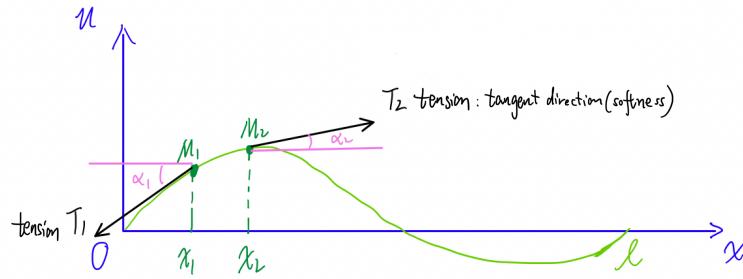


Figure 1.3: String 3

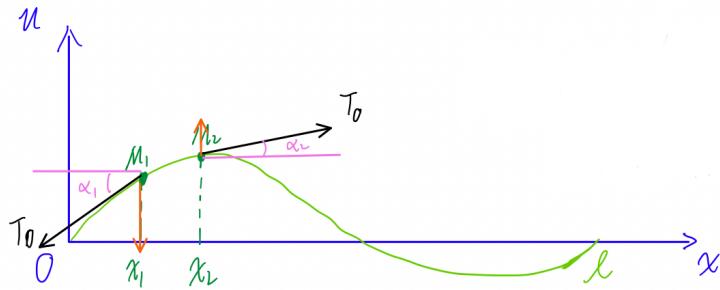


Figure 1.4: String 4

- The **acceleration** is the rate of change in velocity for the segment⁴ $\partial_t^2 u|_{\eta}$.

In fact, solving this equation is essentially an approximation process — a process of simplifying and approximating the equation.

$$\text{Horizontal:} \quad T_2 \cos \alpha_2 - T_1 \cos \alpha_1 = 0, \quad (1.1.1)$$

$$\text{Vertical:} \quad T_2 \sin \alpha_2 - T_1 \sin \alpha_1 = \rho \Delta s \partial_t^2 u|_{\eta}. \quad (1.1.2)$$

- Eq. (1.1.1) \Rightarrow **T is constant in space**: Due to small angles α_1 and α_2 , and since the goal is to linearize the equations, by approximating $\cos(\alpha) = 1 - \frac{1}{2}\alpha^2 + \dots$, i.e., $\cos(\alpha) \approx 1$. Higher-order terms are neglected for simplicity, as we aim to obtain a **linear equation**. Then Eq. (1.1.1) $\Rightarrow T_1(t) = T_2(t)$, which means tension does not depend on spatial position, i.e., $T_0(t) = T_1(t) = T_2(t)$ **is constant in space**⁵.

- Eq. (1.1.2) becomes

$$T_0(t) \sin \alpha_2 - T_0(t) \sin \alpha_1 = \rho \Delta s \partial_t^2 u. \quad (1.1.3)$$

- A similar approach (**aiming the linear equation**) is used with the sine function⁶:

$$\sin(\alpha) \approx \alpha \approx \tan \alpha \approx u_x$$

⁴Since the string segment Δx is very small, the difference in acceleration at each point on it will not be too large. Therefore, the acceleration at one point, denoted as η , can be used to approximate the acceleration at other points.

⁵This conclusion is valid under linear approximation. If higher-order terms in the Taylor expansion of $\cos(\alpha)$ were included, this would no longer hold.

⁶ $\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \dots$ and $\tan \alpha = \alpha + \frac{\alpha^3}{3} + \dots$

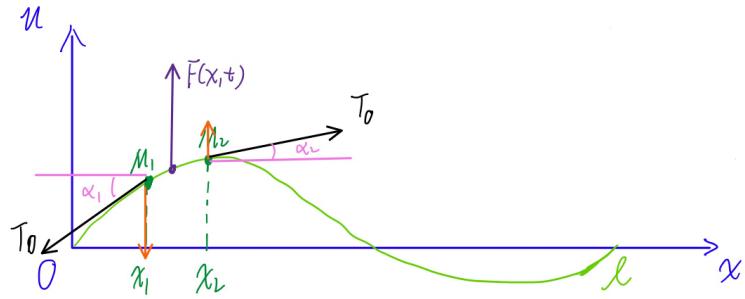


Figure 1.5: String 5

for small angles. Eq. (1.1.3) becomes

$$T_0(t)\partial_x u|_{x_2} - T_0(t)\partial_x u|_{x_1} = T_0(t) \underbrace{(\partial_x u|_{x_2} - \partial_x u|_{x_1})}_{\text{Mean Value Theorem} = \partial_x^2 u|_\xi \cdot \Delta x} = \rho \Delta s \partial_t^2 u|_\eta. \quad (1.1.4)$$

– Using the Mean Value Theorem⁷, We arrive at

$$T_0(t)\partial_x^2 u|_\xi \cdot \Delta x = \rho \Delta s \partial_t^2 u|_\eta. \quad (1.1.5)$$

Question 1.1.2. There's also a tricky part, which is how to handle the arc length⁸ ds ?

A. Using the **arc length formula**, we know:

$$\Delta s = \int_{x_1}^{x_2} \sqrt{1 + u_x^2} dx, \quad u_x = \frac{\partial u}{\partial x} = \tan \alpha \ll 1.$$

Assuming the string only undergoes small vibrations, u_x^2 can be neglected compared to 1, thus The arc length $\Delta s \approx x_2 - x_1 = \Delta x$.

Remark 1.1.4. • Arc length ds is approximated using the **Pythagorean theorem** in the small displacement limit:

$$ds = \sqrt{dx^2 + (du)^2}.$$

- This leads to the approximation:

$$ds \approx dx \left(1 + u_x^2\right)^{1/2}.$$

- For linearization, higher-order terms are discarded, leading to (using $(1 + x)^\alpha = 1 + \alpha x + \dots$):

$$ds \approx dx.$$

⁷A mean value theorem (or Taylor expansion) is used to express the difference in function values at different points:

$$f(x_2) - f(x_1) = \frac{df}{dx}\Big|_\xi \cdot (x_2 - x_1).$$

⁸Since the arc length ds is neither u nor x . In this equation, the independent variables and functions we need are u and x , but how can we express s in terms of u and x ?

- This result implies that the arc length remains constant during vibration, which seems **counterintuitive**⁹. This result holds **only under the assumption of linear approximation**.
- The increase in length includes a second-order small quantity, which means that when the value is small enough, it can be neglected.
- If the small quantity is **not negligible** (e.g., large perturbations), this approximation no longer holds, and the equation would become **nonlinear or variable-coefficient**.

Let us go back to Eq. (1.1.5):

- **T is constant in time:** Since ds does **not change with time**, that is, the deformation is 0 **with time**, by Hooke's law, the **change of tension** $T_0(t)$ is 0 with time. That is, $T_0(t)$ is independent time and thus a constant. The tension T becomes **independent of both space and time**, i.e., $T_0(t) = T_0 = \text{Constant}$.
- (1.1.5) becomes

$$T_0 \partial_x^2 u|_{\xi} \cdot \Delta \tilde{x} = \rho \Delta \tilde{x} \partial_t^2 u|_{\eta} \Rightarrow \frac{T_0}{\rho} \partial_x^2 u|_{\xi} = \partial_t^2 u|_{\eta} \quad (1.1.6)$$

- Assumption: Let Δx be extremely small, tending to zero. As $\Delta x \rightarrow 0$, both terms involving Δx can be cancelled and ξ and $\eta \rightarrow x_1$. (1.1.6) becomes the **free transverse vibration equation** (or **wave equation**¹⁰)

$$\partial_t^2 u = a^2 \partial_x^2 u \quad \text{where } a^2 := \frac{T_0}{\rho}, \quad (\rho = \text{Const. since "homogeneous"}) \quad (1.1.7)$$

- The wave equation can be rewritten as

$$u_{tt} - a^2 u_{xx} = 0, \quad \text{"hyperbolic type".}$$

The negative sign distinguishes wave equations from Laplace equations ("elliptic" type, $u_{xx} + u_{yy} = 0$), highlighting wave phenomena.

Forced Transverse Vibration

If an external force $F(x, t)$ (per unit length) acts vertically:

$$T \frac{\partial^2 u}{\partial x^2} \Delta x + F \Delta x = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

Simplifying gives the **forced vibration equation**:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad \text{where } a^2 := \frac{T_0}{\rho} \quad \text{and} \quad f := \frac{F}{\rho}$$

where $f(x, t)$ is the force density.

- If external forces are present, the equation becomes **non-homogeneous** (i.e., f exists).
- The homogeneous case ($f(x, t) = 0$) represents the **free vibration**.

⁹In a vibration process, the shortest distance between two points should increase if the path is curved, but the theory suggests that it does not, which seems **counterintuitive**.

¹⁰The vibration refers to observing the movement of a point after determining its spatial position, such as vibrating downward. However, if you also consider time and the spatial position simultaneously, the vibration essentially becomes a wave, representing the propagation of a wave. Therefore, this equation is referred to as the one-dimensional free vibration equation. It can also be called the one-dimensional wave equation.

General Wave Equations

The derived equations are **one-dimensional wave equations**. Analogously:

- **2D wave equation** (e.g., membrane vibration):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- **3D wave equation** (e.g., sound or electromagnetic waves):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Remark 1.1.5. The notation “(1+1)D”, “(1+2)D”, and “(1+3)D” is commonly used in relativity, where “1” refers to **time** and the number after “+” refers to **spatial dimensions**.

1.1.2 Boundary and Initial Conditions

For a definite physical process, it is not enough to establish equations that describe the physical quantities involved. Additional conditions are required to fully describe the system’s initial state and the physical conditions at the boundaries.

- The equation derived from physical laws (the equation is at the level of physical laws—universally applicable) may not have a unique solution.
- To ensure a unique solution, *boundary conditions* and *initial conditions* are necessary.
- These conditions help to select one case from the universal physical laws.
- We focus on initial and boundary conditions, though there are other types (e.g., characteristic problems).

Euqation+Condition=the determined solution!

Boundary and initial conditions include:

Initial Conditions

These describe the state of a process at its "initial" moment.

For the problem of string vibration, initial conditions refer to the displacement and velocity of the string at the "initial" moment.

$$\begin{cases} u|_{t=0} = \varphi(x), & \text{Initial displacement} \\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x). & \text{Initial velocity} \end{cases}$$

- Initial conditions specify the state at the initial time (e.g., at $t = 0$).

- For the wave equation, the two required initial conditions are:
 - The initial displacement $u(t = 0)$
 - The initial velocity $\frac{\partial u}{\partial t}(t = 0)$
- The **number of required initial conditions** depends on the **order of the equation**, and formally speaking, it should be **one order lower than the equation**.
 - For the wave equation, the time derivative has two derivatives, so both the initial position and initial velocity need to be specified.
 - For heat equations, which have only a first-order time derivative, only the initial position needs to be specified.

Boundary Conditions (Need to know its math expression, physical meaning, names)

These describe the physical conditions that the physical quantities of a process satisfy at the boundaries of the system.

For the string vibration problem, there are three basic types:

(1). Dirichlet Boundary Conditions (First Type)

If the motion law of one end of the string is known, represented as u , then the boundary condition can be expressed as:

- Non-homogeneous Boundary Condition $u|_{x=0} = \mu_1(t)$;
- In particular, if the end is **fixed**, the corresponding boundary condition is: $u|_{x=0} = 0$ (“Fixed boundary”, “Homogeneous Boundary Condition”).

Remark 1.1.6. • *The value (or behavior) of u itself at the boundary is given.*

- *In this case, u is a function of both space and time, but at the boundary, it is only a function of time (x is fixed due to the boundary).*
- *The physical meaning: The motion of the string at the boundary is specified, i.e., the displacement at the boundary follows a given function.*
- *Special cases: If $u|_{x=0} = 0$, it is called the **fixed boundary**, while if $u \neq 0$, it is a non-fixed boundary.*

(2). Neumann Boundary Conditions (Second Type)

If one end of the string (for example, $x = l$) slides freely along a line perpendicular to the x -axis and is not subject to any external force in the perpendicular direction, this boundary is called a **free boundary**.

- According to the component of the tension at the right end of the boundary element in the perpendicular direction $T_0 \partial_x u$, the condition at the free boundary is:

$$\partial_x u|_{x=0} = 0, \quad \text{Homogeneous Boundary Condition.}$$

- If the component of the boundary tension in the perpendicular direction is a known function $\mu_2(t)$, then the corresponding boundary condition is:

$$\partial_x u|_{x=0} = \mu_2(t), \quad \text{Non-homogeneous Boundary Condition.}$$

Remark 1.1.7. • *The first spatial derivative of u at the boundary is given.*

- *The physical meaning ($u_x = \tan \alpha \sim \sin \alpha$ perpendicular tension $T_0 \sin \alpha$): The value of the tension at the boundary is specified. If the tension is zero at the boundary, it is called a free boundary. If it is non-zero, it is a non-free boundary.*

(3). Robin Boundary Conditions (Third Type)

If one end of the string (for example, $x = 0$) is fixed on an elastic support, and the behavior of the support follows Hooke's Law.

If the position of the support is $u = 0$, then the value of u at the endpoint represents the behavior of the support at that point.

The vertical component $-T_0 \partial_x u$ of the string's force on the support is given by Hooke's Law, thus the boundary condition in the case of an elastic support is:

$$-T_0 \frac{\partial u}{\partial x} \Big|_{x=l} = ku|_{x=l}.$$

- Homogeneous Boundary Condition:

$$\left(\frac{\partial u}{\partial x} + \alpha u \right) \Big|_{x=l} = 0.$$

where α is a known positive number.

- In mathematics, more general boundary conditions can also be considered:

$$\left(\frac{\partial u}{\partial x} + \alpha u \right) \Big|_{x=l} = \mu_3(t), \quad \text{Non-homogeneous Boundary Condition}$$

where $\mu_3(t)$ is a known function of t .

Remark 1.1.8. • *A linear combination of the value of u and its spatial derivative at the boundary is given.*

- *The physical meaning is the behavior of the elastic support on the boundary.*

Summary:

- The boundary conditions for partial differential equations (PDEs) specify how the solution behaves at the boundaries.
- The Dirichlet boundary condition specifies the value of u at the boundary.
- The Neumann boundary condition specifies the derivative of u at the boundary.
- The third type combines the value of u and its derivative in a linear fashion.

A problem consists of a partial differential equation and conditions. For instance, The initial-boundary value problem is given by:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u|_{t=0} = \varphi(x), \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \psi(x), \\ u|_{x=0} = 0, \quad u|_{x=l} = 0. \end{cases}$$

Depending on the type of conditions, the problems are further divided into:

- Initial Value Problem or Cauchy Problem
- Boundary Value Problem
- Mixed Problem or Initial-Boundary Value Problem

1.2 Heat Conduction Equation

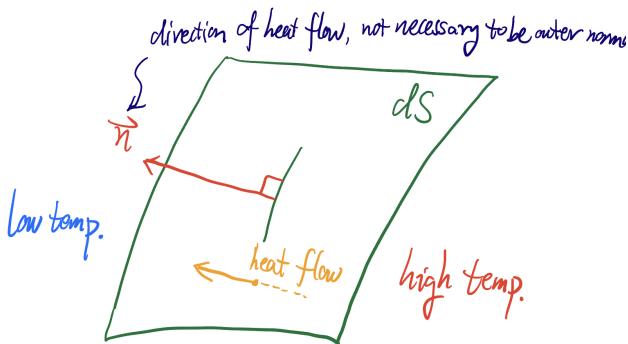
- **Main Topic:** The heat equation is the second of the three key equations in the course, alongside the wave equation and the potential equation. The heat equation differs from the wave equation, and their solutions have fundamentally different properties, necessitating separate study.
- **Physical Meaning:** The heat equation models the transfer of heat. For example, in a classroom, when people enter, they radiate heat, and temperature differences create heat flow from high to low temperature, as described by the second law of thermodynamics (entropy increase).
- **Motivation:** The key question is how the temperature changes over time in response to heat transfer. This dynamic process requires an equation to model the change of temperature, which depends on both space (3D) and time.
- **Approach to Formulating the Equation:**
 - **Selection of Representation Variables:** Temperature $T(t, x)$ is chosen as the primary variable because it represents the physical phenomenon we are interested in (heat transfer).
 - **Choice of Local vs Global Perspective:** A **global (overall) perspective** is used, as we are interested in energy conservation across the entire system, not just at a point.
 - **Physical Law:** **Energy conservation** is the relevant physical law, as heat transfer involves the movement of energy.
 - **Ideal Assumption:** The **empirical Fourier's Law of heat conduction** (in the level of hooke's law, so as assumption instead of physical law) is assumed and **use this empirical formula as an assumption**. This law expresses that the heat flow is proportional to the temperature gradient.

Starting from the heat conduction problem in an object G , we derive the heat conduction equation. If the temperature within the object varies, heat flows from areas of higher temperature to areas of lower temperature. The temperature at any point in the object at a given time is represented as $u(t, x, y, z)$.

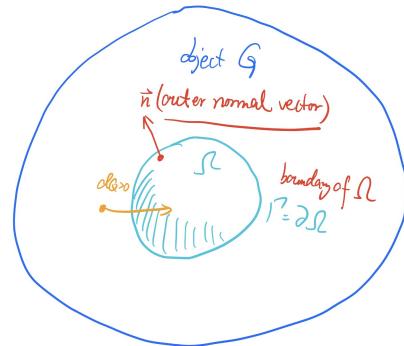
- **Fourier's Law:** The propagation of heat satisfies Fourier's experimental law (see Fig.1.6a): The heat dQ flowing through an infinitesimal area dS in an infinitesimal time dt is proportional to the directional derivative of the temperature u along the normal direction of the surface dS ¹¹, i.e.,

$$dQ = -k(x, y, z) \frac{\partial u}{\partial n} dS dt = -k(x, y, z) n \cdot \nabla u dS dt = -k(x, y, z) \nabla u \cdot d\mathbf{S} dt \Leftrightarrow d\mathbf{q} = -k \nabla u$$

where $k(x, y, z)$ is the thermal conductivity of the material at the point (x, y, z) and is positive. When the material is homogeneous and isotropic, k is a constant. The negative sign in the formula indicates that heat always flows in the direction opposite to the temperature gradient¹² (from higher temperature to lower temperature).



(a) heatflow1



(b) heatflow2

- Fourier's law is typically introduced in thermodynamics or heat studies, but it is not often discussed in the context of equilibrium thermodynamics, which focuses on systems in a state of balance (e.g., Carnot engines).
- The current context involves **non-equilibrium thermodynamics**¹³, which deals with systems that are not in thermal equilibrium¹⁴.

¹¹Note $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma$ this language is more compact and concise $n \cdot \nabla u$, $(\cos \alpha, \cos \beta, \cos \gamma)$ is direction cosine and it is a **unit** directional vector.

¹²The direction of the gradient ∇f is the direction in which the function $f(x, y, z, \dots)$ increases most rapidly. This is why the gradient is often referred to as the **direction of steepest ascent**.

¹³Non-equilibrium thermodynamics is essential for understanding processes like **thermoelectric effects**, where a temperature difference generates electrical energy.

¹⁴In non-equilibrium thermodynamics, a key principle is that:

- * **Thermodynamic Flux is Proportional to Thermodynamic Force.**
- * **Thermodynamic Flux:** Refers to the flow of a physical quantity such as heat, particles, or momentum.
- * **Thermodynamic Force:** The driving force that causes the flux, such as a temperature **gradient** (for heat conduction), concentration **gradient** (for diffusion), or velocity **gradient** (for momentum transfer). For example, gravitational force and electric field intensity are both the negative gradient of a potential.

The relationship can be expressed as:

$$\mathbf{J} = -L \cdot \mathbf{X}$$

where:

- **Empirical Formula:** Fourier's law is an empirical formula that describes heat flow and temperature gradients.

1.2.1 Derivation of Heat Equations

Consider the heat flux entering a region Ω and see Fig.1.6b.

- Heat flux $d\mathbf{q}$ is considered to flow **inward** into a region.
- The direction of heat flow is **opposite** to the outward normal vector of the boundary.

Energy Conservation in Heat Transfer

- The heat flux entering a region Ω increases the temperature of the material inside.
- According to the **principle of energy conservation**, the heat energy entering the region must be equal to the increase in internal energy of the system.
- The **specific heat capacity** (denoted as c) is a physical property of a material that quantifies the amount of heat energy required to raise the temperature of a unit mass of the substance by one degree Celsius (or one Kelvin). It is defined as:

$$c = \frac{1}{m} \frac{dQ}{dT} \quad \Rightarrow \quad dQ = cmdT \quad \Rightarrow \quad \Delta Q = cm\Delta T.$$

where:

- c is the specific heat capacity (in units of $\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$),
- m is the mass of the substance (in kilograms),
- dQ is the infinitesimal amount of heat energy added to the substance (in joules),
- dT is the infinitesimal change in temperature (in Kelvin or Celsius).
- Mathematically, the total heat entering the region is given by:

$$\int_{t_1}^{t_2} \int_{\Gamma} dQ = \int_{t_1}^{t_2} \int_{\Gamma} k\mathbf{n} \cdot \nabla u dS dt = \int_{\Omega} c(x, y, z) \rho(x, y, z) (u(t_2, x, y, z) - u(t_1, x, y, z)) dV \quad (1.2.1)$$

Let us explore possible approaches to simplify (1.2.1).

-
- * J is the thermodynamic flux (e.g., heat flow, mass flux).
 - * X is the thermodynamic force (e.g., temperature **gradient**, concentration **gradient**).
 - * L is the phenomenological coefficient (often called the Onsager coefficient), which determines the proportionality.

This principle is foundational in understanding non-equilibrium processes, such as:

- * Heat conduction
- * Diffusion
- * Electrical currents in thermoelectric materials

1. Objective

- We aim to derive a **differential equation** from a given **integral equation**.
- The key approach is to **eliminate the integral sign** by transforming the equation into a local form.

2. Strategy for Simplification

- Simplify both terms on the left-hand and right-hand sides separately.
- **HOPE**¹⁵: Attempt to merge the integrals into a single integral equation.
- If successful, apply the fundamental theorem of calculus to remove the integral.

3. Handling Different Types of Integrals

- The left-hand side consists of a **double spatial integral** and a **single time integral**.
- The right-hand side consists of a **triple spatial integral** with no time integral.
- To unify these, convert the double integral into a triple integral, then introduce a time integral where necessary.

4. Applying Integral Theorems

- Use mathematical transformations to convert between integral forms.
- The most fundamental theorems for this process are:
 - **Gauss's theorem** (Divergence theorem)

Theorem 1.2.1. *Let Ω be a compact region in \mathbb{R}^n with a smooth boundary $\partial\Omega$, and let \mathbf{F} be a continuously differentiable vector field on Ω . Then:*

$$\int_{\Omega} \nabla \cdot \mathbf{F} dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS \left(= \int_{\partial\Omega} (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS \right),$$

where:

- * $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F} = (P, Q, R)$,
- * dV is the volume element in \mathbb{R}^n ,
- * \mathbf{n} is the outward unit normal to the surface $\partial\Omega$,
- * dS is the surface element on $\partial\Omega$.

– **Green's theorem**

– **Stokes' theorem**

- These three theorems are actually special cases of a unified theorem in vector calculus, Stokes' theorem¹⁶ (see [3, §9.1]).

¹⁵“Hope” drives mathematical thinking, as it formulates conjectures from clues that we can explore.

¹⁶That is,

- **Steps to Derive the Heat Equation (for simplicity if $k = \text{constant}$)¹⁷:**

$$\begin{array}{ccc}
 \underbrace{\int_{t_1}^{t_2} \int_{\partial\Omega} k \mathbf{n} \cdot \nabla u \, dS \, dt}_{\Downarrow \text{Gauss}} & \xrightarrow{\text{energy conservation}} & \underbrace{\int_{\Omega} c(x, y, z) \rho(x, y, z) (u(t_2, x, y, z) - u(t_1, x, y, z)) \, dV}_{\Downarrow \text{N-L}} \\
 \downarrow & & \downarrow \\
 \underbrace{\int_{t_1}^{t_2} \int_{\Omega} k \nabla \cdot \nabla u \, dV \, dt}_{\Downarrow} & & \underbrace{\int_{\Omega} c(x, y, z) \rho(x, y, z) \int_{t_1}^{t_2} \frac{\partial u}{\partial t} \, dt \, dV}_{\Downarrow} \\
 \downarrow & & \downarrow \\
 \int_{t_1}^{t_2} \int_{\Omega} k \Delta u \, dV \, dt & \xrightarrow{\text{energy conservation}} & \int_{t_1}^{t_2} \int_{\Omega} c(x, y, z) \rho(x, y, z) \frac{\partial u}{\partial t} \, dV \, dt
 \end{array}$$

Then

$$\int_{t_1}^{t_2} \left(\int_{\Omega} c(x, y, z) \rho(x, y, z) \frac{\partial u}{\partial t} - k \Delta u \right) \, dV \, dt = 0 \quad (1.2.2)$$

- We ask a question:

Problem 1.2.1. Consider a case where the integral of a function equals zero, i.e.,

$$\int_{\Omega} f(x) \, dx = 0$$

Theorem (Stokes' Theorem). Let M be a smooth manifold with boundary ∂M , and let ω be a smooth differential form on M . Then

$$\int_M d\omega = \int_{\partial M} \omega,$$

where $d\omega$ is the exterior derivative of ω , and the integrals on both sides are taken with respect to the appropriate volume forms on M and ∂M , respectively.

¹⁷Mathematical analysis often involves two fundamental perspectives:

1. Integral (Global) Perspective

- Typically uses the **Newton-Leibniz formula**:

$$\int_a^b f'(x) \, dx = f(b) - f(a).$$

- This perspective is useful for analyzing overall changes and accumulated quantities.

2. Differential (Local) Perspective

- Often relies on the **Mean Value Theorem (MVT)** and **Taylor Expansion**.
- The Mean Value Theorem states that for a differentiable function $f(x)$, there exists a point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Taylor expansion approximates functions locally as:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

3. Summary

- **Integral perspective** is used for global analysis, leveraging the Newton-Leibniz formula.
- **Differential perspective** focuses on local behavior, using the Mean Value Theorem and Taylor expansion.

where Ω is an **arbitrary** region. Can we conclude that $f(x) = 0$ for all x ?

- A simple **counterexample** exists: odd functions (e.g., sine and cosine) integrated over a symmetric interval can result in zero, yet the function itself is not zero.

$$\int_{-a}^a f(x) dx = 0 \quad \text{for odd functions, but} \quad f(x) \neq 0 \text{ for all } x$$

- The **key difference** here lies in the fact that in (1.2.3) and Problem 1.2.1, the integration is done over a region Ω which can be **arbitrarily chosen**.
- If a function's integral equals zero **over any region** Ω , we can conclude that the function must be **identically zero**. This is because (**a proof by contradiction**):
 - * If $f(x)$ is non-zero at some point, say $f(x_0) > 0$, then in a small neighborhood around x_0 , the function would remain positive.
 - * Integrating over this small neighborhood would yield a positive result, leading to a contradiction with the integral being zero.
- Thus, the key insight is the **arbitrary selection** of the integration region Ω . Once we allow for arbitrary regions and time intervals, the result follows that the function must be zero everywhere in the chosen domain.
- Thus, (1.2.3) yields the heat equation (for simplicity, taking ρ, k, c to be constants for homogeneous matter):

$$c\rho \frac{\partial u}{\partial t} = k\Delta u \quad \Leftrightarrow \quad \frac{\partial u}{\partial t} = a^2 \Delta u \quad \text{where } a^2 := \frac{k}{c\rho} \quad (\text{homogeneous}) \quad (1.2.3)$$
- Heat: $\frac{\partial u}{\partial t} = a^2 \Delta u$; Recall wave $\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u$ and compare them:
 - **Similarities and main difference:**
 - * Both equations are similar in form, involving spatial derivatives.
 - * The wave equation involves **second-order time derivatives**, while the heat equation involves a **first-order time derivative**.
 - **Wave Equation:**
$$u_{tt} - a^2 u_{xx} = 0$$
- The wave equation models **oscillations or vibrations** and the solution is often represented by **trigonometric functions**.
- **Heat Equation:**
$$u_t - a^2 u_{xx} = 0$$
- The heat equation models diffusion and the solution involves **exponential decay**, showing how **heat dissipates over time**.

Heat Conduction Equation with Internal Heat Sources

- **Non-Homogeneous Term:**

- Consider a function $f(x, y, z, t)$, where f depends on the independent variables x, y, z, t . This is called a non-homogeneous term.
- Similar to ODEs, the non-homogeneous term f represents an external source or forcing function.

- **Heat Source:**

- In the case of a non-homogeneous heat equation, the right-hand side term $f(x, y, z)$ represents a heat source.
- When there is no heat source, the heat flow into a region balances the heat leaving it, with no additional heating from the inside.
- With a heat source, the heat flow into the region is augmented by the internal heat generation, so the total heat entering the region is the sum of the **external heat** and the **heat generated** within the domain.

- **Conclusion:**

- The derivation for the non-homogeneous case follows the same steps as the homogeneous case, but with an **additional term** representing the **internal heat source**.

If the body contains **internal heat sources** (e.g., due to electric currents or chemical reactions), let $F(x, y, z, t)$ represent the heat source density (the amount of heat generated per unit volume per unit time). The heat generated within Ω from t_1 to t_2 is:

$$Q_3 = \int_{t_1}^{t_2} \int_{\Omega} F(x, y, z, t) dV dt.$$

By conservation of energy, the total heat balance is:

$$Q_1 + Q_3 = Q_2.$$

Combining the previous equations:

$$\int_{t_1}^{t_2} \int_{\Omega} [\nabla \cdot (k \nabla u) + F(x, y, z, t)] dV dt = \int_{\Omega} c\rho[u(x, y, z, t_2) - u(x, y, z, t_1)] dV.$$

Rearranging and using the fundamental theorem of calculus:

$$\int_{t_1}^{t_2} \int_{\Omega} \left[c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) - F(x, y, z, t) \right] dV dt = 0.$$

Since Ω is **arbitrary** and the integrand is **continuous**, we obtain the **non-homogeneous** heat conduction equation:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (k \nabla u) + F(x, y, z, t).$$

For a homogeneous material with $a^2 = \frac{k}{c\rho}$, the equation becomes:

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(x, y, z, t).$$

where $f(x, y, z, t) = \frac{F(x, y, z, t)}{c\rho}$.

1.2.2 Initial and Boundary Conditions for the Heat Conduction Equation

Initial Conditions

The initial condition describes the temperature distribution within the body at the initial time $t = 0$:

$$u(x, y, z, 0) = \phi(x, y, z)$$

where $\phi(x, y, z)$ is a known function representing the initial temperature distribution.

- The number of required initial conditions depends on the order of the time derivative.
- Since the heat equation involves a first-order time derivative (u_t), only the function $u(x, t)$ itself needs to be specified at $t = 0$.

Boundary Conditions (Study: Physical meaning, math form and names)

The boundary conditions describe the physical conditions at the boundary of the body. There are three main types of boundary conditions:

- **First Type (Dirichlet Boundary Condition):**

- Specifies the function value u on the boundary.
- **Physically**, this means the temperature (or other dependent variable) on the boundary is given.
- **Mathematically**: if the boundary surface S of the body G has a known temperature distribution $\mu_1(t)$:

$$u(x, y, z, t)|_S = \mu_1(t).$$

- If the boundary is kept at a constant temperature 0, this is called a homogeneous Dirichlet condition:

$$u(x, y, z, t)|_S = 0.$$

- **Second Type (Neumann Boundary Condition):**

- Specifies the normal derivative of u on the boundary.
- In a 3D setting, this means providing the directional derivative of u along the outward normal \mathbf{n} .
- **Mathematically**:

$$\frac{\partial u}{\partial n}|_S = \mu_2(x, y, z, t).$$

- **Physically**¹⁸, The heat flux q across the boundary S per unit area and per unit time is given. According to Fourier's law,

$$-k \frac{\partial u}{\partial n}|_S = q \Rightarrow \frac{\partial u}{\partial n}|_S = \mu_2(x, y, z, t)$$

where $\mu_2(x, y, z, t) = -q/k$ is a known function defined on S and $t \geq 0$.

¹⁸To understand the physical meaning, one only needs to interpret what $\frac{\partial u}{\partial n}$ represents.

- If the boundary is **insulated** (no heat flux), this is called a homogeneous Neumann condition:

$$\frac{\partial u}{\partial n} \Big|_S = 0.$$

- **Third Type Boundary Condition (Robin Boundary Condition)**

- A linear combination of Dirichlet and Neumann conditions.
- Mathematically:

$$\frac{\partial u}{\partial n} \Big|_S + \alpha u \Big|_S = \mu_3(t)$$

where α is a heat transfer coefficient.

- Consider a body in contact with a surrounding medium. Let u be the temperature of the body and u_1 be the temperature of the surrounding medium. If the temperatures are different, heat exchange occurs at the boundary S . According to **Newton's law of cooling**¹⁹, the heat flux through the boundary is proportional to the temperature difference:

$$dQ = -h(u - u_1)dt dS$$

where h is the **heat transfer coefficient**.

Since heat cannot accumulate on the surface of an object, consider an infinitely close closed surface S_1 inside the object that is tangent to the object's surface S . The heat flux through surface S_1 should be equal to the heat flux through surface S (see Fig.1.7).

The heat flowing through surface S_1 is given by $dQ = -k \frac{\partial u}{\partial n} dS dt$, which leads to the relationship:

$$-k \frac{\partial u}{\partial n} dS dt = h(u - u_1) dS dt \implies k \frac{\partial u}{\partial n} + hu = hu_1$$

¹⁹Heat Transfer in Continuous and Discontinuous Media

Fourier's Law of Heat Conduction

- * Describes heat conduction in a **continuous medium**.
- * States that heat flux \mathbf{q} is proportional to the temperature gradient:

$$d\mathbf{q} = -k \nabla T$$

where k is the thermal conductivity.

- * Applicable within the **same medium**, assuming a **smooth temperature variation**.

Newton's Law of Cooling

- * Describes heat transfer between **two different media** (e.g., a solid surface and surrounding air).
- * The temperature is generally **discontinuous** at the interface.
- * Since temperature is **not continuous**, **derivatives cannot** be directly applied.
- * Empirical law states:

$$\Delta \mathbf{q} = -h(u - u_1)$$

Mathematical Basis

- * Both Fourier's and Newton's laws can be understood as first-order **Taylor expansions**.
- * Fourier's law corresponds to a spatial Taylor expansion within a continuous medium.
- * Newton's law is based on an empirical relationship due to the discontinuity at the interface.

This can be rewritten as:

$$\left(\frac{\partial u}{\partial n} + \alpha u \right) \Big|_S = \mu_3(x, y, z, t)$$

where $\alpha = \frac{h}{k}$, and $f_3(x, y, z, t)$ is a known function defined on $(x, y, z) \in S, t \geq 0$.

This type of boundary condition is particularly useful when the body is in contact with a medium that can exchange heat.

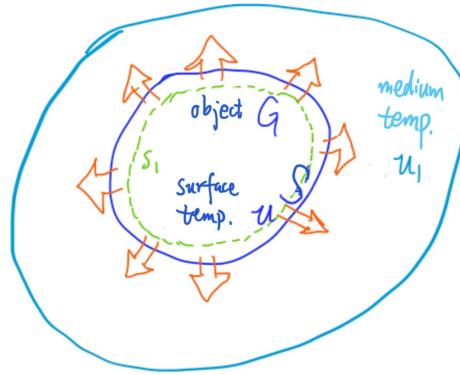


Figure 1.7: Robin condition

1.3 Laplace's Equation and Boundary Value Problems

Laplace's Equation

Laplace's equation (or **harmonic equation**) describes the steady-state distribution of a physical quantity, such as temperature or potential. In three dimensions, it is written as:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

A function $u \in C^2$ that satisfies Laplace's equation is called a **harmonic function**. That is

$$\begin{cases} (1) u \in C^2 \\ (2) \Delta u = 0 \end{cases} \quad (1.3.1)$$

Relationship with Other Equations:

- Laplace's equation can be derived by observing heat and wave equations.
- The **key difference** between Laplace's equation and the heat or wave equation is the **absence of time dependence** $\frac{\partial u}{\partial t} \equiv 0$.
- For example, in the heat equation, the temperature changes over time, while in Laplace's equation, the temperature is independent of t (steady-state).

Physical Interpretations:**1. Wave Interpretation**

- Laplace's equation can be seen as describing the state of a damped wave or oscillation that reaches equilibrium.
- As time progresses, the oscillations decay, and eventually, they stop, reaching a balanced or steady state. This corresponds to the static nature of Laplace's equation.

2. Heat Interpretation

- Laplace's equation also describes heat conduction at equilibrium.
- This steady-state condition, where temperature does not change over time, results in the Laplace equation.

Poisson's Equation

Poisson's equation is a non-homogeneous version of Laplace's equation, where a source term $f(x, y, z)$ is present²⁰:

$$\Delta u = -f(x, y, z)$$

Physical Interpretations:

The physical phenomenon of electrostatic field and potential distribution can also be described by the Laplace equation or Poisson's equation.

The electrostatic field is denoted as \mathbf{E} and the charge density is $f(x, y, z)$, and by Gauss's law:

$$\underbrace{\iint_{\Gamma} \mathbf{E} \cdot \mathbf{n} dS}_{\Downarrow \text{Gauss}} = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} f dV$$

$$\iiint_{\Omega} \nabla \cdot \mathbf{E} dV$$

Assume $u(x, y, z)$ is the electric potential, then $\mathbf{E} = -\nabla u = -(u_x, u_y, u_z)$.

$$-\iiint_{\Omega} \nabla^2 u dV = \frac{1}{4\pi\epsilon_0} \iiint_{\Omega} f dV \Rightarrow \Delta u = -\frac{f}{4\pi\epsilon_0}, \quad (\text{by the arbitrariness of } \Omega)$$

If there is no charge distribution in G , $f = 0$, then u satisfies the Laplace equation:

$$\Delta u = 0.$$

Remark 1.3.1. *The Maxwell's equations can also derive the wave equation - electromagnetic wave.*

²⁰The negative sign represents the eigenvalues of Δ operator is negative, recalling $\mathcal{A}u = \lambda u$ and letting $\mathcal{A} = \Delta$. See Chap. 2.

Boundary Value Problems

Boundary value problems for Laplace's equation involve finding a solution $u(x, y, z)$ that satisfies the equation within a domain Ω and meets certain boundary conditions on the boundary Γ :

- **First Boundary Value Problem (Dirichlet Problem):**

Given a continuous function f on the boundary Γ of a certain region Ω in space, it is required that the function $u(x, y, z)$ is continuous on the closed region $\Omega + \Gamma$ and harmonic within Ω , and coincides with the given function f on the boundary Γ .

$$\underbrace{u}_{\text{conti. in } \Omega + \Gamma; \text{harmonic in } \Omega} \Big|_{\Gamma} = \underbrace{f}_{\text{conti. on } \Gamma}$$

- **Second Boundary Value Problem (Neumann Problem):**

Given a continuous function f on the boundary Γ of a certain region Ω in space, it is required that the function $u(x, y, z)$ is continuous on the closed region $\Omega + \Gamma$, harmonic within Ω , and has a normal derivative $\frac{\partial u}{\partial n}$ existing on the boundary Γ .

$$\underbrace{\frac{\partial u}{\partial n}}_{\text{conti. in } \Omega + \Gamma; \text{harmonic in } \Omega; \frac{\partial u}{\partial n} \text{ exists}} \Big|_{\Gamma} = \underbrace{f}_{\text{conti. on } \Gamma}$$

Remark 1.3.2. • The function u must be continuous not only inside the domain Ω but also on the boundary $\partial\Omega$.

- If u is discontinuous at the boundary, the boundary condition would be meaningless since it could be arbitrarily chosen. This would render the problem ill-posed.
- The boundary conditions must ensure both the continuity of u and the existence of its derivatives at the boundary, especially for problems involving second-type boundary conditions.
- This continuity condition is subtle but essential. Without it, the boundary conditions would not impose proper constraints on the problem, making the solution meaningless.

1.4 Basic Concepts and Knowledge

1.4.1 Basic concepts

- **Reason for Defining Concepts:**

- When discussing equations, it was mentioned that equations with specific characteristics require particular methods for solving.
- The concepts defined below describe these characteristics.
- Understanding these concepts helps determine the appropriate methods to apply.

Definition 1.4.1 (PDE). An equation containing independent variables, unknown functions and their partial derivatives with respect to independent variables is called a partial differential equation (PDE), e.g.,

$$F(\partial^\alpha u(x), \partial^\beta u(x), \dots, \partial u(x), u(x), x) = 0, \quad (1.4.1)$$

where α and β are multi-indices.

- Examples:

- Second Order:

$$U_{tt} = a^2 U_{xx}$$

- Homogeneous:

$$U_{xx} + U_{yy} = 0$$

- Non-homogeneous:

$$\begin{aligned} U_{xxy} + 2xU_{yy} + yU &= xy \\ (U_x)^2 + U_y &= 8x^2 \end{aligned}$$

Definition 1.4.2 (Order). *The highest order of the derivative of the unknown function in the PDE is called the "order of the PDE".*

Definition 1.4.3 (Linearity). *If each term in the PDE is linear with respect to the unknown function and its partial derivatives (including higher-order derivatives), it is called a "linear PDE".*

$$F \text{ is linear on } (\partial^k u, \partial^{k-1} u, \dots, \partial u, u)$$

i.e., it has the form:

$$\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u = f(x),$$

where $\mathcal{A} = (a_\alpha(x), \dots, a(x), a_y)$.

$$\mathcal{A}u = f(x) \quad \text{where } u = (\partial^k u, \dots, \partial u, u).$$

Remark 1.4.1. • A **linear PDE** refers to an equation where the unknown function and its derivatives appear in a linear form.

- In this context, the function F is a linear function of the unknown function u and its derivatives.
- The linearity means that the equation involves linear combinations of u and its derivatives, with no higher powers or nonlinear terms of u .
- Specifically, F is a linear function of u and its partial derivatives, but not necessarily linear in the independent variables.
- The equation can often be expressed in matrix form, where the unknowns are vector functions (i.e., column vectors), and the coefficients depend on the independent variables.

Definition 1.4.4 (Classical Solution). *A function is called a **classical solution** of a partial differential equation if it has all the continuous partial derivatives required by the equation and satisfies the equation.*

Definition 1.4.5 (Free Term, Source Term²¹, Nonhomogeneous Term). *A term in a partial differential equation that does not contain the unknown function or its derivatives (i.e., only include function of independent variables). For example, in the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z)$$

$f(x, y, z)$ is the free term. If $f(x, y, z) = 0$, the equation is homogeneous; otherwise, it is non-homogeneous.

²¹This is because the free term often originates from external forces, heat sources.

- Remark 1.4.2.**
- The free term refers to a term in an equation that does not contain the unknown quantity.
 - Specifically, it is a term that does not involve the unknown function or its derivatives.
 - In other words, this term contains only the independent variables.
 - Such a term is called the “free term” because it is independent of the unknown function and its derivatives.

- Remark 1.4.3.**
- We previously discussed boundary conditions as well.

- If a boundary condition is equal to zero, it is called a **homogeneous boundary condition**.
- If the right-hand side of the boundary condition contains a function of the independent variables, it is called a **non-homogeneous boundary condition**.
- These concepts of homogeneous and non-homogeneous may seem purely mathematical, but as we proceed to solve equations later in this class, you will see that they determine the method you should choose to solve the equation. Thus, these concepts are very important.

1.4.2 Well-Posedness

Definition 1.4.6 (Well-Posedness of Boundary Value Problems). *The well-posedness of a boundary value problem refers to the existence, uniqueness, and stability of the solution. A problem is well-posed if:*

1. (Existence) A solution exists.
2. (Uniqueness) The solution is unique.
3. (Stability) The solution depends continuously on the initial and boundary data.

- Remark 1.4.4.**
- If the three conditions are satisfied, it means that the problem is given just the right conditions, neither too strong nor too weak.

- If the conditions are too strong, there may be no solution, violating the existence of the solution.
- If the conditions are too weak, there may be many solutions, so the solution is not unique.
- Stability means that if you have found a solution to the problem and know a solution, and if the initial conditions deviate very slightly from the initial values of this solution, then the solution will always deviate very slightly from the value of the solution you obtained.
- In other words, if the input error is very small, the output error is also very small. This is what we commonly call stability.
- This is a very good condition because in practical work, you will often involve measurements, and you may calculate a solution through theoretical derivation.
- Then, you need to measure some things through experiments, and measurements will inevitably introduce errors.

- If there is no stability, then even if the input error is very small, the error may deviate significantly from the solution as time progresses.
- In that case, the actual measured solution may differ greatly from the theoretical value, making the theoretical value less reliable.
- Therefore, stability is also a very good requirement.
- However, in recent years, it has been found that instability is also very common in our real life.
- For example, if everything were stable, the universe would not have produced the Earth or humans.
- Theoretical calculations show that the universe would need much longer than its current age to produce the Earth and humans.

1.4.3 Superposition Principle

- This principle of superposition runs through every solution method we will discuss later.
- I want to emphasize that it is called the **linear superposition principle**. The word "linear" is important because we are dealing with **linear** equations in this book.

Consider a second order linear partial differential equation of the form:

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = f_i, \quad (i = 1, 2, \dots) \quad (1.4.2)$$

where A to F , and f_i are known functions of x and y in some region. If u_i is a solution corresponding to the source term f_i , if the series

$$u = \sum_{i=1}^{\infty} c_i u_i \quad (1.4.3)$$

converges, where $c_i (i = 1, 2, \dots)$ are arbitrary constants and it can also be differentiated term by term twice, then the series (1.4.3) is a solution to the following equation

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = \sum_{i=1}^{\infty} c_i f_i$$

In particular, when the free term $f_i = 0$ in equation (1), the corresponding homogeneous equation is

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0. \quad (1.4.4)$$

If $u_i (i = 1, 2, \dots)$ are solutions to equation (1.4.4), then the series (1.4.3) is also a solution to equation (1.4.4).

Ideas for the proof:

1. Let

$$\mathcal{A} = A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F \quad (\text{linear operator})$$

This is a linear operator.

2. From (1.4.2), we have $\mathcal{A}u_i = f_i$; thus, summing up gives $\sum \mathcal{A}u_i = \sum f_i$; hence, $\mathcal{A}(\sum u_i) = \sum f_i$.
3. Letting $u = \sum u_i$, $\mathcal{A}u = f$.

1.4.4 Classification of Second-Order PDEs

Question 1.4.1. *Why classify?*

A. *Phenomena vary greatly. By classifying, we capture commonalities. By studying one problem with commonalities, we can understand the results caused by these commonalities in all such problems.*

A general second-order linear partial differential equation has the following form:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu = f, \quad (1.4.5)$$

where $a_{11}, a_{12}, a_{22}, b_1, b_2, c, f$ etc., are real functions of the independent variables x, y in the region Ω , and it is assumed that they are continuously differentiable.

Eq. (1.4.5) can be rewritten as

$$\underbrace{\left(\partial_x, \partial_y \right) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left(\begin{matrix} \partial_x \\ \partial_y \end{matrix} \right)}_{=:M} u + (b_1, b_2) \left(\begin{matrix} \partial_x \\ \partial_y \end{matrix} \right) u + cu = f \quad (1.4.6)$$

If at some point (x_0, y_0) in the region Ω ,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M > 0,$$

then equation (1.4.5) is said to be **hyperbolic at the point** (x_0, y_0) ; if equation (1.4.5) is hyperbolic at every point in the region Ω , then equation (1.4.5) is said to be **hyperbolic** in this region.

If at some point (x_0, y_0) in the region Ω ,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M = 0,$$

then equation (1.4.5) is said to be **parabolic at the point** (x_0, y_0) ; if equation (1.4.5) is parabolic at every point in the region Ω , then equation (1.4.5) is said to be **parabolic** in this region.

If at some point (x_0, y_0) in the region Ω ,

$$\Delta \equiv a_{12}^2 - a_{11}a_{22} = -\det M < 0,$$

then equation (1.4.5) is said to be **elliptic at the point** (x_0, y_0) ; if equation (4) is elliptic at every point in the region Ω , then equation (1.4.5) is said to be **elliptic** in this region.

- Hyperbolic equations correspond to hyperbolas, and their matrix form has a negative determinant (e.g., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola).
- Elliptic equations correspond to ellipses, and the matrix determinant is positive.
- Parabolic equations correspond to parabolas, where the determinant is zero.
- The classification depends primarily on the second-order terms of the PDE, specifically the determinant of the coefficient matrix.
- The matrix M discussed is known as a **metric matrix** in mathematics. It determines the structure of spacetime.

- The determinant of the matrix is negative for **hyperbolic** form, corresponding to **Lorentz geometry**.
- The classification into **hyperbolic, parabolic, and elliptical** corresponds to the conic sections and is related to **quadratic forms** (see (1.4.6)).
- **Quadratic form normalization** using congruent matrices is a process taught in linear algebra. This process can also be applied to **differential equations** to simplify a second order linear PDE.

An analogy:

The **hyperbola equation** is $a^2x^2 - b^2y^2 = 1$. Using the quadratic form:

$$(x, y) \underbrace{\begin{pmatrix} a^2 & 0 \\ 0 & -b^2 \end{pmatrix}}_{=:M \Rightarrow \det M = -a^2b^2 < 0} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Generalize above hyperbola and take $x \rightarrow \partial_x$ and $y \rightarrow \partial_y$ (in linear algebra, x, y can be “any thing” which obeys certain linear relations, thus they can be differential operators). Equation classification depends on the highest degree:

$$(\partial_x, \partial_y) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

If $\det M < 0$, i.e., $\Delta = -\det M > 0$, then it is a hyperbola.

In addition, $\det M > 0$, it is an ellipse. If $\det M = 0$, it is a parabola, since the parabola has only one quadratic term.

$$\begin{array}{c} \overbrace{ax^2 - by^2 = 1}^{\text{a hyperbola}} \\ \Downarrow \\ \overbrace{ax^2 - by^2 = (x, y) \begin{pmatrix} a & \\ b & \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}^{\text{a hyperbolic polynomial}} \\ \Downarrow \\ \overbrace{x, y \text{ can be any math objects, indeterminate}} \\ \Downarrow \\ a\partial_x^2 - b\partial_y^2 = (\partial_x, \partial_y) \begin{pmatrix} a & \\ b & \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \end{array}$$

Examples of PDE Classification**Wave Equation (Hyperbolic)**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Here, $\Delta = a^2 > 0$, so it is **hyperbolic**.

Heat Equation (Parabolic)

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Here, $\Delta = 0$, so it is **parabolic**.

Laplace's Equation (Elliptic)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Here, $\Delta = -1 < 0$, so it is **elliptic**.

1.5 Fourier Series

Summary of Key Points on Fourier Series

- **Chapters Overview:** Chapters 2 and 5 of the book use **series solution methods** based on Fourier series.
- **New Perspective:** A new viewpoint is introduced that differs from previous calculus. It is recommended to compare this perspective with previous ones.
- **Representation of Fourier Series:**
 - Fourier series can be expressed using trigonometric functions.
 - Alternatively, they can be represented by exponential functions (using the imaginary unit i as in $e^{i\theta}$).
 - In this course, the trigonometric form is primarily used.
- **Applicability:** Fourier series are applicable only to **periodic functions**.
- **Conceptual Note:** The **idea of periodic extension** (due to above applicability) is critical for understanding subsequent material, even if it is not explicitly emphasized in the text.

Remember 4 things:

1. Analogous to linear expansion $y = \sum_{i=0}^n a_i e_i$
2. Odd and Even Functions
3. Orthogonality
4. Completeness and Inner Product Method for Finding Coefficients

1.5.1 Analogous to linear expansion

Suppose the function $f(x)$ with period²² $2l$ can be expanded into a Fourier series, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

where the Fourier coefficients a_n, b_n satisfy

$$\begin{aligned} a_n &= \frac{1}{l} \langle f(x), \cos \frac{n\pi x}{l} \rangle = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \quad (n = 0, 1, 2, \dots), \\ b_n &= \frac{1}{l} \langle f(x), \sin \frac{n\pi x}{l} \rangle = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, 3, \dots). \end{aligned}$$

Analogous to linear expansion $\vec{\xi} = \sum_{i=0}^n a_i \mathbf{e}_i$

$$\vec{\xi} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \dots + a_n \vec{e}_n = \sum_{i=1}^n a_i \vec{e}_i$$

where

$\{\vec{e}_i\}$ is an **orthonormal** basis.

- Vector $\vec{\xi} \leftrightarrow$ Function $f(x)$;
- Basis $\{\vec{e}_i\} \leftrightarrow$ Basis $\{1, \sin, \cos, \dots\}$.

To find the coefficients a_i , use the **inner product** as the **component**.

$$\begin{aligned} \langle \vec{\xi}, \vec{e}_j \rangle &= \sum_{i=1}^n a_i \langle \vec{e}_i, \vec{e}_j \rangle = \sum_{i=1}^n a_i \delta_{ij}, \quad (\text{only orthogonality is needed!!!}) \\ \Rightarrow a_i &= \langle \vec{\xi}, \vec{e}_i \rangle \end{aligned}$$

Then

- **Hope:** A method for computing a_n .
- **Question:** To make such an analogy, further proof of **orthogonality** of the basis $\{1, \sin, \cos\}$ is also needed.
- **Orthogonality** need the **inner product!**
- **Question:** “**What is**” or “**how to define**” the inner product of functions?
A: The **inner product** of two functions $f(x)$ and $g(x)$ over a period $[-l, l]$ is defined as:

$$\langle f, g \rangle = \int_{-l}^l f(x)g(x) dx$$

²²The text does not explicitly state that the function is periodic; it is directly applied as such. However, this inherently involves an extension process, which introduces several important considerations. A clear understanding of this concept is essential for fully comprehending the subsequent material.

- **Ideas for the above inner product:** Compare $a_i = \langle \vec{\xi}, \vec{e}_i \rangle$ in linear algebra with $a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$ in Fourier transformations. They are very similar. By comparing the two formulas, we find that the inner product is likely defined as the integral of the product of two functions. Based on this rough idea, we define it as above.
- Verify orthogonality of $\{1, \sin, \cos, \dots\}$ according to this definitions.

1.5.2 Orthogonality

(1) Orthogonality: Trigonometric Function Series

The functions $\{1, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \cos \frac{2\pi x}{l}, \sin \frac{2\pi x}{l}, \dots, \cos \frac{n\pi x}{l}, \sin \frac{n\pi x}{l}, \dots\}$ are orthogonal on the interval $[-l, l]$ (for any one period), that is:

$$\int_{-l}^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^l \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^l \sin \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = 0,$$

$$\int_{-l}^l \sin \frac{n\pi x}{l} dx = \int_{-l}^l \cos \frac{n\pi x}{l} dx = 0.$$

Similarly,

$$\int_{-l}^l \sin \frac{(2m+1)\pi x}{2l} \sin \frac{(2n+1)\pi x}{2l} dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^l \cos \frac{(2m+1)\pi x}{2l} \cos \frac{(2n+1)\pi x}{2l} dx = \begin{cases} 0, & m \neq n, \\ l, & m = n. \end{cases}$$

$$\int_{-l}^l \sin \frac{(2m+1)\pi x}{2l} \cos \frac{(2n+1)\pi x}{2l} dx = 0,$$

(2) Trigonometric Product-to-Sum Formulas:

$$\sin \alpha \sin \beta = -\frac{1}{2}[\cos(\alpha + \beta) - \cos(\alpha - \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Note

$$\int_{-\pi}^{\pi} \cos(\lambda x) dx = \frac{1}{\lambda} \int_{-\pi}^{\pi} \cos(\lambda x) d\lambda = \frac{1}{\lambda} \left[\frac{\sin(\lambda x)}{\lambda} \right]_{-\pi}^{\pi} = \frac{2}{\lambda} \sin(\lambda \pi)$$

- For $m \neq n$:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos((m+n)x) dx &= \frac{2}{m+n} \sin((m+n)\pi) = 0 \\ \int_{-\pi}^{\pi} \cos((m-n)x) dx &= \frac{2}{m-n} \sin((m-n)\pi) = 0 \end{aligned}$$

- If $m = n$:

$$\int_{-\pi}^{\pi} \cos(mx) dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos(2mx)) dx = \pi$$

$$\begin{aligned} \Rightarrow \|\cos(mx)\| &= \sqrt{\pi} \\ \|\sin(mx)\| &= \sqrt{\pi} \end{aligned}$$

Remark 1.5.1. • If we replace n with a half-integer (e.g., $n + \frac{1}{2}, n + \frac{3}{2}$, etc.), the resulting function set is still **orthogonal** (by similar calculations). That is, the functions

$$\{1, \cos \frac{\frac{1}{2}\pi x}{l}, \sin \frac{\frac{1}{2}\pi x}{l}, \dots, \cos \frac{(\frac{1}{2}+n)\pi x}{l}, \sin \frac{(\frac{1}{2}+n)\pi x}{l}, \dots\}$$

are **orthogonal** on the interval $[-l, l]$.

- In future discussions (e.g., in the heat equation), such function systems will appear naturally.
- One key difference:
 - When using integer indices, the period is $2l$.
 - When using half-integer indices, the period extends to $4l$.
- The verification method remains the same.

1.5.3 Completeness and Inner Product Method for Finding Coefficients

- Analogous to linear algebra, linearly decompose a vector²³.

²³An orthonormal basis means

- If $m \neq n$, then $\langle e_m, e_n \rangle = 0$.
- If $m = n$, then $\langle e_n, e_n \rangle = \|e_n\|^2 = 1$.

- The **difference** lies in that vector decomposition is **finite**, while function decomposition is **infinite**, involving issues of convergence, which should be strictly handled in mathematics.

Recall the vector decomposition:

$$\vec{\xi} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n = \sum_{i=1}^n a_i \vec{e}_i$$

where

$\{\vec{e}_i\}$ is an orthonormal basis.

To find the coefficients a_i , use the **inner product** as the **component**.

$$\begin{aligned} \langle \vec{\xi}, \vec{e}_j \rangle &= \sum_{i=1}^n a_i \langle \vec{e}_i, \vec{e}_j \rangle = \sum_{i=1}^n a_i \delta_{ij}, \quad (\text{only orthogonality is needed!!!}) \\ \Rightarrow a_i &= \langle \vec{\xi}, \vec{e}_i \rangle \end{aligned}$$

In summary,

- Fourier series as a linear combination:** Similar to how a vector can be expressed as a linear combination of basis vectors, a function can be expanded using a set of basis functions.
- Basis Functions:** In Fourier series, the sine and cosine functions serve as basis functions, similar to the basis vectors in linear algebra.
- Coefficients Calculation:** The coefficients in a Fourier series expansion are obtained using an inner product, analogous to the method of projecting a vector onto a basis in linear algebra.

Function Decomposition (Analogous to Vector Decomposition):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

where

$\left\{ 1, \cos \frac{\pi x}{l}, \sin \frac{\pi x}{l}, \dots \right\}$ is the orthonormal basis.

To find the coefficients a_n, b_n , also use the **inner product** as the component of $f(x)$.

$$\begin{aligned} \langle f(x), \cos \frac{m\pi x}{l} \rangle &= \langle a_m \cos \frac{m\pi x}{l}, \cos \frac{m\pi x}{l} \rangle = a_m l \\ \Rightarrow a_m &= \frac{1}{l} \langle f(x), \cos \frac{m\pi x}{l} \rangle \end{aligned}$$

Thus, above a_n, b_n , the inner product is defined as before.

Remark 1.5.2. We only point out there is a significant difference between finite sums and infinite sums. Finite sums are true “=” decomposition. Infinite sums cannot generally be written as $f(x) = \sum^{\infty} \dots$, and

usually only a formal expansion can be obtained. The “=” implies $\sum^{\infty} = \lim_{l \rightarrow \infty} \sum^l$ has a meaning of convergence. Fourier series may diverge. To make this converge to $f(x)$, $f(x)$ must have certain restrictions. See references on Fourier analysis for more details.

In the end, we summarize the above analogues in the following table 1.1:

Vector Decomposition (finite)	Function Decomposition (infinite)
$\xi = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \cdots + a_n \mathbf{e}_n = \sum_{i=1}^n a_i \mathbf{e}_i$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$
$\{\mathbf{e}_i\}$ is an orthonormal basis.	$\left\{ \frac{1}{\sqrt{l}}, \frac{\cos \frac{\pi x}{l}}{\sqrt{l}}, \frac{\sin \frac{\pi x}{l}}{\sqrt{l}}, \dots, \frac{\cos \frac{n\pi x}{l}}{\sqrt{l}}, \frac{\sin \frac{n\pi x}{l}}{\sqrt{l}}, \dots \right\}$ orthonormal basis.
To find the coefficients a_i , use the inner product as the component of ξ . $\langle \xi, \mathbf{e}_i \rangle = \sum_{i=1}^n a_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \sum_{i=1}^n a_i \delta_{ij} \Rightarrow a_i = \langle \xi, \mathbf{e}_i \rangle$	To find coefficients a_n, b_n . Also use the inner product as the component of $f(x)$ in the basis. $\langle f(x), \cos \frac{n\pi x}{l} \rangle = \dots$

Table 1.1: Comparison of Vector and Function Decomposition

1.5.4 Key Differences from Linear Algebra

- **Function vs. Vector:** The analogy replaces vectors with functions.
- **Infinite vs. Finite Dimensions:** Fourier series involves an infinite sum, extending linear algebra concepts to infinite-dimensional function spaces.
- **Functional Analysis Connection:** This extension leads to functional analysis, which generalizes linear algebra to infinite dimensions. We omit these in this lecture.

In summary,

- Fourier series can be understood **through the lens of linear algebra**.
 - Vector $\vec{\xi} \leftrightarrow$ Function $f(x)$;
 - Basis $\{\vec{e}_i\} \leftrightarrow$ Basis $\{1, \sin, \cos, \dots\} \xrightarrow{\text{Chap 5}} \text{Basis } \{\text{Bessel functions}\}$.
- The concept of an **inner product** is crucial for **computing Fourier coefficients**.
- The **transition from finite to infinite** dimensions introduces new challenges, such as **convergence issues**.

1.5.5 Odd and Even Functions

- When $f(x)$ is an **odd** function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, 3, \dots).$$

- When $f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l},$$

where

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad (n = 0, 1, 2, \dots).$$

1.5.6 Generalizations

- (**Orthogonal basis and norms**) The set $\{\varphi_n(x)\}$ is called an **orthogonal basis** on the interval $[a, b]$ with respect to the weight function $q(x)$ ($q(x) = 1$ in the next chapter and $q(x) = x$ in Chapter 5) if

$$\langle \varphi_m(x), \varphi_n(x) \rangle := \int_a^b \varphi_m(x) \varphi_n(x) q(x) dx = 0 \quad (m \neq n).$$

If $m = n$, then

$$\|\varphi_n\| = \left[\int_a^b \varphi_n^2(x) q(x) dx \right]^{\frac{1}{2}} \quad \text{is called the } \mathbf{norm} \text{ of } \varphi_n(x).$$

- (**Orthonormal basis**) For an orthogonal basis $\{\varphi_n(x)\}$, if it satisfies

$$\langle \varphi_m(x), \varphi_n(x) \rangle := \int_a^b \varphi_m(x) \varphi_n(x) q(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

then we have the **orthonormal basis** on $[a, b]$. By normalizing the orthogonal basis $\{\varphi_n(x)\}$, we obtain the orthonormal basis $\left\{ \frac{\varphi_n(x)}{\|\varphi_n\|} \right\}$.

- (**Example**) The set $\{1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots\}$ forms an orthogonal basis (the weight is 1) on the interval $[-\pi, \pi]$.

2

Method of Separation of Variables

Question 2.0.1. *Methods of Solving Equations:* From elementary school to now, you have learned methods for solving equations. **What are the two basic approaches?**

Requirements: Focus on the most fundamental ideas, without discussing specific methods like separation of variables or integrating factors. Just outline the two simplest and most direct concepts.

A. Two Basic Approaches to Solving Equations

1. Trial Method (Guessing)

- This method involves making educated guesses to arrive at a solution.
- It emphasizes discovering clues and refining guesses based on feedback from the results.
- This approach is a fundamental aspect of both the second and fifth chapters of the course.
- Also known as the "method of trial and error," it serves not only as a solution technique but also as a research method.
- If you have limited information for your guess, you need to rely on reasoning. If you guess enough information, reasoning may not be necessary.

2. Inverse Method

- For example, in linear algebra, $Ax = y$ if A is invertible, then $x = A^{-1}y$.
- It involves finding inverses (e.g., matrix inverses) to solve equations.
- This method can be generalized to include solving differential equations through integration, which is viewed as an inverse operation of differential. For example, by the separation of variables for ODEs, for a function $f(x)$, we can integrate $dy = f(x)dx$ to find y .
- Later chapters, such as the third and fourth, discuss methods like wave propagation, integral transform and Green functions, which also involve inversion.

Application of Methods

- *The trial method resembles exploring an ancient tomb, where theorizing and practical experimentation guide the process.*
- *Encountering dead ends requires reevaluation and trying alternative paths to achieve a solution.*

Characteristics of the Separation of Variables Method

1. Equation-Specific Methods

- There is no universal method for solving equations; different types require different approaches.
- Equations must have specific characteristics for certain methods to apply.

2. Conditions for Using the Separation of Variables Method

- **Applicability of Separation of Variables**
 - This method applies to problems involving **linear** wave, heat, or Laplace equations (in order to use the **principle of linear superposition**).
 - The key requirement is that both the equation and the boundary conditions must be **homogeneous** (Initial conditions can be either homogeneous or non-homogeneous).
- **Essential Rule to Remember**

– **Homogeneous equation + homogeneous boundary conditions** → Use separation of variables.

3. Handling Non-homogeneous Cases

- **Non-homogeneous equations:** Solved using the **eigenfunction method** (Section 2.2).
- **Non-homogeneous boundary conditions:** Solved using the **auxiliary function method** (Section 2.3).

Question 2.0.2. In the following solution process, we need to focus on two questions:

- *Where the **homogeneous equation** is used?*
- *Where the **homogeneous boundary conditions** play a role?*

The Role of Intuition and Guessing

- Direct integration is often difficult due to the multi-variable derivatives (Chap. 3 gives some ideas).
- When traditional methods fail, educated guessing is necessary.
- Instead of blind guessing, one should use **physical intuition** to guide the search for solutions.
 - Since this course focuses on **Mathematical Physics Equations**, physical intuition is a valuable tool.

- When mathematical approaches become challenging, **leveraging physical concepts** can provide new insights into the problem.
- For example, in wave equations, considering the physical behavior of vibrating strings helps identify possible solutions.

2.1 Free Vibration of a Bounded String

Consider the free vibration problem of a string fixed at both ends:

$$u_{tt} = a^2 u_{xx}, \quad (0 < x < l, t > 0), \quad (2.1.1)$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad (2.1.2)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (2.1.3)$$

where $\varphi(x)$ and $\psi(x)$ are known functions.

The characteristic of this boundary value problem: Equation (2.1.1) is **linear** and **homogeneous**, hence the **sum of various particular solutions is also a solution** to this equation.

If enough particular solutions to equation (2.1.1) can be found, then their **linear combination** can be used to solve the mixed problem.

Physical Intuition:

To solve the boundary value problem (2.1.1)-(2.1.2), we first examine the **physical model**. From physics, we know that the sound emitted by musical instruments can be decomposed into various **single tones** of **different frequencies**.

Single tones are specific oscillation modes and serve as solutions to this equation.

Each single tone forms a **sine curve** when vibrating, and its amplitude depends on **time**. That is, each single tone can always be expressed as:

$$u(x, t) = c(t) \sin \lambda x, \quad \leftarrow \text{In general, an arbitrary function may not be decomposable in this way.} \quad (2.1.4)$$

The characteristic of this form is that $u(x, t)$ is the product of a function containing **only the variable x** and a function containing **only the variable t** , i.e., it has the **form of separated variables**.

A Priori Assumption in Equation Solving:

1. Starting with an Assumption

- We begin solving the equation based on an **empirical observation**, e.g., (2.1.4).
- The **empirical observation** is an assumption in following PDE solving. This initial assumption is known as an **a priori assumption**.

- Beginners often find it **confusing** why the separation of variables method works and **why one can separate the variables like this at the beginning**. In fact, this is only an **(a priori) assumption**¹.
- **Logical Progression** of a priori assumption:
 - Once the assumption is made, we proceed with mathematical reasoning.
 - If the approach works (i.e., verifiable solution, formal solution becomes the real solution), we obtain a solution.
 - If it fails, we **revise our assumption** and **try another** assumption or approach.
- In nonlinear equations, similar assumptions are also applied.

2. Separation of Variables as an Assumption

- The separation of variables method is fundamentally a **guessing approach**.
- Instead of guessing the exact solution, we assume that the solution can be separated into independent functions (**minimize guesses through more thorough derivations**).
- This assumption is based on physical laws and observed phenomena.

Now, let's try to find **non-trivial solutions** (i.e., not identically zero) of equation (2.1.1)

$$u_{tt} = a^2 u_{xx} \quad (0 < x < l, t > 0),$$

that satisfy the **homogeneous boundary** conditions

$$u(0, t) = 0, \quad u(l, t) = 0,$$

Five Steps of the Method of Separation of Variables:

(1) Separation of Variables: Assume the solution can be expressed as the product

$$u(x, t) = X(x)T(t),$$

where $X(x)$ is a function of the variable x only, and $T(t)$ is a function of the variable t only.

¹The term "a priori assumption" is called "a priori" because it is a hypothesis made before analysis based on **existing experience, knowledge or theoretical frameworks**. This assumption does not rely on specific observations or experimental results but rather arises from established theoretical frameworks or known physical laws. Specifically:

1. **Definition:** An a priori assumption is a **hypothesis proposed before** conducting research or derivations, typically based on existing theories or concepts.
2. **Source:** These assumptions come from an understanding of certain **phenomena or intuition** about a problem, allowing researchers to derive conclusions without complete data.
3. **Importance:** A priori assumptions enable researchers to model and solve problems using existing knowledge in the absence of comprehensive information, thereby advancing theoretical development and application.

Thus, the term "a priori" emphasizes that these **assumptions exist prior to experience or observation**, reflecting a **theoretical preconception**.

Goal: Hope to obtain two ODEs to solve for X and T separately.

(2) PDE → ODEs: Substituting the assumed solution of the separated variable form into equation (2.1.1) gives

$$T''X = a^2 X''T.$$

Transforming, we get:

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad \leftarrow \text{homogeneous equation} \quad (2.1.5)$$

Since the left and right sides of equation (2.2.10) remain constant when their independent variables change, let this constant be $-\lambda$, thus we obtain two ordinary differential equations:

$$T''(t) + \lambda a^2 T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0,$$

λ is a constant:

- **Language Description:** The left function is independent of x and is equal to the right function that is independent of t , then both functions are independent of t and x , then it is a constant.
- **Mathematical Approach:** Define a new function $G(t, x) = \frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$, then

$$\frac{\partial G}{\partial x} = 0 \quad \text{and} \quad \frac{\partial G}{\partial t} = 0 \quad \Rightarrow G \equiv \text{const.}$$

- A negative sign is added to the constant to maintain consistency with the established notation in subsequent sections.

1. **Series Solutions:** The method leads to solutions represented as series.

2. **Convergence Properties:**

- **Different series** can converge to the **same** function.
- This means that different students may arrive at different series forms, but the solution remains the same.

3. **Coefficient Variability:**

- Constants (like a^2) can be placed in different parts of the equations, leading to differences in coefficients.
- Although the resulting series may differ by a coefficient, the convergent function remains unchanged.

Thus, we normalize the steps to ensure consistency in the series solutions in this note.

We can solve these two ordinary differential equations to determine $T(t)$ and $X(x)$, thereby obtaining the particular solution of equation (2.1.1).

$$u(x, t) = X(x)T(t)$$

(3) Solving ODEs: To ensure that $u(x, t) = X(x)T(t)$ satisfies the homogeneous boundary conditions (2.1.2) $u(0, t) = 0, u(l, t) = 0$, it follows that:

$$T(t)X(0) = 0, \quad T(t)X(l) = 0. \quad \leftarrow \text{homogeneous boundary}$$

If $T(t) \equiv 0$, then $u(x, t) \equiv 0$, which is not a non-trivial solution. Therefore, it must be that:

$$X(0) = 0, \quad X(l) = 0.$$

Proof by Contradiction Assume that $X(0)$ is not identically equal to zero. Then $T(t)$ must be identically equal to zero. Substituting this back into the initial assumption leads us to conclude that u is identically equal to zero, resulting in the trivial solution.

Parameter Exploration

- The parameter λ is critical but initially unknown.
- Directly solving the equations will yield general solutions that still depend on λ .

Solution Approach

- Historically, mathematicians employed a trial-and-error method to explore potential solutions.
- The goal is to identify a viable path to the solution through systematic exploration.

Conclusion We will focus on the successful path identified through historical mathematical exploration.

(3a) $X(t)$ -eq. $\xrightarrow{\text{by zero bdry cond.}}$ **SL problem \rightarrow Eigenvalues λ_n and eigenfunctions X_n :**

To find the function $X(x)$, we only need to solve the following boundary value problem for the ordinary differential equation:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0. \quad (2.1.6)$$

If for certain values of λ , non-trivial solutions to problem (2.1.6) exist, then such λ values are called **eigenvalues**, and the corresponding non-trivial solutions $X(x)$ are called **eigenfunctions**, and we seek these. Such problems are commonly referred to as **Sturm-Liouville problems**.

Below we discuss λ under three scenarios:

$$\lambda < 0; \quad \lambda = 0; \quad \lambda > 0.$$

Preparatory Knowledge: General Solution of Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

The general solution formula for the differential equation $y'' + py' + qy = 0$ (*), where p and q are constants. The characteristic equation corresponding to equation (*) is $r^2 + pr + q = 0$.

1. If $\Delta \equiv p^2 - 4q > 0$, the general solution of equation (*) is

$$y = Ae^{r_1 x} + Be^{r_2 x}.$$

2. If $\Delta \equiv p^2 - 4q = 0$, the general solution of equation (*) is

$$y = (A + Bx)e^{rx}.$$

3. If $\Delta \equiv p^2 - 4q < 0$, $r_{1,2} = \alpha \pm i\beta$, the general solution of equation (*) is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

Here, A and B are arbitrary constants.

Consider the differential equation $X''(x) + \lambda X(x) = 0$ with boundary conditions $X(0) = X(l) = 0$ (In fact, from Newtonian mechanics, one know λ determines the frequencies).

1. When $\lambda < 0$, problem (2.1.6) has no non-trivial solutions. In fact, the general solution of the equation is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$

from the boundary conditions, we get

$$\begin{aligned} A + B &= 0, \\ Ae^{\sqrt{-\lambda}l} + Be^{-\sqrt{-\lambda}l} &= 0. \end{aligned}$$

Thus, $A = B = 0$, which implies $X(x) \equiv 0$.

2. When $\lambda = 0$, problem (2.1.6) has no non-trivial solutions. In fact, the general solution of the equation is

$$X(x) = (Ax + B)e^{\lambda x} = Ax + B.$$

From the boundary conditions, we get $A = B = 0$ (from geometric picture, two end points are 0, horizontal line, then $A = B = 0$), so the only solution is the trivial one where $X(x) \equiv 0$.

Consider the differential equation $X''(x) + \lambda X(x) = 0$ with boundary conditions $X(0) = X(l) = 0$.

3. When $\lambda > 0$, the general solution of the equation takes the form

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x.$$

From the boundary conditions, we get

$$X(l) = B \sin \sqrt{\lambda}l = 0, \quad X(0) = A = 0.$$

Assuming $X(x)$ is not identically zero, then $B \neq 0$, which implies $\sin \sqrt{\lambda}l = 0$ ([Find zeros](#)). Thus, we obtain

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots). \quad (\text{eigenvalues})$$

Hence, we find a set of non-zero solutions

$$X_n(x) = B_n \sin \frac{n\pi x}{l} \quad (n = 1, 2, \dots). \quad (\text{eigenfunctions})$$

Some remarks on the discrete values:

- To **ensure** the equation has a solution, the parameter λ must take on **specific discrete values**.
 - These discrete values are referred to as **eigenvalues**, analogous to concepts in linear algebra, i.e., by (2.1.6) and let $\mathcal{A} := \partial_x^2$
- $$X'' = -\lambda X \Rightarrow \mathcal{A}X = -\lambda X$$
- which formally consists with $Ax = \lambda x$ in linear algebra.
- Once λ is determined, substitute it back into the previously obtained solution form.
 - The corresponding functions are called **eigenfunctions**, as they replace “vectors” in the context of functions, i.e., “Vector” → “function”.
 - The **boundary conditions** help to **isolate** the values of λ ; initially, λ could be any parameter, but the **boundary conditions constrain** it to specific **discrete** values.
 - From a physical perspective, λ represents the **wave number**, and the boundary conditions constrain the **number of internal wave modes** allowed. The idea of discrete states is analogous to the behavior of waves, particularly in systems with fixed boundaries, such as standing waves.
 - In quantum mechanics, atoms have discrete energy levels where electrons can exist, but not between these levels. These energy levels arise from solving certain equations and reflect the concept of discrete states.
 - For standing waves, only specific frequencies (or wave numbers) can exist, determined by the boundary conditions.
 - The wave number λ plays a crucial role in defining these specific frequencies, which are essential for the physical behavior of the system.
 - Thus, the discrete values arise from boundary conditions, reflecting a fundamental nature of wave behavior in both quantum mechanics and classical wave mechanics.

(3b) $T(t)$ -eq. $\xrightarrow{\text{substituting } \lambda_n}$ **Find the general solution of T_n :**

Now consider the differential equation

$$T''(t) + \lambda a^2 T(t) = 0,$$

with the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

Substituting these eigenvalues into equation (6), we get

$$T''(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0,$$

and the general solution is

$$T_n(t) = C_n \cos \frac{n\pi at}{l} + D_n \sin \frac{n\pi at}{l} \quad (n = 1, 2, \dots).$$

Thus, we obtain the particular solution of equation (2.1.1) that satisfies the homogeneous boundary conditions (2.1.2) in the form of separated variables $u_n(x, t) = X_n(x)T_n(t)$.

The particular solution is given by

$$u_n(x, t) = \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \quad (n = 1, 2, \dots) \quad (2.1.7)$$

where $a_n = B_n C_n$, $b_n = B_n D_n$ are arbitrary constants.

Note the **initial conditions**

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

where $\varphi(x)$ and $\psi(x)$ are arbitrarily given. Generally speaking, any particular solution in (2.1.7) does **not** satisfy the given initial conditions (Note “any” function can be represented by the infinite sum of sin and cos, i.e., the Fourier series. This inspires us use superposition of u_n).

(4) Superposition of Series Solutions: Since equation (2.1.1) is linear and homogeneous, by the principle of **linear** superposition, the series

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \quad (2.1.8)$$

is still a solution to equation (2.1.1) and satisfies the boundary conditions (2.1.2), i.e.,

$$u_{tt} = a^2 u_{xx}, \quad u(0, t) = 0, \quad u(l, t) = 0.$$

(5) Initial Conditions Determine Coefficients:

Question 2.1.1. Under what conditions on a_n and b_n does equation (14) also satisfy the initial conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Thus, taking the derivative of (2.1.10) with respect to t , we get

$$u_t(x, t) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} \left(-a_n \sin \frac{n\pi at}{l} + b_n \cos \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Key Question

- How do we compute the derivative of an infinite series?
- Can we differentiate term by term and then sum the results?

Potential Issue

- Given that a_n and b_n are undetermined coefficients, we cannot directly judge whether differentiation and summation can be interchanged.

Solution Approach

- Assume (also as a priori assumption) term-by-term differentiation is valid (i.e., assume interchangeability).
- Verify the solution at the end.

Setting $t = 0$ in equation (2.1.10) and its corresponding derivative, and combining with the initial conditions, we have

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

This leads to

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = \varphi(x), \quad \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = \psi(x).$$

1. This is the Fourier expansion of the odd function $\varphi(x)$.
2. The function $\varphi(x)$ is defined on the interval $[0, l]$, but the Fourier series is applicable to **periodic** functions. This implies that $\varphi(x)$ can first be **extended oddly** and then **periodically** to become a **periodic function** before applying the Fourier expansion.

Since $\varphi(x)$ and $\psi(x)$ are defined on the interval $[0, l]$, a_n is the coefficient of the Fourier sine series expansion of $\varphi(x)$, and $b_n \frac{n\pi a}{l}$ is the coefficient of the Fourier sine series expansion of $\psi(x)$. That is,

$$a_n = \frac{2}{l} \int_0^l \underbrace{\varphi(x) \sin \frac{n\pi x}{l}}_{\text{even function} \rightarrow \frac{1}{l} \int_{-l}^l = \frac{2}{l} \int_0^l} dx,$$

$$b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx,$$

You can not write $a_n = \frac{1}{l} \int_{-l}^l \varphi(x) \sin \frac{n\pi x}{l} dx$ since the definition domain of φ is only $[0, l]$.

then the series (2.1.10) satisfies the initial conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$$

Substituting the determined a_n and b_n from (2.1.9) into equation (2.1.10), we obtain the solution to the mixed problem (2.1.1)-(2.1.3).

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

where

$$\begin{aligned} a_n &= \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, \\ b_n &= \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx. \end{aligned} \tag{2.1.9}$$

The method of solving described above is called the **method of separation of variables**.

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Potential Issues with the Solution

1. Reliance on Assumptions:

- The derivation of the series solution involves multiple ac priori assumptions.
- Examples include assuming the validity of variable separation and term-by-term differentiation.
- These assumptions may affect the correctness of the solution.

2. Verification of the Solution:

- To confirm the solution, it must be substituted back into the original equation and see if the derived series truly satisfies the equation.
- It is necessary to verify whether the series converges to a valid function.
- Without verifications, the solution is merely a **formal solution**.

Note: 1. The series solution (2.1.10) may **not necessarily converge**, hence it is sometimes referred to as a **formal solution**. However, under certain conditions in the existence theorem, (2.1.10) can indeed be guaranteed to be the classical solution to the boundary value problem (2.1.1)-(2.1.3).

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l, t > 0), \\ u(0, t) = 0, u(l, t) = 0, \\ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x). \end{cases}$$

Theorem 2.1.1 (Existence Theorem). *If $\varphi(x) \in C^4[0, l]$ (a function with continuous fourth derivatives), $\psi(x) \in C^3[0, l]$, and φ, φ'', ψ are zero at $x = 0$ and $x = l$, then the classical solution to the initial boundary value problem (2.1.1)-(2.1.3) exists and can be expressed as series (2.1.10), with the coefficients determined by (2.1.9).*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l, t > 0), \\ u(0, t) = 0, u(l, t) = 0, \\ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x), \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

$$a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx,$$

$$b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx,$$

Basic Steps of the Method of Separation of Variables:

1. **(Separation of Variables)** Assume separation of variables $u(x, t) = X(x)T(t)$
2. **(PDE → ODEs)** Substitute into the partial differential equation (PDE) to obtain two ordinary differential equations (ODEs) for $X(x)$ and $T(t)$ respectively
3. **(Solving ODEs)** Solve the ODEs:
 - $X(t)$ -eq. $\xrightarrow{\text{by bdry cond.}}$ SL problem → Eigenvalues λ_n and eigenfunctions X_n (Standing waves appear here)
 - $T(t)$ -eq. $\xrightarrow{\text{substituting } \lambda_n}$ Find the general solution of T_n

4. **(Superposition of Series Solutions)** Superpose the series solution $u = \sum^{\infty} u_n$
5. **(Initial Conditions Determine Coefficients)** Determine the Fourier coefficients using initial conditions

Some issues:

- The separation of variables method is initially based on physical observations suggesting that **variables can be separated**.
- However, as the process progresses, **the series solution obtained by superposition may not be variable separated**.
- The superposition of series disrupts the form of separable variables.

Recall:

Basic Steps of the Method of Separation of Variables:

1. **(Separation of Variables)** Assume separation of variables $u(x, t) = X(x)T(t)$
2. **(PDE → ODEs)** Substitute into the partial differential equation (PDE) to obtain two ordinary differential equations (ODEs) for $X(x)$ and $T(t)$ respectively
3. **(Solving ODEs)** Solve the ODEs:
 - $X(t)$ -eq. $\xrightarrow{\text{by bdry condi.}}$ SL problem \rightarrow Eigenvalues λ_n and eigenfunctions X_n (Standing waves appear here)
 - $T(t)$ -eq. $\xrightarrow{\text{substituting } \lambda_n}$ Find the general solution of T_n
4. **(Superposition of Series Solutions)** Superpose the series solution $u = \sum^{\infty} u_n$
5. **(Initial Conditions Determine Coefficients)** Determine the Fourier coefficients using initial conditions

In this lecture:

1. We will discuss 4 to 5 major examples.
2. All examples follow a **structured five-step** separation of variables approach.
3. The class involves a high volume of information, requiring active participation and practice.

Learning Strategy:

To effectively follow along, students should:

1. Memorize the five-step framework.
2. Practice by solving problems alongside the instructor.
3. Compare each new example with the first one, identifying:
 - (a) **Similarities**, which reinforce the core five-step-framework.
 - (b) **Differences**, which determine **necessary modifications** in the steps.
4. **Understanding these differences** is crucial to avoid confusion.
5. You must study through comparison so that you can truly understand the underlying logic and changes.

Conclusion

By systematically comparing examples, students can develop a clear understanding of the separation of variables method and its applications.

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \quad (2.1.10)$$

The physical meaning of the series solution for the boundary value problem

Take a general term of series (14) and transform it as follows:

$$\begin{aligned} u_n(x, t) &= \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \\ &= N_n \sin(\omega_n t + \theta_n) \sin \frac{n\pi x}{l}, \end{aligned} \quad (16)$$

where $N_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \arctan \frac{a_n}{b_n}$, $\omega_n = \frac{n\pi a}{l}$; θ_n is called the initial phase, and ω_n is called the frequency.

The physical meaning of equation (16) can be studied by first fixing the time t to observe the shape of the vibration wave at that moment; then fixing a point on the string to observe the vibration pattern at that point.

When $t = t_0$,

$$u_n(x, t_0) = N'_n \sin \frac{n\pi x}{l},$$

where $N'_n = N_n \sin(\omega_n t_0 + \theta_n)$ is a constant value. This indicates that at any time, the wave shape of $u_n(x, t_0)$ is a sine curve, and its amplitude is related to time t_0 .

When $x = x_0$,

$$u_n(x_0, t) = N''_n \sin(\omega_n t + \theta_n),$$

where $N''_n = N_n \sin \frac{n\pi x_0}{l}$ is a constant value, indicating that each point x_0 on the string is undergoing simple harmonic motion, with an amplitude of $|N_n \sin \frac{n\pi x_0}{l}|$. The frequency is $\omega_n = \frac{n\pi a}{l}$, and the initial phase is θ_n . If another point is taken, the situation is the same, only the amplitude differs.

From the above, it is known that $u_n(x, t)$ represents a vibration wave where all points on the string vibrate with the same frequency in simple harmonic motion, with the same initial phase at each point, and

the amplitude varies with the position of the point. The shape of this vibration wave at any moment is also a sine curve.

The solution is given by

$$u_n(x, t) = N_n \sin(\omega_n t + \theta_n) \sin \frac{n\pi x}{l}, \quad (16)$$

When $x_m = \frac{ml}{n}$ ($m = 0, 1, 2, \dots, n$), $u_n(x, t) = 0$, which indicates that these points remain stationary throughout the vibration process. Such points are referred to as nodes in the physical context of u_n . This implies that the vibration of $u_n(x, t)$ is a segmented vibration on the interval $[0, l]$, and such vibration waves containing nodes are called **standing waves**².

When $x_k = \frac{(2k-1)l}{2n}$ ($k = 1, 2, \dots, n$), the amplitude of the standing wave reaches its maximum value at these points, which are referred to as antinodes.

Thus, we know that $u_1, u_2, \dots, u_n, \dots$ are a series of standing waves, whose frequencies, initial phases, and amplitudes all vary with n . Therefore, it can be said that the solution

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \quad (14)$$

is composed of a series of standing waves with different frequencies, initial phases, and amplitudes. Hence, people also call the method of separation of variables the **method of standing waves**.

Ex 2.1.1 (Examining the Free Vibration Problem of a String Fixed at Both Ends). *Consider the free vibration problem of a string fixed at both ends:*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < 1, t > 0), \\ u(0, t) = 0, u(1, t) = 0, \\ u(x, 0) = \sin 2\pi x, \quad u_t(x, 0) = x(1-x). \end{cases}$$

Solution. Using the Formula

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

Since $l = 1$, the solution to this boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos n\pi at + b_n \sin n\pi at) \sin n\pi x.$$

Using formula again,

$$a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, \quad b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx,$$

we have

$$\begin{aligned} a_n &= 2 \int_0^1 \sin 2\pi x \sin n\pi x dx \\ b_n &= \frac{2}{n\pi a} \int_0^1 x(1-x) \sin n\pi x dx \end{aligned}$$

²In Chap. 3, there is no boundary conditions constrain the wave, thus the wave is traveling instead of standing.

Given

$$a_n = 2 \int_0^1 \sin 2\pi x \sin n\pi x dx = \begin{cases} 0, & n \neq 2, \\ 1, & n = 2. \end{cases} \quad (\text{orthogonality})$$

$$\begin{aligned} b_n &= \frac{2}{n\pi a} \int_0^1 x(1-x) \sin n\pi x dx \\ &= \frac{2}{n\pi a} \left[-\frac{1}{n\pi} x(1-x) \cos n\pi x \Big|_0^1 + \frac{1}{n\pi} \int_0^1 (1-2x) \cos n\pi x dx \right] \\ &= \frac{2}{(n\pi)^2 a} \left[\frac{1}{n\pi} (1-2x) \sin n\pi x \Big|_0^1 + \frac{2}{n\pi} \int_0^1 \sin n\pi x dx \right] \\ &= \frac{4}{(n\pi)^4 a} [1 - (-1)^n]. \end{aligned}$$

(Integration by parts: Differentiation applied to a polynomial can continuously reduce its degree.)

Thus, we have

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos n\pi at + b_n \sin n\pi at) \sin n\pi x,$$

where

$$\begin{aligned} a_n &= 2 \int_0^1 \sin 2\pi x \sin n\pi x dx = \begin{cases} 0, & n \neq 2, \\ 1, & n = 2. \end{cases} \\ b_n &= \frac{4}{(n\pi)^4 a} [1 - (-1)^n] \end{aligned}$$

Therefore, the solution to the boundary value problem is

$$u(x, t) = \cos 2\pi at \sin 2\pi x + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^4 a} [1 - (-1)^n] \sin n\pi at \sin n\pi x.$$

To effectively learn the separation of variables method for differential equations, follow these suggestions:

1. Identify **similarities** with previous problems:
 - Recognizing common structures (the homogeneous equation and boundary) helps determine if the method is applicable.
2. Focus on **differences**:
 - Differences dictate necessary modifications to the separation of variables method.
3. Active practice is essential:
 - Simply watching or reading is insufficient; hands-on problem-solving is required.
4. Fundamental skills in differentiation and integration are crucial:

- These skills form the basis for successfully applying separation of variables.
5. The **eigenvalue problems** from these problems must be memorized.

Ex 2.1.2 (Free Vibration of a String with One End Fixed and the Other Free).

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l, t > 0), \\ u(0, t) = 0, u_x(l, t) = 0, \\ u(x, 0) = x^2 - 2lx, u_t(x, 0) = 3 \sin \frac{3\pi x}{2l}. \end{cases}$$

Similarities

- Both equations are $1+1$ -dimensional (This chapter deals only with 2D; higher dimensions appear in Chapter 5.).
- The primary equation is the same: The wave equation determines that the ODE for T is a second-order oscillatory ODE.
- Two given functions: one is a sine function (trigonometric), and the other is a polynomial function. The coefficient determination follows the same approach:
 - Using orthogonality for integration.
 - Using integration by parts.
- Homogeneous equation + homogeneous boundary conditions \rightarrow separation of variables, and five-step method should be followed for solution.

Differences

- Difference in boundary conditions:
 - First problem: both sides have Dirichlet (first-type) boundary conditions.
 - Second problem: left boundary is Dirichlet (first-type), right boundary is Neumann (second-type).
 - Third problem (as exercise): left boundary is Neumann (second-type), right boundary is Dirichlet (first-type), which is symmetric to the second problem.
 - Fourth problem: both boundaries are Neumann (second-type).
- The boundary conditions affect the eigenvalue problems. The major changes occur in:
 - Step 3: Solving the X -ODE (the eigenvalue problems).
 - Step 5: Determining coefficients due to different eigenfunctions.

Conclusion

Understanding the role of boundary conditions in separation of variables is crucial, as they determine how the problem is solved, particularly in the eigenvalue problems and coefficient determination.

Since the boundary conditions of this problem are different from (2.1.2) $u(0, t) = 0, u(l, t) = 0$, we cannot directly use the formula

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}.$$

However, we can solve it using the method of separation of variables.

Solution. (1. Separation of Variables) Let $u(x, t) = X(x)T(t)$.

(2. PDE → ODEs) Substituting into the equation and separating variables yields two ordinary differential equations

$$T''(t) + \lambda a^2 T(t) = 0, \quad X''(x) + \lambda X(x) = 0,$$

(3. Solving ODEs)

(3a) $X(t)$ -eq. $\xrightarrow{\text{by bdry cond.}}$ **SL problem:** From the boundary conditions, it is easy to obtain (**Difference**)

$$X(0) = 0, \quad X'(l) = 0.$$

Find the non-zero solution of the boundary value problem (**Difference**)

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

(First, solve the ODE corresponding to the homogeneous boundary conditions.)

(1) When $\lambda < 0$, there are no non-trivial solutions (using geometric thinking, the boundary requires $X(0) = 0$ and the slope at l also vanishes).

(2) When $\lambda = 0$, there are also no non-trivial solutions.

(3) When $\lambda > 0$, the general solution of the equation is

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x.$$

From the boundary conditions, we get

$$X(0) = A = 0, \quad X'(l) = B\sqrt{\lambda} \cos \sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $B \neq 0 \implies \cos \sqrt{\lambda}l = 0$, thus

$$\lambda = \lambda_n = \left(\frac{(2n+1)\pi}{2l} \right)^2 \quad (n = 0, 1, 2, \dots). \quad \text{eigenvalues}$$

Thus, we find a set of non-zero solutions

$$X_n(x) = B_n \sin \frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \dots). \quad \text{eigenfunctions}$$

These eigenvalues and eigenfunctions can also be written as

$$\lambda = \lambda_n = \left(\frac{(2n-1)\pi}{2l} \right)^2 \quad \text{and} \quad X_n(x) = B_n \sin \frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \dots).$$

(3b) **T(t)-eq.** $\xrightarrow{\text{substituting } \lambda_n}$ **Find the general solution of T_n :** Now consider

$$T''(t) + \lambda a^2 T(t) = 0,$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{(2n+1)\pi}{2l} \right)^2 \quad (n = 0, 1, 2, \dots),$$

into the equation, we get (*In the heat equation, T-ODE becomes a first-order ODE. Since the heat equation has only a first-order time derivative, it leads to exponential decay.*)

$$T''(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 T(t) = 0,$$

and its general solution is

$$T_n(t) = C_n \cos \frac{(2n+1)\pi at}{2l} + D_n \sin \frac{(2n+1)\pi at}{2l} \quad (n = 0, 1, 2, \dots).$$

(4. Superposition of Series Solutions) Thus, the solution to the problem can be expressed as

$$u(x, t) = \sum_{n=0}^{\infty} \left[a_n \cos \frac{(2n+1)\pi at}{2l} + b_n \sin \frac{(2n+1)\pi at}{2l} \right] \sin \frac{(2n+1)\pi x}{2l},$$

where $a_n = B_n C_n$, $b_n = B_n D_n$ are arbitrary constants.

(5. Initial Conditions Determine Coefficients) In the above formula and its corresponding derivative, let $t = 0$, and combine the initial conditions

$$u(x, 0) = x^2 - 2lx, \quad u_t(x, 0) = 3 \sin \frac{3\pi x}{2l}.$$

We get³

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \sin \frac{(2n+1)\pi x}{2l} &= x^2 - 2lx, \\ \sum_{n=0}^{\infty} b_n \frac{(2n+1)\pi a}{2l} \sin \frac{(2n+1)\pi x}{2l} &= 3 \sin \frac{3\pi x}{2l}. \end{aligned}$$

Thus,

$$a_n = \frac{2}{l} \int_0^l (x^2 - 2lx) \sin \frac{(2n+1)\pi x}{2l} dx = -\frac{32l^2}{(2n+1)^3 \pi^3},$$

$$b_n \frac{(2n+1)\pi a}{2l} = \frac{2}{l} \int_0^l 3 \sin \frac{3\pi x}{2l} \sin \frac{(2n+1)\pi x}{2l} dx = \begin{cases} 0, & n \neq 1, \\ \frac{2l}{\pi a}, & n = 1. \end{cases}$$

Therefore, the (*formal*) solution to the problem is

$$u(x, t) = \sum_{n=0}^{\infty} -\frac{32l^2}{(2n+1)^3 \pi^3} \cos \frac{(2n+1)\pi at}{2l} \sin \frac{(2n+1)\pi x}{2l} + \frac{2l}{\pi a} \sin \frac{3\pi at}{2l} \sin \frac{3\pi x}{2l}.$$

³ $\sin \frac{(2n+1)\pi x}{2l}$ still forms a orthogonal basis, see the last lecture and we need first extend the data functions oddly and then periodically.

Ex 2.1.3 (Free Vibration of a String with Both Ends Free). Consider the free vibration problem of a string with both ends free:

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l, t > 0), \\ u_x(0, t) = 0, u_x(l, t) = 0, \\ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x). \end{cases}$$

Similarities: Homogeneous equations with homogeneous boundary — Try separation of variables.

Differences: Both boundaries are the **second kind** of boundary conditions.

Solution. (1. Separation of Variables) Let $u(x, t) = X(x)T(t)$,

(2. PDE → ODEs) Substituting into the equation and separating variables yields two ordinary differential equations

$$T''(t) + \lambda a^2 T(t) = 0, \quad X''(x) + \lambda X(x) = 0,$$

(3. Solving ODEs) From the boundary conditions, it is easy to obtain $X'(0) = 0$, $X'(l) = 0$. Solve the boundary value problem (**Difference, New S-L prob.**)

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0.$$

for the non-zero solution.

(1) When $\lambda < 0$, there are no non-trivial solutions to this problem.

(2) When $\lambda = 0$, the general solution of the equation is $X_0(x) = A_0x + B_0$, hence $X'_0(x) = A_0$. From the boundary conditions, we get $X'_0(0) = X'_0(l) = A_0 = 0$,

$$\implies X_0(x) = B_0.$$

Major Changes in the Solution

- Previously, for $\lambda = 0$, only the trivial solution existed.
- Now, there exists a **nontrivial constant** solution.
- Reason:
 - Both boundary slopes are zero, forming a horizontal line.
 - This allows the solution to be a constant.
 - Previously, in addition to having zero slope, the solution also had to pass through the origin, restricting it to the trivial case.

Substituting $\lambda = 0$ into the equation $T''(t) + \lambda a^2 T(t) = 0$, we solve to get

$$T_0(t) = C_0 t + D_0.$$

Thus, we obtain a non-trivial solution satisfying the second type of boundary conditions for the original vibration equation

$$u_0(x, t) = \frac{1}{2}(a_0 + b_0 t),$$

where $a_0 = 2B_0D_0$, $b_0 = 2B_0C_0$ are arbitrary constants (*This notation is used to maintain consistency with the Fourier series later on*).

(3) When $\lambda > 0$, the general solution of the equation is of the following form:

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x.$$

From the boundary conditions, we get

$$X'(0) = B\sqrt{\lambda} = 0 \implies B = 0,$$

$$X'(l) = -A\sqrt{\lambda} \sin \sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $A \neq 0$, $\sin \sqrt{\lambda}l = 0$, thus

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

Hence, we find a set of non-zero solutions

$$X_n(x) = A_n \cos \frac{n\pi x}{l} \quad (n = 1, 2, \dots).$$

Now consider

$$T''(t) + \lambda a^2 T(t) = 0,$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots),$$

into the equation, we get

$$T''(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0, \quad (n = 1, 2, \dots),$$

and its general solution is

$$T_n(t) = C_n \cos \frac{n\pi at}{l} + D_n \sin \frac{n\pi at}{l} \quad (n = 1, 2, \dots).$$

(4. Superposition of Series Solutions) Thus, the solution to the problem can be expressed as

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right] \cos \frac{n\pi x}{l} + \frac{1}{2}(a_0 + b_0 t),$$

where $a_n = A_n C_n$, $b_n = A_n D_n$ are arbitrary constants.

- When both boundaries are **second-type boundary conditions**, they only restrict the slope to zero, resulting in **nontrivial constant solutions** for $\lambda = 0$.

- For $\lambda = 0$, the corresponding eigenfunction is a constant $X_0(x) = A_0 = A_0 \cos \frac{0\pi x}{l}$. Thus, the complete eigenvalues and eigenfunctions are

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = A_n \cos \frac{n\pi x}{l}, \quad (n = \underbrace{0}_{\text{Constant solution}}, 1, 2, \dots).$$

Conclusion

- Attention to detail in counting indices is crucial as small errors can lead to incorrect series solutions.
- Ensure to clarify the starting point of indices when completing assignments.

(5. Initial Conditions Determine Coefficients) The derivative of the above formula with respect to t is

$$u_t(x, t) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} \left(-a_n \sin \frac{n\pi at}{l} + b_n \cos \frac{n\pi at}{l} \right) \cos \frac{n\pi x}{l} + \frac{b_0}{2}.$$

Setting $t = 0$ in the above two equations and combining with the initial conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x),$$

we get

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} &= \varphi(x), \\ \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \cos \frac{n\pi x}{l} &= \psi(x). \end{aligned}$$

Thus,

$$\begin{cases} a_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} dx, \\ a_0 = \frac{2}{l} \int_0^l \varphi(x) dx, \\ b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx, \\ b_0 = \frac{2}{l} \int_0^l \psi(x) dx. \end{cases}$$

Substituting the determined a_n and b_n into the series solution expression, we obtain the solution to the original problem.

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \cos \frac{n\pi x}{l} + \frac{1}{2}(a_0 + b_0 t).$$

Summary of the forms of five common eigenfunction series

1. $u(0, t) = 0, u(l, t) = 0; \Rightarrow \{\sin \frac{n\pi x}{l}\} (n = 1, 2, \dots);$
2. $u(0, t) = 0, u_x(l, t) = 0; \Rightarrow \left\{ \sin \frac{(2n-1)\pi x}{2l} \right\} (n = 1, 2, \dots);$
3. $u_x(0, t) = 0, u(l, t) = 0; \Rightarrow \left\{ \cos \frac{(2n-1)\pi x}{2l} \right\} (n = 1, 2, \dots);$
4. $u_x(0, t) = 0, u_x(l, t) = 0; \Rightarrow \{\cos \frac{n\pi x}{l}\} (n = 0, 1, 2, \dots);$

The above forms are applicable to one-dimensional vibration equations, heat conduction equations, and Poisson's equations on rectangular domains.

5. Eigenfunction series corresponding to Poisson's equation on a circular domain

$$\{1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots \cos n\theta, \sin n\theta, \dots\}$$

2.2 Heat Conduction in a Finite Rod

2.2.1 Consider the mixed problem of the homogeneous heat conduction equation (with boundary conditions of the first type)

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l, t > 0), \\ u(0, t) = 0, u(l, t) = 0, \\ u(x, 0) = \varphi(x), \end{cases} \quad (2.2.1)$$

where $\varphi(x)$ is a given known function.

Similarities

- Both involve homogeneous equations and boundary → separation of variables and fundamental five-step approach.
- Both end points satisfy the first-type (Dirichlet) boundary conditions.

Differences

- The equation transforms into the heat equation and the initial condition only needs to specify $u(x, 0)$ due to the heat equation.

We will solve the problem (2.2.1) using the method of separation of variables (also known as the method of standing waves).

(1. Separation of Variables) First, let

$$u(x, t) = X(x)T(t),$$

(2. PDE → ODEs) Substitute it into the equation

$$u_t = a^2 u_{xx}$$

and separate variables to obtain two ordinary differential equations

$$\begin{aligned} T'(t) + \lambda a^2 T(t) &= 0, \quad (\text{Difference: first order for heat; Normalization: } a^2 \text{ in } T\text{-ODE}) \\ X''(x) + \lambda X(x) &= 0, \end{aligned}$$

(3. Solving ODEs)

(3a) $X(t)$ -eq. $\xrightarrow{\text{by bdry condi.}}$ **SL problem:** From the boundary conditions $u(0, t) = 0, u(l, t) = 0$, we get

$$X(0) = 0, X(l) = 0.$$

Solve the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

for the non-zero solution.

1. When $\lambda < 0$, there are no non-trivial solutions to this problem.
2. When $\lambda = 0$, there are also no non-trivial solutions to this problem.
3. When $\lambda > 0$, there are non-trivial solutions to this problem.

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

$$X_n(x) = B_n \sin \frac{n\pi x}{l} \quad (n = 1, 2, \dots).$$

(3b) $T(t)$ -eq. $\xrightarrow{\text{substituting } \lambda_n}$ **Find the general solution of T_n :**

Now Consider

$$T'(t) + \lambda a^2 T(t) = 0, \tag{2.2.2}$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

into the equation, we get

$$T'(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0,$$

and its general solution is

$$T_n(t) = C_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \quad (n = 1, 2, \dots).$$

- In the heat equation, T -ODE becomes a first-order ODE. Since the heat equation has only a first-order time derivative, it leads to exponential decay.
- In the wave equation, T -ODE becomes a second-order ODE. Since the wave equation has a second-order time derivative, it leads to oscillation (sin and cos function).

(4. Superposition of Series Solutions) Thus, the solution to the problem (2.2.1) that satisfies the homogeneous boundary conditions and has the form of variable separation is

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi x}{l}, \quad (2.2.3)$$

where $a_n = B_n C_n$ are arbitrary constants.

(5. Initial Conditions Determine Coefficients) Using the initial condition $u(x, 0) = \varphi(x)$, we get

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} &= \varphi(x), \\ a_n &= \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx. \end{aligned} \quad (2.2.4)$$

Equations (2.3.13) and (2.2.4) together give the particular solution to the problem (2.2.1).

$$\begin{cases} u_t = a^2 u_{xx} \quad (0 < x < l, t > 0), \\ u(0, t) = 0, \quad u(l, t) = 0, \\ u(x, 0) = \varphi(x), \end{cases}$$

Note: If the boundary conditions in the problem are of the second or third kind, the method is similar. Please think for yourselves!

Prerequisite Knowledge: First-Order Linear Ordinary Differential Equation

$$y' + a(x)y = b(x) \quad (2.2.5)$$

The general solution formula. Here $a(x), b(x)$ are known functions.

Let the antiderivative of $a(x)$ be $A(x)$, then the general solution formula for (2.2.5) is

$$y = e^{-A(x)} \left(\int e^{A(x)} b(x) dx + C \right)$$

where C is an arbitrary constant. Specifically, if $b(x) = 0$, then the general solution simplifies to

$$y = Ce^{-A(x)}.$$

Key Ideas

- The method of separation of variables works well for solving homogeneous ODE (2.2.2).
- When the ODE becomes non-homogeneous (2.2.5), we need a different approach.
- Recall that one **common method** is solving the homogeneous equation first and then finding a particular solution.
- Another method is the **variation of parameters**, which is useful for second-order equations.
- The simplest and most direct approach is the **integrating factor method**.

The basic Idea of Integrating Factor Method

- We seek to multiply the equation by a function $F(x)$, called the **integrating factor**, such that the left-hand side becomes the **derivative of a product** by the Leibniz rule.

Finding the Integrating Factor

Question 2.2.1. When encountering forms like $y' + ay$ or $y'' + ay'$, which are **one order less**, the second term hinders direct integration. Is it possible to find a way to still integrate directly?

A. we can consider the **integrating factor**. Essentially, this is similar to the **Leibniz rule**:

$$(Fy)' = Fy' + F'y$$

which is similar to and can be compared with $y' + ay$.

Thus, we can think, by multiplying F on the both sides of (2.2.5):

$$\underbrace{Fy' + aFy}_{\text{Hope it is equal to } Fy' + F'y = (Fy)'} = Fb \quad (2.2.6)$$

We **expect** the left hand of (2.2.6) is equal to $Fy' + F'y = (Fy)'$. Once feasible, what was originally not integrable on the left side (i.e., $y' + ay$) becomes an integrable form $(Fy)'$.

By comparison, we find that as long as $aFy = F'y$, that is, as long as $F' = aF$, Our **expectation** becomes true. Thus, we need

$$(\ln F)' = a \Rightarrow F = e^{\int a dt}$$

Why Use the Integrating Factor?

- One finds the function $e^{\int a(x)dx}$ is a natural and direct choice as an integrating factor and you do not need the above general procedure because the **derivative of exp function remains proportional to itself**: $\frac{d}{dt}e^{\int a dt} = ae^{\int a dt}$, which is a "fixed point" for differentiation, <a useful property!>. sin, cos also have similar properties with respect to $\frac{d^2}{dt^2}$, hence they are useful in integration by parts.
- This idea can be generalized to other general forms, such as $y' + \frac{a}{t}y$. Following the same approach.

- When partial differential equations (PDEs) involve terms like $\frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial x} + au$ and $\frac{\partial u}{\partial t} + au$, this idea of integrating factor can also be applied. For example,

$$\begin{aligned}\partial_t(e^{\int a dt} u) &= e^{\int a dt} \partial_t u + ae^{\int a dt} u \\ \partial_t(e^{\int a dt} \partial_t u) &= e^{\int a dt} \partial_t^2 u + ae^{\int a dt} \partial_t u.\end{aligned}$$

- One idea to solve equations: Transform equations (PDEs or ODEs) that are unfamiliar into ones that are familiar. How to transform? there are two basic ideas:
 - Function transformation
 - Independent variable transformation

The idea of the **integrating factor** mentioned above can be **used for function transformation** in order to achieve a **familiar** equation.

Examples

Ex 2.2.1. For the PDE: $u_{tt} + 2u_t = u_{xx} + 4u_x + 3u$

$$u_{tt} + 2u_t = u_{xx} + 4u_x + 3u \Rightarrow u_{tt} + 2u_t + u = u_{xx} + 4u_x + 4u$$

we let $v(x, t) = e^{t+2x}u(x, t)$, then v satisfies the familiar wave equation $v_{tt} = v_{xx}$.

Ex 2.2.2. For the PDE: $u_t = u_{xx} + 2u_x$

$$\begin{aligned}u_t &= u_{xx} + 2u_x \Rightarrow u_t + u = u_{xx} + 2u_x + u \\ \Rightarrow e^x(e^t u_t + e^t u) &= e^t(e^x u_{xx} + 2e^x u_x + e^x u) \Rightarrow e^x(e^t u)_t = e^t(e^x u)_{xx} \Rightarrow (e^{t+x} u)_t = (e^{t+x} u)_{xx}\end{aligned}$$

We let $v(x, t) = e^{t+x}u$, the v satisfy the familiar heat equation $v_t = v_{xx}$.

Summary of Variable Separation Method

1. Boundary Conditions Modification

- Four types of boundary conditions were discussed:
 - Both ends are of the first kind.
 - One end is of the first kind, the other is of the second kind.
 - One end is of the second kind, the other is of the first kind.
 - Both ends are of the second kind.
- The modification mainly affects the solution of the SL problem in the variable separation method.
- For these four types of boundary conditions, students are required to remember the corresponding eigenvalues, and eigenfunctions.

- One type (left end is of the second kind, right end is of the first kind) was not discussed in class and needs to be calculated by students themselves.

2. Equation Modification

- Two types of equations were discussed: wave equation and heat equation.
- For the heat equation:
 - The time derivative term is reduced to one order.
 - The main modification in the variable separation method is the ODE for T , which becomes a first-order ODE.
 - The solution is an exponential decay function, $e^{-\lambda t}$, which reflects the physical phenomenon of heat transfer from high temperature to low temperature until equilibrium.
- For the wave equation:
 - The time derivative term is of second order.
 - The ODE for T is a second-order ODE.
 - The solution is a combination of trigonometric functions (e.g., $T(t) = A \cos(\omega t) + B \sin(\omega t)$), which reflects the oscillatory nature of wave motion.
- Physical Intuition
 - Heat equation solutions must exhibit **exponential decay** due to **thermal diffusion**.
 - Wave equation solutions must exhibit **oscillations** due to **periodic motion**.
 - Damped oscillations arise when additional terms introduce **decay** in wave equations.

3. Current and Future Work

- This class will complete the discussion of the variable separation method.
- Focus on three main examples:
 - Heat equation with both ends of the second kind boundary conditions.
 - Laplace equation in rectangular coordinates.
 - Laplace equation in circular coordinates (main focus).
- The method for solving the heat equation with boundary conditions is similar to solving the Laplace equation in rectangular coordinates.
- The Laplace equation in circular coordinates will present more challenges and requires special attention.

2.2.2 Consider the Mixed Problem of the Homogeneous Heat Conduction Equation (Boundary Conditions are Both of the Second Type)

Problem 2.2.1. *The heat conduction problem on a finite rod with both ends $x = 0, x = l$ insulated, initial temperature distribution $\varphi(x)$, and no heat source.*

$$u_t = a^2 u_{xx} \quad (0 < x < l, t > 0), \quad (2.2.7)$$

$$u_x(0, t) = 0, \quad u_x(l, t) = 0, \quad (2.2.8)$$

$$u(x, 0) = \varphi(x), \quad (2.2.9)$$

where $\varphi(x)$ is a given known function.

1. Equation Type: Heat Equation

- When the equation becomes a heat equation, the function $T(t)$ is expected to be an exponential function.
- This is because the time-dependent part of the heat equation typically results in exponential decay.

2. Boundary Conditions: Second Kind

- When the boundary conditions are of the second kind, the eigenvalues and eigenfunctions will change accordingly.
- This modification affects the form of the solution, requiring a re-evaluation of the eigenvalues and eigenfunctions.

3. Approach to Solving the Problem

- Before solving, make initial predictions based on the type of equation and boundary conditions.
- Use these predictions to guide the modification process.
- The solution involves combining previously learned concepts and applying them to the specific problem at hand.

Solution. (1. Separation of Variables) Let

$$u(x, t) = X(x)T(t) \quad (2.2.10)$$

(2. PDE → ODEs) Substituting (2.2.10) into equation (2.2.7) to separate variables yields two ordinary differential equations

$$\begin{aligned} T'(t) + \lambda a^2 T(t) &= 0, \\ X''(x) + \lambda X(x) &= 0, \end{aligned}$$

From boundary conditions (2.2.8), we get $X'(0)T(t) = 0$, $X'(l)T(t) = 0$. Then

$$X'(0) = 0, \quad X'(l) = 0.$$

(3. Solving ODEs) Solve the Boundary Value Problem of the Ordinary Differential Equation for Non-zero Solutions.

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0. \quad (2.2.11)$$

1. When $\lambda < 0$, the general solution of the equation is

$$\begin{aligned} X(x) &= Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}, \\ X'(x) &= A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}, \end{aligned}$$

From boundary conditions, we get

$$\begin{aligned}\sqrt{-\lambda}(A - B) &= 0, \\ \sqrt{-\lambda}(Ae^{\sqrt{-\lambda}l} - Be^{-\sqrt{-\lambda}l}) &= 0.\end{aligned}$$

This implies

$A = B = 0, \Rightarrow X(x) \equiv 0$. Therefore, (2.2.11) has no non-zero solutions.

2. When $\lambda = 0$, the general solution of the equation is $X_0(x) = A_0x + B_0$, Then

$$X'_0(x) = A_0.$$

From boundary conditions, we get

$$X'_0(0) = X'_0(l) = A_0 = 0 \Rightarrow X_0(x) = B_0.$$

Substituting $\lambda = 0$ into the equation $T'(t) + \lambda a^2 T(t) = 0$, we solve to get

$$T_0(t) = C_0.$$

This gives a **non-trivial solution** to the conduction equation (2.2.7) satisfying boundary conditions (2.2.8)

$$u_0(x, t) = \frac{1}{2}a_0,$$

where $a_0 = 2B_0C_0$ is an arbitrary constant.

3. When $\lambda > 0$, the general solution of the equation has the following form

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x.$$

Then

$$X'(x) = -A\sqrt{\lambda} \sin \sqrt{\lambda}x + B\sqrt{\lambda} \cos \sqrt{\lambda}x,$$

From boundary conditions, we get

$$X'(0) = B\sqrt{\lambda} = 0 \Rightarrow B = 0 \quad \text{and} \quad X'(l) = -A\sqrt{\lambda} \sin \sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $A \neq 0, \Rightarrow \sin \sqrt{\lambda}l = 0$,

Thus we get

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

Hence, a set of non-zero solutions is found

$$X_n(x) = A_n \cos \frac{n\pi x}{l} \quad (n = 1, 2, \dots).$$

Now consider

$$T'(t) + \lambda a^2 T(t) = 0,$$

Substitute the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \dots).$$

into the equation to get

$$T'(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0, \quad (n = 1, 2, \dots)$$

The general solution is

$$T_n(t) = D_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \quad (n = 1, 2, \dots).$$

Summary of Solving Techniques and Course Insights

1. Solving the T Equation

- The T equation is relatively simple and can be solved using the separation of variables method for ODEs.
- The process involves writing the derivative of $\ln T$ and integrating it directly.
- For homogeneous equations, separation of variables is straightforward.
- For non-homogeneous equations, the integrating factor method is most effective.

2. Importance of the Integrating Factor Method

- The ideas of integrating factor method is crucial for combining terms and simplifying the integration process.
- This technique will be frequently used in future problems and exercises.
- Understanding how to find and apply the integrating factor is essential.

3. Characteristics of the Differential Equations Course

- Unlike other math courses that rely heavily on theorems and formulas, differential equations focus more on methods and underlying ideas.
- The course emphasizes the importance of mastering techniques rather than memorizing specific formulas.
- Methods like separation of variables and the idea of integrating factors are most fundamental and will be reinforced throughout the course.

(4. Superposition of Series Solutions) Thus, the non-zero solution satisfying equation (2.2.7) and boundary conditions (2.2.8) is

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \cos \frac{n\pi x}{l} + \frac{1}{2} a_0, \quad (2.2.12)$$

where $a_n = A_n D_n$, $a_0 = 2B_0 C_0$ are arbitrary constants.

(5. **Initial Conditions Determine Coefficients**) In (2.2.12), let $t = 0$, and combine with the initial condition

$$u(x, 0) = \varphi(x),$$

we get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} = \varphi(x),$$

Then

$$a_0 = \frac{2}{l} \int_0^l \varphi(x) dx \quad \text{and} \quad a_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} dx, \quad (n = 1, 2, \dots). \quad (2.2.13)$$

Thus, the solution to the boundary value problem (2.2.7)-(2.2.9) is given by the series

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi a}{l})^2 t} \cos \frac{n\pi x}{l} + \frac{1}{2} a_0.$$

where the coefficients a_n are determined by (2.2.13):

$$a_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} dx, \quad (n = 0, 1, 2, \dots).$$

Ex 2.2.3. Solve the Following Problem

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < \pi, t > 0), \\ u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \\ u(x, 0) = x^2(\pi - x)^2, \end{cases} \quad (2.2.14)$$

1. Common Integrals in Exams and Homework

- The most common integrals in this course involve:
 - Polynomials multiplied by trigonometric functions.
 - Polynomials multiplied by exponential functions.
- These integrals are typically solved using integration by parts.
- For higher powers (e.g., fourth power), multiple applications of integration by parts may be required. Each application reduces the power by one.

Solution. Using the formula

$$u(x, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi a}{l})^2 t} \cos \frac{n\pi x}{l}.$$

where

$$a_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} dx, \quad (n = 0, 1, 2, \dots).$$

Since $l = \pi$, we have

$$a_n = \frac{2}{\pi} \int_0^\pi x^2(\pi - x)^2 \cos nx dx = -\frac{24[(-1)^n + 1]}{n^4} \quad (n \neq 0),$$

and

$$a_0 = \frac{2}{\pi} \int_0^\pi x^2(\pi - x)^2 dx = \frac{1}{15}\pi^4,$$

Substitute $a_0 = \frac{1}{15}\pi^4$, $a_n = -\frac{24[(-1)^n + 1]}{n^4}$ ($n \neq 0$), and $l = \pi$ into the formula

$$u(x, t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi a}{l})^2 t} \cos \frac{n\pi x}{l}.$$

The solution to the problem is

$$u(x, t) = \frac{1}{30}\pi^4 - \sum_{n=1}^{\infty} \frac{24[(-1)^n + 1]}{n^4} e^{-(na)^2 t} \cos nx.$$

2.3 Boundary Value Problems of the Two-Dimensional Laplace Equation

For certain special regions, the boundary value problems of the Laplace equation can also be solved using the method of separation of variables.

2.3.1 Boundary Value Problem of the Laplace Equation in a Rectangular Domain

Problem 2.3.1. Consider the temperature distribution problem of a rectangular thin plate in a steady state. Assume the top and bottom surfaces of the plate are insulated (**means 2D problem**), and the two sides ($x = 0, x = a$) are always kept at 0 degrees, while the other two sides ($y = 0, y = b$) have temperatures $f(x)$ and $g(x)$ respectively. Find the temperature distribution pattern inside the plate in a steady state.

Summary of Laplace Equation and Variable Separation Method

1. Laplace Equation in Two Dimensions

- We focus on two-dimensional cases in this chapter because they involve two ODEs, which are simpler to handle.
- Higher dimensions (e.g., three-dimensional) would involve more ODEs (two S-L problems) and more complex boundary conditions (see Chapter 5).

2. Applicability of Variable Separation Method

- Variable separation is applicable to specific regions, such as rectangular and circular domains.
- For rectangular regions:
 - The domain is defined by $u(x, y) = 0$ within a rectangle.

- Separation of variables is straightforward because the boundaries are aligned with the coordinate axes.
- For circular regions:
 - The domain is defined by $u(r, \theta) = 0$ within a circle.
 - Separation of variables is easier in polar coordinates due to the local orthogonality (i.e., variable separable) of r and θ .

3. Other Related Regions

- Regions related to rectangles (e.g., infinite strips) can also be solved using variable separation.
- Regions related to circles (e.g., annular and sector regions) can be handled similarly by modifying the circular case.
- These regions will be covered in homework and exams.

4. Limitations and Future Work

- More general regions are difficult to solve using variable separation.
- Different methods may yield different forms of solutions, but they represent the same underlying function (see Chap. 4).

Solution. Use $u(x, y)$ to represent the temperature at point (x, y) on the plate, i.e.,

$$u_{xx} + u_{yy} = 0 \quad (0 < x < a, 0 < y < b), \quad (2.3.1)$$

$$u(x, 0) = f(x), \quad u(x, b) = g(x), \quad (2.3.2)$$

$$u(0, y) = 0, \quad u(a, y) = 0. \quad (2.3.3)$$

1. Boundary Conditions for Laplace Equation

- The problem requires one pair of boundaries to have homogeneous (zero) boundary conditions.
- The other pair of boundaries can have non-homogeneous boundary conditions.

2. Approach to Solving the Problem

- Given the homogeneous equation and boundary conditions, the separation of variables method is a suitable approach.
- For Laplace's equation, there is no time derivative, so there are no initial conditions.
- Instead, the Non-Homogeneous boundary conditions act as the “initial conditions” in the context of the separation of variables method.

3. Role of Non-Homogeneous Boundaries

- The non-homogeneous boundary conditions are used to determine the coefficients in the solution.
- This is different from typical initial value problems, where initial conditions are used to determine coefficients.
- In this problem, the non-homogeneous boundaries play a crucial role in defining the solution.

4. Steps in Separation of Variables

- Apply the separation of variables method to solve the Laplace equation.
- Use the homogeneous boundaries to solve the S-L problem.
- Use the non-homogeneous boundaries to determine the coefficients in the final solution.

(1. Separation of Variables) Applying the method of separation of variables, let

$$u(x, y) = X(x)Y(y), \quad (2.3.4)$$

(2. PDE → ODEs) Substitute (2.3.4) into equation (2.3.1), and separate variables to get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

where λ is a constant.

1. Importance of Homogeneity

- Homogeneous equations are required for the separation of variables method to work in this step.
- In non-homogeneous cases (to be discussed in the next class), the separation of variables method fails from the second step onward.

2. Choosing the Sign of λ

- For homogeneous boundary conditions, especially when X is fixed at the boundaries, λ should be added a negative sign.
- This choice aligns with the form of the Sturm-Liouville problem for X , resulting in the ODE $X'' + \lambda X = 0$.
- Choosing a positive sign in front of λ may lead to an unfamiliar form, requiring additional work to derive the solution.

Thus, we obtain two ordinary differential equations

$$X''(x) + \lambda X(x) = 0, \quad (2.3.5)$$

$$Y''(y) - \lambda Y(y) = 0. \quad (2.3.6)$$

From the homogeneous boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0,$$

we get

$$X(0) = X(a) = 0.$$

(3. Solving ODEs) Now solve the boundary value problem of the ordinary differential equation

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(a) = 0, \quad (2.3.7)$$

for non-zero solutions.

- Students should be proficient in solving problems with all combinations of first and second kind boundary conditions.

1. When $\lambda < 0$, problem (2.3.7) has no non-trivial solutions.

2. When $\lambda = 0$, problem (2.3.7) also has no non-trivial solutions.

3. When $\lambda > 0$, problem (2.3.7) has non-trivial solutions.

At this time,

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2,$$

and the corresponding

$$X_n(x) = B_n \sin \frac{n\pi x}{a} \quad (n = 1, 2, \dots).$$

Next, consider the equation

$$Y''(y) - \lambda Y(y) = 0,$$

Substitute λ_n into equation (2.3.6) to get

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0, \quad (n = 1, 2, \dots).$$

The general solution is

$$Y_n(y) = C_n e^{\frac{n\pi}{a} y} + D_n e^{-\frac{n\pi}{a} y} \quad (n = 1, 2, \dots).$$

Thus, we can obtain a series of particular solutions that satisfy the homogeneous boundary conditions (2.3.3) for equation (2.3.1)

$$u_n(x, y) = (a_n e^{\frac{n\pi}{a} y} + b_n e^{-\frac{n\pi}{a} y}) \sin \frac{n\pi}{a} x \quad (n = 1, 2, \dots),$$

(4. Superposition of Series Solutions) Since equation (2.3.1) and boundary conditions (2.3.3) are homogeneous, therefore

$$u(x, y) = \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{a}y} + b_n e^{-\frac{n\pi}{a}y}) \sin \frac{n\pi}{a}x \quad (2.3.8)$$

still satisfies equation (2.3.1) and the homogeneous boundary conditions (2.3.3).

(5. Initial Conditions Determine Coefficients) Applying the non-homogeneous boundary conditions

$$u(x, 0) = f(x), \quad u(x, b) = g(x),$$

we have the relationships

$$\sum_{n=1}^{\infty} (a_n + b_n) \sin \frac{n\pi}{a}x = f(x),$$

and

$$\sum_{n=1}^{\infty} (a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}}) \sin \frac{n\pi}{a}x = g(x),$$

Using Fourier series coefficients, we get (*first odd extension then periodic extension*)

$$a_n + b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a}x dx,$$

$$a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a}x dx,$$

for $n = 1, 2, \dots$

Solving for a_n and b_n from the above equations and substituting back into (2.3.8) gives the solution to problem (2.3.1)-(2.3.3).

The solution to the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & (0 < x < a, 0 < y < b), \\ u(x, 0) = f(x), \quad u(x, b) = g(x), \\ u(0, y) = 0, \quad u(a, y) = 0. \end{cases}$$

is

$$u(x, y) = \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{a}y} + b_n e^{-\frac{n\pi}{a}y}) \sin \frac{n\pi}{a}x,$$

where

$$\begin{cases} a_n + b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a}x dx, \\ a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a}x dx, \end{cases}$$

for $n = 1, 2, \dots$

2.3.2 Boundary Value Problem of Laplace's Equation in a Circular Domain

- Regions related to circles (e.g., annular and sector regions) can be handled similarly by modifying the circular case.

Problem 2.3.2. Consider the temperature distribution problem of a circular plate with radius r_0 under steady-state conditions. Assume the top and bottom surfaces of the plate are adiabatic, and the temperature on the circular boundary is known as $f(\theta)$ ($0 \leq \theta \leq 2\pi$), and $f(0) = f(2\pi)$. Find the temperature distribution pattern under steady-state conditions.

1. Understanding the Physical Problem

- The term “steady-state” indicates that the problem involves Laplace’s equation: $\Delta u = 0$.

2. Dimensionality of the Problem

- The problem is two-dimensional because the temperature distribution is independent of the z -axis (insulated along z).

3. Boundary Conditions

- The temperature on the circular boundary is given as $u(R, \theta) = f(\theta)$, where $\theta \in [0, 2\pi]$.
- The periodicity condition $f(0) = f(2\pi)$ must be satisfied to ensure the temperature distribution is single-valued.

4. Choice of Coordinate System

- Cartesian coordinates are not suitable for circular regions because they do not separate variables easily (due to the boundary conditions).
- Polar coordinates (r, θ) are preferred due to their orthogonality and simplicity in handling circular boundaries (according to the boundary conditions).
- Polar coordinates are often more suitable, especially for problems involving circular symmetry.

5. Concept of 'Pao Ding Jie Niu'

- 'Pao Ding Jie Niu' (the skill of a master butcher) is a Chinese idiom that emphasizes mastering a task through practice.
- This concept is widely applicable in mathematics and physics.
- In fluid dynamics, choosing the Lagrangian (comoving) coordinate system can simplify complex problems. For example, considering a fluid flow from the perspective of a moving reference frame (like a boat on a river) can make the problem easier to handle.

6. Laplace's Equation in Polar Coordinates

- The Laplace operator in polar coordinates is given by:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

- This can also be written as:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

7. Mathematical Techniques

- The transformation from Cartesian to polar coordinates involves using the chain rule for partial derivatives.
- Care must be taken to distinguish between functions defined in different coordinate systems to avoid confusion.

8. Importance of Practice

- It is essential to practice deriving the Laplace operator in polar coordinates to understand its form and application.
- Remembering the form of the Laplace operator in polar coordinates is crucial for solving problems involving circular regions.

9. Two useful identities:

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = r \frac{\partial^2}{\partial r^2} (ru) \quad \text{and} \quad r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = r^2 u_{rr} + ru_r \quad (\text{Inspiring the Euler ODEs})$$

Since the temperature under steady-state conditions satisfies Laplace's equation, and the region is circular, it is more convenient to use **polar coordinates** (the idea of "cutting up an ox like a master butcher"⁴) for the Laplace equation when applying the method of separation of variables.

Solution. We use $u(r, \theta)$ to represent the temperature at point (r, θ) inside the circular plate. The problem can be formulated as the following boundary value problem:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad (0 < r < r_0), \quad (\text{Not equivalent to } u_{xx} + u_{yy} = 0 \text{ completely}) \quad (2.3.9)$$

$$u|_{r=r_0} = f(\theta). \quad (2.3.10)$$

Exercise: Verify that the Laplace equation $u_{xx} + u_{yy} = 0$ in polar coordinates takes the form $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$.

⁴This is a classic Chinese idiom derived from the ancient text "Zhuangzi." It tells the tale of a skilled butcher who, through years of practice, develops exceptional expertise in his craft. The butcher is able to dismember an ox with such precision and ease that his movements appear to be a dance. His knife glides through the joints and along the natural lines of the ox's body, never needing to hack or force his way through bone or tissue.

Hint: Perform polar coordinate transformation

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \arctan \frac{y}{x}. \end{cases}$$

For $u(x, y) := \bar{u}(r(x, y), \theta(x, y))$ and $\bar{u}(r, \theta) := u(x(r, \theta), y(r, \theta))$

$$u_x = \bar{u}_r \cdot r_x + \bar{u}_\theta \cdot \theta_x, \quad u_y = \bar{u}_r \cdot r_y + \bar{u}_\theta \cdot \theta_y$$

Further, we obtain

$$u_{xx} = (\bar{u}_{rr}r_x + \bar{u}_{r\theta}\theta_x) \cdot r_x + \bar{u}_r \cdot r_{xx} + (\bar{u}_{\theta r}r_x + \bar{u}_{\theta\theta}\theta_x) \cdot \theta_x + \bar{u}_\theta \cdot \theta_{xx}$$

and

$$u_{yy} = (\bar{u}_{rr}r_y + \bar{u}_{r\theta}\theta_y) \cdot r_y + \bar{u}_r \cdot r_{yy} + (\bar{u}_{\theta r}r_y + \bar{u}_{\theta\theta}\theta_y) \cdot \theta_y + \bar{u}_\theta \cdot \theta_{yy}$$

1. Coordinate Transformation Equivalence

- **Question:** Is the transformation from Cartesian to polar coordinates equivalent?
- This question is crucial as it impacts the solution process. Consider it if stuck.

2. Characteristics of the Equation

- **Question:** When solving equations, first examine their characteristics.
- Homogeneous equations with homogeneous boundaries can use the method of separation of variables.
- However, this problem does not satisfy homogeneous boundary conditions. What should be done?

3. Solution Approach

- Proceed with the method of separation of variables despite non-homogeneous boundaries.
- Be prepared to modify and patch the method as necessary.

(1. Separation of Variables) Assume the solution to equation (2.3.9) is

$$u(r, \theta) = R(r)\Phi(\theta),$$

(2. PDE → ODEs) Substitute into equation (2.3.9) to get

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$$

Separation of variables gives

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda$$

where λ is a constant.

1. Formulation of the Equation

- We derive a form where one side is a function of R and the other side a function of θ , implying they must equal a constant.

2. Choice of Constant Sign

- The decision between a positive or negative constant is typically guided by homogeneous boundary conditions and the familiar SL problem.
- In the absence of homogeneous boundary conditions, this guidance is not available, leading to a dilemma.

3. Familiarity with Standard Forms

- It's beneficial to aim for a form we recognize, such as $X'' + \lambda X = 0$, which we are familiar with.
- This approach is driven by the **hope** of obtaining a solvable problem.

4. Selection of the Constant

- We choose the constant with a positive sign to match a familiar form of the equation $X'' + \lambda X = 0$.

5. Challenges in Separation of Variables

- The non-homogeneous boundary condition hinders the separation of variables, making the problem unsolvable in its current form.

From this, two ordinary differential equations can be obtained

$$r^2 R'' + rR' - \lambda R = 0,$$

$$\Phi'' + \lambda \Phi = 0.$$

Question 2.3.1. Which ODE should be solved first?

- Principle: First solve the S-L problem with homogeneous boundary conditions.
- However, now there are only non-homogeneous boundaries, and the boundary variables cannot be separated.

- The idea is to look for **implicit conditions**.

1. Hidden Conditions in Boundary Value Problems:

- A boundary value problem may have hidden conditions that are not explicitly given.
- These hidden conditions can be derived from the given equation and boundary conditions through appropriate reasoning and physical considerations.
- Hidden conditions may serve as **additional boundary conditions** or constraints for solving the problem.

2. Coordinate Transformation and Information Loss:

- When transforming coordinates from Cartesian to polar coordinates, **information is lost**.
- The origin loses its **unique representation** in polar coordinates since $r = 0$ corresponds to an **undefined angle θ** .
- To maintain a **one-to-one** correspondence, the origin is typically **excluded in polar coordinates**.
- **Periodicity issues** arise because a point in Cartesian coordinates may correspond to **multiple angles in polar coordinates** (e.g., θ and $\theta + 2\pi$). For example, $(x, y) = (a, 0)$ becomes $(r, \theta) = (a, 0), (a, 2\pi), \dots$

3. Solving Periodicity Issues:

- **Method 1:** Restrict the domain of θ to $[0, 2\pi]$ and enforce periodic boundary conditions such that

$$\begin{cases} u(r, \theta) \Big|_{\theta=0} = u(r, \theta) \Big|_{\theta=2\pi} \\ u_\theta(r, \theta) \Big|_{\theta=0} = u_\theta(r, \theta) \Big|_{\theta=2\pi} \end{cases}$$

Since $\Phi'' + \lambda\Phi = 0$ is a second order ODE, the data upto u_θ is enough.

- **Method 2:** Allow θ to take real values and enforce a periodic relationship $u(\theta) = u(\theta + 2\pi)$.
- Both methods ensure that the function values remain consistent across the transformation.

4. Equivalence of Equations:

- The transformed equation is **not completely equivalent** to the original Laplace equation in Cartesian coordinates.
- The loss of origin and introduction of periodicity affect the equivalence of the equations.

5. Supplementing Lost Information:

- Lost information includes the **origin** and **periodicity**.

- To address the **lost origin information**, impose a **boundedness condition at the origin**, requiring $|u(0, \theta)| < \infty$ (note $u(0, \theta) = \infty$ is a solution in polar coordinates, but not in Cartesian coordinates).
- To address **periodicity**, introduce **periodic boundary conditions** such as $u(\theta + 2\pi) = u(\theta)$.

6. Extending the Domain:

- The original problem may define θ within $[0, 2\pi]$, but periodic boundary conditions require **extending the domain to include all real values of θ periodically**.
- After extending the function to the entire real axis, it can be restricted back to the interval $[0, 2\pi]$.
- Textbooks often implicitly assume analytic continuation without explicitly stating it.

Since the temperature function $u(r, \theta)$ is single-valued, when θ changes from θ to $\theta + 2\pi$, $u(r, \theta + 2\pi) = u(r, \theta)$ holds, thus we have

$$\Phi(\theta + 2\pi) = \Phi(\theta).$$

At the same time, according to the physical meaning of the problem, the temperature at each point inside the circle should be bounded, hence $|u(0, \theta)| < +\infty$ holds, therefore $R(r)$ should satisfy the condition

$$|R(0)| < +\infty.$$

Thus, we obtain two ordinary differential equations with boundary conditions

$$\begin{cases} \Phi'' + \lambda\Phi = 0, \\ \Phi(\theta + 2\pi) = \Phi(\theta). \end{cases} \quad (\text{The fifth SL problem: the periodic boundary.}) \quad (2.3.11)$$

and

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0, \\ |R(0)| < +\infty. \end{cases} \quad (\text{Do not forget this new restriction!}) \quad (2.3.12)$$

(3. Solving ODEs) We start with problem (2.3.11) and discuss λ in three cases:

1. When $\lambda < 0$, the general solution of the equation is

$$\Phi(\theta) = Ae^{-\sqrt{-\lambda}\theta} + Be^{\sqrt{-\lambda}\theta}, \quad (\text{exp monotonic functions, can be periodic})$$

where A, B are arbitrary constants. Since such functions do not satisfy periodic conditions, λ cannot take negative values.

2. When $\lambda = 0$, the general solution of the equation is

$$\Phi_0(\theta) = A_0\theta + B_0,$$

where A_0, B_0 are arbitrary constants. Only when $A_0 = 0$, the function Φ_0 satisfies periodic conditions. Therefore, when $\lambda = 0$, the solution to problem (2.3.11) is

$$\Phi_0(\theta) = B_0. \quad (\text{horizontal line is periodic. Every number is its period by definition.})$$

(Like the case with both sides second type boundary, there is 0 eigenvalue!)

Substituting $\lambda = 0$ into equation (2.3.12) gives

$$r^2 R'' + rR' - \lambda R = 0,$$

and its general solution is

$$R_0(r) = C_0 \ln r + D_0,$$

where C_0, D_0 are arbitrary constants.

General Solution to a Reducible Second-Order Differential Equation

Problem: Find the general solution to the following reducible second-order differential equation for $R(r)$:

$$r^2 R'' + rR' = 0$$

Solution. Let

$$R' = P(r) \Rightarrow R'' = P'(r)$$

Substitute into the original equation:

$$r^2 P'(r) + rP(r) = 0 \quad \Rightarrow \quad \frac{1}{P} dP = -\frac{1}{r} dr$$

Then

$$\Rightarrow \ln P = -\ln r + C \quad \Rightarrow \quad P(r) = \frac{C}{r}$$

Thus, we have

$$R'(r) = \frac{C}{r}$$

The general solution to the original equation is:

$$R(r) = C_0 \ln r + D_0,$$

where C_0, D_0 are arbitrary constants.

Only when $C_0 = 0$, the function R_0 satisfies the boundedness condition $|R(0)| < +\infty$. Therefore, when $\lambda = 0$, the solution to problem (2.3.12) is

$$R_0(r) = D_0.$$

Thus, a non-zero solution to the original equation (2.3.9) is obtained

$$u_0(r, \theta) = B_0 D_0 = \frac{1}{2} a_0.$$

3. When $\lambda > 0$, the general solution of the equation is

$$\Phi(\theta) = A \cos \sqrt{\lambda} \theta + B \sin \sqrt{\lambda} \theta,$$

where A, B are arbitrary constants. Since $\Phi(\theta + 2\pi) = \Phi(\theta)$,

$$\lambda = n^2 \quad (n = 1, 2, \dots),$$

at this time, the solution to equation (2.3.11) can be expressed as

$$\Phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta.$$

The eigenfunction set corresponding to the Poisson equation in a circular domain (i.e., $\Phi'' + \lambda\Phi = 0$ and $\Phi(\theta + 2\pi) = \Phi(\theta)$) is:

$$\{1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots, \cos n\theta, \sin n\theta, \dots\}$$

Substituting $\lambda = n^2$ into equation (2.3.12) gives the **Euler equation**

$$r^2 R'' + rR' - n^2 R = 0,$$

and its general solution is

$$R_n(r) = C_n r^n + D_n r^{-n}.$$

To ensure $|R(0)| < +\infty$, only $D_n = 0$ ($n = 1, 2, \dots$), so

$$R_n(r) = C_n r^n \quad (n = 1, 2, \dots).$$

General Form of Euler's Equation

The general form of Euler's equation is:

$$x^n y^{(n)} + P_1 x^{n-1} y^{(n-1)} + \dots + P_{n-1} x y' + P_n y = f(x).$$

Here, P_1 to P_n are constants, and $f(x)$ is a known function.

General Solution to a Second-Order Euler Equation

Problem: Find the general solution to the following second-order Euler equation:

$$r^2 R'' + rR' - n^2 R = 0, \quad (n = 1, 2, \dots)$$

Solution. Make the transformation $r = e^t \Rightarrow t = \ln r$.

Thus, we have:

$$R_r = R_t \cdot \frac{1}{r}, \quad R_{rr} = \left(R_{tt} \cdot \frac{1}{r^2} + R_t \cdot \left(-\frac{1}{r^2} \right) \right) = \frac{1}{r^2} R_{tt} - \frac{1}{r^2} R_t.$$

Substitute into the original equation:

$$R_{tt} - R_t + R_t - n^2 R = 0 \Rightarrow R_{tt} - n^2 R = 0$$

$$\Rightarrow R_n = C_n e^{nt} + D_n e^{-nt}$$

Substitute $t = \ln r$ back to get the original form:

$$R_n(r) = C_n r^n + D_n r^{-n} \quad (n = 1, 2, \dots)$$

The general solution to the original equation is:

$$R_n(r) = C_n r^n + D_n r^{-n} \quad (n = 1, 2, \dots)$$

where C_n, D_n are arbitrary constants.

Another method—trial and error: By observation ($r^2 \partial_r^2, r \partial_r$), the solution should be r^α (since derivatives of polynomial reduce its order $r^2 \partial_r^2 r^\alpha \sim r \partial_r r^\alpha \sim r^\alpha$) and substitute it in the equation to determine α .

1. General Approach to Solving Equations:

- Use both **intuitive guessing** and **inverse methods** to solve equations.
- Transform **new equations** into **familiar forms** by **changing** either (1) the **function** or (2) the **variable**.

2. Transformations in Differential Equations:

- Since differential equations only involve two objects: (1) an **unknown function** and (2) its **variables**.
- Transformations can be made on (1) the **function itself** (e.g., the method of integration factor involves merging various functions to form a new function that simplifies the equation.) or (2) the **independent variable** (e.g., the Euler equation).

3. Euler's Equation:

- For Euler's equation, a variable transformation is applied.
- Let $r = e^t$, then $t = \ln r$.
- This leads to a simpler form of the differential equation.

4. Guessing Solutions:

- Assume solutions of the form $R = r^\alpha$.
- Substitute into the equation to determine α .

5. Exploratory Process:

- The process of finding transformations is exploratory.
- Use known forms and patterns to guide the search for new transformations.

6. Chain Rule Application:

- Use the chain rule to relate derivatives with respect to different variables.
- This can simplify the process of finding suitable transformations.

7. Physical Intuition:

- Physicists often use intuitive methods that are not rigorous but efficient.
- These methods can provide quick insights into the behavior of solutions.

8. General Solution for Euler's Equation:

- The general solution involves a linear combination of r^n and r^{-n} .

The ideas for Euler ODE:

We start with the differential identity:

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} = r \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) \stackrel{\text{Hope}}{=} \frac{\partial}{\partial t} \frac{\partial}{\partial t} R$$

Transformation Goal:

Hope a transformation: $r \frac{\partial}{\partial r} \rightarrow \frac{\partial}{\partial t}$ (*)

Simplification:

Then, $\frac{\partial^2 R}{\partial t^2}$ is easy since $\partial_t^2 R - n^2 R = 0 \Rightarrow R_n = C e^{nt} + D e^{-nt}$

How to achieve (*)?

$$\begin{aligned} \frac{dR}{dt} &\stackrel{\text{chain rule (+)}}{=} \frac{dr}{dt} \frac{dR}{dr} = r \frac{dR}{dr} \\ \left(\text{Physicist's approach: } r \frac{dR}{dr} = \frac{dR}{dt} \Rightarrow dt = \frac{1}{r} dr = d \ln r \right) \end{aligned}$$

By chain rule (+) for $R(r(t))$:

$$\frac{dr}{dt} = r \Rightarrow d \ln r = dt \Rightarrow r = e^t$$

Then we find the transformation of variables.

Thus, when $\lambda = n^2$ ($n = 1, 2, \dots$), we obtain a series of particular solutions for equation (2.3.9).

$$u_n(r, \theta) = (a_n \cos n\theta + b_n \sin n\theta) r^n \quad (n = 1, 2, \dots),$$

where $a_n = A_n C_n, b_n = B_n C_n$ are arbitrary constants.

(4. Superposition of Series Solutions) Since equation (2.3.9) is linear and homogeneous, using the principle of superposition, we can obtain the series solution that satisfies the univalence and boundedness conditions for the equation as

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n. \quad (2.3.13)$$

(5. Initial Conditions Determine Coefficients) To determine the coefficients a_n, b_n , use the boundary condition (2.3.10) i.e., $u|_{r=r_0} = f(\theta)$.

$$u(r_0, \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r_0^n = f(\theta), \quad (0 \leq \theta \leq 2\pi)$$

By Fourier series theory, we know

$$\begin{cases} a_n r_0^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n r_0^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

Therefore, the solution to the boundary value problem (2.3.9)-(2.3.10) is given by the series solution (2.3.13).

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)r^n.$$

where the coefficients a_n, b_n are determined by equation (2.3.14),

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases} \quad (2.3.14)$$

1. Determine Coefficients:

- Use non-homogeneous boundary conditions (play the role of initial data) instead of initial values to determine coefficients.

2. Domain Importance:

- The domain $[0, 2\pi]$ (θ restricts to $[0, 2\pi]$ from the extended \mathbb{R}) is crucial for defining the coefficients.

3. Orthogonality and Integration:

- Use orthogonality of trigonometric functions over one period $[0, 2\pi]$ (one period ensures the orthogonality of trigonometric functions).
- Extend the function to the entire real line **periodically** and then restrict it back to $[0, 2\pi]$.

4. Integral Limits:

- Perform integration from 0 to 2π for determining coefficients.

Ex 2.3.1. Solve the Following Problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & (0 < r < R), \\ u(R, \theta) = \theta \sin \theta. \end{cases}$$

Solution. Using the formula

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

we obtain

$$\begin{cases} a_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin \theta \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin \theta \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

Since

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin \theta \cos n\theta d\theta \quad (n = 0, 1, 2, \dots),$$

we have

$$a_n = \frac{1}{2\pi R^n} \int_0^{2\pi} \theta [\sin(1+n)\theta + \sin(1-n)\theta] d\theta = \frac{1}{R^n} \cdot \frac{2}{n^2 - 1}, \quad (n \neq 1)$$

and

$$a_1 = \frac{1}{\pi R} \int_0^{2\pi} \theta \sin \theta \cos \theta d\theta = \frac{1}{2\pi R} \int_0^{2\pi} \theta \sin 2\theta d\theta = -\frac{1}{2R}$$

Especially,

$$a_0 = -2$$

And since

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin \theta \sin n\theta d\theta \quad (n = 1, 2, \dots),$$

we have

$$b_n = \frac{1}{2\pi R^n} \int_0^{2\pi} \theta [\cos(1-n)\theta - \cos(1+n)\theta] d\theta = 0 \quad (n \neq 1)$$

and

$$b_1 = \frac{1}{\pi R} \int_0^{2\pi} \theta \sin^2 \theta d\theta = \frac{1}{2\pi R} \int_0^{2\pi} \theta (1 - \cos 2\theta) d\theta = \frac{\pi}{R}.$$

Substituting the obtained coefficients into the series solution formula

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n.$$

The solution to the given problem is

$$u(r, \theta) = -1 - \frac{r}{2R} \cos \theta + \frac{\pi}{R} r \sin \theta + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \left(\frac{r}{R}\right)^n \cos n\theta.$$

Ex 2.3.2. Solve the Following Problem

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 & (0 < r < r_0), \\ u(r_0, \theta) = A \sin 2\theta. \end{cases} \quad (2.3.15)$$

Solution. Using the formula

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

we obtain

$$\begin{cases} a_n = \frac{1}{\pi R^n} \int_0^{2\pi} A \sin 2\theta \cos n\theta d\theta & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi R^n} \int_0^{2\pi} A \sin 2\theta \sin n\theta d\theta & (n = 1, 2, \dots), \end{cases}$$

Since

$$a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} A \sin 2\theta \cos n\theta d\theta \quad (n = 0, 1, 2, \dots),$$

using the **orthogonality** of trigonometric functions, we get

$$a_n = 0 \quad (n = 0, 1, 2, \dots),$$

and since

$$b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} A \sin 2\theta \sin n\theta d\theta \quad (n = 1, 2, \dots),$$

using the **orthogonality** of trigonometric functions again, we get

$$b_n = 0 \quad (n \neq 2),$$

and

$$b_2 = \frac{A}{\pi r_0^2} \int_0^{2\pi} \sin^2 2\theta d\theta = \frac{A}{2\pi r_0^2} \int_0^{2\pi} (1 - \cos 4\theta) d\theta = \frac{A}{r_0^2}.$$

Substituting the obtained coefficients into the series solution formula

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)r^n.$$

The solution to the given problem is

$$u(r, \theta) = \frac{A}{r_0^2} r^2 \sin 2\theta.$$

This example can also be solved by the **method of trial and error**.

The method of trial and error:

The trial method is very important. For certain practical problems, you can guess the shape of the solution based on the **physical meaning, algebraic structure** or **geometric characteristics** of the problem or the experience, then make an assumption of a specific form, and solve the equation to determine the parameters. (The method of separation of variables is essentially also a trial method, assuming a separable form.)

Ex 2.3.3. Solve the Following Problem

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & (0 < r < r_0), \\ u(r_0, \theta) = A \sin 2\theta. \end{cases}$$

Solution. Since the function $r^2 \sin 2\theta$ is harmonic, the function $C_1 r^2 \sin 2\theta + C_2$ is also harmonic, where C_1, C_2 are arbitrary constants. Let us assume (*a priori* assumption) the solution be

$$u(r, \theta) = C_1 r^2 \sin 2\theta + C_2$$

From the boundary condition, we get

$$u(r_0, \theta) = C_1 r_0^2 \sin 2\theta + C_2 = A \sin 2\theta,$$

Comparing coefficients, we know

$$C_2 = 0, \quad C_1 r_0^2 = A \implies C_1 = \frac{A}{r_0^2}.$$

Thus, the solution to the given problem is

$$u(r, \theta) = \frac{A}{r_0^2} r^2 \sin 2\theta.$$

Summary of the forms of five common eigenfunction series

1. $u(0, t) = 0, u(l, t) = 0; \Rightarrow \{\sin \frac{n\pi x}{l}\} (n = 1, 2, \dots);$
2. $u(0, t) = 0, u_x(l, t) = 0; \Rightarrow \left\{ \sin \frac{(2n-1)\pi x}{2l} \right\} (n = 1, 2, \dots);$
3. $u_x(0, t) = 0, u(l, t) = 0; \Rightarrow \left\{ \cos \frac{(2n-1)\pi x}{2l} \right\} (n = 1, 2, \dots);$
4. $u_x(0, t) = 0, u_x(l, t) = 0; \Rightarrow \{\cos \frac{n\pi x}{l}\} (n = 0, 1, 2, \dots);$

The above forms are applicable to one-dimensional vibration equations, heat conduction equations, and Poisson's equations on rectangular domains.

5. Eigenfunction series corresponding to Poisson's equation on a circular domain

$$\{1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots \cos n\theta, \sin n\theta, \dots\}$$

- **Method Discussed:** Separation of Variables

- General format and steps are similar.
- Differences lie in
 - * The types of equations (determine the T -functions);
 - * Boundary conditions (determine the SL problem and eigenfunctions, i.e., X -function).

- **Types of Equations:** Three main types

- Affect the differentiation of the T (or Y)-function.
- T -function equations can be oscillatory (u_{tt}) or exponentially decaying (u_t).

- **Boundary Conditions:** Five types (including periodic boundary conditions)

- Five SL problems are presented under these boundary conditions.

- These problems need to be memorized.
- **Current Requirement:**
 - Ability to immediately identify eigenvalues and eigenfunctions upon seeing the problem.
 - According to the PDE to guess the T -function and further guess the possible solution before solving the problem.

- **Trial Method Overview**

- Also known as the "guessing method."
- Separation of variables is a type of trial method.
- Trial method allows guessing a more accurate form of the solution with some undetermined parameters.
- Parameters are determined by substituting the guessed form back into the original equation and boundary conditions.

- **Challenges and Strategies for Guessing**

- The difficulty lies in how to make an appropriate guess.
- Common strategies for guessing:
 - * **Physical Phenomena:** Use physical intuition (e.g., vibration of a string) to guide the guess.
 - * **Geometric Properties:** Use symmetry or geometric constraints (e.g., spherical symmetry implies functions depend only on R).
 - * **Non-homogeneous Terms:** Guess based on the **form of boundary conditions** or **non-homogeneous terms** in the equation.

- **Specific Example**

- Given boundary conditions (e.g., $u(r_0, \theta) = A \sin 2\theta$), guess the simplest form of the solution $C_1 r^\alpha \sin 2\theta + C_2$. That is, $\sin 2\theta \rightarrow \sin 2\theta$ and $A \rightarrow C_1 r^\alpha$. Since When r is fixed, start guessing from the simplest functions—polynomial functions r^α .

Transform the Series Solution into an Integral Form

The series solution (2.3.16) can be transformed into an integral form:

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n. \quad (2.3.16)$$

The coefficients a_n, b_n are determined by the following formulas (2.3.17):

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\varphi) \cos n\varphi d\varphi & (n = 0, 1, 2, \dots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\varphi) \sin n\varphi d\varphi & (n = 1, 2, \dots), \end{cases} \quad (2.3.17)$$

- **Objective:**

- Convert the **series solution** of Laplace's equation into an **integral form**.
- Preparation for Chapter 4, where a different method (Green's function method) will be used.
The solution in Chapter 4 will be in integral form without summation symbols.
- To ensure the solution from this chapter matches the form of the solution in Chapter 4.

- **Current Solution Form:**

- Contains both integrals (a_n and b_n) and series summation.

- **Goal of the Transformation:**

- Remove the summation symbol, implying finding the function to which the series converges.

Substituting (2.3.17) into (2.3.16) gives:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^n \int_0^{2\pi} f(\varphi) \underbrace{(\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi)}_{\cos n(\theta - \varphi)} d\varphi.$$

Simplify to obtain

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} \left[\frac{1}{2} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^n \cos n(\theta - \varphi)}_{(*)} \right] f(\varphi) d\varphi \quad (r < r_0).$$

where (*) means this form inspires us to use Geometric Series if $\cos(n\cdot)$ becomes $e^{n\cdot}$ and this can be done by Euler's formula

Using Euler's formula:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \quad (|q| < 1)$$

Perform the following identity transformation: Let $k = \frac{r}{r_0}$

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} k^n \cos n(\theta - \varphi) &= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} k^n [e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}] \\ &= \frac{1}{2} \left[1 + \frac{ke^{i(\theta-\varphi)}}{1 - ke^{i(\theta-\varphi)}} + \frac{ke^{-i(\theta-\varphi)}}{1 - ke^{-i(\theta-\varphi)}} \right] \\ &= \frac{1}{2} \cdot \frac{1 - k^2}{1 - ke^{i(\theta-\varphi)} - ke^{-i(\theta-\varphi)} + k^2} \end{aligned}$$

Then we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} k^n \cos n(\theta - \varphi) = \frac{1}{2} \cdot \frac{1 - k^2}{1 + k^2 - 2k \cos(\theta - \varphi)} \quad (|k| < 1),$$

So the series solution (2.3.16) can be expressed in integral form

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \frac{r_0^2 - r^2}{r_0^2 + r^2 - 2r_0 r \cos n(\theta - \varphi)} d\varphi \quad (r < r_0),$$

This formula is called the **Poisson formula in a circular domain**.

2.4 Non-homogeneous Equations and Their Solving Problems

This section examines the boundary value problems of **Non-homogeneous equations** and introduces a commonly used method: **the method of eigenfunction expansion**.

We will use three types of boundary value problems as examples to illustrate the key points and steps of this method.

1. Forced vibration problems with boundary conditions.
2. Heat conduction problems in finite rods (with heat sources).
3. Poisson's equation (Non-homogeneous Laplace's equation).

2.4.1 Forced Vibration Problem with Boundary Conditions

Firstly, we consider the following problem

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (0 < x < l, t > 0), \\ u(0, t) = 0, \quad u(l, t) = 0, & \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). & \end{cases} \quad (\text{Multiple non-homogeneities}) \quad (2.4.1)$$

At this time, the vibration of the string is caused by two parts of interference:

1. One is the **external forcing force**,
2. and the other is the **initial state** of the string.

From the physical point of view, this vibration can be seen as a combination of the vibration caused solely by the forcing force and the vibration caused solely by the initial state.

General Ideas:

- **Isolate** and separate the **non-homogeneity** in the problem to multiple systems such that one system keeps only one non-homogeneity.

- Express u as the sum of v and w , i.e., $u = v + w$ (**the principle of linear superposition**).
- Each function v and w will handle different aspects of the non-homogeneity. That is,

* v solves

$$\begin{cases} \text{Non-homogeneous equation} \\ \text{Homogeneous boundary} \\ \text{Homogeneous initial data} \end{cases} \rightarrow \text{using the eigenfunction method}$$

* w solves

$$\begin{cases} \text{Homogeneous equation} \\ \text{Homogeneous boundary} \\ \text{Non-homogeneous initial data} \end{cases} \rightarrow \text{using separation of variables}$$

Thus, we can set the solution of problem (2.4.1) as

$$u(x, t) = v(x, t) + w(x, t),$$

where $v(x, t)$ represents the displacement of the string caused **solely by the forcing force**, and $w(x, t)$ represents the displacement caused **solely by the initial state**, such that $v(x, t)$ and $w(x, t)$ satisfy the following boundary value problems respectively:

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x, t) & (0 < x < l, t > 0), \\ v(0, t) = 0, \quad v(l, t) = 0, \\ v(x, 0) = v_t(x, 0) = 0. \end{cases} \quad (2.4.2)$$

and

$$\begin{cases} w_{tt} = a^2 w_{xx} & (0 < x < l, t > 0), \\ w(0, t) = 0, \quad w(l, t) = 0, \\ w(x, 0) = \varphi(x), \quad w_t(x, 0) = \psi(x). \end{cases} \quad (2.4.3)$$

- Use the principle of linear superposition.

To solve the forced vibration problem with homogeneous boundary conditions and zero initial conditions:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (0 < x < l, t > 0), \\ u(0, t) = 0, \quad u(l, t) = 0, \\ u(x, 0) = u_t(x, 0) = 0. \end{cases} \quad (2.4.4)$$

The above problem can be solved using a parameter variation method similar to that used for linear Non-homogeneous ordinary differential equations, and maintain the idea that:

The solution to this boundary value problem can be decomposed into an infinite number of standing waves, and the shape of each standing wave is still determined by the eigenfunctions of the vibrating body.

- **Non-homogeneous equations + homogeneous boundary + homogeneous initial conditions**
→ The eigenfunction method.
- Why these conditions? Consider this question in the following derivations.

How to invent the method of eigenfunctions? (A guessing method on every step)

- “A fair guessing” means one gives **a priori assumption**.
- We currently only have the method of separation of variables to solve PDEs.
- The method of separation of variables was discovered early through physical practice.
- When solving **non-homogeneous** equations, we still start with the separation of variables method to see if it can be solved.
- We need to understand **why a homogeneous equation** is required in the separation of variables method.
- When working on this problem, first **compare the differences**. By understanding these differences, we can identify where the difficulties will arise, and then we can figure out how to overcome them.
- In the five steps of the separation of variables method, the **second step** involves substituting back into the original equation.
- If the original equation contains a source function $f(t, x)$ that **cannot** be separated into a form like $T(t)X(x)$, the separation of variables method **fails**. **Non-homogeneous** terms **prevent** direct application of separation of variables.
- The function f is **generally not** separable into X and T components.
- When encountering this problem, we can consider using the **idea of superposition of solutions**.

- In the **fourth step** of the separation of variables method, we already use **superposition to satisfy initial conditions**.
- Initially, we assume that the solution can be separated into variables. However, in fact the final superposed solution cannot be separated by variables.
- Since the superposed solution already cannot be separated even in the method of separation of variables, we give a priori assumption (a weaker assumption) that directly superposing the solutions, i.e., assume the solution has the form that $u = \sum XT$ instead of $u = XT$.
- This approach allows us to handle cases where the original method fails due to non-separable terms.
- We can try to directly **superpose the solutions** even if they **cannot** be separated into variables. This is a weaker a priori assumption than the separation of variables since a lot of function can be expanded to a series based on some basis functions system.
- If the function f can be expressed in a separable form through summation as well (i.e., $f = \sum f_n(t)g_n(x)$, in fact, you will find that it is benefit to choose $g_n(x) = X_n(x)$ later), we can apply this idea to solve the problem.

Finding the Necessary Conditions: Introduce **a priori assumption**

$$u = \sum u_n(t)X_n(x) \rightarrow \text{to be determined } u_n \text{ and } X_n.$$

- **Goal:** We expect to **choose the appropriate functions $X_n(x)$ and coefficients $u_n(t)$, such that the equation and boundary, initial data are satisfied.**
- **Question:** How to choose or construct $X_n(x)$ and $u_n(t)$? ← This is what we want to construct.

Assume (fair assumption) the expansion of f is denoted as (let $g_n(x) = X_n(x)$ already)

$$f = \sum f_n(t)X_n(x)$$

Let us see what the equation and data become:

- **The equation:**

$$\sum u_n'' X_n = \underbrace{\sum a^2 u_n X_n''}_{(+)} + \sum f_n X_n \quad (2.4.5)$$

- **The boundary:**

$$\begin{cases} u(0, t) = \sum u_n(t)X_n(0) = 0, \\ u(l, t) = \sum u_n(t)X_n(l) = 0. \end{cases}$$

- **The initial data:**

$$\begin{cases} u(x, 0) = \sum u_n(0)X_n(x) = 0, \\ u_t(x, 0) = \sum u'_n(0)X_n(x) = 0. \end{cases} \quad (2.4.6)$$

- By manipulating (2.4.5)–(2.4.6), try to find a way to construct X_n and u_n .

1. **(Using the equation)** By (2.4.5), we obtain

$$\sum u_n'' X_n = \underbrace{\sum a^2 u_n X_n''}_{(+)} + \sum f_n X_n \implies \sum (u_n'' - f_n) X_n - \underbrace{\sum a^2 u_n X_n''}_{(+)} = 0 \quad (2.4.7)$$

where (+)-term involving X_n'' cannot be combined with the other two terms, which involve X_n .

Remark 2.4.1. • If each term in the summation (2.4.7) can be separated into a product of functions of X and T , we may be able to conclude that each term must be zero. That is,

$$0 = \sum (u_n'' - f_n) X_n - \underbrace{\sum a^2 u_n X_n''}_{(+)} \stackrel{HOPES}{=} \sum T_n(t) X_n(x) \leftarrow Goal$$

- If successful, this **may** imply that $T_n(t) = 0$. In fact, this requires $\{X_n\}$ is a complete basis function system. However, in order to find an idea, we just roughly believe this is true currently and later reconsider if this can be satisfied (viewed as **a priori assumption**).

In order to combine it with the other terms, we **hope** (to find the **simplest and familiar** way although there might be infinite ways and freedom).

$$X_n'' = -\lambda X_n \rightarrow \text{the key invent and a new a priori assumption} \quad (2.4.8)$$

- Since X_n is to be determined, we want to choose X_n to satisfy this ODE in order to determine X_n . However, this ODE can not determine X_n completely. We need boundary conditions. On the other hand, we observe this ODE is like the one in S-L problem, so we **hope** to construct **homogeneous boundary conditions**.

Remark 2.4.2. • There are **multiple ways** to achieve (2.4.7), we only pick a **simplest and efficient one**, if it works against all odds, we finish. Otherwise, if it fails in the later steps, we choose another possible way to get through this step.

- **Motivation for the Choice:** The choice of a solution path is like navigating a maze with multiple possible routes. We choose one that works, but others might also be viable.
- **Simplicity and Familiarity:** We choose a path proportional to X_n because it is the simplest and most familiar. This choice implies $X_n'' = \text{constant} \times X_n$.
- **Requirement for Consistency:** To satisfy this choice, we require $X_n'' + \lambda X_n = 0$. This equation is **familiar** and aligns with our previous work: **SL problem**.
- **Extracting X_n :** By choosing this path, we can factor out X_n in the equation, simplifying the problem.
- **Hope and Consistency:** This choice is driven by our **hope** that it aligns with a **familiar form** and can be consistent with our previous results.

2. (**Using the boundary**) We have been only using the equation so far, and now we need to incorporate the **boundary conditions**. Substituting the apriori assumption $u = \sum u_n(t)X_n(x)$ to the homogeneous boundary:

$$\begin{cases} u(0, t) = \sum u_n(t)X_n(0) = 0 \Leftrightarrow X_n(0) = 0 \\ u(l, t) = \sum u_n(t)X_n(l) = 0 \Leftrightarrow X_n(l) = 0 \end{cases} \quad (\text{a simplest condition and a priori assumption})$$

- This means if we choose $X_n(0) = 0$ and $X_n(l) = 0$, it can ensure the boundary condition satisfies the required boundary in (2.4.4). This gives a specific way on X_n to realize the boundary condition.
- By choosing these boundaries and equation of X_n , we have used some freedom of this system and this leaves no other room for u_n but only one choice.

Then we determine X_n by the S-L problem

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n(0) = 0, \quad X_n(l) = 0 \end{cases} \implies \boxed{\text{S-L problem leads to } X_n \text{ and } \lambda_n} \implies \text{determined } X_n(x) \quad (2.4.9)$$

- This means we give a priori assumption on the form of u . $u = \sum u_n X_n$ where X_n satisfy this S-L problem (it is S-L problem for the corresponding homogeneous equation and homogeneous boundary problem).

Remark 2.4.3. • For non-trivial solutions, the simplest condition is $X_n(0) = 0$, which can be viewed as a priori assumption.

- We solve the equation for X_n with these boundary conditions, which leads to the eigenfunctions $\sin\left(\frac{n\pi x}{l}\right)$.
- We design and identify a useful tool—S-L problem—to achieve the solution.
- The X_n -ODE and its boundary conditions are simplest conditions for the system to hold, representing one possibility among many.
- This approach is like choosing one path from multiple possible routes to reach the destination.
- If the chosen path works, the task is completed; if not, the idea is deemed unfeasible.
- We replace X_n with $\sin\left(\frac{n\pi x}{l}\right)$ and consider the **second hope**: finding u_n .
- The coefficients f_n are found by taking the inner product of f with $\sin\left(\frac{n\pi x}{l}\right)$ and integrating.

If $X_n = \sin\frac{n\pi x}{l}$ solves the S-L problem, then (2.4.7) becomes:

$$\sum u_n'' X_n = -\sum a^2 \lambda u_n X_n + \sum f_n X_n \implies \sum (u_n'' - f_n + a^2 \lambda u_n) X_n = 0$$

- **Determining Coefficients:** Can we conclude that each coefficient is zero from the equation being zero? Yes, by using (1) a **Fourier series expansion** by solving the coefficient; (2) $\{X_n\}$ is a complete and linear independent (orthogonal in fact) basis. → we call this “compare the coefficient”.

3. (**Using the initial data**) Comparing coefficients, we get:

$$u_n'' + a^2 \lambda u_n - f_n = 0$$

Using the initial data:

$$\begin{aligned} \left\{ \begin{array}{l} u(x, 0) = \sum u_n(0) X_n(x) = 0 \implies u_n(0) = 0 \\ u_t(x, 0) = \sum u'_n(0) X_n(x) = 0 \implies u'_n(0) = 0 \end{array} \right. \\ \implies \left\{ \begin{array}{l} u_n'' + a^2 \lambda u_n - f_n = 0 \\ u_n(0) = 0, \quad u'_n(0) = 0 \end{array} \right. \implies \text{determined } u_n(t) \end{aligned} \quad (2.4.10)$$

Considering (2.4.9) and (2.4.10), we arrive at

$$u(t, x) = \sum u_n(t) X_n(x) \rightarrow \text{determined}$$

- **Eigenfunction Method:** This process is essentially the eigenfunction method, which has been formalized by mathematicians to provide a clear and systematic approach. We list below.

Summary of the Method of Eigenfunction Expansion

1. Consider the eigenfunction system corresponding to the homogeneous problem (i.e., in (2.4.4), we set $f = 0$ and solve this homogeneous problem first, and we only need). [**Homogeneous Eigenfunction System**]
 - **Approach:** Solve the homogeneous problem by setting $f = 0$ in (2.4.4) and using the separation of variables method upto solving the S-L problem (referred to as the "2.5 step") to obtain the eigenfunctions X_n .
 - **Significance:** Once the eigenfunctions X_n are obtained, they form a **complete set** of eigenfunctions for the homogeneous problem.
2. Assume the solution of the non-homogeneous problem can be expanded using the eigenfunction series. [**Assumed Series Solution**]
3. Expand the free term using the eigenfunction series. [**Expansion of Free Term**]
 - Expand both u and f using the eigenfunctions X_n . This is analogous to expressing vectors in a linear space using a chosen basis, any function related to the equation can be expanded using the eigenfunctions.
4. Substitute the particular solution and the series form of the free term into the non-homogeneous equation, compare coefficients to obtain ODEs. [**Comparison of Coefficients**]

5. Use initial conditions to obtain the initial conditions for the ODEs. **[Initial Value (Boundary Value) Transformation]**
6. Solve the ODEs to obtain the coefficient functions. **[Solving ODEs]**

Basis decomposition method in algebra and PDEs

- In linear algebra and analytic geometry, to describe a linear space, it is essential to first establish a basis or a coordinate system. Only then can we effectively formulate and solve algebraic problems within this framework. Similarly, in our approach here, we first select a basis function system. Subsequently, we formulate all partial differential equation (PDE) problems within this basis or in terms of the corresponding eigenfunctions.

For this, we first discuss the **forced vibration problem** with **homogeneous boundary conditions** and **zero initial conditions**:

$$u_{tt} = a^2 u_{xx} + f(x, t) \quad (0 < x < l, t > 0), \quad (2.4.11)$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad (2.4.12)$$

$$u(x, 0) = u_t(x, 0) = 0. \quad (2.4.13)$$

Solution. **[1. Homogeneous Eigenfunction System]** From the knowledge in Section ??, the corresponding homogeneous equation to (2.4.11) is:

$$u_{tt} = a^2 u_{xx},$$

which satisfies the homogeneous boundary conditions (2.4.12) with the eigenfunctions satisfying:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

The eigenfunctions are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \quad (n = 1, 2, \dots).$$

Thus, the eigenfunction series that satisfies the homogeneous boundary conditions (2.4.12) for the corresponding homogeneous equation to (2.4.11) is $\{\sin\left(\frac{n\pi x}{l}\right)\}$.

[2. Assumed Series Solution] Assume the solution is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad (2.4.14)$$

where $u_n(t)$ is a function to be determined with respect to t .

- **Analogy to Linear Algebra:** Similar to linear algebra, where any vector can be decomposed using an orthogonal basis, functions in the equation can be decomposed using the eigenfunctions.

- **Key Idea:** The eigenfunction system provides a basis for decomposing all relevant functions in the problem.

[3. Expansion of Free Term] Expand the free term $f(x, t)$ in the equation into a Fourier series:

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad (2.4.15)$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin\left(\frac{n\pi x}{l}\right) dx \quad (n = 1, 2, \dots).$$

[4. Comparison of Coefficients] Substituting (2.4.14) and (2.4.15) into equation (2.4.11) gives:

$$\sum_{n=1}^{\infty} \left[u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) - f_n(t) \right] \sin\left(\frac{n\pi x}{l}\right) = 0, \quad (\text{wave equations in series version})$$

from which we obtain⁵:

$$u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \quad (n = 1, 2, \dots).$$

- **Coefficient Comparison:** By “comparing coefficients”, we can determine that the coefficients are zero. When mentioning “comparing coefficients”, we mean
 - **Fourier Expansion Interpretation:** This process is analogous to a **Fourier expansion of zero**, where the coefficients are determined using inner products.
 - The definition of Linear Independence and the completeness of $\{X_n\}$ lead to that the coefficients are zero.
- **Orthogonality and Linear Independence:**
 - Orthogonality implies linear independence.
 - For finite sums, linear independence can be used to conclude that coefficients are zero.
 - For infinite sums, roughly **completeness** is required to ensure that any functions in a function space can be represented by the basis function system. **Analogy to Linear Algebra:**
 - * In linear algebra, a complete set of linearly independent vectors can be used to represent any vector in the space.
 - * If the set is incomplete (e.g., missing one vector), some vectors cannot be represented.

⁵Completeness: refer to https://en.wikipedia.org/wiki/Orthonormal_basis#Incomplete_orthogonal_sets

[5. Initial Value Transformation] Using the initial conditions (2.4.13) in expression (2.4.14) gives:

$$u_n(0) = 0, \quad u'_n(0) = 0. \quad (\text{Coefficient comparison} - \{X_n\} \text{ orthogonality})$$

- Recall that the exchange between differentiation and summation can be treated as an a priori assumption.

[6. Solving ODEs] Thus, we obtain the following initial value problem for the ordinary differential equation:

$$\begin{cases} u''_n(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \\ u_n(0) = u'_n(0) = 0, \quad (n = 1, 2, \dots). \end{cases} \quad (52)$$

By applying the method of variation of parameters or Laplace transform in ordinary differential equations, the solution to problem (52) is

$$u_n(t) = \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin\left(\frac{n\pi a}{l}(t-\tau)\right) d\tau \quad (n = 1, 2, \dots).$$

Substituting u_n into

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right),$$

We obtain the solution to the mixed value problem (2.4.11)-(2.4.13).

Summary of Solving Non-Homogeneous ODEs

- **Solution Form:** The solution to the second order non-homogeneous ODE (52) involves a convolution of the **forcing term** f_n with a sine function, scaled by a coefficient.
- **Physical Interpretation:** This ODE describes vibrations, similar to the wave equation for a vibrating string. The **sine function** in the solution represents the **oscillatory nature** of the system.
- **Convolution with Sine Function:** The non-homogeneous term f_n (representing an external force) is convolved with a sine function. This convolution represents the response of the system to the external force.

Solution Using Laplace Transform

$$\begin{cases} u'(t) + k^2 u(t) = f(t), \\ u(0) = 0. \end{cases} \quad (2.4.16)$$

Solution. Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get

$$L[u'(t)] = sL[u(t)] - u(0)$$

$$sU(s) - u(0) + k^2U(s) = F(s)$$

$$sU(s) + k^2U(s) = F(s)$$

Thus

$$U(s) = \frac{1}{s+k^2}F(s).$$

$$L[e^{-at}] = \frac{1}{s+a}$$

Taking the inverse Laplace transform gives

$$u(t) = f(t) * e^{-k^2t} = \int_0^t f(\tau)e^{-k^2(t-\tau)}d\tau.$$

Remark 2.4.4.

- Advantages of Integral Transforms:

- Derivatives and multipliers are interchanged.
- Convolutions and products are interchanged.
- Integral transformations convert ordinary differential equations (ODEs) into algebraic equations.
- For partial differential equations (PDEs), they are similar and we will discuss this in Chap. 3.

- **Analogy to Heat Equation:** For the ODE (2.4.16), it is similar to the heat equation, the solution involves an **exponential decay** term. The non-homogeneous term f would be convolved with an **exponential function** to describe the heat distribution.
- **General Form:** The solution to non-homogeneous equations can generally be expressed as the convolution of the forcing term f with a characteristic function (e.g., sine for vibrations, exponential for heat).

Another Method to Solve

$$\begin{cases} u'(t) + k^2u(t) = f(t), \\ u(0) = 0. \end{cases}$$

Solution. Directly use the general solution formula of the first-order linear differential equation to get

$$u(t) = e^{-k^2 t} \left(\int_0^t e^{k^2 \tau} f(\tau) d\tau + C \right)$$

Using the condition $u(0) = 0$ gives $C = 0$. So the solution to the original problem can be expressed as

$$u(t) = \int_0^t f(\tau) e^{-k^2(t-\tau)} d\tau.$$

Solution Using Laplace Transform

$$\begin{cases} u''(t) + k^2 u(t) = f(t), \\ u(0) = 0, u'(0) = 0. \end{cases} \quad (2.4.17)$$

Solution: Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get

$$L[u''(t)] = s^2 L[u(t)] - su(0) - u'(0) \quad (2.4.18)$$

$$s^2 U(s) - su(0) - u'(0) + k^2 U(s) = F(s) \quad (2.4.19)$$

$$s^2 U(s) + k^2 U(s) = F(s) \quad (2.4.20)$$

Thus

$$U(s) = \frac{1}{k} \frac{k}{s^2 + k^2} F(s). \quad (2.4.21)$$

$$L[\sin at] = \frac{a}{s^2 + a^2} \quad (2.4.22)$$

Taking the inverse Laplace transform gives

$$u(t) = \frac{1}{k} f(t) * \sin kt = \frac{1}{k} \int_0^t f(\tau) \sin k(t-\tau) d\tau. \quad (2.4.23)$$

Recap on Variation of Parameters Method: Solving Second-Order Linear Non-Homogeneous ODEs

Given the differential equation:

$$y'' + p(x)y' + q(x)y = f(x) \quad (2.4.24)$$

Steps:

1. Write down the corresponding homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0$$

$\Rightarrow y = c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are constants

2. **Assume** (a priori assumption) a particular solution (guess, trial method) for the non-homogeneous equation (2.4.24):

$$y = c_1(x)y_1(x) + c_2(x)y_2(x) \leftarrow \text{any function } y(x) \text{ can be written in this form} \quad (2.4.25)$$

- **Variation of Constants:** Treat constants c_1 and c_2 as functions.
- **Rationale:** Q: Why we have to write it as this form? A: This form and approach simplify the process of solving the differential equation. It is not the only way, but it is an efficient way. This form means y is completely unknown, which does not put any restriction on the solution y since by varying $c_1(x), c_2(x)$, y can be any function.
- **Simplification:** After setting up, you need to take the first and second derivatives of y and substitute them back into the original equation.
- **Advantages:** This method allows for significant simplification by eliminating many terms by using the homogeneous ODE since y_1 and y_2 solve it.
- **Non-uniqueness:** The form (2.4.25) of the solution does not represent a unique decomposition; different forms of $c_1(x)$ and $c_2(x)$ can lead to the same $y(x)$, since for some solution y and the given y_1, y_2 , there are two functions c_1, c_2 unknown but only one equation. Any function can be expressed in such a form.
- **Note:** $y_1(x)$ and $y_2(x)$ are **known functions**, while $c_1(x)$ and $c_2(x)$ are **unknown functions**.

3. **Goal:** Substituting (2.4.25) into (2.4.24), we obtain an equation of $c_1(x)$ and $c_2(x)$.

- **Question:** One equation, two unknowns $c_1(x), c_2(x) \leftarrow$ an **indeterminate system**.
- **Ideas:** We can **arbitrarily add an equation** of $c_1(x)$ and $c_2(x)$ to (1) **simplify calculations** and (2) make the system **well-posed** (two **hopes**).
- **Note:** There's an opportunity to supplement the equation! It's important to grasp this well!!!

By calculations, we obtain

$$y' = c_1(x)y'_1(x) + c_2(x)y'_2(x) + \underbrace{[c'_1(x)y_1(x) + c'_2(x)y_2(x)]}_{(*) \text{ Let it vanish to obtain an extra eq.!}}$$

- Because if $(*) \neq 0$, y'' will introduce c''_1 and c''_2 , making the processing more difficult. Why make things harder for ourselves?

- Recall that, with **two unknown functions** and **one equation**, there is **a degree of freedom**.
- Choosing an Equation:** We can choose any equation for c_1 and c_2 . We select $(*) = c'_1 y_1 + c'_2 y_2 = 0$ as our chosen equation.

Let:

$$c'_1(x)y_1(x) + c'_2(x)y_2(x) = 0. \quad (2.4.26)$$

- Simplification Strategy:** By imposing this new equation, certain terms will cancel out, potentially simplifying the problem.

Thus:

$$y'' = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c'_1(x)y_1'(x) + c'_2(x)y_2'(x)$$

- Differentiation:** After differentiating, the second derivative of y will only involve the first derivatives of c_1 and c_2 , not the second derivatives.

Substituting y' and y'' into the original equation (2.4.24) yields

$$c_1 y_1'' + c_2 y_2'' + c'_1 y_1' + c'_2 y_2' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2) = f$$

Simplifying, we get:

$$\cancel{c_1(y_1'' + py_1' + qy_1)} + \cancel{c_2(y_2'' + py_2' + qy_2)} + c'_1 y_1' + c'_2 y_2' = f$$

- Substitution Back:** Substitute y'' and y' back into the original equation and simplify. This will utilize the fact that y_1 and y_2 are solutions to the homogeneous equations.

$$\Rightarrow c'_1 y_1' + c'_2 y_2' = f. \quad (2.4.27)$$

- Resulting Simplicity:** The process simplifies the equation to just two terms.

From (2.4.24) and (2.4.27):

$$\begin{cases} c'_1 y_1 + c'_2 y_2 = 0 \\ c'_1 y_1' + c'_2 y_2' = f \end{cases} \xrightarrow{y_1, y_2 \text{ are known}} \begin{cases} c'_1(x) \\ c'_2(x) \end{cases} \xrightarrow{\text{Integration}} \begin{cases} c_1(x) \\ c_2(x) \end{cases}$$

- Equation System:** We now have two equations (including the one we imposed) and two unknowns, allowing us to solve for c_1 and c_2 .

Summary

- **Indeterminacy:** With two unknown functions and only one equation, the system is indeterminate, leading to potentially multiple solutions.
- **Freedom of Choice:** You can arbitrarily choose one equation for c_1 and c_2 to solve the system.
- **Utilization:** The freedom in choosing c_1 and c_2 should be used to facilitate the solution process.
- **Purpose of Imposing Equations:** Some students may **question why we impose certain equations.** The solutions c_1 and c_2 that we derive are only **one result among many possibilities.** As mentioned before, y can have many decomposition of the form (2.4.25), the **imposed equations** (2.4.26) **help us pick one specific** c_1, c_2 .
- **Non-uniqueness of Solutions:** The decomposition c_1 and c_2 may not necessarily be the only solutions. There could be other valid solutions.
- **Validation of Solutions:** The solutions we obtain can indeed solve the equation, proving their applicability.
- Using the **degree of freedom** to impose an equation for solving is similar to the idea of **selecting an S-L problem** in the method of eigenfunctions. This is a path we artificially choose that is feasible, but it is not the only way. This represents the intrinsic conceptual connection between the method of eigenfunctions and the variation of parameters method.

Summary

For the initial value problem of ordinary differential equations, we have two cases:

1. Case 1: The differential equation is:

$$u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \quad (2.4.28)$$

with initial conditions:

$$u_n(0) = u_n'(0) = 0, \quad (n = 1, 2, \dots).$$

The solution to this problem is given by:

$$u_n(t) = \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin\left(\frac{n\pi a}{l}(t - \tau)\right) d\tau \quad (n = 1, 2, \dots). \quad (2.4.29)$$

2. Case 2: The differential equation is:

$$u_n'(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t)$$

with initial condition:

$$u_n(0) = 0, \quad (n = 1, 2, \dots).$$

The solution to this problem is given by:

$$u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)} d\tau \quad (n = 1, 2, \dots). \quad (2.4.30)$$

These solutions are derived using the method of variation of parameters and Laplace transforms, which are powerful tools for solving linear differential equations with non-homogeneous terms and initial conditions.

Ex 2.4.1. Solve the following problem

$$\begin{cases} u_{tt} = a^2 u_{xx} + A \sin \omega t \cos \frac{\pi x}{l} & (0 < x < l, t > 0), \\ u_x(0, t) = 0, u_x(l, t) = 0, \\ u(x, 0) = u_t(x, 0) = 0. \end{cases}$$

where A, ω are constants.

- **Similarity:** Non-homogeneous equation + Homogeneous boundary and initial data → the eigenfunction method.
- **Differences:** Both boundaries are second-type boundaries → S-L problem.

Solution. [1. Homogeneous Eigenfunction System] From the knowledge of previous sections, it is known that the corresponding homogeneous equation of the original equation is

$$u_{tt} = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous second type boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0.$$

$$\Rightarrow X_0(x) = B_0, \quad X_n(x) = A_n \cos \frac{n\pi x}{l} \quad (n = 1, 2, \dots).$$

Therefore, it is known that the corresponding homogeneous equation of the original equation and the eigenfunction series that satisfy the homogeneous second type boundary conditions are $\{\cos \frac{n\pi x}{l}\}_{n=0}^{\infty}$.

[2. Assumed Series Solution] Let the solution be

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{n\pi x}{l},$$

where $u_n(t)$ is a function to be determined with respect to t .

[3. Expansion of Free Term] $A \sin \omega t \cos \frac{\pi x}{l}$ has already in the right form.

[4. Comparison of Coefficients] Substitute $u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{n\pi x}{l}$ into the original equation to simplify and obtain

$$\sum_{n=0}^{\infty} \left[u_n'' + \left(\frac{n\pi a}{l} \right)^2 u_n \right] \cos \frac{n\pi x}{l} = A \sin \omega t \cos \frac{\pi x}{l},$$

Comparing the coefficients on both sides of the equation gives

$$u_n'' + \left(\frac{n\pi a}{l} \right)^2 u_n = 0 \quad (n \neq 1), \quad u_1'' + \left(\frac{\pi a}{l} \right)^2 u_1 = A \sin \omega t.$$

[5. Initial Value Transformation] In $u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{n\pi x}{l}$, using the initial conditions gives

$$\begin{cases} \sum_{n=0}^{\infty} u_n(0) \cos \frac{n\pi x}{l} = 0 \implies u_n(0) = 0, & (n = 0, 1, 2, \dots). \\ \sum_{n=0}^{\infty} u_n'(0) \cos \frac{n\pi x}{l} = 0 \implies u_n'(0) = 0. \end{cases}$$

Thus, we obtain two sets of initial value problems for ordinary differential equations

$$\begin{cases} u_n'' + \left(\frac{n\pi a}{l} \right)^2 u_n = 0 & (n \neq 1), \\ u_n(0) = u_n'(0) = 0. \end{cases} \quad \text{and} \quad \begin{cases} u_1'' + \left(\frac{\pi a}{l} \right)^2 u_1 = A \sin \omega t, \\ u_1(0) = u_1'(0) = 0. \end{cases}$$

1. First, when $n \neq 1$, using the general solution formula we have

$$u_n(t) = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t$$

and

$$u_n'(t) = -A_n \frac{n\pi a}{l} \sin \frac{n\pi a}{l} t + B_n \frac{n\pi a}{l} \cos \frac{n\pi a}{l} t$$

Using the conditions $u_n(0) = u_n'(0) = 0$, we get

$$u_n(t) = 0.$$

2. when $n = 1$, by (2.4.29)

$$u_1(t) = \frac{l}{\pi a} \int_0^t A \sin \omega \tau \sin \frac{\pi a(t-\tau)}{l} d\tau.$$

Since

$$\begin{aligned} u_1(t) &= \frac{l}{\pi a} \int_0^t A \sin \omega \tau \sin \frac{\pi a(t-\tau)}{l} d\tau \\ &= \frac{Al}{2\pi a} \left\{ \int_0^t \cos \left[\left(\omega + \frac{\pi a}{l} \right) \tau - \frac{\pi a}{l} t \right] d\tau - \int_0^t \cos \left[\left(\omega - \frac{\pi a}{l} \right) \tau + \frac{\pi a}{l} t \right] d\tau \right\} \\ &= \frac{Al}{2\pi a} \left\{ \frac{1}{\omega + \frac{\pi a}{l}} \sin \left[\left(\omega + \frac{\pi a}{l} \right) \tau - \frac{\pi a}{l} t \right] \Big|_{\tau=0}^{\tau=t} - \frac{1}{\omega - \frac{\pi a}{l}} \sin \left[\left(\omega - \frac{\pi a}{l} \right) \tau + \frac{\pi a}{l} t \right] \Big|_{\tau=0}^{\tau=t} \right\} \\ &= \frac{Al}{2\pi a} \left(\frac{\sin \omega t + \sin \frac{\pi a}{l} t}{\omega + \frac{\pi a}{l}} - \frac{\sin \omega t - \sin \frac{\pi a}{l} t}{\omega - \frac{\pi a}{l}} \right) \\ &= \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - (\frac{\pi a}{l})^2} \left(\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t \right). \end{aligned}$$

Substituting

$$\begin{cases} u_n(t) = 0, & n \neq 1 \\ u_1(t) = \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - (\frac{\pi a}{l})^2} (\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t) \end{cases}$$

into

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{n\pi x}{l},$$

we obtain the solution as

$$u(x, t) = \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - (\frac{\pi a}{l})^2} \left(\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t \right) \cdot \cos \frac{\pi x}{l}.$$

2.4.2 Heat Conduction in a Finite Rod with a Heat Source

First, let's consider the following problem

$$\begin{cases} u_t = a^2 u_{xx} + f(x, t) & (0 < x < l, t > 0), \\ u(0, t) = 0, u(l, t) = 0, \\ u(x, 0) = \varphi(x). \end{cases} \quad (2.4.31)$$

At this point, the heat conduction phenomenon is caused by two parts: one is the internal heat **source**, and the other is the **initial temperature** of the rod. Thus, this heat conduction phenomenon can be regarded as a **combination** of heat conduction caused solely by the internal heat source and heat conduction caused solely by the initial temperature.

Therefore, we can assume the solution to problem (2.4.31) to be

$$u(x, t) = v(x, t) + w(x, t),$$

where $v(x, t)$ represents the temperature function caused solely by the **internal heat source**; and $w(x, t)$ represents the temperature function caused solely by the **initial temperature**; $v(x, t)$ and $w(x, t)$ satisfy the following mixed value problems respectively:

$$\begin{cases} v_t = a^2 v_{xx} + f(x, t) & (0 < x < l, t > 0), \\ v(0, t) = 0, v(l, t) = 0, \\ v(x, 0) = 0. \end{cases}$$

and

$$\begin{cases} w_t = a^2 w_{xx} & (0 < x < l, t > 0), \\ w(0, t) = 0, w(l, t) = 0, \\ w(x, 0) = \varphi(x). \end{cases}$$

- Isolating the non-homogeneity—utilizing the principle of linear superposition.

For this, we first discuss the case of **homogeneous boundary** conditions and **zero initial** conditions, taking the temperature at both ends maintained at 0 degrees as an example:

$$u_t = a^2 u_{xx} + f(x, t) \quad (0 < x < l, t > 0), \quad (2.4.32)$$

$$u(0, t) = 0, \quad u(l, t) = 0, \quad (2.4.33)$$

$$u(x, 0) = 0. \quad (2.4.34)$$

We still use the **method of eigenfunctions** to solve this mixed value problem.

Solution. [1. Homogeneous Eigenfunction System] From the knowledge of Section 2.2, the homogeneous equation corresponding to (2.4.32)

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous first type boundary conditions satisfy (2.4.33)

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation (2.4.32) that also satisfies the homogeneous first type boundary conditions (2.4.33) is $\{\sin \frac{n\pi x}{l}\}$.

$$\begin{cases} u_t = a^2 u_{xx} + f(x, t) & (0 < x < l, t > 0), \\ u(0, t) = 0, \quad u(l, t) = 0, \\ u(x, 0) = 0. \end{cases}$$

[2. Assumed Series Solution] Expand the solution of the boundary value problem in terms of x using the eigenfunction series (Fourier sine series):

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}, \quad (2.4.35)$$

[3. Expansion of Free Term] Expand the free term $f(x, t)$ in the equation using the same eigenfunction series:

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \quad (2.4.36)$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \quad (n = 1, 2, \dots).$$

[4. Comparison of Coefficients] Substituting (2.4.35)-(2.4.36) into equation (2.4.32) yields

$$\sum_{n=1}^{\infty} \left[u'_n(t) + \left(\frac{n\pi a}{l} \right)^2 u_n(t) - f_n(t) \right] \sin \frac{n\pi x}{l} = 0,$$

from which we get

$$u'_n(t) + \left(\frac{n\pi a}{l} \right)^2 u_n(t) = f_n(t) \quad (n = 1, 2, \dots).$$

[5. Initial Value Transformation] Using the initial condition (2.4.34) in expression (2.4.43) gives

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{l} = 0 \quad \Rightarrow \quad u_n(0) = 0, \quad (n = 1, 2, \dots).$$

[6. Solving ODEs] Thus, we obtain the following initial value problem for ordinary differential equations

$$\begin{cases} u'_n(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \\ u_n(0) = 0, \quad (n = 1, 2, \dots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problem (2.4.28) is

$$u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)} d\tau \quad (n = 1, 2, \dots).$$

Substituting

$$u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)} d\tau \quad (n = 1, 2, \dots).$$

into

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l},$$

we obtain the solution to the boundary value problem (2.4.32)-(2.4.34).

Ex 2.4.2. Solve the following problem

$$\begin{cases} u_t = a^2 u_{xx} + A & (0 < x < l, t > 0), \\ u(0, t) = 0, u_x(l, t) = 0, \\ u(x, 0) = 0. \end{cases} \quad (2.4.37)$$

where A is a constant.

Solution. [1. Homogeneous Eigenfunction System] The corresponding homogeneous equation of the original equation

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

Then

$$X_n(x) = B_n \sin \frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \dots).$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation and satisfying the homogeneous boundary conditions is $\left\{ \sin \frac{(2n+1)\pi x}{2l} \right\}$.

[2. Assumed Series Solution] Let the solution be

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi x}{2l}. \quad (2.4.38)$$

[3. Expansion of Free Term] Then expand A into the Fourier sine series according to the above eigenfunction series

$$A = \sum_{n=0}^{\infty} A_n(t) \sin \frac{(2n+1)\pi x}{2l}, \quad (2.4.39)$$

where

$$A_n(t) = \frac{2}{l} \int_0^l A \sin \frac{(2n+1)\pi x}{2l} dx = \frac{4A}{(2n+1)\pi}.$$

[4. Comparison of Coefficients] Substituting (2.4.43)-(2.4.44) into equation (2.4.42) yields

$$\sum_{n=1}^{\infty} \left[u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) - \frac{4A}{(2n+1)\pi} \right] \sin \frac{(2n+1)\pi x}{2l} = 0,$$

from which we get

$$u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) = \frac{4A}{(2n+1)\pi}. \quad (2.4.40)$$

[5. Initial Value Transformation] Using the initial condition in expression (2.4.43) gives

$$\sum_{n=0}^{\infty} u_n(0) \sin \frac{(2n+1)\pi x}{2l} = 0 \Rightarrow u_n(0) = 0, \quad (n = 0, 1, 2, \dots). \quad (2.4.41)$$

[6. Solving ODEs] Thus, we obtain the following initial value problem for ordinary differential equations

$$\begin{cases} u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) = \frac{4A}{(2n+1)\pi} \\ u_n(0) = 0, \quad (n = 0, 1, 2, \dots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problems (2.4.45)-(2.4.46) is

$$\begin{aligned} u_n(t) &= \int_0^t \frac{4A}{(2n+1)\pi} e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2(t-\tau)} d\tau \\ &= \frac{4A}{(2n+1)\pi} \int_0^t e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2(t-\tau)} d\tau \\ &= \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\}. \end{aligned}$$

Substituting

$$u_n(t) = \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\}$$

into

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi}{2l} x,$$

we obtain the solution to the problem

$$u(x, t) = \sum_{n=0}^{\infty} \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\} \sin \frac{(2n+1)\pi}{2l} x.$$

Recall:

- Non-homogeneous equations + homogeneous boundary + homogeneous initial conditions
→ The eigenfunction method.

Ex 2.4.3. Solve the following problem

$$\begin{cases} u_t = a^2 u_{xx} + A & (0 < x < l, t > 0), \\ u(0, t) = 0, u_x(l, t) = 0, \\ u(x, 0) = 0. \end{cases} \quad (2.4.42)$$

where A is a constant.

Solution. [1. Homogeneous Eigenfunction System] The corresponding homogeneous equation of the original equation

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

Then

$$X_n(x) = B_n \sin \frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \dots).$$

(Note:) or written as

$$X_n(x) = B_n \sin \frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \dots).$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation and satisfying the homogeneous boundary conditions is $\left\{ \sin \frac{(2n+1)\pi x}{2l} \right\}$.

[2. Assumed Series Solution] Let the solution be

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi x}{2l}. \quad (2.4.43)$$

[3. Expansion of Free Term] Then expand A into the Fourier sine series according to the above eigenfunction series

$$A = \sum_{n=0}^{\infty} A_n(t) \sin \frac{(2n+1)\pi x}{2l}, \quad (2.4.44)$$

where

$$A_n(t) = \frac{2}{l} \int_0^l A \sin \frac{(2n+1)\pi x}{2l} dx = \frac{4A}{(2n+1)\pi}.$$

Recall:

- The concept is analogous to expressing any vector in terms of a chosen set of basis vectors, similar to how coordinates are used in linear algebra and analytic geometry.

[4. Comparison of Coefficients] Substituting (2.4.43)-(2.4.44) into equation (2.4.42) yields

$$\sum_{n=1}^{\infty} \left[u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) - \frac{4A}{(2n+1)\pi} \right] \sin \frac{(2n+1)\pi x}{2l} = 0,$$

from which we get (*orthogonality of* $\sin \frac{(2n+1)\pi x}{2l}$)

$$u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) = \frac{4A}{(2n+1)\pi}. \quad (2.4.45)$$

Memorizing:

- The first order ODE \rightarrow heat behavior \rightarrow (Source term)* $\exp(-\text{coef.} \times t)$;
- The second order ODE \rightarrow oscillation behavior $\rightarrow \frac{1}{\text{coef.}}(\text{Source term}) * \sin(\text{coef.} \times t)$.

[5. Initial Value Transformation] Using the initial condition in expression (2.4.43) gives

$$\sum_{n=0}^{\infty} u_n(0) \sin \frac{(2n+1)\pi x}{2l} = 0 \Rightarrow u_n(0) = 0, \quad (n = 0, 1, 2, \dots). \quad (2.4.46)$$

[6. Solving ODEs] Thus, we obtain the following initial value problem for ordinary differential equations

$$\begin{cases} u'_n(t) + \left(\frac{(2n+1)\pi a}{2l} \right)^2 u_n(t) = \frac{4A}{(2n+1)\pi} \\ u_n(0) = 0, \quad (n = 0, 1, 2, \dots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problems (2.4.45)-(2.4.46) is

$$\begin{aligned} u_n(t) &= \int_0^t \frac{4A}{(2n+1)\pi} e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2(t-\tau)} d\tau \\ &= \frac{4A}{(2n+1)\pi} \int_0^t e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2(t-\tau)} d\tau \\ &= \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\}. \end{aligned}$$

Substituting

$$u_n(t) = \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\}$$

into

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) \sin \frac{(2n+1)\pi}{2l} x,$$

we obtain the solution to the problem

$$u(x, t) = \sum_{n=0}^{\infty} \frac{16Al^2}{(2n+1)^3\pi^3a^2} \left\{ 1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t} \right\} \sin \frac{(2n+1)\pi}{2l} x.$$

2.4.3 Poisson Equation (Non-homogeneous Laplace Equation)

Use the **method of eigenfunctions** to solve the boundary value problem of the non-homogeneous Laplace equation. We illustrate the key points and steps of solving such problems through examples.

Ex 2.4.4. In a circle centered at the origin with a radius of 1, find the solution to the Poisson equation $u_{xx} + u_{yy} = -2x$ that satisfies the boundary condition $u|_{x^2+y^2=1} = 0$.

Note:

- If the boundary condition is given by $u|_{x^2+y^2=1} = f$, our first step is to separate the non-homogeneous part into two systems. We then solve each system using the method of separation of variables and eigenfunction expansion, respectively.
- The rectangular, strip, and sector domains can be analyzed and solved in a similar manner.

Solution. Since the region is a circular domain, perform a polar coordinate transformation:

$$x = r \cos \theta, \quad y = r \sin \theta,$$

and denote $\bar{u}(r, \theta) = u(r \cos \theta, r \sin \theta)$, then the problem is reduced to:

$$\bar{u}_{rr} + \frac{1}{r} \bar{u}_r + \frac{1}{r^2} \bar{u}_{\theta\theta} = -2r \cos \theta \quad (0 < r < 1), \quad (2.4.47)$$

$$\bar{u}|_{r=1} = 0. \quad (2.4.48)$$

Recall:

- For the problem discussed last time regarding solving the Laplace equation in a circular region, transforming to polar coordinates results in the **loss of two pieces of information**. To compensate, we add two conditions: **boundedness at the origin** and **periodicity**.

[1. Homogeneous Eigenfunction System] From the discussion in Section 2.3, the corresponding homogeneous equation to (2.4.47) that satisfies the single-valued condition has eigenfunctions satisfying:

$$\Phi'' + \lambda \Phi = 0, \quad \Phi(\theta + 2\pi) = \Phi(\theta).$$

Thus, the eigenfunction series corresponding to (2.4.47) that also satisfies the single-valued condition is:

$$1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots, \cos n\theta, \sin n\theta, \dots$$

[2. Assumed Series Solution] By the method of eigenfunctions, assume the solution to equation (2.4.47) is:

$$\bar{u}(r, \theta) = \sum_{n=0}^{\infty} [a_n(r) \cos n\theta + b_n(r) \sin n\theta]. \quad (2.4.49)$$

[4. Comparison of Coefficients] Substituting (2.4.49) into equation (2.4.47) and simplifying, we get:

$$\sum_{n=0}^{\infty} \left[\left(a_n'' + \frac{1}{r} a_n' - \frac{n^2}{r^2} a_n \right) \cos n\theta + \left(b_n'' + \frac{1}{r} b_n' - \frac{n^2}{r^2} b_n \right) \sin n\theta \right] = -2r \cos \theta.$$

Comparing the coefficients of $\cos n\theta$ and $\sin n\theta$ on both sides of the equation, we obtain

$$a_1'' + \frac{1}{r} a_1' - \frac{1}{r^2} a_1 = -2r \quad (n = 1), \quad (2.4.50)$$

$$a_n'' + \frac{1}{r} a_n' - \frac{n^2}{r^2} a_n = 0 \quad (n \neq 1), \quad (\text{Euler}) \quad (2.4.51)$$

$$b_n'' + \frac{1}{r} b_n' - \frac{n^2}{r^2} b_n = 0. \quad (\text{Euler}) \quad (2.4.52)$$

[5. Initial (boundary) Value Transformation] Substitute boundary condition (2.4.48) into equation (2.4.49), then we have (“comparing the coefficient”)

$$a_n(1) = 0, \quad b_n(1) = 0. \quad (2.4.53)$$

According to the **boundedness** of the function $\bar{u}(r, \theta)$, it follows that

$$|a_n(0)| < +\infty, \quad |b_n(0)| < +\infty. \quad (2.4.54)$$

[6. Solving ODEs] Since equations (2.4.51) and (2.4.52) are homogeneous Euler equations, their general solutions are:

$$a_n(r) = A_n r^n + B_n r^{-n} \quad (n \neq 1),$$

and

$$b_n(r) = \bar{A}_n r^n + \bar{B}_n r^{-n}.$$

From condition (2.4.54), we get $B_n = 0$ and $\bar{B}_n = 0$. From condition (2.4.53), we get $A_n = 0$ and $\bar{A}_n = 0$. Therefore, $a_n(r) = 0$ (for $n \neq 1$), $b_n(r) = 0$.

Since equation (2.4.50) is a **non-homogeneous Euler** equation, its general solution is:

$$a_1(r) = c_1 r + c_2 r^{-1} - \frac{1}{4} r^3. \quad (2.4.55)$$

From condition (2.4.54), we get $c_2 = 0$. From condition (2.4.53), we get $c_1 = \frac{1}{4}$. Therefore,

$$a_1(r) = \frac{1}{4} r - \frac{1}{4} r^3.$$

Method of Variation of Parameters

We assume

$$a_1 = C_1(r)r + C_2(r)r^{-1}. \quad (2.4.56)$$

Then calculate

$$a_1' = C_1(r) - C_2(r)\frac{1}{r^2} \quad (2.4.57)$$

and

$$a_1'' = C_1'(r) - \frac{1}{r^2}C_2'(r) + \frac{2}{r^3}C_2(r) \quad (2.4.58)$$

Substituting (2.4.56)-(2.4.58) into (2.4.55), we have

$$\begin{aligned} & \Rightarrow C_1'(r) - C_2'(r)\frac{1}{r^2} + C_2(r)\frac{2}{r^3} + \underbrace{\frac{C_1}{r}}_{\cancel{C_1}} - C_2(r)\frac{1}{r^3} - \cancel{\frac{C_1}{r}} - C_2(r)\frac{1}{r^3} = -2r \\ & \Rightarrow \begin{cases} C_1'(r) - C_2'(r)\frac{1}{r^2} = -2r \\ C_1'(r) + C_2'(r)\frac{1}{r^2} = 0 \end{cases} \\ & \Rightarrow \begin{cases} \frac{C_2'(r)}{r^2} = r \Rightarrow C_2'(r) = r^3 \\ C_1'(r) = -r \end{cases} \\ & \Rightarrow C_1(r) = -\frac{1}{2}r^2, \quad C_2(r) = \frac{1}{4}r^4. \\ & \Rightarrow a_1(r) = -\frac{1}{2}r^3 + \frac{1}{4}r^3 = -\frac{1}{4}r^3. \end{aligned}$$

Trial and errors

(similar to the second method for Euler ODE) According to the **homogeneity** of every term, we know there is a special solution $a_1 = Cr^3$ and take it into (2.4.50), we obtain

$$\begin{aligned} & \begin{cases} a_i' = 3Cr^2 \\ a_i'' = 6Cr \end{cases} \\ & \Rightarrow 6Cr + \frac{1}{r} \cdot 3Cr^2 - \frac{1}{r^2}Cr^3 = -2r \\ & \Rightarrow 6Cr + 3Cr - Cr = -2r \\ & \Rightarrow 8C = -2 \Rightarrow C = -\frac{1}{4} \Rightarrow a_1 = -\frac{1}{4}r^3 \end{aligned}$$

The solution is then expressed as:

$$a_1(r) = \frac{1}{4}r - \frac{1}{4}r^3, \quad a_n(r) = 0 \quad (\text{for } n \neq 1), \quad b_n(r) = 0.$$

Substituting these into the series solution:

$$\bar{u}(r, \theta) = \sum_{n=0}^{\infty} [a_n(r) \cos n\theta + b_n(r) \sin n\theta].$$

The solution to the problem is:

$$\bar{u}(r, \theta) = \frac{1}{4}(1 - r^2)r \cos \theta.$$

Converting to Cartesian coordinates:

$$u(x, y) = \frac{1}{4}[1 - (x^2 + y^2)]x.$$

Summary:

1. **Basic Six Steps of Eigenfunction Method.**
2. **Solution of Non-Homogeneous Euler ODE:**
 - Use the method of undetermined coefficients or variation of parameters.
 - Find a particular solution for the non-homogeneous part.
 - Combine with the solution of the corresponding homogeneous equation.
3. **Polar Coordinates System:**
 - Two conditions are typically missing in polar form expressions.
 - Additional implicit conditions are required to fully specify the system.

Alternative Solution Method (Trial and errors)

The Poisson equation in polar coordinates is given by:

$$\begin{cases} \bar{u}_{rr} + \frac{1}{r}\bar{u}_r + \frac{1}{r^2}\bar{u}_{\theta\theta} = -2r \cos \theta & (0 < r < 1), \\ \bar{u}|_{r=1} = 0. \end{cases}$$

Idea: If we know a particular solution w of the Poisson equation, then by making the function transformation $\bar{u} = v + w$, we can transform the Poisson equation into the Laplace equation. Then, by solving the boundary value problem of the Laplace equation, we can obtain the boundary value problem of the Poisson equation.

The **idea** is to use a known particular solution w to **transform the Poisson equation into a Laplace equation**

The particular solution carries the burden of all the non-homogeneity. Idea: find something to carry the burden of the non-homogeneity → will be used in next section for the non-homogeneity of the boundary.

$$w = -\frac{1}{4}r^3 \cos \theta. \quad (\text{Similar to the previous and using the free term, there is a } \cos \theta \rightarrow Cr^3 \cos \theta)$$

Let $\bar{u}(r, \theta) = v(r, \theta) + w(r, \theta)$, then the problem can be transformed into:

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & (0 < r < 1), \\ v|_{r=1} = \frac{1}{4} \cos \theta. \quad (\text{Cost!}) \end{cases}$$

Assume (trial and errors, by observing $v|_{r=1} = \frac{1}{4} \cos \theta$) the solution to the Laplace equation is:

$$v(r, \theta) = Ar \cos \theta + B.$$

To satisfy the boundary condition:

$$v(1, \theta) = A \cos \theta + B = \frac{1}{4} \cos \theta.$$

Solving for A and B :

$$B = 0, \quad A = \frac{1}{4}, \quad \Rightarrow \quad v(r, \theta) = \frac{1}{4} r \cos \theta.$$

The final solution to the boundary value problem is:

$$\bar{u}(r, \theta) = v(r, \theta) + w(r, \theta) = \frac{1}{4} r \cos \theta - \frac{1}{4} r^3 \cos \theta = \frac{1}{4} (1 - r^2) r \cos \theta.$$

Supplementary Information

For the Poisson equation boundary value problem:

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = F(r, \theta), & (0 < r < r_0), \\ u|_{r=r_0} = f(\theta). \end{cases}$$

Approach 1: Decompose the solution $u(r, \theta)$ into two parts:

$$u(r, \theta) = v(r, \theta) + w(r, \theta),$$

where $v(r, \theta)$ satisfy (use the method of eigenfunctions)

$$\begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = F(r, \theta), & (0 < r < r_0), \\ v|_{r=r_0} = 0. \end{cases}$$

and $w(r, \theta)$ satisfy (use separation of variables)

$$\begin{cases} w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0, & (0 < r < r_0), \\ w|_{r=r_0} = f(\theta). \end{cases}$$

Approach 2: (carry the burden of the non-homogeneity) 1. Find a particular solution $w(r, \theta)$ such that:

$$u(r, \theta) = v(r, \theta) + w(r, \theta),$$

2. Transform the Poisson equation into a Laplace equation:

$$\begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0, & (0 < r < r_0), \\ v|_{r=r_0} = f(\theta) - w(r_0, \theta). \end{cases}$$

Solve this problem using separation of variables or trial methods.

2.5 Problems with Non-homogeneous Boundary Conditions

In this section, we discuss the solution methods for problems with **non-homogeneous boundary** conditions.

Basic principle: Regardless of whether the equation is homogeneous or non-homogeneous, choose an auxiliary function $w(x, t)$, and through the function substitution $u(x, t) = v(x, t) + w(x, t)$, make the **boundary conditions** for the new unknown function $v(x, t)$ **homogeneous**. Using w carry the burden of the non-homogeneity of the boundary.

We will use the following problem as an example to illustrate the method of selecting the function substitution (also known as the **auxiliary function method**).

1. Auxiliary Function Concept:

- The **idea** is straightforward: find a specific function to act as a **scapegoat**, attributing all the **faults** to it, in order to **save the other** terms.

2. Purpose of the Scapegoat Function w :

- The given function w (we should find it) should satisfy boundary conditions u_1 and u_2 .
- It carries all the burdens of all **inhomogeneous** boundary conditions.

3. Decomposition of u :

- Write u as $v + w$.
- Use **linear superposition** to let w (given) handle the inhomogeneous boundaries.

4. Properties of w :

- w is a given function (should find).

5. Boundary Conditions for w :

- **Hope** to find a function w satisfying $w(0, t) = u_1$ and $w(l, t) = u_2$.

6. Simplification for v :

- After subtracting w from u , v satisfies a **homogeneous boundary**.

7. Finding w :

- w can be **any** function that **meets the boundary conditions** (this is the only requirement on w , no other conditions, so w can be chosen quite **freely**).
- For example, w can be selected as **linear** functions or **parabolic curves** passing through the boundary points.
- The auxiliary function is **not unique**.

8. Principle:

- The **simplest** the auxiliary function.

9. Standardization for Exams:

- To ensure consistency, students should use a specific method to choose w .
- Suggested method: use a straight line between the two boundary points.

Summary:

- An auxiliary function as a scapegoat **carries all the non-homogeneous boundary**;
- It can be **chosen freely (not unique)** only require it **passes through the boundary point**. Thus the simplest choice is a **linear** function (**linear interpolation**) or **parabolic** curves passing through the boundary.
- Choosing different homogenization functions w , naturally leads to different boundary value problems for v , and thus the **solutions for v will also differ**. However, due to the **uniqueness of solutions** to mixed value problems, it ensures that the final u provided will be the **same**, even though the forms of the **expressions** might differ.

Consider the boundary value problem:

$$u_{tt} = a^2 u_{xx} + f(x, t) \quad (0 < x < l, t > 0), \quad (2.5.1)$$

$$u(0, t) = u_1(t), \quad u(l, t) = u_2(t), \quad (2.5.2)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \quad (2.5.3)$$

By making a function transformation to make the boundary conditions homogeneous, we let

$$u(x, t) = v(x, t) + w(x, t), \quad (2.5.4)$$

and choose the auxiliary function $w(x, t)$ such that the new unknown function $v(x, t)$ satisfies homogeneous boundary conditions, i.e.,

$$v(0, t) = 0, \quad v(l, t) = 0. \quad (2.5.5)$$

From (2.5.2) and (2.5.4), it is easy to see that to satisfy (2.5.5), it is only necessary that

$$w(0, t) = u_1(t), \quad w(l, t) = u_2(t). \quad (2.5.6)$$

In fact, there are **many functions** $w(x, t)$ that satisfy the two conditions in (2.5.6). For the convenience of future calculations, it is usually taken as a **linear function** of x , i.e., let

$$w(x, t) = A(t)x + B(t), \quad \leftarrow \quad (\text{Algebraic method})$$

Determine $A(t)$ and $B(t)$ from condition (2.5.6) to get

$$B(t) = u_1(t), \quad A(t) = \frac{1}{l}[u_2(t) - u_1(t)],$$

Thus, we have

$$w(t, x) = \underbrace{\frac{u_2(t) - u_1(t)}{l} x}_{\text{slope}} + \underbrace{u_1(t)}_{\text{starting pt.}}$$

Geometric methods for the linear auxiliary functions

- Given boundary values u_1 at the left endpoint and u_2 at the right endpoint.
- The auxiliary function is a **straight line passing through** $(0, u_1)$ and (l, u_2) .
- The **slope** of the line is $(u_2 - u_1)/l$.
- The linear function can be expressed as $u_1 + (x/l) \cdot (u_2 - u_1)$.

Therefore, let

$$u(x, t) = v(x, t) + \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t).$$

Then the problem (2.5.1)-(2.5.3) can be transformed into a boundary value problem for $v(x, t)$:

$$\begin{cases} v_{tt} = a^2 v_{xx} + f_1(x, t) & (0 < x < l, t > 0), \\ v(0, t) = v(l, t) = 0, \\ v(x, 0) = \varphi_1(x), \quad v_t(x, 0) = \psi_1(x). \end{cases} \quad (2.5.7)$$

where the functions f_1 , φ_1 , and ψ_1 are defined as:

$$\begin{cases} f_1(x, t) = f(x, t) - \frac{x}{l}[u_2''(t) - u_1''(t)] - u_1''(t), \\ \varphi_1(x) = \varphi(x) - \frac{x}{l}[u_2(0) - u_1(0)] - u_1(0), \\ \psi_1(x) = \psi(x) - \frac{x}{l}[u_2'(0) - u_1'(0)] - u_1'(0). \end{cases} \quad \leftarrow \begin{cases} \text{If you don't choose a linear function, there will} \\ \text{be more terms, causing unnecessary trouble.} \end{cases}$$

Recall: For the system (2.5.7),

1. **First Step:** Isolate the non-homogeneity.

2. **Solving Methods:**

- Use separation of variables for one part (non-homogeneous initial data).
- Use eigenfunction expansion for another part (non-homogeneous equation).

3. **General Approach:**

- Transform the non-homogeneous boundary into a homogeneous one using an auxiliary function.
- Isolate the non-homogeneous part and solve it using different methods.

Substitute the solution of equation (2.5.7) into:

$$u(x, t) = v(x, t) + \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t).$$

This gives the solution to the original problem (2.5.1)-(2.5.3).

If the boundary conditions are not all of the first kind, similar methods can be used to transform non-homogeneous boundary conditions into homogeneous ones. We provide the corresponding auxiliary function $w(x, t)$ expressions for the following cases of non-homogeneous boundary conditions:

1. $u(0, t) = u_1(t), u(l, t) = u_2(t); w(t, x) = \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t).$
2. $u(0, t) = u_1(t), u_x(l, t) = u_2(t); w(x, t) = u_2(t)x + u_1(t).$
3. $u_x(0, t) = u_1(t), u(l, t) = u_2(t); w(x, t) = u_1(t)x + u_2(t) - lu_1(t).$
4. $u_x(0, t) = u_1(t), u_x(l, t) = u_2(t); w(x, t) = \frac{u_2(t) - u_1(t)}{2l}x^2 + u_1(t)x.$

The above four auxiliary function cases are also applicable to the heat conduction equation.

Finding the Auxiliary Function:

- For the case (left end first kind, right end second kind), we want the auxiliary function to be a linear function.
- The left gives a point $(0, u_1(t))$ and the right gives the **slope** $u_2(t)$ (the derivative $u_x(l, t) = u_2(t)$)
- The linear function can be directly written as $u_1 + u_2x$ according to this geometric translations.
- If $u(0, t) = u_1(t), u_x(l, t) = u_2(t)$, then

$$w(x, t) = \underbrace{u_2(t)}_{\text{slope}} x + \underbrace{u_1(t)}_{\text{starting pt.}}$$

- If $u_x(0, t) = u_1(t), u(l, t) = u_2(t)$, then

$$w(x, t) = \underbrace{u_1(t)}_{\text{slope}} x + \underbrace{u_2(t) - lu_1(t)}_{\text{starting pt., starting } u_2, \text{using } u_1 \text{ slope to go back } l \text{ units}}$$

- If $u_x(0, t) = u_1(t), u_x(l, t) = u_2(t)$, then it is **impossible** to find a **linear auxiliary** function if $u_1 \neq u_2$ because the **slopes** at the two endpoints are **different**.
- In this case, the simplest function is the parabolic function, but how to find it?
- It is not easy to find w , but it is **easy to find w_x to be linear** analog to the first case,
 - Since given the values of u_x at both endpoints, we start by finding the derivative of the auxiliary function, denoted as w_x instead of w .

- Assume w_x is linear.

$$w_x(x, t) = \underbrace{\frac{u_2(t) - u_1(t)}{l} x}_{\text{slope}} + \underbrace{u_1(t)}_{\text{starting pt.}} .$$

Then **integrating** it yields the simplest auxiliary function

$$w(x, t) = \frac{u_2(t) - u_1(t)}{2l} x^2 + u_1(t)x.$$

- This can also be calculated by the method of coefficients to be determined.

Ex 2.5.1. Solve the following problem

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l, t > 0), \\ u(0, t) = t, \quad u(l, t) = 0, \\ u(x, 0) = 0. \end{cases} \quad (2.5.8)$$

Solution. Choose the auxiliary function $w(x, t) = -\frac{t}{l}x + t$. Let

$$u(x, t) = v(x, t) - \frac{t}{l}x + t,$$

then problem (2.5.8) is transformed into

$$\begin{cases} v_t = a^2 v_{xx} + \frac{x}{l} - 1 & (0 < x < l, t > 0), \\ v(0, t) = 0, \quad v(l, t) = 0, \\ v(x, 0) = 0. \end{cases} \quad (2.5.9)$$

To solve problem (2.5.9) using the method of eigenfunctions, we set

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{l} x, \quad (2.5.10)$$

Using the formula derived in [Section 2.4.2 \(formula \(64\)\)](#), we know

$$v_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)} d\tau,$$

Using the formula derived in [Section 2.4.2 \(formula 62\)](#), we know

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^l \left(\frac{x}{l} - 1\right) \sin \frac{n\pi x}{l} dx = -\frac{2}{n\pi}.$$

Substitute $f_n(t) = -\frac{2}{n\pi}$ into $v_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2(t-\tau)} d\tau$,

we get

$$v_n(t) = -\frac{2}{n\pi} \int_0^t e^{-(\frac{n\pi a}{l})^2(t-\tau)} d\tau = \frac{2l^2}{(n\pi)^3 a^2} \left[e^{-(\frac{n\pi a}{l})^2 t} - 1 \right], \quad (2.5.11)$$

Substitute (2.5.11) into (2.5.10) $v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{l} x$, we obtain

$$v(x, t) = \sum_{n=1}^{\infty} \frac{2l^2}{(n\pi)^3 a^2} \left[e^{-(\frac{n\pi a}{l})^2 t} - 1 \right] \sin \frac{n\pi x}{l}.$$

Therefore, the solution to the original problem (2.5.8) is

$$u(x, t) = t \left(1 - \frac{x}{l} \right) + \sum_{n=1}^{\infty} \frac{2l^2}{(n\pi)^3 a^2} \left[e^{-(\frac{n\pi a}{l})^2 t} - 1 \right] \sin \frac{n\pi x}{l}.$$

Simultaneous homogenization

It is particularly important to note that for the given boundary value problem, for example:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t), & (0 < x < l, t > 0), \\ u(0, t) = u_1(t), & u(l, t) = u_2(t), \\ u(x, 0) = \varphi(x), & u_t(x, 0) = \psi(x). \end{cases}$$

\downarrow (If f, u_1, u_2 indep. of t)

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x), & (0 < x < l, t > 0), \\ u(0, t) = u_1, & u(l, t) = u_2, \\ u(x, 0) = \varphi(x), & u_t(x, 0) = \psi(x). \end{cases} \rightarrow \text{Take } w(x) \text{ indep. of } t$$

If the **free term** f in the equation and the **boundary conditions** u_1, u_2 are **independent of the independent variable** t , in this case, we can choose an **auxiliary function** $w(x)$ (**note it is independent of t**), and then take the function substitution $u(x, t) = v(x, t) + w(x)$, to make the equation and the boundary conditions **homogeneous at the same time** (then the separation of variables applies).

Ex 2.5.2. Solve the following problem

$$\begin{cases} u_{tt} = a^2 u_{xx} + \sin \frac{2\pi}{l} x \cos \frac{2\pi}{l} x & (0 < x < l, t > 0), \\ u(0, t) = 3, & u(l, t) = 6, \\ u(x, 0) = 3 \left(1 + \frac{x}{l} \right), & u_t(x, 0) = \sin \frac{4\pi}{l} x. \end{cases} \quad (2.5.12)$$

Solution. Assume the solution to the problem is

$$u(x, t) = v(x, t) + w(x). \quad (2.5.13)$$

Substitute (2.5.13) into the equation of problem (2.5.14), we get

$$v_{tt} = a^2 (v_{xx} + w''(x)) + \sin \frac{2\pi}{l} x \cos \frac{2\pi}{l} x.$$

To make this equation homogeneous, naturally choose $w(x)$ to satisfy

$$a^2 w'' + \sin \frac{2\pi}{l} x \cos \frac{2\pi}{l} x = 0.$$

Substituting (2.5.13) into the boundary conditions of problem (2.5.14), we obtain

$$\begin{cases} v(0, t) + w(0) = 3, & v(l, t) + w(l) = 6, \\ v(x, 0) + w(x) = 3(1 + \frac{x}{l}), & v_t(x, 0) = \sin \frac{4\pi}{l}x. \end{cases}$$

To make the boundary conditions of $v(x, t)$ homogeneous as well, $w(x)$ must satisfy

$$w(0, t) = 3, \quad w(l, t) = 6.$$

By making the substitution $u(x, t) = v(x, t) + w(x)$, problem (2.5.14) is transformed into the following two problems:

For $w(x)$:

$$\begin{cases} a^2 w'' + \sin \frac{2\pi}{l}x \cos \frac{2\pi}{l}x = 0, \\ w(0, t) = 3, \quad w(l, t) = 6. \end{cases} \quad (2.5.14)$$

For $v(x, t)$:

$$\begin{cases} v_{tt} = a^2 v_{xx} & (0 < x < l, t > 0), \\ v(0, t) = 0, \quad v(l, t) = 0, \\ v(x, 0) = 3(1 + \frac{x}{l}) - w(x), \quad v_t(x, 0) = \sin \frac{4\pi}{l}x. \end{cases} \quad (2.5.15)$$

Solving the ODE (2.5.14):

$$\frac{dw'}{dx} = -\frac{1}{a^2} \sin \frac{2\pi x}{l} \cos \frac{2\pi x}{l}$$

By integration, we obtain

$$\begin{aligned} \Rightarrow w'(x) &= -\frac{1}{a^2} \int \sin \frac{2\pi x}{l} \cos \frac{2\pi x}{l} dx + C_1 \\ &= -\frac{1}{2a^2} \int \sin \frac{4\pi x}{l} dx + C_1 \\ &= -\frac{l}{8\pi a^2} \cos \frac{4\pi x}{l} + C_1 \end{aligned}$$

Integrating it yields

$$\begin{aligned} \Rightarrow w(x) &= -\frac{l}{8\pi a^2} \int \cos \frac{4\pi x}{l} dx + C_1 x + C_2 \\ &= -\frac{l}{8\pi a^2} \cdot \frac{l}{4\pi} \sin \frac{4\pi x}{l} + C_1 x + C_2 \\ &= -\frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi x}{l} + C_1 x + C_2 \end{aligned}$$

The boundary condition leads to $C_1, C_2 \Rightarrow C_2 = 3$.

$$w(l) = -\frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi l}{l} + C_1 l + 3 = 6 \quad \Rightarrow \quad C_1 = \frac{3}{l}$$

Then,

$$\Rightarrow w(x) = -\frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi x}{l} + \frac{3x}{l} + 3$$

Problem (2.5.14) is a boundary value problem for an ordinary differential equation, and its solution is:

$$w(x) = \frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi}{l} x + 3 \left(1 + \frac{x}{l}\right).$$

Substitute the obtained $w(x)$ into problem (2.5.15):

$$\begin{cases} v_{tt} = a^2 v_{xx} & (0 < x < l, t > 0), \\ v(0, t) = 0, \quad v(l, t) = 0, \\ v(x, 0) = -\frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi}{l} x, \quad v_t(x, 0) = \sin \frac{4\pi}{l} x. \end{cases}$$

Using the formula:

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l},$$

where the coefficients a_n and b_n satisfy:

$$\begin{cases} a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, \\ b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx. \end{cases}$$

Thus,

$$v(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l},$$

where the coefficients a_n and b_n are calculated as follows:

$$\begin{cases} a_n = \frac{2}{l} \int_0^l -\frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi}{l} x \sin \frac{n\pi}{l} x dx = \begin{cases} 0, & n \neq 4, \\ -\frac{l^2}{32\pi^2 a^2}, & n = 4. \end{cases} \\ b_n = \frac{2}{n\pi a} \int_0^l \sin \frac{4\pi x}{l} \sin \frac{n\pi x}{l} dx = \begin{cases} 0, & n \neq 4, \\ \frac{l}{4\pi a}, & n = 4. \end{cases} \end{cases}$$

Thus, the solution to problem (2.5.15) is:

$$v(x, t) = \left(-\frac{l^2}{32\pi^2 a^2} \cos \frac{4\pi at}{l} + \frac{l}{4\pi a} \sin \frac{4\pi at}{l} \right) \sin \frac{4\pi}{l} x.$$

Therefore, the solution to the original problem (2.5.14) is:

$$u(x, t) = \left(-\frac{l^2}{32\pi^2 a^2} \cos \frac{4\pi at}{l} + \frac{l}{4\pi a} \sin \frac{4\pi at}{l} \right) \sin \frac{4\pi}{l} x + \frac{l^2}{32\pi^2 a^2} \sin \frac{4\pi}{l} x + 3 \left(1 + \frac{x}{l} \right).$$

Solution (Alternative solution, a bad method if you use the linear auxiliary function for the simultaneous homogenization). Choose the auxiliary function $w(x, t) = 3 \left(1 + \frac{x}{l} \right)$, then

$$u(x, t) = v(x, t) + 3 \left(1 + \frac{x}{l} \right)$$

Substitute into problem (2.5.14) to obtain

$$\begin{cases} v_{tt} = a^2 v_{xx} + \sin \frac{2\pi}{l} x \cos \frac{2\pi}{l} x, \\ v(0, t) = v(l, t) = 0, \\ v(x, 0) = 0, \quad v_t(x, 0) = \sin \frac{4\pi}{l} x. \end{cases} \rightarrow (\text{isolate non-homog. by linear superpositions}) \quad (2.5.16)$$

From the analysis in Section 2.4.1, we can set $v(x, t) = \bar{v}(x, t) + \bar{w}(x, t)$ and $\bar{v}(x, t)$ and $\bar{w}(x, t)$ satisfy the following boundary value problems respectively:

$$\begin{cases} \bar{v}_{tt} = a^2 \bar{v}_{xx} + \sin \frac{2\pi}{l} x \cos \frac{2\pi}{l} x, \\ \bar{v}(0, t) = \bar{v}(l, t) = 0, \\ \bar{v}(x, 0) = 0, \quad \bar{v}_t(x, 0) = 0. \end{cases} \rightarrow (\text{eigenfunction}) \quad (2.5.17)$$

and

$$\begin{cases} \bar{w}_{tt} = a^2 \bar{w}_{xx}, \\ \bar{w}(0, t) = \bar{w}(l, t) = 0, \\ \bar{w}(x, 0) = 0, \quad \bar{w}_t(x, 0) = \sin \frac{4\pi}{l} x. \end{cases} \rightarrow (\text{sep. of variables}) \quad (2.5.18)$$

Using formulas (14) and (15) from Section 2.1, we can calculate

$$\bar{w}(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}$$

where the coefficients a_n and b_n are

$$\begin{cases} a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x dx = 0, & n \neq 4, \\ b_n = \frac{2}{n\pi a} \int_0^l \sin \frac{4\pi x}{l} \sin \frac{n\pi x}{l} dx = \begin{cases} 0, & n \neq 4, \\ \frac{l}{4\pi a}, & n = 4. \end{cases} \end{cases}$$

Then the solution to problem (2.5.18) is

$$\bar{w}(x, t) = \frac{l}{4\pi a} \sin \frac{4\pi a}{l} t \sin \frac{4\pi}{l} x.$$

Next, using the method of eigenfunctions to solve problem (2.5.17). Thus, let

$$\bar{v}(x, t) = \sum_{n=1}^{\infty} \bar{v}_n(t) \sin \frac{n\pi}{l} x$$

Using formula (53) derived in Section 2.4.1, we know

$$\bar{v}_n(t) = \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin \frac{n\pi a}{l} (t - \tau) d\tau,$$

Using formula (51) derived in Section 2.4.1, we know

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx = \frac{1}{l} \int_0^l \sin \frac{4\pi x}{l} \sin \frac{n\pi x}{l} dx = \begin{cases} 0, & n \neq 4, \\ \frac{1}{2}, & n = 4. \end{cases}$$

1. When $n \neq 4$, $\bar{v}_n(t) = 0$;

2. When $n = 4$, we obtain

$$\bar{v}_n(t) = \frac{l}{8\pi a} \int_0^t \sin \frac{4\pi a}{l}(t-\tau) d\tau = \frac{l^2}{32\pi^2 a^2} \left(1 - \cos \frac{4\pi a}{l} t \right)$$

The solution to problem (2.5.17) is

$$\bar{v}(x, t) = \frac{l^2}{32\pi^2 a^2} \left(1 - \cos \frac{4\pi a}{l} t \right) \sin \frac{4\pi}{l} x.$$

Combine the solution to problem (2.5.18) $\bar{w}(x, t) = \frac{l}{4\pi a} \sin \frac{4\pi a}{l} t \sin \frac{4\pi}{l} x$ with the auxiliary function $w(x, t) = 3(1 + \frac{x}{l})$ and the solution to problem (2.5.17) to obtain the solution to the original problem (2.5.14):

$$u(x, t) = v(x, t) + 3 \left(1 + \frac{x}{l} \right) = \bar{v}(x, t) + \bar{w}(x, t) + 3 \left(1 + \frac{x}{l} \right)$$

2.6 Eigenvalues and Eigenfunctions (Introduction)

In the first three sections of this chapter, when we applied the method of separation of variables to solve the boundary value problems related to the vibration equation, one-dimensional heat conduction equation, and two-dimensional Laplace equation, we needed to solve a boundary value problem of an ordinary differential equation containing the parameter λ :

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0. \quad \leftarrow \boxed{\text{The core of series solutions!}} \quad (2.6.1)$$

This type of problem is called an eigenvalue problem. It also belongs to the Sturm-Liouville problem.

This equation (2.6.1) is not enough for Chapter 5, we need to generalize it!

The general form of the Sturm-Liouville equation is

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) - q(x)y + \lambda \rho(x)y = 0 \rightarrow \boxed{\text{If } p = \rho = 1 \text{ and } q = 0, (2.6.2) \text{ becomes (2.6.1)}} \quad (2.6.2)$$

where

1. $p(x), p'(x) \in C[a, b], p(x) > 0$ for $a < x < b$;
2. $q(x) \in C[a, b]$, or $q(x) \in C(a, b)$, and **at most one endpoint has a first-order pole** (Bessel equation is in this form), and $q(x) \geq 0$;
3. $\rho(x) \in C[a, b], \rho(x) > 0$.

Equation (2.6.2) with boundary conditions is called the **Sturm-Liouville problem**. Those λ values that make the Sturm-Liouville problem have **non-zero solutions** are called the **eigenvalues** of the problem, and the corresponding **non-zero solutions** are called **eigenfunctions**.

Some conclusions about eigenvalues and eigenfunctions:

1. There are (**coutable**) **infinitely many** real eigenvalues:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \quad \leftarrow \boxed{\text{countable infinite!}}$$

When $q(x) \geq 0$, $\lambda_n \geq 0$ ($n = 1, 2, 3, \dots$); corresponding to these eigenvalues, there are infinitely many eigenfunctions:

$$y_1(x), y_2(x), \dots, y_n(x), \dots.$$

Generalize the case, for example, $\lambda_n = (\frac{n\pi}{l})^2$, $y_n(x) = X_n(x) = \sin \frac{n\pi x}{l}$, $n = 1, 2, \dots$ for $X'' + \lambda X = 0$.

2. If the eigenfunction corresponding to the eigenvalue λ_n is denoted as $y_n(x)$, then all $y_n(x)$ form an **orthogonal** function system with **weight** function $\rho(x)$, that is

$$\langle y_m(x), y_n(x) \rangle_{\rho} := \int_a^b y_m(x) y_n(x) \underbrace{\rho(x) dx}_{\text{Measure}} = 0 \quad (m \neq n). \quad \leftarrow \boxed{\text{Generalize the orthogonality of sin, cos}}$$

For $X'' + \lambda X = 0$, $\rho \equiv 1$, thus $\rho(x)dx = dx$. For general S-L problem, we can not ensure y_n are sin or cos, but we can ensure their **orthogonality**.

3. (**Completeness**) Similar to Fourier series, the **expansion** in terms of **eigenfunctions** has the following convergence properties:

If the function $f(x)$ has continuous first-order derivatives and piecewise continuous second-order derivatives in (a, b) , and satisfies the given boundary conditions, then $f(x)$ can be expanded in terms of eigenfunctions as an absolutely and uniformly convergent series in (a, b) :

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad \leftarrow \boxed{\text{Generalization of Fourier series}} \quad (2.6.3)$$

Replace $\{1, \sin, \cos, \cdot\}$ to the eigenfunction system, and “any” good function can be **expanded by the eigenfunction system**. $\{y_n\}$ is complete.

where

$$c_n = \frac{\int_a^b \rho(x) f(x) y_n(x) dx}{\int_a^b \rho(x) y_n^2(x) dx} \quad (n = 1, 2, 3, \dots); \quad \leftarrow \boxed{\text{Calculated by inner product due to orthogonality}}$$

It is easy to verify

$$\langle y_m(x), f(x) \rangle_{\rho} = \sum_{n=1}^{\infty} c_n \langle y_m(x), y_n(x) \rangle_{\rho} = c_m \langle y_m(x), y_m(x) \rangle_{\rho}$$

- In Chapter 5, when the trigonometric function system is replaced by the Bessel function system, there will also be a Bessel-Fourier expansion. This can be sought in the same way and also satisfies these properties.

If the function $f(x), f'(x)$ are **piecewise continuous** functions in (a, b) , then the series (2.6.3) converges at the discontinuity point x_0 of $f(x)$ to

$$\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)],$$

and loses uniform convergence on (a, b) .

New method in the Exercise Class:

- Once in standard S-L form, we will introduce an unfamiliar method called the **energy method**, not covered in textbooks.
- The energy method is essential for understanding certain aspects of Chapter 5.
- Without understanding the energy method, some parts of Chapter 5 will not be explained in textbooks.
- We will discuss this method in the exercise session.
- The three conclusions can be fully analogized to the properties of $X'' + \lambda X = 0$.

2.7 Exercise class

Ex 2.7.1. Use the method of separation of variables to write out the following problem:

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l, t > 0), \\ (u_x - \alpha u)|_{x=0} = 0, & (\text{Third kind}) \\ u_x|_{x=l} = 0, \\ u(x, 0) = \varphi(x) \end{cases}$$

What is the eigenvalue problem? and write out

- when $\alpha = 0$ in the boundary conditions, the eigenvalues and eigenfunctions;
- when $\alpha \rightarrow \infty$ in the boundary conditions, the eigenvalues and eigenfunctions;

Solution. For the following mixed value problem:

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l, t > 0), \\ (u_x - \alpha u)|_{x=0} = 0, \\ u_x|_{x=l} = 0, \\ u(x, 0) = \varphi(x) \end{cases}$$

The eigenvalue problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ (X' - \alpha X)|_{x=0} = 0, \\ X'|_{x=l} = 0. \end{cases} \quad \text{that is,} \quad \begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = \alpha X(0), \\ X'|_{x=l} = 0. \end{cases}$$

(1) When $\alpha = 0$ in the boundary conditions, the eigenvalue problem simplifies to

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'|_{x=0} = 0, \quad X'|_{x=l} = 0. \end{cases}$$

At this case, the corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l} \quad (n = 0, 1, 2, \dots).$$

(2) When $\alpha \rightarrow \infty$ in the boundary conditions, the eigenvalue problem simplifies to (since $X'(0) = \alpha X(0) \Leftrightarrow X(0) = \frac{1}{\alpha} X'(0)$)

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X|_{x=0} = 0, \quad X'|_{x=l} = 0. \end{cases}$$

At this case, the corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2, \quad X_n(x) = \sin \frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \dots).$$

Ex 2.7.2. Prove for the problem,

$$\begin{cases} x^2 y'' + 3xy' + \lambda y = 0, & (1 < x < e) \\ y(1) = y(e) = 0 \end{cases} \leftarrow \boxed{\begin{array}{l} \text{in the class of the general S-L problem,} \\ \text{play the role of } X'' + \lambda X = 0 \end{array}}$$

The eigenfunction series $\{y_n(x)\}$ is orthogonal on $[1, e]$ with respect to the weight function x . i.e.,

$$\int_1^e xy_n(x)y_m(x)dx = \frac{1}{2}\delta_{mn}$$

The first idea:

1. **Initial Step:** To prove the orthogonality of eigenfunctions, the first step is to find the eigenfunctions.
2. **Equation Substitution:** The current equation replaces the previous one involving $X'' + \lambda X = 0$.
3. **Eigenvalue Problem Approach:** Similar to solving for X , we first find the general solution and then apply boundary conditions to determine parameters.
4. **General Solution for ODE:** The equation is a second-order linear ODE with variable coefficients, unlike the constant coefficient case.
5. **Euler's Equation Analogy:** The form of the equation **resembles Euler's equation**, which involves terms like x^2 and x times the second and first derivative, respectively.
6. **Transformation Method:** Use a variable transformation similar to solving Euler's equation to simplify the variable coefficients into constant coefficients.

7. **Transformation Outcome:** After transformation, the equation becomes a standard constant coefficient form ODE, which simplifies the process of finding solutions.
8. **Calculation of Transformation:** The process of transformation should be calculated, but it naturally follows if you aim for a constant coefficient form.

Solution (Method 1: Direct method). (1) First, find the specific expression of the eigenfunction series $\{y_n(x)\}$.

Make the transformation $x = e^t \iff t = \ln x$,

Then we have

$$y_x = y_t \cdot \frac{1}{x}, \quad y_{xx} = \left(y_{tt} \cdot \frac{1}{x} \right) \cdot \frac{1}{x} + y_t \cdot \left(-\frac{1}{x^2} \right) = \frac{1}{x^2} y_{tt} - \frac{1}{x^2} y_t,$$

Substitute into the original equation to get

$$\begin{aligned} y_{tt} - y_t + 3y_t + \lambda y &= 0 \implies y_{tt} + 2y_t + \lambda y = 0 \\ \implies \begin{cases} y_{tt} + 2y_t + \lambda y = 0 \\ y(1) = y(e) = 0. \end{cases} &\quad \leftrightarrow \text{char. eq. } r^2 + 2r + \lambda = 0 \leftrightarrow r_{\pm} = -1 \pm \sqrt{1 - \lambda} \end{aligned}$$

Method 1: General method

Case (i) $\lambda < 1$

$$\begin{cases} y = Ce^{(1+\sqrt{\lambda})t} + De^{(1-\sqrt{\lambda})t} \\ \begin{cases} C + D = 0 \\ Ce^{-t\sqrt{\lambda}} + De^{t\sqrt{\lambda}} = 0 \end{cases} \end{cases}.$$

This leads to $C = D = 0$, hence no non-trivial solutions.

Case (ii) $\lambda = 1$

$$\begin{cases} y = (At + B)e^{-t} \\ \begin{cases} Be^{-t} = 0 \\ (A + B)e^{-t} = 0 \end{cases} \end{cases}.$$

This leads to $B = 0$ and $A = 0$, hence no non-trivial solutions.

Case (iii) $\lambda > 1$

$$y = (A \cos(\sqrt{\lambda-1}t) + B \sin(\sqrt{\lambda-1}t))e^{-t}.$$

Using boundary conditions:

$$\begin{cases} A = 0, \\ (A \cos(\sqrt{\lambda-1}) + B \sin(\sqrt{\lambda-1}))e^{-1} = 0. \end{cases}$$

This leads to $A = 0$ and $\sin(\sqrt{\lambda - 1}) = 0$, hence $\sqrt{\lambda - 1} = n\pi$. Thus, $\lambda = \lambda_n = 1 + (n\pi)^2$, and $y_n(t) = B_n \sin(n\pi t) \cdot e^{-t}$ for $n = 1, 2, \dots$

Summary of Transformation Techniques for Differential Equations

1. **General Approach:** When encountering *unfamiliar* equations, transform them into *familiar* forms using technical skills.
2. **Two Main Techniques:**
 - **Variable transformation** (e.g., for Euler's equation).
 - **Function transformation** to simplify equations.
3. **Goal:** Simplify the equation to a form where known solutions or conclusions can be directly applied.
4. **Desired Form:** Aim to transform it into a form similar to $X'' + \lambda X = 0$, which has known eigenvalues and eigenfunctions.
5. **Transformation Strategy:**
 - Combine the first two derivative terms into a **single second-order derivative**.
 - Use the **binomial theorem** to handle terms involving $(a + b)^{(n)}$ and their derivatives.
6. **Application of Binomial Theorem:** Use the binomial theorem for derivatives to simplify higher-order terms.

$$(fg)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Take $n = 2$, $f = y$, what is g ? Hope g is a fixed point for derivatives. Then

$$(yg)'' = y''g + 2y'g' + yg'' \stackrel{g \text{ a fixed point}}{=} (y'' + 2y' + y)g \quad (2.7.1)$$

7. **Transformation Example:** Multiply both sides of the equation by e^t (a fixed point for derivation) to utilize the binomial theorem effectively.
8. **Resulting Equation:** After transformation, the equation becomes $X'' + (\lambda - 1)X = 0$, which is similar to the known form.
9. **Conclusion:** Once transformed, the equation can be solved using known conclusions without further case distinctions.

Method 2: Integrating factor and binomial theorem method

$$\begin{aligned} 0 &= e^t (y_{tt} + 2y_t + y + (\lambda - 1)y) \\ &= \underbrace{(e^t y)_{tt}}_{X''} + \underbrace{(\lambda - 1)(e^t y)}_{(\lambda-1)X} \end{aligned}$$

According to the conclusion of $X'' + \lambda X = 0$, we conclude the solution.

$$\begin{aligned} X'' + (\lambda - 1)X &= 0 \\ \Rightarrow \lambda - 1 > 0 \Rightarrow \lambda - 1 &= (n\pi)^2 \Rightarrow \lambda = 1 + (n\pi)^2, \quad (n = 1, 2, \dots) \\ \text{and } X_n &= e^t y_n = \sin(n\pi t). \end{aligned}$$

$$\Rightarrow \lambda_n = (n\pi)^2 + 1, \quad y_n(t) = B_n e^{-t} \sin n\pi t \quad (n = 1, 2, \dots).$$

Substitute $t = \ln x$ to get $y_n(x) = B_n \frac{1}{x} \sin(n\pi \ln x)$, $(n = 1, 2, \dots)$. Then the eigenfunction series of the original problem is

$$\{y_n(x)\} = \left\{ \frac{1}{x} \sin(n\pi \ln x) \right\} \quad (n = 1, 2, \dots)$$

(2) Now verify the orthogonality of the eigenfunction series $\{y_n(x)\}$.

$$\begin{aligned} &\int_1^e xy_n(x)y_m(x)dx \quad \text{Make the transformation } x = e^t \\ &= \int_0^1 e^{2t} y_n(t)y_m(t)dt = \int_0^1 \sin(n\pi t) \sin(m\pi t) dt = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases} \end{aligned}$$

Solution (Method 2: Energy method–Compatible with the nonlinear PDEs). Let λ_n, y_n be the corresponding eigenvalues and eigenfunctions. (assume they exist!)

Then λ_n, y_n satisfy:

$$x^2 y_n'' + 3xy_n' + \lambda_n y_n = 0 \implies x^2 y_n'' + 3x^2 y_n' + \lambda_n x y_n = 0 \implies (x^3 y_n')' + \lambda_n x y_n = 0$$

$$\left. \begin{aligned} &\underbrace{x^2 y'' + 3xy'}_{\Downarrow} + \lambda y = 0 \\ &\underbrace{\frac{1}{x}(x^3 y'' + 3x^2 y')}_{\Downarrow} \\ &\quad \left. \begin{aligned} &\frac{1}{x}(x^3 y')' \end{aligned} \right\} \end{aligned} \right\} \Rightarrow (x^3 y')' + \lambda x y = 0$$

- The concept of **integrating factors** is crucial in transforming differential equations into a solvable form.
- To utilize the **energy method** for proving orthogonality, the equation must be converted into **Sturm-Liouville (S-L) standard form**.
- The orthogonality proof in **Chapter 5** also relies on this energy method.

The physical origin of the concept of energy.

$$ma = \frac{d(mv)}{dt} = F.$$

$$\Rightarrow v \frac{d(mv)}{dt} = F \frac{ds}{dt} \Rightarrow \int \frac{d\left(\frac{1}{2}mv^2\right)}{dt} dt = \int \frac{dW}{dt} dt \Rightarrow \frac{1}{2}mv^2 = W.$$

Mathematical formulation of this “energy concept”.

Extract ideas:

- Use unknown functions to multiply both sides of the differential equation;
- Integration (local to global);
- Symmetric energy obtained by integration by parts.

Using y_m to multiply both sides and integrate from 1 to e , we get:

$$\text{Step 1: eq.} \times y. \quad (x^3 y'_n)' y_m + \lambda_n x y_n y_m = 0$$

$$\text{Step 2: integration.} \quad \underbrace{\int_1^e (x^3 y'_n)' y_m dx}_{\parallel (\text{Integration by Parts})} + \lambda_n \int_1^e x y_n y_m dx = 0$$

$$\text{Step 3: integration by parts.} \quad \underbrace{x^3 y'_n y_m \Big|_1^e}_{\parallel \text{bdry. } y(1)=y(e)=0} - \int_1^e x^3 y'_n y'_m dx = 0$$

1. **Advantage of Integration:** The primary benefit of integration is the ability to move derivatives (integration by parts).
2. **Utilizing Symmetry:** By moving derivatives, we can leverage specific symmetries to derive beneficial conclusions.

3. **Method Used:** Integration by parts is employed to shift derivatives.
4. **Result of Integration by Parts:** After applying integration by parts, a boundary term and an integral term remain.
5. **Boundary Term:** The boundary term involves the product of functions and their derivatives evaluated at the boundaries.
6. **Symmetry:** The resulting expression often exhibits symmetry, which is desirable in mathematical and physical contexts.
7. **Boundary Conditions:** The boundary conditions can be used to simplify the boundary term, often making it zero.
8. **Simplification:** If the boundary term is zero, the expression simplifies significantly.
9. **Final Form:** The simplified expression can be rearranged to a more recognizable form, facilitating further analysis.

$$\Rightarrow \lambda_n \int_1^e xy_n y_m dx = \int_1^e x^3 y'_n y'_m dx. \quad (2.7.2)$$

By swapping m and n , we get:

$$\lambda_m \int_1^e xy_n y_m dx = \int_1^e x^3 y'_n y'_m dx. \quad (2.7.3)$$

Subtracting (2.7.3) from (2.7.2) gives:

$$(\lambda_n - \lambda_m) \int_1^e xy_n y_m dx = 0 \Rightarrow \text{if } n \neq m \text{ then } \int_1^e xy_n y_m dx = 0, \text{ orthogonality is proven.}$$

Idea: Use duality to achieve symmetries internally.

- If dual, there may be information sharing and exchange between dual systems. - The “inner product (dual form)” can be moved to reduce the order of the equation (tool: Integration by parts).

Techniques:

1. Both sides multiply the same unknown function.
2. Integration.
3. Integration by parts (eliminating boundary terms).

Comparison of Two Methods for Orthogonality of Eigenfunctions

1. **First Method:**

- Requires finding the eigenfunction series explicitly.
- After obtaining the specific form, substitute into the integral to verify if it equals zero.

2. Second Method:

- Does not require finding the eigenfunctions Y_n explicitly.
- Directly derives the orthogonality properties without explicit computation of Y_n .

3. Key Difference:

- The first method involves **detailed computation of eigenfunctions**.
- The second method focuses on orthogonality **without detailed computation of eigenfunctions**.

4. Advantages of the Second Method:

- More efficient when detailed computation of eigenfunctions is unnecessary.
- Provides a direct approach to proving orthogonality.

5. Implementation of the Second Method:

- Utilize **properties of differential equations** and **boundary conditions**.
- Apply integration techniques to show orthogonality without explicit eigenfunction computation.

Summary:

1. **Method Overview:** This method, initially seems strange, is known as the **energy method**, which simulates the **structure of kinetic energy**.
2. **Construction of Duality:** By **integrating** with an unknown function, we construct a **dual entity** that is **symmetric** with the original function.
3. **Symmetry and Simplification:** Through **integration by parts**, we achieve a **completely symmetric form**, which helps in simplifying the problem.

Ideas of Duality–Daoist Philosophy on Duality

The concept of duality reflects the idea of cooperation. When something is difficult or ineffective, it can be paired with something easier or more effective. By establishing an internal connection, their difficulty or efficiency can be balanced. The power of a collaborative team is greater than the sum of its individual members.

The Chinese classic *Tao Te Ching* expresses a **similar idea**:

万物负阴而抱阳
冲气以为和
道德经 · 第四十二章

This can be translated as

$\left\{ \begin{array}{l} \text{All things bear the shade on their backs} \\ \text{And the sun in their arms; } (\leftrightarrow \text{multiplying the unknown and integrating it}) \\ \text{By the blending of breath } (\leftrightarrow \text{integration by parts}) \\ \text{From the sun and the shade,} \\ \text{Equilibrium comes to the world. } (\leftrightarrow \text{achieve the symmetric energy form}) \end{array} \right.$

or

“All things carry Yin and embrace Yang. Through their interaction, harmony is achieved.”

This reflects the **natural principle of duality**, where seemingly opposing forces complement each other, achieving **balance through cooperation**.

In other words, **Contradiction** is the fundamental driving force of the development of things. Every entity contains two opposing aspects that are both in conflict and interdependent, and under certain conditions, they can transform into each other. Contradiction drives the development of things, causing them to continuously change, progress, and achieve balance.

- 1. **Basic Idea:** The underlying idea is one of **duality**, which is a **fundamental concept in functional analysis**.
- 2. **Benefits of Duality:** Duality helps **transfer complexity** from a difficult problem to a simpler one, reducing overall difficulty.
- 3. **Application in Differential Equations:** By multiplying by a simple function, integrating and shifting the derivatives, we **reduce the order** of the differential equation.
- 4. **Construction of Symmetry:** Lowering the order of derivatives through integration allows us to **construct symmetry**, a crucial tool in mathematical research.

Aesthetic approach

- 1. **Importance of Symmetry:** **Symmetry construction** is vital in mathematics, often inspired by physical or geometric intuition, or even **aesthetics**.

2. **Aesthetics in Mathematics:** The method reflects an **aesthetic approach** to mathematics, seeking **intrinsic symmetry** and **beauty** in mathematical forms.
3. **Artistic Approach:** This method embodies an **artistic perspective** in mathematics, aiming to construct **elegant structures** to derive solutions.
4. **Recurring Theme:** The concept of constructing symmetry and beauty will reappear in various contexts, emphasizing its importance in mathematical problem-solving.

Ex 2.7.3. Eigenvalue Problem

$$\begin{cases} u'' + \lambda u = 0; & 0 < x < l \\ u'(0) = u(l) = 0. \end{cases}$$

Solution. Usually, the method is to exclude $\lambda \leq 0$, we can consider the general solutions separately and then use boundary conditions to determine the solutions.

We provide another method—the **energy method**:

Multiply u on the both sides and integrate both sides with respect to x , and using **integration by parts**:

$$\begin{aligned} & \underbrace{\int_0^l uu'' dx + \lambda \int_0^l u^2 dx}_\text{II} = 0 \\ & \underbrace{uu' \Big|_0^l - \int_0^l (u')^2 dx}_\text{II(*)} = 0 \\ & 0 \end{aligned}$$

(*) is due to the boundary condition $u(0) = u(l) = 0$, then we have:

$$\Rightarrow \lambda \int_0^l u^2 dx = \int_0^l (u')^2 dx \Rightarrow \lambda = \frac{\int_0^l (u')^2 dx}{\int_0^l u^2 dx} \geq 0$$

When $\lambda = 0$, it implies $u'(x) = 0$, hence $u(x) = \text{constant}$. Using $u(l) = 0$, we get $u(x) \equiv 0$.

Thus, λ can only be positive $\lambda > 0$. In this case, the general solution is $u(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$. Using boundary conditions, we find $\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$, and the corresponding eigenfunctions are $u_n(x) = \cos\left(\frac{(2n-1)\pi x}{2l}\right)$ for $n = 1, 2, \dots$

- The significance is that, although we cannot write the general solution in Chapter 5, this method still works and allows us to directly determine if the result is greater than or equal to zero. Otherwise, the previous method no longer applies.

Ex 2.7.4. Let F and G be twice continuously differentiable functions.

1. Prove that $y(x, t) = F(2x + 5t) + G(2x - 5t)$ is the general solution of the equation $4y_{tt} = 25y_{xx}$.
2. Find the solution that satisfies the conditions $y(0, t) = y(\pi, t) = 0$, $y(x, 0) = \sin 2x$, $y_t(x, 0) = 0$.

- Similar methods to those used in this example will be applied in Chapter 3.

Solution. (2) From the conditions, we have:

$$\begin{cases} y(0, t) = F(5t) + G(-5t) = 0 \\ y(\pi, t) = F(2\pi + 5t) + G(2\pi - 5t) = 0 \\ y(x, 0) = F(2x) + G(2x) = \sin 2x \\ y_t(x, 0) = 5F'(2x) - 5G'(2x) = 0 \end{cases}$$

- There are four conditions. In essence, only two conditions are needed to determine the two functions. The other two conditions are naturally fulfilled.

This implies:

$$F(y) = -G(-y) \quad (2.7.4)$$

$$F(2\pi + y) = -G(2\pi - y) \quad (2.7.5)$$

$$F(y) + G(y) = \sin y \quad (2.7.6)$$

$$F'(y) = G'(y) \Rightarrow F(y) = G(y) + C. \quad (2.7.7)$$

From equations (2.7.6) and (2.7.7), we get:

$$\begin{cases} 2F(y) = C + \sin y \Rightarrow F(y) = \frac{C + \sin y}{2} \\ 2G(y) = \sin y - C \Rightarrow G(y) = \frac{\sin y - C}{2} \end{cases} \quad (2.7.8)$$

From equation (2.7.4), we have:

$$F(y) = \frac{C + \sin y}{2} = -\frac{-\sin y - C}{2} = -G(-y) \leftarrow \text{verifies (2.7.4)}$$

This implies:

$$C + \sin y = \sin y + C \quad (\text{identity})$$

Given:

$$F(2\pi + y) = \frac{C + \sin y}{2}$$

$$G(2\pi - y) = \frac{\sin(2\pi - y) - C}{2} = \frac{-\sin y - C}{2}$$

Thus:

$$F(2\pi + y) = -G(2\pi - y) = \frac{\sin y + C}{2} \quad \leftarrow \text{ verifies (2.7.5)}$$

Using equation (2.7.8), we find:

$$\begin{aligned} y(x, t) &= \frac{\sin(2x + 5t) + C}{2} + \frac{\sin(2x - 5t) - C}{2} \\ &= \frac{1}{2}[\sin(2x + 5t) + \sin(2x - 5t)] \\ &= \sin 2x \cos 5t \end{aligned}$$

Ex 2.7.5. Solve the Initial Boundary Value Problem

$$\begin{cases} u_t - a^2 u_{xx} = 0 & 0 < x < l, t > 0 \\ u_x(0, t) - \sigma u(0, t) = 0 & t \geq 0 \\ u_x(l, t) + \sigma u(l, t) = 0 & t \geq 0 \\ u(x, 0) = \varphi(x) & 0 \leq x \leq l \end{cases}$$

where $\sigma > 0$.

- Homogeneous equation and Homogeneous boundary → Try Separation of variables.

Solution. Use separation of variables, let $u(x, t) = X(x)T(t)$, substitute into the system.

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X'(0) - \sigma X(0) = 0, \quad X'(l) + \sigma X(l) = 0 & \end{cases} \quad (2.7.9)$$

and

$$T' + a^2 \lambda T = 0 \quad (2.7.10)$$

From (2.7.9), we have:

$$\begin{aligned} \int_0^l X X'' dx + \int_0^l \lambda X^2 dx &= 0 \Rightarrow X X' \Big|_0^l - \int_0^l (X')^2 dx + \int_0^l \lambda X^2 dx = 0 \\ &\Rightarrow X(l)X'(l) - X(0)X'(0) + \lambda \int_0^l X^2 dx = \int_0^l (X')^2 dx \end{aligned} \quad (2.7.11)$$

By boundary conditions:

$$X'(0) - \sigma X(0) = 0, \quad X'(l) + \sigma X(l) = 0$$

$$\begin{aligned} &\Rightarrow -\sigma(X(l))^2 - \sigma(X(0))^2 + \lambda \int_0^l X^2 dx = \int_0^l (X')^2 dx \\ &\Rightarrow \lambda \int_0^l X^2 dx = \sigma(X(l))^2 + \sigma(X(0))^2 + \int_0^l (X')^2 dx \geq 0 \end{aligned}$$

$$\Rightarrow \lambda \geq 0, \text{ if } \lambda = 0 \Rightarrow 0 \leq \int_0^l (X')^2 dx = -\sigma(X(l))^2 - \sigma(X(0))^2 \leq 0$$

$$\Rightarrow X' \equiv 0 \Rightarrow X(x) = \text{constant} \Rightarrow \lambda > 0$$

For $\lambda > 0$, the general solution of (2.7.9) is:

$$X(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) \Rightarrow X'(x) = -A\sqrt{\lambda} \sin(\sqrt{\lambda}x) + B\sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

Using boundary conditions:

$$X'(0) - \sigma X(0) = B\sqrt{\lambda} - \sigma A = 0$$

$$X'(l) + \sigma X(l) = -A\sqrt{\lambda} \sin(\sqrt{\lambda}l) + B\sqrt{\lambda} \cos(\sqrt{\lambda}l) + \sigma A \cos(\sqrt{\lambda}l) + \sigma B \sin(\sqrt{\lambda}l) = 0$$

This leads to:

$$\sqrt{\lambda}B - \sigma A = 0 \Rightarrow A = \frac{\sqrt{\lambda}}{\sigma}B$$

For a non-trivial solution, A and B must not both be zero.

$$\begin{aligned} (\sigma B - A\sqrt{\lambda}) \sin(\sqrt{\lambda}l) + (\sqrt{\lambda}B + \sigma A) \cos(\sqrt{\lambda}l) &= 0 \\ \Rightarrow (\sigma - \frac{\lambda}{\sigma}) \sin(\sqrt{\lambda}l) + 2\sqrt{\lambda} \cos(\sqrt{\lambda}l) &= 0 \\ \Rightarrow (\sigma^2 - \lambda) \sin(\sqrt{\lambda}l) + 2\sqrt{\lambda}\sigma \cos(\sqrt{\lambda}l) &= 0 \end{aligned} \quad (2.7.12)$$

(a) If $\lambda = \sigma^2$, then $\cos \sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda}l = \frac{2n-1}{2}\pi \Rightarrow \lambda = \left(\frac{2n-1}{2\ell}\pi\right)^2 = \sigma^2$ (for $n = 1, 2, \dots$) (**a necessary condition!**). If there exists an n_0 (where $n_0 = 1, 2, \dots$) such that $\sigma = \frac{2n_0-1}{2\ell}\pi$, then there is a unique $\lambda_0 = \left(\frac{2n_0-1}{2\ell}\pi\right)^2 = \sigma^2$.

Since σ is given, there can only be one n_0 that makes $\sigma = \sqrt{\lambda_0}$. After σ is given, there is only one n_0 , i.e., one λ_0 .

The corresponding eigenfunction is:

$$\begin{aligned} X_0(x) &= \frac{\sqrt{\lambda_0}}{\sigma} \cos \sqrt{\lambda_0}x + \sin(\sqrt{\lambda_0}x) \\ &= \cos\left(\frac{2n-1}{2\ell}\pi x\right) + \sin\left(\frac{2n-1}{2\ell}\pi x\right) \\ &= \cos(\sigma x) + \sin(\sigma x) = \frac{2}{\sqrt{2}} \sin\left(\sigma x + \frac{\pi}{4}\right). \end{aligned}$$

(b) If $\lambda \neq \sigma^2$, both sides can be divided by $\cos(\sqrt{\lambda}l)(\neq 0)$. (2.7.12) leads to

$$\tan \gamma = \frac{2\sigma l \gamma}{\gamma^2 - (\sigma l)^2} \quad (\gamma = \sqrt{\lambda}l) \quad (2.7.13)$$

- In this case, $\sigma^2 - \lambda \neq 0$, if $(\sigma^2 - \lambda) \sin(\sqrt{\lambda}\ell) = 0 \Rightarrow \sin(\sqrt{\lambda}\ell) = 0$ and $\cos(\sqrt{\lambda}\ell) = 0$, which ensures (2.7.12). However, they cannot both be zero at the same time.

Summary Notes on Transcendental Equations

- **Solving Transcendental Equations:**
 - Generally, solving transcendental equations graphically or using computational methods to find approximate solutions.
- **Observation of Periodic Functions:**
 - The curve of the function on the left hand side $\tan \gamma$ is **periodic** from $-\infty$ to $+\infty$.
- **Behavior of Rational Functions:**
 - The rational function is defined from $-\infty$ to $+\infty$.
 - It consistently spans from **negative to positive infinity**.
- **Intersections and Eigenvalues:**
 - The graph will have **infinitely many intersections** with the x -axis.
 - This implies the existence of **infinitely many, but countable, discrete eigenvalues**.
- **Existence and Distribution of Eigenvalues:**
 - We can establish the existence of eigenvalues and understand their distribution properties.
- **Estimation of Eigenvalues:**
 - Estimating eigenvalues is desirable but often challenging.
- **Sufficiency of Information:**
 - With the knowledge of eigenvalue existence and distribution, we consider ourselves informed.
- **Challenges in Nonlinear Equations:**
 - Achieving precise knowledge in the study of nonlinear equations is difficult.
- **Obtaining Eigenfunctions:**
 - Once eigenvalues are determined, eigenfunctions can be found within the corresponding space.
- **Application in Chapter 5:**
 - In the study of **Bessel functions**, eigenvalues are determined and recorded similarly.

Thus, this equation has infinitely many positive roots, denoted as $\gamma_1, \gamma_2, \gamma_n, \dots$, and the corresponding eigenvalues $\lambda_n = (\frac{\gamma_n}{l})^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are:

$$\begin{aligned} X_n(x) &= A_n \cos(\sqrt{\lambda_n}x) + B_n \sin(\sqrt{\lambda_n}x) \\ &= B_n \left(\sin \frac{\gamma_n x}{l} + \frac{\gamma_n}{\sigma l} \cos \frac{\gamma_n x}{l} \right) \\ &= K_n \sin \left(\frac{\gamma_n x}{l} + \theta_n \right) \end{aligned}$$

where $\theta_n = \arctan \frac{\sqrt{\lambda_n}}{\sigma}$ and $K_n = \frac{B_n}{\sigma l} \sqrt{(\sigma l)^2 + \gamma_n^2}$. Therefore, all the eigenvalues are $\lambda_0 = \sigma^2$, $\lambda_n = (\frac{\gamma_n}{l})^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are $X_0(x) = \sin(\sigma x + \frac{\pi}{4})$, $X_n(x) = \sin(\frac{\gamma_n x}{l} + \theta_n)$, $n = 1, 2, \dots$

If $\lambda = \lambda_n$, substitute into the T-equation to get $T_n(t) = C_n e^{-a^2 \lambda_n t}$, $n = 0, 1, 2, \dots$

The superposition solution is:

$$u(x, t) = D_0 e^{-a^2 \sigma^2 t} \sin \left(\sigma x + \frac{\pi}{4} \right) + \sum_{n=1}^{\infty} D_n e^{-a^2 \lambda_n t} \sin \left(\frac{\gamma_n x}{l} + \theta_n \right).$$

Initial conditions determines the coefficients:

$$D_0 = \frac{\int_0^l \sin(\sigma x + \frac{\pi}{4}) \varphi(x) dx}{\int_0^l \sin^2(\sigma x + \frac{\pi}{4}) dx}.$$

$$D_n = \frac{\int_0^l \sin(\frac{\gamma_n x}{l} + \theta_n) \varphi(x) dx}{\int_0^l \sin^2(\frac{\gamma_n x}{l} + \theta_n) dx} \quad (n = 1, 2, \dots).$$

(b) If for any positive integer n , there exists $\sigma \neq \frac{(2n-1)\pi}{2l}$, then σ is not a root of the transcendental equation (2.7.12). At this time, (2.7.12) and (2.7.13) have the same solution. Compared to case (a), there is one less λ_0 .

The eigenvalue is $\lambda_n = (\frac{\gamma_n}{l})^2$, $n = 1, 2, \dots$

The corresponding eigenfunctions are $X_n(x) = \sin(\frac{\gamma_n x}{l} + \theta_n)$, $n = 1, 2, \dots$ where γ_n is the n -th positive root of the equation (2.7.13), and substitute $\lambda = \lambda_n$ into the equation, we obtain $T_n(t) = C_n e^{-a^2 \lambda_n t}$, $n = 1, 2, \dots$

The superposition solution is: $u(x, t) = \sum_{n=1}^{\infty} D_n e^{-a^2 \lambda_n t} \sin(\frac{\gamma_n x}{l} + \theta_n)$.

$$\text{Initial conditions determines the coefficients: } D_n = \frac{\int_0^l \sin(\frac{\gamma_n x}{l} + \theta_n) \varphi(x) dx}{\int_0^l \sin^2(\frac{\gamma_n x}{l} + \theta_n) dx} \quad (n = 1, 2, \dots).$$

Ex 2.7.6. Solve the initial boundary value problem:

$$\begin{cases} u_t = a^2 u_{xx} - b^2 u \\ u(0, t) = u(l, t) = 0 \\ u(x, 0) = \varphi(x) \end{cases}$$

where $0 < x < l$, $t > 0$ and b is the known constant.

- Question: Is the equation homogeneous or non-homogeneous?
- Conclusion: The equation is homogeneous.
- Reason: The term $-b^2 u$ in question is related to u , not just the independent variable. Therefore, it is not a free term.
- The standard method is to directly use the separation of variables in five steps.
- The idea: This kind of adjacent order form can always be combined into one by introducing a new function $v = e^{bt}u$, thus transforming the equation into a simpler form.
- Note that

$$e^{b^2 t} (u_t + b^2 u) = e^{b^2 t} u_t + b^2 e^{b^2 t} u = \underbrace{(e^{b^2 t} u)_t}_{v_t} = a^2 \underbrace{(e^{b^2 t} u)_{xx}}_{v_{xx}}$$

Solution. Let $u(x, t) = e^{-b^2 t} v(x, t)$. The original problem can be transformed into:

$$\begin{cases} v_t = a^2 v_{xx} & 0 < x < l, t > 0 \\ v(0, t) = v(l, t) = 0 & t \geq 0 \\ v(x, 0) = \varphi(x) & 0 \leq x \leq l \end{cases}$$

This becomes a classical heat conduction (mixed) problem.

Using the result from section 8.2, we get:

$$v(x, t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2 \pi^2}{l^2} t} \sin\left(\frac{n\pi x}{l}\right)$$

where

$$C_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad n = 1, 2, \dots$$

The solution to the original problem is:

$$u(x, t) = e^{-b^2 t} \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) dx \right) e^{-\frac{n^2 \pi^2}{l^2} t} \sin\left(\frac{n\pi x}{l}\right).$$

Ex 2.7.7. A rod of length l has an initial uniform temperature of 0°C . At $x = 0$, it maintains a constant temperature u_0 , while at the ends $x = l$ and the lateral surface, it has heat exchange with the surrounding medium at 0°C . The temperature distribution $u(x, t)$ satisfies:

$$\begin{cases} u_t = a^2 u_{xx} \boxed{-b^2 u} & \leftarrow (\text{Integ. factor}) \\ u(0, t) = u_0 & \leftarrow (\text{Non-homog. bdry}) \leftarrow (\text{Simultaneous homogenization \& auxiliary func.}) \\ (u_x + \sigma u)|_{x=l} = 0 & \leftarrow (\text{Third kind}) \\ u(x, 0) = 0 \end{cases}$$

for $0 < x < l$, $t > 0$ and $\sigma > 0$. Solve for $u(x, t)$.

Solution. • Auxiliary function method. Let $u = v + w$. However, by transforming v , the equation can be simplified (simplified equation).

Let $u(x, t) = e^{-b^2 t} v(x, t) + w(\textcolor{red}{x})$. We calculate

$$\begin{aligned} \left\{ \begin{array}{l} u_t = e^{-b^2 t} v_t - b^2 e^{-b^2 t} v \\ u_{xx} = e^{-b^2 t} v_{xx} + w'' \end{array} \right. \\ \Rightarrow e^{-b^2 t} v_t - b^2 e^{-b^2 t} v = a^2 e^{-b^2 t} v_{xx} + \underbrace{a^2 w'' - b^2 w}_{\text{Let it }=0} = b^2 e^{-b^2 t} v \end{aligned}$$

Then we arrive at

$$\begin{cases} v_t - a^2 v_{xx} = 0 & 0 < x < \ell, t > 0 \\ v(0, t) = (u_x + \sigma v)|_{x=\ell} = 0 & t \geq 0 \\ v(x, 0) = -w & \end{cases} \quad (2.7.14)$$

and

$$\begin{cases} a^2 w'' - b^2 w = 0, & 0 < x < \ell \\ w(0) = u_0 \\ w'(\ell) + \sigma w(\ell) = 0 \end{cases} \quad (2.7.15)$$

The general solution of (2.7.15) is $w(x) = A e^{\frac{b}{a}x} + B e^{-\frac{b}{a}x}$ and $w'(x) = A \frac{b}{a} e^{\frac{b}{a}x} - B \frac{b}{a} e^{-\frac{b}{a}x}$, using boundary conditions.

$$\begin{aligned} \left\{ \begin{array}{l} w(0) = A + B = u_0 \\ w'(\ell) + \sigma w(\ell) = A \frac{b}{a} e^{\frac{b}{a}\ell} - B \frac{b}{a} e^{-\frac{b}{a}\ell} + \sigma A e^{\frac{b}{a}\ell} + \sigma B e^{-\frac{b}{a}\ell} = 0 \end{array} \right. \\ \Rightarrow \begin{cases} A = \frac{u_0 (\frac{b}{a} - \sigma) e^{-\frac{b}{a}\ell}}{2(\frac{b}{a} \cosh \frac{b}{a}\ell + \sigma \sinh \frac{b}{a}\ell)} \\ B = \frac{u_0 (\frac{b}{a} + \sigma) e^{\frac{b}{a}\ell}}{2(\frac{b}{a} \cosh \frac{b}{a}\ell + \sigma \sinh \frac{b}{a}\ell)} \end{cases} \\ \Rightarrow w(x) = \frac{\frac{1}{a} \cosh \frac{b}{a}(x - \ell) - \sigma \sinh \frac{b}{a}(x - \ell)}{\frac{b}{a} \cosh \frac{b}{a}\ell + \sigma \sinh \frac{b}{a}\ell} u_0 \end{aligned}$$

The system (2.7.14) can be solved by separation of variables. Let $u(x, t) = X(x)T(t)$, we get

$$\begin{aligned} XT' - a^2 X'' T = 0 & \Rightarrow \frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda \\ & \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + a^2 \lambda T = 0 \end{cases} \end{aligned}$$

Boundary conditions

$$\begin{cases} X(0) = 0 \\ X'(\ell) + \sigma X(\ell) = 0 \end{cases}$$

Then the Sturm-Liouville problem is

$$\begin{cases} X'' + \lambda X = 0 & (0 < x < \ell) \\ X(0) = X'(\ell) + \sigma X(\ell) = 0 \end{cases}$$

By the energy method,

$$\begin{aligned} \int_0^\ell X X'' dx + \int_0^\ell \lambda X^2 dx &= 0 \quad \xrightarrow{\text{Integration by parts}} \quad X X' \Big|_0^\ell - \int_0^\ell (X')^2 dx + \int_0^\ell \lambda X^2 dx = 0 \\ \Rightarrow X(\ell) X'(\ell) + \lambda \int_0^\ell X^2 dx &= \int_0^\ell (X')^2 dx \\ \Rightarrow -\sigma(X(\ell))^2 + \lambda \int_0^\ell x^2 dx &= \int_0^\ell (X')^2 dx \\ \Rightarrow \lambda \int_0^\ell X^2 dx &= \sigma(X(\ell))^2 + \int_0^\ell (X')^2 dx \geq 0 \\ \Rightarrow \lambda \geq 0, \text{ if } \lambda = 0 \Rightarrow \int_0^\ell (X')^2 dx &= -\sigma(X(\ell))^2 \leq 0 \\ \Rightarrow X' \equiv 0 \quad \Rightarrow \quad X(x) = \text{const.} \quad \Rightarrow \quad X(x) \equiv 0 \quad \Rightarrow \quad \lambda > 0. \end{aligned}$$

If $\lambda > 0$, then the general solution is $X = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x$. Therefore $X'(x) = -A\sqrt{\lambda} \sin \sqrt{\lambda}x + B\sqrt{\lambda} \cos \sqrt{\lambda}x$.

Then $X(0) = A = 0$ and $X'(\ell) + \sigma X(\ell) = B\sqrt{\lambda} \cos \sqrt{\lambda}\ell + \sigma B \sin \sqrt{\lambda}\ell = 0$.

$\Rightarrow \sqrt{\lambda} \cos \sqrt{\lambda}\ell = -\sigma \sin \sqrt{\lambda}\ell$ ($\neq 0$ unless cos and sin are zero at the same time, impossible).

$$\Rightarrow \sqrt{\lambda} + \sigma \tan \sqrt{\lambda}\ell = 0. \quad (2.7.16)$$

The eigenvalues are the n -th positive roots of the transcendental equation (denoted as λ_n), and the eigenfunctions are $X_n(x) = \sin \sqrt{\lambda_n}x$. Putting λ_n into the T-equation leads to $T_n(t) = C_n e^{-a^2 \lambda_n t}$.

Then the superposition series solution is

$$v(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 \lambda_n t} \sin \sqrt{\lambda_n}x.$$

Initial data determine the coefficients

$$C_n = \frac{- \int_0^\ell w(x) \sin \sqrt{\lambda_n}x dx}{\int_0^\ell \sin^2 \sqrt{\lambda_n}x dx} = \frac{- \int_0^\ell \frac{1}{a} \cosh \frac{b}{a}(x-\ell) - \sigma \sinh \frac{b}{a}(x-\ell)}{\frac{b}{a} \cosh \frac{b}{a}\ell + \sigma \sinh \frac{b}{a}\ell} u_0 \sin \sqrt{\lambda_n}x dx.$$

Ex 2.7.8. The longitudinal vibration equation for a frustum with height l , top radius r , and bottom radius R is

$$\left(1 - \frac{x}{l}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2}{\partial x^2} \left[\left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial x} \right]$$

where $a^2 = \frac{E}{\rho}$ is the modulus of elasticity, ρ is the density, and $h = \ell R / (R - r)$. If the ends of the frustum are rigidly fixed, with initial displacement $f(x)$ and initial velocity 0, find the longitudinal displacement $u(x, t)$.

Solution. The corresponding spatial problem is

$$\begin{cases} \left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x} \left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x} \right] & 0 < x < \ell, t > 0 \\ u(x, 0) = f(x), u_t(x, 0) = 0 & 0 \leq x \leq \ell \\ u(0, t) = u(\ell, t) = 0 & t \geq 0 \end{cases}$$

Integrating factor:

Step-by-step analysis of

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{(h-x)^2} \frac{\partial}{\partial x} \left((h-x)^2 \frac{\partial u}{\partial x} \right).$$

Focus on term

$$\frac{1}{(h-x)^2} \frac{\partial}{\partial x} \left((h-x)^2 \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{(h-x)^2}{(h-x)^2} \frac{\partial u}{\partial x} \right) - \frac{2}{(h-x)^3} \cdot (h-x)^2 \frac{\partial u}{\partial x} = u_{xx} - \frac{2}{h-x} u_x.$$

The equation becomes

$$u_{tt} = a^2 \left(u_{xx} - \frac{2}{h-x} u_x \right).$$

Ideas:

- $u_{xx} + Au_x$ can be combined into a derivative of a single term using the exponential function.
- The form $u_{xx} + \frac{A}{x} u_x$ also has a similar integrating factor.

Beginners can use the method of the function to be determined to find the integrating factor.

Let a function $A(x)$, guess from the form that it is independent of t , and hope that the u_t term does not appear.

Multiply $A(x)$ to both sides of the above equation.

$$(A(x)u)_{tt} = a^2 \underbrace{\left(A(x)u_{xx} - \frac{2}{h-x} u_x A(x) \right)}_{\text{Hope}=(A(x)u)_{xx}}$$

First we calculate

$$(A(x)u)_x = A'u + A(x)u_x \quad \text{and} \quad (A(x)u)_{xx} = A''u + 2A'u_x + Au_{xx}.$$

From the hope $(A(x)u)_{xx} = A(x)u_{xx} - \frac{2}{h-x} u_x A(x)$, we get $A''u + 2A'u_x + Au_{xx} = Au_{xx} - \frac{2}{h-x} Au_x$,

$$\Rightarrow A''u + 2 \left(A' + \frac{1}{h-x} \right) u_x = 0.$$

Try $A'' = 0$ (under this enforcing condition, we choose an integrating factor). We only need to require

$$A' + \frac{1}{h-x} A = 0.$$

$$\Rightarrow (\ln A)' = -\frac{1}{h-x} \Rightarrow \ln A = \int \frac{1}{x-h} dx + C = \ln(x-h) + C.$$

$$\Rightarrow A = C(x-h) \quad \text{while satisfying } A'' = 0.$$

For simplicity, we take $A(x) = x-h$.

Then the equation can be written as $((x-h)u)_{tt} = a^2((x-h)u)_{xx}$ and let $v = (x-h)u$. Then the equation becomes $v_{tt} = a^2v_{xx}$. The problem becomes:

$$\begin{cases} v_{tt} = a^2v_{xx} \\ v(x, 0) = (x-h)u(x, 0) = (x-h)f(x), v_t(x, 0) = (x-h)u_t(x, 0) = 0 \\ v(0, t) = -hu(0, t) = 0, v(\ell, t) = (\ell-h)u(\ell, t) = 0 \end{cases}$$

That is

$$\begin{cases} v_{tt} = a^2v_{xx} \\ v(x, 0) = (x-h)f(x), v_t(x, 0) = 0 \\ v(0, t) = v(\ell, t) = 0 \end{cases}$$

This is a form that everyone is familiar with, and can be found in §2.1.

$$v(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{\ell} + b_n \sin \frac{n\pi at}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

where

$$\begin{cases} a_n = \frac{2}{\ell} \int_0^{\ell} (x-h)f(x) \sin \frac{n\pi x}{\ell} dx \\ b_n = 0 \end{cases}.$$

$$\Rightarrow v(x, t) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell} \int_0^{\ell} (x-h)f(x) \sin \frac{n\pi x}{\ell} dx \right) \cos \frac{n\pi at}{\ell} \sin \frac{n\pi x}{\ell}.$$

$$\Rightarrow u(x, t) = \frac{v(x, t)}{x-h} = \frac{2}{(x-h)\ell} \sum_{n=1}^{\infty} \left(\int_0^{\ell} (\xi-h)f(\xi) \sin \frac{n\pi \xi}{\ell} d\xi \right) \cos \frac{n\pi at}{\ell} \sin \frac{n\pi x}{\ell}.$$

Ex 2.7.9. Consider a circular ring-shaped heat conductor with an inner radius r_1 and an outer radius r_2 , insulated on the top and bottom. If the temperature of the inner circle is maintained at zero degrees, and the temperature of the outer circle is maintained at u_0 ($u_0 > 0$), find the temperature distribution $u(r, \theta)$ in the steady state. The problem is reduced to solving the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ in the steady state, which is a boundary value problem in polar coordinates:

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, & r_1 < r < r_2, 0 < \theta < 2\pi, \\ u(r_1, \theta) = 0, u(r_2, \theta) = u_0, & 0 < \theta < 2\pi, \\ u(r, \theta) = u(r, \theta + 2\pi) & (\text{natural boundary condition}). \end{cases}$$

- This problem does **not include the origin**, so there is **no need** to add a **boundedness condition at the origin**. A **periodicity condition** is sufficient.

Solution (Method 1: Ignore the spherical symmetry). 1. Assume the separated solution form is:

$$u(r, \theta) = R(r)\Phi(\theta) \quad (\text{separation of variables})$$

2. PDE → ODEs

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0 \Rightarrow -\frac{rR'' + rR'}{R} = \frac{\Phi''}{\Phi} = -\lambda$$

Thus, we have the following system of equations:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ r^2R'' + rR' - \lambda R = 0 \end{cases}$$

This implies:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(\theta) = \Phi(\theta + 2\pi) \end{cases} \quad (\text{S-L problem})$$

and

$$r^2R'' + rR' - \lambda R = 0$$

3. Solve ODEs

- When $\lambda < 0$, the general solution is $\Phi(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}$, which does not satisfy periodic boundary conditions, hence $A = B = 0$, no non-trivial solution.
- When $\lambda = 0$, the solution is $\Phi(\theta) = Ax + B$. To satisfy the periodic boundary conditions $\Phi(0) = \Phi(\theta + 2\pi)$,

$$\Rightarrow A = 0 \Rightarrow \Phi_0(\theta) = B_0$$

Solve ODE of R : $r^2R'' + rR' = 0$. The solution is $R(r) = C_0 \ln r + D_0$. Therefore, $u_0(r, \theta) = a_0 \ln r + b_0$.

- When $\lambda > 0$, the solution is

$$\Phi(\theta) = A \cos(\sqrt{\lambda}\theta) + B \sin(\sqrt{\lambda}\theta).$$

If $\Phi(\theta)$ is periodic with period 2π , then

$$\sqrt{\lambda}(\theta + 2\pi) = \sqrt{\lambda}\theta + 2n\pi \Rightarrow 2\sqrt{\lambda}\pi = 2n\pi \Rightarrow \lambda_n = n^2 \quad (n = 1, 2, \dots)$$

Thus,

$$\Phi_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

Solve ODE of R :

$$\begin{aligned} r^2 R'' + r R' - n^2 R = 0 \quad (\text{Euler equation}) \Rightarrow R_n(r) = C_n r^n + D_n \frac{1}{r^n} \quad (n = 1, 2, \dots) \\ \Rightarrow u_n(r, \theta) = (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r^n + D_n \frac{1}{r^n} \right) \end{aligned}$$

4. Superposition of solutions

$$u(r, \theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r^n + D_n \frac{1}{r^n} \right)$$

5. Determine coefficients

$$\begin{cases} u(r_1, \theta) = C_0 \ln r_1 + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \\ u(r_2, \theta) = C_0 \ln r_2 + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = u_0 \end{cases}$$

This leads to:

$$\begin{cases} C_0 \ln r_1 + D_0 = 0 \\ A_n \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \\ B_n \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \end{cases} \quad \text{and} \quad \begin{cases} C_0 \ln r_2 + D_0 = u_0 \\ A_n \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = 0 \\ B_n \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = 0 \end{cases}$$

Thus, we have the system of equations:

$$\begin{cases} C_0 \ln r_1 + D_0 = 0 \\ C_0 \ln r_1 + D_0 = u_0 \end{cases} \Rightarrow \begin{cases} C_0 = -\frac{u_0}{\ln \frac{r_1}{r_2}} = \frac{u_0}{\ln \frac{r_2}{r_1}} \\ D_0 = -\frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} \end{cases}$$

For the coefficients:

$$\begin{cases} A_n C_n r_1^n + A_n D_n \frac{1}{r_1^n} = 0 \\ A_n C_n r_2^n + A_n D_n \frac{1}{r_2^n} = 0 \end{cases} \Rightarrow \begin{cases} A_n C_n = 0 \\ A_n D_n = 0 \end{cases}$$

and similarly,

$$\begin{cases} B_n C_n = 0 \\ B_n D_n = 0 \end{cases}$$

Finally, the solution is:

$$u(r, \theta) = \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r - \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r_1 = u_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}$$

Solution (Method 2: Note the spherical symmetry). From the boundary conditions, it is known that the circular ring has spherical symmetry, hence $u(r, \theta) = u(r)$, which is independent of θ .

The equation becomes

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0 \Rightarrow \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$\Rightarrow r \frac{du}{dr} = A \Rightarrow \frac{du}{dr} = \frac{A}{r} \Rightarrow du = \frac{A}{r} dr \Rightarrow u(r) = A \ln r + B$$

Using boundary conditions $u(r_1) = A \ln r_1 + B = 0$ and $u(r_2) = A \ln r_2 + B = u_0$, we get:

$$\begin{cases} A = \frac{u_0}{\ln \frac{r_2}{r_1}} \\ B = -\frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} \end{cases}.$$

Thus, the solution is:

$$u(r) = \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r - \frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} = u_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}.$$

Ex 2.7.10. Find the solution to the following boundary value problem:

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, & 0 < r < 1, 0 < \theta < \frac{\pi}{2}, \\ u(r, 0) = 0, u(r, \frac{\pi}{2}) = 0, & 0 < r < 1, \\ u(1, \theta) = \theta (\frac{\pi}{2} - \theta), & 0 < \theta < \frac{\pi}{2}. \end{cases}$$

- Note that the problem has $0 < \theta < \frac{\pi}{2}$, hence it does **not have periodic boundary conditions** but has boundary conditions. However, the problem has **homogeneous boundary conditions** regarding θ , which are the conditions we need to establish a Sturm-Liouville (S-L) problem.

Solution. 1. Assume the separated solution form:

$$u(r, \theta) = R(r)\Phi(\theta).$$

(separation of variables)

2. PDE transforms into ODEs:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ r^2 R'' + rR' - \lambda R = 0. \end{cases}$$

From the boundary conditions $u(r, 0) = 0$, $u(r, \frac{\pi}{2}) = 0$, we get:

$$\begin{aligned} R(r)\Phi(0) &= 0, & R(r)\Phi(\frac{\pi}{2}) &= 0 \\ \Rightarrow \Phi(0) &= 0, & \Phi(\frac{\pi}{2}) &= 0 \end{aligned}$$

otherwise $R(r) = 0$ has no non-trivial solution.

3. Solving ODEs: (a) The S-L problem is given by:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = 0, \Phi(\frac{\pi}{2}) = 0 \end{cases} \leftarrow \boxed{\text{Homog. bdry instead of periodic bdry!}}$$

The eigenvalues and corresponding eigenfunctions are (with $l = \frac{\pi}{2}$):

$$\lambda = \lambda_n = \left(\frac{n\pi}{\pi/2} \right)^2 = (2n)^2 = 4n^2 \quad (n = 1, 2, \dots)$$

$$\Phi_n(\theta) = \sin(2n\theta) \quad (n = 1, 2, \dots)$$

(b) The ODE for R becomes:

$$r^2 R'' + rR' - 4n^2 R = 0$$

The general solution is:

$$R_n(r) = C_n r^{2n} + D_n r^{-2n} \quad (n = 1, 2, \dots)$$

Using the boundedness condition $|R_n(0)| < +\infty$ implies $D_n = 0$.

$$R_n(r) = C_n r^{2n}.$$

This implies

$$u_n(r, \theta) = C_n r^{2n} \sin(2n\theta) \quad (n = 1, 2, \dots)$$

4. Superposition of solutions:

$$u(r, \theta) = \sum_{n=1}^{\infty} C_n r^{2n} \sin(2n\theta)$$

5. Determine coefficients:

$$u(1, \theta) = \sum_{n=1}^{\infty} C_n \sin(2n\theta) = \theta \left(\frac{\pi}{2} - \theta \right)$$

Using Fourier series to find C_n ,

$$\begin{aligned} C_n &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \theta \left(\frac{\pi}{2} - \theta \right) \sin(2n\theta) d\theta = -\frac{2}{n\pi} \int_0^{\frac{\pi}{2}} \theta \left(\frac{\pi}{2} - \theta \right) d\cos(2n\theta) \\ &= -\frac{2}{n\pi} \left[\theta \left(\frac{\pi}{2} - \theta \right) \cos(2n\theta) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(2n\theta) \left(\frac{\pi}{2} - 2\theta \right) d\theta \right] \\ &= \frac{2}{n\pi} \int_0^{\frac{\pi}{2}} \cos(2n\theta) \cdot \left(\frac{\pi}{2} - 2\theta \right) d\theta \\ &= \frac{\pi}{n\pi} \int_0^{\frac{\pi}{2}} \cos(2n\theta) d\theta - \frac{2 \cdot 2}{n\pi} \int_0^{\frac{\pi}{2}} \theta \cos(2n\theta) d\theta \\ &= \frac{1}{2n\pi} \sin(2n\theta) \Big|_0^{\frac{\pi}{2}} - \frac{2}{n^2\pi} \int_0^{\frac{\pi}{2}} \theta d\sin(2n\theta) \\ &= -\frac{2}{n^2\pi} \left[\theta \sin(2n\theta) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(2n\theta) d\theta \right] \\ &= \frac{2}{n^2\pi} \int_0^{\frac{\pi}{2}} \sin(2n\theta) d\theta = -\frac{1}{n^3\pi} \cos(2n\theta) \Big|_0^{\frac{\pi}{2}} = -\frac{1}{n^3\pi} (\cos(n\pi) - 1) \\ &= -\frac{1}{n^3\pi} ((-1)^n - 1) = \frac{1}{n^3\pi} [1 - (-1)^n]. \end{aligned}$$

Thus,

$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{n^3\pi} [1 - (-1)^n] r^{2n} \sin(2n\theta).$$

3

Method of Traveling Waves and Integral Transform Method

Structure of Chapter 3

- **Section 1:** Solution of the (1+1)D wave equation using the **traveling wave method**.
- **Section 2:** Solutions for (1+2)D and (1+3)D wave equations:

$$\left. \begin{array}{l} \text{1D: Traveling wave method.} \\ \text{2D: Dimensional reduction method.} \\ \text{3D: Spherical averaging method.} \end{array} \right\} \text{Wave methods}$$

- These methods apply **only to wave equations** with **initial data**, unlike previous methods that work for various types of equations. i.e.,

wave equations + initial data (no boundary) → wave methods

- **Section 3:** Introduces the **integral transform method**, which is independent of the previous sections.

Boundary Conditions and Initial Value Problems

- The problems addressed in Sections 1 and 2 are **initial value problems** (Cauchy problems).
- There are **no boundary conditions** since the spatial domain extends to infinity.
- This assumption is valid when boundaries are far from the region of interest.

In this chapter, we will introduce two other methods for solving boundary value problems: one is the

method of traveling waves (also known as d'Alembert's method), and the other is the integral transform method.

The method of traveling waves can only be used to solve the problems of wave equations within **unbounded regions**. Although it has significant limitations, it has special advantages for **wave problems**, making it one of the fundamental methods for solving differential equations.

The **integral transform method** is **not limited** by the type of equation and is mainly used for unbounded regions, but it can also be applied to bounded regions.

3.1 D'Alembert's Formula: Wave Propagation

3.1.1 D'Alembert's Solution for the String Vibration Equation

If the length of the string we are examining is very long, and what we need to know is only the vibration situation within a relatively short time and far from the boundary, then the influence of the boundary conditions can be ignored. We might as well consider the length of the string we are examining as infinite, and what we need to know is only the vibration situation within a finite range.

At this point, the boundary value problem is summarized in the following form:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), & u_t(x, 0) = \psi(x) \end{cases} \quad (3.1.1)$$

Solution Strategy:

- **Isolate non-homogeneous terms** and solve individually.
- Use **Homogenization** for non-homogeneous parts.

Isolate non-homogeneous terms: For the above initial value problem, since the differential equation and the initial conditions are both **linear**, the **superposition principle** also holds. That is, if $u_1(x, t)$ and $u_2(x, t)$ are respectively the solutions to the following initial value problems:

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), & u_t(x, 0) = \psi(x) \end{cases} \quad (3.1.2)$$

and

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = 0, & u_t(x, 0) = 0 \end{cases} \quad (3.1.3)$$

then $u = u_1(x, t) + u_2(x, t)$ is the solution to the original problem (3.1.1).

This indicates that the comprehensive effect of the external factors represented by $f(x, t)$ and the initial vibration state represented by $\varphi(x), \psi(x)$ on the entire vibration process can be decomposed into the superposition of the effects produced by considering **only the external factors** and **only the initial vibration state** on the vibration process.

Consider the wave equation and initial conditions (3.1.2), that is,

$$u_{tt} = a^2 u_{xx} \quad (-\infty < x < +\infty, t > 0), \quad (3.1.4)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \quad (3.1.5)$$

Recall: Two Basic Approaches to Solving Equations

- **Trial Method (Guessing):** Chapter 2;
- **Inverse Method:** Chapter 3 and 4.

Core Idea

- A differential equation can be solved by finding its **inverse operation**, which is **integration**.
- Direct integration of the wave equation is difficult due to the presence of two independent variables $u_{tt} - a^2 u_{xx}$.
- The challenge: **which variable to integrate first** (time t or space x)? Can not get rid of second order derivatives in each variable.

Solution Strategy

- To simplify integration, a **variable transformation** is required. Since the principle

$$\text{unfamiliar eqs.} \xrightarrow[\text{(2) variable transf.}]{\text{(1) unknown transf.}} \text{familiar eqs.}$$

- The goal is to convert the equation into a form that allows **separable integration**, i.e., hope transform it to $u_{\xi\eta} = 0 \Rightarrow u_\xi = f_0(\xi) \Rightarrow u = f(\xi) + g(\eta)$.
- If successful, the equation can be integrated twice to obtain the solution, i.e., $u(\xi, \eta) = f(\xi) + g(\eta)$.
- The presence of two arbitrary functions corresponds to the **second-order nature** of the wave equation.
- The question becomes:

- How to transform $u_{tt} = a^2 u_{xx} \rightarrow u_{\xi\eta} = 0$?

- **Variable Transformation:**

- Use coordinate transformations (e.g., $\xi = x - ct$, $\eta = x + ct$) to transform the wave equation (**Why?**).

- Method 1: Matrix Method

$$\begin{pmatrix} \partial_t, \partial_x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -a^2 \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_t, \partial_x \end{pmatrix} Q^{-1}}_{= \begin{pmatrix} \partial_\xi, \partial_\eta \end{pmatrix}} Q \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -a^2 \end{pmatrix}}_{\text{Hope} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}} Q^T \underbrace{(Q^T)^{-1}}_{=: \begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}$$

By the knowledge of eigenvalues and eigenvectors, we conclude $\mu = -2a^2$, and

$$Q = \begin{pmatrix} a & 1 \\ -a & 1 \end{pmatrix}.$$

The inverse of Q can be computed as:

$$Q^{-1} = \begin{pmatrix} \frac{1}{2a} & \frac{1}{2} \\ -\frac{1}{2a} & \frac{1}{2} \end{pmatrix}.$$

Conclusion

- The key to solving the wave equation lies in finding an appropriate transformation.
- Once transformed, the equation can be integrated sequentially to determine the solution.

General solutions of waves: First, we examine the problem through a **variable transformation** (see Fig. 3.1, $\xi = \text{Constant}$ and $\eta = \text{Constant}$, the coordinate lines):

$$\boxed{\xi = x - at, \quad \eta = x + at}, \quad \leftarrow \quad \boxed{\text{How to find it?}} \quad (3.1.6)$$

with the inverse transformation:

$$x = \frac{\xi + \eta}{2}, \quad t = \frac{\eta - \xi}{2a}.$$

Let $\bar{u} = \bar{u}(\xi, \eta)$ be the new unknown function, then:

$$u(x, t) = \bar{u}\left(\frac{\xi + \eta}{2}, \frac{\eta - \xi}{2a}\right) = \bar{u}(\xi, \eta).$$

Using the **chain rule** for composite functions, we obtain:

$$u_x = \bar{u}_\xi \cdot \xi_x + \bar{u}_\eta \cdot \eta_x = \bar{u}_\xi + \bar{u}_\eta,$$

$$u_{xx} = (\bar{u}_{\xi\xi} \cdot \xi_x + \bar{u}_{\xi\eta} \cdot \eta_x) + (\bar{u}_{\eta\xi} \cdot \xi_x + \bar{u}_{\eta\eta} \cdot \eta_x) = \bar{u}_{\xi\xi} + 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta}, \quad (3.1.7)$$

and similarly,

$$u_{tt} = a^2(\bar{u}_{\xi\xi} - 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta}). \quad (3.1.8)$$

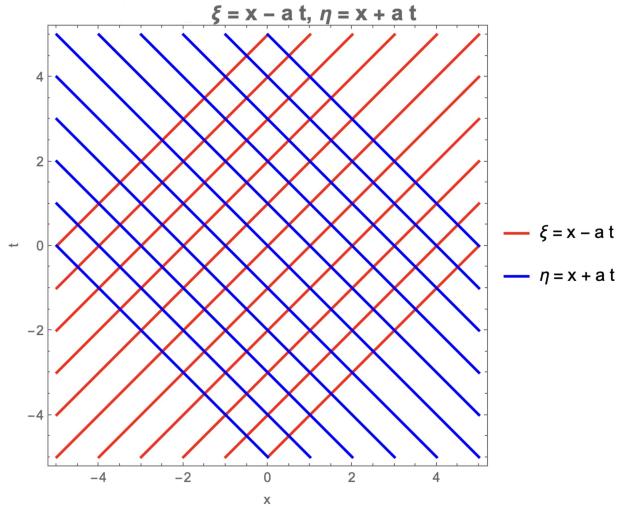


Figure 3.1: Coordinate $\xi = x - at$, $\eta = x + at$ for $a = 1$

Substituting (3.1.7) and (3.1.8) into equation (3.1.4) simplifies to:

$$\bar{u}_{\xi\eta} = 0. \leftarrow \boxed{\text{This is our expecting form!}} \quad (3.1.9)$$

Equation (3.1.9) can be solved directly by integration. Integrate once with respect to η , then integrate once with respect to ξ , to obtain the general solution of equation (3.1.9):

$$\bar{u}(\xi, \eta) = f(\xi) + g(\eta), \quad (3.1.10)$$

where f and g are arbitrary functions with continuous second derivatives.

Substituting the variable transformation (3.1.6) into (3.1.10) gives:

$$u(x, t) = f(x - at) + g(x + at). \leftarrow \boxed{\text{general solutions, traveling wave solutions}} \quad (3.1.11)$$

- The main technique used is **variable substitution**.
- After substitution, differentiation is performed using the **chain rule**.

Why we select the variable transformation (3.1.6) and how to find it:

There are many methods:

Method 2: Chain rules The idea is to transform the wave equation $u_{tt} = a^2 u_{xx}$ using a change of variables. If there is **only one second-order term** instead of two second-order terms, direct integration is possible. Target: one second-order term.

$$\partial_t^2 - a^2 \partial_x^2 = (\partial_t + a\partial_x)(\partial_t - a\partial_x) \quad \text{since (analog to)} \quad A^2 - B^2 = (A+B)(A-B), \quad (A = \partial_t, B = a\partial_x)$$

Then

$$(\partial_t^2 - a^2 \partial_x^2)u = \underbrace{(\partial_t + a\partial_x)}_{\text{Hope } \sim \partial_\eta} \underbrace{(\partial_t - a\partial_x)}_{\text{Hope } \sim \partial_\xi} u$$

This implies

$$\begin{aligned} \underbrace{\partial_\xi u}_{\substack{\text{chain rule} \\ = \frac{\partial t}{\partial \xi} \partial_t u + \frac{\partial x}{\partial \xi} \partial_x u}} &= l\partial_t u - al\partial_x u \quad \text{and} \quad \underbrace{\partial_\eta v}_{\substack{\text{chain rule} \\ = \frac{\partial t}{\partial \eta} \partial_t v + \frac{\partial x}{\partial \eta} \partial_x v}} = k\partial_t v - ak\partial_x v \end{aligned}$$

Then

$$\begin{cases} \frac{\partial t}{\partial \xi} = l \Leftrightarrow t = l\xi + C_1(\eta) \\ \frac{\partial x}{\partial \xi} = -al \Leftrightarrow x = -al\xi + C_3(\eta) \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial t}{\partial \eta} = k \Leftrightarrow t = k\eta + C_2(\xi) \\ \frac{\partial x}{\partial \eta} = ak \Leftrightarrow x = ak\eta + C_4(\xi) \end{cases}$$

Then

$$\begin{aligned} t &= l\xi + C_1(\eta) = k\eta + C_2(\xi) \Rightarrow t = l\xi + k\eta \\ x &= -al\xi + C_3(\eta) = ak\eta + C_4(\xi) \Rightarrow x = -al\xi + ak\eta \end{aligned}$$

If $l = k = 1$,

$$\begin{cases} x = -a\xi + a\eta \\ t = \xi + \eta \end{cases} \Rightarrow \begin{cases} \eta = \frac{1}{2}(t + \frac{x}{a}) \\ \xi = \frac{1}{2}(t - \frac{x}{a}) \end{cases}$$

If $k = \frac{1}{2a}, l = -\frac{1}{2a}$,

$$\begin{cases} x = \frac{1}{2}\xi + \frac{1}{2}\eta \\ t = -\frac{1}{2a}\xi + \frac{1}{2a}\eta \end{cases} \Rightarrow \begin{cases} \eta = x + at \\ \xi = x - at \end{cases}$$

These choices all work.

In the characteristic plane, the equation is an internal differential operator, which only restricts u and does not provide all derivative information.

Initial values determine f and g : Using the initial conditions (3.1.5) to determine the arbitrary functions f and g in the general solution (3.1.11), we substitute (3.1.5) into (3.1.11) to get:

$$f(x) + g(x) = \varphi(x), \tag{3.1.12}$$

$$-af'(x) + ag'(x) = \psi(x). \tag{3.1.13}$$

There are two methods to determine f and g :

- Integrating (3.1.13) (use this in the followings);
- Differentiating (3.1.12) (idea used in the next section).

Integrating (3.1.13) gives:

$$a(-f(x) + g(x)) + c = \int_{x_0}^x \psi(\alpha) d\alpha, \quad (3.1.14)$$

where x_0 is any point and c is the integration constant.

From (3.1.12) and (3.1.14), we derive:

$$\begin{cases} f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a} \int_{x_0}^x \psi(\alpha) d\alpha + \frac{c}{2a}, \\ g(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a} \int_{x_0}^x \psi(\alpha) d\alpha - \frac{c}{2a}. \end{cases} \quad (3.1.15)$$

Substituting (3.1.15) into the general solution (3.1.11) gives the solution to the initial value problem (3.1.4) and (3.1.5):

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha. \quad (3.1.16)$$

This formula is known as **D'Alembert's formula** for the **free vibration** of an **infinitely long string**, or simply **D'Alembert's solution**. This method of solving is called **D'Alembert's method**.

Ex 3.1.1. Use the method of characteristics to solve the following initial-boundary value problem:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, x - at < 0, x > 0 \\ u|_{x-at=0} = \varphi(x), & u|_{x=0} = h(t). \end{cases}$$

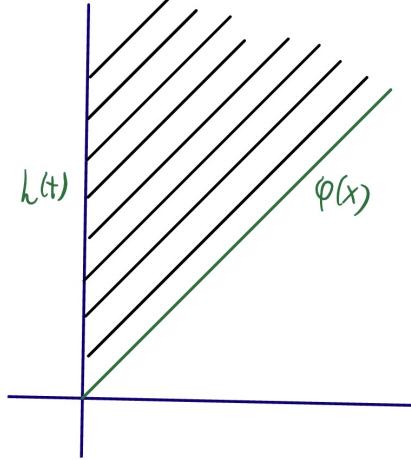


Figure 3.2: Goursat Problem

Solution. The general solution: The general solution of the wave equation is:

$$u(x, t) = f(x - at) + g(x + at). \leftarrow \boxed{\text{Memorize the general solution of wave}}$$

where f, g are arbitrary functions with continuous second derivatives.

Next, we use **boundary conditions** to determine the arbitrary functions f, g . First, from the condition:

$$u|_{x-at=0} = \varphi(x) \implies f(0) + g(2x) = \varphi(x).$$

Let $\eta = 2x$,

$$\implies g(\eta) = \varphi\left(\frac{\eta}{2}\right) - f(0). \leftarrow \boxed{\text{Know how to determine } f \text{ and } g \text{ by conditions}}$$

Using the condition:

$$u|_{x=0} = h(t) \implies f(-at) + g(at) = h(t).$$

Let $\xi = -at$

$$\implies f(\xi) = h\left(-\frac{\xi}{a}\right) - g(-\xi) = h\left(-\frac{\xi}{a}\right) - \varphi\left(-\frac{\xi}{2}\right) + f(0).$$

Substituting $f(\xi)$, $g(\eta)$ into the general solution formula gives the solution to the characteristic boundary problem as:

$$\begin{aligned} u(x, t) &= h\left(-\frac{x-at}{a}\right) - \varphi\left(-\frac{x-at}{a}\right) + f(0) + \varphi\left(\frac{x+at}{2}\right) - f(0) \\ &= \varphi\left(\frac{x+at}{2}\right) - \varphi\left(\frac{at-x}{2}\right) + h\left(\frac{at-x}{a}\right). \end{aligned}$$

Summary:

1. Write down the **general solution** of the differential equation.
2. Apply the **specific boundary conditions** to the general solution.
3. Solve the resulting system of equations, which typically involves **two unknowns** and **two equations**.

Note on D'Alembert's Formula

Consider the wave equation solution:

$$u(x, t) = \frac{1}{2} [\varphi(x - at) + \varphi(x + at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha$$

To understand $u(x_0, t_0)$, observe the geometric interpretation through characteristics (see Fig. 3.3):

$$u(x_0, t_0) = \underbrace{\frac{1}{2} [\varphi(A) + \varphi(B)]}_{\text{Arithmetic mean of the initial positions}} + \underbrace{t_0 \cdot \frac{1}{2at_0} \int_A^B \psi(\alpha) d\alpha}_{t_0 \times \text{Integral average of the initial velocity}}$$

where $A = x_0 - at_0$ and $B = x_0 + at_0$.

This formula represents the **average of φ at points A and B** , plus t_0 times the **average of ψ between A and B** .

- **Multiplication by t_0 :** The multiplication by t_0 in the second term is to ensure **dimensional consistency** and to aid memory. It is crucial not to forget this multiplication.
- **Dimensional Analysis:** The term ψ represents initial velocity. When velocity is integrated, it results in displacement, which has a dimension of length. Multiplying by t_0 (time) ensures that the dimensions on both sides of the equation are consistent (length = speed \times time).

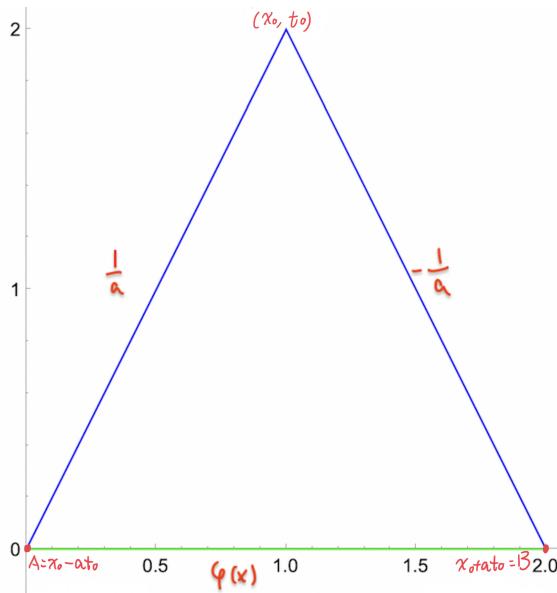


Figure 3.3: D'Alembert's formula

- **Formula Significance:** The formula indicates that the value of u **depends solely** on the information within the interval from $x - at$ to $x + at$. Information **outside** this interval has **no influence** on u .
- **Isolation of Information:** No matter what happens outside the interval $x - at$ to $x + at$, it does not affect the value of u at a specific point t . This demonstrates that u is isolated from external influences beyond this interval.
- **Importance of the Formula:** This formula is significant because it encapsulates the principle that the value of u is determined by local information within a specific segment, unaffected by external events.

3.1.2 Solving Non-Homogeneous Wave Equations

Consider the following wave equations and initial conditions (3.1.3), that is,

$$u_{tt} = a^2 u_{xx} + f(x, t) \quad (-\infty < x < +\infty, t > 0), \quad (3.1.17)$$

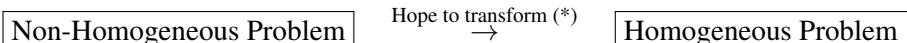
$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (3.1.18)$$

To solve problems (3.1.17) and (3.1.18), we use the **principle of homogenization** (or **Duhamel's principle**¹), transforming the **non-homogeneous** equation into a corresponding **homogeneous equation**, thus directly **utilizing previous results** on homogeneous equations.

Homogenization Principle (Duhamel's principle)

We have solved part of the problem (homogeneous equation + non-homogeneous initial conditions) in the previous section. Next, we need to solve the non-homogeneous equation + homogeneous initial conditions part.

The previous section discussed the initial value problem for the homogeneous equation, which only involved Cauchy problems without boundary conditions. Here, we introduce another method called the homogenization principle, which transforms the non-homogeneous equation problem into a homogeneous equation problem, allowing us to directly use the results from the homogeneous equation solutions.



- **Question:** How to transform?
- **Ideas:** By **physical insight** + mathematical **proof**.
- **Answer:** (*) is **Homogenization Principle**

- **Challenge without Boundary Conditions:** We currently lack the means to solve the problem as separation of variables and the method of eigenfunctions requires boundary conditions (due to S-L problem), which are not available.
- **Shift to Physical Insight:** Since mathematical methods are not directly applicable, we turn to physics. We interpret the physical meaning of the problem and use observations to guess what the solution might look like.
- **Verification with Mathematics:** Once a physical guess is made, we use mathematical methods to verify the solution. This approach combines physical intuition with mathematical rigor.

¹See https://en.wikipedia.org/wiki/Duhamel%27s_principle

- **Importance of Physical Reasoning:** The physical reasoning, though **involving approximations**, provides a well-founded basis for guessing solutions. However, mathematical proof is still necessary to confirm the validity of these solutions.
- **Historical Use of This Method:** This method has been used before, similar to how separation of variables starts with physical insights followed by mathematical confirmation. Here, the role of physics is more prominent.
- **Restoring Physical Context:** To apply physical reasoning, we must first understand the physical meaning of this problem.

We use the principle of impulse and the concept of definite integral to derive the homogenization principle for the vibration of an infinitely long string.

Physical Intuition

Simply speaking, Divide the time interval $[0, t]$ into infinitely many small time intervals. In each small time interval, the external force f can be concentrated at one point, such as the initial velocity of the time interval (where fdt is the impulse), and other parts of this time interval are considered to be free of external forces.

- The wave equation describes the vibration of a string.
- The source term f represents an external force density, meaning force per unit mass.
- Initial conditions:
 - Initial position is at equilibrium.
 - Initial velocity is zero (object is at rest).
- Without external force, the object remains at equilibrium without vibration.
- **Continuous external force density causes vibrations.**

Methodology: Discretization in Time

- Previous analysis used **spatial discretization** with Newton's Second Law.
- Now, we **discretize time instead of space**:
 - Imagine taking snapshots at discrete time intervals.
 - Divide time into segments: $t_0, t_1, t_2, \dots, t_n$.
- External force f is decomposed into **discrete force impulses** f_i :

- Defined as a **piecewise** function.
- f_i is **nonzero** only within specific time intervals $[t_{i-1}, t_i]$.

$$f_i(x, t) = \begin{cases} f(x, t) & \text{if } t_{i-1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

- Summing all f_i reconstructs the original force f .

Equation Decomposition and Solution

- Decomposing force f into impulses f_i , the equation transforms into:

$$\begin{aligned} u_{i,tt} &= u_{i,xx} + f_i \\ u_i|_{t=t_i} &= 0, \quad u_{i,t}|_{t=t_i} = 0. \end{aligned}$$

- The solution u is obtained by summing solutions of each impulse component (the principle of linear superposition):

$$u = \sum u_i$$

Three Phases of Motion

1. **Before the impulse:** $t \leq t_{i-1}$
 - **No external force**, system remains in equilibrium.
2. **During the impulse:** $t_{i-1} < t \leq t_i$
 - **External force impulse** f_i is applied, causing vibration.
3. **After the impulse:** $t > t_i$
 - **External force returns to zero**, system evolves dynamically.

Phase 1: Initial Equilibrium

- Time interval: $t_0 \leq t \leq t_{i-1}$.
- The equation simplifies as the external force $f_\ell = 0$.
- The system **remains at rest**: $u_\ell = 0$.
- The **final state** of this phase is the **initial state** for the **next phase**:
 - $u_{i-1} = 0$.
 - Initial velocity $u'_{i-1} = 0$.

Phase 2: Impulse Effect

- Time interval: $t_{i-1} \leq t \leq t_i$.
- The equation now includes an external force: $f \neq 0$.
- Initial conditions:
 - Position: $u(t_{i-1}) = 0$.
 - Velocity: $u'(t_{i-1}) = 0$.
- Non-homogeneous equation + homogeneous data \rightarrow No mathematical method to solve it since it is exactly (3.1.17)–(3.1.18). That is, the equation in Phase 2 is difficult to solve directly due to its non-homogeneous nature.
- The **differences** between Phase 2 and (3.1.17)–(3.1.18): The **time interval** is very **short** (t_{i-1} to t_i), thus an **approximation** can be applied.
- Instead of using mathematical methods, we **apply physical principles**.
- Use **impulse theorem**:
 - Impulse: $f \cdot dt$.
 - Resulting velocity: $u'(t_i) = f \cdot dt$.
 - Displacement approximation (linear assumption): $u(t_i) = f \cdot dt^2$ (second-order small term ignored, so $u(t_i) \approx 0$).

Phase 3: Solving the Wave Equation

- Time interval: $t_i \leq t$.
- The equation returns to a homogeneous wave equation ($f = 0$).
- Initial conditions:
 - $u(t_i) = 0$.
 - $u'(t_i) = f \cdot dt$.
- Solve using **d'Alembert's formula**.
- Introduce new variable transformation: $w_i = w_i \cdot dt$ to eliminate dt ($dt \approx \Delta t$ the difference of two constant time and treat Δt as a constant so that it can be moved out of the derivative).
- Reformulate the equation in terms of w_i and solve.

$$\begin{cases} w_{i,tt}\Delta t = w_{i,xx}\Delta t \\ w_i\Delta t|_{t=t_i} = 0, \quad w_{i,t}\Delta t|_{t=t_i} = f\Delta t. \end{cases} \quad \leftarrow \text{denote } \tau = t_i.$$

- Use the **linear superposition principle**:

$$u(t) = \sum_{i=0}^{\infty} w_i \cdot dt \quad \Rightarrow \quad u(t) = \int_0^t w_i d\tau.$$

Conclusion

- The solution process follows three phases:
 1. **Initial equilibrium** (trivial solution).
 2. Application of **impulse theorem** to obtain **new initial conditions**.
 3. Solution of the wave equation using **d'Alembert's formula** and **integration**.
- This method is called the **Homogenization Principle (Duhamel's principle)**.

Approximation in the Physical Analysis

- The derivation was based on physical principles with several approximations.
- One key approximation was neglecting second-order small quantities like $(\Delta t)^2$.
- Thus, the obtained solution needs verification to confirm its correctness.

Verification by Substitution

- To verify whether the solution is correct, it must be substituted back into the original equation.
- This requires solving for w first.
- The function w is obtained by substituting u and solving the equation using the d'Alembert formula.

Challenges in the Verification Process

- The verification process involves derivatives of the variable limit integral.

Problem Overview

- The physical approach provides an approximate solution, but mathematical verification is required.
- The verification process involves:
 1. Solving for w .
 2. Substituting w to obtain u .

3. Validating u as the solution to the original equation.

Mathematical verifications

Recall the initial value problem (3.1.17)–(3.1.18):

$$\begin{aligned} u_{tt} &= a^2 u_{xx} + f(x, t) \quad (-\infty < x < +\infty, t > 0), \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0. \end{aligned}$$

Theorem 3.1.1 (Homogenization Principle, Duhamel's Principle). *If $w(x, t; \tau)$ is the solution of the initial value problem:*

$$\boxed{\text{Eq. of Phase 3}} \rightarrow \begin{cases} w_{tt} = a^2 w_{xx} & (t > \tau), \\ w|_{t=\tau} = 0, \quad w_t|_{t=\tau} = f(x, \tau) & \leftarrow \boxed{\text{Impluse from Phase 2}} \end{cases} \quad (3.1.19)$$

where τ is a parameter, then the solution to the initial value problem (3.1.17)–(3.1.18) is:

$$\boxed{u(x, t) = \int_0^t w(x, t; \tau) d\tau.} \quad (3.1.20)$$

Transforming the Initial Condition

- The standard d'Alembert solution assumes initial conditions at $t = 0$.
- Our initial condition is at $t = \tau$.
- Introduce a **shifted time** variable:

$$t' = t - \tau.$$

- This transforms the equation into a form with **initial conditions at $t' = 0$** .

Proof. Let $t' = t - \tau$, and define $\bar{w}(x, t'; \tau) = w(x, t' + \tau; \tau) = w(x, t; \tau)$, then problem (3.1.21) can be transformed into:

$$\begin{cases} \bar{w}_{t't'} = a^2 \bar{w}_{xx} & (t' > 0), \\ \bar{w}|_{t'=0} = 0, \quad \bar{w}_{t'}|_{t'=0} = f(x, \tau). \end{cases} \quad (3.1.21)$$

Using D'Alembert's formula (3.1.16), the solution to problem (3.1.21) is:

$$\bar{w}(x, t'; \tau) = \frac{1}{2a} \int_{x-at'}^{x+at'} f(\xi, \tau) d\xi.$$

By letting $t' = t - \tau$, we transform the variables back to obtain:

$$w(x, t; \tau) = \frac{1}{2a} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi. \quad (3.1.22)$$

Substituting (3.1.22) into equation (3.1.20) gives the solution to the initial value problem (3.1.17)–(3.1.18):

$$u(x, t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi d\tau. \quad (3.1.23)$$

Verifying the Solution

- Verify by substitution:
 1. Ensure u satisfies the wave equation.
 2. Check initial conditions:
$$u(0, x) = 0, \quad u_t(0, x) = 0.$$
- The verification requires differentiation under the integral sign due to the presence of variable limit integrals.

Two Derivative Formulas

(D1) The simplest:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

(D2) m Derivative formula for integrals involving a parameter in a univariate function:

$$\begin{aligned} \frac{d}{dx} & \underbrace{\left[\int_{u(x)}^{v(x)} f(t) dt \right]}_{= \int_0^{v(x)} f(t) dt - \int_0^{u(x)} f(t) dt} = f[v(x)]v'(x) - f[u(x)]u'(x) \end{aligned}$$

(D3) Derivative formula for integrals involving a parameter in a bivariate function (Use this):

$$\frac{d}{dx} \underbrace{\left[\int_{\alpha(x)}^{\beta(x)} f(\textcolor{blue}{x}, \textcolor{red}{y}) dy \right]}_{=:F(\alpha(x), \beta(x), x)} = f(x, \beta(x))\beta'(x) - f(x, \alpha(x))\alpha'(x) + \underbrace{\int_{\alpha(x)}^{\beta(x)} f_x(x, y) dy}_{\text{extra term}} \leftarrow \boxed{\text{Replace } y!}$$

Proof. By **chain rules**,

$$\frac{d}{dx} F(\alpha(x), \beta(x), x) = \underbrace{\frac{\partial F}{\partial \alpha}}_{\text{Fix } \beta, x} \alpha' + \underbrace{\frac{\partial F}{\partial \beta}}_{\text{Fix } \alpha, x} \beta' + \underbrace{\frac{\partial F}{\partial x}}_{\text{Fix } \alpha, \beta} = -f(x, \alpha)\alpha' + f(x, \beta)\beta' + \int_{\alpha}^{\beta} f_x dy.$$

We complete the proof. □

In fact, the function determined by (3.1.23) is indeed the solution to problem (3.1.17)–(3.1.18). When f has continuous first derivatives, from (3.1.23) we have:

$$\begin{aligned} u_t &= \frac{1}{2a} \int_{x-a(t-t)}^{x+a(t-t)} f(\xi, t) d\xi + \frac{1}{2} \int_0^t f(x + a(t - \tau), \tau) d\tau + \frac{1}{2} \int_0^t f(x - a(t - \tau), \tau) d\tau \\ &= \frac{1}{2} \int_0^t [f(x + a(t - \tau), \tau) + f(x - a(t - \tau), \tau)] d\tau. \end{aligned}$$

Example on the computations

Note

$$u(x, t) = \frac{1}{2a} \int_0^t \underbrace{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi}_{=:G(x, t; \tau)} d\tau.$$

Then by (D2),

$$\begin{aligned} u_t(x, t) &= \frac{1}{2a} G(x, t; t) + \frac{1}{2a} \int_0^t G_t(x, t; \tau) d\tau \\ &= \frac{1}{2a} \underbrace{\int_{x-a(t-t)}^{x+a(t-t)} f(\xi, t) d\xi}_{=:G_t(x, t; t)} + \frac{1}{2a} \int_0^t G_t(x, t; \tau) d\tau \end{aligned}$$

Then by (D3),

$$G_t(x, t; \tau) = f(x + a(t - \tau), \tau) \cdot a - f(x - a(t - \tau), \tau) \cdot (-a)$$

We then have:

$$\begin{aligned} u_{tt} &= f(x, t) + \frac{a}{2} \int_0^t [f'(x + a(t - \tau), \tau) - f'(x - a(t - \tau), \tau)] d\tau, \\ u_x &= \frac{1}{2a} \int_0^t [f(x + a(t - \tau), \tau) - f(x - a(t - \tau), \tau)] d\tau, \\ u_{xx} &= \frac{1}{2a} \int_0^t [f'(x + a(t - \tau), \tau) - f'(x - a(t - \tau), \tau)] d\tau. \end{aligned}$$

Thus, $u_{tt} = a^2 u_{xx} + f(x, t)$, i.e., (3.1.23) satisfies equation (3.1.17). Verify the initial conditions (3.1.18). From (3.1.23) and the expression for u_t , we have:

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0.$$

This proves that the function determined by (3.1.23) is indeed the solution to the initial value problem (3.1.17)–(3.1.18). \square

Equation (3.1.16) describes the general solution without external forces (3.1.2). That is,

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha$$

solves

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \end{cases}$$

Equation (3.1.22) represents the effect of an external force integrated over time(3.1.3). That is,

$$u(x, t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi d\tau$$

solves

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = 0, \quad u_t(x, 0) = 0 \end{cases}$$

By the **principle of superposition**, the solution to the problem (3.1.1),

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x, t) & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \end{cases}$$

can be expressed as:

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha + \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi d\tau. \quad (3.1.24)$$

and

$$u(x, t) = \underbrace{\frac{\varphi(x - at) + \varphi(x + at)}{2}}_{\text{Arithmetic mean of the initial positions}} + \underbrace{\frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha}_{t \times \text{Integral average of the initial velocity}} + \underbrace{\frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi d\tau}_{t^2 \times \text{the average of the integral of the area of the triangle}}.$$

Ex 3.1.2. Solve the following initial value problem

$$\begin{cases} u_{tt} = u_{xx} + 2x & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \sin x, \quad u_t(x, 0) = x. \end{cases}$$

Solution. Using formula (3.1.24), we get

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\sin(x + t) + \sin(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} \alpha d\alpha + \frac{1}{2} \int_0^t \left(\int_{x-(t-\tau)}^{x+(t-\tau)} 2\xi d\xi \right) d\tau \\ &= \sin x \cos t + xt + xt^2. \end{aligned}$$

3.1.3 Physical Meaning of D'Alembert's Solution

Physical intuitions → mathematical models → Physical interpretations

From the general solution (3.1.11), the solution to the free vibration equation can be expressed as the sum of two functions $f(x - at)$ and $g(x + at)$. Through them, the nature of wave propagation can be clearly seen.

First, consider:

$$u_1 = f(x - at),$$

which is obviously a solution to equation (3.1.4). By giving t different values, one can see the vibration state of the string at each moment.

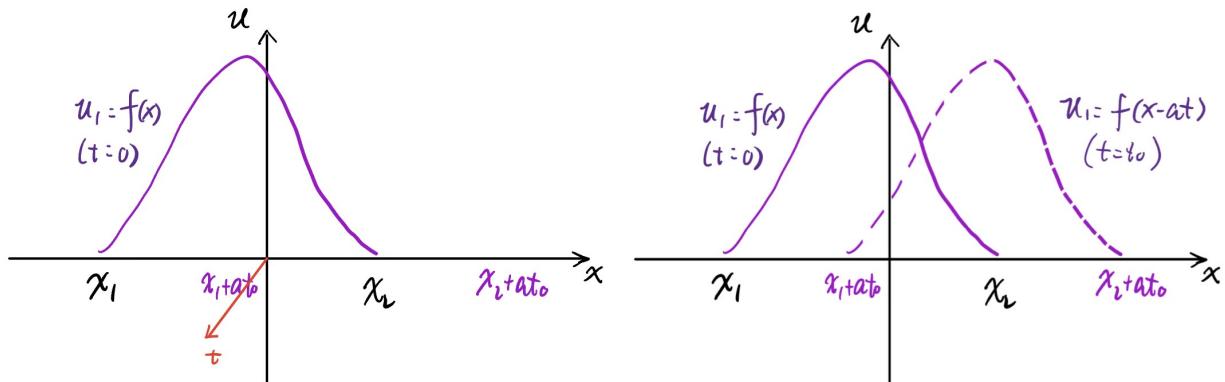


Figure 3.4: Wave propagation

- At $t = 0$, $u_1(x, 0) = f(x)$, which corresponds to the vibration state at the initial moment (equivalent to the displacement state of the string at each point at the **initial moment**), as shown in Fig. 3.4.
- After time t_0 , $u_1(x, t_0) = f(x - at_0)$, which is equivalent to the original shape $u_1 = f(x)$ **shifted to the right** by a distance at_0 on the (x, u) plane.
- As time progresses, this diagram will **continue to move to the right**, indicating that when the solution to equation (3.1.4) is expressed in the form $f(x - at)$, the wave shape of the vibration propagates **to the right** at a **constant speed** a .
- Thus, the vibration pattern described by the function $f(x - at)$ is called a **right-propagating wave**. Similarly, the solution in the form of $g(x + at)$ is called a **left-propagating wave**, which describes a wave pattern propagating to the **left** at a **constant speed** a .
- From this, it is evident that the general solution (3.1.11) represents any disturbance on the string propagating in the form of **traveling waves in both directions**, with the **propagation speed** being the constant a appearing in equation (3.1.4). **D'Alembert's solution method** is also known as the **method of traveling waves**.

3.1.4 Dependency Region, Determination Region, and Influence Region

Recall

$$u(x, t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha.$$

solves

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), & u_t(x, 0) = \psi(x) \end{cases}$$

Question 3.1.1. The solution of the initial value problem (3.1.2) at a point (x, t) is related to the values of the initial conditions on which points on the x -axis?

- **(Dependency region of a point)** From D'Alembert's formula (3.1.16), it can be seen that the value of the solution at the point (x, t) depends only on the initial conditions on the interval $[x - at, x + at]$ on the x -axis, and is independent of the initial conditions at other points. This interval $[x - at, x + at]$ is called the **dependency region** of the point (x, t) (see Fig. 3.7).
- It is the interval intersected by the x -axis with the lines passing through the point (x, t) with slopes $\pm\frac{1}{a}$, as shown in Fig. 3.7.
- Consider an interval $[x_1, x_2]$ on the initial axis $t = 0$, draw a line through point x_1 with a slope of $\frac{1}{a}$, $x = x_1 + at$, and a line through point x_2 with a slope of $-\frac{1}{a}$, $x = x_2 - at$, together with the interval $[x_1, x_2]$ they form a **triangular region**, as shown in Fig. 3.7.
- In this triangular region, the dependency region of any point (x, t) falls within the interval $[x_1, x_2]$, therefore, the value of the solution in the triangular region is completely determined by the initial conditions on the interval $[x_1, x_2]$, and is independent of the initial conditions outside this interval.
- **(Determination region of an interval)** This triangular region is called the **determination region** of the **interval** $[x_1, x_2]$ (see Fig. 3.5 and 3.7). Given the initial conditions on the interval $[x_1, x_2]$, the solution to the initial value problem (3.1.2) can be determined within its determination region.

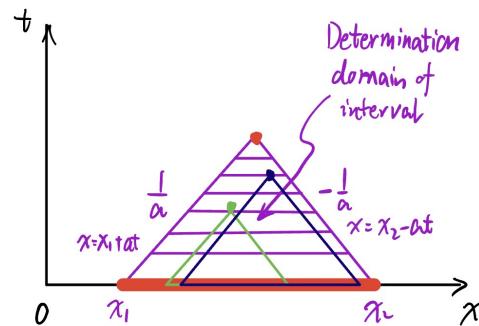


Figure 3.5: determination domains

Question 3.1.2. If at the initial moment $t = 0$, the disturbance exists only in a finite interval $[x_1, x_2]$, what is the range it affects after time t ?

- We know that the wave propagates in **both directions** at a certain **speed a** .
- Therefore, after time t , the range it propagates to (the range affected by the initial disturbance) is limited by the inequality

$$x_1 - at \leq x \leq x_2 + at \quad (t > 0) \quad (3.1.25)$$

and **outside** this range, it **remains** in a state of **rest**.

- (**Influence region of an interval**) On the (x, t) plane, the region represented by equation (3.1.25) is called the **influence region** of the **interval** $[x_1, x_2]$ (see Fig. 3.6 and 3.7). In this region, the solution $u(x, t)$ of the initial value problem is affected by the initial conditions on the interval $[x_1, x_2]$.

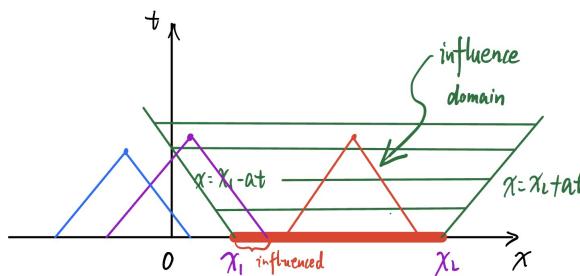


Figure 3.6: Influence domains

- Outside this region, the solution $u(x, t)$ of the initial value problem is not affected by the initial conditions on the interval $[x_1, x_2]$.
- In particular, shrinking the interval $[x_1, x_2]$ to a single point x_0 , we can obtain the influence region of a single point x_0 , as shown in Fig. 3.7. This influence region is the triangular area formed by the two lines passing through this point with slopes $\pm\frac{1}{a}$, $x = x_0 \pm at$.

Extended Materials:

Characteristics of Linear Wave Equations

- In a linear wave equation, the wave speed a is a constant.
- The solution is constructed using characteristic lines with slopes $\frac{1}{a}$ and $-\frac{1}{a}$, forming triangular influence regions.
- This structure results in straight-line characteristics, ensuring well-defined propagation.

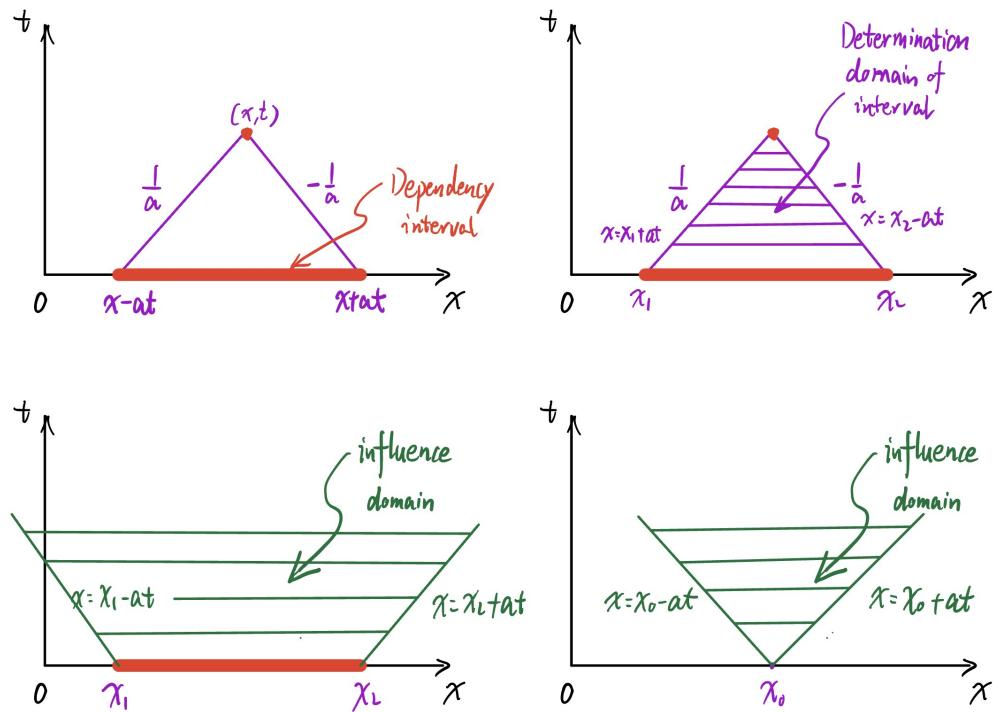


Figure 3.7: Domains

Nonlinear Wave Equations and Curved Characteristics

- In nonlinear wave equations, the wave speed a may depend on t , x , or u .
- This leads to variable coefficients, altering the propagation behavior.
- Characteristic lines may become curved instead of straight.
- The influence region is no longer a simple triangle but a more complex shape.

Finite Speed of Wave Propagation

- Waves propagate at a finite speed, meaning disturbances take time to reach different locations.
- If an event (e.g., an explosion) occurs at a point, the effects will not be felt instantaneously elsewhere.
- This principle ensures that causality is preserved:
 - Causes precede effects.
 - No information can propagate faster than the wave speed.

Causality in Wave Equations

- The finite speed of propagation enforces a strict order of events.
- No effect can precede its cause.
- This property is fundamental in physics and ensures consistency in signal transmission and interactions.

3.2 Initial Value Problems for Higher Dimensional Wave Equations

Key Ideas

- The approach starts by considering the 3D wave equation and then applying dimensional reduction to handle the 2D case.
- The main difficulty lies in the 3D wave equation, while the 2D case is relatively easier.
- The nature of solutions differs significantly between even and odd spatial dimensions for wave equations.
- Despite deriving 2D results from the 3D case, their solution properties remain fundamentally different.

Physical Analogy: Gravity vs. Wave Propagation

- Gravity is governed by Poisson's equation, which does not evolve over time.
- A thought experiment: Suppose ϕ satisfies the 3D wave equation instead.
- General Relativity suggests that gravity behaves like waves, though in a much more complex manner.
- This leads to an intuitive approach: **body thinking**.
- If gravity were governed by a wave equation, it would exhibit oscillatory behavior.
- This would mean that gravitational strength at a point would fluctuate over time, akin to experiencing a roller-coaster effect.

Simplest 3D Wave: Spherical Symmetry

- The simplest 3D wave solution is a **spherically symmetric wave**.
- Spherically symmetric waves simplify the problem by **reducing dependence on angular coordinates**.
- This allows for a transformation of the problem into an effectively 1D equation with a **radial coordinate**.

Body Thinking: A Learning Strategy

- Body thinking involves associating abstract concepts with physical sensations.
- Example: A chemist mentally links chemical reactions to body movements, responding physically to conceptual changes.
- Applying this idea: Imagine being surrounded by a fluctuating gravitational field.
- In this model, gravity varies dynamically like a wave:
 - You feel gravitational strength increasing and decreasing.
 - The sensation is similar to riding a roller coaster while sitting still.

Approach to the 3D Wave Equation

- Direct solution methods such as separation of variables are not feasible without boundary conditions.
- The only viable approach is to transform the 3D problem into a simpler 1D form.
- However, reducing 3D to 1D is challenging due to the greater degrees of freedom in 3D.
- To solve the 3D wave equation, we begin by identifying the simplest case: **spherical symmetry**.
- The problem-solving approach resembles detective work:
 - Collect key clues.
 - Formulate bold hypotheses.
 - Attempt to construct a viable solution.
- The first clue: consider the simplest wave solution, a spherically symmetric wave.
- Using the **method of dissecting complexity** (similar to *Pao Ding's Butchering the Ox* analogy), we rewrite the wave equation in spherical coordinates.
- This transformation simplifies the problem and helps in deriving an explicit solution.

In the previous section, we discussed the initial value problem of the **one-dimensional** wave equation and obtained **D'Alembert's formula**. For the **three-dimensional** wave equation, the solution can be expressed in a **spherical mean form**, which is commonly referred to as **Kirchhoff's formula**.

3.2.1 Kirchhoff's Formula for the Three-Dimensional Wave Equation

Now, let's consider the initial value problem of the three-dimensional wave equation

$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \quad (-\infty < x, y, z < +\infty, t > 0), \quad (3.2.1)$$

$$u(x, y, z, 0) = \varphi(x, y, z), \quad u_t(x, y, z, 0) = \psi(x, y, z), \quad (3.2.2)$$

where $\varphi(x, y, z)$ and $\psi(x, y, z)$ are known functions.

How to invent the method of spherical mean

Clue 1:

To explain how the spherical means method is conceived, let's first look at the **first clue**: the spherically symmetric solution of the three-dimensional wave equation.

Using spherical coordinates:

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

The **wave equation in spherical coordinates** (by chain rules) is expressed as:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

Since u is **independent of θ and ϕ** (due to the spherical symmetry)

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

- We currently have the 1D wave equation describing string vibrations.
- The only available approach is the **traveling wave method**, which provides solutions in the form of moving waves.
- Our goal: Transform this equation into the **1D wave equation** to simplify analysis.

Recall a useful relation

$$\partial_r(r^2 \partial_r u) = r \partial_r^2(ru) \quad (3.2.3)$$

Then we arrive at

$$\frac{\partial^2(ru)}{\partial r^2} = \frac{1}{a^2} \frac{\partial^2(ru)}{\partial t^2}$$

The general solution gives:

$$ru = f(r - at) + g(r + at)$$

That is,

$$u = \frac{f(r - at)}{r} + \frac{g(r + at)}{r}.$$

This is the **spherically symmetric** solution of the three-dimensional wave equation.

Clue 2:

Using the D'Alembert's formula to **guess** the 3D formula,

- The goal is to rewrite the formula to unify two types of averages:
 - **Arithmetic Mean**
 - **Integral Mean**
- Mathematicians prefer **symmetry** and **consistency**, leading to the search for a **unified form**
 \leftarrow **Aesthetic Criterion**.
- The **arithmetic mean** can be expressed as an **integral mean** by integration followed by differentiation.

$$\begin{aligned} u(x, t) &= \frac{1}{2} (\varphi(x - at) + \varphi(x + at)) + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha \\ &= \frac{1}{2a} \partial_t \left(\int_{x-at}^{x+at} \varphi(\alpha) d\alpha \right) + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha) d\alpha \\ &= \frac{\partial}{\partial t} \left(\frac{t}{2at} \int_{x-at}^{x+at} \varphi(\alpha) d\alpha \right) + \frac{t}{2at} \int_{x-at}^{x+at} \psi(\alpha) d\alpha \end{aligned}$$

- The rewritten formula contains:
 - The integral mean of the initial displacement.
 - The integral mean of the initial velocity, scaled by time.
- Since the 1D and 3D cases are special cases of an n -dimensional wave equation, there must be an underlying relationship.
- Extending the 1D case to 3D leads to a natural question:
 - What type of averaging should be used in 3D?
 - Possibilities: **Spherical volume average** vs. **Spherical surface average**.
- Experimental verification (trial and errors) shows that **spherical surface averaging** is the appropriate choice for solving the 3D wave equation.

Analogy and guess (see Fig. 3.8):

$$u(M, t) = \frac{\partial}{\partial t} \left(\frac{t}{4\pi(at)^2} \int_{S_{at}^M} \varphi(\xi, \eta, \zeta) dS \right) + \frac{t}{4\pi(at)^2} \int_{S_{at}^M} \psi(\xi, \eta, \zeta) dS \leftarrow \boxed{\text{Aesthetic Criteria}} \quad (3.2.4)$$

This may be the solution to the three-dimensional wave equation, and

- It suggests that the solution **at a given point** can be expressed using the **spherical average around that point**.

- Conclusion:
 - **Spherical symmetry** is a crucial property in solving the 3D wave equation.
 - **Spherical surface averaging around every point** plays a fundamental role in the solution process.

About (3.2.4)

- From the conjectured formula, in order to determine $u(t_0, x_0)$, we need to know the data on the sphere at any arbitrary time t_1 . However, in reality, we only need the spherical average data at

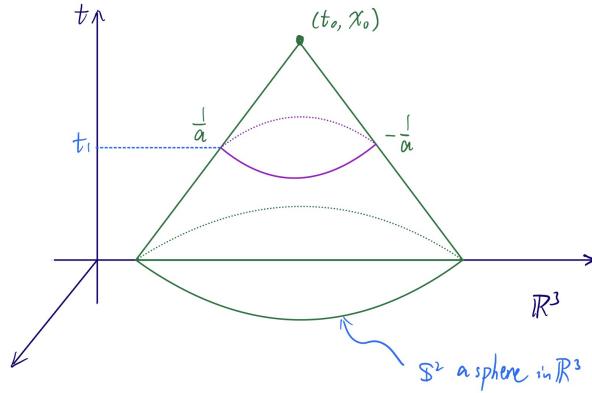


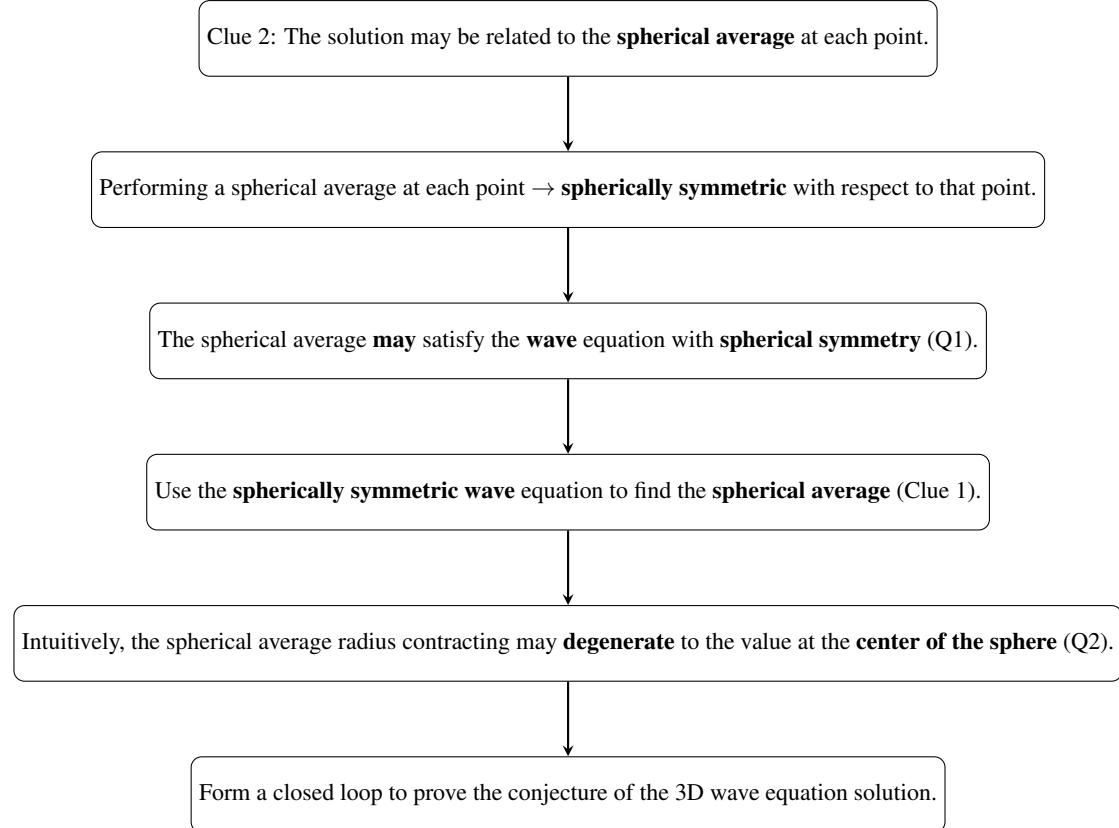
Figure 3.8: 3D wave analogy

any given time t_1 . This means that if we know the spherical average at any time t_1 , it is sufficient to determine $u(t_0, x_0)$.

- However, the direct result of the spherical average data is the spherical average solution \bar{u} instead of u . Therefore, we pose the following questions:
 1. What is the relationship between u and \bar{u} ?
 - The information of u and \bar{u} should be equivalent.
 - * If u is known, \bar{u} can be determined.
 - * Conversely, knowing \bar{u} , intuition suggests that as the radius approaches zero, $\bar{u} \rightarrow u$.
 - * This follows from the fact that as the concentric spheres around point M shrink to a single point, the average over the sphere reduces to the value at M .
 - This equivalence allows us to study u via \bar{u} .
 - * u is difficult to determine directly.
 - * However, \bar{u} depends only on a single spatial variable r , making it potentially solvable using a one-dimensional wave equation.
 - From the conjectured solution:
 - * u does not distinguish specific initial values but only their spherical averages.
 - * This suggests redistributing values over the sphere such that every point takes the spherical average.
 - * This transformation leads to a spherically symmetric problem, which may connect to **Clue 1**.
- 2. What equation does \bar{u} satisfy?
 - The information of u and \bar{u} is equivalent (a conjecture that needs to be proven). Since the information of u is equal to the information contained in the wave equation, it follows that the information of \bar{u} should also be equal to the information of the wave equation. Therefore, the wave equation should be able to derive the equation for \bar{u} .

- The above considerations are purely based on the conjectures. Based on this idea, we can further refine and develop a mathematical method.

According to these clues, we conceive the ideas:



where (Q1) and (Q2) are conjectures that we have to prove. Next, we first prove these two questions.

Key Questions and Solutions

How does the spherical average relate to the original solution? (about (Q2))

- Consider a sphere of radius r centered at any given point.
- Compute the spherical average over this sphere.
- As $r \rightarrow 0$, the spherical average approaches the value of the function at the center.
- This suggests that the spherical averaging method can be used to approximate the solution at any point.

Does the spherical average satisfy a wave equation? (about (Q1))

- Two possible ways to check:
 1. **Assume the solution exists:** If the solution to the wave equation exists everywhere, we can explicitly compute its spherical average.
 2. **Use the wave equation itself:** The wave equation contains all necessary information about the solution, making it a valid tool to derive properties of the spherical average.

Wave Eqs $\xrightarrow{\text{directly derive}}$ Eq. of Spherical Average (spherical symmetric wave eq.)

- Since the wave equation is **equivalent to the information** contained in the solution, if properly **transformed**, it should **hold for the spherical average** as well (like the energy method).

First, fix any point $M = (x, y, z)$, S_r^M represents the sphere with center M and radius r . Using spherical coordinates, a point on the sphere is given by:

$$P \equiv (\xi, \eta, \zeta) = (x + r \sin \theta \cos \phi, y + r \sin \theta \sin \phi, z + r \cos \theta).$$

Let $\omega = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ represent the unit outward normal vector to the sphere S_r^M , then a point on the sphere S_r^M can be simply written as $M + r\omega$. At the same time, ω can also be considered as a point on the unit sphere. Therefore, we also denote the surface element on the sphere as:

$$dS_r^M = r^2 \sin \theta d\theta d\phi \quad \text{and} \quad d\omega = \sin \theta d\theta d\phi.$$

Note the normalization relation ($d\omega$ is independent of r and it leads to some convenience).

$$dS_r^M = r^2 d\omega.$$

Now introduce the **spherical average** of u (note $dS_r^M = r^2 d\omega$):

$$\bar{u}(r, t) \equiv \frac{1}{4\pi r^2} \iint_{S_r^M} u(P, t) dS_r^M = \frac{1}{4\pi} \iint_{S_1^M} u(M + r\omega, t) d\omega.$$

(Proof of (Q2)) Taking the limit as $r \rightarrow 0$ on both sides of the above equation, we get:

$$\lim_{r \rightarrow 0} \bar{u}(r, t) = \frac{1}{4\pi} \iint_{S_1^M} u(M, t) d\omega = u(M, t) \leftarrow \boxed{\text{indep. of } \theta \text{ and } \phi}.$$

Furthermore, let V_r^M denote the ball with center M and radius r , then the volume integral on V_r^M can be expressed in spherical coordinates as:

$$\iiint_{V_r^M} f dV_r^M \stackrel{\text{Onion-peeling for integration}}{=} \int_0^r dr_1 \iint_{S_{r_1}^M} f dS_{r_1}^M = \int_0^r dr_1 \iint_{S_1^M} f(M + r_1\omega) r_1^2 d\omega.$$

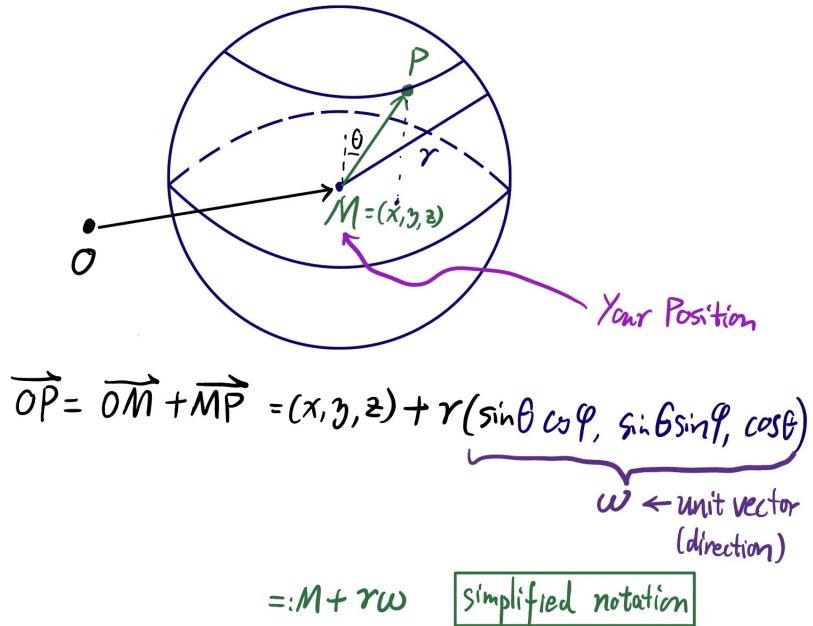


Figure 3.9: Notation on Sphere

Claim (Proof of (Q1)). u solves $\partial_t^2 u = \Delta u \Rightarrow r\bar{u}$ solves $\partial_t^2(r\bar{u}) = \partial_r^2(r\bar{u})$. (For simplicity, take $a = 1$)

Proof. First, we integrate the wave equation on V_r^M .

$$\underbrace{\int_{V_r^M} \partial_t^2 u \, dV}_{\text{LHS}} = \underbrace{\int_{V_r^M} \Delta u \, dV}_{\text{RHS}}$$

Then proceed the RHS by **Guass formula**, while the LHS by **onion-peeling for integration**:

$$\begin{aligned} \text{RHS} &\stackrel{\text{Guass}}{=} \int_{\partial V_r^M} \mathbf{n} \cdot \nabla u \, dS = \int_{S_r^M} \partial_r u \, dS = \int_{S_1^M} \partial_r u \cdot r^2 d\omega = r^2 \partial_r \left(\int_{S_1^M} u(M + r\omega) d\omega \right) = 4\pi r^2 \partial_r \bar{u}(r, t); \\ \text{LHS} &= \partial_t^2 \int_{V_r^M} u \, dV \stackrel{\text{Onion-peeling for integration}}{=} \partial_t^2 \left(\int_0^r \int_{S_1^M} u(M + r_1\omega) r_1^2 d\omega dr_1 \right) = 4\pi \partial_t^2 \int_0^r r_1^2 \bar{u}(r_1, t) dr_1. \end{aligned}$$

where \mathbf{n} is the outward normal. This impliles

$$4\pi \partial_t^2 \int_0^r r_1^2 \bar{u}(r_1, t) dr_1 = 4\pi r^2 \partial_r \bar{u}(r, t)$$

Taking the derivative on both sides with respect to r

$$\underbrace{\Rightarrow \partial_t^2 \partial_r \left(\int_0^r r_1^2 \bar{u}(r_1, t) dr_1 \right)}_{= r^2 \partial_t^2 \bar{u}} = \underbrace{\partial_r (r^2 \partial_r \bar{u}(r, t))}_{\stackrel{\text{by (3.2.3)}}{=} r \partial_r^2(r\bar{u})}$$

Cancel one r

$$\Rightarrow \partial_t^2(r\bar{u}) = \partial_r^2(r\bar{u})$$

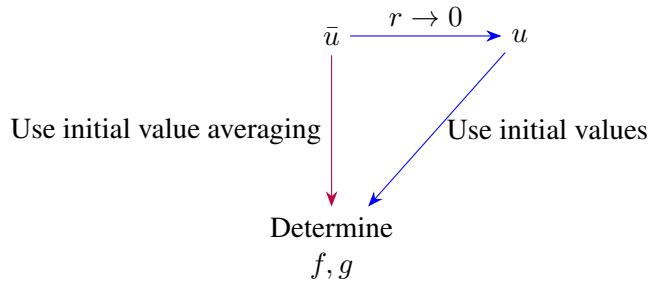
We finish the proof of this claim. \square

That is, we arrive at $(r\bar{u})_{tt} = a^2(r\bar{u})_{rr}$. Therefore, the general solution for $r\bar{u}$ is

$$r\bar{u} = f(r + at) + g(r - at), \quad (3.2.5)$$

where f and g are twice continuously differentiable functions.

There are two approaches that can utilize initial values to determine f and g according to the following flowchart.



(M1) **The blue line:** Feasible, use L'Hospital's rule to make preliminary assumptions about f and g .

(M2) **The red line:** More straightforward.

(Determine f' and g') To apply Method (M2), we need to compute the **spherical average** of the **initial values**. First, we need to satisfy the initial conditions. From (3.2.5), i.e.,

$$r\bar{u} = f(r + at) + g(r - at),$$

we obtain

$$(r\bar{u})_t = r\bar{u}_t = af'(r + at) - ag'(r - at). \quad (3.2.6)$$

- In the form of $f + g$, $f' - g'$, to find f and g , we only need to integrate $f' - g'$, or differentiate $f + g$. We use the method of differentiation for $f + g$, otherwise it's not easy to do.
- To differentiate $f + g$, we can only differentiate $f' + g'$ with respect to r .

Let use differentiate (3.2.5) with respect to r ,

$$(r\bar{u})_r = \bar{u} + r\bar{u}_r = f'(r + at) + g'(r - at) \quad (3.2.7)$$

By (3.2.6) and (3.2.7), we calculate

$$\begin{aligned} 2f'(r+at) &= (r\bar{u})_r + \frac{1}{a}(r\bar{u})_t = \frac{\partial}{\partial r} \left(\frac{r}{4\pi} \int_{S_1^m} u d\omega \right) + \frac{1}{a} \frac{r}{4\pi} \int_{S_1^m} u_t d\omega \\ 2g'(r-at) &= (r\bar{u})_r - \frac{1}{a}(r\bar{u})_t = \frac{\partial}{\partial r} \left(\frac{r}{4\pi} \int_{S_1^m} u d\omega \right) - \frac{1}{a} \frac{r}{4\pi} \int_{S_1^m} u_t d\omega \end{aligned}$$

If $t = 0$, we obtain

$$\begin{aligned} 2f'(r) &= \frac{\partial}{\partial r} \left(\frac{r}{4\pi} \int_{S_1^m} \varphi(M+r\omega) d\omega \right) + \frac{1}{a} \frac{r}{4\pi} \int_{S_1^m} \psi(M+r\omega) d\omega \\ 2g'(r) &= \frac{\partial}{\partial r} \left(\frac{r}{4\pi} \int_{S_1^m} \varphi(M+r\omega) d\omega \right) - \frac{1}{a} \frac{r}{4\pi} \int_{S_1^m} \psi(M+r\omega) d\omega \end{aligned}$$

(Obtain u) On the other hand, using (3.2.7) (since we are considering the classical solution, $|\bar{u}_r| < \infty$ and $|\bar{u}_t| < \infty$),

$$\begin{aligned} u(M, t) &= \lim_{r \rightarrow \infty} \bar{u}(r, t) = f'(at) + g'(-at) \leftarrow \boxed{\text{or L'Hospital law by (3.2.5)}}$$

$$\begin{aligned} &= \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \int_{S_1^m} \varphi(M+at\omega) d\omega \right) + \frac{1}{2} \frac{t}{4\pi} \int_{S_1^m} \psi(M+at\omega) d\omega \\ &\quad + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \int_{S_1^m} \varphi(M+at\omega) d\omega \right) + \frac{1}{2} \frac{t}{4\pi} \int_{S_1^m} \psi(M+at\omega) d\omega \\ &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \int_{S_1^m} \varphi(M+at\omega) d\omega \right) + \frac{t}{4\pi} \int_{S_1^m} \psi(M+at\omega) d\omega \\ &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \int_{S_{at}^M} \varphi(\xi, \eta, \xi) dS \right) + \frac{t}{4\pi a^2 t^2} \int_{S_{at}^M} \psi(\xi, \eta, \xi) dS. \end{aligned} \tag{3.2.8}$$

Then

$$\begin{aligned} u(M, t) &= \underbrace{\frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \int_{S_{at}^M} \varphi(\xi, \eta, \xi) dS \right)}_{\partial_t(t \times \text{the spherical average of the initial displacement on the } at \text{ sphere})} \\ &\quad + \underbrace{\frac{t}{4\pi a^2 t^2} \int_{S_{at}^M} \psi(\xi, \eta, \xi) dS}_{t \times \text{the spherical average of the initial velocity on the sphere of radius } at} \end{aligned}$$

Another way which is not quite direct

In (3.2.6) and (3.2.7), let $r \rightarrow 0$, we obtain

$$f'(at) = g'(-at),$$

$$u(M, t) = \lim_{r \rightarrow 0} \bar{u}(r, t) = f'(at) + g'(-at) = 2f'(at).$$

In equations (3.2.6) and (3.2.7), taking $t = 0$ gives

$$(r\bar{u})_t |_{t=0} = af'(r) - ag'(r), \quad (r\bar{u})_r |_{t=0} = f'(r) + g'(r).$$

Then we arrive at

$$\begin{aligned} 2f'(r) &= (r\bar{u})_r |_{t=0} + \frac{1}{a}(r\bar{u})_t |_{t=0} \\ &= \frac{\partial}{\partial r} \left(\frac{r}{4\pi r^2} \iint_{S_r^M} u |_{t=0} dS_r^M \right) + \frac{r}{a} \left(\frac{1}{4\pi r^2} \iint_{S_r^M} u_t |_{t=0} dS_r^M \right) \\ &= \frac{\partial}{\partial r} \left(\frac{r}{4\pi r^2} \iint_{S_r^M} \varphi(P) dS_r^M \right) + \frac{r}{a} \left(\frac{1}{4\pi r^2} \iint_{S_r^M} \psi(P) dS_r^M \right) \end{aligned}$$

Taking $r = at$ and substituting $u(M, t) = 2f'(at)$ gives

$$\begin{aligned} u(M, t) = 2f'(at) &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \varphi(\xi, \eta, \xi) dS \right) + \frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \psi(\xi, \eta, \xi) dS \\ &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi} \iint_{S_1^M} \varphi(M + at\omega) d\omega \right) + \frac{t}{4\pi} \iint_{S_1^M} \psi(M + at\omega) d\omega. \end{aligned}$$

When the initial functions are sufficiently smooth, it is easy to verify that the function $u(x, y, z, t)$ represented by formula (3.2.8) is indeed the solution to problem (3.2.1)–(3.2.2).

Ex 3.2.1. Solve the following initial value problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz} & (-\infty < x, y, z < +\infty, t > 0), \\ u(x, y, z, 0) = 0, & u_t(x, y, z, 0) = 2xy, \end{cases}$$

Solution. From formula (3.2.8), we get

$$\begin{aligned} u(\underbrace{x, y, z, t}_{\text{Your position}}) &= \frac{t}{2\pi} \int_0^{2\pi} \int_0^\pi \underbrace{(x + t \sin \theta \cos \varphi)(y + t \sin \theta \sin \varphi) \sin \theta}_{\text{Sphere around you}} d\theta d\varphi \\ &= \frac{t}{2\pi} \int_0^{2\pi} \int_0^\pi (xy \sin \theta + xt \sin^2 \theta \sin \varphi + yt \sin^2 \theta \cos \varphi + t^2 \sin^3 \theta \cos \varphi \sin \varphi) d\theta d\varphi \\ &= \frac{t}{2\pi} \int_0^{2\pi} \int_0^\pi (x + t \sin \theta \cos \varphi)(y + t \sin \theta \sin \varphi) \sin \theta d\theta d\varphi \\ &= \frac{xyt}{2\pi} \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\varphi = 2xyt. \end{aligned}$$

3.2.2 Dimension Reduction Method

Using the dimension reduction method to solve the initial value problem of the two-dimensional wave equation.

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy}) & (-\infty < x, y < +\infty, t > 0), \\ u|_{t=0} = \varphi(x, y), \\ u_t|_{t=0} = \psi(x, y). \end{cases} \quad (3.2.9)$$

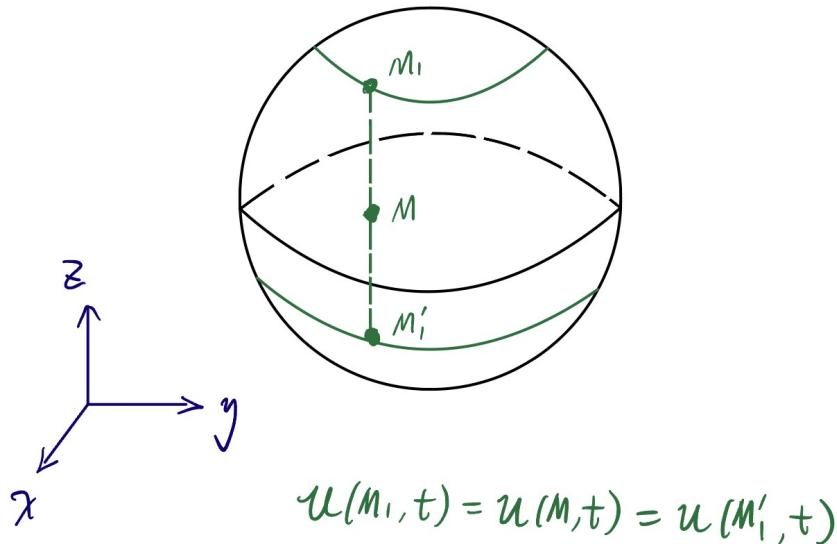


Figure 3.10: 2D wave equations

Since the initial value problem of the two-dimensional wave equation can be considered a special case of the initial value problem of the three-dimensional wave equation (see Fig. 3.10), the Poisson formula for the three-dimensional wave equation can be used to represent the solution of the initial value problem of the two-dimensional wave equation, and thus derive another form of the solution representation for the two-dimensional problem.

- The 2D problem is represented on a plane. Imagine a large trampoline with no boundaries, vibrating in two dimensions.
- To convert this 2D problem into a 3D case, we **extend the 2D surface along the z -axis**.
- This extension involves replicating the 2D surface infinitely along the z -axis, keeping the value of u constant across different z levels.
- After the surface is stacked infinitely, it forms a 3D structure, and we can apply 3D formulas such as Kirchhoff's equations.
- The **key idea** is that along the z -axis, $u_{zz} = 0$ because the value of u does not change in the z -direction.

- This extension allows us to use the 3D Kirchhoff equation to solve the problem.
- To calculate any point in the stacked structure, we apply the second-order wave equation, averaging over a spherical surface surrounding the point.
- Finally, we need to project the result back onto the original 2D trampoline.

Mathematical Interpretation

- The value of u is constant along the z -axis, leading to $u_{zz} = 0$.

Claim.

Initial data is independent of z , implies u is independent of z for all $t > 0$.

Proof. Given $u_z|_{t=0} = 0$, $\partial_t u_z|_{t=0} = 0$, differentiating the wave equation with respect to z ,

$$(u_z)_{tt} = a^2 ((u_z)_{xx} + (u_z)_{yy}) \Rightarrow u_z \equiv 0.$$

□

- Once we reduce the problem to 3D, we can apply the 3D Kirchhoff equation directly to solve for $u(t)$.

Using Equation (3.2.8)

$$u(M, t) = \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \varphi(\xi, \eta, \zeta) dS \right) + \frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \psi(\xi, \eta, \zeta) dS$$

The solution to the initial value problem of the two-dimensional wave equation (3.2.9) can be obtained as

$$u(x, y, t) = \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \varphi dS \right) + \frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \psi dS,$$

where the integration is performed on the sphere S_{at}^M in the three-dimensional space (x, y, z) .

- After extending the 2D surface to 3D, the task is to project the 3D spherical surface integral back onto the 2D trampoline surface.
- This involves using a projection formula for surface integrals that you learned in calculus.

- The formula for projecting a surface integral is:

$$\int_{\Sigma} f \, d\sigma = \int_S f \cos \gamma \, dS$$

where γ is the angle between the surface normals.

- This projection simplifies the surface integral from 3D to 2D.
- Alternatively, using the **standard method of calculating surface integrals** in calculus. If $z = \varphi(x, y)$, the integral can be transformed to:

$$\int_S f(x, y) \, dS = \int_{\Sigma} f(x, y) \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2} \, dx \, dy$$

where $f(x, y)$ represents the integrand function and the square root term adjusts for the curvature of the surface. For a sphere, $z = \varphi(x, y) = \sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}$, then

$$(\varphi_x)^2 = \frac{(\xi - x)^2}{(at)^2 - (\xi - x)^2 - (\eta - y)^2} \quad \text{and} \quad (\varphi_y)^2 = \frac{(\eta - y)^2}{(at)^2 - (\xi - x)^2 - (\eta - y)^2}$$

Then (see Fig. 3.10)

$$dS = \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2} \, d\sigma = \frac{at}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \, d\sigma$$

- This formula is derived from standard techniques in multivariable calculus and is applicable for transforming integrals from a curved surface to a flat 2D plane.

Key Concepts

- Projection of 3D integrals onto 2D involves transforming the integrand using standard surface integration formulas.
- The formula for surface integrals in calculus allows us to compute the projected integral over a 2D region from a 3D surface.
- A key aspect is recognizing how surface curvature and the normal vector impact the projection of the integral.

Since φ and ψ are functions independent of z , the integral on the spherical surface can be transformed into the integral on its projection on the plane $z = \text{constant}$: $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \leq a^2 t^2$. Since the area element dS on the spherical surface and its projection area element $d\sigma$ satisfy the following relationship (see Fig. 3.10):

$$d\sigma = \cos \gamma \cdot dS,$$

where γ is the angle between the normal directions of these two surface elements. Therefore, we have:

$$\cos \gamma = \frac{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}{at}.$$

Note that the integrals over the upper and lower hemispheres both transform into integrals over the same circle, hence the integral over the circle Σ_{at}^M should be **taken as twice the integral over the upper hemisphere**.

Thus,

$$\begin{aligned} u(x, y, t) &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \varphi dS \right) + \frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \psi dS \\ &= \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{\Sigma_{at}^M} \frac{2\varphi}{\cos \gamma} d\sigma \right) + \frac{t}{4\pi a^2 t^2} \iint_{\Sigma_{at}^M} \frac{2\psi}{\cos \gamma} d\sigma, \\ u(x, y, t) &= \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\Sigma_{at}^M} \frac{\varphi(\xi, \eta) d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \right] + \frac{1}{2\pi a} \iint_{\Sigma_{at}^M} \frac{\psi(\xi, \eta) d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}. \end{aligned} \quad (3.2.10)$$

The above equation is called the **Poisson formula** for the initial value problem of the two-dimensional wave equation. Since the integration region $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \leq a^2 t^2$ is a circular domain centered at M with radius at , we usually use **polar coordinates** to calculate the integral in equation (3.2.10).

Ex 3.2.2. Solve the following problem

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & (-\infty < x, y < +\infty, t > 0), \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 2xy. \end{cases}$$

Solution. From equation (3.2.10), we get

$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi} \int_0^t \int_0^{2\pi} \frac{2(x + \rho \cos \theta)(y + \rho \sin \theta)}{\sqrt{t^2 - \rho^2}} \rho d\rho d\theta \\ &= \frac{xy}{\pi} \int_0^t \int_0^{2\pi} \frac{\rho d\rho d\theta}{\sqrt{t^2 - \rho^2}} = 2xyt. \end{aligned}$$

Key Differences Between 2D and 3D Solution Formulas

- In 3D, the integral is taken over a **hollow spherical surface**, meaning the integration surface is a sphere with an empty interior.
- In 2D, the integral is taken over a **solid circular region**, meaning the entire disk, including its interior, contributes to the integration.
- This **difference** is a fundamental characteristic to remember when working with wave equations in different dimensions.

Modification of the Integrand

- In 3D, the integrand function is simply φ, ψ .
- In 2D, the integrand includes an additional factor due to the transformation of coordinates.
- As time increases, the denominator of the modified integrand grows, leading to a decrease in the integral's value.
- This results in **attenuation over time** in 2D wave propagation.

3.2.3 Physical meaning of the Solution

Physical Interpretation

- Consider a region Ω where an initial perturbation is introduced.
- Outside Ω , the initial state is zero.
- Observing from a point M , one will detect the perturbation after some time as the wave propagates.
- Wave propagation speed is finite, meaning there is a delay before the perturbation reaches M .
- At time T_1 , the value at M is determined by an **average over a spherical shell (3D) or a circular disk (2D)**.
- The integration method used depends on whether the problem is in 2D or 3D.

Recall Solution (3.2.8)

$$u(M, t) = \frac{\partial}{\partial t} \left(\frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \varphi(\xi, \eta, \zeta) dS \right) + \frac{t}{4\pi a^2 t^2} \iint_{S_{at}^M} \psi(\xi, \eta, \zeta) dS.$$

Assume that the initial disturbance occurs only within a finite region Ω in space. Outside of region Ω , consider any point M , and examine the situation of the influence of the initial disturbance on point M at different times (see Fig. 3.11).

We know that the value of the solution u at point M and time t , $u(M, t)$, is determined by the values of the initial functions φ and ψ on the sphere S_{at}^M . Therefore, the integral in equation (3.2.8) is not zero **only** when the sphere S_{at}^M intersects with region Ω , hence $u(M, t) \neq 0$.

Let d and D represent the **closest** and **farthest** distances from point M to region Ω , respectively, as shown in the figure. When $at < d$, the sphere S_{at}^M is still at a distance from region Ω , so the values of φ

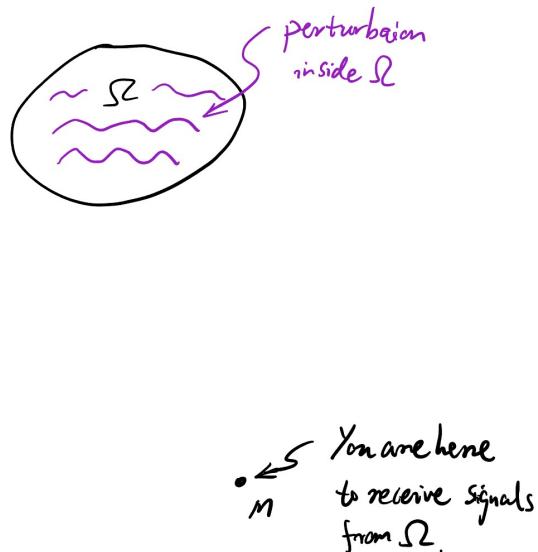


Figure 3.11: Physical meaning 1

and ψ on this sphere are 0, the integral is 0, and thus $u(M, t) = 0$. At this time, the disturbance has not yet reached point M (see Fig. 3.12).

When $d \leq at \leq D$, the sphere S_{at}^M continuously intersects with region Ω , the value of the integral is generally not 0, and the value of $u(M, t)$ is generally not 0 either. At this time, point M is in a disturbed state. The initial disturbance reaches point M instantaneously at $t = d/a$ (see Fig. 3.13)..

When $at > D$, the sphere S_{at}^M has already passed the initial disturbance region Ω and no longer intersects with it. Starting from $t = D/a$, $u(M, t)$ takes the zero value again, indicating that the disturbance has already passed point M , and point M returns to its original state of rest (see Fig. 3.14).

In a bounded region Ω , any disturbance caused by a point propagates outward at speed a . Therefore, at time t , the **region affected by the initial disturbance** in Ω is **all of the spheres** centered at $p \in \Omega$ with radius at . When t is sufficiently large, these spherical surfaces have **two envelopes**. The **outer envelope** is called the **front wavefront**, and the **inner envelope** is called the **rear wavefront**. The middle part between these two wavefronts is the region affected by the initial disturbance.

From the above analysis, it can be seen that the wave generated by a point disturbance propagates as a spherical surface passing through M . A large number of point disturbances form the entirety of these spherical surfaces, which together constitute the disturbance region. Between the disturbance regions, there exist a leading envelope and a trailing envelope.

The part outside the front wavefront indicates the region where the wave has **not yet reached**, while the part inside the rear wavefront indicates the region where the wave has **passed and returned to its original state**. Therefore, when the initial disturbance is confined to a certain local area in space, the wave propagation has a clear front and rear wavefront. This phenomenon is known in physics as **Huygens' principle** or the **non-retroactive phenomenon**. Since the disturbance at point $M_0 \in \Omega$ at $t = t_0$ affects the sphere $S_{at_0}^{M_0}$ centered at M_0 with radius at_0 , solution (3.2.8) is referred to as a **spherical wave**.

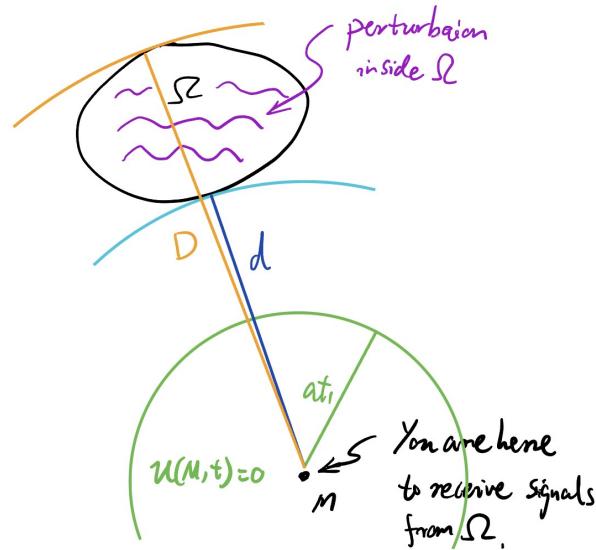


Figure 3.12: Physical meaning 2

Recall Solution (3.2.10):

$$u(x, y, t) = \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\Sigma_{at}^M} \frac{\varphi(\xi, \eta) d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \right] + \frac{1}{2\pi a} \iint_{\Sigma_{at}^M} \frac{\psi(\xi, \eta) d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}.$$

For the solution of the initial value problem of the two-dimensional wave equation (3.2.10), a similar discussion can be made. However, it is important to note that since the **integration is performed over the circular domain $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \leq a^2 t^2$** . Thus for any point M , once $u(M, t)$ changes from 0 to non-zero as time t increases, it will **not gradually decrease back to 0** as in the spatial case, but it will **gradually decrease** from a certain moment onward. Thus, there are **significant differences** between two-dimensional and three-dimensional situations (see Fig. 3.15).

For the two-dimensional case, the wave propagation **only has a front wavefront and no rear wavefront**, and **Huygens' principle no longer holds**. This phenomenon is called **wave diffusion**, or in other words, this type of wave exhibits a **retroactive effect**. For the two-dimensional problem, it can be considered that the initial disturbance occurs within an **infinitely long cylinder** and is **independent** of the z -coordinate. Thus, the initial disturbance at point M_0 should be considered as an initial disturbance along an infinitely long straight line passing through point M_0 and parallel to the z -axis. At $t = t_0$, its influence is within a cylindrical surface with this line as the axis and at_0 as the radius. Therefore, solution (3.2.10) is referred to as a **cylindrical wave**.

Summary:

Wave Propagation and Initial Disturbance (see Fig. 3.11)

- Consider a region Ω with an initial disturbance.

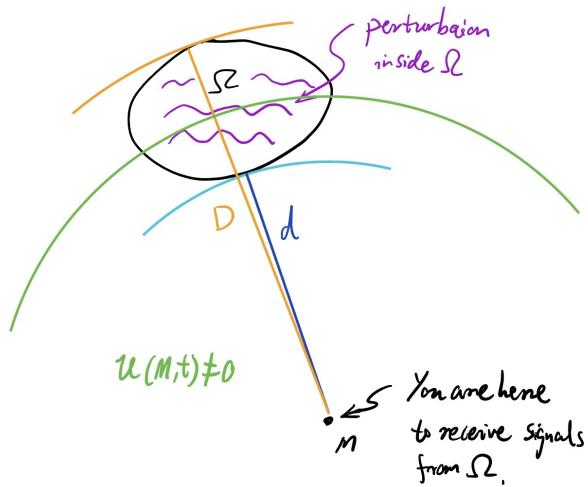


Figure 3.13: Physical meaning 3

- Outside Ω , the initial condition is zero.
- Observation point M is used to analyze wave propagation.
- As time evolves, point M will receive the wave signal after a certain delay due to finite propagation speed.

Three Stages of Wave Reception

First Stage: No Signal (see Fig. 3.12)

- At time t_1 , construct a spherical surface centered at M with radius at_1 .
- If this sphere does not intersect Ω , the integral over the surface is zero.
- Thus, $u(M, t_1) = 0$, meaning no signal is received at M .

Second Stage: Signal Reception (see Fig. 3.13)

- At time t_2 , the spherical surface intersects Ω .
- Since the integral now includes nonzero contributions from Ω , $u(M, t_2) \neq 0$.
- This indicates that the signal has reached M .

Third Stage: Key Differences Between 2D and 3D

For 3D Waves (see Fig. 3.14):

- At time t_3 , the sphere extends beyond Ω .

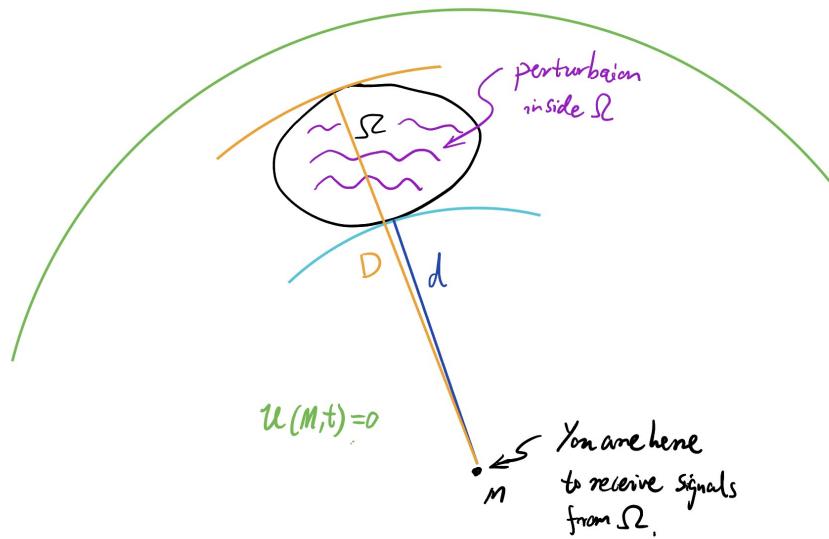


Figure 3.14: Physical meaning 4

- The integral is again zero, implying that $u(M, t_3) = 0$.
- This demonstrates the principle of **Huygens' Principle** (no after-effect phenomenon).

For 2D Waves (see Fig. 3.15):

- In 2D, the integration region is a solid disk instead of a spherical shell.
- At time t_3 , the integral remains nonzero since the solid region still overlaps with Ω .
- However, the signal weakens over time due to a decay factor in the denominator.
- This results in a **persistence effect** and **wave dispersion**, unlike the 3D case.

Key Differences Between 2D and 3D Wave Propagation

- **3D waves exhibit no after-effect:** once the signal passes, the region remains undisturbed (“Let bygones be bygones”).
- **2D waves exhibit after-effect:** signals persist beyond initial interaction but decay over time (“What’s done is done”).
- The difference arises from the geometry of integration: spherical surface vs. solid disk.

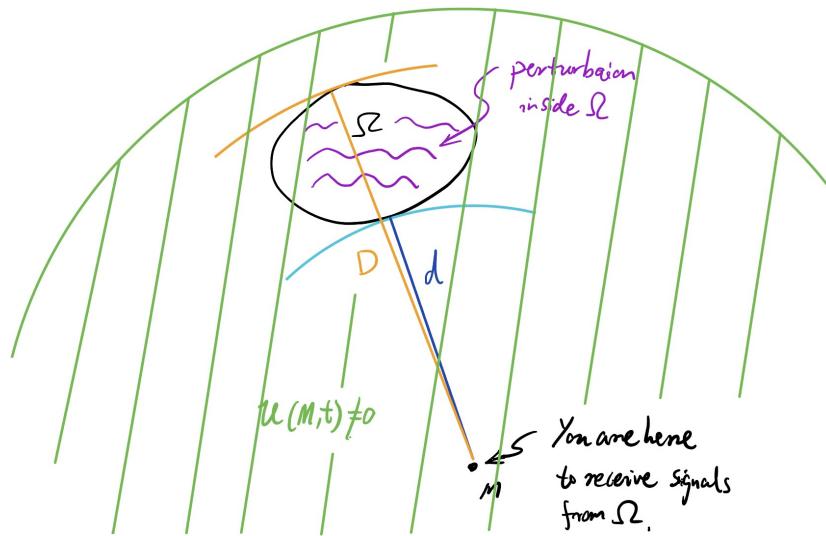


Figure 3.15: Physical meaning 5

3.3 Integral Transforms

3.3.1 Integral Transforms and Their Properties (Review)

If a function $f(x)$ is continuous and differentiable on $(-\infty, +\infty)$ and **absolutely integrable** (make sure the existence of the integral), then it has the **Fourier Transform**

$$\hat{f}(\lambda) = F(f) = \int_{-\infty}^{+\infty} f(x)e^{-ix\lambda} dx$$

and its **inverse Fourier Transform**

$$f(x) = F^{-1}(\hat{f}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\lambda)e^{ix\lambda} d\lambda.$$

If a function $f(t)$ does not grow faster than an exponential² on $(0, +\infty)$, then its **Laplace Transform** is defined as

$$F(s) = L(f) = \int_0^{+\infty} f(t)e^{-st} dt, \quad (\text{Res } > C)$$

Compared to the Fourier transform, the conditions for the existence of the Laplace transform are much weaker because the exponential factor $\exp(-\beta t)$ makes the integral converge more easily. However, this does not mean that any function has a Laplace transform without any conditions. In fact, the **sufficient conditions** for the existence of the **Laplace transform** can be described as follows:

1. The function $f(t)$ is piecewise continuous on the interval $[0, \infty)$;

²There exist constants $M > 0, C > 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t > 0$.

2. There exist positive constants M and α such that for all $t \geq 0$, $|f(t)| \leq M \exp(Ct)$ holds, then the function $f(t)$ has a Laplace transform for all $\text{Re}s > C$, i.e.,

$$\left| \int_0^\infty f(t)e^{-st} dt \right| < \infty$$

Hint: $|F(s)| \leq \int_0^{+\infty} |f(t)|e^{-\text{Re}(s)t} dt \leq \int_0^{+\infty} M e^{(C-\text{Re}(s))t} dt < +\infty$

Laplace Transform is denoted as $f(t) = L^{-1}(F(s))$. Laplace Transform can be viewed as the special case of Fourier Transform. Because, (let $s = \alpha + i\beta$) then $\text{Re}(s) > 0$,

$$\begin{aligned} F(s) = L(f) &= \int_0^\infty f(t)e^{-st} dt = \int_0^\infty f(t)e^{-\alpha t - i\beta t} dt \\ &= \int_0^\infty [f(t)e^{-\alpha t}] e^{-i\beta t} dt = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-i\beta t} dt \end{aligned}$$

where $\tilde{f}(t) \equiv \begin{cases} f(t)e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$.

Using the inverse Fourier Transform, (when $t \geq 0$)

$$\begin{aligned} \tilde{f}(t) &= f(t)e^{-\alpha t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha + i\beta) e^{i\beta t} d\beta \stackrel{\text{let } s=\alpha+i\beta}{=} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} F(s) e^{st} e^{-\alpha t} ds \\ &\Rightarrow f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} F(s) e^{st} ds \end{aligned}$$

This form of calculation is usually more difficult, but when $F(s)$ satisfies certain conditions, it can be calculated using the residue theorem.

Laplace inverse transform is denoted as

$$f(t) = L^{-1}(F(s))$$

Theorem 3.3.1 (Using the residue theorem). Suppose $F(s)$ is analytic except only a finite number of isolated singular points s_1, s_2, \dots, s_n in the half-plane $\text{Re}s \leq c$, and when $s \rightarrow \infty$, $F(s) \rightarrow 0$, then

$$f(t) = \sum_{k=1}^n \text{Res}[F(s)e^{st}, s_k].$$

Basic idea: A differential equation (DE) (in a rough form): $Au = f$ where A is a **differential operator**.

Using Fourier (or Laplace) transform

$$\mathcal{F}(Au) = \mathcal{F}f \Rightarrow B\hat{u} = \mathcal{F}f \quad (\text{one equation can solve multiple unknowns})$$

$$\Rightarrow \hat{u} = B^{-1}\mathcal{F}f \Rightarrow u = \mathcal{F}^{-1}(B^{-1}\mathcal{F}f) \text{ to find the solution.}$$

Key: Hope that the integral transform can **transform a differential equation** into an **algebraic equation**. Fourier and Laplace transforms have this effect.

Convergence Region of Laplace Transform

The existence of the Laplace transform $U(x, s)$ requires s to be within a certain region of convergence. For physical problems, it is generally assumed that the real part of s is sufficiently large to ensure the convergence of the transform. Specifically for this topic:

- The initial conditions and boundary conditions of the Laplace transform require $\operatorname{Re}(s)$ to be large enough to ensure the integral $\int_0^\infty e^{-st}u(x, t) dt$ converges.
- To ensure $U(x, s)$ converges as $x \rightarrow +\infty$, we implicitly assume $\operatorname{Re}(s) > 1$.

Why can we assume $\operatorname{Re}(s) > 1$?

- The convergence region of the Laplace transform is usually determined by the growth of the function. For most physical problems (especially decay problems), the real part of s can be chosen to be large enough.
- During the solution process, we implicitly assume s is within the convergence region that makes all transforms and inverse transforms valid. Therefore, it is reasonable to assume $\operatorname{Re}(s) > 1$.

Physical Meaning

- The assumption $\operatorname{Re}(s) > 1$ corresponds to the decay in time domain of the solution. If $\operatorname{Re}(s)$ is too small, the solution may grow (unstable), which contradicts the boundary condition $\lim_{x \rightarrow +\infty} u(x, t) = 0$.
- Starting from the physical problem, usually only the case where $\operatorname{Re}(s)$ is large enough is of concern.

Integral Transform Properties

(1) Linearity

Integral transforms exhibit linearity, which means:

$$\begin{aligned} F[af + bg] &= aF[f] + bF[g], \\ L[af + bg] &= aL[f] + bL[g], \end{aligned}$$

where a and b are arbitrary constants.

(2) Differential Theorem 1

If f and f' can both have Fourier or Laplace transforms and are zero at infinity, then:

$$F[f'(x)] = i\lambda F[f(x)], \quad F[f''(x)] = (i\lambda)^2 F[f(x)],$$

$$F[f^{(n)}(x)] = (i\lambda)^n F[f(x)],$$

and

$$L[f'(t)] = sL[f(t)] - f(0),$$

$$L[f''(t)] = s^2 \underbrace{L[f(t)]}_{\text{viewed as } -1 \text{ order derivative}} - sf(0) - f'(0), \leftarrow \boxed{\text{order of polynomial} + \text{order of derivative} = 1}$$

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

- The proof involves using integration by parts to transfer the derivative to the exponential function.

(3) Differential Theorem 2

If $\hat{f}(\lambda) = F[f(x)]$, $F(s) = L[f(t)]$, then:

$$\hat{f}'(\lambda) = F[-ixf] \quad (\text{Fourier transform}),$$

$$F'(s) = -L[tf(t)] \quad (\text{Laplace transform}).$$

(4) Convolution Theorem

If the convolution of f and g is defined as:

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)dy,$$

and can undergo Fourier transform, then:

$$F[f * g] = F[f] \cdot F[g],$$

and thus:

$$F^{-1}[\hat{f} \cdot \hat{g}] = f * g.$$

For Laplace transforms, there is a similar convolution theorem:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau. \leftarrow \boxed{\text{According to Laplace transf. } \tau > 0 \text{ and } t-\tau > 0 \text{ imply } 0 < \tau < t}.$$

Definition and Origin

- Convolution is not originally defined in integral form but arises naturally in series multiplication.
- Given two infinite series:

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{m=0}^{\infty} b_m,$$

their product can be computed in different ways.

Series Multiplication Methods

- Element-wise multiplication results in an infinite matrix:

$$\begin{bmatrix} a_0b_0 & a_0b_1 & a_0b_2 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Different summation methods yield different products:
 - Summing row by row.
 - Summing column by column.
 - Summing along diagonals (Cauchy product).

Cauchy Product

- Diagonal summation gives:

$$\sum_{n=0}^{\infty} \sum_{i=0}^n a_i b_{n-i},$$

which is known as the Cauchy product formula.

Connection to Convolution

- Summation is a **discrete form** of integration.
- Replacing sums with integrals generalizes the concept to continuous convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

- The index transformation in series multiplication corresponds to variable substitution in convolution.
- Convolution originates as a redistribution method in multiplication but has many useful properties in mathematics.

(5) Frequency Shift Theorem (Translation Theorem)

For the variable of the transform, if $\hat{f}(\lambda) = F[f(x)]$, $F(s) = L[f(t)]$, then:

$$F[f(x)e^{-i\lambda_0 x}] = \hat{f}(\lambda + \lambda_0) \quad (\text{Fourier transform}),$$

$$L[f(t)e^{-at}] = F(s + a) \quad (\text{Laplace transform}).$$

- By definition, it can be proven directly.

(6) Delay Theorem

For the variable of the transform, if $\hat{f}(\lambda) = F[f(x)]$, $F(s) = L[f(t)]$, then:

$$F[f(x - x_0)] = \hat{f}(\lambda)e^{-i\lambda x_0} \quad (\text{Fourier transform}),$$

$$L[f(t - t_0)u(t - t_0)] = F(s)e^{-st_0} \quad (\text{Laplace transform}),$$

where

$$u(t - t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases} \leftarrow \boxed{\text{Heaviside function}}$$

This can be simplified to:

$$L[f(t - t_0)] = F(s)e^{-st_0} \quad (t > t_0).$$

Proof of the Laplace Transform Delay Theorem. By the definition of the Laplace transform,

$$L[f(t - t_0)u(t - t_0)] = \int_0^\infty f(t - t_0)u(t - t_0)e^{-st}dt = \int_{t_0}^\infty f(t - t_0)e^{-st}dt$$

Let $y = t - t_0$, then the equation becomes

$$\text{Left side} = \int_0^\infty f(y)e^{-s(y+t_0)}dy = e^{-st_0} \int_0^\infty f(y)e^{-sy}dy = F(s)e^{-st_0} = \text{Right side}$$

□

Supplementary: Definition and Properties of the Dirac Delta Function

The Dirac delta function is a mathematical model abstracted from some physical phenomena, such as the impact force in mechanics, the explosion of a hydrogen bomb, etc. These physical phenomena have a common characteristic: the action time is extremely short, but the action intensity is extremely large. (Impulse function)

(1) Definition of the Dirac Delta Function: A function satisfying the following two conditions

$$1. \delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$2. \int_{-\infty}^{+\infty} \delta(x) dx = 1$$

If the impulse action does not occur at $x = 0$, but occurs at $x = x_0$, then the function is denoted as $\delta(x - x_0)$, and satisfies

$$\delta(x - x_0) = \begin{cases} \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x - x_0) dx = 1$$

(2) Properties of the Dirac Delta Function:

- Sampling Property:

$$\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

Specifically,

$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0)$$

- The sampling property considers the function F at the point where δ is infinite.

- Symmetry: $\delta(x)$ is an even function, then

$$\delta(x - x_0) = \delta(x_0 - x)$$

Specifically,

$$\delta(x) = \delta(-x)$$

Naturally, there is also

$$\int_{-\infty}^{+\infty} f(x) \delta(x_0 - x) dx = f(x_0) = f * \delta(x_0)$$

Ex 3.3.1. Find the Fourier Transform of $\delta(x + a)$, where a is a constant independent of the variable x .

Solution. By definition,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-ix\lambda} dx = \int_{-\infty}^{\infty} \delta(x + a) e^{-ix\lambda} dx$$

Using the properties of the δ function,

$$\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

Thus,

$$F[\delta(x + a)] = e^{ia\lambda}$$

Similarly,

$$F[\delta(x - a)] = e^{-ia\lambda}$$

Using the linearity of the Fourier transform,

$$F\left[\frac{1}{2}[\delta(x + a) + \delta(x - a)]\right] = \frac{e^{ia\lambda} + e^{-ia\lambda}}{2} = \cos a\lambda$$

$$F\left[\frac{1}{2i}[\delta(x + a) - \delta(x - a)]\right] = \frac{e^{ia\lambda} - e^{-ia\lambda}}{2i} = \sin a\lambda$$

Thus, we have the formulas:

$$F^{-1}[\cos a\lambda] = \frac{1}{2}[\delta(x + a) + \delta(x - a)]$$

$$F^{-1}[\sin a\lambda] = \frac{1}{2i}[\delta(x + a) - \delta(x - a)]$$

Ex 3.3.2. Find the Fourier Transform of

$$f(x) = \begin{cases} 1, & |x| \leq m \\ 0, & |x| > m \end{cases}$$

where $m > 0$.

Solution. By definition and using $e^{i\theta} = \cos \theta + i \sin \theta$,

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-ix\lambda} dx = \int_{-m}^m e^{-ix\lambda} dx = \int_{-m}^m (\cos x\lambda - i \sin x\lambda) dx = 2 \int_0^m \cos x\lambda dx = \frac{2 \sin m\lambda}{\lambda}$$

From this, we can conclude:

$$F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \leq m.$$

Ex 3.3.3. Find the Inverse Fourier Transform of $\hat{f}(\lambda) = e^{-\lambda^2 t}$, where $t > 0$.

Solution. By definition,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{ix\lambda} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 t} e^{ix\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 t} (\cos x\lambda + i \sin x\lambda) d\lambda = \frac{1}{\pi} \int_0^{\infty} e^{-\lambda^2 t} \cos x\lambda d\lambda \end{aligned} \tag{3.3.1}$$

Differentiate $f(x)$ and use integration by parts once to obtain:

$$\frac{df(x)}{dx} + \frac{x}{2t} f(x) = 0 \Rightarrow f(x) = f(0) e^{-\frac{x^2}{4t}}$$

Note (REVISE: According to my new lectures)

$$\frac{df(x)}{dx} = -\frac{1}{\pi} \int_0^\infty \lambda e^{-\lambda^2 t} \sin(\lambda x) d\lambda$$

and

$$\begin{aligned} \frac{xf(x)}{2t} &= \frac{1}{\pi} \int_0^\infty \frac{x}{2t} e^{-\lambda^2 t} \cos(\lambda x) d\lambda = \frac{1}{\pi} \int_0^\infty \frac{1}{2t} e^{-\lambda^2 t} d(\sin(\lambda x)) \\ &= \frac{1}{\pi} \left[\frac{1}{2t} e^{-\lambda^2 t} \sin(\lambda x) \right]_0^\infty - \frac{1}{\pi} \int_0^\infty \frac{1}{2t} \sin(\lambda x) d(e^{-\lambda^2 t}) \\ &= -\frac{1}{\pi} \int_0^\infty \frac{1}{2t} e^{-\lambda^2 t} \cdot (-t) 2\lambda \sin(\lambda x) d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \lambda e^{-\lambda^2 t} \sin(\lambda x) d\lambda \\ &\Rightarrow \frac{df(x)}{dx} + \frac{x}{2t} f(x) = 0. \end{aligned}$$

By (3.3.1),

$$f(0) = \frac{1}{\pi} \int_0^\infty e^{-\lambda^2 t} d\lambda$$

Using the Gaussian integral,

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

we conclude $f(0) = \frac{1}{2\sqrt{\pi t}}$, and further

$$f(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

From this example, we conclude:

$$F^{-1}[e^{-\lambda^2 t}] = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad (t > 0)$$

A novel approach is introduced to **solve integral problems** by transforming them into **differential equation problems**. This method, referred to as the "Spotlight Method," involves viewing a problem from a higher perspective to simplify the solution process.

Introduction

A problem-solving approach, inspired by a description of John Nash's thinking, is introduced. Instead of solving a problem directly, this method suggests gaining insight from a seemingly unrelated but broader perspective.

Core Idea

- Traditional approach: Solve the problem directly, like climbing a mountain to reach its peak.
- Nash's approach: Instead of climbing the given mountain, he first ascends a different, higher but gently sloping mountain (see Fig. 3.16).
- At the peak of the higher mountain, he uses a "searchlight" to illuminate the original mountain, gaining a clear view of its structure.
- This method suggests that solving a more general or advanced problem may provide clarity on a specific one.

Key Insights

- Reformulating a problem by looking at a broader or higher-level problem can simplify the original challenge.
- A shift in perspective often reveals hidden structures and deeper connections.
- Tackling an abstract or seemingly unrelated problem can offer new insights into the original one.

Key Ideas

- Instead of solving the integral directly, transform it into a differential equation.
- Exponential and trigonometric functions are useful due to their well-behaved derivatives:
 1. $\frac{d}{dx} e^x = e^x$
 2. $\frac{d^2}{dx^2} \cos x = -\cos x$
- The **goal** is to construct a function $f(x)$ satisfying a differential equation of the form:

$$\frac{d}{dx} f(x) = k(x)f(x)$$

which simplifies the integration process.

Solution Strategy

1. Define $f(x)$ based on the given integral.
2. Compute its derivative to form a differential equation.
3. Utilize integration by parts to transform terms into a solvable form.



Figure 3.16: Mountain analogy

4. Ensure boundary conditions eliminate unnecessary terms.
5. Solve the differential equation, leading to an explicit solution.

Gaussian Integral Application

- The method confirms that the Gaussian integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

is valid.

Ex 3.3.4. Find the Inverse Fourier Transform of $\hat{f}(\lambda) = e^{-|\lambda|y}$, where $y > 0$.

Solution. By definition,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{ix\lambda} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\lambda|y} e^{ix\lambda} d\lambda \\ &= \frac{1}{2\pi} \left(\int_0^{\infty} e^{\lambda(ix-y)} d\lambda + \int_{-\infty}^0 e^{\lambda(ix+y)} d\lambda \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{y-ix} + \frac{1}{y+ix} \right) = \frac{1}{\pi} \frac{y}{y^2+x^2} \end{aligned}$$

From this example, we conclude:

$$F^{-1}[e^{-|\lambda|y}] = \frac{1}{\pi} \frac{y}{y^2+x^2} \quad (y > 0)$$

Common Fourier Transforms and Inverse Transforms

1. $F[\delta(x+a)] = e^{ia\lambda}$, $F[\delta(x-a)] = e^{-ia\lambda}$ and $F(\delta(x)) = 1$
2. $F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \leq m$
3. $F^{-1}[e^{-\lambda^2 t}] = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad (t > 0)$
4. $F^{-1}[e^{-|\lambda|y}] = \frac{1}{\pi} \frac{y}{y^2+x^2} \quad (y > 0)$
5. $F^{-1}[\cos a\lambda] = \frac{1}{2}[\delta(x+a) + \delta(x-a)]$ and $F^{-1}[\sin a\lambda] = \frac{1}{2i}[\delta(x+a) - \delta(x-a)].$

Common Laplace Transforms and Inverse Transforms ($\text{Re } s > 0$)

1. $L[\delta(t)] = 1$
2. $L[e^{-at}] = \frac{1}{s+a}$ (especially, $L[1] = \frac{1}{s}$)
3. $L[t^n] = \frac{n!}{s^{n+1}}$ and $L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$
4. $L[\sin at] = \frac{a}{s^2+a^2}$ and $L[\cos at] = \frac{s}{s^2+a^2}$
5. $L[e^{-at} \sin at] = \frac{a}{(s+a)^2+a^2}$ and $L[e^{-at} \cos at] = \frac{s+a}{(s+a)^2+a^2}$
6. $L^{-1}[F(s)e^{-sa}] = f(t-a) \quad (t > a)$ (Inverse Transform Form of the Delay Theorem)
7. $L^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{2}{\sqrt{\pi}} \int_a^{\infty} \frac{e^{-y^2}}{2\sqrt{t}} dy \quad (\text{Error Function})$
8. $L^{-1}[e^{-a\sqrt{s}}] = \frac{a}{2\sqrt{\pi t^3}} e^{-\frac{a^2}{4t}}$. In fact, $L^{-1}[e^{-a\sqrt{s}}] = L^{-1}\left[s \cdot \frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{d}{dt} \left[\frac{2}{\sqrt{\pi}} \int_a^{\infty} \frac{e^{-y^2}}{2\sqrt{t}} dy \right] = \frac{a}{2\sqrt{\pi t^3}} e^{-\frac{a^2}{4t}}$

Ex 3.3.5. Use Laplace Transform to solve

$$\begin{cases} u''(t) + k^2 u(t) = f(t), \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

Solution. Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get:

$$\begin{aligned}s^2U(s) - su(0) - u'(0) + k^2U(s) &= F(s) \\ \Rightarrow s^2U(s) + k^2U(s) &= f(s)\end{aligned}$$

Thus,

$$\begin{aligned}U(s) &= \frac{1}{k} \frac{k}{s^2 + k^2} \cdot F(s) \\ L[\sin at] &= \frac{a}{s^2 + a^2}\end{aligned}$$

Taking the inverse Laplace transform of the above equation gives:

$$u(t) = \frac{1}{k} f(t) * \sin kt = \frac{1}{k} \int_0^t f(\tau) \sin k(t-\tau) d\tau$$

3.3.2 Examples of Integral Transform Methods

The Integral transform is a powerful technique for solving differential equations, particularly linear ODEs and PDEs. It simplifies the problem by converting differential equations into algebraic equations, which are easier to solve.

Three-Step Method

The solution process consists of three clear steps:

1. **Transform to Frequency Domain:** Apply the Laplace transform to all functions in the equation. Use the differentiation property of the Laplace transform:

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

If the initial conditions are zero, the additional terms disappear.

2. **Solve the Algebraic Equation:** After transformation, the differential equation becomes an algebraic equation in terms of $F(s)$. Solve for $F(s)$.
3. **Inverse Laplace Transform:** Convert $F(s)$ back to the time domain using the inverse Laplace transform. This often involves convolution:

$$L^{-1}\{G(s)H(s)\} = (g * h)(t)$$

Key Insights

- **Dimensional Reduction:** Each application of the Integral transform reduces the number of the derivatives of independent variables:
 - An ODE becomes an algebraic equation.
 - A PDE with two variables becomes an ODE.

- **Analogy to a Thought Experiment:**

- The process is compared to a classic riddle: "How do you put an elephant in a refrigerator?"
- Step 1: Open the refrigerator door (*transform to frequency space*).
- Step 2: Put the elephant inside (*solve the equation*).
- Step 3: Close the door (*inverse transform back to time space*).

- **Hidden Complexity:** While the process seems straightforward, practical applications reveal underlying difficulties that require careful handling.

Conclusion

The Laplace transform provides a structured and efficient way to solve differential equations. Its strength lies in reducing complex differential operations into algebraic manipulations, making it a widely used tool in engineering and applied mathematics.

The advantage of integral transform methods is that they simplify the original equation into a simpler form, facilitating the solution.

In applications, for **initial value problems**, the **Fourier transform** (for **spatial variables**) is commonly used, while for problems with **boundary conditions**, the **Laplace transform** (for **time variables**) is used.

When solving partial differential equations (PDEs) using integral transforms, a key question arises:

- Should we use the **Fourier transform** or the **Laplace transform**?
- How do we choose the appropriate transform?

Understanding the selection criteria is essential, as real-world problems do not explicitly specify which transform to use.

Criteria for Choosing the Transform

Step 1: Determine Which Variable to Transform

- The **goal** of an integral transform is to **eliminate derivatives in one variable**, converting the PDE into an ordinary differential equation (ODE).
- The choice should align with the **given boundary or initial conditions**.
- If the problem provides **initial conditions in time** ($t = 0$), it is generally better to eliminate x , reducing the PDE to an ODE in t .
- If the problem provides **boundary conditions in space** (x), it is generally better to eliminate t .

Step 2: Decide Between Fourier and Laplace Transforms

- The choice depends on the domain of the variable:
 - The **Fourier transform** is defined for $x \in (-\infty, \infty)$.
 - The **Laplace transform** is typically used for functions defined on $x \geq 0$ or $t \geq 0$.
- If x extends over the entire real line, use the **Fourier transform**.
- If x is restricted to $x \geq 0$, use the **Laplace transform**.

Application to a Specific Problem

For a given PDE, follow these steps:

1. **Select the variable to transform** based on initial or boundary conditions.
2. **Choose the transform type** based on the domain of the variable.
3. **Apply the transform** to all relevant functions in the equation.
4. **Solve the transformed equation** in the frequency domain.
5. **Use the inverse transform** to return to the original domain.

Key Insight

- **Dimensional Reduction:** Each application of the Integral transform reduces the number of the derivatives of independent variables;
- In some cases, it is not necessary to explicitly compute certain transforms of the **free terms** and **data** since they will **transform back** in the final solution.

In application, we give a general principle (deviating from it may introduce additional complexity or difficulties, but alternative methods may still yield a solution but could be less efficient)

- Initial value problem → Use Fourier transform (for spatial variables $x \in (-\infty, +\infty)$)
 ← Because initial conditions are given, so retain ∂_t derivative (solving ∂_t -eq needs initial data);
- Boundary value problem → Use Laplace transform (for temporal variables $t > 0$) ← Because boundary conditions are given, so retain ∂_x derivative (solving ∂_x -eq needs boundary data).

Otherwise, additional **hidden conditions** need to be specified to determine the parameters.

Conclusion

By systematically selecting the appropriate transform and variable, PDEs can be simplified effectively. The choice of integral transform plays a crucial role in ensuring the problem remains solvable with minimal additional complexity.

Ex 3.3.6. Solve the following problem:

$$u_t = a^2 u_{xx} + f(x, t) \quad (-\infty < x < +\infty, t > 0), \quad (3.3.2)$$

$$u|_{t=0} = \varphi(x). \quad (3.3.3)$$

Key Considerations

- Two key questions must be addressed:
 1. Which variable should be transformed?
 2. Which integral transform should be used?
- The variable to be transformed should align with the given boundary or initial conditions to simplify solving the ODE.
- If initial conditions are provided, it is generally preferable to transform the spatial variable x .

Choosing the Right Transform

- The choice of transform depends on the domain of the variable:
 - **Fourier Transform:** Suitable for variables defined over the entire real line $(-\infty, \infty)$.
 - **Laplace Transform:** Suitable for variables defined for non-negative values $t \geq 0$.
- The definition domain of the variable helps determine whether to use the Laplace or Fourier transform.

Solution. First, take the Fourier transform with respect to x , denote:

$$F[u(x, t)] = U(\lambda, t), \quad F[f(x, t)] = G(\lambda, t), \quad F[\varphi(x)] = \Phi(\lambda)$$

- Apply the Fourier transform to both sides of the given PDE.
- By linearity and using the differentiation property of the Fourier transform, the transform can be

applied term by term

$$\underbrace{F(\partial_t u)}_{=\partial_t U} = \underbrace{a^2 F(\partial_x^2 u)}_{=-a^2 \lambda^2 U} + \underbrace{F(f)}_{=G}$$

where

$$F(\partial_t u) = \int_{-\infty}^{\infty} \partial_t u \cdot e^{-i\lambda x} dx = \int_{-\infty}^{\infty} \partial_t(u \cdot e^{-i\lambda x}) dx = \partial_t \int_{-\infty}^{\infty} u \cdot e^{-i\lambda x} dx = \partial_t F(u) = \partial_t U$$

- Using this property, the PDE is transformed into an ODE.

Taking the Fourier transform of equation (3.3.2) with respect to x , we get:

$$\frac{dU(\lambda, t)}{dt} = -a^2 \lambda^2 U(\lambda, t) + G(\lambda, t), \quad (3.3.4)$$

which satisfies the initial condition:

$$U(\lambda, t)|_{t=0} = \Phi(\lambda). \quad (3.3.5)$$

- Various methods can be used to solve the ODE, such as:
 - Integrating factor method.
 - Method of variation of parameters.
 - Laplace transform.

To solve the ordinary differential equation initial value problem (3.3.4)–(3.3.5), denote:

$$L[U(\lambda, t)] = \bar{U}(\lambda, s), \quad L[G(\lambda, t)] = \bar{G}(\lambda, s).$$

Taking the Laplace transform of equation (3.3.4) with respect to t , and combining with condition (3.3.5), we get:

$$s\bar{U}(\lambda, s) - \Phi(\lambda) = -a^2 \lambda^2 \bar{U}(\lambda, s) + \bar{G}(\lambda, s),$$

$$\Rightarrow \bar{U}(\lambda, s) = \frac{1}{s + \lambda^2 a^2} \Phi(\lambda) + \frac{1}{s + \lambda^2 a^2} \bar{G}(\lambda, s). \quad (3.3.6)$$

Applying the Inverse Transforms

- After obtaining the solution in the frequency space, we must transform it back to the original space.
- This requires applying two inverse transforms:
 - Since two integral transforms have been applied, two **integral variables** appear in the solution.
 - It is **crucial to distinguish** which variable each inverse transform is applied to.
 - First, the inverse Laplace transform L^{-1} to revert from s -domain to t -domain.
 - Second, the inverse Fourier transform F^{-1} to revert from λ -domain to x -domain.

Taking the inverse Laplace transform of both sides of equation (3.3.6), we get:

$$U(\lambda, t) = \Phi(\lambda)L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \right] + L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \bar{G}(\lambda, s) \right].$$

Taking the inverse Laplace transform of equation (3.3.6) and using $L[e^{-at}] = \frac{1}{s+a}$, we get:

$$\begin{aligned} U(\lambda, t) &= \Phi(\lambda)L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \right] + L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \bar{G}(\lambda, s) \right] \\ &= \Phi(\lambda)e^{-a^2 \lambda^2 t} + G(\lambda, t) * e^{-a^2 \lambda^2 t} \\ &= \Phi(\lambda)e^{-a^2 \lambda^2 t} + \int_0^t G(\lambda, \tau)e^{-a^2 \lambda^2 (t-\tau)} d\tau. \end{aligned} \quad (3.3.7)$$

Handling the Inverse Laplace Transform

- The solution \tilde{U} is expressed in terms of s and λ .
- Applying the inverse Laplace transform, we use the convolution property:

$$L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \bar{G}(\lambda, s) \right] = L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \right] * G$$

where $*$ denotes convolution.

- If a term **does not depend on** s , it acts as a constant under the inverse Laplace transform.
- This results in:

$$L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \Phi(\lambda) \right] = L^{-1} \left[\text{constant}(\lambda) \cdot \frac{1}{s + \lambda^2 a^2} \right] = \text{constant}(\lambda) \cdot L^{-1} \left[\frac{1}{s + \lambda^2 a^2} \right]$$

Final Step: Applying the Inverse Fourier Transform

- After obtaining $U(x, t)$, apply the inverse Fourier transform:

$$u(x, t) = F^{-1}[U(\lambda, t)] \quad (3.3.8)$$

- This step restores the solution to its original spatial form.

To find the solution to problem (3.3.2)–(3.3.3), we still need to take the inverse Fourier transform of $U(\lambda, t)$. Taking the inverse Fourier transform of equation (3.3.7), we get:

$$u(x, t) = F^{-1}[\Phi(\lambda)e^{-a^2\lambda^2t}] + \int_0^t F^{-1}[G(\lambda, \tau)e^{-a^2\lambda^2(t-\tau)}] d\tau. \leftarrow \boxed{\text{Since } F^{-1} \text{ is an integral for } x, \text{ commutable.}}$$

Using the convolution theorem, we get:

$$u(x, t) = \varphi(x) * F^{-1}[e^{-a^2\lambda^2t}] + \int_0^t f(x, \tau) * F^{-1}[e^{-a^2\lambda^2(t-\tau)}] d\tau.$$

Using the conclusion:

$$F^{-1}[e^{-\lambda^2t}] = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \quad (t > 0)$$

we know

$$F^{-1}[e^{-a^2\lambda^2t}] = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}.$$

Thus, we obtain:

$$u(x, t) = \varphi(x) * \left(\frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \right) + \int_0^t f(x, \tau) * \left(\frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-\frac{x^2}{4a^2(t-\tau)}} \right) d\tau.$$

This is the solution to the original problem.

Ex 3.3.7. Use Fourier Transform to Solve the Following Problem

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \end{cases} \leftarrow \boxed{\text{initial data: tranf. } x \text{ by Fourier}} \quad (3.3.9)$$

↑ solved by the traveling wave method

Solution. Assume

$$\lim_{|x| \rightarrow \infty} u(x, t) = \lim_{|x| \rightarrow \infty} u_t(x, t) = 0.$$

(One might also try Laplace transform, but this hidden condition is needed.)

- When applying the Laplace transform to t (since $t > 0$), derivatives transform as follows:
 - The second derivative u_{tt} transforms into $s^2 L[u]$ with initial condition terms.
 - Initial conditions introduce additional terms, but **boundary conditions** may be lost.
- The resulting equation is a second-order ODE in x , but **lacks explicit boundary conditions for x** .
- To solve for x , implicit conditions must be identified, such as:
 - A common assumption: u and u_x approach zero at infinity.
 - However, specific problems may require different assumptions.
- If Laplace transform is used instead of the standard approach, additional **hidden conditions** must be extracted.
- Therefore, in such cases, it is often **preferable** to use the Fourier transform instead.

Taking the Fourier transform of both sides of equation (3.3.9) with respect to x , denote:

$$F[u(x, t)] = U(\lambda, t), \quad F[\varphi(x)] = \Phi(\lambda), \quad F[\psi(x)] = \Psi(\lambda).$$

Thus, we obtain:

$$\begin{cases} \frac{d^2U}{dt^2} = -a^2\lambda^2U, \\ U(\lambda, t)|_{t=0} = \Phi(\lambda), \quad \frac{dU}{dt}(\lambda, t)|_{t=0} = \Psi(\lambda). \end{cases} \quad \leftarrow \boxed{\partial_t \text{ and } \int \text{(i.e., } F\text{) are commutable.}} \quad (3.3.10)$$

Since $u_t|_{t=0} = \psi$, we obtain

$$F[\psi] = \int_{-\infty}^{+\infty} u_t(x, t) \Big|_{t=0} e^{-i\lambda x} dx = \left(\int_{-\infty}^{+\infty} u_t e^{-i\lambda x} dx \right) \Big|_{t=0} = \partial_t \left(\int_{-\infty}^{+\infty} u e^{-i\lambda x} dx \right) \Big|_{t=0}$$

Equation (3.3.10) is an ordinary differential equation with parameter λ . The solution to this initial value problem is:

$$U(\lambda, t) = \Phi(\lambda) \cos(a\lambda t) + \frac{\Psi(\lambda)}{a\lambda} \sin(a\lambda t). \quad (3.3.11)$$

Taking the inverse Fourier transform of equation (3.3.11), we get:

$$\begin{aligned} u(x, t) &= F^{-1}[\Phi(\lambda) \cos(a\lambda t)] + F^{-1}\left[\frac{\Psi(\lambda)}{a\lambda} \sin(a\lambda t)\right] \\ &= \varphi(x) * F^{-1}[\cos(a\lambda t)] + \frac{1}{a} \psi(x) * F^{-1}\left[\frac{\sin(a\lambda t)}{\lambda}\right]. \end{aligned} \quad (3.3.12)$$

Using the conclusions:

$$F^{-1}[\cos(a\lambda)] = \frac{1}{2}[\delta(x+a) + \delta(x-a)] \quad \text{and} \quad F^{-1}\left[\frac{\sin m\lambda}{\lambda}\right] = \frac{1}{2}, \quad |x| \leq m.$$

Thus, we obtain:

$$F^{-1}[\Phi(\lambda) \cos(a\lambda t)] = \varphi(x) * \frac{1}{2}[\delta(x+at) + \delta(x-at)] = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)].$$

$$\begin{aligned} & F^{-1}[\Phi(\lambda)] * F^{-1}[\cos(a\lambda t)] \\ &= \varphi(x) * \frac{1}{2}[\delta(x+at) + \delta(x-at)] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \varphi(x-\xi)[\delta(\xi+at) + \delta(\xi-at)]d\xi \leftarrow \boxed{\text{use } \int_{-\infty}^{\infty} f(\xi)\delta(\xi-x_0)d\xi = f(x_0)} \\ &= \frac{1}{2}[\varphi(x+at) + \varphi(x-at)]. \end{aligned}$$

Substituting the results into equation (3.3.12), the solution to the original problem (3.3.9) is:

$$u(x, t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha)d\alpha.$$

Ex 3.3.8. Solve the Following Problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & (-\infty < x < +\infty, y > 0), \\ u|_{y=0} = g(x), & \lim_{x^2+y^2 \rightarrow \infty} u(x, y) = 0. \end{cases} \quad (3.3.13)$$

Choice of Transformation

- Based on the first condition, since the given boundary condition involves y , we should transform with respect to x , a Fourier transform with respect to x , accordingly.
- However, the second condition complicates the choice, making it difficult to determine the proper action.
- Despite this, we prioritize the first condition, as transforming x ensures a boundary condition first, ensuring that **at least one** boundary condition is **properly handled**.
- Next, analyze the given second condition and determine **necessary modifications on the second condition** to facilitate solving the problem.

Solution. Take the Fourier transform of equation (3.3.13) with respect to x , denote:

$$F[u(x, y)] = U(\lambda, y), \quad F[g(x)] = G(\lambda).$$

Equation (3.3.13) becomes:

$$\begin{cases} \frac{d^2U}{dy^2} - \lambda^2 U = 0, \\ U(\lambda, 0) = G(\lambda), \quad \lim_{y \rightarrow \infty} U(\lambda, y) = 0. \end{cases} \quad (3.3.14)$$

Handling the Limit Condition

- The second condition is crucial. Since we have chosen to transform x , we need to ensure that we can transform (Fourier on x) the second boundary condition as well.
- To achieve this, x should not appear within the limit of the second condition; otherwise, the transformation cannot proceed.

$$\begin{cases} F\left(\underbrace{\lim_{x^2+y^2 \rightarrow \infty} u(x, y)}_{\downarrow}\right) = 0 \\ \int_{-\infty}^{+\infty} \lim_{x^2+y^2 \rightarrow \infty} u \cdot e^{-i\lambda x} dx \stackrel{\text{weaken}}{\Longrightarrow} \int \lim_{y \rightarrow \infty} u \cdot e^{-i\lambda x} dx = \lim_{y \rightarrow \infty} F(u) = 0 \end{cases}$$

- This transformation allows us to **replace the original condition** with a **necessary but weaker form**.
- This original condition is stronger because it requires convergence to zero in all radial directions as $r \rightarrow \infty$, while the weaker condition only applies along the y -axis.
- The weaker condition does not guarantee that the function tends to zero along, for instance, the x -axis, making it insufficient in certain cases.
- Therefore, we replace the stronger condition with a weaker necessary condition to facilitate the transformation.

Uniqueness of the Solution

- A **condition that is too strong** may lead to **no solution**, while a **too-weak** condition may result in **non-uniqueness**.
- Using the **weakest necessary condition** does **not alter the uniqueness** of the solution.
- This can be understood via the “**bucket theory**” analogy: **weakening a non-determining condition does not change** the final solution (see Fig. 3.17).
- Since we can obtain a solution under the **weaker condition** which ensures the existence of the solution, the uniqueness of the solution are mathematically provable.

- Moreover, the existence and uniqueness of the original problem can also be established.
- Since the weaker condition still leads to a unique solution, the solution to the weaker condition problem is identical to that of the original problem.

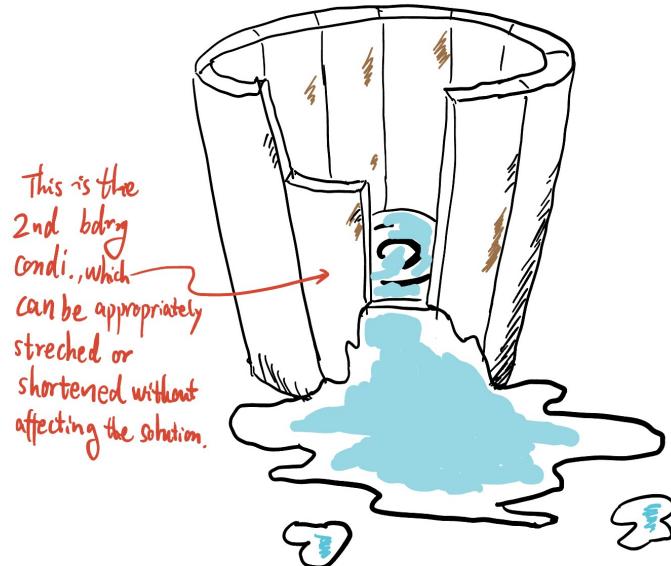


Figure 3.17: Bucket theory

Solving equation (3.3.14) gives:

$$U(\lambda, y) = G(\lambda)e^{-|\lambda|y}.$$

Taking the inverse Fourier transform of the above equation, we get:

$$u(x, y) = F^{-1}[G(\lambda)e^{-|\lambda|y}] = g(x) * F^{-1}[e^{-|\lambda|y}]$$

Using the conclusion:

$$F^{-1}[e^{-|\lambda|y}] = \frac{y}{\pi(y^2 + x^2)} \quad (y > 0).$$

Thus, the solution to the original problem (3.3.13) is:

$$u(x, y) = \frac{y}{\pi} g(x) * \frac{1}{y^2 + x^2} = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{y^2 + (x - \xi)^2} d\xi.$$

Ex 3.3.9. Solve the Following Problem

$$u_t = a^2 u_{xx}, \quad (x > 0, t > 0), \quad (3.3.15)$$

$$u|_{t=0} = 0, \quad (3.3.16)$$

$$u|_{x=0} = f(t), \quad (3.3.17)$$

$$|u(x, t)| < M. \quad (3.3.18)$$

- This problem involves both initial and boundary conditions.
- The key question is: which transformation should be used in this case?
- However, only the Laplace transform can be applied because $x > 0$ and $t > 0$, which align with its definition.
- The best approach is to apply the Laplace transform with respect to t since its initial data is zero.
- Transforming with respect to t is preferable because it simplifies the equation.

Solution. Take the Laplace transform of equations (3.3.15), (3.3.17) and (3.3.18) with respect to t , denote:

$$L[u(x, t)] = U(x, s), \quad L[f(t)] = F(s).$$

Equations (3.3.15)–(3.3.18) become:

$$a^2 \frac{d^2 U}{dx^2} - \underbrace{s}_{\text{Note } s \in \mathbb{C}} U = 0, \quad (3.3.19)$$

$$U(x, s)|_{x=0} = F(s), \quad (3.3.20)$$

$$|U(x, s)| < \overline{M} \leftarrow \boxed{\text{different from } M, \text{ bucket theory}}. \quad (3.3.21)$$

- Condition (3.3.21) weakens Condition (3.3.18) like the previous example. That is, (3.3.21) is a necessary condition of (3.3.18).
- More specifically,

$$|U(x, s)| = \left| \int_0^{+\infty} u(x, t) e^{-st} dt \right| \leq M \int_0^{+\infty} e^{-Re(s)t} dt < \infty.$$

- **Question:** Can the general solution be found using methods for second-order constant coefficient ODEs?
- The key issue: $s \in \mathbb{C}$ is a **complex number**, whereas standard methods are typically applied in the **real number**, i.e., $s \in \mathbb{R}$.
- In fact, the derivation of the general solution formula also applies to $s \in \mathbb{C}$. This derivation of the general solution formula holds identically in the complex domain, meaning the same process applies. That is, assume a **formal solution** $U = e^{\beta x}$ (trial method). Substituting it into (3.3.19) yields

$$a^2 \beta^2 e^{\beta x} - s e^{\beta x} = 0 \Rightarrow a^2 \beta^2 - s = 0 \Rightarrow \beta^2 = \frac{s}{a^2} \Rightarrow \beta = \pm \frac{\sqrt{s}}{a}.$$

- Therefore, $s \in \mathbb{C}$ can be **treated as a real number**, and the general solution formula can be used as before.

The general solution to equation (3.3.19) is:

$$U(x, s) = c_1 e^{-\frac{\sqrt{s}}{a}x} + c_2 e^{\frac{\sqrt{s}}{a}x}.$$

From condition (3.3.21), we know $c_2 = 0$. From condition (3.3.20), we know $c_1 = F(s)$. Thus, we have:

$$U(x, s) = F(s)e^{-\frac{\sqrt{s}}{a}x}. \quad (3.3.22)$$

Taking the inverse Laplace transform of equation (3.3.22), we get:

$$u(x, t) = L^{-1}[F(s)e^{-\frac{\sqrt{s}}{a}x}] = \underbrace{f(t)}_{\text{do not need transf. explicitly since it will be transformed back}} * L^{-1}[e^{-\frac{\sqrt{s}}{a}x}].$$

Using the conclusion:

$$L^{-1}\left[\frac{1}{s}e^{-a\sqrt{s}}\right] = \frac{2}{\sqrt{\pi}} \int_a^\infty \frac{e^{-y^2}}{2\sqrt{t}} dy.$$

Thus, we have:

$$L^{-1}\left[\frac{1}{s}e^{-\frac{x\sqrt{s}}{a}}\right] = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^\infty e^{-y^2} dy.$$

Using the first differentiation theorem of the Laplace transform again, we have

$$L^{-1}\left[e^{-\frac{x}{a}\sqrt{s}}\right] = L^{-1}\left[s \cdot \frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}\right] = \frac{d}{dt} \left[\frac{2}{\sqrt{\pi}} \int_t^{+\infty} \frac{1}{2a\sqrt{y}} e^{-y^2} dy \right] = \frac{x}{2a\sqrt{\pi}t^{3/2}} e^{-\frac{x^2}{4a^2t}}.$$

Thus, the solution to the original problem (3.3.15)–(3.3.18) is:

$$u(x, t) = f(t) * \frac{x}{2a\sqrt{\pi}t^{3/2}} e^{-\frac{x^2}{4a^2t}} = \frac{x}{2a\sqrt{\pi}} \int_0^t f(\tau) \frac{1}{(t-\tau)^{3/2}} e^{-\frac{x^2}{4a^2(t-\tau)}} d\tau.$$

Ex 3.3.10. Solve the Free Vibration Problem of a Semi-Infinite String

$$u_{tt} = a^2 u_{xx}, \quad (x > 0, t > 0), \quad (3.3.23)$$

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad (3.3.24)$$

$$u(0, t) = f(t), \quad \lim_{x \rightarrow +\infty} u(x, t) = 0. \quad (3.3.25)$$

where $f(t)$ is a known function (satisfying the Laplace transform conditions), and $f(0) = 0$.

Choosing Between Laplace and Fourier Transforms

- If the domain is $x > 0, t > 0$, using the Laplace transform is often the simplest choice.
- Fourier transform can still be used, but it requires an extension and later restriction, which complicates the process.

Domain and Boundary Conditions

- If the given boundary conditions involve zero initial values, the Laplace transform is usually preferable.
- When applying the Laplace transform to a second-order equation, initial conditions appear naturally in the transformed equation. Given that the initial conditions are zero, these terms vanish, simplifying the equation.
- If transforming in the x variable instead of t , some boundary conditions may be missing, requiring additional conditions to be inferred.

Properties of the Laplace Transform

- The Laplace transform is an integral with respect to t , meaning it does not interfere with differentiation in x .
- Similarly, limits on x and integrals (Laplace integral for t) can be interchanged when taking limits at infinity.

Solution. Take the Laplace transform of equations (3.3.23) and (3.3.25) with respect to t , denote:

$$L[u(x, t)] = U(x, s), \quad L[f(t)] = F(s).$$

Equations (3.3.23)–(3.3.25) become:

$$a^2 \frac{d^2 U}{dx^2} - s^2 U = 0, \quad (3.3.26)$$

$$U(0, s) = F(s), \quad \lim_{x \rightarrow +\infty} U(x, s) = 0. \quad (3.3.27)$$

The general solution to equation (3.3.26) is:

$$U(x, s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x},$$

Given a second-order constant coefficient ODE:

$$ay'' + by' + cy = 0,$$

where a, b, c are real or complex constants, the solution can be found using the following steps:

- Assume a trial solution of the form $y = e^{\lambda x}$.
- Compute derivatives: $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$.
- Substituting into the ODE gives the characteristic equation:

$$a\lambda^2 + b\lambda + c = 0.$$

Key Insights

- The approach is based on the trial solution $e^{\lambda x}$, leveraging its property under differentiation.
- The characteristic equation arises naturally from substituting the assumed solution.

From condition (3.3.27), we know $c_1 = 0$, $c_2 = F(s)$, thus we have:

$$U(x, s) = F(s)e^{-\frac{s}{a}x}.$$

Taking the inverse Laplace transform of the above equation, we get:

$$u(x, t) = L^{-1}[F(s)e^{-\frac{s}{a}x}]. \quad (3.3.28)$$

Using the inverse transform formula of the delay theorem of Laplace transform:

$$L^{-1}[F(s)e^{-sa}] = f(t - a) \quad (t > a).$$

Thus, equation (3.3.28) can be simplified to:

$$u(x, t) = \begin{cases} 0, & t < \frac{x}{a}, \\ f(t - \frac{x}{a}), & t > \frac{x}{a}. \end{cases}$$

This is the solution to the free vibration problem (3.3.23)–(3.3.25) of the semi-infinite string.

Ex 3.3.11. Solve the Following Problem

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < 1, \textcolor{red}{t > 0} \leftarrow \boxed{\text{Laplace transf.}}), \\ u(x, 0) = 4 \sin \pi x, \\ u(0, t) = 0, \quad u(1, t) = 0. \end{cases} \quad (3.3.29)$$

Solution. Obviously, take the Laplace transform with respect to t , denote:

$$L[u(x, t)] = U(x, s),$$

Equation (3.3.29) can be transformed into:

$$a^2 \frac{d^2 U}{dx^2} - sU = -4 \sin \pi x, \quad (3.3.30)$$

with boundary conditions:

$$U(0, s) = 0, \quad U(1, s) = 0. \quad (3.3.31)$$

The general solution to equation (3.3.30) is:

$$U(x, s) = c_1 e^{\frac{\sqrt{s}}{a}x} + c_2 e^{-\frac{\sqrt{s}}{a}x} + \frac{4 \sin \pi x}{s + a^2 \pi^2},$$

- **Type of Equation:** A second-order linear non-homogeneous ODE.
- **General Solution Method:**
 - First, treat the ODE as homogeneous and find the general solution.
 - Then, add a particular solution for the non-homogeneous part.
- **Methods for Particular Solution:**
 - Trial and Error Method: This is highly recommended for ODEs. Functions like exponential, sine, and cosine are useful due to their derivative invariance.
 - Variation of Parameters: Can be used but may be time-consuming.
 - Integral Transform: Not suitable for this ODE due to the domain issues (e.g., zero to one) and the need for extension and restriction.
- **Conclusion:** The trial and error method is the most efficient for solving this ODE. We calculate it below:

Let

$$U = A \sin \pi x. \quad (A \text{ to be determined}).$$

Then

$$U'' = -A\pi^2 \sin \pi x$$

From

$$a^2 U'' - sU = -4 \sin \pi x \Rightarrow -Aa^2\pi^2 \sin \pi x - sA \sin \pi x = -4 \sin \pi x$$

From

$$\begin{aligned} \Rightarrow -A \sin \pi x (a^2\pi^2 + s) &= -4 \sin \pi x \Rightarrow A = \frac{4}{a^2\pi^2 + s} \\ \Rightarrow U &= \frac{4}{a^2\pi^2 + s} \sin \pi x. \end{aligned}$$

From conditions (3.3.31),

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\frac{\sqrt{3}}{2}} + c_2 e^{-\frac{\sqrt{3}}{2}} + \frac{4 \sin \pi}{5+4\pi^2} = 0 \end{cases} \Rightarrow c_1 = c_2 = 0.$$

we know $c_1 = 0, c_2 = 0$, thus we have:

$$U(x, s) = \frac{4 \sin \pi x}{s + a^2\pi^2}.$$

Taking the inverse Laplace transform of the above equation, we get the solution to problem (3.3.29):

$$u(x, t) = 4 \sin \pi x L^{-1} \left[\frac{1}{s + a^2 \pi^2} \right] = 4e^{-a^2 \pi^2 t} \sin \pi x.$$

This solution is exactly the same as the one obtained by separation of variables.

3.4 Exercise class

Ex 3.4.1. Use the method of characteristics to solve the following initial-boundary value problem:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, x - at < 0, x > 0 \\ u|_{x=at=0} = \varphi(x), & u|_{x=0} = h(t). \end{cases}$$

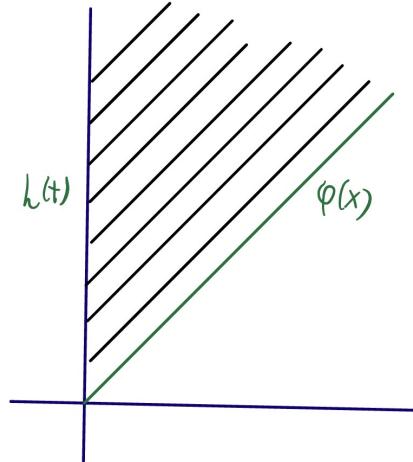


Figure 3.18: Goursat Problem

Solution. The general solution: The general solution of the wave equation is:

$$u(x, t) = f(x - at) + g(x + at). \leftarrow \boxed{\text{Memorize the general solution of wave}}$$

where f, g are arbitrary functions with continuous second derivatives.

Next, we use **boundary conditions** to determine the arbitrary functions f, g . First, from the condition:

$$u|_{x=at=0} = \varphi(x) \implies f(0) + g(2x) = \varphi(x).$$

Let $\eta = 2x$,

$$\implies g(\eta) = \varphi\left(\frac{\eta}{2}\right) - f(0). \leftarrow \boxed{\text{Know how to determine } f \text{ and } g \text{ by conditions}}$$

Using the condition:

$$u|_{x=0} = h(t) \implies f(-at) + g(at) = h(t).$$

Let $\xi = -at$

$$\implies f(\xi) = h\left(-\frac{\xi}{a}\right) - g(-\xi) = h\left(-\frac{\xi}{a}\right) - \varphi\left(-\frac{\xi}{2}\right) + f(0).$$

Substituting $f(\xi)$, $g(\eta)$ into the general solution formula gives the solution to the characteristic boundary problem as:

$$\begin{aligned} u(x, t) &= h\left(-\frac{x-at}{a}\right) - \varphi\left(-\frac{x-at}{a}\right) + f(0) + \varphi\left(\frac{x+at}{2}\right) - f(0) \\ &= \varphi\left(\frac{x+at}{2}\right) - \varphi\left(\frac{at-x}{2}\right) + h\left(\frac{at-x}{a}\right). \end{aligned}$$

Ex 3.4.2. Goursat problem

$$\begin{cases} u_{tt} = u_{xx} & (-\infty < x < +\infty, t > 0) \\ u|_{t=-x} = \varphi(x), \quad u|_{t=x} = \psi(x) \end{cases}$$

Solution. General solution:

$$u(x, t) = f_1(x+t) + f_2(x-t)$$

By initial data,

$$\begin{cases} f_1(0) + f_2(2x) = \varphi(x) \\ f_1(2x) + f_2(0) = \psi(x) \end{cases}$$

Let $y := 2x$, then

$$\begin{aligned} &\begin{cases} f_1(0) + f_2(y) = \varphi\left(\frac{y}{2}\right) \\ f_1(y) + f_2(0) = \psi\left(\frac{y}{2}\right) \end{cases} \\ &\Rightarrow \begin{cases} f_2(y) = \varphi\left(\frac{y}{2}\right) - f_1(0) \\ f_1(y) = \psi\left(\frac{y}{2}\right) - f_2(0) \end{cases} \end{aligned} \tag{3.4.1}$$

$$\Rightarrow u(x, t) = \psi\left(\frac{x+t}{2}\right) - f_2(0) + \varphi\left(\frac{x-t}{2}\right) - f_1(0) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - (f_1(0) + f_2(0))$$

Let $y = 0$ in (3.4.1) $\Rightarrow f_1(0) + f_2(0) = \varphi(0) = \psi(0) = \frac{1}{2}(\varphi(0) + \psi(0))$

$$\Rightarrow u(x, t) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - \underbrace{\frac{1}{2}(\varphi(0) + \psi(0))}_{\text{or } -\varphi(0) \text{ or } -\psi(0)}$$

Ex 3.4.3. Let a be a positive constant. Use the method of the traveling wave to solve the following problem:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & x > 0, x - at < 0, t > 0, \\ u|_{x-at=0} = \varphi(x), \quad u_x|_{x=0} = \phi(t). \end{cases}$$

Solution. Let the general solution

$$u(x, y) = f(x+at) + g(x-at).$$

This implies

$$u_x = f'(x + at) + g'(x - at).$$

Using the boundary conditions

$$\begin{aligned} u|_{x=at} &= f(2x) + g(0) = \varphi(x) \implies f'(2x) = \frac{\varphi'(x)}{2} \implies f'(x) = \frac{\varphi'(x/2)}{2} \\ u_x|_{x=0} &= f'(at) + g'(-at) = \phi(t) \implies f'(x) + g'(-x) = \phi\left(\frac{x}{a}\right) \end{aligned}$$

This implies

$$\implies g'(-x) = \phi\left(\frac{x}{a}\right) - \frac{1}{2}\varphi'\left(\frac{x}{2}\right)$$

Gathering these together,

$$\begin{cases} g'(x) &= \phi\left(-\frac{x}{a}\right) - \frac{1}{2}\varphi'\left(-\frac{x}{2}\right) \\ f(x) &= \frac{1}{2}\varphi\left(\frac{x}{2}\right) \end{cases}$$

Then

$$\begin{aligned} \implies g(x) &= \int_0^x \phi\left(-\frac{\xi}{a}\right) d\xi - \frac{1}{2} \int_0^x \varphi'\left(-\frac{\xi}{a}\right) d\xi + C \\ &= \int_0^x \phi\left(-\frac{\xi}{a}\right) d\xi + \int_0^{-\frac{x}{2}} \varphi'(y) dy + C \quad \left(y = -\frac{x}{2}\right) \\ &= \int_0^x \phi\left(-\frac{\xi}{a}\right) d\xi + \varphi\left(-\frac{x}{2}\right) - \varphi(0) + C \\ f(x) &= \frac{1}{2} \int_0^x \varphi'\left(\frac{\xi}{2}\right) d\xi + C \\ &= \int_0^{\frac{x}{2}} \varphi'(y) dy + C = \varphi\left(\frac{x}{2}\right) - \varphi(0) + C. \quad \left(y = \frac{x}{2}\right) \end{aligned}$$

Then

$$u = f(x + at) + g(x - at) = \varphi\left(\frac{x + at}{2}\right) + \int_0^{x-at} \phi\left(-\frac{\xi}{a}\right) d\xi + \varphi\left(\frac{at - x}{2}\right) - 2\varphi(0) + C.$$

$$\begin{aligned} u|_{x-at=0} &= \varphi(x) + \varphi(0) - 2\varphi(0) + C = \varphi(x) \implies C = \varphi(0) \\ \implies u &= \varphi\left(\frac{x + at}{2}\right) + \int_0^{x-at} \phi\left(-\frac{\xi}{a}\right) d\xi + \varphi\left(\frac{at - x}{2}\right) - \varphi(0). \end{aligned}$$

Ex 3.4.4. Solve the following problem

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad (-\infty < x < \infty, t > 0) \quad (3.4.2)$$

$$u(x, 0) = \varphi(x) \quad (3.4.3)$$

Solution (Method 1: like the derivations of the traveling wave method). Let $\xi = x - at$, $\eta = x$, solve this problem in the (ξ, η) coordinate system. Let

$$u(x, t) = \bar{u}(\xi(x, t), \eta(x, t)).$$

By the chain rules,

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial \bar{u}}{\partial \eta} = -a \frac{\partial \bar{u}}{\partial \xi}, \\ \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial \bar{u}}{\partial \eta} = \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{u}}{\partial \eta}. \end{cases}$$

By (3.4.2), we obtain

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = -a \frac{\partial \bar{u}}{\partial \xi} + a \frac{\partial \bar{u}}{\partial \eta} + a \frac{\partial \bar{u}}{\partial \eta} = 0.$$

That is,

$$\frac{\partial \bar{u}}{\partial \eta} = 0 \Rightarrow \bar{u}(\xi, \eta) = f(\xi) \Rightarrow u(x, t) = f(x - at).$$

Using the initial condition $u(x, 0) = f(x) = \varphi(x)$,

$$\Rightarrow u(x, t) = \varphi(x - at).$$

Solution (Method 2: Characteristic methods). Define a curve $x(t)$ in (x, t) plane by

$$\frac{dx(t)}{dt} = a \Leftrightarrow x(t) = x(0) + at \leftarrow \boxed{\text{called the characteristic curves}}$$

Define a function $\bar{u}(t) := u(x(t), t)$, then by the chain rule and the equation (3.4.2), we obtain

$$\frac{d\bar{u}}{dt} = \frac{dx}{dt} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0.$$

Then integrating this equation, we arrive at

$$\bar{u}(t) = \text{constant } C \Leftrightarrow u(x(t), t) = C$$

By the initial data, we have $u(x(0), 0) = C = \varphi(x(0))$. Thus

$$u(x(t), t) = \varphi(x(0)) = \varphi(x(t) - at).$$

Then

$$u(x, t) = \varphi(x(0)) = \varphi(x - at).$$

Solution (Method 3: Transforming to a traveling wave). Differentiate equation (3.4.2) with respect to t and x respectively, we get

$$u_{tt} + au_{tx} = 0 \tag{3.4.4}$$

$$u_{tx} + au_{xx} = 0 \tag{3.4.5}$$

If u solves (3.4.2), then it solves (3.4.6): Multiplying equation (3.4.5) by a and subtracting equation (3.4.4) from it, we obtain a one-dimensional wave equation

$$u_{tt} - a^2 u_{xx} = 0 \tag{3.4.6}$$

The general solution of this equation is

$$u(x, t) = f_1(x + at) + f_2(x - at) \quad (3.4.7)$$

To find the particular solutions of equations (3.4.2) and (3.4.3), substitute equation (3.4.7) into equation (3.4.2) to get

$$af'_1(x + at) - af'_2(x - at) + af'_1(x + at) + af'_2(x - at) = 0$$

which simplifies to

$$2af'_1(x + at) = 0 \quad (3.4.8)$$

Thus,

$$f_1(x + at) = C$$

Substituting equation (3.4.7) into equation (3.4.3), we get

$$f_1(x) + f_2(x) = \varphi(x)$$

Solving, we find

$$f_2(x) = \varphi(x) - f_1(x)$$

Therefore,

$$f_2(x - at) = \varphi(x - at) - C. \quad (3.4.9)$$

Substituting equations (3.4.8) and (3.4.9) into the general solution (3.4.7), we obtain the solution to the initial value problem for the wave equation

$$u(x, t) = \varphi(x - at)$$

Ex 3.4.5. Vibration of a semi-infinite rod with one end free

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < \infty, t > 0) \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) & (0 \leq x < \infty) \\ u_x(0, t) = 0 & (\text{second type; tension vanishes at } x = 0) \end{cases}$$

Solution (Method 1: Extension methods). Extend the original problem to $-\infty < x < +\infty$, then the solution must satisfy

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < \infty, t > 0) \\ u(x, 0) = \Phi(x) = \begin{cases} \varphi(x) & 0 \leq x < \infty \\ f(x) & -\infty < x < 0 \end{cases} \\ u_t(x, 0) = \Psi(x) = \begin{cases} \psi(x) & 0 \leq x < \infty \\ g(x) & -\infty < x < 0 \end{cases} \end{cases}$$

where f and g are to be determined.

D'Alembert's formula implies

$$u(x, t) = \frac{1}{2} [\Phi(x + at) + \Phi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(\xi) d\xi. \quad (3.4.10)$$

Only if (3.4.10) satisfies the original problem boundary conditions, it is the solution to the original problem by limiting to $x \geq 0$.

The boundary condition verifies that the only extension is the even extension: The problem is transformed into using the boundary conditions to determine f and g . By boundary conditions, we have

$$u_x(0, t) = \frac{1}{2} [\Phi'(at) + \Phi'(-at)] + \frac{1}{2a} [\Psi(at) - \Psi(-at)] = 0$$

Since Φ and Ψ are independent,

$$\begin{cases} \Phi'(at) = -\Phi'(-at) \\ \Psi(at) = \Psi(-at) \end{cases}$$

Then

$$\begin{cases} -\Phi'(x) = \Phi'(-x) \Rightarrow \int_0^x \Phi'(\alpha) d\alpha = \int_0^x (-\Phi'(-\alpha)) d\alpha \Rightarrow \Phi(x) - \Phi(0) = \Phi(-x) - \Phi(0) \Rightarrow \Phi(x) = \Phi(-x) \\ \Psi(x) = \Psi(-x) \end{cases}$$

This means the Φ and Ψ satisfying the boundary condition must both be even functions. Thus,

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x), & x < 0 \end{cases} \quad \text{and} \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x), & x < 0 \end{cases} \quad (3.4.11)$$

Note that $x + at$ is always greater than or equal to zero, thus from equation (9) we have:

$$\Phi(x + at) = \varphi(x + at)$$

$$\int_0^{x+at} \Psi(\xi) d\xi = \int_0^{x+at} \psi(\xi) d\xi$$

Since $x - at$ could be greater than, equal to, or less than zero:

1. If $x - at \geq 0$, then from equation (3.4.11) we have:

$$\Phi(x - at) = \varphi(x - at)$$

$$\int_{x-at}^0 \Psi(\xi) d\xi = \int_{x-at}^0 \psi(\xi) d\xi$$

2. If $x - at < 0$, then from equation (3.4.11) we have:

$$\Phi(x - at) = \varphi[-(x - at)] = \varphi(at - x)$$

$$\int_{x-at}^0 \Psi(\xi) d\xi = \int_{x-at}^0 \psi(-\xi) d\xi = - \int_{at-x}^0 \psi(\eta) d\eta$$

Thus:

$$\int_{x-at}^0 \Psi(\xi) d\xi = \int_0^{at-x} \psi(\xi) d\xi$$

By D'Alembert's formula, solve (3.4.10) to get. When $x \geq at$

$$u(x, t) = \frac{1}{2} [\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi.$$

When $0 < x < at$

$$u(x, t) = \frac{1}{2} [\varphi(x + at) + \varphi(at - x)] + \frac{1}{2a} \int_0^{x+at} \psi(\xi) d\xi + \frac{1}{2a} \int_0^{at-x} \psi(\xi) d\xi.$$

- From this example, it can be seen that for the one-dimensional wave problem in a semi-infinite domain, it can be extended to an infinite domain and then solved using d'Alembert's formula.
- If the endpoint has a **second-type homogeneous boundary condition**, then the initial conditions need to be extended as even extensions;
- If the endpoint has a **first-type homogeneous boundary condition**, a similar method can be used to deduce that the initial conditions need to be extended as odd extensions:

$$\Phi(x) = \begin{cases} \varphi(x), & 0 \leq x < \infty \\ -\varphi(-x), & -\infty < x < 0 \end{cases} \quad \text{and} \quad \Psi(x) = \begin{cases} \psi(x), & 0 \leq x < \infty \\ -\psi(-x), & -\infty < x < 0 \end{cases}$$

Solution (Method 2). The general solution of equation $u_{tt} = a^2 u_{xx}$ is:

$$u(x, t) = f_1(x + at) + f_2(x - at).$$

Thus, from initial data, we have:

$$f_1(x) + f_2(x) = \varphi(x) \quad \text{and} \quad af'_1(x) - af'_2(x) = \psi(x). \quad (3.4.12)$$

That is:

$$f_1(x) - f_2(x) = \frac{1}{a} \int_0^x \psi(\xi) d\xi + C \quad (3.4.13)$$

Where $C = f_1(0) - f_2(0)$. Solving equations (3.4.12) and (3.4.13) gives:

$$f_1(x) = \frac{1}{2} \varphi(x) + \frac{1}{2a} \int_0^x \psi(\xi) d\xi + \frac{C}{2} \quad (0 \leq x < \infty); \quad (3.4.14)$$

$$f_2(x) = \frac{1}{2} \varphi(x) - \frac{1}{2a} \int_0^x \psi(\xi) d\xi - \frac{C}{2} \quad (0 \leq x < \infty). \quad (3.4.15)$$

The above two equations are derived under the premise of $0 \leq x < \infty$ since (3.4.12) holds for $0 \leq x < \infty$ only by the initial data. Since $x + at$ is always greater than or equal to zero, from equation (3.4.14) we have:

$$f_1(x + at) = \frac{1}{2} \varphi(x + at) + \frac{1}{2a} \int_0^{x+at} \psi(\xi) d\xi + \frac{C}{2}. \quad (3.4.16)$$

As for $x - at$, it is not necessarily greater than zero.

(1) If $x - at \geq 0$, then from equation (3.4.15) we have:

$$f_2(x - at) = \frac{1}{2} \varphi(x - at) - \frac{1}{2a} \int_0^{x-at} \psi(\xi) d\xi - \frac{C}{2}. \quad (3.4.17)$$

(2) (**Using the boundary to obtain the negative parts**) If $x - at < 0$, then (3.4.15) cannot be used. However, substituting the **boundary condition** into the general solution gives:

$$f'_1(at) + f'_2(-at) = 0.$$

Let $x = at$, and integrate the above it from 0 to x to get:

$$f_1(x) - f_2(-x) = C$$

That is:

$$f_2(-x) = f_1(x) - C \quad (x \geq 0).$$

Thus:

$$f_2(x - at) = f_2[-(at - x)] = f_1(at - x) - C = \frac{1}{2}\varphi(at - x) + \frac{1}{2a} \int_0^{at-x} \psi(\xi) d\xi - C \quad (at - x \geq 0). \quad (3.4.18)$$

Substituting equations (3.4.16), (3.4.17), and (3.4.18) into the general solution, we get:

$$u(x, t) = \begin{cases} \frac{1}{2}[\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi, & x - at \geq 0 \\ \frac{1}{2}[\varphi(x + at) + \varphi(at - x)] + \frac{1}{2a} \left[\int_0^{x+at} \psi(\xi) d\xi + \int_0^{at-x} \psi(\xi) d\xi \right], & x - at < 0 \end{cases}$$

where $C = 0$ can be determined by data.

Ex 3.4.6.

$$\begin{cases} u_t - a^2 u_{xx} - bu_x - cu = f(x, t), & -\infty < x < \infty, t > 0 \\ u(x, 0) = 0 \end{cases}$$

where a, b, c are constants. Solve using the method of integral transform.

Solution. For x , use the Fourier transform:

$$\hat{u}(\lambda, t) = F[u], \quad \hat{f}(\lambda, t) = F[f(x, t)]$$

Then

$$\begin{aligned} \frac{d\hat{u}}{dt} + a^2 \lambda^2 \hat{u} - ib\lambda \hat{u} - c\hat{u} &= \hat{f} \quad \Rightarrow \quad \frac{d\hat{u}}{dt} + (a^2 \lambda^2 - ib\lambda - c)\hat{u} = \hat{f} \\ \begin{cases} \frac{d\hat{u}}{dt} + (a^2 \lambda^2 - ib\lambda - c)\hat{u} = \hat{f} \\ \hat{u}(\lambda, 0) = 0 \end{cases} \end{aligned}$$

There are multiple method to solve this ODE. I use the integrating factor method. $e^{(a^2 \lambda^2 - ib\lambda - c)t}$ is the integrating factor.

$$\frac{d}{dt} \left(e^{(a^2 \lambda^2 - ib\lambda - c)t} \hat{u} \right) = e^{(a^2 \lambda^2 - ib\lambda - c)t} \hat{f}(\lambda, t).$$

Integrate both sides:

$$e^{(a^2 \lambda^2 - ib\lambda - c)t} \hat{u}(\lambda, t) = \hat{u}(\lambda, 0) + \int_0^t e^{(a^2 \lambda^2 - ib\lambda - c)\tau} \hat{f}(\lambda, \tau) d\tau.$$

$$\Rightarrow \hat{u}(\lambda, t) = \int_0^t e^{(a^2\lambda^2 - ib\lambda - c)(\tau-t)} \hat{f}(\lambda, \tau) d\tau.$$

The inverse Fourier transform is used to obtain:

$$u(x, t) = \int_0^t e^{-c(\tau-t)} F^{-1} [e^{(a^2\lambda^2 - ib\lambda)(\tau-t)}] * f(x, \tau) d\tau$$

Using the Fourier transform table:

$$\begin{aligned} F^{-1} [e^{-a^2\lambda^2 t}] &= \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \\ F^{-1} [e^{(a^2\lambda^2 - ib\lambda)(\tau-t)}] &= \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{\frac{(x-b(\tau-t))^2}{4a^2(\tau-t)}} \\ \Rightarrow u(x, t) &= \int_0^t e^{-c(\tau-t)} \frac{1}{2a\sqrt{\pi(t-\tau)}} \int_{-\infty}^{+\infty} e^{\frac{(x-y-b(\tau-t))^2}{4a^2(\tau-t)}} f(y, \tau) dy d\tau \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{\frac{(x-y-b(\tau-t))^2}{4a^2(\tau-t)} - c(\tau-t)} f(y, \tau) dy d\tau. \end{aligned}$$

Ex 3.4.7. Solve the initial value problem:

$$\begin{cases} u_{xx} - au_{tt} - bu_t - cu = 0 & (x > 0, t > 0) \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 & (\text{Initial conditions}) \\ u(0, t) = \phi(t), \quad \lim_{x \rightarrow \infty} u(x, t) = 0 & (\text{Boundary conditions}) \end{cases}$$

where a, b, c are constants, and $b^2 = 4ac$.

Solution. Directly apply Laplace transform to t , let

$$L[u(x, t)] = U(x, s), \quad L[\phi(t)] = \Phi(s)$$

Then the problem transforms into:

$$\begin{cases} U_{xx} - as^2 U - bsU - cU = 0 \\ U(0, s) = \Phi(s), \quad \lim_{x \rightarrow \infty} U(x, s) = 0 \end{cases}$$

Since $b^2 = 4ac$, we have $as^2 + bs + c = \left(\sqrt{as} + \frac{b}{2\sqrt{a}}\right)^2$

$$\Rightarrow \text{The solution is } U(x, s) = c_1 e^{\left(\sqrt{as} + \frac{b}{2\sqrt{a}}\right)x} + c_2 e^{-\left(\sqrt{as} + \frac{b}{2\sqrt{a}}\right)x}.$$

By the boundary conditions, $c_1 = 0$ and $c_2 = \Phi(s)$

$$\Rightarrow U(x, s) = \Phi(s) e^{-\left(\sqrt{as} + \frac{b}{2\sqrt{a}}\right)x}.$$

Using the inverse Laplace transform:

$$\begin{aligned} u(x, t) &= L^{-1}[U(x, s)] = L^{-1} [\Phi(s) e^{-\sqrt{as}x}] e^{-\frac{b}{2\sqrt{a}}x} \\ &= \begin{cases} \int_0^t \phi(t - \sqrt{ax}) e^{-\frac{b}{2\sqrt{a}}x} dt & t > \sqrt{ax} \\ 0 & t < \sqrt{ax} \end{cases}. \end{aligned}$$

Ex 3.4.8. When the initial value $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$ satisfy what conditions, does the solution of the one-dimensional wave equation consist only of a right-traveling wave?

Solution (Method 1: D'Alembert solution). Use D'Alembert's formula, which is a special case of the general solution.

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\varphi(x + at) + \varphi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi \leftarrow \boxed{\text{Newton-Leibniz formula}} \\ &= \underbrace{\frac{1}{2} \left[\varphi(x + at) + \frac{1}{2a} \Psi(x + at) \right]}_{\text{left-traveling wave}} + \underbrace{\frac{1}{2} \left[\varphi(x - at) - \frac{1}{2a} \Psi(x - at) \right]}_{\text{right-traveling wave}} \end{aligned}$$

The left-traveling wave: the term independent of x and t is a constant.

$$\Rightarrow a\varphi(x + at) + \Psi(x + at) = C \quad \text{Let } z = x + at \Rightarrow a\varphi(z) + \Psi(z) = C'$$

In other words, we find that $a\varphi'(z) + \psi(z) = 0$.

Solution (Method 2: Traveling wave solution). Suppose the solution of the initial value problem has only a right-traveling wave, i.e., the solution $u(x, t) = f(x - at)$. Then the initial conditions imply:

$$\begin{cases} u(x, 0) = f(x) = \varphi(x) \\ u_t(x, 0) = -af'(x) = \psi(x) \end{cases} \Rightarrow -a\varphi'(x) = \psi(x)$$

This is consistent with Method 1.

Ex 3.4.9. Suppose $f(x, y)$ is a harmonic function (i.e., $\Delta f = 0$), $g(z) \in C^2(\mathbb{R})$, solve the Cauchy problem

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz}) = 0 & (x, y, z) \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = f(x, y)g(z), \leftarrow \boxed{\text{inspiring the form of the solution}} \\ u_t|_{t=0} = 0 \end{cases}$$

Inspired by the form of the initial data, find a good variable transformations, which promotes this problem.

Solution (Ideas: Trial and Errors). Let $u(x, y, z, t) = f(x, y)v(z, t) \leftarrow \text{guess this form of the solution according to its data.}$

$$u_{xx} + u_{yy} + u_{zz} = f_{xx}v + f_{yy}v + fv_{zz} = \underbrace{(f_{xx} + f_{yy})}_= v + fv_{zz} = fv_{zz}$$

This leads to

$$u_{tt} = a^2fv_{tt}$$

Thus, the original problem transforms into

$$\begin{cases} v_{tt} - a^2v_{zz} = 0 \\ v(z, 0) = g(z), \quad v_t|_{t=0} = 0 \end{cases}$$

The solution can be obtained using the one-dimensional wave equation D'Alembert's formula.

$$v = \frac{1}{2} [g(z + at) + g(z - at)].$$

$$\Rightarrow u(x, y, z, t) = f(x, y)v(z, t) = \frac{1}{2}f(x, y)[g(z + at) + g(z - at)].$$

Ex 3.4.10. Solve the initial value problem.

$$\begin{cases} u_{tt} - 8(u_{xx} + u_{yy} + u_{zz}) = t^2 x^2, & (x, y, z) \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = y^2, \quad u_t|_{t=0} = z^2 \end{cases}$$

- Note these non-homogeneous terms are all just related to x , y and z , respectively. That is, they only involve one variable.
- Isolate the non-homogeneities.

Solution (Method of descent). Transform it into three one-dimensional problems, linearly superimpose them to solve the problem. By the principle of superposition, let

$$u(x, y, z, t) = q(x, t) + p(y, t) + w(z, t)$$

Divide into three problems:

$$\begin{cases} q_{tt} - 8q_{xx} = t^2 x^2 \\ q|_{t=0} = 0, \quad q_t|_{t=0} = 0 \end{cases}$$

$$\begin{cases} p_{tt} - 8p_{yy} = 0 \\ p|_{t=0} = y^2, \quad p_t|_{t=0} = 0 \end{cases}$$

$$\begin{cases} w_{tt} - 8w_{zz} = 0 \\ w|_{t=0} = 0, \quad w_t|_{t=0} = z^3 \end{cases}$$

Using D'Alembert's formula, we get:

$$q(x, t) = \frac{1}{12a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \tau^2 \xi^2 d\xi d\tau = \frac{1}{12} t^4 x^2 + \frac{2}{45} t^6$$

$$p(y, t) = \frac{1}{2} [(y + at)^2 + (y - at)^2] = y^2 + 8t^2$$

$$w(z, t) = \frac{1}{2a} \int_{z-at}^{z+at} \xi^2 d\xi = tz^2 + \frac{8}{3} t^3$$

$$\Rightarrow u(x, y, z) = y^2 + 8t^2 + tz^2 + \frac{8}{3} t^3 + \frac{1}{12} t^4 x^2 + \frac{2}{45} t^6$$

Ex 3.4.11. Solve the initial value problem

$$\begin{cases} u_{tt} - (u_{x_1 x_1} + u_{x_2 x_2}) = t \sin x_2, & (x_1, x_2) \in \mathbb{R}^2, t > 0. \\ u|_{t=0} = x_1^2, \quad u_t|_{t=0} = \sin x_2. \end{cases}$$

Solution. Use the same method as the previous problem. Let $u(x_1, x_2, t) = v(x_1, t) + w(x_2, t)$, by superposition, the problem transforms into two one-dimensional problems. It can be verified that $u_{x_1 x_1} = v_{x_1 x_1}$ and $u_{x_2 x_2} = w_{x_2 x_2}$.

(I)

$$\begin{cases} v_{tt} - v_{x_1 x_1} = 0 \\ v|_{t=0} = x_1^2, \quad v_t|_{t=0} = 0 \end{cases}$$

and (II)

$$\begin{cases} w_{tt} - w_{x_2 x_2} = t \sin x_2 \\ w|_{t=0} = 0, \quad w_t|_{t=0} = \sin x_2 \end{cases}$$

Using D'Alembert's formula:

$$\begin{aligned} v &= \frac{1}{2} [(x_1 - at)^2 + (x_1 + at)^2] = x_1^2 + t^2 \\ w &= \frac{1}{2} \int_{x_2 - at}^{x_2 + at} \sin \xi d\xi + \frac{1}{2} \int_0^t \int_{x_2 - a(t-\tau)}^{x_2 + a(t-\tau)} \tau \sin \xi d\xi d\tau = t \sin x_2 \\ \Rightarrow u(x_1, x_2, t) &= v + w = x_1^2 + t^2 + t \sin x_2 \end{aligned}$$

Ex 3.4.12. Solve the initial value problem.

$$\begin{cases} u_{tt} - a^2(u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}) = 0 & (x_1, x_2, x_3) \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = f(x_1) + g(x_2) \\ u_t|_{t=0} = \varphi(x_2) + \psi(x_3) \end{cases}$$

Solution. Let $u(x_1, x_2, x_3, t) = p(x_1, t) + q(x_2, t) + w(x_3, t)$. The original problem can be transformed into three one-dimensional wave equations.

(I)

$$\begin{cases} p_{tt} - a^2 p_{x_1 x_1} = 0 \\ p|_{t=0} = f(x_1) \\ p_t|_{t=0} = 0 \end{cases}$$

(II)

$$\begin{cases} q_{tt} - a^2 q_{x_2 x_2} = 0 \\ q|_{t=0} = g(x_2) \\ q_t|_{t=0} = \varphi(x_2) \end{cases}$$

(III)

$$\begin{cases} w_{tt} - a^2 w_{x_3 x_3} = 0 \\ w|_{t=0} = 0 \\ w_t|_{t=0} = \psi(x_3) \end{cases}$$

Using D'Alembert's formula, we obtain:

$$\begin{aligned}
 p &= \frac{1}{2} [f(x_1 + at) + f(x_1 - at)] \\
 q &= \frac{1}{2} [g(x_2 + at) + g(x_2 - at)] + \frac{1}{2a} \int_{x_2 - at}^{x_2 + at} \varphi(\xi) d\xi \\
 w &= \frac{1}{2a} \int_{x_3 - at}^{x_3 + at} \psi(\xi) d\xi \\
 \Rightarrow u(x_1, x_2, x_3, t) &= p + q + w \\
 &= \frac{1}{2} [f(x_1 + at) + f(x_1 - at) + g(x_2 + at) + g(x_2 - at)] \\
 &\quad + \frac{1}{2a} \left[\int_{x_2 - at}^{x_2 + at} \varphi(\xi) d\xi + \int_{x_3 - at}^{x_3 + at} \psi(\xi) d\xi \right]
 \end{aligned}$$

Ex 3.4.13. Solve the initial value problem.

$$\begin{cases} u_{tt} - a^2(u_{x_1 x_1} + u_{x_2 x_2}) = \underbrace{c^2 u}_{\text{damping}}, & (x_1, x_2) \in \mathbb{R}^2, t > 0 \\ u|_{t=0} = \varphi(x_1, x_2), \quad u_t|_{t=0} = \psi(x_1, x_2) \end{cases}$$

where c is a constant.

- **Ideas:** Introduce the third variable x_3 , hoping that the damping $c^2 u$ is the second derivative of some new function with respect to x^3 .
- **Tools:** Exponential functions help finish the above idea.
- **Strategy: Hope** find a function $w(x_1, x_2, x_3, t)$ such that:

$$c^2 u \sim a^2 w_{x_3 x_3}, \quad u_{tt} \sim w_{tt}$$

$$u_{x_1 x_1} \sim w_{x_1 x_1}, \quad u_{x_2 x_2} \sim w_{x_2 x_2}$$

Try a variable separable form $w(x_1, x_2, x_3, t) = f(x_3)u(x_1, x_2, t)$. Then we need $a^2 w_{x_3 x_3} = a^2 f''(x_3)u = c^2 u f$

Thus, we can write:

$$\begin{aligned}
 w_{tt} &= f u_{tt}, \quad w_{x_1 x_1} = f u_{x_1 x_1}, \quad w_{x_2 x_2} = f u_{x_2 x_2} \\
 \Rightarrow \underbrace{f u_{tt}}_{w_{tt}} - a^2 \underbrace{(f u_{x_1 x_1} + f u_{x_2 x_2})}_{w_{x_1 x_1} + w_{x_2 x_2}} &= \underbrace{c^2 u f}_{a^2 w_{x_3 x_3} = a^2 f''(x_3)u}
 \end{aligned}$$

If $c^2 f = a^2 f''$, then the equation can be simplified as:

$$w_{tt} - a^2(w_{x_1 x_1} + w_{x_2 x_2} + w_{x_3 x_3}) = 0$$

In order to find f , we solve this equation $c^2 f = a^2 f''$. Then $f = e^{\frac{c}{a} x_3}$ (any solution will work)

- In fact, If one have good intuition, and use the exponential function is invariant for derivations, it is easy to guess the for of w . Organize the above ideas to get the solution.

Solution (Method of Dimensional Lifting). Let $w(x_1, x_2, x_3, t) = e^{\frac{c}{a}x_3}u(x_1, x_2, t)$, then w satisfies a three-dimensional wave equation. The initial value problem for w (where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$) is:

$$\begin{cases} w_{tt} - a^2\Delta w = 0 \\ w|_{t=0} = e^{\frac{c}{a}x_3}\varphi(x_1, x_2), \quad w_t|_{t=0} = e^{\frac{c}{a}x_3}\psi(x_1, x_2) \end{cases}$$

By the three-dimensional Poisson formula (Kirchhoff formula):

$$\begin{aligned} w(x_1, x_2, x_3, t) &= \frac{1}{4\pi} \frac{\partial}{\partial t} \int_0^{2\pi} \int_0^\pi t e^{\frac{c}{a}(x_3+at\cos\theta)} \varphi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi) \sin\theta d\theta d\phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \int_0^\pi e^{\frac{c}{a}(x_3+at\cos\theta)} \psi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi) \sin\theta d\theta d\phi. \\ \Rightarrow u(x_1, x_2, t) &= e^{-\frac{c}{a}x_3}w(x_1, x_2, x_3, t) \\ &= \frac{1}{4\pi} \frac{\partial}{\partial t} \int_0^{2\pi} \int_0^\pi t e^{ct\cos\theta} \varphi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi) \sin\theta d\theta d\phi \\ &\quad + \frac{t}{4\pi} \int_0^{2\pi} \int_0^\pi e^{ct\cos\theta} \psi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi) \sin\theta d\theta d\phi. \end{aligned}$$

Ex 3.4.14. Solve the following initial value problem using the method of characteristics:

$$\begin{cases} u_{tt} + 2u_t = u_{xx} + 2u_x, & -\infty < x < +\infty, t > 0, \\ u(x, 0) = e^{-x} \cos x, & u_t(x, 0) = e^{-x}(\sin x - \cos x). \end{cases}$$

[Hint: Choose appropriate constants α and β as unknown function substitutions $v(x, t) = e^{\alpha t + \beta x}u(x, t)$ to transform the equation into a homogeneous string vibration equation.]

- When **derivatives of adjacent orders appear**, an integrating factor can be used to consolidate multiple terms into a single expression.
- The primary tools employed are the **integrating factor** and the **binomial theorem** for higher-order derivatives.
- (Method 1) By directly comparing with the binomial theorem, it is evident that we should multiply by $e^t e^x$.

The coefficients of t -derivatives terms 1 2 1 $\leftarrow (e^t u)_{tt} \cdot e^x$

The coefficients of x -derivatives terms 1 2 1 $\leftarrow (e^x u)_{xx} \cdot e^t$

Let $(\underbrace{e^{t+x} u}_{v})_{tt} = (\underbrace{e^{x+t} u}_{v})_{xx}$

- (Method 2) Assume $u = e^{\alpha t + \beta x}v$

$$u_{tt} = \alpha^2 e^{\alpha t + \beta x} v + 2\alpha e^{\alpha t + \beta x} u_t + e^{\alpha t + \beta x} u_{tt}$$

$$\begin{aligned} u_{xx} &= \beta^2 e^{\alpha t + \beta x} v + 2\beta e^{\alpha t + \beta x} u_x + e^{\alpha t + \beta x} u_{xx} \\ 2u_t &= 2\alpha e^{\alpha t + \beta x} v + 2e^{\alpha t + \beta x} u_t \\ 2u_x &= 2\beta e^{\alpha t + \beta x} v + 2e^{\alpha t + \beta x} u_x \end{aligned}$$

Substitute into the original equation to get $u_t = u_x$, α and β should be determined.

Solution. Let $v(x, t) = e^{t+x}u(x, t)$, then v satisfies

$$\begin{cases} v_{tt} = v_{xx}, & -\infty < x < +\infty, t > 0, \\ v(x, 0) = \cos x, & u_t(x, 0) = \sin x. \end{cases}$$

Using D'Alembert's formula to solve, we get

$$v(x, t) = \frac{1}{2}[\cos(x-t) + \cos(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} \sin \alpha d\alpha = \cos(x-t).$$

Ex 3.4.15. Solve the following initial value problem using the method of traveling waves:

$$\begin{cases} u_{tt} + 2u_t = u_{xx} + 4u_x + 3u, & -\infty < x < +\infty, t > 0, \\ u(x, 0) = e^{-2x} \cos x, & u_t(x, 0) = e^{-2x}(\sin x - \cos x). \end{cases}$$

[Hint: Choose appropriate constants α and β as unknown function substitutions $v(x, t) = e^{\alpha t + \beta x}u(x, t)$ to transform the equation into a homogeneous string vibration equation.]

The coefficients of t -derivatives terms The coefficients of x -derivatives terms	$1 \quad 2 \quad 1$ $1 \quad \underbrace{2\alpha}_4 \quad \underbrace{\alpha^2}_4$	$\leftarrow (e^t u)_{tt} \cdot e^{\alpha x}$ $\leftarrow (e^{\alpha x} u)_{xx} \cdot e^t$
	Then $\alpha = 2$, Let $(\underbrace{e^{t+2x} u}_{v})_{tt} = (\underbrace{e^{2x+t} u}_{v})_{xx}$	

Solution. Let $v(x, t) = e^{t+2x}u(x, t)$, then v satisfies

$$\begin{cases} v_{tt} = v_{xx}, & -\infty < x < +\infty, t > 0, \\ v(x, 0) = \cos x, & u_t(x, 0) = \sin x. \end{cases}$$

Using D'Alembert's formula to solve, we get

$$v(x, t) = \frac{1}{2}[\cos(x-t) + \cos(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} \sin \alpha d\alpha = \cos(x-t).$$

Thus,

$$u(x, t) = e^{-t-2x} \cos(x-t).$$

Ex 3.4.16. Use the method of traveling waves to solve the following unbounded vibration problem:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, \\ u(0, t) = t^2 + 3, & u_t|_{x=at} = 2x. \end{cases}$$

where a is a positive constant.

Solution. The general solution of the wave equation is

$$u(x, t) = f(x - at) + g(x + at)$$

Substitute the initial values

$$-af'(0) + ag'(2x) = 2x, \quad f(-at) + g(at) = t^2 + 3,$$

we get

$$ag'(x) = x + af'(0), \quad g(x) = \frac{x^2}{2a} + f'(0)x + c$$

Thus,

$$f(x) = \left(\frac{1}{a^2} - \frac{1}{2a}\right)x^2 + f'(0)x + 3 - c.$$

Therefore,

$$\begin{aligned} u(x, t) &= f(x - at) + g(x + at) \\ &= \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x - at)^2 + f'(0)(x - at) + 3 - c + \frac{(x + at)^2}{2a} + f'(0)(x + at) + c \\ &= \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x - at)^2 + \frac{(x + at)^2}{2a} + 2f'(0)x + 3. \end{aligned}$$

From $u_t(0, 0) = 0$, we get $f'(0) = 0$. The solution to the equation is

$$u(x, t) = \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x - at)^2 + \frac{(x + at)^2}{2a} + 3.$$

Ex 3.4.17. Solve the following initial value problem using the method of traveling waves:

$$\begin{cases} u_{tt} = u_{xx} - 2, & -\infty < x < +\infty, t > 0, \\ u(x, 0) = x^2, & u_t(x, 0) = \sin x, \quad -\infty < x < +\infty. \end{cases}$$

Solution. From D'Alembert's formula, we know

$$\begin{aligned} u(x, t) &= \frac{1}{2} \left((x + t)^2 + (x - t)^2 \right) + \frac{1}{2} \int_{x-t}^{x+t} \sin \xi d\xi + \frac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} (-2) d\xi d\tau \\ &= x^2 + t^2 - \frac{1}{2} (\cos(x + t) - \cos(x - t)) - t^2 \\ &= x^2 + \sin x \sin t. \end{aligned}$$

Ex 3.4.18. Solve the following initial value problem using the method of integral transforms:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{4} \frac{\partial^2 u}{\partial x^2}, & -\infty < x < +\infty, t > 0, \\ u(x, 0) = e^{-x^2}, & -\infty < x < +\infty. \end{cases}$$

(Hint: $F^{-1}(e^{-\lambda^2 t}) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$)

Solution. Perform the Fourier transform with respect to x , let $U(\lambda, t) = F(u(x, t))$, then we have

$$\begin{aligned} U_t(\lambda, t) &= -\frac{1}{4} \lambda^2 U(\lambda, t), \\ U(\lambda, 0) &= F(e^{-x^2}). \end{aligned}$$

Thus,

$$U(\lambda, t) = e^{-\frac{1}{4}\lambda^2 t} F(e^{-x^2}).$$

Perform the inverse Fourier transform to obtain

$$\begin{aligned} u(x, t) &= F^{-1}(e^{-\frac{1}{4}\lambda^2 t}) * e^{-x^2} \\ &= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}} * e^{-x^2} \\ &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2} e^{-\frac{\xi^2}{t}} d\xi \\ &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{1+t}} \exp\left(\sqrt{\frac{t}{1+t}}\xi - \sqrt{\frac{1+t}{t}}x\right)^2 d\xi \\ &= \sqrt{1+t} e^{-\frac{x^2}{1+t}}. \end{aligned}$$

Ex 3.4.19. Solve the following initial value problem using the method of integral transforms:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, t > 0, \\ u(x, 0) = 0, u_t(x, 0) = 0, & 0 \leq x < +\infty, \\ u(0, t) = e^{-t} \sin(\omega t), & t \geq 0, \\ \lim_{x \rightarrow +\infty} |u(x, t)| < +\infty, & t \geq 0. \end{cases}$$

(Hint: If $F(s) = L[f(t)]$, then $L^{-1}[F(s)e^{-sa}] = \begin{cases} f(t-a), & t > a, \\ 0, & t < a. \end{cases}$)

Solution. Perform the Laplace transform with respect to t , let $U(x, s) = L(u(x, t))$, then we have

$$s^2 U(x, s) = a^2 U_{xx}(x, s).$$

The general solution of this ODE is

$$U(x, s) = c_1(s) e^{-\frac{s}{a}x} + c_2(s) e^{\frac{s}{a}x}.$$

From the condition $u(0, t) = e^{-t} \sin(\omega t)$, we get

$$U(0, s) = L(e^{-t} \sin(\omega t)).$$

And since $\lim_{x \rightarrow +\infty} |u(x, t)| < +\infty$, for any $s > 0$, there exists $M < +\infty$ such that

$$\sup_{x \in (0, \infty)} |U(x, s)| < M.$$

Thus,

$$U(x, s) = L(e^{-t} \sin(at))e^{-\frac{s}{a}x}.$$

Perform the inverse Laplace transform to obtain

$$u(x, t) = \begin{cases} e^{-(t-\frac{x}{a})} \sin(\omega(t - \frac{x}{a})), & t > \frac{x}{a}, \\ 0, & t < \frac{x}{a}. \end{cases}$$

Ex 3.4.20. Solve the following initial value problem using the method of integral transforms:

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, t > 0, \\ u(0, t) = e^{-t} - 1, \lim_{x \rightarrow +\infty} |u(x, t)| < +\infty, & t \geq 0, \\ u(x, 0) = 0, u_t(x, 0) = 0, & 0 \leq x < +\infty. \end{cases}$$

Solution. Let $U(x, s) = L(u(x, t))$, perform the Laplace transform with respect to t to obtain

$$s^2 U(x, s) = a^2 U_{xx}(x, s).$$

Solve to get

$$U(x, s) = A(s)e^{\frac{s}{a}x} + B(s)e^{-\frac{s}{a}x}.$$

From $\lim_{x \rightarrow +\infty} |u(x, t)| < +\infty$, we know $A(s) \equiv 0$. From $u(0, t) = e^{-t} - 1$, we know $B(s) = L(e^{-t} - 1)$. Thus,

$$U(x, s) = L(e^{-t} - 1)e^{-\frac{s}{a}x}.$$

Perform the inverse Laplace transform and use the delay theorem to obtain

$$u(x, t) = \begin{cases} e^{-(t-\frac{x}{a})} - 1, & x \leq at, \\ 0, & x > at. \end{cases}$$

4

Green's Function Method

Why is PDE Courses Difficult?

- The lecture “Equations of Mathematical Physics (PDEs)” is difficult because:
 - Different sections introduce different problem-solving methods.
 - Each new method changes the perspective and approach to solving equations.
- Example of evolving techniques:
 - **Initial Simplicity in Chapter 2:** In Chapter 2, when learning the method of separation of variables, it seems manageable, and students feel they’ve grasped the main idea.
 - **Changing Approaches:** However, in Chapter 3, a new method is introduced—wave motion and averaging methods—requiring students to adjust their thinking.
 - **Continuous Shifts in Methodology:** After mastering one approach, such as the Green’s function method, students find that previous methods no longer work, necessitating further adjustments.
 - **Adapting Learning Strategies:** The course continuously shifts its teaching methods, and students must constantly adapt their learning approach to keep up.

Structure of Chapter 4

- **Main Problem:** The chapter focuses on solving a boundary value problem represented by the equation

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \end{cases}$$

which is the primary issue to be addressed in the first three sections.

- **First harmonic equations in §4.1:** In section 4.1, we focus only on the equation

$$\Delta u(x, y) = 0$$

without considering boundary conditions, aiming to find a **general solution** in the **form of an integral**. This step is crucial as the integral form helps explain many aspects of the problem.

- **Adding Boundary Conditions §4.2:** Section 4.2 introduces boundary conditions and focus on

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \end{cases} \quad (4.0.1)$$

However, we do not solve the problem directly. Instead, we **find a transformation** that transform it into a simpler boundary value problem for a new variable v .

$$\begin{cases} \Delta v = 0, & (x, y, z) \in \Omega, \\ v|_{\Gamma} = \frac{1}{4\pi r_{MM_0}}|_{\Gamma} \end{cases} \quad (4.0.2)$$

- **Why the Problem (4.0.2) Seems Simpler:** The problem (4.0.2) appears simpler because the function f in (4.0.1) can take many forms, but we only need to solve one common problem in (4.0.2) if (4.0.1) and (4.0.2) share the same domain and boundary.
- **Key Approach:** The key to solving the problem is to find the solution for v (§4.3 on solving v), which is a transformed version of the original problem.
- **Role of Green's Function:** Once v is solved, we can convert it back to solve for u , effectively solving the original problem.
- **Solution Strategy (Section 4.2):** Section 4.2 **only** demonstrates and gives the **transformation** between u and v , but does **not solve** v .
- **Green's Function:** In section 4.2, v is identified as the Green's function, which helps solve the problem. The section also introduces a specific type of Green's function G , with another variation discussed later.
- **Solving for v §4.3:** In section 4.3, the goal is to **solve for** v , but this can **only** be done in **certain regions** (e.g., **half-space** or **spherical regions**). The solution uses the **method of images**, a physical approach, although the separation of variables method can also be applied.
- **Final Solution §4.3:** Once the Green's function v is solved, we can convert it back to find u and complete the solution to the original problem.
- **Trial methods in §4.4.**

Main Ideas

- **Interwoven Concepts:** Each section introduces additional related properties. For example, while studying Green's function, its properties must also be analyzed.

- **Harmonic Function:**

- In Section 4.1, the function u is identified as a *harmonic function* and study the properties of harmonic functions.
 - A harmonic function satisfies $\Delta u = 0$ and must be C^2 to ensure the existence of second-order derivatives.
- **Understanding the Structure:** Without a clear grasp of the overall structure, students may struggle to distinguish between harmonic functions and Green's functions.

Why This Chapter Is Challenging

- The chapter focuses on solving the simplest partial differential equation: $\Delta u = 0$.
- **Paradox of Simplicity:** Despite its simplicity, it is difficult because:
 - The equation is well understood, leading to numerous useful properties.
- **Exam Expectations:**
 - Proofs are not tested directly.
 - However, students must understand the underlying ideas and apply them flexibly in problems ← a defining feature of the study of PDEs.

In this chapter, we will introduce the key points and steps of using Green's function method to solve the boundary value problems of Laplace's equation. The solution of the first type boundary value problem of Laplace's equation will be expressed in the form of an integral through Green's function.

4.1 Green's Formula and Its Applications

Overview of the Approach

- The **goal** of Section 4.1 is to study the Laplace equation $\Delta u = 0$ and derive its general solution in integral form.
- The process is divided into four steps:
 1. **Step 1: Understanding the Raw Material**
 - The first step provides fundamental Raw Material: **spherically and circularly symmetric** solutions.
 - This approach is similar to studying wave equations in three dimensions by first analyzing a spherical solution.

2. Step 2: Introducing a Mathematical Tool

- The second step presents an important mathematical tool: **the first and second Green's identities**—In fact, it is Gauss Formula.
- Although it may seem unrelated at first, it is essential for the next step.

3. Step 3: Combining Raw Material and Mathematical Tool

- By embedding the raw material into the tool, the integral form of the general solution is obtained naturally.

4. Step 4: Studying Properties of the Solution

- Since u is a **harmonic function**, its **properties** are analyzed using the **integral representation**.
- This step extends beyond the differential equation itself and focuses on the nature of harmonic functions.

Key Concepts

- **Green's Identity vs. Green's Function:**

- The Green's identities used here should not be confused with Green's functions.
- Green's identities are derived from Gauss's theorem, which relates surface integrals and volume integrals.

- **Main Thought Process:**

1. Study fundamental solutions (raw material).
2. Introduce Green's identities (mathematical tool).
3. Combine both to derive the general solution.
4. Analyze harmonic function properties based on this solution.

4.1.1 Spherically Symmetric Solutions (Raw Material)

Here, we first introduce the **circularly symmetric** solutions of the **two dimensional** Laplace's equation

$$u_{xx} + u_{yy} = 0$$

1. **Recognizing the Symmetry:** Since we are looking for a circularly symmetric solution, it is natural to consider a polar coordinate transformation.
2. **Coordinate Transformation:** Convert Cartesian coordinates to polar coordinates to exploit the symmetry.

3. Laplace Operator in Polar Coordinates: The Laplacian in two-dimensional polar coordinates is well known:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

For a circularly symmetric function $u(r)$, the **dependence on θ vanishes**, simplifying to:

$$\Delta u = \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = 0.$$

4. Solving the ODE: The above equation can be solved by standard techniques for ordinary differential equations.

The two dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ in polar coordinates is expressed as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (4.1.1)$$

We seek the circularly symmetric solution $u = U(r)$ (i.e., the solution where $u = U(r)$ does not depend on θ) of equation (4.1.1). At this time, the above equation can be simplified to

$$\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} = 0$$

- Important note:** This equation is **NOT a Euler** equation for which we have a general solution formula, although it may look like it, but it does not meet the strict conditions for applying the standard Euler formula.

- Recall:** The second order Euler equation

$$r^2 R_{rr} + r R_r - n^2 R = 0, \quad (n = 1, 2, \dots)$$

- For the Euler equation, make the transformation $r = e^t \Leftrightarrow t = \ln r$ ($n \neq 0$), then

$$\begin{aligned} R_{tt} - R_t + R_t - n^2 R &= 0 \Leftrightarrow R_{tt} - n^2 R = 0 \\ \Leftrightarrow R_n &= C_n e^{nt} + D_n e^{-nt} \Leftrightarrow R_n(r) = C_n r^n + D_n r^{-n}, \quad (n = 1, 2, \dots) \end{aligned}$$

- If $n = 0$, we need use the different general solution for one real root.

- $\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} = 0$ is an ODE, which can be solved. (There are many solution methods, such as the **method of separation of variables**). For example, let $F = \frac{dU}{dr}$, then $\frac{d \ln F}{dr} = -\frac{1}{r}$, integrating gives $\ln F = -\ln r + C$, so $F = \frac{C}{r} \Rightarrow \frac{dU}{dr} = \frac{C}{r} \Rightarrow U = C_1 \ln r + C_2$. Or transform it into $r^2 \frac{d^2 U}{dr^2} + r \frac{dU}{dr} = 0$ (This is not an Euler equation, but it can be solved by $r = e^t$).

Its solution is

$$U = c_1 \ln r + c_2 \quad (r \neq 0)$$

where c_1 and c_2 are **arbitrary constants** (since there is no boundary condition). If we let $c_1 = -1$ and $c_2 = 0$ (choose **arbitrarily**), we can obtain

$$U_0 = \ln \frac{1}{r} \quad (r \neq 0)$$

It is usually called the **fundamental solution** of the **two dimensional** Laplace's equation.

Now we introduce the **spherically symmetric** solutions of the three - dimensional Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Make the spherical coordinate transformation

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases}$$

By the chain rule of composite functions

$$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x + u_\varphi \cdot \varphi_x$$

$$u_{xx} = u_{rr}r_x^2 + u_{\theta\theta}\theta_x^2 + u_{\varphi\varphi}\varphi_x^2 + u_r \cdot r_{xx} + u_\theta \cdot \theta_{xx} + u_\varphi \cdot \varphi_{xx} + 2u_{r\theta}\theta_xr_x + 2u_{r\varphi}r_x\varphi_x + 2u_{\varphi\theta}\varphi_x\theta_x$$

We can transform the three dimensional Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ into the following form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} = 0. \quad (4.1.2)$$

We seek the spherically symmetric solution $u = U(r)$ (i.e., the solution where $u = U(r)$ does **not depend on θ and φ**) of equation (4.1.2). At this time, the above equation (4.1.2) can be simplified to

$$\frac{d}{dr} \left(r^2 \frac{dU}{dr} \right) = 0 \quad \Rightarrow \quad \frac{dU}{dr} = \frac{c_1}{r^2}. \leftarrow \boxed{\text{Gravitational force}}$$

Its solution is

$$U = \frac{c_1}{r} + c_2 \quad (r \neq 0)$$

where c_1 and c_2 are arbitrary constants. If we let $c_1 = 1$ and $c_2 = 0$, we can obtain

$$U_0 = \frac{1}{r} \quad (r \neq 0)$$

It is usually called the **fundamental solution** of the three dimensional Laplace's equation.

Physical Meaning of Laplace's Equation

- The function u in Laplace's equation can represent a **gravitational potential** or an **electrostatic potential**.
- The **gradient** of u with respect to r corresponds to **field strength** (e.g., gravitational field or electric field).
- The force is given by the **negative gradient**:

$$\mathbf{F} = -\nabla u$$

- The solution follows an **inverse-square law**, consistent with **Newton's law of gravitation** and **Coulomb's law**.

Determining the Potential Function

- The potential function u is obtained by integrating the field strength.
- A general solution involves integration constants c_1 and c_2 .
- The choice of these constants depends on **selecting a reference point where potential is zero**.
- Typically, the **potential is set to zero at infinity**:

$$u(\infty) = 0.$$

- This selection of zero potential is a form of **gauge choice**.

Is the Derived Solution a True Solution?

- The obtained solution satisfies Laplace's equation except at $r = 0$.
- At the origin $r = 0$, the solution diverges to infinity.
- Since Laplace's equation is originally defined in Cartesian coordinates, it should hold everywhere in the domain, including the origin.
- However, the transformation to spherical coordinates can lead to a loss of information at the origin.

Addressing the Singularity at the Origin

- The classical solution should not have singularities.

- To ensure well-posedness, we define the solution in the punctured domain:

$$\Omega' = \Omega \setminus \{M_0\},$$

where M_0 is the origin.

- In this restricted domain, the solution satisfies Laplace's equation (or the Laplace's equation in polar coordinate).

Extending the Solution to Include the Origin

- To formally include M_0 , the equation must be modified.
- The right-hand side of the equation should account for the singularity at M_0 .
- A gravitational potential of this form suggests the presence of a **point mass**.
- Recall Poisson's equation in three dimensions:

$$\Delta u = \rho,$$

where ρ represents the mass (or charge) density.

- A **point mass** located at M_0 corresponds to a singular mass distribution:

$$\rho(\mathbf{x}) = m\delta(\mathbf{x} - M_0),$$

where $\delta(\mathbf{x} - M_0)$ is the Dirac delta function.

- This can be done by introducing the Dirac delta function $\delta(\mathbf{x} - M_0)$:

$$\Delta u = \kappa\delta(\mathbf{x} - M_0).$$

- This equation defines the **fundamental solution** of Laplace's equation.

Claim 4.1.1. Suppose $u = \frac{1}{r}$ solve the Poission equation

$$\Delta u = \kappa\delta(\mathbf{r} - \mathbf{r}_M)$$

then

$$\Delta u = -4\pi\delta$$

which implies $\kappa = -4\pi$.

Proof. Since κ is a constant, we can calculate it by taking integration on any region, for simplicity, we take a ball B_a ,

$$\int_{B_a} \Delta u dV = \kappa \int_{B_a} \delta(\mathbf{r} - \mathbf{r}_M) dV = \kappa \leftarrow \boxed{\delta \text{ require integration to become meaningful.}}$$

Using the divergence theorem:

$$\int_{\partial B_a} \mathbf{n} \cdot \nabla u \, d\sigma = \int_{\partial B_a} \frac{\partial u}{\partial r} \, d\sigma = \kappa.$$

For $u = \frac{C}{r}$, we compute:

$$\frac{\partial u}{\partial r} = -\frac{C}{r^2}.$$

Thus, integrating over the sphere:

$$\int_{\partial B_a} \left(-\frac{C}{r^2} \right) d\sigma = -\frac{C}{a^2} \cdot 4\pi a^2 = -4\pi C.$$

Thus, we conclude: $\kappa = -4\pi C$. □

Summary

Fundamental Solution of the Two Dimensional Laplace Equation:

$$U_0 = \ln \frac{1}{r} \quad (r \neq 0)$$

Fundamental Solution of the Three Dimensional Laplace Equation:

$$U_0 = \frac{1}{r} \quad (r \neq 0)$$

Especially in the study of the three-dimensional Laplace equation, the **fundamental solution** plays an extremely important role.

It is easy to verify that when $r \neq 0$, the functions $\frac{1}{r}$ and $\ln \frac{1}{r}$ respectively satisfy the three-dimensional and two-dimensional Laplace equations.

In the rest of this Chapter, we focus on the problems in 3D case.

- **Parallel Extension to Two Dimensions (2D):**

- Every 3D derivation can be directly extended to 2D.
- In 2D, replace 3D fundamental solutions with 2D fundamental solutions.
- Corresponding modifications should be made accordingly.

4.1.2 Green's Formulas (Mathematical Tool)

Green's formulas are a direct **corollary** of the **Ostrogradsky-Gauss** formula.

Let Ω be a bounded region with a sufficiently smooth surface $\partial\Omega$ (also denoted as $\Gamma := \partial\Omega$) as its boundary, and $u = u(x, y, z)$ (works for any n -dimensional as well) be an arbitrary function that is continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$ and has continuous partial derivatives in Ω . Then the following **Ostrogradsky-Gauss** formula holds

$$\int_{\Omega} \nabla \cdot \mathbf{F} dV = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{F} dS \quad (4.1.3)$$

where $F = (P, Q, R)$ is a vector, dV is the volume element, \mathbf{n} is the **outer normal direction** of $\partial\Omega$, and dS is the surface element on $\partial\Omega$.

Let the functions u and v and all their first order partial derivatives be continuous on $\bar{\Omega}$, and have continuous second order partial derivatives in Ω . In **Gauss formula** (4.1.3), let $F = u\nabla v$, then we get:

$$\begin{aligned} & \underbrace{\int_{\Omega} \nabla \cdot (u\nabla v) dV}_{\Downarrow \text{Leibniz}} \stackrel{\text{Gauss}}{=} \underbrace{\int_{\partial\Omega} u\nabla v \cdot \mathbf{n} dS}_{\Downarrow \text{Direction derivative}} \\ & \int_{\Omega} u\Delta v dV + \int_{\Omega} \nabla u \cdot \nabla v dV = \int_{\partial\Omega} u \frac{\partial v}{\partial n} dS \end{aligned}$$

Then we get **Green's first formula**:

$$\int_{\Omega} u\Delta v dV = \int_{\partial\Omega} u \frac{\partial v}{\partial n} dS - \int_{\Omega} \nabla u \cdot \nabla v dV \quad (4.1.4)$$

In formula (4.1.4), exchange the positions of the functions u and v , we get

$$\int_{\Omega} v\Delta u dV = \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS - \int_{\Omega} \nabla v \cdot \nabla u dV \quad (4.1.5)$$

Subtract (4.1.5) from (4.1.4), then we get **Green's second formula**:

$$\int_{\Omega} (u\Delta v - v\Delta u) dV = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS \quad (4.1.6)$$

Formula (4.1.12) holds for any functions u and v that have **first order continuous partial derivatives** on $\bar{\Omega}$ and **second order continuous partial derivatives** on Ω .

- **Generalization:** u and v can be Dirac δ function.
- In application, if one need $\int_{\Omega} u\Delta v dV$, it can be calculated directly by

$$\begin{aligned} \int_{\Omega} u\Delta v dV & \stackrel{\text{Leibniz}}{=} - \int_{\Omega} \nabla \cdot (u\nabla v) dV - \int_{\Omega} \nabla u \cdot \nabla v dV \\ & \stackrel{\text{Gauss}}{=} \underbrace{\int_{\partial\Omega} u\nabla v \cdot \mathbf{n} dS}_{\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS} - \int_{\Omega} \nabla u \cdot \nabla v dV. \end{aligned}$$

There is no need to memorize the formula.

4.1.3 Integral Expressions of Harmonic Functions

We use Green's formula to derive the integral expressions of harmonic functions.

First, note that the function

$$\frac{1}{r_{MM_0}} = \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}},$$

where $M_0(x_0, y_0, z_0)$ is a fixed point in the region Ω . Except at M_0 , this function satisfies Laplace's equation everywhere. If the function u has first order continuous partial derivatives on $\Omega + \Gamma$ and is harmonic in Ω , then

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS. \quad (4.1.7)$$

To derive the integral representation of harmonic functions, we utilize the **fundamental solution** obtained in the first part and apply **Green's second identity**. This section builds on the previous discussions, demonstrating how the solution structure naturally leads to the integral formula.

Choice of Functions

- Since u is **harmonic**, it satisfies $\Delta u = 0$ and belongs to $C^2(\Omega)$.
- We set v as the **fundamental solution**, given by:

$$v = \frac{1}{r_{MM_0}},$$

where r_{MM_0} represents the Euclidean distance between a point M and a fixed point M_0 .

Challenges and Resolution

While substituting into Green's second identity, we face an issue:

- v is singular at M_0 , meaning $v \notin C^2(\Omega)$ and not even $C^1(\Omega)$.
- Direct substitution of v into Green's identity is not valid due to the singularity at M_0 .

Resolution:

To address this, we adopt a standard idea from complex analysis:

1. Remove the singularity by excluding a **small ball** around M_0 .
2. Consider a **modified domain** $\Omega \setminus B_\epsilon(M_0)$.
3. Apply Green's identity in the **punctured domain** and then take the **limit** as $\epsilon \rightarrow 0$.

This process allows for a well-defined integral representation of harmonic functions.

Proof. In formula (4.1.12), let u be a harmonic function and take $v = \frac{1}{r}$. Since the function v becomes infinite at the point M_0 , Green's second formula (4.1.12) cannot be directly applied to the region Ω . However, if we remove a sphere $K_\varepsilon^{M_0}$ centered at M_0 with a sufficiently small positive radius ε from the region Ω , then the function v is continuously differentiable in the remaining region $\Omega - K_\varepsilon^{M_0}$ (as shown in Fig. 4.1).

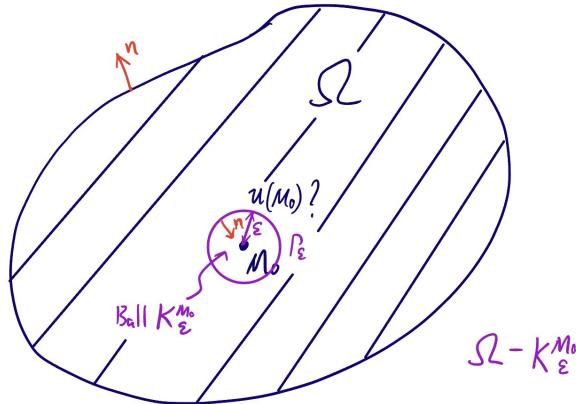


Figure 4.1: Domain of $\Omega - K_\varepsilon^{M_0}$

Apply formula (4.1.12) to the above mentioned **harmonic function** u and $v = \frac{1}{r}$ in the region $\Omega - K_\varepsilon^{M_0}$, we get

$$\iiint_{\Omega - K_\varepsilon^{M_0}} \left(u \Delta \frac{1}{r} - \frac{1}{r} \Delta u \right) d\Omega = \iint_{\Gamma + \Gamma_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS, \quad (4.1.8)$$

where Γ_ε is the spherical surface of the sphere $K_\varepsilon^{M_0}$.

Because in the region $\Omega - K_\varepsilon^{M_0}$, $\Delta u = 0$ and $\Delta \frac{1}{r} = 0$ (due to the **fundamental solution** $\Delta(\frac{1}{r}) = \delta(r - M_0)$), then formula (4.1.8) is transformed into

$$\iint_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS + \iint_{\Gamma_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS = 0. \quad (4.1.9)$$

On the spherical surface Γ_ε ,

$$\frac{\partial}{\partial n} \left(\frac{1}{r} \right) = -\frac{\partial}{\partial r} \left(\frac{1}{r} \right) = \frac{1}{r^2} = \frac{1}{\varepsilon^2},$$

- The outward normal direction of $\Omega - K_\varepsilon^{M_0}$ points towards the center M_0 of the ball, which is opposite to the direction of the radial direction r . Thus $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$.

from which we can obtain

$$\iint_{\Gamma_\varepsilon} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = \frac{1}{\varepsilon^2} \iint_{\Gamma_\varepsilon} u dS = \frac{1}{\varepsilon^2} \cdot 4\pi \varepsilon^2 \bar{u} = 4\pi \bar{u},$$

where \bar{u} is the average value of the function u on the spherical surface Γ_ε .

On the other hand, since on the spherical surface Γ_ε

- Note \mathbf{n} is the outward normal for $\Omega \setminus K_\varepsilon^{M_0}$, thus \mathbf{n} is pointed to M_0 which is the inward normal for the ball $K_\varepsilon^{M_0}$;
- But for the Gauss formula, the outward normal should be $-\mathbf{n}$.

$$\iint_{\Gamma_\varepsilon} \frac{1}{r} \frac{\partial u}{\partial n} dS = \frac{1}{\varepsilon} \iint_{\Gamma_\varepsilon} \frac{\partial u}{\partial n} dS = \frac{1}{\varepsilon} \iint_{\Gamma_\varepsilon} \mathbf{n} \cdot \nabla u dS \stackrel{\text{Gauss}}{=} -\frac{1}{\varepsilon} \iint_{K_\varepsilon^{M_0}} \Delta u d\Omega = 0.$$

Then, substituting

$$\iint_{\Gamma_\varepsilon} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = 4\pi \bar{u}, \quad \iint_{\Gamma_\varepsilon} \frac{1}{r} \frac{\partial u}{\partial n} dS = 0$$

into formula (4.1.9), we can get

$$\iint_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS + 4\pi \bar{u} = 0.$$

Now, let $\varepsilon \rightarrow 0$. Since $\lim_{\varepsilon \rightarrow 0} \bar{u} = u(M_0)$, from the above formula, we can obtain the integral expression (4.1.13) of the harmonic function u . \square

This integral expression (4.1.13) shows that: for a harmonic function u with continuous first order partial derivatives on $\Omega + \Gamma$, its value at any point M_0 in the region Ω can be expressed by the integral expression (4.1.13) using the **values of this harmonic function** and its **normal derivative on the boundary** Γ of the region.

Key Insights from the Derivation

- The derivation of the integral representation involves a crucial technique: the “**removal of a small ball**” **around a singularity**.
- Understanding the idea of this method is essential since it appears frequently in problem-solving.

Interpretation of the Formula

- The formula is **slightly analogous to the Poisson formula** in the “3D wave equation”.
- To determine u at a point M_0 , one must use:
 1. The values of u on the **boundary** Γ .

2. The **normal derivative** $\frac{\partial u}{\partial n}$ on the **boundary**.

- The integral representation (4.1.13)

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

shows that the function's **value** at any **interior point** "depends **only** on **boundary data**".

$$\boxed{\text{boundary data on } \Gamma \text{ surface}} \xrightarrow{\text{completely determined}} \boxed{\text{all interior information}}$$

Physical and Mathematical Implications

- The formula states that to determine the value of u at any point inside Ω , knowledge of boundary conditions is sufficient.
- This highlights a fundamental property of "harmonic functions": their internal values are entirely determined by boundary values.

When the point M_0 is taken outside the region Ω or on its boundary Γ , two other formulas can also be derived in the same way:

$$\begin{aligned} & - \iint_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS \\ &= \begin{cases} 0, & \text{if } M_0 \text{ is outside } \Omega, \leftarrow \boxed{\text{No need for a ball of singularity, Green 2nd directly implies}} \\ 2\pi u(M_0), & \text{if } M_0 \text{ is on } \Gamma, \leftarrow \boxed{\text{Need for a half ball of singularity, Green 2nd directly implies}} \\ 4\pi u(M_0), & \text{if } M_0 \text{ is inside } \Omega. \end{cases} \end{aligned}$$

Method 2: δ function method

Only two tools required:

- Fundamental solution in 3D:** $\Delta \frac{1}{r} = -4\pi\delta(r - r_0)$ ($r = r_{MM_0}, r_0 = r_{M_0 O}$)
- Green's second formula:**

$$\int_{\Omega} u \Delta v - v \Delta u dV = \int_{\partial\Omega} u \partial_n v - v \partial_n u dS$$

Proof. Let $v = \frac{1}{r}$ and u is **harmonic**, i.e., $\Delta u = 0$. Inserting these into the Green's second formula,

$$\Rightarrow \int_{\Omega} \left(u \underbrace{\Delta \frac{1}{r}}_{=-4\pi\delta(r-r_0)} - \cancel{\frac{1}{r} \Delta u} \right) dV = \int_{\partial\Omega} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} dS.$$

$$\Rightarrow -4\pi u(r_0) = \int_{\partial\Omega} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} dS.$$

Then we obtain

$$u(M_0) = -\frac{1}{4\pi} \iint \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS,$$

and complete the proof. \square

If u is not a harmonic function, as long as it has first order continuous partial derivatives on $\Omega + \Gamma$ and $\Delta u = F$ in the region Ω , a formula similar to (4.1.13) can also be obtained

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS - \frac{1}{4\pi} \iiint_{\Omega} \frac{F}{r_{MM_0}} d\Omega.$$

↑ It is easy to use the δ method to prove by replacing $\Delta u = F$.

Supplementary 2: Integral expression of harmonic functions in two dimensional cases

First, consider the function

$$\ln \frac{1}{r_{MM_0}} = \ln \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

where $M_0(x_0, y_0)$ is a fixed point in the region D . Except at M_0 , this function satisfies Laplace's equation everywhere. If the function u has continuous first order partial derivatives on $D + C$ and is **harmonic** in D , then

$$u(M_0) = -\underbrace{\frac{1}{2\pi}}_{\text{2D circumference is } 2\pi} \int_C \left[u(M) \frac{\partial}{\partial n} \left(\underbrace{\ln \frac{1}{r_{MM_0}}}_{\text{2D fund. solution}} \right) - \ln \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS \quad (4.1.10)$$

Proof. In (4.1.12), let u be a harmonic function and take $v = \ln \frac{1}{r}$. Since the function v becomes infinite at the point M_0 , Green's formula (4.1.12) cannot be directly applied to the region D . However, if a circle $K_\varepsilon^{M_0}$ with M_0 as the center and a sufficiently small positive number ε as the radius is removed from the region D , then the function v is continuously differentiable in the remaining region $D - K_\varepsilon^{M_0}$.

Apply formula (4.1.12) to the above mentioned harmonic function u and $v = \ln \frac{1}{r}$ in the region $D - K_\varepsilon^{M_0}$, we get

$$0 = \int_{C+C_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \ln \frac{1}{r} \frac{\partial u}{\partial n} \right) dS$$

where C_ε is the circumference of the circle $K_\varepsilon^{M_0}$.

$$\int_C \left(u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \ln \frac{1}{r} \frac{\partial u}{\partial n} \right) dS + \int_{C_\varepsilon} \left(u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \ln \frac{1}{r} \frac{\partial u}{\partial n} \right) dS = 0. \quad (4.1.11)$$

On the circumference C_ε ,

$$\frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) = -\frac{\partial}{\partial r} \left(\ln \frac{1}{r} \right) = \frac{1}{r} = \frac{1}{\varepsilon}.$$

From this, we can obtain

$$\int_{C_\varepsilon} u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) dS = \frac{1}{\varepsilon} \int_{C_\varepsilon} u dS = \frac{1}{\varepsilon} \cdot 2\pi \varepsilon \bar{u} = 2\pi \bar{u}$$

where \bar{u} is the average value of the function u on the circumference C_ε .

On the other hand, since on the circumference C_ε

$$\int_{C_\varepsilon} \ln \frac{1}{r} \frac{\partial u}{\partial n} dS = \ln \frac{1}{\varepsilon} \int_{C_\varepsilon} \frac{\partial u}{\partial n} dS = -\ln \frac{1}{\varepsilon} \int_{C_\varepsilon} \frac{\partial u}{\partial r} dS = -\ln \frac{1}{\varepsilon} \iint_{K_\varepsilon^{M_0}} \Delta u d\sigma = 0$$

Then, substituting $\int_{C_\varepsilon} u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) dS = 2\pi \bar{u}$ and $\int_{C_\varepsilon} \ln \frac{1}{r} \frac{\partial u}{\partial n} dS = 0$ into formula (4.1.11), we can get

$$\int_C \left(u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \ln \frac{1}{r} \frac{\partial u}{\partial n} \right) dS + 2\pi \bar{u} = 0.$$

Now, let $\varepsilon \rightarrow 0$. Since $\lim_{\varepsilon \rightarrow 0} \bar{u} = u(M_0)$, from the above formula, we can obtain the integral expression (4.1.10) of the harmonic function u in two dimensional cases. \square

4.1.4 Basic Properties of Harmonic Functions

Recall:

$$\int_{\Omega} (u \Delta v - v \Delta u) dV = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS \quad (4.1.12)$$

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS. \quad (4.1.13)$$

Theorem 4.1.1 (Flux Properties). *Let the function $u(x, y, z)$ be a harmonic function in the region Ω , and it has first order continuous partial derivatives on $\bar{\Omega}$. Then*

$$\iint_{\Gamma} \frac{\partial u}{\partial n} dS = 0, \quad (4.1.14)$$

where Γ is the boundary of the region Ω , and \mathbf{n} is the direction of the outer normal of Γ .

Proof. Just take u as a harmonic function and $v = 1$ in equation (4.1.12), then formula (4.1.14) can be obtained. \square

Alternative Proof Using Gauss's Theorem.

$$\int_{\Gamma} \frac{\partial u}{\partial n} dS = \int_{\Gamma} \mathbf{n} \cdot \nabla u dS = \int_{\Omega} \Delta u dV = 0.$$

\square

Formula (4.1.14) shows that the integral of the normal derivative of a harmonic function along the boundary of the region is 0. For a stable temperature field, this means that the amount of heat flowing into and out of the object through the object interface is equal. Otherwise, the thermal dynamic equilibrium cannot be maintained, and the temperature will be unstable.

Physical Interpretation via Electrostatics

- The equation $\Delta u = 0$ describes a **steady-state** condition.
- In electrostatics, u can be interpreted as an **electric potential** in a charge-free region.
- Gauss' flux theorem in electrostatics states:

$$\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \leftarrow \boxed{\text{For Green function in next section, one positive charge!}}$$

where $Q = 0$ in a charge-free domain.

- Hence, due to $\mathbf{E} = -\nabla u$, from physical perspective, using the Gauss' flux theorem, the net flux of $\int_{\Gamma} \frac{\partial u}{\partial n} dS = \int_{\Gamma} \mathbf{n} \cdot \nabla u dS = -\int_{\Gamma} \mathbf{n} \cdot \mathbf{E} dS = 0$ through the boundary is zero.
- “The electric field lines entering are equal to the electric field lines exiting” (see Fig. 4.2).

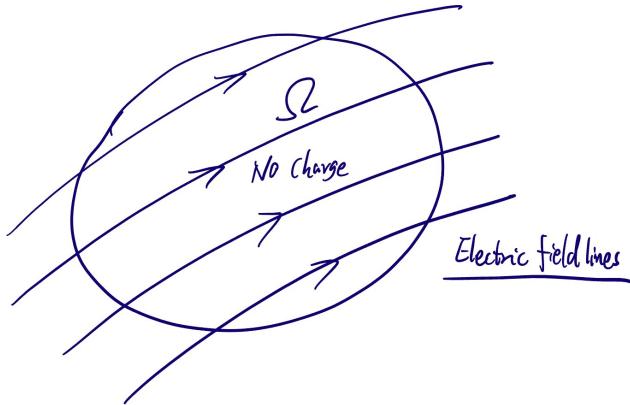


Figure 4.2: Gauss' flux theorem

From formula (4.1.14), it can be deduced that a **necessary condition** for the Neumann problem

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ \frac{\partial u}{\partial n} \Big|_{\Gamma} = f(x, y, z) \end{cases}$$

to have a solution is $\iint_{\Gamma} f(x, y, z) dS = 0$.

Application to Boundary Value Problems

- If $\int_{\Gamma} f dS \neq 0$, the boundary value problem has **no solution**.

Theorem 4.1.2 (Mean Value Theorem). *Let the function $u(M)$ be **harmonic** in the region Ω , and M_0 be any point in the region Ω . If Γ_a is a **spherical surface** centered at M_0 with radius a , and this sphere is completely inside the region Ω , then*

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\Gamma_a} u dS. \quad (4.1.15)$$

Conditions: $\begin{cases} (1) u \text{ is harmonic} \\ (2) \Gamma_a \text{ a spherical surface (not every surface possesses such a good property)} \end{cases}$

Comparison with the General Case

- For a general function (may not be harmonic), the spherical mean approximates the function's value at the center only in the limit as the radius tends to zero.
- For harmonic functions, this equality holds for any radius, making the condition much stronger.
- This demonstrates that harmonic functions are uniquely determined once boundary values are given.

Key Conditions for Applying the Theorem

- The function must be harmonic, meaning it satisfies Laplace's equation.
- The averaging surface must be a sphere.
- A common mistake is applying the mean value property to non-spherical regions, where it does not hold.

Proof. Apply formula (4.1.13) to the spherical surface Γ_a , we get

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma_a} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

Since

$$\iint_{\Gamma_a} \frac{1}{r} \frac{\partial u}{\partial n} dS = \frac{1}{a} \iint_{\Gamma_a} \frac{\partial u}{\partial n} dS = 0 \quad (\text{using Theorem 4.1.1})$$

and

$$\iint_{\Gamma_a} u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = \iint_{\Gamma_a} u \frac{\partial}{\partial r} \left(\frac{1}{r} \right) dS = \iint_{\Gamma_a} u \left(-\frac{1}{r^2} \right) dS = -\frac{1}{a^2} \iint_{\Gamma_a} u dS,$$

then equation (4.1.15) is proved. \square

Theorem 4.1.3 (Extremum Principle). *Let the function $u(x, y, z)$ be a harmonic function in the region Ω , and it is continuous on $\Omega + \Gamma$ and not a constant function. Then its maximum and minimum values can only be achieved on the boundary Γ .*

Continuity Condition

- A function u is harmonic in a domain Ω if it satisfies the Laplace equation $\Delta u = 0$ in Ω and $u \in C^2(\Omega)$.
- Additionally, u must be continuous up to the boundary Γ of Ω .
- If u is not continuous up to the boundary Γ , then it may have jumps or discontinuities at the boundary.
- In this case, boundary conditions such as $u|_{\Gamma} = f$ lose their meaning, as the function inside Ω is not constrained by the boundary values.
- For boundary conditions to meaningfully affect the interior solution, u must be continuously extendable to Γ .

Implicit Assumption of Continuity

- From now on, whenever we refer to a harmonic function in Ω , we assume it is continuous up to the boundary.
- This implicit assumption ensures that boundary values play a role in determining the solution inside Ω .

Maximum and Minimum Value Principle

- If u is harmonic in Ω and not a constant function, then u
 - can attain its maximum and minimum values;
 - only on the boundary Γ .
- This is a **stronger condition** than the general case of **continuous** functions, which are **only guaranteed to attain extreme values** in bounded domains.
- The principle remains valid even if Ω is unbounded or partially unbounded.

Strong Maximum Principle

- The maximum and minimum values of u cannot be attained in the interior of Ω , unless u is a constant function.
- If u is a harmonic function in a domain Ω , then only two possibilities exist:
 - u is a **constant** function.
 - If u is **not constant**, then its maximum and minimum values must occur on the **boundary** Γ .

Significance of the Principle

- The principle highlights the restrictive nature of harmonic functions compared to general continuous functions.
- It ensures that boundary values fully determine the behavior of a harmonic function in the interior.

Proof. The extremum principle can be easily proved by the mean value theorem. We only need to prove the case of the maximum value (by replacing u with $-u$, the case of the minimum value can be reduced to the case of the maximum value).

We use the method of proof by contradiction. Suppose the function u achieves its maximum value at a certain point $M_1 \in \Omega$. Then it can be deduced that u must be **identically equal to a constant**, and $u = u(M_1)$. This contradicts the condition that u is **not** a constant function.

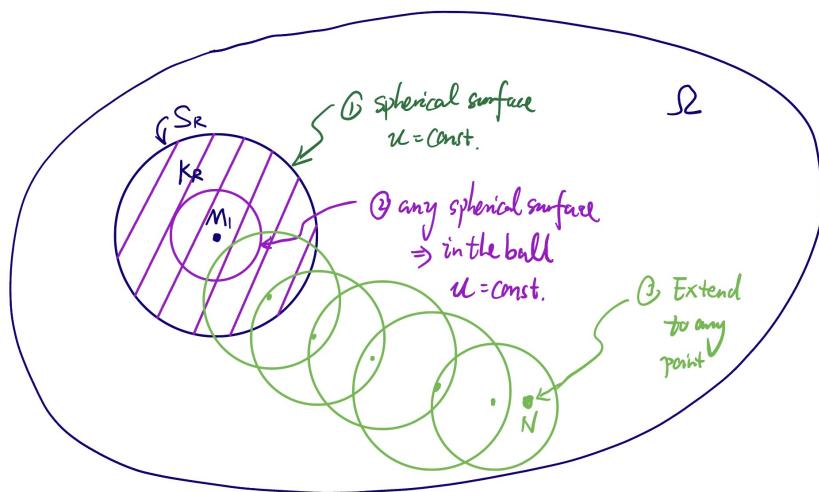


Figure 4.3: Extrmum Theorem

Take M_1 as the center and make a sphere K_R with an arbitrary radius R such that it is completely contained in the region Ω . Denote the spherical surface of K_R as S_R . Then on S_R , we have $u(M) = u(M_1)$.

(1) On S_R : In fact, if there exists a point M such that $u(M) < u(M_1)$, then by the **continuity** of the function, we can surely find a neighborhood of this point on the spherical surface S_R where also $u(M) < u(M_1)$. Therefore, even if $u(M) = u(M_1)$ on the remaining part of the spherical surface S_R , we still have

$$\iint_{S_R} u(M)dS < \iint_{S_R} u(M_1)dS$$

In fact,

$$\frac{1}{4\pi R^2} \iint_{S_R} u(M)dS < \frac{1}{4\pi R^2} \iint_{S_R} u(M_1)dS = u(M_1)$$

But by the **mean value formula** (4.1.15), we have

$$\frac{1}{4\pi R^2} \iint_{S_R} u(M)dS = u(M_1)$$

This is a contradiction. Then on the spherical surface S_R , we have $u(M) \equiv u(M_1)$.

(2) On K_R : Similarly, on the spherical surface with M_1 as the center and an arbitrary radius r ($r \leq R$), we also have $u(M) \equiv u(M_1)$. Therefore, in the entire sphere K_R , we always have $u(M) \equiv u(M_1)$.

(3) Any point $N \in \Omega$: Now we prove that $u(M) \equiv u(M_1)$ holds for all points in Ω . Arbitrarily take a point $N \in \Omega$. In the region Ω , make a polyline l connecting the two points M_1 and N . Denote the minimum distance from the polyline l to the boundary Γ of the region Ω as d . Due to the arbitrariness of the point N , we obtain that $u(M) \equiv u(M_1)$ holds in the entire region Ω . This contradicts the problem assumption. Then the extremum principle is proved. \square

Corollary 4.1.1 (Principle of Comparison). *Let u and v be both harmonic functions in the region Ω , and they are continuous on $\Omega + \Gamma$. If the inequality $u \leq v$ holds on the boundary Γ of Ω , then the inequality also holds in Ω , and the equality holds in Ω if and only if $u \equiv v$.*

Statement of the Theorem

- Suppose u and v are harmonic functions in a domain Ω .
- If $u \leq v$ (i.e., $u - v \leq 0$) on the boundary Γ , then $u \leq v$ (i.e., $u - v \leq 0$) in the entire domain Ω . Therefore, inspire us to define $w = u - v$ (one function can apply Extreme Theorem).

Proof Outline

- Define $w = u - v$.
- Since u and v are harmonic, we have:

$$\Delta u = 0, \quad \Delta v = 0.$$

- Subtracting the two equations gives:

$$\Delta w = \Delta u - \Delta v = 0.$$

Thus, w is also harmonic in Ω .

- By assumption, $w \leq 0$ on Γ .
- Apply the maximum principle:
 - The maximum and minimum of a harmonic function in Ω must occur on the boundary unless the function is constant.
 - If $w > 0$ at some interior point, this contradicts the maximum principle, since the boundary values are non-positive.
- Therefore, $w \leq 0$ in Ω , which implies $u \leq v$ in Ω .

Remarks

- This result is a direct consequence of the maximum principle.
- The standard method of proof involves contradiction: assuming $w > 0$ somewhere inside leads to a contradiction with the maximum principle.
- The comparison principle is frequently used in solving boundary value problems for harmonic functions.

Corollary 4.1.2 (Application: Uniqueness Theorem). *Proving the uniqueness of solutions to the Dirichlet problem using the extremum principle*

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \end{cases} \quad (4.1.16)$$

Well-posedness (Recall)

To ensure **well-posedness** of (4.1.16), we require the solution to satisfy three properties:

- **Existence:** A solution must exist.
- **Uniqueness:** The solution must be unique.
- **Stability:** Small changes in the boundary data should result in small changes in the solution.

In this section, we focus on proving the uniqueness of the solution using the maximum principle.

Summary of Uniqueness Proof in Differential Equations

The common approach to proving the uniqueness of solutions in differential equations is as follows:

- We use a contradiction argument.

- Assume there are two solutions u_1 and u_2 .
- Define $w = u_1 - u_2$.
- Prove that $w = 0$, which implies $u_1 = u_2$.

Thus, the proof of uniqueness is reduced to demonstrating that the associated homogeneous problem for w has only the trivial solution. Various tools can be employed for this purpose, including:

- Maximum and Minimum Value Principle,
- Energy Methods (e.g. wave equations, there is no Extreme Theorems), and
- Other analytical tools.

Uniqueness Proof

Analysis of w

- Since both u_1 and u_2 satisfy the Laplace equation:

$$\Delta u_1 = 0, \quad \Delta u_2 = 0,$$

subtracting these equations gives:

$$\Delta w = \Delta(u_1 - u_2) = 0.$$

This implies that w is a harmonic function in Ω .

- On the boundary Γ , we have:

$$w = u_1 - u_2 = f - f = 0.$$

That is, $w = 0$ on Γ .

Application of the Maximum Principle

- The strong maximum principle states that a non-constant harmonic function attains its maximum and minimum on the boundary.
- Since w is harmonic and vanishes on Γ , the only possibility is $w \equiv 0$ in Ω .
- Thus, $u_1 = u_2$, proving uniqueness.

Proof. Let u_1 and u_2 be two solutions of problem (4.1.16). Then $u = u_1 - u_2$ is a harmonic function in Ω , that is, $\Delta u = 0$, and $u|_{\Gamma} = 0$. By the **extremum principle**, u can neither be greater than 0 nor less than 0 in Ω . So on $\Omega + \Gamma$, we have $u \equiv 0$, that is, $u_1 \equiv u_2$. This proves the uniqueness of solutions to the Dirichlet problem. \square

4.2 Green's Function

- In Section 4.1, we solved the Laplace equation **without** considering **boundary** conditions. The solution was a **harmonic function**, expressed in an **integral form**.
- In Section 4.2, we **aim** to **incorporate** boundary conditions and investigate the form of the solution.
- However, we **cannot solve** the equation with boundary conditions in this section. Instead, we aim to **find a transformation** (**the main objective in this section**) between u and a function v .
- Through this transformation, we convert the boundary value problem for u into a problem for v , i.e.,

$$\Delta v = 0, \quad \text{and} \quad v = \frac{1}{4\pi r}.$$

For a function u that is a harmonic function in the region Ω and has first order continuous partial derivatives on $\Omega + \Gamma$, we have the equality

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

This integral expression shows that the value of the function u **inside the region** Ω can be expressed in terms of the **values of the function** u and its **normal derivative** $\frac{\partial u}{\partial n}$ on the **boundary** Γ . However, the solutions to the Dirichlet problem or the Neumann problem **cannot** be directly obtained from formula (4.1.13).

For example, for the Dirichlet problem, the value of u on Γ is given, while the value of $\frac{\partial u}{\partial n}$ on the boundary Γ is unknown. Since the solution to the Dirichlet problem is unique, the value of $\frac{\partial u}{\partial n}$ on the boundary Γ **cannot** be **arbitrarily specified**.

So, in order to solve the Dirichlet problem, we naturally first think of eliminating $\frac{\partial u}{\partial n}$ from formula (4.1.13). For this purpose, we need to introduce the concept of Green's function. We also need to rely on Green's second formula

$$\iiint_{\Omega} (u \Delta v - v \Delta u) d\Omega = \iint_{\Gamma} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \quad (6)$$

- The integral expression of a harmonic function has a geometric meaning: it describes the value of the harmonic function u at any point inside a region Ω , depended on the boundary values.
- Specifically, for a harmonic function u inside Ω , its value at any point depends on:

$$u = \text{Boundary value of } u \quad \text{and} \quad \text{the normal derivative of } u \text{ on the boundary.}$$

- This property suggests that the boundary values of u and its normal derivative determine all internal values. In other words, the boundary conditions determine the solution inside the region.

- If we consider a system as follows,

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \\ \frac{\partial u}{\partial n}|_{\Gamma} = g(x, y, z) \end{cases} \leftarrow \boxed{\text{Extra bdry. may break the wellposedness, leading to no solution}}$$

- Since (4.1.16) leads to the unique solution u and this u yields a specific $\frac{\partial u}{\partial n}|_{\Gamma}$. If it is not equal to g , then there is no solution.
- **Idea:** Add one term A and subtract its variant B . Consider the form $A = \int_{\partial\Omega} kv \frac{\partial u}{\partial n}$ since we want to use A to **eliminate** $\frac{\partial u}{\partial n}$ in (4.1.13). The requirement is $A = B$ on the boundary, but actually the functions $A \neq B$. On careful consideration: (1) $A \neq B$ is only in the forms of expression. (2) As long as they are **equal on the boundary**. Green's second formula helps us find the answer (detailed idea can be found in Fig. 4.16).

Summary:

For a harmonic function $u \in C^1(\Omega \cup \Gamma)$ in the region Ω , there is the integral representation of u .

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS$$

The value of u inside Ω is expressed in terms of the value of u and the normal derivative $\frac{\partial u}{\partial n}$ of u on the boundary Γ .

However, the solutions to the Dirichlet or Neumann problems cannot be directly obtained from (4.1.13). Because for the value of u and the normal derivative $\frac{\partial u}{\partial n}$ of u on the boundary Γ , only one of them can be prescribed.

For example, for the Dirichlet problem, the value of u on the boundary Γ is given, and $\frac{\partial u}{\partial n}|_{\Gamma}$ is unknown. Since the solution to the Dirichlet problem is unique, $\frac{\partial u}{\partial n}|_{\Gamma}$ cannot be arbitrarily specified (otherwise, there may be no solution, redundant information).

To solve the Dirichlet problem, naturally, the first idea is to try to eliminate $\frac{\partial u}{\partial n}$ from the formula (4.1.13). For this purpose, the concept of the Green's function needs to be introduced.

In Green's second formula (4.1.12), if we take both u and v as harmonic functions in the region Ω and they have continuous first order partial derivatives on $\Omega + \Gamma$, then we get

$$0 = \iint_{\Gamma} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS.$$

Adding the above formula to formula (4.1.13), we obtain

$$u(M_0) = \iint_{\Gamma} \left\{ u \cdot \frac{\partial}{\partial n} \left(v - \frac{1}{4\pi r_{MM_0}} \right) + \left(\frac{1}{4\pi r_{MM_0}} - v \right) \frac{\partial u}{\partial n} \right\} dS. \quad (4.2.1)$$

If we choose a harmonic function v such that $v|_{\Gamma} = \frac{1}{4\pi r_{MM_0}}|_{\Gamma}$, then the $\frac{\partial u}{\partial n}$ term in formula (4.2.1) disappears, and we have

$$u(M_0) = - \iint_{\Gamma} u \frac{\partial}{\partial n} \left(\frac{1}{4\pi r_{MM_0}} - v \right) dS. \quad (4.2.2)$$

Let

$$G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v, \quad (4.2.3)$$

Then formula (4.2.2) can be expressed as

$$u(M_0) = - \iint_{\Gamma} u \frac{\partial G}{\partial n} dS. \quad (18)$$

where $G(M, M_0)$ is called the **Green's function** for Laplace's equation (or the source function for the Dirichlet problem). And $G(M, M_0)$ is identically equal to 0 on the boundary Γ .

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $\Omega + \Gamma$, for the Dirichlet problem of Laplace's equation

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \end{cases} \quad (4.2.4)$$

If a solution with first order continuous partial derivatives exists on $\Omega + \Gamma$, then the solution to problem (4.2.4) can be expressed as

$$u(M_0) = - \iint_{\Gamma} f(x, y, z) \frac{\partial G}{\partial n} dS. \quad (4.2.5)$$

Since

$$G = \frac{1}{4\pi r} - v,$$

$$\Delta \left(\frac{1}{r} \right) = -4\pi \delta(r - r_0) \Rightarrow \Delta \left(\frac{1}{4\pi r} \right) = -\delta(r - r_0).$$

Since $\Delta u = 0$

$$\Rightarrow \Delta G = \Delta \left(\frac{1}{4\pi r} - v \right) = -\delta(r - r_0).$$

$$\Rightarrow \begin{cases} \Delta G = -\delta(r - r_0) \\ G|_{\Gamma} = 0 \end{cases}.$$

G is the second kind Green's function

- In fact, the role of $u(M_0) = - \iint_{\Gamma} f \frac{\partial G}{\partial n} ds$ is to provide a variable transformation between u and v . Through this variable transformation, the problem of

$$\begin{cases} \Delta u = 0 \\ u|_{\Gamma} = f \end{cases}$$

can be transformed into the problem of

$$\begin{cases} \Delta v = 0 \\ v|_{\Gamma} = \frac{1}{4\pi r} \end{cases} \leftarrow [v \text{ harmonic}] \quad \text{or} \quad \begin{cases} \Delta G = -\delta(r - r_0) \\ G|_{\Gamma} = 0 \end{cases} \leftarrow [G \sim \frac{1}{r_{MM_0}} \text{ singular at } M_0]$$

↑ [Do not means we have solved u since v and G have to be solved in next section]

The second problem is simpler and independent of f .

- Simplification of the Problem:**

- For different f , only the same v problem needs to be solved if the domain is the same.
- This **simplifies** or **categorizes** the problem (4.2.3).
- Recall to convert a difficult problem into a familiar or simpler one:
 - Function Transformation
 - Variable Transformation

Then transformations $u(M_0) = - \iint_{\Gamma} f \frac{\partial G}{\partial n} ds$ plays a role of such transformation.

Mathematical Approach solving the Green functions

- When solving a problem involving Green functions G or v , we can use mathematical methods such as the separation of variables.
 - Separation of Variables.
 - Eigenfunction Expansion.

Semi-Physical Approach (introduced in Section 4.3)

- Instead of purely mathematical methods, we will use a semi-physical approach.
- This method provides a deeper understanding of the physical significance of G .
- The key idea is to interpret the differential equation in a physical context.

Given the Problems:

$$\begin{cases} \Delta u = 0 \\ u|_{\Gamma} = f \end{cases} \quad \text{and} \quad \begin{cases} \Delta G = -\delta(r - r_0) \\ G|_{\Gamma} = 0 \end{cases} \quad (\text{where } \Gamma = \partial\Omega)$$

By integrating $G\Delta u - u\Delta G$ over Ω :

$$\int_{\Omega} \underbrace{G\Delta u}_{=0} - \underbrace{u\Delta G}_{-u\delta(r-r_0)} dV = \int_{\partial\Omega} \underbrace{G \frac{\partial u}{\partial n}}_{=0} - \underbrace{u \frac{\partial G}{\partial n}}_{f \frac{\partial G}{\partial n}} dS.$$

Then,

$$\int_{\Omega} u\delta(r - r_0) dV = - \int_{\partial\Omega} f \frac{\partial G}{\partial n} dS \quad \Rightarrow \quad u(r_0) = - \int_{\partial\Omega} f \frac{\partial G}{\partial n} dS.$$

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $\Omega + \Gamma$, for the Dirichlet problem of Poisson's equation

$$\begin{cases} \Delta u(x, y, z) = F, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) \end{cases}$$

If a solution with first order continuous partial derivatives exists on $\Omega + \Gamma$, then this solution must be expressible as

$$u(M_0) = - \iint_{\Gamma} f \frac{\partial G}{\partial n} dS - \iiint_{\Omega} FG d\Omega.$$

This is because, similar to before, according to Green's second identity, we have $\Delta V = 0$, $\Delta u = F$.

$$-\iiint_{\Omega} v F d\Omega = \iint_{\Omega} (u \Delta v - v \Delta u) d\Omega = \iint_{\Gamma} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds \quad ①$$

$$u(M_0) = -\frac{1}{4\pi} \iint_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS - \frac{1}{4\pi} \iiint_{\Omega} \frac{F(M)}{r} d\Omega \quad ②$$

Adding ① and ② gives

$$u(M_0) = \iint_{\Gamma} \left[u \frac{\partial}{\partial n} \left(v - \frac{1}{4\pi r} \right) - \left(v - \frac{1}{4\pi r} \right) \frac{\partial u}{\partial n} \right] dS + \iiint_{\Omega} F \left(v - \frac{1}{4\pi r} \right) d\Omega$$

Let $(v - \frac{1}{4\pi r})|_{\Gamma} = 0$, then

$$u(M_0) = - \iint_{\Gamma} u \frac{\partial G}{\partial n} dS - \iiint_{\Omega} FG d\Omega.$$

Application of formula (4.2.5) to solve the Dirichlet problem for Laplace's equation When applying formula (4.2.5) to solve the Dirichlet problem for Laplace's equation, the key lies in finding the Green's function $G(M, M_0)$ in formula (4.2.3), where v is the solution to the following special Dirichlet problem:

$$\begin{cases} \Delta v = 0, & (x, y, z) \in \Omega \\ v|_{\Gamma} = \frac{1}{4\pi r_{MM_0}} \Big|_{\Gamma} \end{cases} \quad (21)$$

The Green's function determined by this function v is called the Green's function for the first boundary value problem. (For some special regions, such as spherical regions and half-spaces, the Green's function can be obtained.)

Physical meaning of Green's function in electrostatics

Suppose a **unit positive charge** is placed at point M_0 . Then in free space, the **electric potential** it generates is $\frac{1}{4\pi r_{MM_0}}$. If the point charge at M_0 is enclosed within a **closed conducting surface** that is **grounded**, then the **electric potential** inside the conducting surface can be represented by the function

$$G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v,$$

This function is identically equal to 0 on the conducting surface. The function $-v$ exactly represents the electric potential generated by the induced charges on the conducting surface (see Fig. 4.4).

Physical Interpretation of Green's Function

To understand the meaning of G , we consider translating the equation into a physical problem.

Electrostatic Analogy

In electrostatics, Laplace's equation with a delta function source describes the potential due to a point charge:

$$\nabla^2 G = -\delta(r - r_0)$$

This equation represents:

- A unit point charge located at a specific position.
- The resulting electrostatic potential field in a given domain Ω .
- Conducting boundaries modifying the potential due to induced charges.

Boundary Conditions and Physical Setup

The boundary condition $G = 0$ on the boundary translates to:

- A grounded conductor surrounding the domain.
- Induced charges appearing on the conductor.
- The resulting electrostatic field being different from the free-space solution.

The problem consists of three key elements

- A conducting shell.
- A unit positive charge.
- A grounded connection.

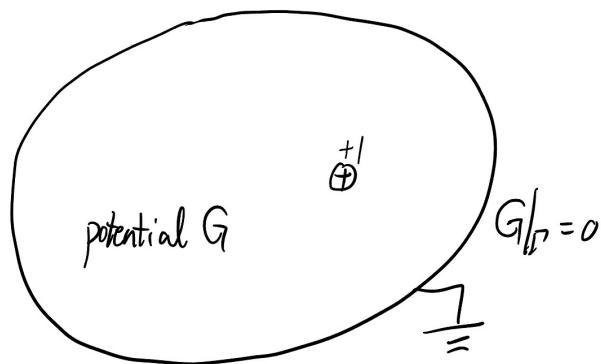


Figure 4.4: Physical meaning of Green functions G

Supplementary 3: Define the Green's function for the first boundary value problem in the plane and derive the integral expression for the solution of this problem

Differences

- In 3D: Green's function is solved in a given region Ω .
- In 2D: The boundary becomes a line or a curve, reducing the problem dimension.
- 4π (3D) \rightarrow 2π (2D).
- The fundamental solution $\frac{1}{r}$ (3D) $\rightarrow \ln\left(\frac{1}{r}\right)$ (2D).

The solution involves finding relationships between functions u and G and their derivatives.

For this purpose, we need to rely on the formula

$$u(M_0) = -\frac{1}{2\pi} \int_C \left[u(M) \frac{\partial}{\partial n} \left(\ln \frac{1}{r_{MM_0}} \right) - \ln \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS. \quad (4.2.6)$$

and the second Green's formula in the plane

$$\iint_D (u\Delta v - v\Delta u) d\sigma = \int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \quad (4.2.7)$$

In Green's formula (4.2.7), if we take both u and v as harmonic functions in the region D and they have continuous first order partial derivatives on $D + C$, then we get

$$0 = \int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS.$$

Adding the above formula to formula (4.2.6) gives

$$u(M_0) = \int_C \left\{ u \frac{\partial}{\partial n} \left(v - \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} \right) + \left(\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v \right) \frac{\partial u}{\partial n} \right\} dS. \quad (4.2.8)$$

If we choose a harmonic function v such that $v|_C = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}|_C$, then the $\frac{\partial u}{\partial n}$ term in formula (4.2.8) disappears, and we have

$$u(M_0) = - \int_C u \frac{\partial}{\partial n} \left(\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v \right) dS. \quad (4.2.9)$$

Let

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v, \quad (4.2.10)$$

Then formula (4.2.9) can be expressed as

$$u(M_0) = - \int_C u \frac{\partial G}{\partial n} dS. \quad (4.2.11)$$

where $G(M, M_0)$ is called the Green's function for the two dimensional Laplace's equation (or the source function for the Dirichlet problem). And $G(M, M_0)$ is identically equal to 0 on the boundary C .

Therefore, if the Green's function $G(M, M_0)$ is known and it has first order continuous partial derivatives on $D + C$, for the Dirichlet problem of the two dimensional Laplace's equation

$$\begin{cases} \Delta u(x, y) = 0, & (x, y) \in D \\ u|_C = f(x, y) \end{cases} \quad (4.2.12)$$

If a solution with first - order continuous partial derivatives exists on $D + C$, then the solution to problem (4.2.12) can be expressed as

$$u(M_0) = - \int_C f(x, y) \frac{\partial G}{\partial n} dS.$$

When applying formula (4.2.11) to solve the Dirichlet problem for Laplace's equation, the key lies in finding the Green's function $G(M, M_0)$ in formula (4.2.10), where v is the solution to the following special Dirichlet problem:

$$\begin{cases} \Delta v = 0, & (x, y) \in D \\ v|_C = \left(\frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} \right)|_C \end{cases}$$

The Green's function determined by this function v is called the Green's function for the first boundary value problem. (For some special regions, such as circular regions and half-planes, the Green's function can be obtained.)

Properties of Green's function

Theorem 4.2.1. *The Green's function $G(M, M_0)$ satisfies Laplace's equation everywhere except at the point $M = M_0$. When $M \rightarrow M_0$, $G(M, M_0)$ tends to infinity, and its order is the same as that of $\frac{1}{r_{MM_0}}$.*

Theorem 4.2.2. *The Green's function $G(M, M_0)$ is identically equal to 0 on the boundary Γ .*

These two Theorems precisely describe

$$\begin{cases} \Delta G = -\delta(r - r_0) \\ G|_{\Gamma} = 0 \end{cases}$$

Theorem 4.2.3. *The Green's function $G(M, M_0)$ satisfies Laplace's equation everywhere except at the point $M = M_0$. When $M \rightarrow M_0$, $G(M, M_0)$ tends to infinity, and its order is the same as that of $\frac{1}{r_{MM_0}}$.*

Theorem 4.2.4. *The Green's function $G(M, M_0)$ is identically equal to 0 on the boundary Γ .*

- The theorem is simple but has two crucial points that are very useful in proofs of theorems.
- The **first key point** is that the function v is a harmonic function. This means v has the property of being C^2 inside the domain Ω , which is very useful for solving boundary value problems.
- The **second key point** is about the function G :
 - G is infinite at the point M_0 , indicating a singularity at M_0 , and has a growth order of $\frac{1}{r}$ near M_0 .
 - G is zero on the boundary.
- v and G each have their own advantages:
 - v has good behavior inside Ω (**harmonic**), but boundary behavior is less ideal.
 - G has a **singularity** at M_0 but its boundary behavior is very good.

Proof. In fact, these theorems state the problem of G :

$$\begin{cases} \Delta G = -\delta(r - r_0) \\ G|_{\Gamma} = 0 \end{cases}$$

where r and r_0 give the position of M and M_0 , respectively. In addition, the second statement of Theorem 4.2.3 can be proven by

$$G(M, M_0) = \underbrace{\frac{1}{4\pi r_{MM_0}}}_{\text{dominate}} - \underbrace{v}_{\text{bounded}}$$

and v is harmonic (thus is bounded in Ω since $v \in C^2$).

Another way:

$$\lim_{M \rightarrow M_0} \left(\frac{G(M, M_0)}{\frac{1}{r_{MM_0}}} \right) = \lim_{M \rightarrow M_0} \left(\frac{1}{4\pi} - r_{MM_0} v \right) = \frac{1}{4\pi}.$$

It completes the proof. \square

- **Green's Function Notation:**

- Green's function $G(M, M_0)$ is written with two indices: M (the **true variable**) and M_0 (a parameter indicating the **position of the unit positive charge**).
- M represents the point inside the **conductor shell** where you place a probe to measure the potential.
- M_0 is the position where the unit positive charge is placed.

- **Integral Variable and Charge Position:**

$$u(\textcolor{red}{M}_0) = - \iint_{\Gamma} f(\textcolor{blue}{M}) \frac{\partial G(\textcolor{blue}{M}, \textcolor{red}{M}_0)}{\partial n} dS$$

- The integral variable in the expression for G is M , which is taken over the **boundary of Ω** .
- M_0 is the **position of the unit positive charge** and is the **independent variable** you are solving for in $u(M_0)$.

- **Summary:**

- The key is to understand that M_0 is the point where the **charge is placed** and the **independent variable of u** of the left hand side, and M is the **variable of integration** over the boundary and the **independent variable of $G(M, M_0)$** .

Theorem 4.2.5. Inside the region Ω , the following inequality holds

$$0 < G(M, M_0) < \frac{1}{4\pi r_{MM_0}}.$$

- Each function, v and G , has distinct advantages.
- Utilize the individual strengths of v and G .
- Perform transformations back and forth between v and G .

Proof. 1. Note

$$G < \frac{1}{4\pi r} \Leftrightarrow v > 0.$$

Let us focus on the system of v . Since

$$\begin{cases} \Delta v = 0 \\ v|_{\Gamma} = \frac{1}{4\pi r_{MM_0}}|_{\Gamma} > 0 \end{cases}, \leftarrow \boxed{\text{Using } v \text{ is harmonic!}}$$

by the **maximum principle**, we get $v > 0$.

$$\Rightarrow G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v < \frac{1}{4\pi r_{MM_0}}.$$

2. Prove $G(M, M_0) > 0$, that is, $v(M) < \frac{1}{4\pi r_{MM_0}}$. Note $G \sim \frac{1}{r_{MM_0}}$, when $M \rightarrow M_0$ there is a singularity.

We cannot directly use the maximum principle to get this conclusion (see Fig. 4.5). Applying the maximum principle to v , we get

$$\min_{M_1 \in \Gamma} \frac{1}{4\pi r_{M_1 M_0}} < v(M) < \max_{M_1 \in \Gamma} \frac{1}{4\pi r_{M_1 M_0}} = \frac{1}{4\pi r_{\min}},$$

which cannot give the result we want since what we want is

$$v(M) < \frac{1}{4\pi r_{MM_0}} \leftarrow \boxed{\text{Here } M \text{ is not always on the boundary. In fact } M \in \Omega}$$

- Here M is not always on the boundary. In fact $M \in \Omega$;
- In fact, from the example of Fig. 4.5, since $r_{MM_0} > r_{\min}$,

$$v(M) < \frac{1}{4\pi r_{MM_0}} < \frac{1}{4\pi r_{\min}}.$$

That is, what we want is stronger than the one given by the maximum principle.

- **Idea:** Excavate a small ball $K_{\varepsilon}^{M_0}$ (a small ball with M_0 as the center and ε as the radius). By taking $\varepsilon > 0$ small enough, $\frac{1}{4\pi\varepsilon} \rightarrow +\infty$ can be arbitrarily large.

Let $\Omega_{\varepsilon} = \Omega \setminus \overline{K_{\varepsilon}^{M_0}}$

Case 1: If $M \in \Omega_{\varepsilon}$, G is a harmonic function in Ω_{ε} and $G|_{\Gamma} = 0$. By the maximum principle, $G(M, M_0) > 0$. And we can take ε small enough ($\frac{1}{4\pi\varepsilon}$ can be large enough) such that $G|_{\partial K_{\varepsilon}^{M_0}} = \frac{1}{4\pi\varepsilon} - v > 0$ since $v \in C^2$ is harmonic and this implies v is bounded.

Case 2: If $M \notin \Omega_{\varepsilon}$, that is, $M \in K_{\varepsilon}^{M_0}$ (G has singularity, thus using v is better than G in this domain), by taking ε small enough, we can make

$$v < \frac{1}{4\pi\varepsilon} \leftarrow \boxed{v \text{ is harmonic and thus bounded by } C \text{ in } K_{\varepsilon}^{M_0}. \text{ Hence, there is } \varepsilon \text{ such that } \frac{1}{4\pi\varepsilon} > C > v}$$

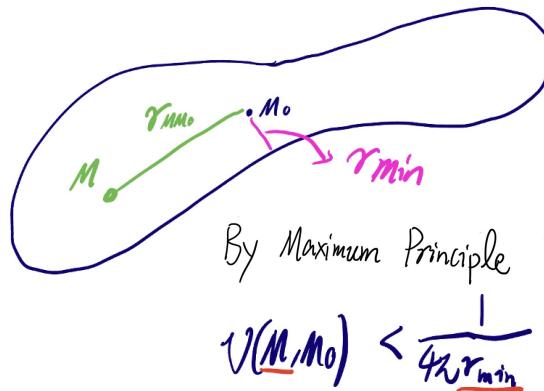


Figure 4.5: Theorem 4.2.5

Because $\frac{1}{4\pi\varepsilon}$ can be arbitrarily large, $v \in C^2 \Rightarrow v$ is bounded. Also, because $r_{MM_0} < \varepsilon \Rightarrow v < \frac{1}{4\pi\varepsilon} < \frac{1}{4\pi r_{MM_0}}$ (Question: Compare with Fig.4.5 and think about the differences between the case in a ball with the one in Ω). \square

Theorem 4.2.6 (Symmetry). *The Green's function $G(M, M_0)$ has a symmetric property with respect to the independent variable M and the parameter variable M_0 . That is, if $M_1, M_2 \in \Omega$, then*

$$G(M_1, M_2) = G(M_2, M_1).$$

- **Symmetry Property:**

- Mathematically expressed as $G(M_1, M_2) = G(M_2, M_1)$.
- Indicates that the Green's function remains unchanged when its arguments are swapped.

- **Physical Interpretation:**

- Placing a positive charge at M_2 and measuring potential at M_1 gives the same result as placing a charge at M_1 and measuring at M_2 .
- Despite the positions of charges and measurement points being reversed, the potential remains the same.

The significance of this property in electrostatics can be described as follows: The electric potential generated at M_2 by a unit point charge at M_1 is equal to the electric potential generated at M_1 by a unit point charge at M_2 . A principle similar to this is called the reciprocity principle in physics.

Method 1: (δ functions).

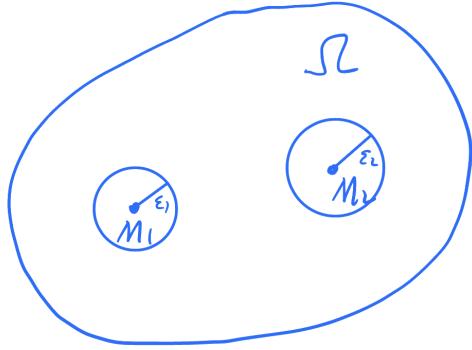
$$\begin{cases} \Delta G(M, M_i) = -\kappa\delta(\mathbf{x} - \mathbf{x}_i) & (\text{where } \mathbf{OM} = \mathbf{x}, \mathbf{OM}_i = \mathbf{x}_i, i = 1, 2) \\ G|_\Gamma = 0 \end{cases}$$

$$\begin{aligned}
& \Rightarrow \int_{\Omega} [G(M, M_1) \Delta G(M, M_2) - G(M, M_2) \Delta G(M, M_1)] d\Omega \\
& = -\kappa \int_{\Omega} G(M, M_1) \delta(\mathbf{x} - \mathbf{x}_2) d\Omega + \kappa \int_{\Omega} G(M, M_2) \delta(\mathbf{x} - \mathbf{x}_1) d\Omega \\
& = -\kappa G(M_2, M_1) + \kappa G(M_1, M_2) \leftarrow \boxed{\text{Property of } \delta \text{ function}}
\end{aligned} \tag{4.2.13}$$

Using the **2nd Green's identity** and the **boundary condition** $G|_{\Gamma} = 0$

$$(4.2.13) = \int_{\Gamma} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial n} \right] dS = 0 \tag{4.2.14}$$

Combining (4.2.13) and (4.2.14) gives $G(M_2, M_1) = G(M_1, M_2)$. \square



$G(M, M_2)$ is harmonic in $B_{\varepsilon_1}^{M_1}$

Figure 4.6: Theorem 4.2.6

Method 2: (Rigorous method. Excavate a small ball). Let's consider the region Ω'' obtained by removing two small balls $B_{\varepsilon_1}^{M_1}, B_{\varepsilon_2}^{M_2}$ centered at M_1 and M_2 from Ω (see Fig. 4.6). Applying the 2nd Green's identity on Ω'' , we get:

$$\int_{\partial\Omega''} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial n} \right] dS = 0.$$

Here, $\partial\Omega''$ consists of $\partial B_{\varepsilon_1}^{M_1}, \partial B_{\varepsilon_2}^{M_2}$ and Γ . Because the boundary condition $G|_{\Gamma} = 0$, the integral along Γ is 0. Thus,

$$\int_{\partial B_{\varepsilon_1}^{M_1} + \partial B_{\varepsilon_2}^{M_2}} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial n} \right] dS = 0$$

By Theorem 4.2.3, $G(M, M_2) \sim \frac{1}{r_{MM_2}}$, thus $\lim_{M \rightarrow M_1} \frac{G(M, M_1)}{1/r_{MM_1}} = A$.

$$\begin{aligned}
& \Rightarrow \lim_{\varepsilon_1 \rightarrow 0} \int_{\partial B_{\varepsilon_1}^{M_1}} G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} dS \\
& = \lim_{\varepsilon_1 \rightarrow 0} \int_{\partial B_{\varepsilon_1}^{M_1}} \frac{A}{r_{MM_1}} \frac{\partial G(M, M_2)}{\partial n} dS = \lim_{\varepsilon_1 \rightarrow 0} \frac{A}{\varepsilon_1} \int_{\partial B_{\varepsilon_1}^{M_1}} \frac{\partial G(M, M_2)}{\partial n} dS \\
& \stackrel{\text{Gauss}}{=} \lim_{\varepsilon_1 \rightarrow 0} \frac{A}{\varepsilon_1} \int_{B_{\varepsilon_1}^{M_1}} \Delta G(M, M_2) dS = 0
\end{aligned} \tag{4.2.15}$$

In the last step, in $B_{\varepsilon_1}^{M_1}$, when ε_1 is small enough, such that $M_2 \notin B_{\varepsilon_1}^{M_1}$. Additionally,

$$\begin{aligned}
& \lim_{\varepsilon_2 \rightarrow 0} \int_{\partial B_{\varepsilon_2}^{M_2}} G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} dS \\
& = A \lim_{\varepsilon_2 \rightarrow 0} \int_{\partial B_{\varepsilon_2}^{M_2}} G(M, M_1) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_2}} \right) dS \quad \left(n \text{ is inward normal, then } \frac{\partial}{\partial n} = -\frac{\partial}{\partial r} \right) \\
& = A \lim_{\varepsilon_2 \rightarrow 0} \int_{\partial B_{\varepsilon_2}^{M_2}} G(M, M_1) \frac{1}{r_{MM_2}^2} dS \\
& = A \lim_{\varepsilon_2 \rightarrow 0} \frac{1}{\varepsilon_2^2} \int_{\partial B_{\varepsilon_2}^{M_2}} G(M, M_1) dS \\
& = A \lim_{\varepsilon_2 \rightarrow 0} \frac{1}{\varepsilon_2^2} G(M_2, M_1) \cdot 4\pi\varepsilon_2^2 \quad (\text{using mean value theorem, or mean value theorem for integrals}) \\
& = 4\pi AG(M_2, M_1).
\end{aligned} \tag{4.2.16}$$

By a similar calculation of $\int_{\partial B_{\varepsilon_1}^{M_1} + \partial B_{\varepsilon_2}^{M_2}} G(M, M_2) \frac{\partial G(M, M_1)}{\partial n} dS$, and combining (4.2.15) and (4.2.16), we get:

$$\begin{aligned}
0 &= \int_{\partial B_{\varepsilon_1}^{M_1} + \partial B_{\varepsilon_2}^{M_2}} \left[G(M, M_1) \frac{\partial G(M, M_2)}{\partial n} - G(M, M_2) \frac{\partial G(M, M_1)}{\partial n} \right] dS \\
&= 4\pi AG(M_2, M_1) - 4\pi AG(M_1, M_2)
\end{aligned}$$

Then $G(M_2, M_1) = G(M_1, M_2)$

□

Theorem 4.2.7.

$$\iint_{\Gamma} \frac{\partial G(M, M_0)}{\partial n} dS_M = -1.$$

1. (**Method 1: Physical meaning**) This property seems familiar. When we talked about the first property of harmonic functions, it was similar, but with a harmonic function u instead of G . If we replace G with the harmonic function u , the value should be zero. This can be judged directly from the **physical meaning**. For harmonic functions, the **flux is zero** because there are **no charges inside**.

2. **(Method 1: Physical meaning)** For the Green's function, the **physical meaning** is different. It represents a grounded conducting shell with a unit positive charge inside. Therefore, its **flux** should be equal to 1.
3. **(Method 1: Physical meaning)** The **negative sign** in the flux calculation comes from the fact that the electric field \mathbf{E} is related to the potential G as a **negative gradient**. According to **Gauss's flux theorem**, the integral of $\mathbf{E} \cdot d\mathbf{S}$ over a closed surface is equal to the charge enclosed, which is q .
4. **(Method 1: Physical meaning)** The relationship between \mathbf{E} and G is given by $\mathbf{E} = -\nabla G$. The flux can be rewritten in terms of the outward normal vector \mathbf{n} and the gradient of G :

$$1 = q = - \int_{\Gamma} \mathbf{n} \cdot \mathbf{E} dS = - \int_{\Gamma} \mathbf{n} \cdot \nabla G dS = - \int_{\Gamma} \frac{\partial G}{\partial n} dS.$$

5. **(Method 2: Gauss's theorem)** Using **Gauss's theorem**, this surface integral can be converted into a volume integral involving ΔG . For a Green's function with a point charge, the Laplacian of G is related to the delta function:

$$\Delta G = -\delta(\mathbf{r}).$$

6. **(Method 3: Green's first identity)** The flux calculation using the Green's function can be related to the divergence theorem (**Gauss's theorem**) and the **Green's first identity**. These methods are **equivalent** and can be used to prove the same results.
7. **(Method 4: Transformation between u and G -Nash's mountain-climbing-like metaphorical idea)**

- Another interesting proof method mentioned is to transform the problem into solving a differential equation.
- By comparing this relationship with the given equation, according to the form $\int \frac{\partial G}{\partial n} dS$, we make a connection to $\int f \frac{\partial G}{\partial n} dS = -u(M_0)$, that is, we want to prove

$$-u(M_0) = \int f \frac{\partial G}{\partial n} dS = \int \frac{\partial G}{\partial n} dS = -1.$$

providing $\Delta u = 0$ and $f = 1$ where $u|_{\Gamma} = f$.

- Therefore, the problem is transformed to

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) = 1 \end{cases}$$

and we have to solve u .

Proof: (Method 4). Consider the following Dirichlet problem

$$\begin{cases} \Delta u(x, y, z) = 0, & (x, y, z) \in \Omega \\ u|_{\Gamma} = f(x, y, z) = 1 \end{cases}$$

On the one hand, using the relation $u(M_0) = - \iint_{\Gamma} f(x, y, z) \frac{\partial G}{\partial n} dS$, we can obtain

$$u(M_0) = - \iint_{\Gamma} f(x, y, z) \frac{\partial G}{\partial n} dS = - \iint_{\Gamma} \frac{\partial G}{\partial n} dS,$$

On the other hand, by the extremum principle, the solution to this problem is $u \equiv 1$. According to the uniqueness of the solution to the Dirichlet problem, this theorem holds. \square

4.3 Applications of Green's Function–Solve Green functions G

From the formula

$$u(M_0) = - \iint_{\Gamma} f(M) \frac{\partial G(M, M_0)}{\partial n} dS, \quad (4.3.1)$$

it can be seen that for a region Ω bounded by a surface Γ , as long as its **Green's function is found**, then within this region, the solution to the **Dirichlet problem** of **Laplace's equation** can be expressed by this integral.

- In the previous section, we transformed the problem of finding u into the problem of finding the Green's functions G and v , but we did not solve for them. In this section, we will solve for the Green's functions G and v .
- The choice of the domain for the Green's function is crucial; arbitrary boundaries can make the Green's function difficult to determine.
- Two specific regions are discussed: half space and ball, using a physical approach combining physical thinking and mathematical derivation.
- The method used is known as the method of images or electrostatic analogy.

For some special regions, its Green's function can be obtained by the **method of electrical images** (**method of images–Physical Method**).

The so called **method of images** is to find the **image point** (**symmetric point**) M_1 of the point M_0 with respect to the boundary Γ **outside the region** Ω . Then, an **appropriate negative charge** is placed at this image point M_1 . The **negative** electric potential generated by it and the **positive** electric potential generated by the unit positive charge at the point M_0 **cancel each other out** on the surface Γ . At this time, the **electric potential** of the electric field formed by the two inside Γ is **equivalent** to the required **Green's function**.

Remarks on the Method of Images

1. The fundamental **physical idea** of the **method of images**:
 - The essence of the method of images: **Equivalence** (a tool for physicists)
 - A fictitious negative **charge** is used to replace the **effect** of a **conducting shell**.
2. Determining the information of the **fictitious charge**:
 - (a) **Charge amount**
 - (b) **Position**
3. Principle of Determination:
 - The **electric potential** on the **conducting shell** is **0**. The fictitious charge has the **same effect** as the shell.
 - In other words, the criterion for finding the fictitious charge is that the electric fields produced by the fictitious charge and the conducting shell should cancel each other out at the boundary.
4. **Question:** Why must the **fictitious charge** be **equivalent** to the **induced charge** on the **conducting shell**? \Leftrightarrow Just because the **electric potential** on the **conducting shell Γ is 0**, can it be guaranteed that the **internal electric potential** is also the **same**?
5. Since for the system of G :

$$\begin{cases} \Delta G = -\delta \\ G|_{\Gamma} = 0, \end{cases} \quad (4.3.2)$$

the solution exists and is unique.

\Leftrightarrow For the electric field generated by a single electric charge, once the boundary electric potential is determined, there is only one distribution inside, which is also determined!

- The uniqueness of the solution is ensured by the uniqueness of the Green's function, which is related to the maximum principle for the first Green function v .
6. Because from the problem (4.3.2) of G , it is not apparent why the potential is zero on the boundary. **Mathematically**, as long as there is such a boundary condition, the solution is unique. This offers a great deal of **freedom** in physics to employ **various manageable physical models** to simulate such mathematical boundary conditions.

4.3.1 Green's Function in the Half Space and the Dirichlet Problem

We aim to solve the Dirichlet problem in the upper half space $z > 0$. The problem is formulated as:

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad (z > 0), \quad (4.3.3)$$

$$u|_{z=0} = f(x, y), \quad -\infty < x, y < +\infty. \quad (4.3.4)$$

Three steps of the method of images:

1. Determine the image point (the charge quantity and position. Place M on the boundary for verification).
2. Write out the Green's function (Place M inside Ω).
3. Calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again).

Step 1: Determine the image point (the charge quantity and position. Place M on the boundary for verification.)

First, we need to find the Green's function $G(M, M_0)$ (see Fig. 4.7). To this end, we place a unit positive charge at the point $M_0(x_0, y_0, z_0)$ in the upper half space $z > 0$, and a unit negative charge at the symmetric point $M_1(x_0, y_0, -z_0)$ of M_0 with respect to the plane $z = 0$ (**based on the symmetry, see the following discussions and other methods**).

- The location of the unit positive charge can be **arbitrary**, but if one wants to know the **value of u at point M_0** , then place a unit positive **charge at M_0** .

$$u(\textcolor{red}{M}_0) = - \iint_{\Gamma} f(\textcolor{blue}{M}) \frac{\partial G(\textcolor{blue}{M}, \textcolor{red}{M}_0)}{\partial n} dS.$$

- It is crucial to remember that the **independent variable of u** indicates the **location of the unit positive charge** in the Green's function.

- The goal is to find an **equivalent point**, also known as a **mirror point** or **fictitious charge**.
- On the boundary, the electric potential generated by the mirror point should **cancel out** the potential at the point of interest.
- This cancellation is the **criterion** for determining the mirror point.

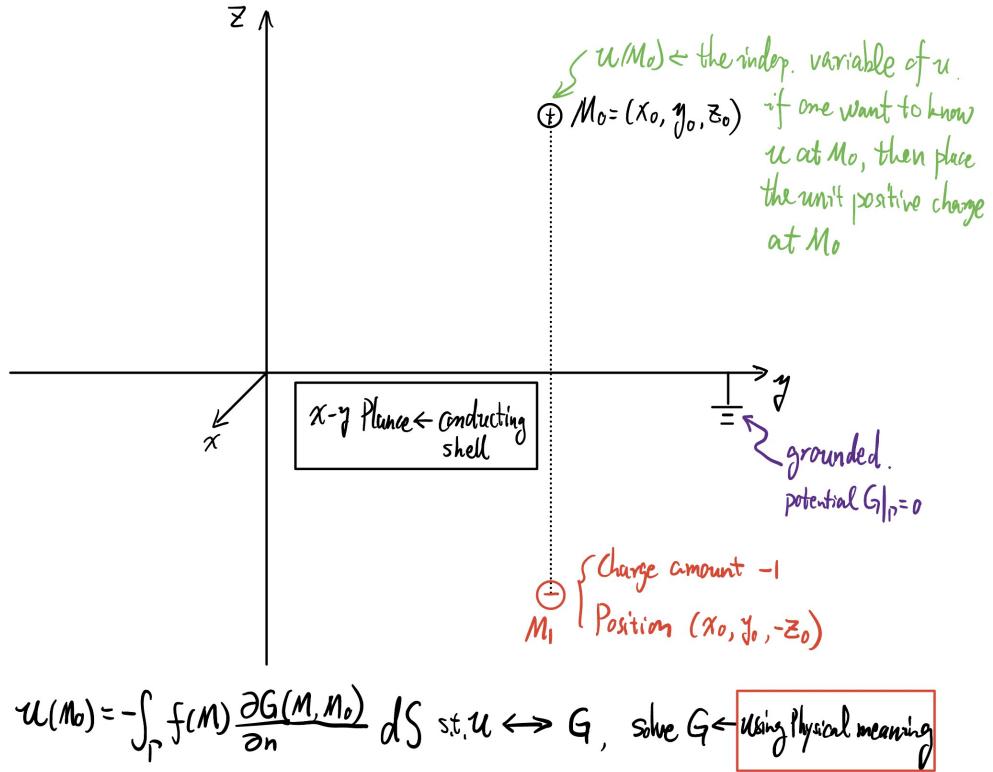


Figure 4.7: Green's function in the half space

- To find the mirror point, we need to determine its **position** and **charge**.
- (Method 1: Symmetry)** The mirror point can often be **guessed** based on **symmetry**. The **symmetry** of this problem allows for a reasonable guess about the location of the mirror point.
- (Method 2: A General method determining the mirror point–Not recommended)** Using the cancellation on the boundary, one can obtain **algebraic equations** to determine the **position** and **charge amount** of the mirror point. This is very difficult, the idea is given as follows (see Fig. 4.8):

$$\frac{1}{4\pi r} - \frac{q_1}{4\pi r_1} = 0 \Leftrightarrow \text{Criterion of cancellations.}$$

$$\Rightarrow \frac{r_1}{r} = q_1 \Rightarrow F(x, y) := \frac{(x_1 - x)^2 + (y_1 - y)^2 + z^2}{(x_0 - x)^2 + (y_0 - y)^2 + z_0^2} = q_1^2$$

Differentiate with respect to x and y respectively, because no matter how x and y move in the xy plane, the cancellation criterion is always satisfied. That is,

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0 \quad (\text{because } q = \text{const.})$$

Additionally,

$$\begin{cases} \frac{\partial x_1}{\partial x} = 0 \\ \frac{\partial x_1}{\partial y} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial y_1}{\partial x} = 0 \\ \frac{\partial y_1}{\partial y} = 0 \end{cases}$$

- **(Method 3: Based on the specific (x, y) values)** Due to the criterion of cancellations,

$$\frac{(x_1 - x)^2 + (y_1 - y)^2 + z^2}{(x_0 - x)^2 + (y_0 - y)^2 + z^2} = q^2 \quad \forall x, y \in \mathbb{R}^2$$

- There are 4 unknowns (x_1, y_1, z_1) and q , with only one equation.
- To determine these 4 unknowns, we need 4 equations.
- Since x, y can be chosen arbitrarily, we can obtain 4 equations by choosing 4 specific (x, y) values.

$$1. (x, y) = (x_0, y_0),$$

$$\Rightarrow \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2 + z^2}{z_0^2} = q^2.$$

$$2. (x, y) = (x_1, y_1),$$

$$\Rightarrow \frac{z_1^2}{(x_0 - x_1)^2 + (y_0 - y_1)^2 + z_0^2} = q^2$$

$$3. (x, y) = (0, 0),$$

$$\Rightarrow \frac{x_1^2 + y_1^2 + z_1^2}{x_0^2 + y_0^2 + z_0^2} = q^2$$

$$4. (x, y) = (x_0, 0),$$

$$\Rightarrow \frac{(x_1 - x_0)^2 + y_1^2 + z_1^2}{y_0^2 + z_0^2} = q^2$$

With these 4 equations, the 4 unknowns can be solved.

The electric potential of the electrostatic field formed by them is exactly 0 on the plane $z = 0$.

- **Verification of Mirror Point:** We need to verify if the electric potential at the boundary (see Fig. 4.9).
- **Criterion for Position:** The sum of the potentials from the original charge and the mirror charge at the boundary should be zero.
- **Boundary Point Selection:** The point M on the boundary is selected for verification. That is, place M on the boundary for verification.
- **Dynamic Point M :** The selection of point M will change multiple times during the following calculation process (see Fig. 4.9, 4.10 and 4.11).

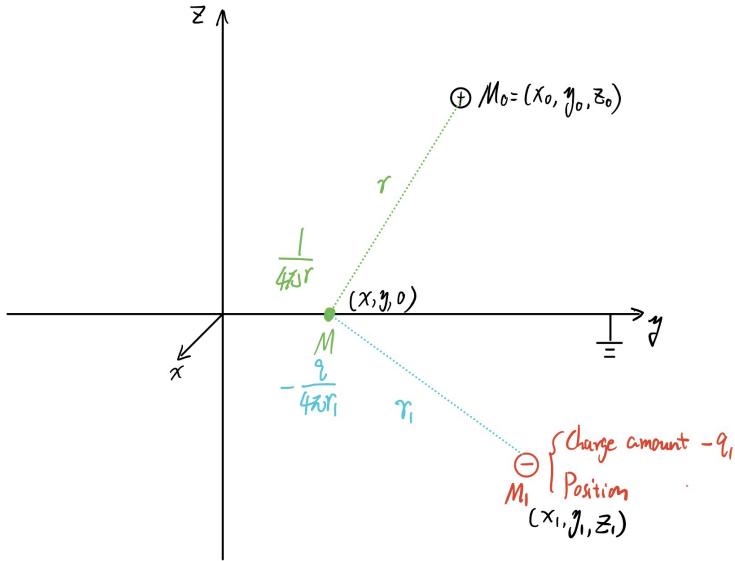


Figure 4.8: General idea for the mirror point

- (Fig. 4.9) Take M on the boundary to verify the selected mirror point is correct. The criterion of this verification is the cancellation of potentials.

Step 2: Write out the Green's function (Place M inside Ω)

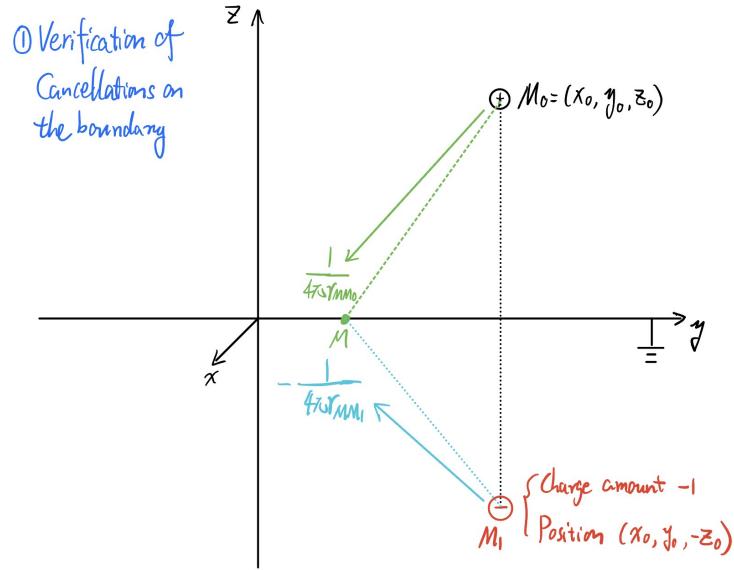
Therefore, by placing M inside Ω , the Green's function in the upper half space is

$$G(M, M_0) = \frac{1}{4\pi} \left(\frac{1}{r_{MM_0}} - \frac{1}{r_{MM_1}} \right),$$

where $r_{MM_0} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ and $r_{MM_1} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}$.

Step 3: Calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again)

To solve problems (4.3.3) and (4.3.4), we need to calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$ (need to place M on the boundary again since the integration in (4.3.1) is on the boundary Γ , see Fig. 4.11). Since the outward normal direction on the plane $z = 0$ is the negative direction of the oz -axis, we have $\frac{\partial G}{\partial n} \Big|_{z=0} = -\frac{\partial G}{\partial z} \Big|_{z=0}$ (n is the direction of $-z$, the outward normal).



Potential on the boundary point $M \in x-y$ Plane.

$$= \frac{1}{4\pi r_{MM_0}} - \frac{1}{4\pi r_{MM_1}} = 0 \quad \text{since } r_{MM_0} = r_{MM_1}, \text{ due to the symmetry.}$$

Figure 4.9: Step 1: Verification of cancellations

We first calculate the partial derivative:

$$\begin{aligned}
 \frac{\partial G}{\partial n} \Big|_{z=0} &= - \frac{\partial G}{\partial z} \Big|_{z=0} \\
 &= - \frac{1}{4\pi} \left\{ \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \right) \right. \\
 &\quad \left. - \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \right) \right\} \Big|_{z=0} \\
 &= \frac{1}{4\pi} \left\{ \frac{z-z_0}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \right. \\
 &\quad \left. - \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \right) \right\} \Big|_{z=0} \\
 &= \frac{1}{4\pi} \left\{ \frac{z-z_0}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} - \frac{z+z_0}{[(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2]^{3/2}} \right\} \Big|_{z=0} \\
 &= - \frac{1}{2\pi} \frac{z_0}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}}, \tag{4.3.5}
 \end{aligned}$$

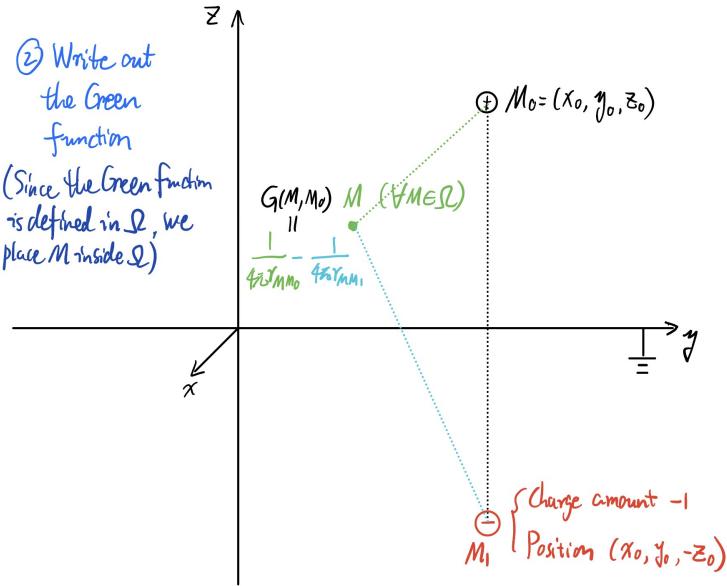


Figure 4.10: Step 2: Write out the Green function

Substituting (4.3.5) into

$$u(M_0) = - \iint_{\Gamma} f(M) \frac{\partial G}{\partial n} dS,$$

we obtain the solution to the problems (4.3.3) and (4.3.4):

$$u(M_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f(x, y) z_0 dx dy}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}}. \quad (4.3.6)$$

Ex 4.3.1. Suppose that on the boundary of a homogeneous half space, the steady state temperature is maintained. It is equal to 1 inside the circle $K : x^2 + y^2 < 1$ and 0 outside it. We want to find the steady state temperature distribution in the half space.

Solution. This problem can be reduced to the following well posed problem:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 & (z > 0) \\ u|_{z=0} = f(x, y) = \begin{cases} 1, & x^2 + y^2 < 1 \\ 0, & x^2 + y^2 \geq 1 \end{cases} \end{cases}$$

According to formula (4.3.6), we have

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \iint_K \frac{dx dy}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}}$$

In particular, on the positive semi-axis of the oz-axis ($x_0 = 0, y_0 = 0$), we have

$$u(0, 0, z_0) = \frac{z_0}{2\pi} \iint_K \frac{dx dy}{(x^2 + y^2 + z_0^2)^{3/2}}$$

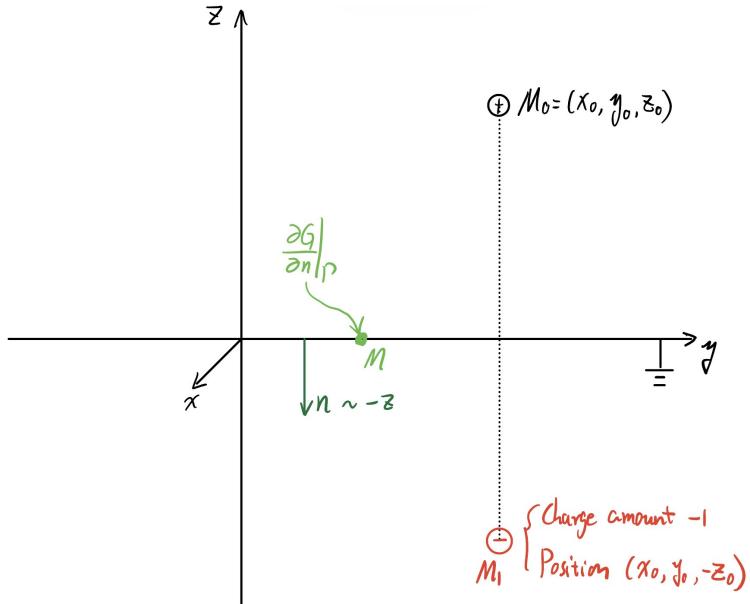


Figure 4.11: Step 3: Calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$

Since the integration region K is a circular region, we use polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$. Then

$$\begin{aligned} u(0, 0, z_0) &= \frac{z_0}{2\pi} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{(r^2 + z_0^2)^{3/2}} \\ &= z_0 \int_0^1 \frac{r dr}{(r^2 + z_0^2)^{3/2}} \\ &= -z_0 \frac{1}{(r^2 + z_0^2)^{1/2}} \Big|_{r=0}^{r=1} \\ &= 1 - \frac{z_0}{(1 + z_0^2)^{1/2}} \end{aligned}$$

When $(0, 0, z_0)$ approaches infinity along the positive semi-axis of the oz -axis, $u(0, 0, z_0) \rightarrow 0$.

Supplementary: Green's function in the half plane and the Dirichlet problem

We aim to solve the Dirichlet problem in the upper half plane $y > 0$. The problem is formulated as:

$$u_{xx} + u_{yy} = 0 \quad (y > 0), \quad (4.3.7)$$

$$u|_{y=0} = f(x), \quad -\infty < x < +\infty. \quad (4.3.8)$$

First, we find the Green's function $G(M, M_0)$. To this end, we place a unit positive charge at the point $M_0(x_0, y_0)$ in the upper half plane $y > 0$, and a unit negative charge at the symmetric point $M_1(x_0, -y_0)$ of M_0 with respect to the boundary $y = 0$. The electric potential of the electrostatic field formed by them is

exactly 0 on the boundary $y = 0$. Therefore, the Green's function in the upper half plane is

$$G(M, M_0) = \underbrace{\frac{1}{2\pi}}_{4\pi \rightarrow 2\pi} \left(\underbrace{\ln \frac{1}{r_{MM_0}}}_{\frac{1}{r_{MM_0}} \rightarrow \ln \frac{1}{r_{MM_0}}} - \ln \frac{1}{r_{MM_1}} \right),$$

To solve problems (4.3.7) and (4.3.8), we need to calculate $\frac{\partial G}{\partial n} \Big|_{y=0}$. Since the outward normal direction on the boundary $y = 0$ is the negative direction of the oy -axis, we have $\frac{\partial G}{\partial n} \Big|_{y=0} = -\frac{\partial G}{\partial y} \Big|_{y=0}$.

$$\begin{aligned} \frac{\partial G}{\partial n} \Big|_{y=0} &= -\frac{\partial G}{\partial y} \Big|_{y=0} \\ &= -\frac{1}{2\pi} \left\{ \frac{\partial}{\partial y} \left(\ln \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right) - \frac{\partial}{\partial y} \left(\ln \frac{1}{\sqrt{(x-x_0)^2 + (y+y_0)^2}} \right) \right\} \Big|_{y=0} \\ &= \frac{1}{2\pi} \left\{ \frac{y-y_0}{[(x-x_0)^2 + (y-y_0)^2]} - \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{(x-x_0)^2 + (y+y_0)^2}} \right) \right\} \Big|_{y=0} \\ &= \frac{1}{2\pi} \left\{ \frac{y-y_0}{[(x-x_0)^2 + (y-y_0)^2]} - \frac{y+y_0}{[(x-x_0)^2 + (y+y_0)^2]} \right\} \Big|_{y=0} \\ &= -\frac{1}{\pi} \frac{y_0}{[(x-x_0)^2 + y_0^2]}, \end{aligned} \tag{4.3.9}$$

Substituting (4.3.9) into

$$u(M_0) = - \int_C f(x, y) \frac{\partial G}{\partial n} dS,$$

we obtain the integral expression for the solution of the Dirichlet problem of the Laplace equation in the half plane (4.3.7) and (4.3.8):

$$u(M_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(x) y_0 dx}{[(x-x_0)^2 + y_0^2]}.$$

4.3.2 Green's Function in a Spherical Domain and the Dirichlet Problem

We aim to solve the Dirichlet problem in a **spherical domain**:

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in \Omega, \tag{4.3.10}$$

$$u|_{\Gamma} = f(x, y, z), \tag{4.3.11}$$

where Ω is a spherical domain centered at o with radius R , and Γ is its boundary.

Reviews

- Three key elements of the physical meanings of G
 - A conducting shell.
 - A unit positive charge.
 - A grounded connection.
- The location of the unit positive charge can be **arbitrary**, but if one wants to know the **value of u at point M_0** , then place a unit positive **charge at M_0** .

$$u(\textcolor{red}{M}_0) = - \iint_{\Gamma} f(\textcolor{blue}{M}) \frac{\partial G(\textcolor{blue}{M}, \textcolor{red}{M}_0)}{\partial n} dS. \quad (4.3.12)$$

- Three steps of the method of images:

1. Determine the image point (the charge quantity and position. Place M on the boundary for verification).
 - **Criterion for Position:** The sum of the potentials from the original charge and the mirror charge at the boundary should be zero.
2. Write out the Green's function (Place M inside Ω).
3. Calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again).

Step 1: Determine the image point (the charge quantity and position. Place M on the boundary for verification).

Now, we use the **method of electrical images** to find the Green's function of the sphere. To this end, we take an arbitrary point $M_0(x_0, y_0, z_0)$ inside the sphere. On the semiray oM_0 , we intercept a line segment oM_1 such that

$$r_{OM_0} \cdot r_{OM_1} = R^2. \leftarrow \boxed{\text{gives the position of the image point}} \quad (4.3.13)$$

The point M_1 is called the **inversion point** or **symmetric point** of the point M_0 with respect to the spherical surface Γ (see Fig. 4.12).

- The image point must lie on the line passing through the center of the sphere and the charge point.
- This is due to **symmetry**; if it were not on this line, the system would be asymmetric.
- **Using Similar Triangles:**
 - The triangle \triangle_{oM_0P} should be similar to the triangle \triangle_{oPM_1} .
 - This similarity gives the condition $R^2 = r_{OM_0} \cdot r_{OM_1}$.

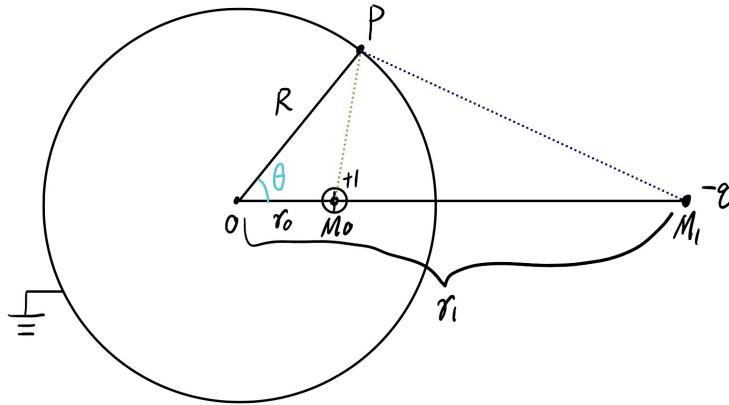


Figure 4.12: Verification of cancellations

- The potentials due to the charges at M_0 and M_1 should cancel out at the boundary.

To find the Green's function $G(M, M_0)$, we place a **unit positive charge** at the point M_0 and a charge of q **units** of negative charge **at the point** M_1 . We need to appropriately choose the **value of q** so that the electric potentials generated by these two point charges **exactly cancel each other out** on the **spherical surface Γ** . Let P be an arbitrary point **on the spherical surface**. Then we should have

$$\frac{1}{4\pi r_{M_0} P} = \frac{q}{4\pi r_{M_1} P} \implies q = \frac{r_{M_1} P}{r_{M_0} P}. \leftarrow \boxed{\text{gives the } \mathbf{\text{charge quantity}} \text{ of the image point}}$$

Since $\triangle OM_0P$ and $\triangle OPM_1$ have a common angle at point O , and the corresponding two sides enclosing this angle are proportional according to formula (4.3.13), these two triangles are similar. Thus, we have

$$\frac{r_{M_1} P}{r_{M_0} P} = \frac{R}{r_{OM_0}} \implies q = \frac{R}{r_{OM_0}},$$

That is to say, we must place a negative charge of R/r_0 units at the point M_1 . From this, we get

$$v = \frac{1}{4\pi} \frac{R}{r_{OM_0}} \frac{1}{r_{MM_1}}.$$

How to select the position of the image point–Method 1

$$\begin{aligned}
 |M_0 P|^2 &= |OM_0|^2 + |OP|^2 - 2|OP||OM_0| \cos \theta \\
 |M_1 P|^2 &= |OP|^2 + |OM_1|^2 - 2|OP||OM_1| \cos \theta \\
 \text{const. } \left(\frac{1}{q}\right)^2 &= \frac{|M_0 P|^2}{|M_1 P|^2} = \frac{|OM_0|^2 + |OP|^2 - 2|OP||OM_0| \cos \theta}{|OP|^2 + |OM_1|^2 - 2|OP||OM_1| \cos \theta} = \frac{r_0^2 + R^2 - 2Rr_0 \cos \theta}{R^2 + r_1^2 - 2Rr_1 \cos \theta} =: f(\theta)
 \end{aligned} \tag{4.3.14}$$

for all $\theta \in [0, \pi]$. Once r_1 and q are selected, this equation (4.3.14) should hold for every θ . Differentiating (4.3.14) yields (q is a constant if it is selected)

$$\begin{aligned} 0 = f'(\theta) &= \frac{2Rr_0 \sin \theta (R^2 + r_1^2 - 2Rr_1 \cos \theta) - (r_0^2 + R^2 - 2Rr_0 \cos \theta) \cdot 2Rr_1 \sin \theta}{(R^2 + r_1^2 - 2Rr_1 \cos \theta)^2} \\ &\Rightarrow r_0(R^2 + r_1^2 - 2Rr_1 \cos \theta) = r(r_0^2 + R^2 - 2Rr_0 \cos \theta) \\ &\Rightarrow r_0R^2 + r_0r^2 = r_1r_0^2 + r_1R^2 \\ &\Rightarrow (r_0 - r_1)R^2 = r_1r_0(r_0 - r_1) \Rightarrow R^2 = r_0r_1 \Rightarrow \frac{R}{r_1} = \frac{r_0}{R} \text{ (similar triangles)} \end{aligned}$$

How to select the position of the image point–Method 2

Since

$$\left(\frac{1}{q}\right)^2 = \frac{|M_0P|^2}{|M_1P|^2} = \frac{|OM_0|^2 + |OP|^2 - 2|OP||OM_0| \cos \theta}{|OP|^2 + |OM_1|^2 - 2|OP||OM_1| \cos \theta} = \frac{r_0^2 + R^2 - 2Rr_0 \cos \theta}{R^2 + r_1^2 - 2Rr_1 \cos \theta}$$

There are two unknowns q and r_1 and we need two equations to solve them. Since this identity holds for all $\theta \in [0, \pi]$, we choose $\theta = 0$ and $\theta = \pi$ to obtain two equations

$$\begin{aligned} \left(\frac{1}{q}\right)^2 &= \frac{r_0^2 + R^2 - 2Rr_0}{R^2 + r_1^2 - 2Rr_1} \\ \left(\frac{1}{q}\right)^2 &= \frac{r_0^2 + R^2 + 2Rr_0}{R^2 + r_1^2 + 2Rr_1} \end{aligned}$$

Solving these equations, we arrive at

$$r_1 = \frac{R^2}{r_0} \quad \text{and} \quad q = \frac{r_1}{R} = \frac{R}{r_0}.$$

Step 2: Write out the Green's function (Place M inside Ω).

Then, the Green's function of the spherical domain with Γ as the spherical surface is (see Fig. 4.13)

$$G(M, M_0) = \frac{1}{4\pi} \left(\frac{1}{r_{MM_0}} - \frac{R}{r_{OM_0}} \frac{1}{r_{MM_1}} \right). \quad (4.3.15)$$

Let $r_0 = r_{OM_0}$, $r = r_{OM}$, $r_1 = r_{OM_1}$, and γ be the angle between OM_0 and OM . Then formula (4.3.15) can be transformed into

$$G(M, M_0) = \frac{1}{4\pi} \left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0r \cos \gamma}} - \frac{R}{r_0} \frac{1}{\sqrt{r_1^2 + r^2 - 2r_1r \cos \gamma}} \right]$$

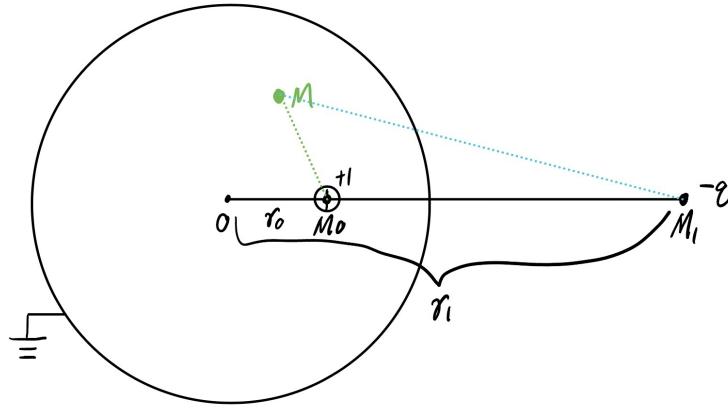


Figure 4.13: Green's function of the sphere

Using the relation (4.3.13) $r_{OM_0} \cdot r_{OM_1} = R^2$, we can obtain

$$G(M, M_0) = \frac{1}{4\pi} \left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r \cos \gamma}} - \frac{R}{\sqrt{r^2 r_0^2 - 2R^2 r_0 r \cos \gamma + R^4}} \right]$$

Step 3: Calculate $\frac{\partial G}{\partial n} \Big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again).

To solve the original problems (4.3.10) and (4.3.11), we also need to calculate $\frac{\partial G}{\partial n} \Big|_{\Gamma}$. On the spherical surface Γ ,

$$\begin{aligned} \frac{\partial G}{\partial n} \Big|_{\Gamma} &= \frac{\partial G}{\partial r} \Big|_{r=R} \\ &= -\frac{1}{4\pi} \left[\frac{r - r_0 \cos \gamma}{(r_0^2 + r^2 - 2r_0 r \cos \gamma)^{3/2}} - \frac{(rr_0^2 - R^2 r_0 \cos \gamma)R}{(r^2 r_0^2 - 2R^2 r_0 r \cos \gamma + R^4)^{3/2}} \right]_{r=R} \\ &= -\frac{1}{4\pi R} \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} \end{aligned}$$

Therefore, from (4.3.12), the expression for the solution of problems (4.3.10) and (4.3.11) is

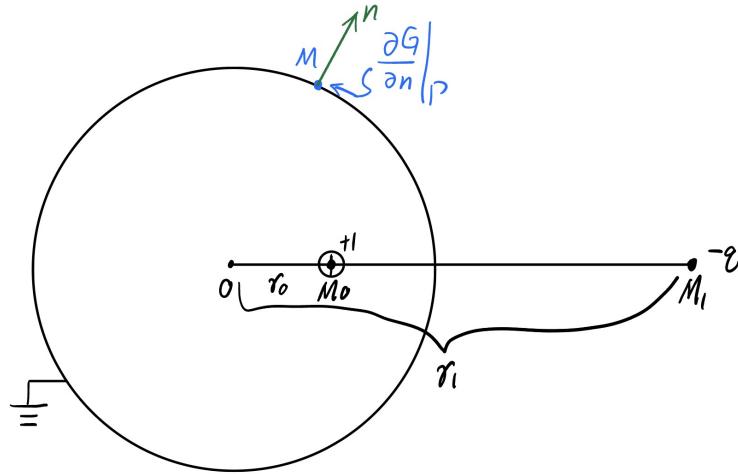
$$u(M_0) = \frac{1}{4\pi R} \iint_{\Gamma} f(x, y, z) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} dS. \quad (4.3.16)$$

In the spherical coordinate system, expression (4.3.16) becomes

$$u(r_0, \theta_0, \varphi_0) = \frac{R}{4\pi} \int_0^{2\pi} \int_0^\pi f(R, \theta, \varphi) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} \sin \theta d\theta d\varphi, \quad (4.3.17)$$

Formulas (4.3.16) or (4.3.17) are called **Poisson's formulas for the spherical domain**.

Here, $(r_0, \theta_0, \varphi_0)$ are the spherical coordinates of the point M_0 , (R, θ, φ) are the moving coordinates of points on the spherical surface Γ , and $\cos \gamma$ is the cosine of the angle between $\overrightarrow{OM_0}$ and \overrightarrow{OP} .

Figure 4.14: Calculate $\frac{\partial G}{\partial n}|_{z=0}$

Since

$$\begin{aligned}\overrightarrow{OM_0} \cdot \overrightarrow{OP} &= |\overrightarrow{OM_0}| |\overrightarrow{OP}| \cos \gamma \\ \overrightarrow{OM_0} &= (r_0 \sin \theta_0 \cos \varphi_0, r_0 \sin \theta_0 \sin \varphi_0, r_0 \cos \theta_0) \\ \overrightarrow{OP} &= (R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta),\end{aligned}$$

we obtain

$$\begin{aligned}\cos \gamma &= \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 (\cos \varphi \cos \varphi_0 + \sin \varphi \sin \varphi_0) \\ &= \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\varphi - \varphi_0)\end{aligned}$$

Ex 4.3.2. Suppose there is a homogeneous sphere of radius R . The temperature of the upper hemisphere surface is maintained at $0^\circ C$, and the temperature of the lower hemisphere surface is maintained at $1^\circ C$. We want to find the steady state temperature distribution inside the sphere.

Solution. This problem can be reduced to the following well-posed problem:

$$\begin{cases} \Delta u(r, \theta, \varphi) = 0 & (0 < r < R) \\ u|_{r=R} = \begin{cases} 0, & 0 < \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta < \pi \end{cases} \end{cases}$$

Using formula (4.3.17), we have

$$u(r_0, \theta_0, \varphi_0) = \frac{R}{4\pi} \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} \sin \theta d\theta d\varphi.$$

In particular, we want to find the temperature distribution on the vertical diameter of the sphere: $\theta_0 = 0$ (the upper half of the diameter) and $\theta_0 = \pi$ (the lower half of the diameter).

When $\theta_0 = 0$, $\cos \gamma = \cos \theta$, so

$$\begin{aligned} u(r_0, 0, \varphi_0) &= \frac{R}{4\pi} \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \theta)^{3/2}} \sin \theta d\theta d\varphi \\ &= \frac{R}{2} \left[-\frac{R^2 - r_0^2}{Rr_0(R^2 + r_0^2 - 2Rr_0 \cos \theta)^{1/2}} \right]_{\theta=\frac{\pi}{2}}^{\theta=\pi} \\ &= \frac{1}{2} \frac{R^2 - r_0^2}{r_0} \left(\frac{1}{\sqrt{R^2 + r_0^2}} - \frac{1}{R + r_0} \right) \end{aligned}$$

When $\theta_0 = \pi$, $\cos \gamma = -\cos \theta$, so

$$u(r_0, 0, \varphi_0) = \frac{1}{2} \frac{R^2 - r_0^2}{r_0^2} \left(\frac{1}{R - r_0} - \frac{1}{\sqrt{R^2 + r_0^2}} \right)$$

Supplementary: Green's Function and Dirichlet Problem in a Circular Domain (2D)

Firstly according to the previous results, the Green's function and the solution to the Dirichlet problem in a circular domain are

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}} - v, \quad (4.3.18)$$

$$u(M_0) = - \int_C f(M) \frac{\partial G}{\partial n} dS. \quad (4.3.19)$$

Solving the Dirichlet Problem in a Circular Domain We aim to solve the Dirichlet problem in a circular domain:

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D, \quad (4.3.20)$$

$$u|_{x^2 + y^2 = R^2} = f(x, y), \quad (4.3.21)$$

where D is a circular domain centered at o with radius R , and C is its boundary.

Using the method of images, the Green's function for the circular domain can be obtained as

$$G(M, M_0) = \frac{1}{2\pi} \left[\ln \frac{1}{r_{MM_0}} - \underbrace{\ln \left(\frac{R}{r_0} \frac{1}{r_{MM_1}} \right)}_{(*)} \right], \quad (4.3.22)$$

where $r_0 = r_{OM_0}$, and the point M_1 is called the inversion point or symmetric point of the point M_0 with respect to the circular boundary C .

- (*) term is actually not due to the charge quantity, but rather an additional term resulting from the choice of the zero point of electric potential.

Differences between 3D and 2D

- To solve the 2D problem, one might initially think of following the same procedure as in 3D, which involves finding the Green's function and the image charge. However, there are **two significant differences** between the 2D and 3D cases.
 1. At first glance, Equation (4.3.22) appears to be very similar to Equation (4.3.15). One might be tempted to think that it is derived through a straightforward analogy. However, things are not that simple. First of all, what is the electric **potential in 2D case**?
 - In 3D, the potential at a point due to a charge q is given by $\frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}$, where r is the distance from the charge. In 2D, the fundamental solution for the potential is $\frac{1}{2\pi} \ln\left(\frac{1}{r}\right)$.
 - A common mistake is to directly apply the 3D method to the 2D case, leading to an incorrect expression for the potential. For example, one might mistakenly write the potential as $-\frac{1}{2\pi} \ln\left(\frac{q}{r}\right)$, but this is wrong.
 - The **correct expression** for the **potential in 2D** due to a charge q should be $-\frac{q}{2\pi} \ln\left(\frac{1}{r}\right)$. This form ensures that the potential satisfies the 2D Laplace's equation.

Proof. Consider the equation:

$$\nabla^2 u = \kappa \delta(\mathbf{r} - \mathbf{r}_0)$$

$\kappa = -1$ Potential of unit positive charge

$\kappa = -q$ Potential of point charge with charge q

Let $u = C_1 \ln \frac{C_2}{r}$, Just like before, calculate κ in the same way, integrate over a circle,

$$\int_{S_a} \Delta u dS = \kappa \int_{S_a} \delta(\mathbf{r} - \mathbf{r}_0) dS = \kappa$$

Using Gauss's theorem:

$$\int_{\partial S_a} \mathbf{n} \cdot \nabla u dl = \int_{\partial S_a} \frac{\partial u}{\partial r} dl = \kappa = -q$$

Since

$$\frac{\partial u}{\partial r} = -\frac{C_1}{r} \Rightarrow -C_1 \int_{\partial S_a} \frac{1}{r} dl = -q$$

$$\Rightarrow -\frac{C_1}{a} \cdot 2\pi a = -q \Rightarrow C_1 = \frac{q}{2\pi} \Rightarrow u = \frac{q}{2\pi} \ln \frac{C_2}{r}$$

where C_2 is related to the **gauge (choice of zero potential point)**. □

2. The **error in the direct analogy** arises because in 2D, the **potential cannot be zero at infinity** as it is in 3D. The correct derivation involves **additional terms** and considerations that are not present in the 3D case.

Additional reading: about the 2D Green function from [5, Page 384-386]

The following content comes from [5, Page 384-386], we refer readers to [5, Page 384-386] or [7, Page 352] for details.

Let us consider the problem of G :

$$\Delta G = \partial_x^2 G + \partial_y^2 G = -\delta(r - r_0) \quad (4.3.23)$$

$$G|_{r=R} = 0. \quad (4.3.24)$$

An equivalent way to make the potential of the cylinder zero is that, in addition to the point source at M_0 , there is an image charge (an infinitely long line charge parallel to the z -axis with linear density ρ) at a certain point M_1 . According to symmetry analysis, M_1 should be on the extension of OM_0 . Let the potential at point M be

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{r_{M_0 M}} + \frac{\rho}{2\pi} \ln \frac{1}{r_{M_1 M}} + \frac{1}{2\pi} \ln \frac{r_0}{R} \leftarrow \boxed{\text{related to the zero point of the potential}} \quad (4.3.25)$$

- Note $r = \infty$ is not the zero potential for 2D case due to the potential $\frac{1}{2\pi}q \ln \frac{1}{r}$.

where

$$r_{M_0 M} = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)},$$

$$r_{M_1 M} = \sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_0)}.$$

Since the value of the potential is relative and can differ by a constant, the **third term on the right hand side of equation (4.3.25)** is an **additional constant** [this constant is added to satisfy the boundary condition (4.3.24) subsequently]. Now we use the boundary condition (4.3.24), that is, $G(P, M_0) = 0$, to determine the linear density ρ of the image charge and its position r_1 in equation (4.3.25). When $r \rightarrow R$, $M \rightarrow P$, $G(M, M_0) \rightarrow G(P, M_0) = 0$, and we have

$$\begin{aligned} G(P, M_0) &= \frac{1}{2\pi} \ln \frac{1}{r_{M_0 P}} + \frac{\rho}{2\pi} \ln \frac{1}{r_{M_1 P}} + \frac{1}{2\pi} \ln \frac{r_0}{R} \\ &= -\frac{1}{4\pi} \ln \left[R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0) \right] \\ &\quad - \frac{\rho}{4\pi} \ln \left[R^2 + r_1^2 - 2Rr_1 \cos(\theta - \theta_0) \right] + \frac{1}{2\pi} \ln \frac{r_0}{R} \\ &= 0 \end{aligned}$$

In the above formula, θ is a variable. From $\frac{\partial G(P, M_0)}{\partial \theta} = 0$, we obtain

$$\frac{r_0}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} + \frac{\rho r_1}{R^2 + r_1^2 - 2Rr_1 \cos(\theta - \theta_0)} = 0.$$

That is

$$r_0(R^2 + r_1^2) - 2r_0r_1R\cos(\theta - \theta_0) = -\rho r_1(R^2 + r_0^2) + 2\rho r_0r_1R\cos(\theta - \theta_0) \quad (4.3.26)$$

Equation (4.3.26) holds for any θ . Comparing the constant terms and the coefficients of $\cos(\theta - \theta_0)$ on both sides, we get

$$r_0(R^2 + r_1^2) = -\rho r_1(R^2 + r_0^2) \quad (4.3.27)$$

$$\rho = -1 \quad (4.3.28)$$

Substituting equation (4.3.28) into equation (4.3.27), we obtain

$$r_1 = \frac{R^2}{r_0}$$

Then equation (4.3.25) becomes

$$\begin{aligned} G(M, M_0) &= \frac{1}{2\pi} \ln \frac{1}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}} \\ &\quad - \frac{1}{2\pi} \ln \frac{1}{\sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_0)}} + \frac{1}{2\pi} \ln \frac{r_0}{R} \\ &= \frac{1}{4\pi} \ln \left[\frac{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_0)}{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)} \cdot \frac{r_0^2}{R^2} \right] \\ &= \frac{1}{4\pi} \ln \frac{r^2 r_0^2 + R^4 - 2rr_0 R^2 \cos(\theta - \theta_0)}{R^2 [r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)]} \end{aligned} \quad (4.3.29)$$

Equation (4.3.29) only contains the coordinates r_0, θ_0 of the point source and the coordinates r, θ of the observation point. It is the Green's function we seek. In particular, we can see that due to the introduction of the additional constant $\frac{1}{2\pi} \ln \frac{r_0}{R}$, equation (4.3.29) gives $G(P, M_0) = 0$ when $r \rightarrow R$, satisfying the boundary condition (4.3.24).

On the circle $x^2 + y^2 = R^2$,

$$\frac{\partial G}{\partial n} \Big|_C = \frac{\partial G}{\partial r} \Big|_{r=R} = -\frac{1}{2\pi R} \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos \gamma}$$

Therefore, from (4.3.19), the integral expression for the solution of problems (4.3.20) and (4.3.21) is

$$u(M_0) = \frac{1}{2\pi R} \int_C f(x, y) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos \gamma} dS. \quad (4.3.30)$$

In polar coordinates, expression (4.3.30) becomes ($r_0 < R$)

$$u(r_0, \theta_0) = \frac{1}{2\pi} \int_0^{2\pi} f(R, \theta) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos \gamma} d\theta, \quad (4.3.31)$$

where $\cos \gamma = \cos(\theta - \theta_0)$.

Formulas (4.3.30) or (4.3.31) are called Poisson's formulas for the circular domain. Formula (4.3.31) is the same as formula in Section 2.3 of Chapter 2.

4.4 Trial and Error Method and Solution of Poisson's Equation

4.4.1 Trial and Error Method

For some well posed problems in practice, based on the **physical meaning** and **geometric characteristics** of the problems, we can assume that the solution has a certain form and substitute it for trial. This is called the **trial and error method**.

Ex 4.4.1. Find the electric potential in a capacitor made of two concentric spherical conductors $r = r_1$ and $r = r_2$, where the inner spherical surface $r = r_1$ maintains a constant potential v_0 and the outer spherical surface is grounded.

- **Trial Method:** Guessing should be based on
 - physical meaning,
 - geometric features,
 - the form of inhomogeneous terms.
- **Geometric Features:** For problems with boundary conditions independent of θ, ϕ , the solution may also be independent of θ, ϕ .
- **Spherical Symmetry:** Symmetric solutions—we have previously discussed spherical symmetric solutions in Section 4.1.
- **General Solution:** The general solution for symmetric cases is often of the form $\frac{C_1}{r} + C_2$. The fundamental solution can be chosen by setting $C_1 = 1$ and $C_2 = 0$ for simplicity.

Solution. Since the region is a spherical shell, it is more convenient to use spherical coordinates. In the spherical coordinate system, the above problem is reduced to

$$\begin{cases} \Delta u(r, \theta, \varphi) = 0 & (r_1 < r < r_2) \\ u|_{r=r_1} = v_0, u|_{r=r_2} = 0 \end{cases}$$

From the boundary conditions, we know that the distribution of the electric potential inside the sphere is only related to r , that is, the potential function is spherically symmetric. Using the general form of the spherically symmetric solution in Section 4.1, we can assume

$$u(r) = \frac{A}{r} + B$$

where A and B are undetermined constants. To determine A and B , from the boundary conditions, we have

$$\begin{aligned} \frac{A}{r_1} + B &= v_0, & \frac{A}{r_2} + B &= 0 \\ A &= \frac{r_1 r_2}{r_2 - r_1} v_0, & B &= -\frac{r_1}{r_2 - r_1} v_0 \end{aligned}$$

So the required electric potential is

$$u = \frac{r_1 r_2}{r_2 - r_1} \left(\frac{1}{r} - \frac{1}{r_2} \right) v_0$$

Ex 4.4.2. Suppose there is an infinitely long homogeneous cylinder of radius R . Given that the temperature distribution on its cylindrical surface is xy , find the steady - state temperature distribution inside the cylinder.

Solution. Since the given temperature on the cylindrical surface is independent of z , the circular discs perpendicular to the z -axis have the same temperature distribution. Therefore, the given spatial problem can be reduced to a planar problem. Because the boundary shape is a circle, it is more convenient to use polar coordinates. So we solve the problem in the polar coordinate system:

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, & (r < R) \\ u|_{r=R} = \frac{1}{2} R^2 \sin 2\theta. \end{cases} \leftarrow \boxed{\text{Guessing according to the inhomogeneous term}}$$

Let its solution be $u(r, \theta) = Ar^2 \sin 2\theta + B$, where A and B are undetermined constants. It is easy to verify that the function $u(r, \theta) = Ar^2 \sin 2\theta + B$ satisfies the equation. Now, we determine the constants A and B from the boundary conditions. Since

$$u|_{r=R} = AR^2 \sin 2\theta + B = \frac{1}{2} R^2 \sin 2\theta$$

we get $A = \frac{1}{2}$ and $B = 0$. So the solution to the original problem is

$$u(r, \theta) = \frac{1}{2} r^2 \sin 2\theta, \text{ i.e., } u(x, y) = xy$$

Ex 4.4.3. Find the electric potential u in the cylindrical domain $r < R$ such that the normal component of the electric field strength on the cylindrical surface is given as:

$$\frac{\partial u}{\partial n} \Big|_{r=R} = (x + y)|_{r=R}$$

Solution. This problem can be reduced to

$$\begin{cases} \Delta u(r, \theta) = 0 & (r < R) \\ \frac{\partial u}{\partial n} \Big|_{r=R} = \underbrace{\frac{\partial u}{\partial r}}_{\substack{\text{may come from the constant } A, B \text{ or } r}} \Big|_{r=R} = R \cos \theta + R \sin \theta \end{cases} \leftarrow \boxed{\text{the second boundary}}$$

Let the solution of this problem be $u(r, \theta) = Ar \cos \theta + Br \sin \theta + C$, where A , B , and C are specific constants. It is obvious that this function satisfies the equation. To make it satisfy the boundary conditions, from the boundary conditions, we have

$$\frac{\partial u}{\partial n} \Big|_{r=R} = \frac{\partial u}{\partial r} \Big|_{r=R} = A \cos \theta + B \sin \theta = R \cos \theta + R \sin \theta$$

Thus, $A = B = R$. So the solution to the original problem is

$$u(r, \theta) = Rr(\cos \theta + \sin \theta) + C \leftarrow \boxed{\text{The solution is not unique for the second boundary}}$$

- **Boundary Conditions:** The variable R could originate from r^2 or a constant term A, B , making it challenging to ascertain its origin.
- **Setting Up the Solution:** One might initially try solutions of the form $R^2 \cos(\theta)$ or $R^2 \sin(\theta)$, but these may not yield a solution. A more general approach is to assume a solution of the form $Ar^\alpha \cos(\theta) + Br^\alpha \sin(\theta) + C$, where α is a constant to be determined.
- **Determining Parameters:** By substituting the assumed solution into the boundary conditions and the differential equation, one can determine the parameters A, B , and α .
- **Non-uniqueness of Solutions:** The presence of an arbitrary constant C in the solution indicates that the solution is not unique, as C can take any real value. This implies that the second type of boundary value problem may have infinitely many solutions. The non-uniqueness is because the problem does **not involve the function u itself** but only its derivatives.
- **Well-posed Problems:** The concept of well-posed problems is discussed. For the Laplace equation, the first type of boundary conditions ensures a unique solution, while the second type may not. The third type will be shown to have a unique solution through an example later.
- **Conclusion:** The second type of boundary conditions for the Laplace equation may lead to an ill-posed problem, which is a point of attention.

4.4.2 Solution of Poisson's Equation

If we know a particular solution of Poisson's equation, then through a **function substitution**, we can transform the boundary value problem of **Poisson's equation** into a boundary value problem of **Laplace's equation**—*Find a function to carry the inhomogeneous term, the rest part is a harmonic function.*

If the **free term** in Poisson's equation is an n -th degree polynomial of the independent variables, then we can take a **particular solution** of the equation as an $(n+2)$ -th degree polynomial of the independent variables. Substitute it into Poisson's equation and compare the coefficients of the corresponding terms on both sides of the equation to determine the constants.

Ex 4.4.4. Find a particular solution of the equation $u_{xx} + u_{yy} = xy$.

Solution. Since $f(x, y) = xy$ is a quadratic polynomial of the independent variables x and y , for the convenience of calculation, we may take the particular solution as

$$w(x, y) = Ax^3y + Bxy^3$$

Substitute it into the equation, we get

$$6(A + B)xy = xy \implies 6(A + B) = 1$$

$$\begin{cases} B = 0, A = \frac{1}{6}, & w(x, y) = \frac{1}{6}x^3y \\ A = 0, B = \frac{1}{6}, & w(x, y) = \frac{1}{6}xy^3 \\ A = B = \frac{1}{12}, & w(x, y) = \frac{1}{12}xy(x^2 + y^2) \end{cases}$$

Ex 4.4.5. Find the solution of the following problem

$$\begin{cases} \Delta u(x, y) = -4 & (x^2 + y^2 < a^2) \\ u|_{x^2+y^2=a^2} = 0 \end{cases}$$

Solution. Obviously, a particular solution of Poisson's equation is

$$w(x, y) = -(x^2 + y^2).$$

Make a function substitution $u = v(x, y) - (x^2 + y^2)$, then the above problem is transformed into

$$\begin{cases} \Delta v(x, y) = 0 & (x^2 + y^2 < a^2) \\ v|_{x^2+y^2=a^2} = a^2 \end{cases} \leftarrow \boxed{\text{Consider the boundary } x^2 + y^2 = a^2}$$

By the extreme value principle, the solution of the above problem is $v(x, y) = a^2$, that is, the solution of the original problem is

$$u(x, y) = a^2 - (x^2 + y^2).$$

Ex 4.4.6. Solve

$$\begin{cases} \Delta u(x, y, z) = -6 & (x^2 + y^2 + z^2 < a^2) \\ u|_{x^2+y^2+z^2=a^2} = a^2 \end{cases}$$

Solution. Obviously, we can take $w(x, y, z) = -(x^2 + y^2 + z^2)$. Let $u = v(x, y, z) - (x^2 + y^2 + z^2)$. Then the above problem can be transformed into

$$\begin{cases} \Delta v(x, y, z) = 0 \\ v|_{x^2+y^2+z^2=a^2} = 2a^2 \end{cases}$$

By the extreme - value principle, the solution of this problem is $v(x, y, z) = 2a^2$, so the solution of the original problem is

$$u(x, y, z) = 2a^2 - (x^2 + y^2 + z^2).$$

External Dirichlet Problem

Given a continuous function $f(x, y, z)$ on a closed surface Γ in the space (x, y, z) , we require the function $u(x, y, z)$ to satisfy Laplace's equation in the external region Ω^c of Γ (except for the **point at infinity**), be continuous on $\Omega^c \cup \Gamma$, and satisfy the conditions

$$\lim_{r \rightarrow \infty} u(x, y, z) = 0 \quad (r = \sqrt{x^2 + y^2 + z^2}) \leftarrow \boxed{\text{Infinity is a boundary}}$$

and the condition

$$u|_{\Gamma} = f(x, y, z)$$

The original Dirichlet problem is called the internal Dirichlet problem.

External Neumann Problem

Given a continuous function $f(x, y, z)$ on a smooth closed surface Γ in the space (x, y, z) , we require the function $u(x, y, z)$ to satisfy Laplace's equation in the external region Ω^c of Γ (except for the point at infinity), be continuous on $\Omega^c \cup \Gamma$, the normal derivative $\frac{\partial u}{\partial n'}$ exists at any point on Γ , and satisfy the conditions

$$\lim_{r \rightarrow \infty} u(x, y, z) = 0 \quad (r = \sqrt{x^2 + y^2 + z^2}) \quad (33)$$

and the condition

$$\left. \frac{\partial u}{\partial n'} \right|_{\Gamma} = f(x, y, z)$$

where n' is the direction of the **inner normal** of Γ .

4.5 Exercise Class

Ex 4.5.1. Prove that the following functions are all harmonic functions.

1. $x^3 - 3xy^2$;
2. $r^n \cos n\theta$.

Solution. We only need to verify that they satisfy Laplace's equation $\Delta u = 0$.

1. Let $u = x^3 - 3xy^2$. Using Laplace's equation in **Cartesian coordinates**:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

First, calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x, & \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(-6xy) = -6x \\ &\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.} \end{aligned}$$

2. Let $u = r^n \cos n\theta$. Using Laplace's equation in **polar coordinates**:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

First, calculate the partial derivatives:

$$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (nr^n \cos n\theta) = n^2 r^{n-2} \cos n\theta \quad (4.5.1)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= -r^n n^2 \cos n\theta \Rightarrow \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -n^2 r^{n-2} \cos n\theta \\ &\Rightarrow (4.5.1) + (4.5.2) = 0 \Rightarrow u \text{ is harmonic.} \end{aligned} \quad (4.5.2)$$

Ex 4.5.2. Use the first Green's formula to prove the uniqueness of the solution of the Robin boundary value problem for the three dimensional Laplace equation:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, & (x, y, z) \in \Omega \\ \left(\frac{\partial u}{\partial n} + \sigma u \right) |_{\partial\Omega} = f, & \sigma > 0 \text{ is a constant} \end{cases} \leftarrow \boxed{\text{The third boundary}}$$

- **Uniqueness Theorem:** The basic idea behind all uniqueness theorems is the same: assume there are two distinct solutions and prove that their difference is identically zero.
- **Third Type Boundary Conditions:** Unlike the first type, where the maximum principle can be applied, the third type boundary conditions do not allow for the use of the maximum principle directly.
- **Energy Method replace the extreme principle:** Use the energy method (similar to the one discussed in Chapter 2) to prove uniqueness.
 - Multiply the equation by w ;
 - Integrate;
 - **Integration by Parts:** Apply integration by parts to transform the integral into boundary terms and an internal term involving Δw .

Proof. We only need to prove that the homogeneous problem corresponding to $f = 0$ has only the zero solution. Suppose there are two solutions $u_1 \neq u_2$, then

$$\begin{cases} \Delta u_1 = 0, & \left(\frac{\partial u_1}{\partial n} + \sigma u_1 \right) |_{\partial\Omega} = f \\ \Delta u_2 = 0, & \left(\frac{\partial u_2}{\partial n} + \sigma u_2 \right) |_{\partial\Omega} = f \end{cases} \implies \begin{cases} \Delta(u_1 - u_2) = 0 \\ \left[\frac{\partial(u_1 - u_2)}{\partial n} + \sigma(u_1 - u_2) \right] |_{\partial\Omega} = 0 \end{cases}$$

Let $u = u_1 - u_2$, then

$$\begin{cases} \Delta u = 0 \\ \left(\frac{\partial u}{\partial n} + \sigma u \right) |_{\partial\Omega} = 0 \end{cases}$$

By the first Green's formula (or Gauss's formula) \leftarrow the higher dimensional version of "Integration by Parts",

$$\int_{\Omega} v \Delta u d\Omega = \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS - \int_{\Omega} \nabla v \cdot \nabla u d\Omega.$$

Let $v = u$, we get

$$\int_{\partial\Omega} u \frac{\partial u}{\partial n} dS = \int_{\Omega} |\nabla u|^2 d\Omega.$$

Using $\left(\frac{\partial u}{\partial n} + \sigma u \right) |_{\partial\Omega} = 0$, we have

$$\int_{\Omega} |\nabla u|^2 d\Omega + \sigma \int_{\partial\Omega} u^2 dS = 0 \Rightarrow \int_{\Omega} |\nabla u|^2 d\Omega = \int_{\partial\Omega} u^2 dS = 0.$$

$\Rightarrow u \equiv \text{constant}$. Since $u \equiv 0$ on $\partial\Omega \Rightarrow u \equiv 0$ in $\bar{\Omega}$. □

Ex 4.5.3. If $u(r, \theta)$ is a **harmonic function** on the **unit circle** and $u(1, \theta) = \cos \theta$, find the value of u at the origin.

- “**harmonic function**”+“**unit circle**”→ “**the mean value theorem**”!

Solution. By the mean value theorem for harmonic functions, (let C denote the unit circle)

$$u(0, 0) = \frac{1}{2\pi} \int_C u(r, \theta) ds = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta d\theta = 0$$

due to the orthogonality of trigonometric functions.

Ex 4.5.4. Let u be harmonic in the region $\Omega \subset \mathbb{R}^3$, and the ball $B(a, R)$ with center a and radius R is a subset of Ω . Prove that for any $0 < \rho < R$, the following **ball average formula** holds:

$$u(a) = \frac{3}{4\pi\rho^3} \int_{B_\rho(a)} u(x, y, z) dx dy dz$$

The idea is inspired by

- the spherical average formula for harmonic function on a spherical surface;
- Tools for transforming spherical surface integrals into spherical volume integrals:
 - the onion peeling integration:

$$\int_{B_r^{M_0}} f dV = \int_{B_r^{M_0}} f r^2 \sin \theta dr d\theta d\varphi = \int_0^R \int_{S_r^{M_0}} f \underbrace{r^2 \sin \theta d\theta d\varphi}_{dS \text{ (surface element)}} = \int_0^R \int_{S_r^{M_0}} f dS$$

- Gauss's theorem.

Here, obviously only the onion peeling integration can be used

Solution. By the spherical average formula, we have:

$$\int_{B_\rho(a)} u(x, y, z) dx dy dz = \int_0^\rho \left(\int_{\Gamma_r(a)} u(x, y, z) dS \right) dr = \int_0^\rho 4\pi r^2 u(a) dr = \frac{4}{3}\pi\rho^3 u(a).$$

So $u(a) = \frac{3}{4\pi\rho^3} \int_{B_\rho(a)} u(x, y, z) dx dy dz$.

Ex 4.5.5. Let $u(x)$ be harmonic in the disk B_R centered at the origin with radius R , and continuous on $\overline{B_R}$. Denote $M = \int_{B_R} u^2 d\Omega$. Prove that:

$$|u(0)| \leq \frac{1}{R} \left(\frac{M}{\pi} \right)^{\frac{1}{2}}.$$

Note

$$\underbrace{|u(0)|}_{\text{value of the center of } B_R} \leq \frac{1}{R} \underbrace{\left(\frac{M}{\pi}\right)^{\frac{1}{2}}}_{\text{the mean value of } u^2 \text{ on } B_R}.$$

- Related to the mean value theorem for harmonic functions;
- Need to build a relation $\int u^2 \rightarrow \int u$. The tools for this goal are:

– The **Cauchy-Schwarz inequality**:

$$(\sum_k a_k b_k)^2 \leq \sum_k a_k^2 \sum_k b_k^2 \Leftrightarrow \langle \mathbf{a}, \mathbf{b} \rangle \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|.$$

– The **Hölder inequality** (A continuous version of the **Cauchy-Schwarz inequality**):

$$\left(\int f g \, dx\right)^2 \leq \int f^2 dx \int g^2 dx \Leftrightarrow \langle f, g \rangle \leq \|f\| \cdot \|g\|.$$

Proof. By the **mean value formula** for harmonic functions (the mean value theorem),

$$u(0) = \frac{1}{|B_R|} \int_{B_R} u \, d\Omega = \frac{1}{|\partial B_R|} \int_{\partial B_R} u \, ds$$

Using the **Hölder inequality** $(\int f g \, dx)^2 \leq \int f^2 dx \int g^2 dx$, we have

$$u^2(0) = \frac{1}{|B_R|^2} \left(\int_{B_R} u \, d\Omega\right)^2 \leq \frac{1}{|B_R|^2} \int_{B_R} u^2 d\Omega \int_{B_R} d\Omega = \frac{M}{|B_R|}$$

Since $|B_R| = \pi R^2$, we get

$$|u(0)| \leq \left(\frac{M}{|B_R|}\right)^{\frac{1}{2}} = \frac{1}{R} \left(\frac{M}{\pi}\right)^{\frac{1}{2}}.$$

□

Ex 4.5.6. Determine whether the following problem has a solution:

$$\begin{cases} \Delta u = 1, & r < 1 \\ \frac{\partial u}{\partial n} = 0, & r = 1 \end{cases} \leftarrow \boxed{\text{the second type boundary}}$$

- **Recall of Second Type Boundary Conditions:** For harmonic functions, $\int_{\Gamma} \frac{\partial u}{\partial n} dS = 0$ This is a necessary condition derived using the flux theorem.
- **Necessity of Understanding Methods:** Not all theorems learned may be directly applicable. It's crucial to understand the underlying methods. The flux theorem was used to prove a proposition previously; a similar approach is needed here.

Solution. If u is a solution of $\Delta u = 1$, then by Gauss's formula, we have

$$\frac{4}{3}\pi = \int_{\Omega} \Delta u \, d\Omega = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS = 0$$

This is a contradiction. So the problem has no solution.

Ex 4.5.7. If $u = u(r, \theta)$ is a harmonic function, prove that ru_r is also a harmonic function, and thereby prove the solution of the second kind boundary value problem:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, & 0 < r < R \\ \left. \frac{\partial u}{\partial r} \right|_{r=R} = \varphi(\theta) \end{cases}$$

is given by $u(r, \theta) = -\frac{R}{2\pi} \int_0^{2\pi} \varphi(\beta) \ln(R^2 + r^2 - 2Rr \cos(\beta - \theta)) \, d\beta + C$ when $\int_0^{2\pi} \varphi(\theta) \, d\theta = 0$, where C is an arbitrary constant.

Proof. Since $u = u(r, \theta)$ is a harmonic function, we have

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Let $v = ru_r$. Then

$$\begin{aligned} \Delta v &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (ru_r) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (ru_r) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (ru_r) + \frac{\partial^2 u}{\partial \theta^2} \right] \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left[r^2 \left(\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \right] \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r^2 \Delta u) = 0 \end{aligned}$$

So v is also a harmonic function.

In this way, the second kind boundary value problem of u

$$\begin{cases} \Delta u = 0 \\ \left. \frac{\partial u}{\partial r} \right|_{r=R} = \varphi(\theta) \end{cases}$$

is transformed into the first kind boundary value problem of v

$$\begin{cases} \Delta v = 0 \\ v|_{r=R} = R\varphi(\theta) \end{cases}$$

Using Poisson's formula, we have

$$v(r, \theta) = \frac{R}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)\varphi(\beta)}{R^2 + r^2 - 2Rr \cos(\beta - \theta)} \, d\beta$$

Using the known condition $\int_0^{2\pi} \varphi(\theta) d\theta = 0$, we can rewrite v as

$$\begin{aligned} v(r, \theta) &= \frac{R}{2\pi} \int_0^{2\pi} \left(\frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\beta - \theta)} - 1 \right) \varphi(\beta) d\beta \\ &= -\frac{R}{\pi} \int_0^{2\pi} \frac{r^2 - Rr \cos(\beta - \theta)}{R^2 + r^2 - 2Rr \cos(\beta - \theta)} \varphi(\beta) d\beta \end{aligned}$$

Then the solution of the original problem is:

$$\begin{aligned} u(r, \theta) &= u(0, \theta) + \int_0^r \frac{v(\rho, \theta)}{\rho} d\rho \\ &= u(0, \theta) - \frac{R}{\pi} \int_0^{2\pi} \left(\int_0^r \frac{\rho - R \cos(\beta - \theta)}{R^2 + \rho^2 - 2R\rho \cos(\beta - \theta)} d\rho \right) \varphi(\beta) d\beta \\ &= u(0, \theta) - \frac{R}{2\pi} \int_0^{2\pi} \varphi(\beta) \ln \left(R^2 + r^2 - 2Rr \cos(\beta - \theta) \right) d\beta \end{aligned}$$

where $u(0, \theta)$ is an arbitrary constant. \square

Ex 4.5.8. 1. Let $r = \sqrt{x^2 + y^2}$, and $r_0 < 1$ be a positive constant. Denote $\Omega = \{(x, y) | r_0 < r < 1\}$, which is an annulus in the two dimensional plane. Given $a > 0$, if the function $u(x, y)$ satisfies:

$$\begin{cases} \Delta u = 0, & r_0 < r < 1 \\ u|_{r=1} = 0, & u|_{r=r_0} \leq a \ln \frac{1}{r_0} \end{cases}$$

Prove by the maximum principle or comparison principle that:

$$u(x, y) \leq a \ln \frac{1}{r} \quad \text{for any } (x, y) \in \Omega$$

- The extremum principle cannot be directly applied, as it yields $u \leq a \ln \frac{1}{r_0}$ (where r_0 is a constant), instead of the desired $u \leq a \ln \frac{1}{r}$ (where r is the variable).
- The problem involves comparing two functions.
- When discussing the comparison principle, it was mentioned that the extremum principle only applies to a single function.
- To handle two functions, construct a new function by taking their difference, and then apply the extremum principle.
- Note $a \ln \frac{1}{r}$ is a fundamental solution in the two-dimensional case satisfying $\Delta a \ln \frac{1}{r} = -2\pi a \delta(r - r_0)$.

2. If $u(x, y)$ is continuous in $\{(x, y) | 0 < r \leq 1\}$ and satisfies:

$$\begin{cases} \Delta u = 0, & 0 < r < 1 \\ u|_{r=1} = 0, & \lim_{r \rightarrow 0} \frac{u(x, y)}{\ln r} = 0 \end{cases}$$

Prove that:

$$u(x, y) \equiv 0, \quad 0 < r \leq 1$$

Proof of 1. Denote $w = a \ln \frac{1}{r} - u$. Then w satisfies:

$$\begin{cases} \Delta w = 0, & r_0 < r < 1 \\ w|_{r=1} = 0, & w|_{r=r_0} \geq 0 \end{cases}$$

By the maximum principle, $w \geq 0$, that is $u(x, y) \leq a \ln \frac{1}{r}$ when $r_0 \leq r \leq 1$. \square

- **Comparison:** Both questions appear similar, but the difference lies in the inner boundary conditions.
- **Inner Boundary:**
 - In the first scenario, the inner radius r_0 is fixed.
 - In the second scenario, the inner boundary radius r approaches zero.
- **Behavior at Zero:**
 - As $r \rightarrow 0$, $\ln r \rightarrow -\infty$.
 - The condition $r \rightarrow 0$ implies u approaches zero slower than $\ln r$.
- **Theorem Implication:**
 - If a function approaches zero slower than $\ln r$ on the boundary, it must be zero.
 - This is a **strong** property of harmonic functions.
- **Proof Strategy:**
 - Transform the problem into a form similar to Question 1.
 - Use the limit definition to express boundary conditions.
- **Boundary Behavior:**
 - The limit definition only applies to boundary points.

Proof of 2. (1) For any given $M = (x_1, y_1) \neq (0, 0)$, denote $r_1 = \sqrt{x_1^2 + y_1^2} \in (0, 1)$ (see Fig. 4.15).

(2) Since $\lim_{r \rightarrow 0} \frac{u(x, y)}{\ln r} = 0$ (this condition can be viewed as the the inner boundary, i.e., r and (x, y) are on the boundary! a series boundary condition satisfies this limit), for any sufficient small $\epsilon > 0$, there exists a sufficient small constant $\delta \in (0, r_1)$ such that when

$$0 < r \text{ (the radius of the inner boundary)} < \delta < r_1,$$

we have $\left| \frac{u(x, y)}{\ln r} \right| \leq \epsilon$, which is equivalent to $|u(x, y)| \leq \epsilon |\ln r| = -\epsilon \ln r = \epsilon \ln \frac{1}{r}$ (since $\ln r < 0$ when $r < 1$). That is, there exists an inner boundary radius $r = \delta_1 < \delta$ such that $|u(x, y)| \leq \epsilon \ln \frac{1}{\delta_1}$ for $\sqrt{x^2 + y^2} = \delta_1$.

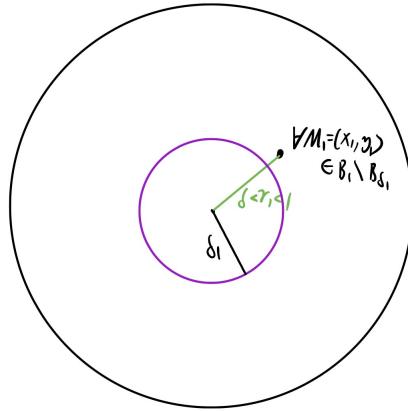


Figure 4.15: Example 1

(3) So we have

$$\begin{cases} \Delta u = 0, & \delta_1 < r < 1 \\ u|_{r=1} = 0, & u|_{r=\delta_1} \leq \epsilon \ln \frac{1}{\delta_1} \end{cases}$$

We can obtain, for any $r_1 > \delta_1$ (which means r_1 is in the annulus), $u(M) \leq \epsilon \ln \frac{1}{r_1}$ by 1. Similarly, we can get $u(M) \geq -\epsilon \ln \frac{1}{r_1}$, that is $|u(M)| \leq \epsilon \ln \frac{1}{r_1}$. Fix a r_1 in the annulus. Since ϵ can be arbitrarily small and $\frac{1}{r_1}$ is fixed, by the squeeze theorem, letting $\epsilon \rightarrow 0$, we can get $u(M) = 0$ for any $M \neq (0, 0)$.

- **Transformation to Series of Problems:**

- The problem can be transformed into a series of boundary value problems: If u solves the original problem

$$\begin{cases} \Delta u = 0, & 0 < r < 1 \\ u|_{r=1} = 0, & \lim_{r \rightarrow 0} \frac{u(x,y)}{\ln r} = 0 \end{cases}.$$

It is equivalent to, for any sufficiently small $\epsilon > 0$, there is a constant $\delta_1 > 0$, such that u satisfies

$$\begin{cases} \Delta u = 0 \\ u|_{r=r_0} = 0, & |u(x,y)|_{r=\delta_1} \leq \epsilon \ln \frac{1}{\delta_1} \end{cases}$$

- Each problem in the series has a parameter δ_1 .
- Some students are confused about $-\epsilon \ln \frac{1}{r_1} < u(x_1, y_1) < \epsilon \ln \frac{1}{r_1}$, they misunderstand that $\epsilon \rightarrow 0$ leads to $r_1 \rightarrow 0$, thinking $\epsilon \ln \frac{1}{r_1}$ is an indeterminate form $0 \cdot \infty$. In fact, $\epsilon \rightarrow 0$ leads to $\delta_1 \rightarrow 0$, but r_1 is an arbitrary fixed value in the annulus greater than δ_1 , and r_1 does not necessarily change with δ_1 (it is fixed in the annulus).
- Therefore, we emphasize M is in the annulus and thus r_1 is fixed for ϵ .

- This is why we use the result of the first question. If we do not use the strong conclusion $u(x, y) < \epsilon \ln \frac{1}{r_1}$ from the first question and directly use the maximum principle, we can only get $u(x, y) < \epsilon \ln \frac{1}{\delta_1}$. This conclusion is obviously weaker than $u(x, y) < \epsilon \ln \frac{1}{r_1}$. We cannot deduce $u(x, y) = 0$ from $u(x, y) < \epsilon \ln \frac{1}{\delta_1}$ because $\epsilon \ln \frac{1}{\delta_1}$ is an indeterminate form as $\epsilon \rightarrow 0$ and $\delta_1 \rightarrow 0$. We need the strong result $u(x, y) < \epsilon \ln \frac{1}{r_1}$ to deduce $u(x, y) = 0$ since $\epsilon \ln \frac{1}{r_1} \rightarrow 0 \times (\text{a fixed and bounded value } \ln \frac{1}{r_1}) = 0$ since M ($r = r_1$) is fixed in the annulus.

□

Ex 4.5.9. It is known that a harmonic function on the plane satisfies the mean value formula:

$$u(M) = \frac{1}{2\pi r} \int_{S_r^M} u \, dS$$

where S_r^M is a circle centered at M with radius r .

1. Prove that the function $u(x, y) = e^y \cos x$ is a harmonic function.
2. Use the mean value formula to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\sin t} \cos(\cos t) dt = 1$$

3. Write the Green's function for the half plane $\{(x, y) \in \mathbb{R}^2 | x - y > 0\}$. (Hint: The symmetric point of (x, y) with respect to the line $y = x$ is (y, x))

Proof. **Proof of 1:** $u_{xx} + u_{yy} = -e^y \cos x + e^y \cos x = 0$

Proof of 2: By the mean value formula,

$$1 = u(0, 0) = \frac{1}{2\pi} \int_{S_1^0} u(x, y) dS = \frac{1}{2\pi} \int_{S_1^0} e^y \cos x dS$$

Let $x = r \cos t, y = r \sin t$, then

$$1 = \frac{1}{2\pi} \int_{S_1^0} e^y \cos x dS = \frac{1}{2\pi} \int_0^{2\pi} e^{\sin t} \cos(\cos t) dt$$

Solution of 3: Denote $M_0 = (x_0, y_0), M_1 = (y_0, x_0), M = (x, y)$, then

$$\begin{aligned} G(M, M_0) &= \frac{1}{2\pi} \left(\ln \frac{1}{r_{MM_0}} - \ln \frac{1}{r_{MM_1}} \right) \\ &= \frac{1}{2\pi} \left(\ln \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} - \ln \frac{1}{\sqrt{(x - y_0)^2 + (y - x_0)^2}} \right) \end{aligned}$$

□

Ex 4.5.10. 1. Let $u(x, y, z)$ be harmonic in the cube $\Omega = \{(x, y, z) : |x| < 1, |y| < 1, |z| < 1\}$, continuous on $\bar{\Omega}$, and satisfy $u(x, y, z)|_{\Gamma} = e^{-(x^2+y^2+z^2)}$ on the boundary Γ of Ω . Prove that $e^{-3} < u(x, y, z) < e^{-1}$ in Ω .

2. Let u be a smooth function in \mathbb{R}^3 . If $\Delta u \geq 0$, then u is called **subharmonic**. Prove that the following two propositions are equivalent:

(a) u is subharmonic in \mathbb{R}^3 .

(b) For any closed sphere Γ_r , $\iint_{\Gamma_r} \frac{\partial u}{\partial n} dS \geq 0$ holds, where n is the unit outward normal vector of Γ_r .

Proof. **Proof of 1:** The maximum value of u on the boundary is e^{-1} , and the minimum value is e^{-3} . Obviously, u is not a constant. According to the maximum principle, the maximum and minimum values of u can only be achieved on the boundary. Therefore, $e^{-3} < u(x, y, z) < e^{-1}$.

Proof of 2:

- “ \Rightarrow ” Let V_r be the sphere enclosed by Γ_r . In the first Green's formula

$$\iiint_{V_r} v \Delta u \, d\Omega = \iint_{\Gamma_r} v \frac{\partial u}{\partial n} \, dS - \iiint_{V_r} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial v} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} \right) \, d\Omega$$

Take $v \equiv 1$, then

$$\iint_{\Gamma_r} \frac{\partial u}{\partial n} \, dS = \iiint_{V_r} \Delta u \, d\Omega \geq 0$$

- “ \Leftarrow ” Arbitrarily take a sphere V_r and denote its boundary as Γ_r . By the first Green's formula,

$$\iiint_{V_r} \Delta u \, d\Omega = \iint_{\Gamma_r} \frac{\partial u}{\partial n} \, dS \geq 0$$

Due to the arbitrariness of V_r and the continuity of u , we know that $\Delta u \geq 0$.

Note: Gauss's formula or the second Green's formula can also be used. □

Ex 4.5.11. 1. State the mean value theorem for harmonic functions.

2. If u and v are harmonic in $\Omega = \{(x, y), x^2 + y^2 < 1\}$, and satisfy

$u|_{\Gamma} = |xy|, \quad v|_{\Gamma} = x^3y - xy^3 \leftarrow \boxed{\text{One can change these two functions to obtain new exercises}}$

on the boundary Γ of Ω . Prove that $u(x, y) > v(x, y)$ for any $(x, y) \in \Omega$.

Proof. **Solution of 1:** Let the function $u(M)$ be harmonic in the region Ω , and M_0 be any point in Ω . If Γ_a is a sphere centered at M_0 with radius a , and this sphere is completely inside the region Ω , then

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\Gamma_a} u \, dS.$$

Proof of 2: Let $w = u - v$, then w is harmonic in Ω and on the boundary

$$w|_{\Gamma} = |xy| - x^3y + xy^3 \geq 0.$$

And obviously w is not a constant. By the maximum principle, $w > 0$ in Ω , so $u > v$. □

Ex 4.5.12. Denote $B_R(M_0)$ as the sphere in \mathbb{R}^3 centered at M_0 with radius R , and its boundary is denoted as $\Gamma_R(M_0)$.

1. If u is harmonic in $B_R(M_0)$, continuous on the boundary $\Gamma_R(M_0)$ and $u \geq 0$, let $M \in B_R(M_0)$, and denote $r = |M - M_0| < R$. Using the Poisson formula for a spherical domain

$$u(M) = \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(P)(R^2 - r^2)}{(R^2 + r^2 - 2Rr \cos \gamma)^{3/2}} dS$$

prove the following Harnack's inequality:

$$\frac{R(R-r)}{(R+r)^2} u(M_0) \leq u(M) \leq \frac{R(R+r)}{(R-r)^2} u(M_0)$$

2. Prove Liouville's theorem: If $u \geq 0$ is a harmonic function in \mathbb{R}^3 , then u is a constant.

Proof. **Proof of 1:**

$$u(M_0) = \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS$$

Since $u \geq 0$ (Ensure the direction of the inequality sign) and $(R-r)^2 \leq R^2 + r^2 - 2Rr \cos \gamma \leq (R+r)^2$, we have

$$\begin{aligned} u(M) &\leq \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(y)(R^2 - r^2)}{(R-r)^3} dS = \frac{R(R+r)}{(R-r)^2} \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS = \frac{R(R+r)}{(R-r)^2} u(M_0) \\ u(M) &\geq \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(y)(R^2 - r^2)}{(R+r)^3} dS = \frac{R(R-r)}{(R+r)^2} \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS = \frac{R(R-r)}{(R+r)^2} u(M_0) \end{aligned}$$

Proof of 2: Without loss of generality, assume that u has a lower bound (if u has an upper bound, then consider the function $-u$), that is, there exists $M \in \mathbb{R}^3$ such that $u \geq M$. Let $v = u - M$, then $v \geq 0$, and v is a harmonic function in \mathbb{R}^3 . For any $x \in \mathbb{R}^3$, $R > r = |x|$, by Harnack's inequality,

$$\frac{R(R-r)}{(R+r)^2} v(O) \leq v(M) \leq \frac{R(R+r)}{(R-r)^2} v(O).$$

Let $R \rightarrow +\infty$, then $v(M) = v(O)$. Due to the arbitrariness of x , v is a constant, and thus u is a constant. \square

Ex 4.5.13. Use the method of images to find the Green's function for the first quadrant $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$.

Proof. Omitted. See, for instance, [1]. \square

IDEAS: We hope

$$-\frac{1}{4\pi} \int_{\partial\Omega} (-B + A) dS = 0, \quad (*)$$

where two functions A, B to be determined.

Then

$$U(M_0) = -\frac{1}{4\pi} \int_{\partial\Omega} \left[U(M) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - B + \boxed{A - \frac{1}{r} \frac{\partial u}{\partial n}} \right] dS \quad (**)$$

have common term $\frac{\partial u}{\partial n}$ → take $A = 4\pi U \frac{\partial u}{\partial n}$ to eliminate $\frac{1}{r} \frac{\partial u}{\partial n}$

This requires

$$U|_P = \frac{1}{4\pi r_{MM_0}} \quad (1)$$

$$\text{since } A = 4\pi U \frac{\partial u}{\partial n}, (*) \Rightarrow \int_{\partial\Omega} -B + 4\pi U \frac{\partial u}{\partial n} dS = 0$$

|| In view of symmetries

|| Aesthetic criterion

take $B = 4\pi U \frac{\partial u}{\partial n}$

$$4\pi \int_{\partial\Omega} -u \frac{\partial v}{\partial n} + v \frac{\partial u}{\partial n} dS = 0$$

|| Green's second formula

$$4\pi \int_{\Omega} -u \Delta v + v \Delta u \stackrel{!}{=} 0$$

Green 1st function

We need v is also harmonic, i.e. $\Delta v = 0$! (2)

Therefore, we need take v as below

$$\begin{cases} \Delta v = 0 \\ v|_P = \frac{1}{4\pi r_{MM_0}} \end{cases}$$

$A, B \rightarrow (**)$

$$U(M_0) = - \int_{\partial\Omega} U(M) \frac{\partial}{\partial n} \left(\frac{1}{4\pi r} - v \right) dS$$

bdry condit. f

denoted as G
Green 2nd function.

Figure 4.16: Ideas of Green functions

5

Bessel Functions

- **Bessel Functions (Chapter 5):** Also known as **cylindrical functions**, primarily used for studying the **two-dimensional Laplace operator**.
- **Importance of Bessel Functions:**
 - Common in electrical engineering, especially in designing circuit boards with cylindrical capacitors.
- **Special Functions and PDEs:**
 - **Special functions** are introduced to **solve partial differential equations** (PDEs).
 - Bessel functions are directly linked to the **method of separation of variables**.
- **Separation of Variables Method (SVM for short):**
 - Used in solving PDEs, involves solving ordinary differential equations (ODEs) that arise in the process.
 - In higher dimensions, new types of ODEs appear, which are solved using **Bessel functions**.
- **Bessel Equations and Functions–Revise the third step (S-L problem) of SVM:**
 - The solutions to these new ODEs (**Bessel equations**) are Bessel functions.
 - **Bessel functions** play a role similar to **trigonometric functions** in Chapter 2.
- **Initial Value Problems and Coefficients–Revise the fifth step of SVM:**
 - After introducing Bessel functions, further discussion is needed on how they are applied in solving initial value problems and determining coefficients.

Motivation and Problem Statement:

1. (Old) Due to limitations of the separation of variables method in Chapter 2 ($1+1 = 2$ dimensions), for higher dimensions, special functions need to be introduced.
2. (New) In practical applications, problems such as:
 - 3-dimensional Newtonian gravitational potential (Poisson's equation) or the distribution of electric potential
 - Electromagnetic waves in cylindrical waveguides
 - Vibration phenomena of drum membranes or speaker diaphragms in headphones
 - Heat conduction phenomena in cylindrical objects
3. **Nature of the problem:** Extension from low dimensional to high dimensional cases.

New Problems and Solutions:

- **Analogous extension:** Try to extend the separation of variables method for $1 + 1 = 2$ dimensions and see where problems occur, and attempt to remedy them.
- **Problems encountered in the analogy:**
 - The number of variables increases.
 - When separating variables successively, multiple ordinary differential equations (ODEs) will appear, and new Sturm-Liouville (S-L) problems will be encountered (the original separation of variables method has problems in solving S-L problems).
- **Solution:** Study and solve the new S-L problems for ODEs, replace the solution of S-L problems in the separation of variables method, obtain a new system of eigenfunctions, and study their properties to determine coefficients.

Review and outline of this Chapter:

- **Chapter Outline:** Understanding the structure of this chapter is crucial for grasping the content of Sections 5.1–5.3.
- **Chapter Basis:** Builds upon the five-step separation of variables method.
- **Unchanged Steps:** The overall steps of the separation of variables method remain the same.
- **Modification Goals (Sections 5.1-5.3):** Aim to adjust parts of the steps to facilitate solving higher-dimensional cases.

- **Specific Steps to Modify:** Only the third and fifth steps of the separation of variables method need alteration.
- **Basic steps of the separation of variables method**
 1. First, assume $u(x, t) = X(x)T(t)$.
 2. Transform the partial differential equation (PDE) into ordinary differential equations (ODEs).
 3. **(Revised) Solve the ODEs: Use the boundary conditions to solve the Sturm-Liouville (S-L) problem;** substitute the obtained eigenvalues λ_n into the $T(t)$ equation.
 - **Bessel Equation Emergence:** A Bessel equation, an ordinary differential equation (ODE), appears as a Sturm-Liouville (S-L) problem.
 - **Section 5.1:** Introduces Bessel equations and their solutions, known as Bessel functions, to solve the **new S-L Problem** (see Table 5.1).
 - **Solution Role:** The series of Bessel functions serve a role analogous to the trigonometric functions series, acting as eigenfunctions in the solution process.
 4. Superpose the series solution $u(x, t)$.
 5. **(Revised) Determine the coefficients a_n, b_n from the initial values.**
 - **Coefficient Determination (Step 5):** Involves using inner products to find coefficients, previously used with trigonometric orthogonality.
 - **Zeros, Orthogonality, Norms and completeness (leading to the formula of the coefficients, Section 5.3):** For Bessel functions, orthogonality and norms must be established, which is the focus of Section 5.3.
 - **Recurrence Relations (Section 5.2):** Teaches how to calculate integrals involving Bessel functions in the formula of the coefficients, essential for determining coefficients. That is, by employing recurrence relations, the complexity of integrating higher-order Bessel functions is systematically reduced to more manageable, lower-order integrals.
 - * **Integration and Differentiation Relationship:** Integration and differentiation are inverse processes.
 - * **Use of Trigonometric Derivatives in Integration:** When integrating trigonometric functions, we often use their derivatives, such as $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\sin x) = \cos x$, which are crucial for integration by parts.
 - * **Recurrence Relations in Trigonometry:** These derivatives are a form of recurrence relations, showing how derivatives of trigonometric functions cycle back to the original function with a sign change after two derivatives.
 - * **Recurrence Relations in Bessel Functions:** Bessel functions also exhibit recurrence relations, where the derivative of a Bessel function relates to other Bessel functions of different orders.
 - * **Application in Integration:** These recurrence relations are useful in integration, allowing us to transform integrals involving Bessel functions into more manageable forms.

* **Integral Calculation Using Recurrence Relations:** By understanding how derivatives of Bessel functions relate to each other, we can simplify the process of integrating these functions.

- **Connection Between Sections:**

- **Chapter Overview:** This chapter focuses on modifying steps in the separation of variables method for solving higher-dimensional PDEs.
 - **Sections 5.1-5.3:** These sections explain how to adjust steps 3 and 5 of the separation of variables method. Sections 5.1, 5.2, and 5.3 are interconnected, with 5.1 solving for Bessel functions, 5.2 for integrals, and 5.3 for orthogonality.
 - **Application to PDEs:** Section 5.4 integrates the knowledge from the first three sections to solve PDEs, using the modified five-step method, including non-homogeneous equations and non-homogeneous boundary conditions.
 - **Auxiliary Function Method:** Not covered in detail in lectures but included in exercises and homework; involves finding a function to handle boundary conditions, reducing the problem to separation of variables or eigenfunction methods.
 - **Eigenfunction Method:** Covered with an example; slightly more complex than previously studied due to non-homogeneous conditions.
- Review the process of the separation of variables method for Laplace's equation in a circular domain and its system of eigenfunctions: $1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots, \cos n\theta, \sin n\theta, \dots$

- **Methodology for Special Functions:**

- **Analogous Approach:** The methodology for studying other special functions is similar to that used for Bessel functions.
- **Applicability of Chapter's Framework:** The approach outlined in this chapter can be similarly applied to the study of other special functions.

When using the **separation of variables method** to solve the boundary value problems of other partial differential equations, boundary value problems of other forms of ordinary differential equations will also be derived, thus introducing various systems of **coordinate functions**. These systems of **coordinate functions** are what people usually call **special functions**. In this chapter, we will derive the Bessel equation through separation of variables for boundary value problems in the **cylindrical coordinate system**; then discuss the solution method of this equation and the properties of its solutions; finally, introduce some applications of Bessel functions in solving problems of mathematical physics equations.

Chapter 2	Chapter 5
S-L problem: $X'' + \lambda X = 0$ with homogeneous boundary	S-L problem: Bessel equations with $F(R) = 0$ (Step 3; §5.1)
Eigenfunctions (basis): {trigonometric functions}	Eigenfunctions (basis): {Bessel functions} (Step 3; §5.1)
Infinite but countable eigenvalues; orthogonality; Fourier expansions	Infinite but countable eigenvalues; orthogonality; Fourier-Bessel expansions (Step 5; §5.3)
How to calculate the integral $\int f \sin$ to get the Fourier coefficient	How to calculate the integral $\int f \times$ Bessel function to get the Fourier-Bessel coefficient (Step 5; §5.2)

Table 5.1: Comparison of Bessel functions and trigonometric functions

5.1 Bessel Equation and Bessel Functions

5.1.1 Derivation of the Bessel Equation

When using the **separation of variables method** to solve the **vibration problem** of a **circular membrane** or the law of **instantaneous temperature distribution** on a thin **circular disk**, we will encounter the **Bessel equation**. Below, we will derive the Bessel equation taking the instantaneous temperature distribution of a circular disk as an example.

Consider a thin circular disk of radius R with adiabatic upper and lower surfaces. The temperature on the boundary of the disk is always kept at 0 degrees, and the initial temperature distribution is known. We want to find the law of instantaneous temperature distribution inside the disk. Let $u(x, y, t)$ represent the temperature at point (x, y) on the disk at time t .

This problem is reduced to solving the following boundary value problem:

$$u_t = a^2(u_{xx} + u_{yy}) \quad (0 < x^2 + y^2 < R^2) \quad (5.1.1)$$

$$u|_{x^2+y^2=R^2} = 0 \quad (5.1.2)$$

$$u|_{t=0} = \varphi(x, y) \quad (5.1.3)$$

Use the **separation of variables method** to solve this problem. Let $u(x, y, t) = V(x, y)T(t)$ and substitute it into equation (5.1.1), we get

$$VT' = a^2(V_{xx} + V_{yy})T.$$

Multiply both sides by $\frac{1}{a^2VT}$, we obtain

$$\frac{T'}{a^2T} = \frac{V_{xx} + V_{yy}}{V} = -\lambda, \quad (\lambda > 0) \leftarrow \boxed{\text{Why? Explain later.}}$$

Then we have

$$T' + \lambda a^2 T = 0 \quad (5.1.4)$$

$$V_{xx} + V_{yy} + \lambda V = 0 \leftarrow \boxed{\text{Helmholtz equation } \Delta V + \lambda V = 0} \quad (5.1.5)$$

Equation (5.1.4) is the **Helmholtz equation**.

- **General n -Dimensional Form:** The n -Dimensional Helmholtz equation is generally written as $\Delta V + \lambda V = 0$.
- **One-Dimensional Case:** For $n = 1$, the Helmholtz equation reduces to the familiar one-dimensional form $V'' + \lambda V = 0$, where $\lambda > 0$ has been proven using various methods. For instance, by **classification of λ** :
 - For $\lambda < 0$, the solution is a linear combination of exponentials.
 - For $\lambda = 0$, the solution is linear, which does not satisfy non-trivial boundary conditions.
 - Therefore, λ must be greater than zero.
- **Proof of $\lambda > 0$:** We have demonstrated that $\lambda > 0$ using various methods for one-dimensional case, with the **most robust method** being the **energy method**.
- **Proof Without General Solution:** In higher dimensions without a general solution formula, we use the energy method to prove $\lambda > 0$.
- **Energy Method:**
 - **Multiply** the equation by V
 - **Integrate** over the domain.
 - Apply **integration by parts** (Green's first identity in higher dimensions).
 - The **boundary terms** imply that λ must be positive for non-trivial solutions.

The solution of equation (5.1.4) is

$$T(t) = Ae^{-\lambda a^2 t}$$

From the boundary condition (5.1.2), we have

$$T(t)V(x, y)|_{x^2+y^2=R^2} = 0$$

That is

$$V|_{x^2+y^2=R^2} = 0. \quad (5.1.6)$$

To find the non zero solutions of equation (5.1.5) that satisfy condition (5.1.6),

$$\begin{cases} V_{xx} + V_{yy} + \lambda V = 0 \\ V|_{x^2+y^2=R^2} = 0 \end{cases}$$

We use the polar coordinate system in the plane, and then the boundary value problem becomes

$$\frac{\partial^2 \bar{V}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{V}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{V}}{\partial \theta^2} + \lambda \bar{V} = 0 \quad (0 < r < R), \quad (5.1.7)$$

$$\bar{V}|_{r=R} = 0. \quad (5.1.8)$$

Why $\lambda > 0$?

Multiply equation (5.1.5) by V on both sides, we get

$$VV_{xx} + VV_{yy} + \lambda V^2 = 0$$

that is $V\Delta V + \lambda V^2 = 0$. Then integrating it yields

$$\iint_D (V\Delta V + \lambda V^2) d\sigma = 0,$$

where $D = \{(x, y) | x^2 + y^2 \leq R^2\}$ and $\partial D = \{(x, y) | x^2 + y^2 = R^2\}$. By **Gauss's formula** or **Green's first formula**, we have

$$\underbrace{\oint_{\partial D} V \frac{\partial V}{\partial n} ds}_{\text{Due to the boundary condition}} - \iint_D [|\nabla V|^2 - \lambda V^2] d\sigma = 0.$$

Then

$$\iint_D [|\nabla V|^2 - \lambda V^2] d\sigma = 0. \quad (5.1.9)$$

Then

$$\lambda = \frac{\iint_D |\nabla V|^2 d\sigma}{\iint_D V^2 d\sigma} \geq 0.$$

Thus, $\lambda \geq 0$ or $V \equiv 0$.

If $\lambda = 0$, then

- (Method 1) by (5.1.9), $\nabla V \equiv 0 \implies V \equiv \text{Constant}$. With the help of the boundary condition, $V \equiv 0$.
- (Method 2) the boundary value problem becomes $\Delta V = 0$. From the boundary condition $V|_{\partial D} = 0$, according to the **extreme value theorem**, $V \equiv 0$. So there is only the trivial solution when $\lambda = 0$.

Let $\bar{V}(r, \theta) = F(r)G(\theta)$ and substitute it into equation (5.1.7), we get

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' + \lambda FG = 0 \leftarrow \boxed{\text{If } \lambda = 0, \text{ it reduces to the Laplace equation (see Chapter 2)}}$$

- The following derivation, if $\lambda = 0$, reduces to the separation of variables method for the Laplace equation on a circular domain as discussed in Chapter 2, involving **boundedness conditions** at the center and **periodicity conditions**. We have highlighted the changes in red font.

Multiply both sides by $\frac{r^2}{FG}$ and rearrange the terms, we obtain (it is similar to the procedures of the Laplace equation, see Chapter 2)

$$\frac{G''}{G} = -\frac{r^2 F'' + rF' + \lambda r^2 F}{F} = -\mu \leftarrow \boxed{\text{taking minus sign since it leads to the familiar } G'' + \mu G = 0}$$

Then we have

$$G'' + \mu G = 0, \quad (5.1.10)$$

$$r^2 F'' + rF' + (\lambda r^2 - \mu)F = 0. \leftarrow \boxed{\text{If } \lambda = 0, \text{ it becomes Euler equation}} \quad (5.1.11)$$

Since the temperature function $u(x, y, t)$ is single valued, $V(x, y)$ must also be a single valued function, that is, $\bar{V}(r, \theta) = \bar{V}(r, \theta + 2\pi)$.

$$\implies G(\theta) = G(\theta + 2\pi) \leftarrow \boxed{\text{Periodic condition}}$$

Solving the boundary value problem of the ordinary differential equation

$$G'' + \mu G = 0, \quad G(\theta) = G(\theta + 2\pi)$$

we can obtain

$$\mu = n^2 \quad (n = 0, 1, 2, \dots)$$

$$G_0(\theta) = \frac{1}{2}a_0 \quad G_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \quad (n = 1, 2, \dots)$$

Substitute $\mu = n^2$ into equation (5.1.11), we get

$$\boxed{r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0,} \quad (5.1.12)$$

This equation is called the **n -th order Bessel equation**.

- The method for solving Euler's equation cannot be applied because $\lambda r^2 F$ is not well-behaved for $r = e^t$, and it cannot be transformed into an ordinary differential equation (ODE) with constant coefficients.

From the boundary condition (5.1.8) $\bar{V}|_{r=R} = 0$, we know that $\bar{V}(R, \theta) = F(R)G(\theta) = 0$.

$$\implies F(R) = 0$$

In addition, since the temperature on the disk is finite, especially at the center of the disk, we can obtain

$$|F(0)| < +\infty \leftarrow \boxed{\text{Boundness condition}}$$

Therefore, the final solution of the original boundary value problem is reduced to finding the eigenvalues and eigenfunctions of the problem

$$\begin{cases} r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0, \\ F(R) = 0, \quad |F(0)| < +\infty \end{cases} \leftarrow \boxed{\text{ODEs}}$$

- **Next Section Focus:** On solving ODEs using Power Series Method and Generalized Power Series Method.
- **Forgetting Physical Context:** From now on, we will focus solely on analyzing the Bessel equation, disregarding the physical background.
- **General Study of Bessel Equation:** We will study the general Bessel equation for **general values of n , not just integers**.
- **Eliminating the Parameter λ :** We will perform a transformation to simplify the equation by eliminating the parameter λ . Perform a variable rescaling to eliminate the parameter λ .

$$\underbrace{(\sqrt{\lambda})^2 r^2}_{x^2} \frac{d^2 F}{d(\underbrace{(\sqrt{\lambda})^2 r^2}_{x^2})} + \underbrace{(\sqrt{\lambda})r}_{x} \frac{dF}{d(\underbrace{(\sqrt{\lambda})r}_{x})} + (\underbrace{(\sqrt{\lambda})^2 r^2 - n^2}_{x^2}) F = 0$$

- **Importance of n :** n determines the **order** of the Bessel equation, which is crucial for different solutions.

If we let $x = \sqrt{\lambda}r$, and denote $F(r) = F\left(\frac{x}{\sqrt{\lambda}}\right) = y(x)$, then

$$F_r = y_x \cdot \sqrt{\lambda}, \quad F_{rr} = (y_{xx} \cdot \sqrt{\lambda}) \cdot \sqrt{\lambda} = y_{xx} \cdot \lambda$$

Substitute the above formulas into equation (5.1.12), we can get

$$x^2 y'' + xy' + (x^2 - n^2)y = 0. \quad (5.1.13)$$

Equation (5.1.14) is a second order linear ordinary differential equation with variable coefficients, and its solutions are called **Bessel functions**. (Sometimes they are called **cylindrical functions**).

Consider the Bessel equation

$$\underbrace{x^2 y'' + xy'}_{=x\partial_x(x\partial_x y)} + (x^2 - n^2)y = 0 \implies \partial_x(x\partial_x y) - \frac{n^2}{x}y + xy = 0 \quad (\text{S-L problem}) \quad (5.1.14)$$

↑ Memorize this useful operator as mentioned before

- The equation is closely related to the parameter n ; if n changes, the equation changes.
- The value of n determines the form of the solution, specifically the Bessel function, making it a crucial parameter.
- After normalization, it is found that the Bessel equation only contains the parameter n .
- λ is not a parameter.

5.1.2 Bessel Functions

According to the theory of solutions of differential equations, equation (5.1.14) has a **generalized power series solution** in the following form:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{s+k} \quad (a_0 \neq 0), \quad (5.1.15)$$

where s is a constant.

- The power series method is inspired by the resemblance of the Bessel equation to the Euler equation, but the presence of an x^2y term in the Bessel equation makes the Euler equation's solution methods inapplicable.
- The Euler equation has two solution methods:
 - Variable transformation: Replace x with e^t to convert derivatives with respect to x into derivatives with respect to t , making coefficients constant.
 - Trial method: Guess the solution as $y = x^s$ and determine s by substituting it into the equation.
- For the Bessel equation, the trial method fails due to the x^2y term, which cannot be simplified by the usual substitution.
- The idea of using a power series comes from the observation that the solution should be related to polynomials, but a simple polynomial form is insufficient.
 - (**Ideas**) In a manner similar to the Euler equation, if we assume a solution of the form x^n , substituting it into the differential equation typically produces an extra term involving x^{n+2} . To eliminate this term, we introduce a new term proportional to x^{n+2} as part of the solution. However, this modification in turn generates an x^{n+4} term. Repeating this process iteratively, we are led to introduce successively higher powers of x . This recursive structure suggests that the solution should naturally take the form of a power series in x .
 - The idea of assuming the solution as a power series is also analogous to the method of eigenfunctions. In this context, the set of monomials $\{x^n\}$ can be regarded as a system of basis functions.
- The power series method assumes the solution is a power series of the form $\sum_{n=0}^{\infty} a_n x^n$ and attempts to determine the coefficients a_n .
- However, for the Bessel equation, one would find that the power series method alone cannot determine the coefficients a_n .
- The **generalized power series method** is introduced:
 - It assumes the solution starts from $a_0 x^s$ instead of a_0 , i.e., $\sum_{k=0}^{\infty} a_k x^{k+s}$.

- For an ordinary differential equation (ODE) whose coefficients are free of singularities and are analytic, the power series method can be used to solve it (see ODE references, e.g., [6]).
- If the coefficients have a second order singularity (a regular singularity), the generalized power series method can be used to solve it (see ODE references, e.g., [6]).
- The coefficients of the Bessel equation have a regular singularity, and the generalized power series method can be used to solve it.

Next, we will determine s and a_k ($k = 0, 1, 2, \dots$). To this end, substitute (5.1.15) and

$$y' = \sum_{k=0}^{\infty} a_k (s+k)x^{s+k-1}, \quad y'' = \sum_{k=0}^{\infty} a_k (s+k-1)(s+k)x^{s+k-2}$$

into equation (5.1.14). We can obtain

$$\sum_{k=0}^{\infty} a_k (s+k-1)(s+k)x^{s+k} + \sum_{k=0}^{\infty} a_k (s+k)x^{s+k} - n^2 \sum_{k=0}^{\infty} a_k x^{s+k} + \sum_{k=0}^{\infty} a_k x^{s+k+2} = 0,$$

Let $k+2 = l \Rightarrow k = l-2$. Then,

$$\sum_{k=0}^{\infty} a_k x^{s+k+2} = \sum_{l=2}^{\infty} a_{l-2} x^{s+l} = \sum_{k=2}^{\infty} a_{k-2} x^{s+k}$$

k is a dummy variable, similar to the variable in integration, which can be replaced.

$$\begin{aligned} & \sum_{k=0}^{\infty} a_k [(s+k-1)(s+k) + (s+k) - n^2] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0, \\ & \sum_{k=0}^{\infty} a_k [(s+k)^2 - n^2] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0, \\ & (s^2 - n^2)a_0 x^s + [(s+1)^2 - n^2] a_1 x^{s+1} + \sum_{k=2}^{\infty} a_k [(s+k)^2 - n^2] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0, \\ & (s^2 - n^2)a_0 x^s + [(s+1)^2 - n^2] a_1 x^{s+1} + \sum_{k=2}^{\infty} \{([(s+k)^2 - n^2] a_k + a_{k-2}\} x^{s+k} = 0 \end{aligned}$$

By comparing the coefficients (due to the basis functions $\{x^{s+k}\}$ are linearly independent) on both sides of the above formula, we have **indicial equations**

$$(s^2 - n^2)a_0 = 0, \tag{5.1.16}$$

$$[(s+1)^2 - n^2] a_1 = 0, \tag{5.1.17}$$

$$[(s+k)^2 - n^2] a_k + a_{k-2} = 0 \quad (k = 2, 3, \dots) \tag{5.1.18}$$

Since $a_0 \neq 0$, from (5.1.16) we can get $s_1 = n, s_2 = -n$.

Question 1: The variable s has two values. Is this advantageous or problematic?—This is advantageous!

- The parameter s can take two values, $s = \pm n$, which implies the existence of two potential solutions.
- Having two values for s is **beneficial** because it suggests the **possibility of two linearly independent solutions**.
- For a second-order ordinary differential equation (ODE), the general solution is formed by **two linearly independent** solutions.
- If two solutions y_1 and y_2 are **linearly independent**, the general solution can be expressed as $C_1y_1 + C_2y_2$, where C_1 and C_2 are arbitrary constants.
- The **presence of two distinct values** for s (roots) is **advantageous** and aligns with the expected structure of solutions for a second-order ODE.
- Linear dependence between the two potential solutions ($s = \pm n$) would be problematic, as it is not enough for the structure of the general solution for a second-order ODE.
- If only one solution were found (i.e., if $s = \pm n$ is corresponding to two linearly dependent solutions), it would be necessary to seek additional solutions.

First, take $s_1 = n$, substituting it into (5.1.17), we get $a_1 = 0$. Substituting it into (5.1.18), we get

$$a_k = -\frac{a_{k-2}}{k(2n+k)}. \quad (k = 2, 3, \dots). \quad (5.1.19)$$

From (5.1.19), we know that

$$a_1 = a_3 = a_5 = \dots = 0$$

In addition,

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2 \cdot 1 \cdot (n+1)},$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{2 \cdot 4(2n+2)(2n+4)} = \frac{a_0}{2^4 \cdot 2 \cdot 1(n+1)(n+2)},$$

.....

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} \cdot m! \underbrace{(n+1)(n+2)\cdots(n+m)}_{\text{If } n, m \text{ are integers, then this term is } (n+m)!/n!}},$$

If n, m are integers, then this term is $(n+m)!/n!$

Question 2: The coefficient a_0 cannot be determined—is this advantageous or problematic? –It is advantageous.

- It is **advantageous** because the undetermined a_0 represents a **free parameter**, allowing for a **general solution**.
- The presence of a_0 in the solution is natural because if y is a solution to the differential equation (5.1.14), then a_0y is also a solution for any non-zero a_0 .
- From the **structure of the general solution**, a_0 plays the role of an **arbitrary constant**, just like C_1 and C_2 in $C_1y_1 + C_2y_2$.
- The **flexibility** of a_0 can be used to **simplify the expression** of the solution, making it more manageable and memorable.
- The **complexity** of the solution can be **reduced** by **strategically choosing** the value of a_0 to simplify the expression, particularly when dealing with factorials and powers of 2.

Basic knowledge of the Γ function (Review)

1. **Definition:**

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \quad (x > 0),$$

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

2. **Recurrence formula of the Γ function:**

$$\Gamma(x+1) = x\Gamma(x).$$

In particular, when x is a positive integer n , we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n! \Gamma(1) = n!.$$

↑ The gamma function can be regarded as a **generalization of the factorial**.

3. When $n = 0, -1, -2, \dots$

$$\frac{1}{\Gamma(n)} = 0 \quad (\text{or } \Gamma(n) = \infty).$$

Since a_0 is an arbitrary constant, we can choose its value in the following way: make the power of 2 in the coefficient of the general term the same as that of x , and simplify the denominator at the same time. For

this purpose, take

$$a_0 = \frac{1}{2^n \Gamma(n+1)}.$$

Consider one term (if n, m are integers)

$$\begin{aligned} a_{2m}x^{n+2m} &= (-1)^m \frac{a_0}{2^{2m} \cdot m!(n+1)(n+2) \cdots (n+m)} x^{n+2m} \\ &= (-1)^m \frac{a_0}{m! \frac{(n+m)!}{n!}} \frac{x^{n+2m}}{2^{2m}} \\ &= \underbrace{(-1)^m \frac{1}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m}}_{\text{This is a nice form}} \underbrace{(a_0 2^n n!)}_{\text{let this term to be 1}}. \end{aligned}$$

Thus, we take $a_0 = \frac{1}{2^n n!}$. If n and m are not the integer, this becomes $a_0 = \frac{1}{2^n \Gamma(n+1)}$ and

$$a_{2m}x^{n+2m} = (-1)^m \frac{1}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}.$$

Using the recurrence formula $n\Gamma(n) = \Gamma(n+1)$, the coefficient of the general term becomes

$$a_{2m} = (-1)^m \frac{1}{2^{n+2m} \cdot m! \Gamma(n+m+1)}$$

Substitute this coefficient expression back into (5.1.15). We get a particular solution of equation (5.1.14), denoted as $J_n(x)$

$$J_n(x) = \sum_{m=0}^{\infty} a_{2m}x^{n+2m} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m! \Gamma(n+m+1)}, \quad (5.1.20)$$

$J_n(x)$ is called the **Bessel function of the first kind of order n** .

- The solution obtained is a **formal solution**, which is based on assumptions about the interchangeability of differentiation and summation.
- To confirm that the **formal solution is indeed a solution**, two verifications are necessary:
 - **Verification of Identity:** Substitute the solution back into the original differential equation to check if it satisfies the equation (i.e., turns it into an identity).
 - **Convergence Verification:** Ensure that the series solution converges, as a divergent series cannot represent a valid function, let alone a solution.
- The coefficients in the series are now specific, allowing for convergence verification.

- The simplest method to verify convergence is the **ratio test** (also known as **D'Alembert's ratio test**), which involves comparing successive terms:
 - Compute the limit of the ratio of successive terms.
 - If the limit is less than 1, the series is convergent.
- This method is essentially a **comparison test**, comparing the series to a **geometric series** where the common ratio q is less than 1, ensuring convergence.

Also, since

$$\lim_{m \rightarrow \infty} \frac{|u_{m+1}|}{|u_m|} = \lim_{m \rightarrow \infty} \frac{x^2}{4(m+1)(n+m+1)} = 0 < 1.$$

Then, by the **ratio test (D'Alembert's test)**, the series (5.1.20) is **absolutely convergent** on the entire real axis.

Then let $s_2 = -n$, substituting it into (5.1.17), we get $a_1 = 0$, then substituting it into (5.1.18), we get

$$a_k = -\frac{a_{k-2}}{k(-2n+k)}. \quad (k = 2, 3, \dots).$$

From the above formula, we know that

$$a_1 = a_3 = a_5 = \dots = 0$$

In addition

$$a_2 = -\frac{a_0}{2(-2n+2)} = -\frac{a_0}{2^2 \cdot 1 \cdot (-n+1)},$$

$$a_4 = -\frac{a_2}{4(-2n+4)} = \frac{a_0}{2 \cdot 4(-2n+2)(-2n+4)} = \frac{a_0}{2^4 \cdot 2 \cdot 1(-n+1)(-n+2)},$$

.....

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} \cdot m!(-n+1)(-n+2) \cdots (-n+m)},$$

Since a_0 is an arbitrary constant, we can choose its value in the following way: make the power of 2 in the coefficient of the general term the same as that of x , and simplify the denominator at the same time. For this purpose, take

$$a_0 = \frac{1}{2^{-n}\Gamma(-n+1)}.$$

Using the recurrence formula $\Gamma(-n+1) = -n\Gamma(-n)$, the coefficient of the general term becomes

$$a_{2m} = (-1)^m \frac{1}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}$$

Substitute this coefficient expression back into (5.1.15). We get another particular solution of equation (5.1.14), denoted as $J_{-n}(x)$

$$J_{-n}(x) = \sum_{m=0}^{\infty} a_{2m} x^{-n+2m} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}, \quad (5.1.21)$$

$J_{-n}(x)$ is called the **Bessel function of the first kind of order $-n$** .

- The **goal** is to generate the general solution using the two particular solutions obtained.
- A **general solution** requires the **linear combination** of two **linearly independent particular solutions**.
- The **task** is to determine whether the solutions J_n and J_{-n} are **linearly dependent or independent**.
- The analysis depends on the nature of n (whether it is an integer or not, and whether it is a half-integer).
- For n that is neither an integer nor a half-integer, J_n and J_{-n} are proven to be linearly independent.

Next, we will discuss in **three cases**.

Case 1: If n is **neither an integer (including 0) nor a half-odd integer**, then $s_1 - s_2 = 2n$ is also **not an integer**. Since $n \neq -n$, $J_n(x)$ and $J_{-n}(x)$ are **linearly independent**.

- The **reason for linear independence** is the **presence of negative exponents** in J_{-n} (i.e., terms like x^{-n}) and **only positive exponents** in J_n (i.e., terms like x^{2m}).

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m! \Gamma(n+m+1)},$$

$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m! \Gamma(-n+m+1)}.$$

- **Linear dependence** would imply a **proportional relationship** between the two functions, which is **impossible** given their different exponent characteristics of J_n and J_{-n} .
- It is crucial to ensure that the **coefficients of the negative exponent terms** in J_{-n} are **non-zero** to confirm **linear independence**; otherwise, there might be a possibility of linear dependence.

According to the structure theorem of solutions of homogeneous linear ordinary differential equations, the general solution of equation (5.1.14) is

$$y = AJ_n(x) + BJ_{-n}(x), \quad (5.1.22)$$

where A and B are two arbitrary constants.

In (5.1.22), take (because n is neither an integer nor a half-odd integer)

$$A = \cot n\pi, \quad B = -\csc n\pi,$$

Then we get another particular solution of equation (5.1.14) that is linearly independent of $J_n(x)$, denoted as

$$Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}. \quad (5.1.23)$$

$\uparrow [n \neq \text{integer}, \text{thus } \sin n\pi \neq 0]$

Therefore, the general solution of equation (5.1.14) can be written as

$$y = C J_n(x) + D Y_n(x), \quad (5.1.24)$$

$Y_n(x)$ is called the **Bessel function of the second kind** or the **Neumann function**.

- Introduction of a second type of Bessel function is necessary due to the **linear dependence** of J_n and J_{-n} when n is an **integer** (see below).
- The second type of Bessel function provides a **linearly independent solution** when n is an integer, which is crucial for forming the general solution.
- The construction of a new solution involves a **linear combination** of J_n and J_{-n} , denoted as Y_n , which is linearly independent from J_n .
- The general solution can be expressed as a linear combination of J_n and Y_n , which is **valid for all n , including integers**.
- The approach of using J_n and Y_n simplifies the solution process and provides a unified form for the general solution of Bessel's equation.
- It is recommended to focus on the linear combination of J_n and Y_n as the general solution for Bessel's equation, as it covers all cases, including when n is an integer.

Case 2: If n is an **integer (including 0)**, then $s_1 - s_2 = 2n$ is also an integer. Following the previous approach, we can also obtain two particular solutions of equation (5.1.14)

$$s_1 = n, \quad J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m! \Gamma(n+m+1)},$$

$$s_2 = -n, \quad J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m! \Gamma(-n+m+1)}, \quad (5.1.25)$$

Note that when $n \geq 0$ is an **integer**, using the recurrence formula of the Γ function, we can get $\Gamma(n+m+1) = (n+m)!$. Thus, one of the particular solutions (5.1.20) can be transformed into

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m! (n+m)!}, \quad (5.1.26)$$

And at this time, the function $J_{-n}(x)$ and $J_n(x)$ are linearly dependent.

In fact, without loss of generality, let n be a positive integer N .

- If $-N + m + 1 \leq 0$ (i.e., $0 \leq m \leq N - 1$), then $\Gamma(-N + m + 1) = \infty$ and the coefficients vanish.

When $m = 0, 1, 2, \dots, (N - 1)$, $-n + m + 1 = -N + m + 1$ will be a negative integer or 0. For these values, $\Gamma(-N + m + 1)$ is infinite. So

$$J_{-N}(x) = \sum_{m=N}^{\infty} (-1)^m \frac{x^{-N+2m}}{2^{-N+2m} \cdot m! \Gamma(-N + m + 1)},$$

Let $m = N + k$, ($k = 0, 1, 2, \dots$), we get

$$J_{-N}(x) = \sum_{k=0}^{\infty} (-1)^{N+k} \frac{x^{N+2k}}{2^{N+2k} (N+k)! \Gamma(k+1)}$$

Then, after simplification, we have

$$\begin{aligned} J_{-N}(x) &= \sum_{k=0}^{\infty} (-1)^{N+k} \frac{x^{N+2k}}{2^{N+2k} \cdot k!(N+k)!} \\ &= (-1)^N \sum_{k=0}^{\infty} (-1)^k \frac{x^{N+2k}}{2^{N+2k} \cdot k!(N+k)!} \\ &= (-1)^N \sum_{m=0}^{\infty} (-1)^m \frac{x^{N+2m}}{2^{N+2m} \cdot m!(N+m)!} = (-1)^N J_N(x). \end{aligned}$$

This shows that when $n(n \geq 0)$ is an integer, $J_{-n}(x)$ and $J_n(x)$ are **linearly dependent**. In order to find the general solution of the Bessel equation, we still **need to find a particular solution** that is **linearly independent of $J_n(x)$** .

$$Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}. \quad (5.1.27)$$

From formula (5.1.27), when n is not an integer, $Y_n(x)$ and $J_n(x)$ are linearly independent. When n is an integer, since

$$J_{-n} = (-1)^n J_n(x), \quad \cos n\pi = (-1)^n,$$

Then the right hand side of formula (5.1.27) becomes an indeterminate form of the type " $\frac{0}{0}$ ". At this time, we naturally define

$$Y_n(x) = \lim_{\alpha \rightarrow n} Y_\alpha(x) = \lim_{\alpha \rightarrow n} \frac{J_\alpha(x) \cos \alpha\pi - J_{-\alpha}(x)}{\sin \alpha\pi},$$

where n is an integer and α is not an integer.

$$Y_n(x) = \lim_{\alpha \rightarrow n} Y_\alpha(x) = \lim_{\alpha \rightarrow n} \frac{J_\alpha(x) \cos \alpha\pi - J_{-\alpha}(x)}{\sin \alpha\pi},$$

Applying L'Hopital's rule and through lengthy derivations (refer to [2]), we get

$$Y_0(x) = \frac{2}{\pi} J_0(x) \left(\ln \frac{x}{2} + C \right) - \frac{2}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2} \right)^{2m} \sum_{k=0}^{m-1} \frac{1}{k+1},$$

$$\begin{aligned}
Y_n(x) = & \frac{2}{\pi} J_n(x) \left(\ln \frac{x}{2} + C \right) - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2} \right)^{-n+2m} \\
& - \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(n+m)!} \left(\frac{x}{2} \right)^{n+2m} \cdot \left(\sum_{k=0}^{n+m-1} \frac{1}{k+1} + \sum_{k=0}^{m-1} \frac{1}{k+1} \right) \\
(n = 1, 2, 3, \dots),
\end{aligned}$$

$$Y_n(0) \text{ is infinite } (n = 1, 2, 3, \dots),$$

where $C = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right) = 0.5772 \dots$, which is called the **Euler constant**. Obviously, $Y_n(x)$ is a **particular solution** of the n -th order Bessel equation that is **linearly independent** of $J_n(x)$ (since $Y_n(x)$ includes $\ln \frac{x}{2}$).

Memorize that $\ln \frac{x}{2}$ appears in Y_n , then remember two key points:

- J_n and Y_n are linearly independent;
- $Y_n(0) = \infty$.

- (Intuitive ideas) Y_n is derived from a **linear combination** of two solutions, but after taking the limit, it becomes **linearly independent** from them.
- (Intuitive ideas) This phenomenon is peculiar and highlights a significant difference between **finite-dimensional and infinite-dimensional** linear spaces.
- (Intuitive ideas) In finite-dimensional spaces, any linear combination remains within the space, whereas in infinite-dimensional spaces, taking limits can result in outcomes that lie outside the original space.
- (Intuitive ideas) This can lead to the creation of linearly independent solutions that were not present in the original space.
- Multiple Roots and Logarithmic-Type Solutions
 - When the roots differ by an integer, the two power series solutions obtained by the direct substitution method usually yield only one pure power series solution, while the other one **often involves logarithmic terms**. This phenomenon is common in the theory of ordinary differential equations. For example, the second solution may take the form

$$y_2(x) = y_1(x) \ln x + \text{other terms in the power series expansion.}$$

For the Bessel equation, when we seek a second independent solution, a solution **containing a $\ln x$ term naturally emerges**, which is precisely the characteristic of $Y_n(x)$.

- The introduction of $Y_n(x)$ was initially motivated by the need for **completeness of the solution space** from a mathematical perspective. Since the Bessel equation for integer order produces only one “good” solution using conventional methods, it becomes necessary to obtain a **second solution with logarithmic terms** through **analytic continuation** and a **limiting process**. Although this definition may appear **non-intuitive**, it actually reflects two key aspects:
 - Theoretical Aspect:** By constructing a second independent solution via the **limiting definition**, the **completeness** of the solution space is ensured.
 - Practical Aspect:** The introduction of $Y_n(x)$ facilitates a better adjustment and matching of boundary conditions in physical problems, thereby capturing the finer details of actual phenomena.

This thought process, driven jointly by mathematical analysis and physical requirements, is a vivid illustration of the continual refinement and evolution of mathematical methods.

- It is necessary to verify whether Y_n is a solution to the original Bessel’s equation.
- The verification process involves substituting Y_n back into the original differential equation.
- Since Y_n is expressed as a series, one can assess its convergence and the behavior of its coefficients.

$$\lim_{\alpha \rightarrow n} \underbrace{[x^2 Y_\alpha'' + x Y_\alpha' + (x^2 - n^2) Y_\alpha]}_{=0} = x^2 Y_n'' + x Y_n' + (x^2 - n^2) Y_n. \quad \text{We omit the details}$$

- The conclusion is that Y_n is indeed a solution to Bessel’s equation and is linearly independent from J_n .
- With these two conclusions, the general solution of Bessel’s equation can be constructed as a linear combination of J_n and Y_n .

In conclusion, regardless of whether n is an integer or not, the general solution of the Bessel equation (5.1.14) can be expressed as

$$y(x) = C J_n(x) + D Y_n(x),$$

where C and D are arbitrary real numbers, and n is an arbitrary real number. In addition, when n is an **integer**, from $J_n(-x) = (-1)^n J_n(x)$, it can be deduced that

- when n is an **even** number, $J_n(x)$ is an **even function** ← it resembles the **cosine** function.;
- when n is an **odd** number, $J_n(x)$ is an **odd function** ← it resembles the **sine** function..

Case 3: When n is a **half-odd integer**, it will be discussed in the next section.

Summary

- The primary focus of this subsection was on understanding the **form of Bessel's equation solutions**.
- Students should be **able to identify the parameter n** in a **Bessel equation** (**Memorize this equation**) and directly write down the solutions J_n and Y_n without intermediate derivations.
- Similar to how one would directly apply the general solution formula for second-order linear differential equations with constant coefficients, the **goal** is to **recognize n -Bessel's equation** and its solutions J_n and Y_n immediately.
- It is crucial to remember that $Y_n(0) = \infty$ due to the presence of $\ln(\frac{x}{2})$ in its series representation.
- This infinity at $x = 0$ **violates the boundedness condition** often required in physical problems, leading to the exclusion of Y_n from the solution set in such cases.
- The **main takeaway** is
 - to recognize n -Bessel's equation and its solutions $CJ_n + DY_n$
 - to understand the implications of $Y_n(0) = \infty$ on the solution's applicability.

5.2 Recurrence Formulas of Bessel Functions

- This section focuses on fixing the fifth step in the method of separation of variables, which involves determining coefficients.
- To calculate these coefficients, one needs to compute integrals, specifically inner products, which often involve integrals over functions.
- Calculating these integrals requires the use of recursive relationships of functions.

There are certain relationships among Bessel functions of different orders. In this section, we will establish **recurrence formulas** that reflect these relationships. From the expression (5.1.20) of $J_n(x)$, the following two basic recurrence formulas can be derived:

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x), \quad (5.2.1)$$

$$\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x). \quad (5.2.2)$$

Proof. In fact, multiply both sides of formula (5.1.20) by x^{-n} , and then differentiate with respect to x . We get

$$\begin{aligned}\frac{d}{dx} [x^{-n} J_n(x)] &= \frac{d}{dx} \left[\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{n+2m} \cdot m! \Gamma(n+m+1)} \right] \\ &= \sum_{m=1}^{\infty} (-1)^m \frac{x^{2m-1}}{2^{n+2m-1} \cdot (m-1)! \Gamma(n+m+1)}\end{aligned}$$

Let $m = 1 + k$, ($k = 0, 1, 2, \dots$), then

$$\begin{aligned}\frac{d}{dx} [x^{-n} J_n(x)] &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{2^{n+2k+1} \cdot k! \Gamma(n+1+k+1)} \\ &= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \frac{x^{n+1+2k}}{2^{n+1+2k} \cdot k! \Gamma(n+1+k+1)} \\ &= -x^{-n} J_{n+1}(x)\end{aligned}$$

Similarly, formula (5.2.1) can be proved. \square

- The key to using the recursive relationships is flexibility in application.
- **Derivatives** are provided because they are the **inverse operations of integrals**.
- The actual calculations used involve **integration**, which is the **reverse process of differentiation**.
- The first formula provided is for the derivative of $x^n J_n$, but the useful formula for integration is $\int x^n J_{n-1} dx = x^n J_n$.
- Similarly, for the second formula, the integral of $x^{-n} J_{n+1}$ is $x^{-n} J_n$.

If we express the derivatives on the left hand sides of the above two formulas and simplify, we obtain

$$x J'_n(x) + n J_n(x) = x J_{n-1}(x),$$

$$x J'_n(x) - n J_n(x) = -x J_{n+1}(x).$$

Eliminating $J'_n(x)$ and $J_n(x)$ successively, we get

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \quad (5.2.3)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2 J'_n(x). \quad (5.2.4)$$

Obviously, formulas (5.2.1), (5.2.2) are equivalent to formulas (5.2.3), (5.2.4).

How to memorize?

- The first formula describes the relationship among J_{n-1} , J_n , and J_{n+1} .
- It involves an **arithmetic mean**: $\frac{J_{n+1} + J_{n-1}}{2} \sim J_n$ with a correction factor $\frac{n}{x}$.
- The second formula is more conveniently remembered as $\frac{J_{n+1} - J_{n-1}}{2} = -J'_n$.
- The left side represents a finite difference ("difference quotient"), and the right side is a derivative ("differential quotient" with a sign change).

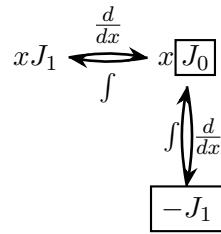
If the values of $J_{n-1}(x)$ and $J_n(x)$ are known, the value of $J_{n+1}(x)$ can be calculated from formula (5.2.3). In this way, through formula (5.2.3), Bessel functions of any positive integer order can be expressed in terms of Bessel functions of order 0 and order 1. In particular, when $n = 0$, from formula (5.2.2), we have

$$J'_0(x) = -J_1(x);$$

when $n = 1$, from formula (5.2.1), we have

$$\frac{d}{dx} [xJ_1(x)] = xJ_0(x). \quad (5.2.5)$$

How to memorize?



Ex 5.2.1. Calculate $\int xJ_2(x)dx$.

Solution (Method 1). From formula (5.2.3), we know that $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$. Then

$$xJ_2(x) = 2J_1(x) - xJ_0(x)$$

$$\begin{aligned} \int xJ_2(x)dx &= 2 \int J_1(x)dx - \int xJ_0(x)dx \\ &= -2J_0(x) - xJ_1(x) + c. \end{aligned}$$

Solution (Method 2). By (5.2.4), $J_2(x) = J_0(x) - 2J'_1(x)$, then

$$\begin{aligned}\int xJ_2(x)dx &= \int xJ_0(x)dx - 2 \int xJ'_1(x)dx \\ &= xJ_1(x) - 2 \left(xJ_1(x) - \int J_1(x)dx \right) \\ &= -xJ_1(x) - 2J_0(x) + c.\end{aligned}$$

Remark

- The purpose of calculating the integral is threefold:
 1. To **eliminate the integral sign** as much as possible.
 2. To **reduce the order of J_n** to lower values, since J_0 and J_1 are the simplest and most readily available from tables and graphs.
 3. To **minimize the polynomial orders**, as many problems involve polynomials multiplied by Bessel functions.
- The actual goal is not just to compute the integral but to **simplify it by reducing the indices of J_n and x^n** , and **removing the integral sign**.

Highlight:

- Using **different recurrence formulas** may lead to **different expressions**, but the **results are the same**, just with **different forms of expression**.
- When should we stop integrating? Simplify to the point where **no further simplification** is possible. For example, $\int J_0 dx$ (called **generalized hypergeometric function**) is left as it is!
- Try to express the results **in terms of J_0 and J_1** . Sometimes, it can also contain J_2 , J_3 , etc. In short, **simplify as much as possible**.

For the second kind Bessel functions, the following recurrence formulas also hold:

$$\begin{cases} \frac{d}{dx} [x^n Y_n(x)] = x^n Y_{n-1}(x), \\ \frac{d}{dx} [x^{-n} Y_n(x)] = -x^{-n} Y_{n+1}(x), \\ Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x), \\ Y_{n-1}(x) - Y_{n+1}(x) = 2Y'_n(x). \end{cases}$$

- The relations of Y_n is often not used in this course.
- The reason for not using Y_n is that $Y_n(0) = \infty$, which violates the boundedness condition typically required in physical problems.

- Due to the unbounded nature of Y_n at $x = 0$, it is usually discarded.
- The relationships involving Y_n are thus not frequently applied in this course.

An **important feature** of Bessel functions when n is a **half-odd integer** is that they can be **expressed in terms of elementary functions**. First, calculate $J_{\frac{1}{2}}(x)$. From formula (5.1.20), we have

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{\frac{1}{2}+2m} \cdot m! \Gamma(\frac{1}{2} + m + 1)}$$

Using the properties of the Γ function, we get

$$\begin{aligned} \Gamma\left(\frac{3}{2} + m\right) &= \left(\frac{1}{2} + m\right)\Gamma\left(\frac{1}{2} + m\right) \\ &= \left(\frac{1}{2} + m\right)\left(m - \frac{1}{2}\right)\Gamma\left(m - \frac{1}{2}\right) \\ &= \dots \\ &= \left(\frac{1}{2} + m\right)\left(m - \frac{1}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2m+1)}{2^{m+1}} \sqrt{\pi} \end{aligned}$$

Calculations:

$$\begin{aligned} J_{\frac{1}{2}} &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{\frac{1}{2}+2m} m! \frac{1 \cdot 3 \cdot 5 \cdots (2m+1)}{2^{m+1}} \sqrt{\pi}} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{-\frac{1}{2}+m} m! \cdot 1 \cdot 3 \cdot 5 \cdots (2m+1) \sqrt{\pi}} \\ &= \sqrt{2} \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{\underbrace{(2^m m!) \cdot 1 \cdot 3 \cdot 5 \cdots (2m+1)}_{(2m)!!} \underbrace{\sqrt{\pi}}_{(2m+1)!!}} \\ &= \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{(2m+1)!} \\ &= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{1+2m}}{(2m+1)!} \end{aligned}$$

Thus

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}} \sin x. \quad (5.2.6)$$

A sine function with decreasing amplitude as x increases ↑

Similarly, we can obtain

$$J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cos x. \quad (5.2.7)$$

Applying formula (5.2.3), we have

$$\begin{aligned} J_{\frac{3}{2}}(x) &= \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \\ &= \sqrt{\frac{2}{\pi x}} \left(-\cos x + \frac{1}{x} \sin x \right) \\ &= -\sqrt{\frac{2}{\pi}} x^{\frac{3}{2}} \cdot \left(\frac{1}{x} \frac{d}{dx} \right) \left(\frac{\sin x}{x} \right). \end{aligned}$$

Similarly, applying formula (5.2.3), we get

$$\begin{aligned} J_{-\frac{3}{2}}(x) &= \frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) \\ &= \sqrt{\frac{2}{\pi x}} \left(-\sin x + \frac{1}{x} \cos x \right) \\ &= \sqrt{\frac{2}{\pi}} x^{\frac{3}{2}} \cdot \left(\frac{1}{x} \frac{d}{dx} \right) \left(\frac{\cos x}{x} \right). \end{aligned}$$

In general, we have

$$\begin{aligned} J_{\frac{2m+1}{2}}(x) &= (-1)^m \sqrt{\frac{2}{\pi}} x^{m+\frac{1}{2}} \left(\frac{1}{x} \frac{d}{dx} \right)^m \left(\frac{\sin x}{x} \right), \\ J_{-\frac{2m+1}{2}}(x) &= \sqrt{\frac{2}{\pi}} x^{m+\frac{1}{2}} \left(\frac{1}{x} \frac{d}{dx} \right)^m \left(\frac{\cos x}{x} \right). \end{aligned}$$

Here, for convenience, we use the differential operator $\left(\frac{1}{x} \frac{d}{dx}\right)^m$, which is an abbreviation for the operator $\frac{1}{x} \frac{d}{dx}$ acting m times successively. For example,

$$\left(\frac{1}{x} \frac{d}{dx} \right)^2 \left(\frac{\sin x}{x} \right) = \frac{1}{x} \frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} \left(\frac{\sin x}{x} \right) \right].$$

5.3 Expansion into Series in Terms of Bessel Functions

- This section is about modifying the **fifth step** in the method of separation of variables, which is **determining the coefficients** using initial values.

- Previously, a Fourier series was written, with unknown coefficients determined by taking the inner product of the Fourier series with sine or cosine functions, utilizing their **orthogonality and norms**.
- Now, the method of separation of variables is modified by **replacing the trigonometric functions with Bessel functions**.
- This section focuses on studying the **orthogonality and norms of Bessel functions**.
- Similar to **Fourier series expansion** using trigonometric functions, many functions can be **expanded** using the **Bessel function system** under certain conditions.
- The resulting series after expansion is called the **Fourier-Bessel series**, which is a **generalization of the Fourier series**, replacing the basis function system with the **Bessel function system**.

When using Bessel functions to solve the definite solution problems of mathematical physics equations, we ultimately need to expand the known function into a series in terms of the Bessel function system. In this section, we will discuss this problem.

- We draw an analogy with the S-L problem solved in Chapter 2 to help understand the current problem (see Table 5.2).
- In the upcoming Section 5.4, we will solve the PDEs and see that the S-L problem is replaced by this Bessel equation.
- In Section 5.4, the primary focus will be on the method of separation of variables.
- It is essential to learn to use analogies to transfer the methods of eigenfunction and auxiliary function methods that you have previously learned to PDE problems involving Bessel equations.

At the beginning of this chapter, from the definite solution problem of the temperature distribution in a thin circular disk, we derived the **eigenvalue problem of the Bessel equation**:

$$r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0, \quad (5.3.1)$$

$$F(R) = 0, \quad |F(0)| < +\infty. \quad (5.3.2)$$

The **general solution** of equation (5.3.1) is

$$F(r) = CJ_n(\sqrt{\lambda}r) + DY_n(\sqrt{\lambda}r)$$

Since the general solution of the Bessel equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

is

$$y = CJ_n(x) + DY_n(x)$$

By the coordinate transformation $x = \sqrt{\lambda}r$ and $y(x) = F(r)$, we have

$$F(r) = CJ_n(\sqrt{\lambda}r) + DY_n(\sqrt{\lambda}r)$$

Since $Y_n(0)$ is infinite, according to the boundedness condition in the boundary condition (5.3.2), we know that $D = 0$. Thus

$$F(r) = CJ_n(\sqrt{\lambda}r)$$

In addition, by using the condition $F(R) = 0$ in (5.3.2), we obtain

$$J_n(\sqrt{\lambda}R) = 0. \quad (5.3.3)$$

5.3.1 Zeros of Bessel Functions

Equation (5.3.3) indicates that in order to find the eigenvalue λ of the eigenvalue problem (5.3.1)-(5.3.2), we need to determine whether the zeros of $J_n(x)$ exist. The so called **zeros of the Bessel function** refer to the values of x that make $J_n(x) = 0$.

There is a series of theorems about the **zeros of Bessel functions**.

Distribution of Zeros of Bessel Functions (Important Conclusions):

1. $J_n(x)$ has infinitely many simple real zeros. These zeros are symmetrically distributed about the origin on the x -axis. Therefore, $J_n(x)$ has **infinitely many positive zeros (countable)**.
2. The zeros of $J_n(x)$ and $J_{n+1}(x)$ are **interlaced** with each other. Moreover, the zero with the smallest absolute value of $J_n(x)$ is closer to 0 than the zero with the smallest absolute value of $J_{n+1}(x)$. Naturally, $J_n(x)$ and $J_{n+1}(x)$ have **no common zeros**.
3. When x is large enough, the **distance between two adjacent zeros** of $J_n(x)$ is **close to π** .

- Bessel functions exhibit oscillatory behavior similar to trigonometric functions.
- Only J_0 reaches a maximum value of 1 at its peak, while others do not reach 1.
- Trigonometric functions always peak at 1, unlike higher-order Bessel functions.
- Bessel functions exhibit parity properties:
 - J_n is an **odd function** if n is **odd**.
 - J_n is an **even function** if n is **even**.
- J_0 , being even, **resembles the cosine function** due to its symmetry about the y -axis.

- The shape of Bessel functions can be **visualized** by **distorting trigonometric functions**:
 - Imagine holding a trigonometric function and pulling the sides while keeping the middle high.
 - This distortion results in the characteristic shape of Bessel functions.
- Bessel functions can be thought of as a **twisted version** of trigonometric functions.
- Trigonometric functions like sine and cosine have a period of 2π , with **zeros spaced π apart**.
- As Bessel functions **approach infinity**, their behavior and **shape increasingly resemble trigonometric functions**.
- **Qualitative descriptions** of zeros provide insight into their **distribution**.
- **Specific values of zeros** can be **approximated** using **computational methods or numerical algorithms**.
- For nonlinear PDEs, exact solutions are often intractable; instead, we **describe their properties in this fashion**.
- We can **estimate** maximum and minimum values, frequency of oscillations, and asymptotic behavior.
- **Asymptotically**, solutions may **resemble known functions**, such as trigonometric functions.
- These **descriptive methods** are fundamental in PDE theory when **exact forms are not obtainable**.

Integer order Bessel functions are more widely used, especially $J_0(x)$ and $J_1(x)$.

Applying the above conclusions about the zeros of Bessel functions, let $\mu_m^{(n)}$ ($m = 1, 2, \dots$) be the positive zeros of $J_n(x)$. Then from equation (5.3.3), we have

$$\sqrt{\lambda}R = \mu_m^{(n)} = \mu \begin{cases} (n) & \text{in the Bessel equation ("n-th order")} \\ m & \text{the } m\text{-th positive zero} \end{cases} \quad (m = 1, 2, \dots)$$

$$\lambda_m^{(n)} = \left(\frac{\mu_m^{(n)}}{R} \right)^2 \quad (m = 1, 2, \dots),$$

The eigenfunctions corresponding to these eigenvalues are (this form has a very strong correspondence with $\sin(\frac{n\pi}{l}x)$)

$$F_m(r) = J_n \left(\sqrt{\lambda_m^{(n)}} r \right) = \underbrace{J_n}_{\sin} \left(\underbrace{\frac{\mu_m^{(n)}}{R}}_{\frac{n\pi}{l}} \underbrace{r}_x \right) \quad (m = 1, 2, \dots). \quad (5.3.4)$$

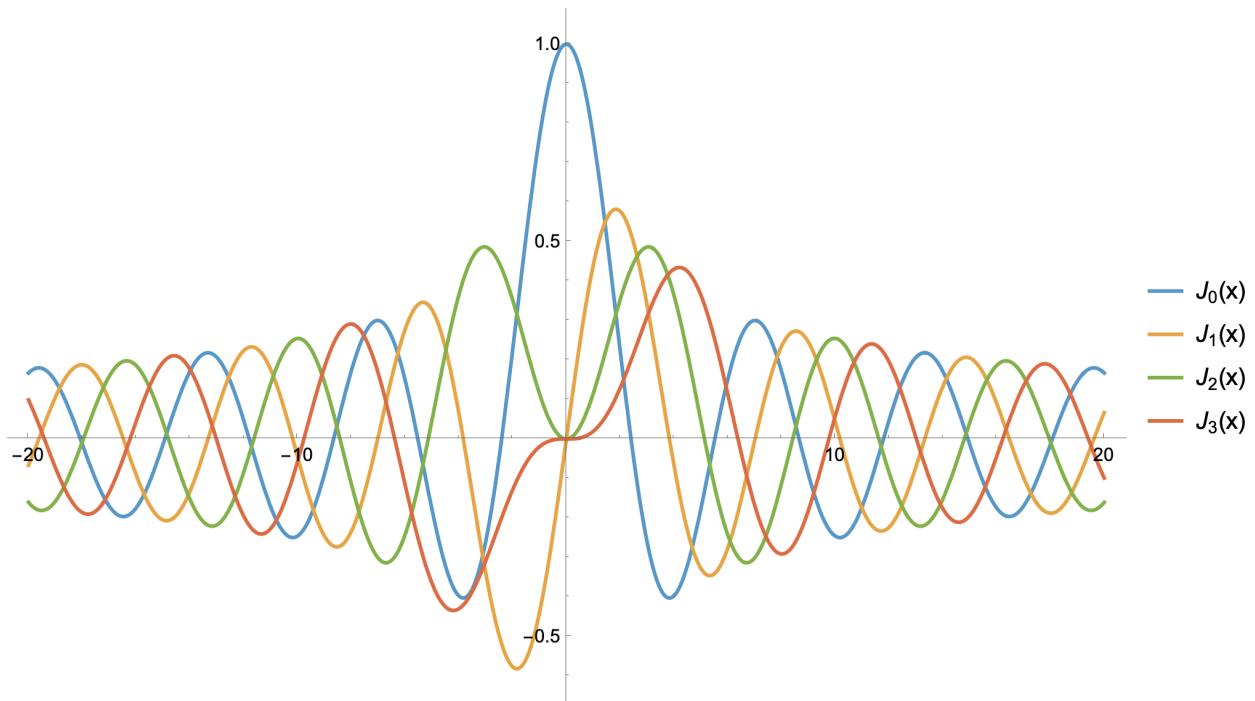


Figure 5.1: Zeros of Bessel functions 1

Note the relations

$$J_n(\mu_m^{(n)}) = 0 \quad \leftarrow \boxed{\text{Definition of zeros}}$$

$$\left. \begin{array}{l} J_{n+1}(\mu_m^{(n)}) \neq 0 \\ J_{n-1}(\mu_m^{(n)}) \neq 0 \end{array} \right\} \leftarrow \boxed{\text{Zeros must occur alternately, and there are no common zeros}}$$

5.3.2 Orthogonality of the Bessel Function System

Recall the Bessel equation and §2.6

$$\underbrace{r^2 F'' + rF'}_{=r\partial_r(r\partial_r F)} + (\lambda r^2 - n^2)F = 0 \implies \partial_r(r\partial_r F) - \frac{n^2}{r}F + \lambda \underbrace{r}_{\text{Weight}} F = 0 \quad (\text{S-L problem})$$

↑ Memorize this useful operator as mentioned before

- Recall a general form of a S-L problem discussed in Section 2.6, which provides three key conclusions:

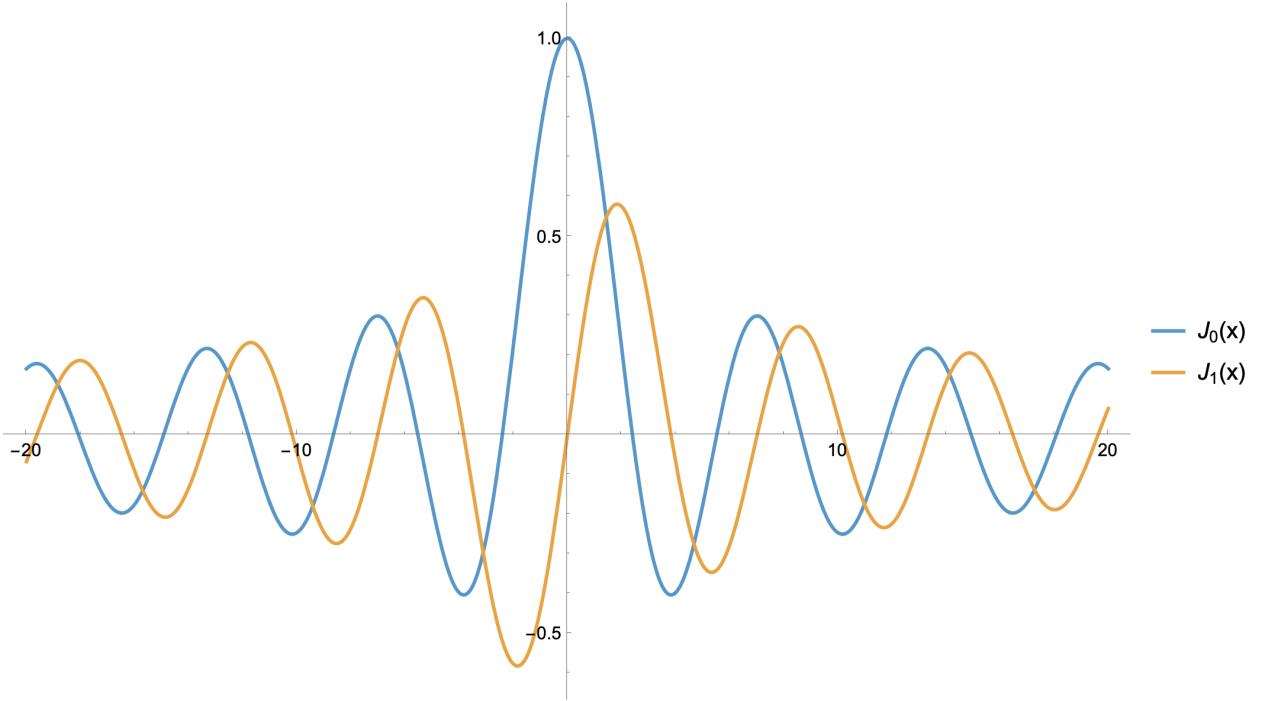


Figure 5.2: Zeros of Bessel functions 2

1. There are **countably infinite** eigenvalues and eigenfunctions.
 - Transform it into the problem of **finding the zeros** of special functions (such as J_n , \sin).
2. Eigenfunctions exhibit **orthogonality**.
 - If the eigenfunctions corresponding to the eigenvalue λ_n are denoted as $y_n(x)$, then all $y_n(x)$ form an orthogonal set with respect to **the weight function** $\rho(x)$, that is,
$$\int_a^b \rho(x) y_m(x) y_n(x) dx = 0 \quad (m \neq n).$$
 - Below we will use Bessel functions as an example to prove this conclusion.
 - The inner product must be replaced with the **weighted inner product**, that is, $\langle f, g \rangle_\rho = \int f g \rho dx$. For the Bessel equation, the **weight** function $\rho = r$.
3. Any function can be **expanded using these eigenfunctions**.

Theorem 5.3.1. *The sequence of n -th order Bessel functions (5.3.4) is **orthogonal** with weight r on the interval $(0, R)$, that is*

$$\left\langle J_n \left(\frac{\mu_m^{(n)}}{R} r \right), J_n \left(\frac{\mu_k^{(n)}}{R} r \right) \right\rangle_r = \int_0^R r \underbrace{J_n}_{n\text{-th order Bessel}} \left(\frac{\mu_m^{(n)}}{R} r \right) J_n \left(\frac{\mu_k^{(n)}}{R} r \right) dr = 0, \quad m \neq k. \quad (5.3.5)$$

- The following proof applies to general special functions, as long as the Bessel functions are replaced with other eigenfunction systems.
- Recalling the Exercise class Ex 2.10.2 in Chapter 2, the proof is the same.
- **Energy Method:**
 - **Multiply** the equation by the unknown function.
 - **Integrate** over the domain.
 - Apply **integration by parts** (Green's first identity in higher dimensions).
 - The **boundary terms** imply that λ must be positive for non-trivial solutions.

Proof. Rewrite the Bessel equation (5.3.1) as follows (to prove the orthogonality, we must use the standard form of S-L problem)

$$\frac{d}{dr} \left(r \frac{dF}{dr} \right) + \left(\lambda r - \frac{n^2}{r} \right) F = 0 \quad \text{and} \quad F(R) = 0.$$

For the convenience of writing, let

$$F_1(r) = J_n(\alpha_1 r), \quad F_2(r) = J_n(\alpha_2 r),$$

where α_1 and α_2 are arbitrary parameters. Note, by taking $\alpha_1 = \mu_m^{(n)}/R$, $\alpha_2 = \mu_k^{(n)}/R$,

$$F_1(R) = J_n(\mu_m^{(n)}) = 0, \quad F_2(R) = J_n(\mu_k^{(n)}) = 0, \quad \mu_m^{(n)} \neq \mu_k^{(n)}. \quad (5.3.6)$$

That is, the Bessel solutions satisfy the boundary condition.

Then

$$\begin{aligned} \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) + \left(\alpha_1^2 r - \frac{n^2}{r} \right) F_1 &= 0, \\ \frac{d}{dr} \left(r \frac{dF_2}{dr} \right) + \left(\alpha_2^2 r - \frac{n^2}{r} \right) F_2 &= 0. \end{aligned}$$

Step 1: Multiply the equation by the unknown function. Multiply the above two equations by F_2 and F_1 respectively. Then

$$F_2 \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) + \left(\alpha_1^2 r - \frac{n^2}{r} \right) F_1 F_2 = 0 \Rightarrow \int_0^R F_2 \frac{d}{dr} \left(r \frac{dF_1}{dr} \right) + \left(\alpha_1^2 r - \frac{n^2}{r} \right) F_1 F_2 dr = 0,$$

$$F_1 \frac{d}{dr} \left(r \frac{dF_2}{dr} \right) + \left(\alpha_2^2 r - \frac{n^2}{r} \right) F_2 F_1 = 0 \Rightarrow \int_0^R F_1 \frac{d}{dr} \left(r \frac{dF_2}{dr} \right) + \left(\alpha_2^2 r - \frac{n^2}{r} \right) F_2 F_1 dr = 0.$$

Step 2 and 3: Integrate, Apply integration by parts and obtain the symmetric forms, then subtract the symmetric forms: Integrate both sides of the above equation with respect to r from 0 to R

$$(\alpha_1^2 - \alpha_2^2) \int_0^R r F_1(r) F_2(r) dr + \left[r F_2 \frac{dF_1}{dr} - r F_1 \frac{dF_2}{dr} \right]_0^R = 0. \quad (5.3.7)$$

By taking $\alpha_1 = \mu_m^{(n)}/R$, $\alpha_2 = \mu_k^{(n)}/R$ and using (5.3.6), we can immediately obtain that equation (5.3.5) holds. The sequence of n -th order Bessel functions (5.3.4) is **orthogonal** with weight r on the interval $(0, R)$. \square

5.3.3 The Norm of Bessel Functions

Theorem 5.3.2. *The square root of the definite integral*

$$\left\langle J_n \left(\frac{\mu_m^{(n)}}{R} r \right), J_n \left(\frac{\mu_m^{(n)}}{R} r \right) \right\rangle_r = \int_0^R r J_n^2 \left(\frac{\mu_m^{(n)}}{R} r \right) dr = \frac{R^2}{2} J_{n-1}^2(\mu_m^{(n)}) = \frac{R^2}{2} J_{n+1}^2(\mu_m^{(n)})$$

is called the **norm of the Bessel function** $J_n \left(\frac{\mu_m^{(n)}}{R} r \right)$.

Proof. When $\alpha_1 \neq \alpha_2$, from equation (5.3.7), we have

$$\int_0^R r F_1(r) F_2(r) dr = - \frac{\left[r F_2 \frac{dF_1}{dr} - r F_1 \frac{dF_2}{dr} \right]_0^R}{\alpha_1^2 - \alpha_2^2}. \quad (5.3.8)$$

In the above formula, let $\alpha_1 = \mu_m^{(n)}/R$ fixed, and α_2 remain an arbitrary parameter (α_2 is not a eigenvalue, but it is close to α_1). Since

$$F_1(R) = J_n(\mu_m^{(n)}) = 0 \quad \text{and} \quad \left. \frac{dF_1}{dr} \right|_R = \alpha_1 J'_n(\alpha_1 R) = \frac{\mu_m^{(n)}}{R} J'_n(\mu_m^{(n)}),$$

the above formula (5.3.8) is transformed into

$$\mathcal{G}(\alpha_2) := \underbrace{\int_0^R r J_n \left(\frac{\mu_m^{(n)}}{R} r \right) J_n(\alpha_2 r) dr}_{\text{Continuous function of } \alpha_2 \text{ since } J_n \text{ is continuous}} = - \frac{\mu_m^{(n)} J_n(\alpha_2 R) J'_n(\mu_m^{(n)})}{(\mu_m^{(n)}/R)^2 - \alpha_2^2}.$$

Since $\mathcal{G}(\alpha_2)$ is a continuous function and thus

$$\lim_{\alpha_2 \rightarrow \mu_m^{(n)}/R} \mathcal{G}(\alpha_2) = \mathcal{G}(\mu_m^{(n)}/R) = \int_0^R r J_n^2 \left(\frac{\mu_m^{(n)}}{R} r \right) dr.$$

However, when $\alpha_2 \rightarrow \mu_m^{(n)}/R$, the right-hand side of the above formula is an indeterminate form of the type $\frac{0}{0}$ (since $\lim_{\alpha_2 \rightarrow \mu_m^{(n)}/R} J_n(\alpha_2 R) = J_n(\mu_m^{(n)}) = 0$). Applying L'Hopital's rule, we get

$$\begin{aligned} \int_0^R r J_n^2 \left(\frac{\mu_m^{(n)}}{R} r \right) dr &= \lim_{\alpha_2 \rightarrow \mu_m^{(n)}/R} - \frac{\mu_m^{(n)} J'_n(\mu_m^{(n)}) J'_n(\alpha_2 R) R}{-2\alpha_2} \\ &= \frac{\mu_m^{(n)} J'_n(\mu_m^{(n)}) J'_n(\mu_m^{(n)}) R}{2\mu_m^{(n)}/R} = \frac{R^2}{2} [J'_n(\mu_m^{(n)})]^2. \end{aligned} \quad (5.3.9)$$

From the recurrence formulas

$$xJ'_n(x) + nJ_n(x) = xJ_{n-1}(x), \quad xJ'_n(x) - nJ_n(x) = -xJ_{n+1}(x),$$

and

$$J_n(\mu_m^{(n)}) = 0,$$

we obtain

$$J'_n(\mu_m^{(n)}) = J_{n-1}(\mu_m^{(n)}), \quad J'_n(\mu_m^{(n)}) = -J_{n+1}(\mu_m^{(n)}).$$

Then equation (5.3.9) becomes

$$\int_0^R r J_n^2 \left(\frac{\mu_m^{(n)}}{R} r \right) dr = \frac{R^2}{2} J_{n-1}^2(\mu_m^{(n)}) = \frac{R^2}{2} J_{n+1}^2(\mu_m^{(n)}) \quad (5.3.10)$$

Since the Bessel functions $J_n(x)$ and $J_{n+1}(x)$ have **no common zeros**, from equation (5.3.10), we know that the norm of the Bessel function is not zero. \square

5.3.4 Fourier-Bessel Series

- **Definition:** The Fourier-Bessel series is a type of series expansion **similar to the Fourier series**.
- **Function System Replacement:** It **replaces the trigonometric function system** used in Fourier series with a **new system of Bessel functions**.

When using Bessel functions to solve the definite solution problems of mathematical physics equations, it is often necessary to **expand** the known function into **a series in terms of the Bessel function system**. It can be proved that: if $f(r)$ is a piecewise smooth function defined in the interval $(0, R)$, and the integral

$$\int_0^R r^{\frac{1}{2}} |f(r)| dr \leftarrow \boxed{\text{ensures the integrability of the coefficients below, using Hölder's inequality}}$$

has a **finite value**, then it can be expanded into a series of the following form:

$$f(r) = \sum_{m=1}^{\infty} C_m J_n \left(\frac{\mu_m^{(n)}}{R} r \right), \quad (5.3.11)$$

Moreover, at the continuous points of $f(r)$, the series (5.3.11) converges to $f(r)$.

- **Starting Index:** The series typically starts from $m = 1$ because λ (the eigenvalue) is greater than 0, excluding the possibility of including a constant term.

At the discontinuous point r_0 of $f(r)$, the series converges to the average of the left hand and right hand limits at the point r_0 , that is, it converges to $[f(r_0 + 0) + f(r_0 - 0)]/2$. The coefficient C_m is determined by the following formula

$$C_m = \frac{\int_0^R r f(r) J_n\left(\frac{\mu_m^{(n)}}{R}r\right) dr}{\frac{R^2}{2} J_{n+1}^2(\mu_m^{(n)})}. \quad (5.3.12)$$

The C_m determined by formula (5.3.12) is called the Fourier-Bessel coefficient, and the series (5.3.11) is called the **Fourier-Bessel series**.

In fact, multiply both sides of equation (5.3.11) by $r J_n\left(\frac{\mu_k^{(n)}}{R}r\right)$ and integrate with respect to r from 0 to R , we get

$$\int_0^R r f(r) J_n\left(\frac{\mu_k^{(n)}}{R}r\right) dr = \sum_{m=1}^{\infty} C_m \int_0^R r J_n\left(\frac{\mu_k^{(n)}}{R}r\right) J_n\left(\frac{\mu_m^{(n)}}{R}r\right) dr.$$

By the orthogonality of the Bessel function system,

$$\int_0^R r f(r) J_n\left(\frac{\mu_m^{(n)}}{R}r\right) dr = C_m \int_0^R r J_n^2\left(\frac{\mu_m^{(n)}}{R}r\right) dr.$$

Therefore,

$$C_m = \frac{\int_0^R r f(r) J_n\left(\frac{\mu_m^{(n)}}{R}r\right) dr}{\int_0^R r J_n^2\left(\frac{\mu_m^{(n)}}{R}r\right) dr}.$$

Or simply,

$$\left\langle f(r), J_n\left(\frac{\mu_k^{(n)}}{R}r\right) \right\rangle_r = \sum_{m=1}^{\infty} C_m \left\langle J_n\left(\frac{\mu_m^{(n)}}{R}r\right), J_n\left(\frac{\mu_k^{(n)}}{R}r\right) \right\rangle_r = C_k \left\langle J_n\left(\frac{\mu_k^{(n)}}{R}r\right), J_n\left(\frac{\mu_k^{(n)}}{R}r\right) \right\rangle_r.$$

Then

$$C_m = \frac{\left\langle f(r), J_n\left(\frac{\mu_k^{(n)}}{R}r\right) \right\rangle_r}{\left\langle J_n\left(\frac{\mu_m^{(n)}}{R}r\right), J_n\left(\frac{\mu_m^{(n)}}{R}r\right) \right\rangle_r}.$$

Ex 5.3.1. Let $\mu_m^{(0)}$ ($m = 1, 2, \dots$) be the positive zeros of the function $J_0(x)$. Try to expand the function $f(x) = 1$ into a Fourier-Bessel series in terms of $J_0(\mu_m^{(0)}x)$ on the interval $(0, 1)$.

Solution. According to equations (5.3.11) and (5.3.12), we have

$$1 = \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(0)}x), \quad C_m = \frac{\int_0^1 x J_0(\mu_m^{(0)}x) dx}{\frac{1}{2} J_1^2(\mu_m^{(0)})}.$$

First, calculate the numerator. Let $\mu_m^{(0)}x = r$, then

$$\int_0^1 x J_0(\mu_m^{(0)}x) dx = \frac{1}{(\mu_m^{(0)})^2} \int_0^{\mu_m^{(0)}} r J_0(r) dr = \frac{1}{(\mu_m^{(0)})^2} [r J_1(r)]_0^{\mu_m^{(0)}} = \frac{1}{\mu_m^{(0)}} J_1(\mu_m^{(0)}).$$

Substitute it into C_m , we get

$$C_m = \frac{2}{\mu_m^{(0)} J_1(\mu_m^{(0)})}.$$

Therefore,

$$1 = \sum_{m=1}^{\infty} \frac{2}{\mu_m^{(0)} J_1(\mu_m^{(0)})} J_0(\mu_m^{(0)}x).$$

	Chapter 2	Chapter 5
S-L problem:	$X'' + \lambda X = 0$ $X(0) = X(\ell) = 0$	$r^2 F'' + r F' + (\lambda r^2 - n^2) F = 0$ $F(R) = 0 \quad F(0) < \infty$
Zeros (Solve a transcendental eq. to find λ)	$\sin \sqrt{\lambda} \ell = 0$ $\Rightarrow \sqrt{\lambda} \ell = n\pi (n = 1, 2, \dots)$	$J_n(\sqrt{\lambda} R) = 0$ $\Rightarrow \sqrt{\lambda} R = \mu_m^{(n)}$
Eigenvalues	$\lambda_n = \left(\frac{\mu_m^{(n)}}{R}\right)^2$	$\lambda_n = \left(\frac{\mu_m^{(n)}}{R}\right)^2$
Eigenfunctions	$\sin \frac{n\pi x}{\ell}$	$J_n\left(\frac{\mu_m^{(n)}}{R}r\right)$
Orthogonality	$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx = 0$ $(n \neq m)$	$\int_0^R r J_n\left(\frac{\mu_m^{(n)}}{R}r\right) J_n\left(\frac{\mu_k^{(n)}}{R}r\right) dr = 0$ $(m \neq k)$
Norms	$\int_{-\ell}^{\ell} \left(\sin\left(\frac{n\pi x}{\ell}\right)\right)^2 dx = \ell$	$\int_0^R r J_n^2\left(\frac{\mu_m^{(n)}}{R}r\right) dr = \frac{R^2}{2} \left(J'_n(\mu_m^{(n)})\right)^2$ $= \frac{R^2}{2} J_{n+1}^2(\mu_m^{(n)})$
Expansions	$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$ $b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$	$f(r) = \sum_{m=1}^{\infty} C_m J_n\left(\frac{\mu_m^{(n)}}{R}r\right)$ $C_m = \frac{\int_0^R r f(r) J_n\left(\frac{\mu_m^{(n)}}{R}r\right) dr}{\frac{R^2}{2} J_{n+1}^2(\mu_m^{(n)})}$

Table 5.2: Summary of Orthogonality, Norms, and Expansions

5.4 Applications of Bessel Functions

Bessel functions have extremely wide applications. In this section, we only choose the simplest problems to illustrate the **key points and steps** of using Bessel functions to solve mathematical physics problems.

- Note: In higher-dimensional problems involving the **Laplace operator** on a **cylindrical domain** or **circular disk**, Bessel equations and functions arise.

- However, to **simplify calculations**, this section typically **assumes axisymmetry, reducing** the problem to a **two-dimensional** one.
- If axisymmetry is **not** present, the approach outlined in Section 5.1 can be followed.

Ex 5.4.1 (Heat Conduction Problem). Consider a uniform thin circular disk with a radius of 1. The temperature on the circumference is maintained at 0 degrees, and the initial temperature distribution inside the disk is $1 - r^2$, where r is the polar radius of any point inside the disk. Try to find the temperature distribution inside the disk.

Solution. The temperature u to be found satisfies the two dimensional homogeneous heat conduction equation. Since the solution domain is a circular domain, **polar coordinates** are used. Because the definite solution conditions are **independent of θ** , so $u = u(r, t)$. Then the definite solution problem is as follows:

$$u_t = a^2(u_{rr} + \frac{1}{r}u_r), \quad (0 < r < 1), \leftarrow \boxed{\text{Homogeneous equation}} \quad (5.4.1)$$

$$u|_{r=1} = 0, \leftarrow \boxed{\text{Homogeneous boundary}} \quad (5.4.2)$$

$$u|_{t=0} = 1 - r^2. \quad (5.4.3)$$

Apply the **method of separation of variables**. Let $u(r, t) = R(r)T(t)$ and substitute it into (5.4.1), we get

$$RT' = a^2(R'' + \frac{1}{r}R')T.$$

$$\Rightarrow \frac{T'}{a^2T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda.$$

From this, we obtain

$$T' + \lambda a^2 T = 0 \quad (5.4.4)$$

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad (5.4.5)$$

From the physical meaning of the problem, we know that the temperature function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy the natural boundary condition

$$|R(0)| < +\infty \quad (5.4.6)$$

And from the **homogeneous boundary condition** (5.4.2), we can get

$$R(1) = 0 \quad (5.4.7)$$

Equations (5.4.5)-(5.4.7) form the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} r^2 R'' + rR' + \lambda r^2 R = 0 \leftarrow \boxed{\text{Identify 0-order Bessel equation}} \\ |R(0)| < +\infty, R(1) = 0 \end{cases}$$

The general solution of the Bessel equation of **order 0** (5.4.5) is

$$R(r) = C J_0(\sqrt{\lambda}r) + D Y_0(\sqrt{\lambda}r)$$

From the condition (5.4.6) $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (5.4.7) $R(1) = 0$, we get $J_0(\sqrt{\lambda}) = 0$, that is, $\sqrt{\lambda}$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (5.4.5) under the conditions (5.4.6) and (5.4.7) are

$$\begin{cases} \lambda_m^{(0)} = (\mu_m^{(0)})^2 & (m = 1, 2, \dots) \\ R_m(r) = J_0(\mu_m^{(0)} r) \end{cases}$$

Now consider the equation

$$T' + \lambda a^2 T = 0$$

Substitute $\lambda_m^{(0)}$ into equation (5.4.4) to obtain its general solution

$$T_m(t) = C_m e^{-(\mu_m^{(0)} a)^2 t}$$

Then, using $u(r, t) = R(r)T(t)$, we can get

$$u_m(r, t) = C_m e^{-(\mu_m^{(0)} a)^2 t} J_0(\mu_m^{(0)} r)$$

According to the superposition principle, the solution of equation (5.4.1) that satisfies condition (5.4.2) is

$$u(r, t) = \sum_{m=1}^{\infty} C_m e^{-(\mu_m^{(0)} a)^2 t} J_0(\mu_m^{(0)} r) \quad (5.4.8)$$

Then, from the initial condition (5.4.4), we have

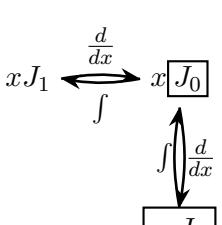
$$u(r, 0) = \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(0)} r) = 1 - r^2$$

Using the Fourier-Bessel coefficient formula (5.3.12), we have

$$C_m = \frac{\int_0^1 r (1 - r^2) J_0(\mu_m^{(0)} r) dr}{\frac{1}{2} J_1^2(\mu_m^{(0)})} \leftarrow \boxed{\text{Don't forget the weight } r}$$

First, calculate the numerator. Let $\mu_m^{(0)} r = x$, then

$$\begin{aligned} \int_0^1 r J_0(\mu_m^{(0)} r) dr &= \frac{1}{(\mu_m^{(0)})^2} \int_0^{\mu_m^{(0)}} x J_0(x) dx \\ &= \frac{1}{(\mu_m^{(0)})^2} [x J_1(x)]_0^{\mu_m^{(0)}} = \frac{1}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) \end{aligned}$$


 Recall $\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$ and

$$\begin{aligned}
\int_0^1 r^3 J_0(\mu_m^{(0)} r) dr &= \frac{1}{(\mu_m^{(0)})^4} \int_0^{\mu_m^{(0)}} x^3 J_0(x) dx \\
&= \frac{1}{(\mu_m^{(0)})^4} \int_0^{\mu_m^{(0)}} x^2 \cdot x J_0(x) dx \\
&= \frac{1}{(\mu_m^{(0)})^4} \left[x^3 J_1(x) \Big|_0^{\mu_m^{(0)}} - 2 \int_0^{\mu_m^{(0)}} x^2 J_1(x) dx \right] \\
&\quad \boxed{\uparrow \text{Another method: } \int_0^{\mu_m^{(0)}} x^2 J_1(x) dx = - \int_0^{\mu_m^{(0)}} x^2 dJ_0(x)} \\
&= \frac{1}{(\mu_m^{(0)})^4} \left[(\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2x^2 J_2(x) \Big|_0^{\mu_m^{(0)}} \right] \\
&= \frac{1}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) - \frac{2}{(\mu_m^{(0)})^2} J_2(\mu_m^{(0)})
\end{aligned}$$

- The problem involves integrating the product of polynomials and Bessel functions.
- To integrate such products, **polynomials** are typically **decomposed**.
- The approach often involves **breaking down the polynomial into simpler terms** that can be more easily integrated with Bessel functions. That is, to generate xJ_0 , $x^n J_{n-1}$ or $x^{-n} J_{n+1}$ terms which can be integrated.

Substitute into C_m to get

$$C_m = \frac{4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})}$$

Substitute C_m into (5.4.8), and the solution of the problem (5.4.1)-(5.4.3) is

$$u(r, t) = \sum_{m=1}^{\infty} \frac{4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} J_0(\mu_m^{(0)} r) e^{-(\mu_m^{(0)} a)^2 t}$$

Ex 5.4.2 (Electric Potential in a Cylindrical Domain). For an empty cylinder composed of conductor walls, with the height of the cylinder being h and the radius being b . Suppose the electric potential of the upper base is U , and the electric potentials of the side surface and the lower base are 0. Try to find the electric potential inside the cylinder.

Solution. Since the region is cylindrical, the cylindrical coordinate system is used. Because the boundary conditions are **independent of the angle φ** , the electric potential u to be found is only a function of two

variables ρ and z , that is $u = u(\rho, z)$. Then the definite solution problem is as follows:

$$u_{\rho\rho} + \frac{1}{\rho}u_\rho + u_{zz} = 0 \quad (0 < \rho < b, 0 < z < h), \leftarrow \boxed{\text{Homogeneous equation}} \quad (5.4.9)$$

$$u|_{z=0} = 0, u|_{z=h} = U, \leftarrow \boxed{\text{determine the coefficients}} \quad (5.4.10)$$

$$u|_{\rho=b} = 0, \leftarrow \boxed{\text{Homogeneous boundary}} \quad (5.4.11)$$

where U is a constant.

Apply the **method of separation of variables**. Let $u(\rho, z) = R(\rho)Z(z)$ and substitute it into (5.4.9), we get

$$R''Z + \frac{1}{\rho}R'Z + RZ'' = 0$$

$$\frac{R'' + \frac{1}{\rho}R'}{R} = -\frac{Z''}{Z} = -\lambda$$

From this, we obtain

$$Z'' - \lambda Z = 0 \quad (5.4.12)$$

$$\rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \quad (5.4.13)$$

From the physical meaning of the problem, we know that the electric potential function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy

$$|R(0)| < +\infty. \quad (5.4.14)$$

And from the homogeneous boundary condition (5.4.11), we can get

$$R(b) = 0 \quad (5.4.15)$$

Equations (5.4.13)-(5.4.15) form the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} \rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \\ |R(0)| < +\infty, R(b) = 0 \end{cases}$$

The general solution of the Bessel equation of order 0 (5.4.13) is

$$R(\rho) = C J_0(\sqrt{\lambda}\rho) + D Y_0(\sqrt{\lambda}\rho)$$

From the condition (5.4.14) $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (5.4.15) $R(b) = 0$, we get $J_0(\sqrt{\lambda}b) = 0$, that is, $\sqrt{\lambda}b$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (5.4.13) under the conditions (5.4.14) and (5.4.15) are

$$\lambda_m^{(0)} = \left(\frac{\mu_m^{(0)}}{b} \right)^2, R_m(\rho) = J_0 \left(\frac{\mu_m^{(0)}}{b} \rho \right) \quad (m = 1, 2, \dots)$$

Now consider the equation

$$Z'' - \lambda Z = 0 \quad (5.4.16)$$

Substitute $\lambda_m^{(0)}$ into equation (5.4.16) to obtain its general solution

$$Z_m(z) = C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}.$$

Thus

$$u_m(\rho, z) = (C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right).$$

According to the superposition principle, the solution of equation (5.4.9) that satisfies condition (5.4.11) is

$$u(\rho, z) = \sum_{m=1}^{\infty} (C_m e^{\frac{\mu_m^{(0)}}{b}z} + D_m e^{-\frac{\mu_m^{(0)}}{b}z}) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) \quad (5.4.17)$$

From the first formula in condition (5.4.10), we have

$$u(\rho, 0) = \sum_{m=1}^{\infty} (C_m + D_m) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) = 0$$

So we get (comparing the coefficients—orthogonality or linear independency)

$$C_m + D_m = 0 \quad (m = 1, 2, \dots) \quad (5.4.18)$$

From the second formula in condition (5.4.10), we have

$$u(\rho, h) = \sum_{m=1}^{\infty} (C_m e^{\frac{\mu_m^{(0)}}{b}h} + D_m e^{-\frac{\mu_m^{(0)}}{b}h}) J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) = U$$

Using the Fourier-Bessel coefficient formula (5.3.12), we have

$$C_m e^{\frac{\mu_m^{(0)}}{b}h} + D_m e^{-\frac{\mu_m^{(0)}}{b}h} = \frac{\int_0^b \rho U J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) d\rho}{\frac{b^2}{2} J_1^2(\mu_m^{(0)})}. \quad (5.4.19)$$

First, calculate the numerator. Let $\frac{\mu_m^{(0)}}{b}\rho = x$, then

$$\begin{aligned} \int_0^b \rho U J_0\left(\frac{\mu_m^{(0)}}{b}\rho\right) d\rho &= \frac{b^2 U}{(\mu_m^{(0)})^2} \int_0^{\mu_m^{(0)}} x J_0(x) dx \\ &= \frac{b^2 U}{(\mu_m^{(0)})^2} [x J_1(x)]_0^{\mu_m^{(0)}} = \frac{b^2 U}{\mu_m^{(0)}} J_1(\mu_m^{(0)}). \end{aligned}$$

Then formula (5.4.20) simplifies to

$$C_m e^{\frac{\mu_m^{(0)}}{b}h} + D_m e^{-\frac{\mu_m^{(0)}}{b}h} = \frac{2U}{\mu_m^{(0)} J_1(\mu_m^{(0)})}.$$

Solve equations (5.4.18) and (5.4.19) simultaneously, with $shx = \frac{e^x - e^{-x}}{2}$, we get

$$C_m = \frac{U}{\mu_m^{(0)} J_1(\mu_m^{(0)}) sh \frac{\mu_m^{(0)}}{b}h}, D_m = -\frac{U}{\mu_m^{(0)} J_1(\mu_m^{(0)}) sh \frac{\mu_m^{(0)}}{b}h}$$

Substitute C_m and D_m into (5.4.17), and the solution of the problem (5.4.9)-(5.4.22) is

$$u(\rho, z) = \sum_{m=1}^{\infty} \frac{2U}{\mu_m^{(0)} \operatorname{sh} \frac{\mu_m^{(0)}}{b} h J_1(\mu_m^{(0)})} \operatorname{sh} \frac{\mu_m^{(0)}}{b} z J_0 \left(\frac{\mu_m^{(0)}}{b} \rho \right).$$

Ex 5.4.3 (Axisymmetric Vibration Problem of a Circular Membrane). Consider a circular membrane with a radius of B . The circumference is fixed. If a very small height $h > 0$ is lifted at the center of the membrane and then held stationary, and suddenly released to let it vibrate, try to find the vibration law of the membrane.

Solution. Since the equation is homogeneous and the definite - solution conditions are independent of the angle θ , in the polar coordinate system, the displacement function u is only a function of two variables r and t , that is $u = u(r, t)$. Then the definite solution problem is as follows:

$$u_{tt} = a^2(u_{rr} + \frac{1}{r}u_r) \quad (0 < r < B), \quad (5.4.20)$$

$$u|_{r=B} = 0, \quad (5.4.21)$$

$$u|_{t=0} = h(1 - \frac{r}{B}), \quad u_t|_{t=0} = 0. \quad (5.4.22)$$

Apply the method of separation of variables. Let $u(r, t) = R(r)T(t)$ and substitute it into (5.4.20), we get

$$\begin{aligned} RT'' &= a^2(R'' + \frac{1}{r}R')T \\ \Rightarrow \frac{T''}{a^2 T} &= \frac{R'' + \frac{1}{r}R'}{R} = -\lambda \end{aligned}$$

From this, we obtain

$$T'' + a^2\lambda T = 0 \quad (5.4.23)$$

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad (5.4.24)$$

From the physical meaning of the problem, we know that the displacement function u should satisfy the condition $|u| < +\infty$. Therefore, the function R should satisfy

$$|R(0)| < +\infty$$

And from the homogeneous boundary condition (5.4.21), we can get

$$R(B) = 0 \quad (5.4.25)$$

This constitutes the eigenvalue problem of the Bessel equation of order 0:

$$\begin{cases} r^2 R'' + rR' + \lambda r^2 R = 0 \\ |R(0)| < +\infty, \quad R(B) = 0 \end{cases}$$

The general solution of the Bessel equation of order 0 (5.4.24) is

$$R(r) = C J_0(\sqrt{\lambda}r) + D Y_0(\sqrt{\lambda}r)$$

From the boundedness condition $|R(0)| < +\infty$, we know that $D = 0$. Then, using the condition (5.4.25) $R(B) = 0$, we get $J_0(\sqrt{\lambda}B) = 0$, that is, $\sqrt{\lambda}B$ is a zero of $J_0(x) = 0$.

Let $\mu_m^{(0)}$ represent the positive zeros of $J_0(x)$, that is, $J_0(\mu_m^{(0)}) = 0$. Then the eigenvalues and corresponding eigenfunctions of equation (5.4.24) under the boundedness condition and (5.4.25) are

$$\lambda_m^{(0)} = \left(\frac{\mu_m^{(0)}}{B} \right)^2, R_m(r) = J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) \quad (m = 1, 2, \dots)$$

Now consider the equation

$$T'' + a^2 \lambda T = 0$$

Substitute $\lambda_m^{(0)}$ into equation (5.4.23) to obtain its general solution

$$T_m(t) = a_m \cos \frac{a\mu_m^{(0)}}{B} t + b_m \sin \frac{a\mu_m^{(0)}}{B} t$$

Thus

$$u_m(r, t) = (a_m \cos \frac{a\mu_m^{(0)}}{B} t + b_m \sin \frac{a\mu_m^{(0)}}{B} t) J_0 \left(\frac{\mu_m^{(0)}}{B} r \right)$$

According to the superposition principle, the solution of equation (5.4.20) that satisfies condition (5.4.21) is

$$u(r, t) = \sum_{m=1}^{\infty} (a_m \cos \frac{a\mu_m^{(0)}}{B} t + b_m \sin \frac{a\mu_m^{(0)}}{B} t) J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) \quad (5.4.26)$$

From the second formula in the initial condition (5.4.22), we have

$$\sum_{m=1}^{\infty} \frac{a\mu_m^{(0)}}{B} b_m J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) = 0$$

So we get

$$b_m = 0 \quad (m = 1, 2, \dots)$$

From the first formula in the initial condition (5.4.22), we have

$$u(r, 0) = \sum_{m=1}^{\infty} a_m J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) = h \left(1 - \frac{r}{B} \right)$$

Using the Fourier-Bessel coefficient formula (5.3.12), we have

$$a_m = \frac{\int_0^B r h \left(1 - \frac{r}{B} \right) J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) dr}{\frac{B^2}{2} J_1^2(\mu_m^{(0)})} \quad (5.4.27)$$

First, calculate the numerator. Let $\frac{\mu_m^{(0)}}{B} r = x$, then

$$\begin{aligned} h \int_0^B r J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) dr &= \frac{h B^2}{\left(\frac{\mu_m^{(0)}}{B} \right)^2} \int_0^{\mu_m^{(0)}} x J_0(x) dx \\ &= \frac{h B^2}{\left(\frac{\mu_m^{(0)}}{B} \right)^2} [x J_1(x)]_0^{\mu_m^{(0)}} = \frac{h B^2}{\mu_m^{(0)}} J_1 \left(\mu_m^{(0)} \right) \end{aligned}$$

and

$$\begin{aligned}
\frac{h}{B} \int_0^B r^2 J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) dr &= \frac{hB^2}{(\mu_m^{(0)})^3} \int_0^{\mu_m^{(0)}} x^2 J_0(x) dx \\
&= \frac{hB^2}{(\mu_m^{(0)})^3} \int_0^{\mu_m^{(0)}} x \cdot x J_0(x) dx \\
&= \frac{hB^2}{(\mu_m^{(0)})^3} \left[x^2 J_1(x) \Big|_0^{\mu_m^{(0)}} - \int_0^{\mu_m^{(0)}} x J_1(x) dx \right] \\
&= \frac{hB^2}{(\mu_m^{(0)})^3} \left[(\mu_m^{(0)})^2 J_1(\mu_m^{(0)}) + x J_0(x) \Big|_0^{\mu_m^{(0)}} - \int_0^{\mu_m^{(0)}} J_0(x) dx \right] \\
&= \frac{hB^2}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) - \frac{hB^2}{(\mu_m^{(0)})^3} \underbrace{\int_0^{\mu_m^{(0)}} J_0(x) dx}_{\text{Generalized Hypergeometric Function}}.
\end{aligned}$$

Using the Fourier-Bessel coefficient formula (5.3.12), we have:

$$\begin{aligned}
h \int_0^B r J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) dr &= \frac{hB^2}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) \\
\frac{h}{B} \int_0^B r^2 J_0 \left(\frac{\mu_m^{(0)}}{B} r \right) dr &= \frac{hB^2}{\mu_m^{(0)}} J_1(\mu_m^{(0)}) - \frac{hB^2}{(\mu_m^{(0)})^3} \int_0^{\mu_m^{(0)}} J_0(x) dx
\end{aligned}$$

Substitute the above results into formula (5.4.27):

$$\begin{aligned}
a_m &= \frac{2h}{(\mu_m^{(0)})^3 J_1^2(\mu_m^{(0)})} \int_0^{\mu_m^{(0)}} J_0(x) dx \\
a_m &= \frac{2h}{(\mu_m^{(0)})^3 J_1^2(\mu_m^{(0)})} \int_0^{\mu_m^{(0)}} J_0(x) dx, b_m = 0 \quad (m = 1, 2, \dots)
\end{aligned}$$

Substitute a_m and b_m into (5.4.26), and the solution of the problem (5.4.20)-(5.4.22) is:

$$u(r, t) = \sum_{m=1}^{\infty} \left[\frac{2h}{(\mu_m^{(0)})^3 J_1^2(\mu_m^{(0)})} \int_0^{\mu_m^{(0)}} J_0(x) dx \right] \cos \frac{a\mu_m^{(0)}t}{B} J_0 \left(\frac{\mu_m^{(0)}}{B} r \right).$$

Ex 5.4.4. Solve the following definite solution problem:

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) \quad (0 < r < 1), \leftarrow \boxed{\text{leads to 1st oder Bessel equation}} \quad (5.4.28)$$

$$u|_{r=1} = 0, |u(r, t)| < +\infty, \quad (5.4.29)$$

$$u|_{t=0} = 1 - r. \quad (5.4.30)$$

- Replacing u_t with u_{tt} transforms the equation into a wave equation.
- Replacing u_t with $-u_{zz}$ transforms the equation into a potential equation in cylindrical coordinates.
- These substitutions allow for the creation of various new problems in PDEs.
- The corresponding solutions can be found by substituting the appropriate parts of the five steps of the method of separation of variables.

Solution. Apply the method of separation of variables. Let $u(r, t) = R(r)T(t)$ and substitute it into (5.4.28), we get

$$RT' = a^2 \left(R'' + \frac{1}{r}R' - \frac{1}{r^2}R \right) T$$

or

$$\frac{T'}{a^2 T} = \frac{R'' + \frac{1}{r}R' - \frac{1}{r^2}R}{R} = -\lambda$$

From this, we obtain

$$T' + \lambda a^2 T = 0 \quad (5.4.31)$$

$$r^2 R'' + r R' + (\lambda r^2 - 1)R = 0 \quad (5.4.32)$$

Note that this equation is a Bessel equation of order one. Using the definite solution conditions (5.4.29), we can get

$$R(1) = 0, |R(0)| < +\infty \quad (5.4.33)$$

(5.4.32) and (5.4.33) form the eigenvalue problem of the Bessel equation of order one. The corresponding eigenvalues and eigenfunctions are respectively

$$\lambda_m^{(1)} = (\mu_m^{(1)})^2, R_m(r) = J_1(\mu_m^{(1)}r) \quad (m = 1, 2, \dots)$$

Substitute the eigenvalues into (5.4.31), we can get

$$T_m(t) = C_m e^{-(\mu_m^{(1)}a)^2 t}$$

Then, using $u(r, t) = R(r)T(t)$, we can get

$$u_m(r, t) = C_m e^{-(\mu_m^{(1)}a)^2 t} J_1(\mu_m^{(1)}r) \quad (m = 1, 2, \dots)$$

According to the superposition principle, the general solution that satisfies equation (5.4.28) and condition (5.4.29) is

$$u(r, t) = \sum_{m=1}^{\infty} C_m e^{-(\mu_m^{(1)}a)^2 t} J_1(\mu_m^{(1)}r). \quad (5.4.34)$$

Finally, from the initial condition (5.4.30), we have

$$1 - r = \sum_{m=1}^{\infty} C_m J_1(\mu_m^{(1)}r)$$

Using the Fourier-Bessel coefficient formula (5.3.12), we have

$$C_m = \frac{\int_0^1 r(1-r)J_1(\mu_m^{(1)}r)dr}{\frac{1}{2}J_1^2(\mu_m^{(1)})} \quad (5.4.35)$$

First, calculate the numerator. Let $\mu_m^{(1)}r = x$, then

$$\begin{aligned} \int_0^1 rJ_1(\mu_m^{(1)}r)dr &= \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} xJ_1(x)dx \\ &= -\frac{1}{(\mu_m^{(1)})^2} [xJ_0(x)]_0^{\mu_m^{(1)}} + \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} J_0(x)dx \\ &= -\frac{J_0(\mu_m^{(1)})}{\mu_m^{(1)}} + \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} J_0(x)dx \end{aligned}$$

and

$$\begin{aligned} \int_0^1 r^2 J_1(\mu_m^{(1)}r)dr &= \frac{1}{(\mu_m^{(1)})^3} \int_0^{\mu_m^{(1)}} x^2 J_1(x)dx \\ &= \frac{1}{(\mu_m^{(1)})^3} [x^2 J_2(x)]_0^{\mu_m^{(1)}} = \frac{J_2(\mu_m^{(1)})}{\mu_m^{(1)}} \end{aligned}$$

Note that, from the recurrence formula, we can get

$$J_0(\mu_m^{(1)}) + J_2(\mu_m^{(1)}) = \frac{2}{\mu_m^{(1)}} J_1(\mu_m^{(1)}) = 0$$

Substitute the above results into (5.4.35) and simplify to get

$$C_m = \frac{2}{(\mu_m^{(1)})^2 J_1^2(\mu_m^{(1)})} \int_0^{\mu_m^{(1)}} J_0(x)dx$$

Substitute the value of C_m into expression (5.4.34) to obtain the solution of the original problem (5.4.28)-(5.4.30).

Ex 5.4.5. Solve the following definite solution problem:

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r \right) + A \quad (0 < r < 1), \leftarrow \boxed{\text{Non-homogeneous eq.}} \quad (5.4.36)$$

$$u|_{r=1} = 0, |u(r, t)| < +\infty, \leftarrow \boxed{\text{Homogeneous bdry.}} \quad (5.4.37)$$

$$u|_{t=0} = 0, \leftarrow \boxed{\text{Homogeneous initial data}} \quad (5.4.38)$$

where A is a constant.

Solution. Apply the *method of eigenfunctions* (recall §2.4).

Step 1: First, for the non-homogeneous equation (5.4.36), the corresponding homogeneous equation is

$$u_t = a^2 \left(u_{rr} + \frac{1}{r} u_r \right).$$

The system of eigenfunctions that simultaneously satisfies the homogeneous boundary condition (5.4.37) is the Bessel function system $\{J_0(\mu_m^{(0)} r)\}_{m=1}^{\infty}$.

Step 2: Assume the solution is

$$u(r, t) = \sum_{m=1}^{\infty} u_m(t) J_0(\mu_m^{(0)} r) \quad (5.4.39)$$

where $u_m(t)$ is a function of t to be determined.

Step 3: Expand the free term A in the equation into a Fourier-Bessel series according to the corresponding Bessel function system:

$$A = \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)} r) \quad (5.4.40)$$

where the coefficients are

$$f_m(t) = \frac{\int_0^1 r \cdot A \cdot J_0(\mu_m^{(0)} r) dr}{\frac{1}{2} J_1^2(\mu_m^{(0)})} = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})} \quad (m = 1, 2, \dots).$$

Step 4: Substitute (5.4.39) and (5.4.40) into (5.4.36) and simplify to get

$$\begin{aligned} & \sum_{m=1}^{\infty} u'_m(t) J_0(\mu_m^{(0)} r) - \sum_{m=1}^{\infty} a^2 u_m(t) \underbrace{\left\{ [J_0(\mu_m^{(0)} r)]'' + \frac{1}{r} [J_0(\mu_m^{(0)} r)]' \right\}}_{\text{Hope to replace it to } J_0(\mu_m^{(0)} r), \text{ then one can compare the coef.}} \\ &= \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)} r). \end{aligned} \quad (5.4.41)$$

- This step features a significant innovation: the derivative terms are replaced by the Bessel equation.
- Only by making this substitution so that it satisfies the Bessel equation, does the first step you took earlier amount to solving the corresponding homogeneous Sturm-Liouville (S-L) problem.
- Similar situations will also occur in Chapter 2, for example, $(\sin \frac{n\pi}{l} x)'' \propto \sin \frac{n\pi}{l} x$. (Recall and compare the relevant content and calculations from Chapter 2 “How to invent the method of eigenfunctions?”)
- This process can be substituted in because it is designed to use the S-L equation to handle (Recall Chapter 2 “How to invent the method of eigenfunctions?”).

From the Bessel equation of order zero, we know that:

$$r^2[J_0(\mu_m^{(0)}r)]'' + r[J_0(\mu_m^{(0)}r)]' + (\mu_m^{(0)})^2 r^2 J_0(\mu_m^{(0)}r) = 0$$

Naturally, we have

$$[J_0(\mu_m^{(0)}r)]'' + \frac{1}{r}[J_0(\mu_m^{(0)}r)]' = -(\mu_m^{(0)})^2 J_0(\mu_m^{(0)}r)$$

Substitute the above formula into (5.4.41) and simplify to get

$$\sum_{m=1}^{\infty} [u'_m(t) + (\mu_m^{(0)}a)^2 u_m(t)] J_0(\mu_m^{(0)}r) = \sum_{m=1}^{\infty} f_m(t) J_0(\mu_m^{(0)}r).$$

By comparing the coefficients of the like terms on both sides, we can get

$$u'_m(t) + (\mu_m^{(0)}a)^2 u_m(t) = f_m(t) = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})}.$$

Step 5: From the initial condition (5.4.38), we can get

$$u_m(0) = 0.$$

Step 6: Apply the general solution formula of the first order linear differential equation or the Laplace transform method to obtain

$$u_m(t) = \frac{2A}{\mu_m^{(0)} J_1(\mu_m^{(0)})} \int_0^t e^{-(\mu_m^{(0)}a)^2(t-\tau)} d\tau = \frac{2A}{(\mu_m^{(0)})^3 a^2 J_1(\mu_m^{(0)})} (1 - e^{-(\mu_m^{(0)}a)^2 t})$$

Finally, substitute the value of $u_m(t)$ into formula (5.4.39) to obtain the solution of the definite solution problem (5.4.36)-(5.4.38):

$$u(r, t) = \frac{2A}{a^2} \sum_{m=1}^{\infty} \frac{(1 - e^{-(\mu_m^{(0)}a)^2 t})}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} J_0(\mu_m^{(0)}r)$$

Notes on Solving Non-homogeneous Boundary Value Problems

- Students may ask how to use auxiliary functions to handle non-homogeneous boundary conditions when encountering non-homogeneous boundary.
- The approach involves constructing an auxiliary function.
- After constructing the auxiliary function, remove it from the variable to transform the problem into one that can be solved using eigenfunction methods and separation of variables.
- Although we haven't covered auxiliary functions explicitly, the method from Chapter 2 still applies.

5.5 Exercise Class

Ex 5.5.1. Calculate definite integrals and indefinite integrals.

1. $\int_0^1 x^3 J_0(\alpha x) dx$, where α is a positive zero point of the Bessel function $J_0(x)$ of order zero, (that is $J_0(\alpha) = 0 \Rightarrow J_1(\alpha) \neq 0$).
2. $\int x^4 J_1(x) dx$
3. $\int J_3(x) dx$ (Hint: Express it in terms of J_0 , J_1 and J_2 , use the recurrence relations $x^n J_n \xrightarrow{\text{diff}} x^n J_{n-1}$, $x^{-n} J_n \xrightarrow{\text{diff}} -x^{-n} J_{n+1}$)

Solution. (1) We will obtain two different expressions of the same result by different recurrence relations.
Let $t = \alpha x$, then $dx = \frac{1}{\alpha} dt$.

$$\begin{aligned} \int_0^1 x^3 J_0(\alpha x) dx &= \frac{1}{\alpha^4} \int_0^\alpha t^3 J_0(t) dt \\ &= \frac{1}{\alpha^4} \int_0^\alpha t^2 \cdot t J_0(t) dt \\ &= \frac{1}{\alpha^4} \int_0^\alpha t^2 d(t J_1(t)) \leftarrow \boxed{\text{Integrate by parts}} \\ &= \frac{1}{\alpha^4} [\alpha^3 J_1(\alpha) - 2 \int_0^\alpha t^2 J_1(t) dt] \end{aligned}$$

Method 1: Using $x^n J_n(x) \xrightarrow{\text{diff}} x^n J_{n-1}(x)$, we obtain

$$\begin{aligned} &= \frac{1}{\alpha^4} [\alpha^3 J_1(\alpha) - 2[t^2 J_2(t)]_0^\alpha] \\ &= \frac{1}{\alpha^4} [\alpha^3 J_1(\alpha) - 2\alpha^2 J_2(\alpha)] \\ &= \frac{1}{\alpha} J_1(\alpha) - \frac{2}{\alpha^2} J_2(\alpha) \end{aligned}$$

Method 2:

$$= \frac{1}{\alpha^4} [\alpha^3 J_1(\alpha) + 2 \int_0^\alpha t^2 J'_0(t) dt]$$

Since $J_0(\alpha) = 0$, then $\int_0^\alpha t^2 J'_0(t) dt = [t^2 J_0(t)]_0^\alpha - 2 \int_0^\alpha t J_0(t) dt$

$$\begin{aligned} &= \frac{1}{\alpha^4} [\alpha^3 J_1(\alpha) + 2[t^2 J_0(t)]_0^\alpha - 4 \int_0^\alpha t J_0(t) dt] \\ &= \frac{1}{\alpha} J_1(\alpha) - \frac{4}{\alpha^3} J_1(\alpha) \end{aligned}$$

Remark 5.5.1. • Compare Method 1 and Method 2: $\frac{2}{\alpha^2} J_2(\alpha) = \frac{4}{\alpha^3} J_1(\alpha) \Rightarrow J_2(\alpha) = \frac{2}{\alpha} J_1(\alpha)$.

- This can be proven by the recurrence relations $J_2(\alpha) + J_0(\alpha) = \frac{2}{\alpha} J_1(\alpha)$ and $J_0(\alpha) = 0$.

(2) **Method 1:** Using $x^n J_n(x) \xrightarrow{\text{diff}} x^n J_{n-1}(x)$,

$$\begin{aligned} \int x^4 J_1(x) dx &= \int x^2 \cdot x^2 J_1(x) dx \\ &= \int x^2 d(x^2 J_2(x)) \\ &= x^2 \cdot x^2 J_2(x) - \int x^2 J_2(x) dx^2 \\ &= x^4 J_2(x) - 2 \int x^3 J_2(x) dx \\ &= x^4 J_2(x) - 2 \int d(x^3 J_3(x)) \\ &= x^4 J_2(x) - 2x^3 J_3(x) + C \end{aligned}$$

Method 2: By $J_1(x) = -J'_0(x)$ and $x^n J_n(x) \xrightarrow{\text{diff}} x^n J_{n-1}(x)$,

$$\begin{aligned} \int x^4 J_1(x) dx &= - \int x^4 dJ_0(x) \\ &= -x^4 J_0(x) + \int J_0(x) dx^4 \\ &= -x^4 J_0(x) + 4 \int x^3 J_0(x) dx \\ &= -x^4 J_0(x) + 4 \int x^2 d(x J_1(x)) \\ &= -x^4 J_0(x) + 4x^3 J_1(x) - 4 \int x J_1(x) dx^2 \\ &= -x^4 J_0(x) + 4x^3 J_1(x) - 8 \int x^2 J_1(x) dx \\ &= -x^4 J_0(x) + 4x^3 J_1(x) - 8x^2 J_2(x) + C. \end{aligned}$$

(3) Idea: reduce the order of J_3 to J_0 and J_1 , use relations $x^{-n} J_n(x) \xrightarrow{\text{diff}} -x^{-n} J_{n+1}(x)$, then $-x^{-1} J_1 \longrightarrow x^{-1} J_2$.

$$\begin{aligned} \int J_3(x) dx &= \int x^2 \cdot x^{-2} J_3(x) dx \leftarrow \boxed{\text{introduce } x^{-n} \text{ to help}} \\ &= - \int x^2 d(x^{-2} J_2(x)) \\ &= -x^2 \cdot x^{-2} J_2(x) + \int x^{-2} J_2(x) dx^2 \\ &= -J_2(x) + 2 \int x^{-1} J_2(x) dx \\ &= -J_2(x) - 2 \int d(x^{-1} J_1(x)) \\ &= -J_2(x) - 2x^{-1} J_1(x) + C. \end{aligned}$$

Ex 5.5.2. 1. Let $\mu_m^{(0)}$ be the m -th positive zero of the Bessel function $J_0(x)$. Try to expand the function $f(x) = x^2 - 1$ into a Fourier-Bessel series of $J_0(\mu_m^{(0)} x)$ on the interval $(0, 1)$. (Hint: Use the recurrence formulas: $\frac{d}{dx}[J_0(x)] = -J_1(x)$, $\frac{d}{dx}[x J_1(x)] = x J_0(x)$)

2. Solve the following boundary value problem:

$$\begin{cases} u_{tt} = a^2(u_{rr} + \frac{1}{r}u_r), & 0 \leq r < 1, t > 0 \leftarrow \boxed{\text{change } u_{tt} \text{ to } u_t, -u_{zz} \text{ to create new exercise}} \\ u(1, t) = 0 \\ u(r, 0) = f(r), \quad f(1) = 0 \leftarrow \boxed{\text{Compatibility condition--initial data is compatible with boundary}} \\ u_t(r, 0) = 0 \end{cases}$$

Solution. 1. $x^2 - 1 = \sum_{n=1}^{\infty} C_m J_0(\mu_m^{(0)} x)$, where

$$C_m = \frac{\int_0^1 x(x^2 - 1)J_0(\mu_m^{(0)} x)dx}{\frac{1}{2}J_1^2(\mu_m^{(0)})}$$

$$\begin{aligned} \int_0^{\mu_m^{(0)}} x^3 J_0(x)dx &= \int_0^{\mu_m^{(0)}} x^2(xJ_1(x))' dx \\ &= (\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2 \int_0^{\mu_m^{(0)}} x^2 J_1(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)})) + 2 \int_0^{\mu_m^{(0)}} x^2 J'_0(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)})) - 4 \int_0^{\mu_m^{(0)}} x J_0(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)})) - 4\mu_m^{(0)} J_1(\mu_m^{(0)}) \end{aligned}$$

or

$$\begin{aligned} \int_0^{\mu_m^{(0)}} x^3 J_0(x)dx &= \int_0^{\mu_m^{(0)}} x^2(xJ_1(x))' dx \\ &= (\mu_m^{(0)})^3 J_1(\mu_m^{(0)}) - 2 \int_0^{\mu_m^{(0)}} x^2 J_1(x)dx \\ &= ((\mu_m^{(0)})^3 J_1(\mu_m^{(0)})) - 2(\mu_m^{(0)})^2 J_2(\mu_m^{(0)}) \end{aligned}$$

Therefore,

$$C_m = \frac{-8}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} \quad \text{or} \quad C_m = \frac{-4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})}.$$

2. Apply the **method of separation of variables**. Let $u(x, t) = R(r)Z(z)$. Substitute it into the equation and separate variables to get

$$\begin{aligned} T'' - \lambda a^2 T &= 0 \\ r^2 R'' + rR' + \lambda r^2 R &= 0 \end{aligned}$$

From $|u(0, t)| < +\infty$ and $u(1, t) = 0$, we know $|R(0)| < +\infty$ and $R(1) = 0$. Solving the zero - order Bessel equation, we get the general solution

$$R(r) = C J_0(\sqrt{\lambda}r) + D Y_0(\sqrt{\lambda}r)$$

From the condition $|R(0)| < +\infty$, we know $D = 0$. Denote $\mu_m^{(0)}$ as the m -th positive zero of $J_0(x)$. Then from the condition $R(1) = 0$, we have $J_0(\sqrt{\lambda}) = 0$, so

$$\begin{cases} \lambda_m = (\mu_m^{(0)})^2 \\ R_m(r) = J_0(\mu_m^{(0)}r) \end{cases}$$

Substitute λ_m into the equation of T to get

$$T_m(t) = a_m \cos(\mu_m^{(0)}at) + b_m \sin(\mu_m^{(0)}at)$$

Then

$$u_m(r, t) = [a_m \cos(\mu_m^{(0)}at) + b_m \sin(\mu_m^{(0)}at)]J_0(\mu_m^{(0)}r)$$

According to the superposition principle

$$u(x, t) = \sum_{m=1}^{+\infty} [a_m \cos(\mu_m^{(0)}at) + b_m \sin(\mu_m^{(0)}at)]J_0(\mu_m^{(0)}r)$$

From the initial value $u(r, 0) = \phi(r)$, we get

$$a_m = \frac{\int_0^1 r\phi(r)J_0(\mu_m^{(0)}r)dr}{\frac{1}{2}J_1^2(\mu_m^{(0)})}, \quad b_m = 0$$

Ex 5.5.3. 1. Calculate the definite integral $\int_0^1 x^3 J_0(\alpha x)dx$, where α is a positive zero of the Bessel function $J_0(x)$ of order zero. (Hint: Use the recurrence formulas: $J'_0(x) = -J_1(x)$, $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$)

2. Solve the following boundary value problem:

$$\begin{cases} u_t = u_{rr} + \frac{1}{r}u_r, & 0 \leq r < 2, t > 0 \\ u(2, t) = 0, |u(0, t)| < +\infty, & t > 0 \\ u(r, 0) = 4 - r^2, & 0 \leq r \leq 2 \end{cases}$$

Solution. 1.

$$\begin{aligned} \int_0^1 x^3 J_0(\alpha x)dx &= \frac{1}{\alpha^4} \int_0^\alpha t^3 J_0(t)dt \\ &= \frac{1}{\alpha^4} \int_0^\alpha t^2 (tJ_1(t))' dt \\ &= \frac{1}{\alpha^4} (\alpha^3 J_1(\alpha) - 2 \int_0^\alpha t^2 J_1(t)dt) \\ &= \frac{J_1(\alpha)}{\alpha} + \frac{2}{\alpha^4} \int_0^\alpha t^2 J'_0(t)dt \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{4}{\alpha^4} \int_0^\alpha t J_0(t)dt \\ &= \frac{J_1(\alpha)}{\alpha} - \frac{4J_1(\alpha)}{\alpha^3} \end{aligned}$$

or

$$\begin{aligned}
\int_0^1 x^3 J_0(\alpha x) dx &= \frac{1}{\alpha^4} \int_0^\alpha t^3 J_0(t) dt \\
&= \frac{1}{\alpha^4} \int_0^\alpha t^2 (t J_1(t))' dt \\
&= \frac{1}{\alpha^4} (\alpha^3 J_1(\alpha) - 2 \int_0^\alpha t^2 J_1(t) dt) \\
&= \frac{J_1(\alpha)}{\alpha} - \frac{2}{\alpha^4} \int_0^\alpha (t^2 J_2)'(t) dt \\
&= \frac{J_1(\alpha)}{\alpha} - \frac{2J_2(\alpha)}{\alpha^2}
\end{aligned}$$

2. Apply the **method of separation of variables**. Let $u(x, t) = R(r)T(t)$. Substitute it into the equation and separate variables to get

$$\begin{aligned}
T' + \lambda T &= 0 \\
r^2 R'' + r R' + \lambda r^2 R &= 0
\end{aligned}$$

From $|u(0, t)| < +\infty$, we have $|R(0)| < +\infty$, and from $u(2, t) = 0$, we have $R(2) = 0$. Solving the zero order Bessel equation, we get the general solution

$$R(r) = C J_0(\sqrt{\lambda} r) + D Y_0(\sqrt{\lambda} r)$$

From the condition $|R(0)| < +\infty$, we know $D = 0$. Denote $\mu_m^{(0)}$ as the m -th positive zero of $J_0(x)$. Then from the condition $R(2) = 0$, we have $J_0(2\sqrt{\lambda}) = 0$, so

$$\begin{cases} \lambda_m = \frac{(\mu_m^{(0)})^2}{4} \\ R_m(r) = J_0\left(\frac{\mu_m^{(0)} r}{2}\right) \end{cases}$$

Substitute λ_m into the equation of T to get

$$T_m(t) = C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}}$$

Then

$$u_m(r, t) = C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0\left(\frac{\mu_m^{(0)} r}{2}\right)$$

According to the superposition principle

$$u(x, t) = \sum_{m=1}^{+\infty} C_m e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0\left(\frac{\mu_m^{(0)} r}{2}\right)$$

From the initial condition, we have

$$\begin{aligned}
 C_m &= \frac{\int_0^2 r(4-r^2)J_0\left(\frac{\mu_m^{(0)}r}{2}\right)dr}{2J_1^2(\mu_m^{(0)})} \\
 &= \frac{8\int_0^{\mu_m^{(0)}} xJ_0(x)dx}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} - \frac{8\int_0^{\mu_m^{(0)}} x^3 J_0(x)dx}{(\mu_m^{(0)})^4 J_1^2(\mu_m^{(0)})} \\
 &= \frac{8}{\mu_m^{(0)} J_1(\mu_m^{(0)})} - \frac{8}{\mu_m^{(0)} J_1(\mu_m^{(0)})} + \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} \\
 &= \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})}.
 \end{aligned}$$

Therefore,

$$u(x, t) = \sum_{m=1}^{+\infty} \frac{32}{(\mu_m^{(0)})^3 J_1(\mu_m^{(0)})} e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0\left(\frac{\mu_m^{(0)} r}{2}\right)$$

or

$$u(x, t) = \sum_{m=1}^{+\infty} \frac{16 J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2 J_1^2(\mu_m^{(0)})} e^{-\frac{(\mu_m^{(0)})^2 t}{4}} J_0\left(\frac{\mu_m^{(0)} r}{2}\right)$$

Ex 5.5.4. Suppose there is a cylinder with radius a and height h , which is adiabatic (heat-insulated) from the outside. The initial temperature is $u_0(1 - \frac{\rho^2}{a^2})$. Find the temperature distribution and variation inside this cylinder.

Solution. Since the initial temperature is **independent** of θ and z , this problem is **independent** of θ and z . The problem can be translated as:

$$\begin{cases} \frac{\partial u}{\partial t} - k \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial u}{\partial \rho}) = 0 & \Leftrightarrow u_t = k(u_{\rho\rho} + \frac{1}{\rho} u_\rho) \\ \frac{\partial u}{\partial \rho} \Big|_{\rho=a} = 0, \quad u|_{\rho=0} \text{ is bounded} & \leftarrow \boxed{\text{The second kind boundary}} \\ u|_{t=0} = u_0(1 - \frac{\rho^2}{a^2}) \end{cases}$$

Using the **method of separation of variables**.

The separation of variables: Let $u(\rho, t) = R(\rho)T(t)$. Then

$$RT' - k \frac{1}{\rho} \frac{d}{d\rho} (\rho \frac{dR}{d\rho}) = RT' - k \frac{1}{\rho} (R' + \rho R'') = 0.$$

Dividing both sides by kRT , we get

$$\frac{T'}{kT} = \frac{R' + \rho R''}{\rho R} = -\lambda,$$

that is $R' + \rho R'' + \lambda \rho R = 0$ and $T' + k\lambda T = 0$.

Transform PDE to ODEs:

$$\begin{cases} T' + k\lambda T = 0 \\ \rho^2 R'' + \rho R' + \lambda \rho^2 R = 0 \end{cases}$$

The boundary conditions can be written as $R'(a)T(t) = 0$. To obtain a nontrivial solution, we have $R'(a) = 0$ and $|R(0)| < +\infty$.

Solve ODEs: The equation $\rho^2 R'' + \rho R' + \lambda \rho^2 R = 0$ is a Bessel equation of order 0.

Supplementary proof of $\lambda \geq 0$

The Bessel's equation is

$$\rho R'' + R' + \lambda \rho R = 0$$

Multiply both sides by R and integrate:

$$\int_0^a \rho R'' R d\rho + \int_0^a R R' d\rho + \lambda \int_0^a \rho R^2 d\rho = 0.$$

Using integration by parts:

$$\int_0^a \rho R'' R d\rho = \underbrace{[\rho R' R]_0^a}_{\text{by bdry condition}=0} - \int_0^a R R' d\rho - \int_0^a \rho (R')^2 d\rho$$

It follows that: $\lambda \int_0^a \rho R^2 d\rho = \int_0^a \rho (R')^2 d\rho$, and then $\lambda \geq 0$.

- Note: Compared with the first type boundary problem, λ can be 0 here.
- If $\lambda = 0 \Rightarrow R' = 0, \Rightarrow R = \text{constant}$. Unlike the first type problem, in the second type problem, $R(\rho) \not\equiv 0$, i.e., $R(\rho)$ could be a nonzero constant.
- Because in the first type problem, $\lambda = 0$ leads to $R(a) = 0$ and $R' = 0 \Rightarrow R \equiv 0$, while the second type boundary problem has no such issue.

Case 1: when $\lambda > 0$. The general solution of the Bessel equation of order 0 is $R(\rho) = C J_0(\sqrt{\lambda}\rho) + D Y_0(\sqrt{\lambda}\rho)$. From the boundedness $|R(0)| < +\infty$, we have $D = 0$. Then using $R'(a) = 0$, since $R'(\rho) = C \sqrt{\lambda} J'_0(\sqrt{\lambda}\rho) \implies R'(a) = C \sqrt{\lambda} J'_0(\sqrt{\lambda}a) = 0$, and $J'_0(\sqrt{\lambda}a) = -J_1(\sqrt{\lambda}a) = 0$ (**recall in the first type boundary, here is $J_0(\sqrt{\lambda}a) = 0$. This is a main different!**). Therefore, $R'(a) = 0 \implies J_1(\sqrt{\lambda}a) = 0$.

Let $\mu_m^{(1)}$ be the positive zero of $J_1(x)$ ($m = 1, 2, \dots$). Then the eigenvalues are

$$\lambda_m = \left(\frac{\mu_m^{(1)}}{a}\right)^2,$$

and the eigenfunctions are

$$R_m(\rho) = J_0\left(\frac{\mu_m^{(1)}}{a}\rho\right) \leftarrow \boxed{\text{It is not } J_0\left(\frac{\mu_m^{(0)}}{a}\rho\right)}, \quad (m = 1, 2, \dots).$$

For the ODE of T , $T_m(t) = C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t}$. So $u_m(\rho, t) = C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0\left(\frac{\mu_m^{(1)}}{a}\rho\right)$.

Case 2 when $\lambda = 0$. Solve $\rho^2 R'' + \rho R' = 0$, we get $R(\rho) = A + B \ln \rho$, then $R'(\rho) = \frac{B}{\rho}$. Using the boundary conditions $R'(a) = 0$ and $|R(0)| < +\infty$, we have $B = 0$, so $R_0(\rho) = A_0$. For the ODE of T , $T_0(t) = A_0^*$, so $u_0(\rho, t) = A_0^* A_0 \equiv C_0$.

Superposition series solution

$$u(\rho, t) = C_0 + \sum_{m=1}^{\infty} C_m e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0\left(\frac{\mu_m^{(1)}}{a} \rho\right)$$

Determine coefficients from initial values

$$u(\rho, 0) = C_0 + \sum_{m=1}^{\infty} C_m J_0\left(\frac{\mu_m^{(1)}}{a} \rho\right) = u_0\left(1 - \frac{\rho^2}{a^2}\right)$$

- We cannot use the Fourier-Bessel coefficients given previously because $\mu_m^{(1)}$ is a zero of J_1 , not a zero of J_0 .

Proposition 5.5.1. Prove that $\int_0^1 J_0(\mu_i^{(1)} x) x dx = 0$, where $\mu_i^{(1)}$ is a zero of $J_1(x)$, that is $J_1(\mu_i^{(1)}) = 0$.

Proof.

$$\int_0^1 J_0(\mu_i^{(1)} x) x dx = \frac{1}{(\mu_i^{(1)})^2} \int_0^{\mu_i^{(1)}} J_0(t) t dt = \frac{1}{(\mu_i^{(1)})^2} [t J_1(t)]_0^{\mu_i^{(1)}} = \frac{1}{(\mu_i^{(1)})^2} \mu_i^{(1)} J_1(\mu_i^{(1)}) = 0$$

This shows that 1 and $J_0(\mu_i^{(1)} x)$ are **orthogonal**. □

Theorem 5.5.1. Let μ_i be a positive zero of $J'_n(r)$. Then

1. Orthogonality: $\int_0^1 J_n(\mu_i r) J_n(\mu_k r) r dr = 0$ ($i \neq k$)
2. And the norm: $\int_0^1 J_n^2(\mu_i r) r dr = \frac{1}{2}(1 - \frac{n^2}{\mu_i^2}) J_n^2(\mu_i)$

Proof. 1. From the previous theorem, we know $\int_0^R r J_n(\frac{\mu_m^{(n)}}{R} r) J_n(\frac{\mu_k^{(n)}}{R} r) dr = 0$ ($m \neq k$). Using the notation in the proof of the orthogonality theorem,

$$(\alpha_1^2 - \alpha_2^2) \int_0^R r F_1(r) F_2(r) dr + \left[r F_2 \frac{dF_1}{dr} - r F_1 \frac{dF_2}{dr} \right]_0^R = 0. \quad (5.5.1)$$

Let $R = 1$, $\alpha_1 = \mu_i$, $\alpha_2 = \mu_k$, then $\frac{dF_1}{dr} \Big|_{r=1} = \mu_i J'_n(\mu_i) = 0$, $\frac{dF_2}{dr} \Big|_{r=1} = \mu_k J'_n(\mu_k) = 0$. So

$$(\alpha_1^2 - \alpha_2^2) \int_0^1 r F_1(r) F_2(r) dr = 0.$$

When $i \neq k$, $\int_0^1 r J_n(\mu_i r) J_n(\mu_k r) dr = 0$, and the orthogonality is proved.

2. Calculate $\int_0^1 J_n^2(\mu_i r) r dr$. Using (5.5.1),

$$\int_0^1 r F_1(r) F_2(r) dr = \frac{[r F_2 \frac{dF_1}{dr} - r F_1 \frac{dF_2}{dr}]_0^1}{\alpha_1^2 - \alpha_2^2}.$$

Let $\alpha_1 = \mu_i$, then $\frac{dF_1}{dr}|_{r=1} = 0$. We further arrive at

$$\underbrace{\int_0^1 r J_n(\mu_i r) J_n(\alpha_2 r) dr}_{\text{a continuous function of } \alpha_2} = \frac{\alpha_2 J_n(\mu_i) J'_n(\alpha_2)}{\mu_i^2 - \alpha_2^2}$$

Then

$$\begin{aligned} \int_0^1 J_n^2(\mu_i r) r dr &= \lim_{\alpha_2 \rightarrow \mu_i} \int_0^1 r J_n(\mu_i r) J_n(\alpha_2 r) dr \\ &= \lim_{\alpha_2 \rightarrow \mu_i} \frac{\alpha_2 J_n(\mu_i) J'_n(\alpha_2)}{\mu_i^2 - \alpha_2^2} \\ &= \lim_{\alpha_2 \rightarrow \mu_i} \frac{\alpha_2 J_n(\mu_i) J''_n(\alpha_2) + J_n(\mu_i) J'_n(\alpha_2)}{-2\alpha_2} \\ &= -\frac{\mu_i J_n(\mu_i) J''_n(\mu_i)}{2\mu_i} = -\frac{1}{2} J_n(\mu_i) J''_n(\mu_i) \end{aligned}$$

Using the Bessel equation $J''_n(x) + \frac{1}{x} J'_n(x) + (1 - \frac{n^2}{x^2}) J_n(x) = 0$ with $x = \mu_i$, $J''_n(\mu_i) = -(1 - \frac{n^2}{\mu_i^2}) J_n(\mu_i)$.

Then

$$\int_0^1 J_n^2(\mu_i r) r dr = \frac{1}{2} (1 - \frac{n^2}{\mu_i^2}) J_n^2(\mu_i).$$

□

Let $a = 1$, $u(\rho, 0) = C_0 + \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(1)} \rho) = u_0(1 - \rho^2)$

$$C_0 = \frac{\int_0^1 u_0(1 - \rho^2) \rho d\rho}{\int_0^1 \rho d\rho} = \frac{u_0 \int_0^1 (\rho - \rho^3) d\rho}{\frac{1}{2}} = \frac{1}{2} u_0,$$

$$C_m = \frac{\int_0^1 u_0(1 - \rho^2) \rho J_0(\mu_m^{(1)} \rho) d\rho}{\int_0^1 J_0^2(\mu_m^{(1)} \rho) \rho d\rho} = \frac{\mu_0 \int_0^1 [\rho J_0(\mu_m^{(1)} \rho) - \rho^3 J_0(\mu_m^{(1)} \rho)] d\rho}{\frac{1}{2} \int_0^1 J_0^2(\mu_m^{(1)}) \rho d\rho}.$$

Calculate

$$\int_0^1 \rho J_0(\mu_m^{(1)} \rho) d\rho = \frac{1}{(\mu_m^{(1)})^2} \int_0^{\mu_m^{(1)}} t J_0(t) dt = \frac{1}{(\mu_m^{(1)})^2} [t J_1(t)]_0^{\mu_m^{(1)}} = 0$$

and

$$\begin{aligned}
\int_0^1 \rho^3 J_0(\mu_m^{(1)} \rho) d\rho &= \frac{1}{(\mu_m^{(1)})^4} \int_0^{\mu_m^{(1)}} t^3 J_0(t) dt \\
&= \frac{1}{(\mu_m^{(1)})^4} \int_0^{\mu_m^{(1)}} t^2 d(t J_1(t)) \\
&= \frac{1}{(\mu_m^{(1)})^4} \left([t^3 J_1(t)]_0^{\mu_m^{(1)}} - 2 \int_0^{\mu_m^{(1)}} t J_1(t) dt \right) \\
&= -\frac{2(\mu_m^{(1)})^2}{(\mu_m^{(1)})^4} J_2(\mu_m^{(1)}) = -\frac{2J_2(\mu_m^{(1)})}{(\mu_m^{(1)})^2}.
\end{aligned}$$

Substitute into C_m , and using $J_0(\mu_m^{(1)}) + J_2(\mu_m^{(1)}) = \frac{2}{\mu_m^{(1)}} J_1(\mu_m^{(1)}) = 0 \Rightarrow J_2(\mu_m^{(1)}) = -J_0(\mu_m^{(1)})$

$$C_m = -\frac{4u_0 J_0(\mu_m^{(1)})}{(\mu_m^{(1)})^2 J_0^2(\mu_m^{(1)})} = -\frac{4u_0}{(\mu_m^{(1)})^2 J_0(\mu_m^{(1)})}.$$

So

$$u(\rho, t) = \frac{1}{2} u_0 - \sum_{m=1}^{\infty} \frac{4u_0}{(\mu_m^{(1)})^2 J_0(\mu_m^{(1)})} e^{-k(\frac{\mu_m^{(1)}}{a})^2 t} J_0\left(\frac{\mu_m^{(1)}}{a} \rho\right).$$

Bibliography

- [1] Yao Duanzheng, *Mathematical physics methods: Learning guide (chinese edition)*, Science Press, 2001.
- [2] Nikolaj Nikolaevič Lebedev, *Special functions and their applications*, 1. publ., unabridged and corr. rep. of the work 1965, Prentice Hall (Richard A. Silverman, ed.), Dover books on mathematics, Dover, New York, NY, 1972.
- [3] Jeffrey M. Lee, *Manifolds and differential geometry*, Graduate Studies in Mathematics, American Mathematical Society, Providence, Rhode Island, 2009. Includes bibliographical references and index. Description based on print version record.
- [4] School of Mathematics, Huazhong University of Science Statistics, and Technology, *Mathematical physics equations and special functions*, 3rd Edition, Higher Education Press, 2019.
- [5] Gu Qiao, *Methods of mathematical physics (chinese edition)*, Science Press, 2000.
- [6] Ding Tongren and Li Chengzhi, *Ordinary differential equations: A textbook (chinese edition)*, 3rd ed., Higher Education Press, 2022.
- [7] Chongshi Wu and Chunyuan Gao, *Methods of mathematical physics (chinese edition)*, 3rd ed., Peking University Press, 2019.