# Lecture Notes: Mathematical Physics Equations and Special Functions

Chao Liu

Week 8-2

### 0.0.1 Green's Function in a Spherical Domain and the Dirichlet Problem

We aim to solve the Dirichlet problem in a **spherical domain**:

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in \Omega, \tag{1}$$

$$u|_\Gamma = f(x, y, z), \tag{2}$$

where $\Omega$ is a spherical domain centered at $o$ with radius $R$, and $\Gamma$ is its boundary.

**Reviews**

- **Three key elements of the physical meanings of $G$**
  - A conducting shell.
  - A unit positive charge.
  - A grounded connection.
- The location of the unit positive charge can be **arbitrary**, but if one wants to know the **value of $u$ at point** $M_0$, then place a unit positive **charge at** $M_0$.

$$u(M_0) = -\iint_\Gamma f(M) \frac{\partial G(M, M_0)}{\partial n} dS. \tag{3}$$

- **Three steps of the method of images:**
  1. Determine the image point (the charge quantity and position. Place $M$ on the boundary for verification).
     - **Criterion for Position:** The sum of the potentials from the original charge and the mirror charge at the boundary should be zero.
  2. Write out the Green's function (Place $M$ inside $\Omega$).
  3. Calculate $\frac{\partial G}{\partial n}\big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place $M$ on the boundary again).

**Step 1: Determine the image point (the charge quantity and position. Place $M$ on the boundary for verification).**

Now, we use the **method of electrical images** to find the Green's function of the sphere. To this end, we take an arbitrary point $M_0(x_0, y_0, z_0)$ inside the sphere. On the semiray $oM_0$, we intercept a line segment $oM_1$ such that

$$r_{OM_0} \cdot r_{OM_1} = R^2. \leftarrow \boxed{\text{gives the } \textbf{position} \text{ of the image point}} \tag{4}$$

The point $M_1$ is called the **inversion point** or **symmetric point** of the point $M_0$ with respect to the spherical surface $\Gamma$ (see Fig. 1).

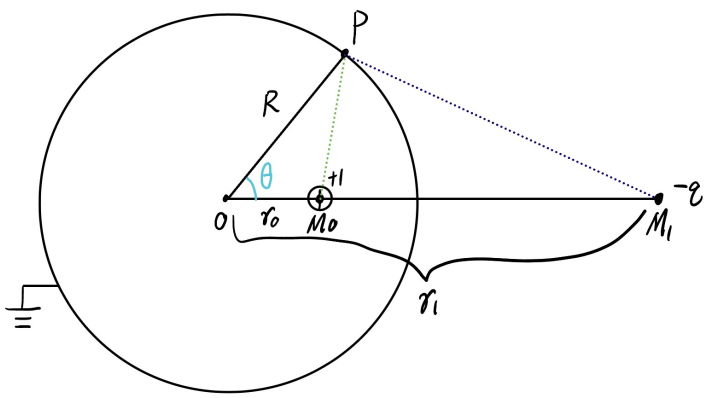


Figure 1: Verification of cancellations

- The image point must lie on the line passing through the center of the sphere and the charge point.
- This is due to **symmetry**; if it were not on this line, the system would be asymmetric.
- **Using Similar Triangles:**
  - The triangle $\triangle_{oM_0P}$ should be similar to the triangle $\triangle_{oPM_1}$.
  - This similarity gives the condition $R^2 = r_{oM_0} \cdot r_{oM_1}$.
- The potentials due to the charges at $M_0$ and $M_1$ should cancel out at the boundary.

To find the Green's function $G(M, M_0)$, we place a **unit positive charge** at the point $M_0$ and a charge of $q$ **units** of negative charge **at the point** $M_1$. We need to appropriately choose the **value of** $q$ so that the electric potentials generated by these two point charges **exactly cancel each other out** on the **spherical surface** $\Gamma$. Let $P$ be an arbitrary point **on the spherical surface**. Then we should have

$$\frac{1}{4\pi r_{M_0P}} = \frac{q}{4\pi r_{M_1P}} \implies q = \frac{r_{M_1P}}{r_{M_0P}}. \leftarrow \boxed{\text{gives the } \textbf{charge quantity} \text{ of the image point}}$$

Since $\triangle OM_0P$ and $\triangle OPM_1$ have a common angle at point $O$, and the corresponding two sides enclosing this angle are proportional according to formula (4), these two triangles are similar. Thus, we have

$$\frac{r_{M_1P}}{r_{M_0P}} = \frac{R}{r_{OM_0}} \implies q = \frac{R}{r_{OM_0}},$$

That is to say, we must place a negative charge of $R/r_0$ units at the point $M_1$. From this, we get

$$v = \frac{1}{4\pi}\frac{R}{r_{OM_0}}\frac{1}{r_{MM_1}}.$$

**How to select the position of the image point–Method 1**

$$\begin{aligned}
|M_0P|^2 &= |OM_0|^2 + |OP|^2 - 2|OP||OM_0|\cos\theta \\
|M_1P|^2 &= |OP|^2 + |OM_1|^2 - 2|OP||OM_1|\cos\theta \\
\text{const.}\left(\frac{1}{q}\right)^2 &= \frac{|M_0P|^2}{|M_1P|^2} = \frac{|OM_0|^2 + |OP|^2 - 2|OP||OM_0|\cos\theta}{|OP|^2 + |OM_1|^2 - 2|OP||OM_1|\cos\theta} = \frac{r_0^2 + R^2 - 2Rr_0\cos\theta}{R^2 + r_1^2 - 2Rr_1\cos\theta} =: f(\theta)
\end{aligned} \tag{5}$$

for all $\theta \in [0, \pi]$ Once $r_1$ and $q$ are selected, this equation (5) should hold for every $\theta$. Differentiating (5) yields ($q$ is a constant if it is selected)

$$
\begin{aligned}
0 = f'(\theta) &= \frac{2Rr_0 \sin\theta(R^2 + r_1^2 - 2Rr_1 \cos\theta) - (r_0^2 + R^2 - 2Rr_0 \cos\theta) \cdot 2Rr_1 \sin\theta}{(R^2 + r_1^2 - 2Rr_1 \cos\theta)^2} \\
&\Rightarrow r_0(R^2 + r_1^2 - 2Rr_1 \cos\theta) = r(r_0^2 + R^2 - 2Rr_0 \cos\theta) \\
&\Rightarrow r_0 R^2 + r_0 r^2 = r_1 r_0^2 + r_1 R^2 \\
&\Rightarrow (r_0 - r_1)R^2 = r_1 r_0 (r_0 - r_1) \Rightarrow R^2 = r_0 r_1 \Rightarrow \frac{R}{r_1} = \frac{r_0}{R} \text{ (similar triangles)}
\end{aligned}
$$

**How to select the position of the image point–Method 2**

Since

$$
\left(\frac{1}{q}\right)^2 = \frac{|M_0P|^2}{|M_1P|^2} = \frac{|OM_0|^2 + |OP|^2 - 2|OP||OM_0|\cos\theta}{|OP|^2 + |OM_1|^2 - 2|OP||OM_1|\cos\theta} = \frac{r_0^2 + R^2 - 2Rr_0\cos\theta}{R^2 + r_1^2 - 2Rr_1\cos\theta}
$$

There are two unknowns $q$ and $r_1$ and we need two equations to solve them. Since this identity holds for all $\theta \in [0, \pi]$, we choose $\theta = 0$ and $\theta = \pi$ to obtain two equations

$$
\left(\frac{1}{q}\right)^2 = \frac{r_0^2 + R^2 - 2Rr_0}{R^2 + r_1^2 - 2Rr_1}
$$

$$
\left(\frac{1}{q}\right)^2 = \frac{r_0^2 + R^2 + 2Rr_0}{R^2 + r_1^2 + 2Rr_1}
$$

Solving these equation, we arrive at

$$
r_1 = \frac{R^2}{r_0} \quad \text{and} \quad q = \frac{r_1}{R} = \frac{R}{r_0}.
$$

**Step 2: Write out the Green's function (Place $M$ inside $\Omega$).**

Then, the Green's function of the spherical domain with $\Gamma$ as the spherical surface is (see Fig. 2)

$$
G(M, M_0) = \frac{1}{4\pi}\left(\frac{1}{r_{MM_0}} - \frac{R}{r_{OM_0}}\frac{1}{r_{MM_1}}\right). \tag{6}
$$

Let $r_0 = r_{OM_0}$, $r = r_{OM}$, $r_1 = r_{OM_1}$, and $\gamma$ be the angle between $OM_0$ and $OM$. Then formula (6) can be transformed into

$$
G(M, M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\gamma}} - \frac{R}{r_0}\frac{1}{\sqrt{r_1^2 + r^2 - 2r_1 r\cos\gamma}}\right]
$$

Using the relation (4) $r_{OM_0} \cdot r_{OM_1} = R^2$, we can obtain

$$
G(M, M_0) = \frac{1}{4\pi}\left[\frac{1}{\sqrt{r_0^2 + r^2 - 2r_0 r\cos\gamma}} - \frac{R}{\sqrt{r^2 r_0^2 - 2R^2 r_0 r\cos\gamma + R^4}}\right]
$$

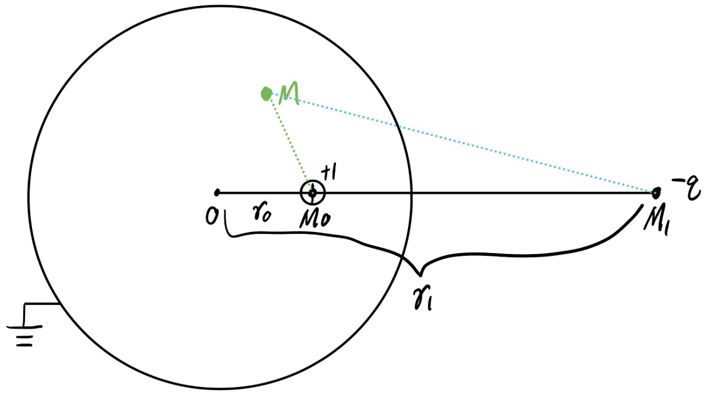


Figure 2: Green's function of the sphere

**Step 3: Calculate $\frac{\partial G}{\partial n}\big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place $M$ on the boundary again).**

To solve the original problems (1) and (2), we also need to calculate $\frac{\partial G}{\partial n}\big|_\Gamma$. On the spherical surface $\Gamma$,

$$\begin{aligned}\frac{\partial G}{\partial n}\bigg|_\Gamma &= \frac{\partial G}{\partial r}\bigg|_{r=R}\\ &= -\frac{1}{4\pi}\left[\frac{r - r_0\cos\gamma}{(r_0^2 + r^2 - 2r_0 r\cos\gamma)^{3/2}} - \frac{(rr_0^2 - R^2 r_0\cos\gamma)R}{(r^2 r_0^2 - 2R^2 r_0 r\cos\gamma + R^4)^{3/2}}\right]_{r=R}\\ &= -\frac{1}{4\pi R}\frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0\cos\gamma)^{3/2}}\end{aligned}$$

Therefore, from (3), the expression for the solution of problems (1) and (2) is

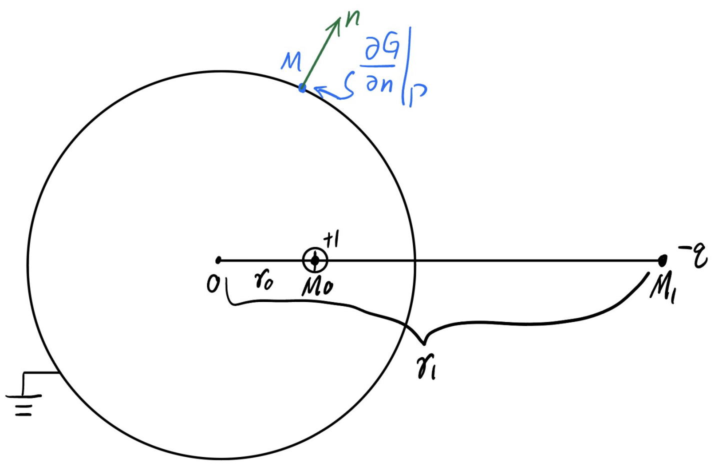


Figure 3: Calculate $\frac{\partial G}{\partial n}\big|_{z=0}$

$$u(M_0) = \frac{1}{4\pi R}\iint\limits_\Gamma f(x,y,z)\frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0\cos\gamma)^{3/2}}dS. \tag{7}$$

In the spherical coordinate system, expression (7) becomes

$$u(r_0,\theta_0,\varphi_0) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(R,\theta,\varphi)\frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0\cos\gamma)^{3/2}}\sin\theta d\theta d\varphi, \tag{8}$$

Formulas (7) or (8) are called **Poisson's formulas for the spherical domain**.

Here, $(r_0,\theta_0,\varphi_0)$ are the spherical coordinates of the point $M_0$, $(R,\theta,\varphi)$ are the moving coordinates of points on the spherical surface $\Gamma$, and $\cos\gamma$ is the cosine of the angle between $\overrightarrow{OM_0}$ and $\overrightarrow{OP}$.

Since

$$\begin{aligned}\overrightarrow{OM_0}\cdot\overrightarrow{OP} &= |\overrightarrow{OM_0}||\overrightarrow{OP}|\cos\gamma\\ \overrightarrow{OM_0} &= (r_0\sin\theta_0\cos\varphi_0, r_0\sin\theta_0\sin\varphi_0, r_0\cos\theta_0)\\ \overrightarrow{OP} &= (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta),\end{aligned}$$

we obtain

$$\begin{aligned}\cos\gamma &= \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0(\cos\varphi\cos\varphi_0 + \sin\varphi\sin\varphi_0)\\ &= \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi-\varphi_0)\end{aligned}$$

**Ex 0.1.** *Suppose there is a homogeneous sphere of radius $R$. The temperature of the upper hemisphere surface is maintained at $0°C$, and the temperature of the lower hemisphere surface is maintained at $1°C$. We want to find the steady state temperature distribution inside the sphere.*

**Solution.** *This problem can be reduced to the following well - posed problem:*

$$\begin{cases}\Delta u(r,\theta,\varphi) = 0 & (0<r<R)\\ u|_{r=R} = \begin{cases}0, & 0<\theta<\frac{\pi}{2}\\ 1, & \frac{\pi}{2}<\theta<\pi\end{cases}\end{cases}$$

*Using formula (8), we have*

$$u(r_0,\theta_0,\varphi_0) = \frac{R}{4\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}}\sin\theta d\theta d\varphi.$$

*In particular, we want to find the temperature distribution on the vertical diameter of the sphere: $\theta_0 = 0$ (the upper half of the diameter) and $\theta_0 = \pi$ (the lower half of the diameter).*

*When $\theta_0 = 0$, $\cos\gamma = \cos\theta$, so*

$$\begin{aligned}u(r_0,0,\varphi_0) &= \frac{R}{4\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\theta)^{3/2}}\sin\theta d\theta d\varphi\\ &= \frac{R}{2}\left[-\frac{R^2-r_0^2}{Rr_0(R^2+r_0^2-2Rr_0\cos\theta)^{1/2}}\right]_{\theta=\frac{\pi}{2}}^{\theta=\pi}\\ &= \frac{1}{2}\frac{R^2-r_0^2}{r_0}\left(\frac{1}{\sqrt{R^2+r_0^2}}-\frac{1}{R+r_0}\right)\end{aligned}$$

*When $\theta_0 = \pi$, $\cos\gamma = -\cos\theta$, so*

$$u(r_0,0,\varphi_0) = \frac{1}{2}\frac{R^2-r_0^2}{r_0^2}\left(\frac{1}{R-r_0}-\frac{1}{\sqrt{R^2+r_0^2}}\right)$$

### Supplementary: Green's Function and Dirichlet Problem in a Circular Domain (2D)

Firstly according to the previous results, the Green's function and the solution to the Dirichlet problem in a circular domain are

$$G(M,M_0) = \frac{1}{2\pi}\ln\frac{1}{r_{MM_0}} - v, \tag{9}$$

$$u(M_0) = -\int_C f(M)\frac{\partial G}{\partial n}dS. \tag{10}$$

Solving the Dirichlet Problem in a Circular Domain We aim to solve the Dirichlet problem in a circular domain:

$$u_{xx}+u_{yy} = 0, \quad (x,y)\in D, \tag{11}$$

$$u|_{x^2+y^2=R^2} = f(x,y), \tag{12}$$

where $D$ is a circular domain centered at $o$ with radius $R$, and $C$ is its boundary.

Using the method of images, the Green's function for the circular domain can be obtained as

$$G(M, M_0) = \frac{1}{2\pi}\left[\ln\frac{1}{r_{MM_0}} - \ln\Big(\underbrace{\frac{R}{r_0}}_{(*)}\frac{1}{r_{MM_1}}\Big)\right], \tag{13}$$

where $r_0 = r_{OM_0}$, and the point $M_1$ is called the inversion point or symmetric point of the point $M_0$ with respect to the circular boundary $C$.

- (*) term is actually not due to the charge quantity, but rather an additional term resulting from the choice of the zero point of electric potential.

**Differences between 3D and 2D**

- To solve the 2D problem, one might initially think of following the same procedure as in 3D, which involves finding the Green's function and the image charge. However, there are **two significant differences** between the 2D and 3D cases.
  1. At first glance, Equation (13) appears to be very similar to Equation (6). One might be tempted to think that it is derived through a straightforward analogy. However, things are not that simple. First of all, what is the electric **potential in 2D case**?
     - In 3D, the potential at a point due to a charge $q$ is given by $\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r}$, where $r$ is the distance from the charge. In 2D, the fundamental solution for the potential is $\frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$.
     - A common mistake is to directly apply the 3D method to the 2D case, leading to an incorrect expression for the potential. For example, one might mistakenly write the potential as $-\frac{1}{2\pi}\ln\left(\frac{q}{r}\right)$, but this is wrong.
     - The **correct expression** for the **potential in 2D** due to a charge $q$ should be $-\frac{q}{2\pi}\ln\left(\frac{1}{r}\right)$. This form ensures that the potential satisfies the 2D Laplace's equation.

       *Proof.* Consider the equation:

       $$\begin{aligned} \nabla^2 u &= \kappa\delta(\mathbf{r}-\mathbf{r}_0) \\ \kappa &= -1 \quad \text{Potential of unit positive charge} \\ \kappa &= -q \quad \text{Potential of point charge with charge } q \end{aligned}$$

       Let $u = C_1 \ln\frac{C_2}{r}$, Just like before, calculate $\kappa$ in the same way, integrate over a circle,

       $$\int_{S_a} \Delta u dS = \kappa\int_{S_a}\delta(\mathbf{r}-\mathbf{r}_0)dS = \kappa$$

       Using Gauss's theorem:

       $$\int_{\partial S_a}\mathbf{n}\cdot\nabla u dl = \int_{\partial S_a}\frac{\partial u}{\partial r}dl = \kappa = -q$$

       Since

       $$\frac{\partial u}{\partial r} = -\frac{C_1}{r} \Rightarrow -C_1\int_{\partial S_a}\frac{1}{r}dl = -q$$

       $$\Rightarrow -\frac{C_1}{a}\cdot 2\pi a = -q \Rightarrow C_1 = \frac{q}{2\pi} \Rightarrow u = \frac{q}{2\pi}\ln\frac{C_2}{r}$$

       where $C_2$ is related to the **gauge** (**choice** of **zero potential** point). □
  2. The **error in the direct analogy** arises because in 2D, the **potential cannot be zero at infinity** as it is in 3D. The correct derivation involves **additional terms** and considerations that are not present in the 3D case.

## Additional reading: about the 2D Green function from [2, Page 384-386]

The following content comes from [2, Page 384-386], we refer readers to [2, Page 384-386] or [1, Page 352]for details.

Let us consider the problem of $G$:

$$\Delta G = \partial_x^2 G + \partial_y^2 G = -\delta(r - r_0) \tag{14}$$

$$G|_{r=R} = 0. \tag{15}$$

An equivalent way to make the potential of the cylinder zero is that, in addition to the point source at $M_0$, there is an image charge (an infinitely long line charge parallel to the $z$-axis with linear density $\rho$) at a certain point $M_1$. According to symmetry analysis, $M_1$ should be on the extension of $OM_0$. Let the potential at point $M$ be

$$G(M, M_0) = \frac{1}{2\pi}\ln\frac{1}{r_{M_0M}} + \frac{\rho}{2\pi}\ln\frac{1}{r_{M_1M}} + \frac{1}{2\pi}\ln\frac{r_0}{R} \leftarrow \boxed{\text{related to the zero point of the potential}} \tag{16}$$

- Note $r = \infty$ is not the zero potential for 2D case due to the potential $\frac{1}{2\pi}q\ln\frac{1}{r}$.

where

$$r_{M_0M} = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)},$$

$$r_{M_1M} = \sqrt{r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_0)}.$$

Since the value of the potential is relative and can differ by a constant, the **third term on the right** hand side of equation (16) is an **additional constant** [this constant is added to satisfy the boundary condition (15) subsequently]. Now we use the boundary condition (15), that is, $G(P, M_0) = 0$, to determine the linear density $\rho$ of the image charge and its position $r_1$ in equation (16). When $r \to R$, $M \to P$, $G(M, M_0) \to G(P, M_0) = 0$, and we have

$$\begin{aligned}
G(P, M_0) &= \frac{1}{2\pi}\ln\frac{1}{r_{M_0P}} + \frac{\rho}{2\pi}\ln\frac{1}{r_{M_1P}} + \frac{1}{2\pi}\ln\frac{r_0}{R} \\
&= -\frac{1}{4\pi}\ln\left[R^2 + r_0^2 - 2Rr_0\cos(\theta - \theta_0)\right] \\
&- \frac{\rho}{4\pi}\ln\left[R^2 + r_1^2 - 2Rr_1\cos(\theta - \theta_0)\right] + \frac{1}{2\pi}\ln\frac{r_0}{R} \\
&= 0
\end{aligned}$$

In the above formula, $\theta$ is a variable. From $\frac{\partial G(P, M_0)}{\partial\theta} = 0$, we obtain

$$\frac{r_0}{R^2 + r_0^2 - 2Rr_0\cos(\theta - \theta_0)} + \frac{\rho r_1}{R^2 + r_1^2 - 2Rr_1\cos(\theta - \theta_0)} = 0.$$

That is

$$r_0\left(R^2 + r_1^2\right) - 2r_0r_1R\cos(\theta - \theta_0) = -\rho r_1\left(R^2 + r_0^2\right) + 2\rho r_0r_1R\cos(\theta - \theta_0) \tag{17}$$

Equation (17) holds for any $\theta$. Comparing the constant terms and the coefficients of $\cos(\theta - \theta_0)$ on both sides, we get

$$r_0\left(R^2 + r_1^2\right) = -\rho r_1\left(R^2 + r_0^2\right) \tag{18}$$

$$\rho = -1 \tag{19}$$

Substituting equation (19) into equation (18), we obtain

$$r_1 = \frac{R^2}{r_0}$$

Then equation (16) becomes

$$\begin{aligned}
G(M, M_0) &= \frac{1}{2\pi} \ln \frac{1}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}} \\
&- \frac{1}{2\pi} \ln \frac{1}{\sqrt{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_0)}} + \frac{1}{2\pi} \ln \frac{r_0}{R} \\
&= \frac{1}{4\pi} \ln \left[ \frac{r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_0)}{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)} \cdot \frac{r_0^2}{R^2} \right] \\
&= \frac{1}{4\pi} \ln \frac{r^2 r_0^2 + R^4 - 2rr_0 R^2 \cos(\theta - \theta_0)}{R^2 \left[ r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) \right]}
\end{aligned} \tag{20}$$

Equation (20) only contains the coordinates $r_0, \theta_0$ of the point source and the coordinates $r, \theta$ of the observation point. It is the Green's function we seek. In particular, we can see that due to the introduction of the additional constant $\frac{1}{2\pi} \ln \frac{r_0}{R}$, equation (20) gives $G(P, M_0) = 0$ when $r \to R$, satisfying the boundary condition (15).

On the circle $x^2 + y^2 = R^2$,

$$\left. \frac{\partial G}{\partial n} \right|_C = \left. \frac{\partial G}{\partial r} \right|_{r=R} = -\frac{1}{2\pi R} \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos\gamma}$$

Therefore, from (10), the integral expression for the solution of problems (11) and (12) is

$$u(M_0) = \frac{1}{2\pi R} \int_C f(x, y) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos\gamma} dS. \tag{21}$$

In polar coordinates, expression (21) becomes ($r_0 < R$)

$$u(r_0, \theta_0) = \frac{1}{2\pi} \int_0^{2\pi} f(R, \theta) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos\gamma} d\theta, \tag{22}$$

where $\cos\gamma = \cos(\theta - \theta_0)$.

Formulas (21) or (22) are called Poisson's formulas for the circular domain. Formula (22) is the same as formula in Section 2.3 of Chapter 2.

# 1 Trial and Error Method and Solution of Poisson's Equation

## 1.1 Trial and Error Method

For some well posed problems in practice, based on the **physical meaning** and **geometric characteristics** of the problems, we can assume that the solution has a certain form and substitute it for trial. This is called the **trial and error method**.

**Ex 1.1.** *Find the electric potential in a capacitor made of two concentric spherical conductors $r = r_1$ and $r = r_2$, where the inner spherical surface $r = r_1$ maintains a constant potential $v_0$ and the outer spherical surface is grounded.*

- **Trial Method:** Guessing should be based on
  - physical meaning,
  - geometric features,
  - the form of inhomogeneous terms.
- **Geometric Features:** For problems with boundary conditions independent of $\theta, \phi$, the solution may also be independent of $\theta, \phi$.

- **Spherical Symmetry:** Symmetric solutions–we have previously discussed spherical symmetric solutions in Section 4.1.
- **General Solution:** The general solution for symmetric cases is often of the form $\frac{C_1}{r} + C_2$. The fundamental solution can be chosen by setting $C_1 = 1$ and $C_2 = 0$ for simplicity.

**Solution.** *Since the region is a spherical shell, it is more convenient to use spherical coordinates. In the spherical coordinate system, the above problem is reduced to*

$$\begin{cases} \Delta u(r,\theta,\varphi) = 0 & (r_1 < r < r_2) \\ u|_{r=r_1} = v_0,\ u|_{r=r_2} = 0 \end{cases}$$

*From the boundary conditions, we know that the distribution of the electric potential inside the sphere is only related to $r$, that is, the potential function is spherically symmetric. Using the general form of the spherically symmetric solution in Section 4.1, we can assume*

$$u(r) = \frac{A}{r} + B$$

*where $A$ and $B$ are undetermined constants. To determine $A$ and $B$, from the boundary conditions, we have*

$$\frac{A}{r_1} + B = v_0, \quad \frac{A}{r_2} + B = 0$$

$$A = \frac{r_1 r_2}{r_2 - r_1} v_0, \quad B = -\frac{r_1}{r_2 - r_1} v_0$$

*So the required electric potential is*

$$u = \frac{r_1 r_2}{r_2 - r_1}\left(\frac{1}{r} - \frac{1}{r_2}\right) v_0$$

**Ex 1.2.** *Suppose there is an infinitely long homogeneous cylinder of radius $R$. Given that the temperature distribution on its cylindrical surface is $xy$, find the steady - state temperature distribution inside the cylinder.*

**Solution.** *Since the given temperature on the cylindrical surface is independent of $z$, the circular discs perpendicular to the $z$-axis have the same temperature distribution. Therefore, the given spatial problem can be reduced to a planar problem. Because the boundary shape is a circle, it is more convenient to use polar coordinates. So we solve the problem in the polar coordinate system:*

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & (r < R) \\ u|_{r=R} = \frac{1}{2}R^2 \sin 2\theta. \leftarrow \boxed{\textit{Guessing according to the inhomogeneous term}} \end{cases}$$

*Let its solution be $u(r,\theta) = Ar^2 \sin 2\theta + B$, where $A$ and $B$ are undetermined constants. It is easy to verify that the function $u(r,\theta) = Ar^2 \sin 2\theta + B$ satisfies the equation. Now, we determine the constants $A$ and $B$ from the boundary conditions. Since*

$$u|_{r=R} = AR^2 \sin 2\theta + B = \frac{1}{2}R^2 \sin 2\theta$$

*we get $A = \frac{1}{2}$ and $B = 0$. So the solution to the original problem is*

$$u(r,\theta) = \frac{1}{2}r^2 \sin 2\theta, \ \textit{i.e.}, \ u(x,y) = xy$$

**Ex 1.3.** *Find the electric potential $u$ in the cylindrical domain $r < R$ such that the normal component of the electric field strength on the cylindrical surface is given as:*

$$\left.\frac{\partial u}{\partial n}\right|_{r=R} = (x+y)|_{r=R}$$

**Solution.** *This problem can be reduced to*

$$\begin{cases} \Delta u(r,\theta) = 0 \qquad (r < R) \\ \frac{\partial u}{\partial n}\Big|_{r=R} = \frac{\partial u}{\partial r}\Big|_{r=R} = \underbrace{R}_{\text{may come from the constant } A,B \text{ or } r} \cos\theta + R\sin\theta \leftarrow \boxed{\text{the second boundary}} \end{cases}$$

*Let the solution of this problem be $u(r,\theta) = Ar\cos\theta + Br\sin\theta + C$, where $A$, $B$, and $C$ are specific constants. It is obvious that this function satisfies the equation. To make it satisfy the boundary conditions, from the boundary conditions, we have*

$$\frac{\partial u}{\partial n}\bigg|_{r=R} = \frac{\partial u}{\partial r}\bigg|_{r=R} = A\cos\theta + B\sin\theta = R\cos\theta + R\sin\theta$$

*Thus, $A = B = R$. So the solution to the original problem is*

$$u(r,\theta) = Rr(\cos\theta + \sin\theta) + C \leftarrow \boxed{\text{The solution is not unique for the second boundary}}$$

- **Boundary Conditions:** The variable $R$ could originate from $r^2$ or a constant term $A, B$, making it challenging to ascertain its origin.
- **Setting Up the Solution:** One might initially try solutions of the form $R^2\cos(\theta)$ or $R^2\sin(\theta)$, but these may not yield a solution. A more general approach is to assume a solution of the form $Ar^\alpha\cos(\theta) + Br^\alpha\sin(\theta) + C$, where $\alpha$ is a constant to be determined.
- **Determining Parameters:** By substituting the assumed solution into the boundary conditions and the differential equation, one can determine the parameters $A$, $B$, and $\alpha$.
- **Non-uniqueness of Solutions:** The presence of an arbitrary constant $C$ in the solution indicates that the solution is not unique, as $C$ can take any real value. This implies that the second type of boundary value problem may have infinitely many solutions. The non-uniqueness is because the problem does **not involve the function $u$ itself** but only its derivatives.
- **Well-posed Problems:** The concept of well-posed problems is discussed. For the Laplace equation, the first type of boundary conditions ensures a unique solution, while the second type may not. The third type will be shown to have a unique solution through an example later.
- **Conclusion:** The second type of boundary conditions for the Laplace equation may lead to an ill-posed problem, which is a point of attention.

## 1.2 Solution of Poisson's Equation

If we know a particular solution of Poisson's equation, then through a **function substitution**, we can transform the boundary value problem of **Poisson's equation** into a boundary value problem of **Laplace's equation**← *Find a function to carry the inhomogeneous term, the rest part is a harmonic function.*

If the **free term** in Poisson's equation is an **$n$-th degree polynomial** of the independent variables, then we can take a **particular solution** of the equation as an **$(n+2)$-th degree polynomial** of the independent variables. Substitute it into Poisson's equation and compare the coefficients of the corresponding terms on both sides of the equation to determine the constants.

**Ex 1.4.** *Find a particular solution of the equation $u_{xx} + u_{yy} = xy$.*

**Solution.** *Since $f(x,y) = xy$ is a quadratic polynomial of the independent variables $x$ and $y$, for the convenience of calculation, we may take the particular solution as*

$$w(x,y) = Ax^3y + Bxy^3$$

*Substitute it into the equation, we get*

$$6(A+B)xy = xy \implies 6(A+B) = 1$$

$$\begin{cases} B=0, A=\frac{1}{6}, & w(x,y)=\frac{1}{6}x^3y \\ A=0, B=\frac{1}{6}, & w(x,y)=\frac{1}{6}xy^3 \\ A=B=\frac{1}{12}, & w(x,y)=\frac{1}{12}xy(x^2+y^2) \end{cases}$$

**Ex 1.5.** *Find the solution of the following problem*

$$\begin{cases} \Delta u(x,y) = -4 & (x^2+y^2<a^2) \\ u|_{x^2+y^2=a^2} = 0 \end{cases}$$

**Solution.** *Obviously, a particular solution of Poisson's equation is*

$$w(x,y) = -(x^2+y^2).$$

*Make a function substitution $u = v(x,y) - (x^2+y^2)$, then the above problem is transformed into*

$$\begin{cases} \Delta v(x,y) = 0 & (x^2+y^2<a^2) \\ v|_{x^2+y^2=a^2} = a^2 \end{cases} \leftarrow \boxed{\textit{Consider the boundary } x^2+y^2=a^2}$$

*By the extreme value principle, the solution of the above problem is $v(x,y) = a^2$, that is, the solution of the original problem is*

$$u(x,y) = a^2 - (x^2+y^2).$$

**Ex 1.6.** *Solve*

$$\begin{cases} \Delta u(x,y,z) = -6 & (x^2+y^2+z^2<a^2) \\ u|_{x^2+y^2+z^2=a^2} = a^2 \end{cases}$$

**Solution.** *Obviously, we can take $w(x,y,z) = -(x^2+y^2+z^2)$. Let $u = v(x,y,z) - (x^2+y^2+z^2)$. Then the above problem can be transformed into*

$$\begin{cases} \Delta v(x,y,z) = 0 \\ v|_{x^2+y^2+z^2=a^2} = 2a^2 \end{cases}$$

*By the extreme - value principle, the solution of this problem is $v(x,y,z) = 2a^2$, so the solution of the original problem is*

$$u(x,y,z) = 2a^2 - (x^2+y^2+z^2).$$

### 1.2.1 External Dirichlet Problem

Given a continuous function $f(x,y,z)$ on a closed surface $\Gamma$ in the space $(x,y,z)$, we require the function $u(x,y,z)$ to satisfy Laplace's equation in the external region $\Omega^c$ of $\Gamma$ (except for the **point at infinity**), be continuous on $\Omega^c \cup \Gamma$, and satisfy the conditions

$$\lim_{r\to\infty} u(x,y,z) = 0 \quad (r=\sqrt{x^2+y^2+z^2}) \leftarrow \boxed{\text{Infinity is a boundary}}$$

and the condition

$$u|_\Gamma = f(x,y,z)$$

The original Dirichlet problem is called the internal Dirichlet problem.

### 1.2.2 External Neumann Problem

Given a continuous function $f(x,y,z)$ on a smooth closed surface $\Gamma$ in the space $(x,y,z)$, we require the function $u(x,y,z)$ to satisfy Laplace's equation in the external region $\Omega^c$ of $\Gamma$ (except for the point at infinity), be continuous on $\Omega^c \cup \Gamma$, the normal derivative $\frac{\partial u}{\partial n'}$ exists at any point on $\Gamma$, and satisfy the conditions

$$\lim_{r\to\infty} u(x,y,z) = 0 \quad (r=\sqrt{x^2+y^2+z^2}) \tag{33}$$

and the condition

$$\left.\frac{\partial u}{\partial n'}\right|_\Gamma = f(x,y,z)$$

where $n'$ is the direction of the **inner normal** of $\Gamma$.

## 2 Exercise Class

**Ex 2.1.** *Prove that the following functions are all harmonic functions.*

1. $x^3 - 3xy^2$;
2. $r^n \cos n\theta$.

**Solution.** *We only need to verify that they satisfy Laplace's equation $\Delta u = 0$.*

1. *Let $u = x^3 - 3xy^2$. Using Laplace's equation in* ***Cartesian coordinates****:*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

*First, calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$:*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy) = -6x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \textit{ is harmonic.}$$

2. *Let $u = r^n \cos n\theta$. Using Laplace's equation in* ***polar coordinates****:*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

*First, calculate the partial derivatives:*

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(nr^n \cos n\theta) = n^2 r^{n-2}\cos n\theta \tag{23}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r^n n^2 \cos n\theta \Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -n^2 r^{n-2}\cos n\theta \tag{24}$$

$$\Rightarrow (23) + (24) = 0 \Rightarrow u \textit{ is harmonic.}$$

**Ex 2.2.** *Use the first Green's formula to prove the uniqueness of the solution of the Robin boundary value problem for the three dimensional Laplace equation:*

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, & (x, y, z) \in \Omega \\ \left(\frac{\partial u}{\partial n} + \sigma u\right)\big|_{\partial\Omega} = f, & \sigma > 0 \textit{ is a constant} \leftarrow \boxed{\textit{The third boundary}} \end{cases}$$

- **Uniqueness Theorem:** The basic idea behind all uniqueness theorems is the same: assume there are two distinct solutions and prove that their difference is identically zero.
- **Third Type Boundary Conditions:** Unlike the first type, where the maximum principle can be applied, the third type boundary conditions do not allow for the use of the maximum principle directly.
- **Energy Method replace the extreme principle:** Use the energy method (similar to the one discussed in Chapter 2) to prove uniqueness.
  - Multiply the equation by $w$;
  - Integrate;
  - **Integration by Parts:** Apply integration by parts to transform the integral into boundary terms and an internal term involving $\Delta w$.

*Proof.* We only need to prove that the homogeneous problem corresponding to $f = 0$ has only the zero solution. Suppose there are two solutions $u_1 \neq u_2$, then

$$\begin{cases} \Delta u_1 = 0, & \left(\frac{\partial u_1}{\partial n} + \sigma u_1\right)\big|_{\partial\Omega} = f \\ \Delta u_2 = 0, & \left(\frac{\partial u_2}{\partial n} + \sigma u_2\right)\big|_{\partial\Omega} = f \end{cases} \Longrightarrow \begin{cases} \Delta(u_1 - u_2) = 0 \\ \left[\frac{\partial(u_1-u_2)}{\partial n} + \sigma(u_1 - u_2)\right]\big|_{\partial\Omega} = 0 \end{cases}$$

Let $u = u_1 - u_2$, then

$$\begin{cases} \Delta u = 0 \\ \left(\frac{\partial u}{\partial n} + \sigma u\right)\big|_{\partial\Omega} = 0 \end{cases}$$

By the first Green's formula (or Gauss's formula) $\leftarrow$ *the higher dimensional version of "Integration by Parts"*,

$$\cancel{\int_\Omega v\Delta u d\Omega} = \int_{\partial\Omega} v\frac{\partial u}{\partial n}dS - \int_\Omega \nabla u \cdot \nabla v d\Omega.$$

Let $v = u$, we get

$$\int_{\partial\Omega} u\frac{\partial u}{\partial n}dS = \int_\Omega |\nabla u|^2 d\Omega.$$

Using $\left(\frac{\partial u}{\partial n} + \sigma u\right)\big|_{\partial\Omega} = 0$, we have

$$\int_\Omega |\nabla u|^2 d\Omega + \sigma\int_{\partial\Omega} u^2 dS = 0 \Rightarrow \int_\Omega |\nabla u|^2 d\Omega = \int_{\partial\Omega} u^2 dS = 0.$$

$\Rightarrow u \equiv$ constant. Since $u \equiv 0$ on $\partial\Omega \Rightarrow u \equiv 0$ in $\overline{\Omega}$. □

**Ex 2.3.** *If $u(r,\theta)$ is a* ***harmonic function*** *on the* ***unit circle*** *and $u(1,\theta) = \cos\theta$, find the value of $u$ at the origin.*

- "**harmonic function**"+"**unit circle**"→ "**the mean value theorem**"!

**Solution.** *By the mean value theorem for harmonic functions, (let $C$ denote the unit circle)*

$$u(0,0) = \frac{1}{2\pi}\int_C u(r,\theta)ds = \frac{1}{2\pi}\int_0^{2\pi}\cos\theta d\theta = 0$$

*due to the orthogonality of trigonometric functions.*

**Ex 2.4.** *Let $u$ be harmonic in the region $\Omega \subset \mathbb{R}^3$, and the ball $B(a,R)$ with center $a$ and radius $R$ is a subset of $\Omega$. Prove that for any $0 < \rho < R$, the following* ***ball average formula*** *holds:*

$$u(a) = \frac{3}{4\pi\rho^3}\int_{B_\rho(a)} u(x,y,z)dxdydz$$

The idea is inspired by

- the spherical average formula for harmonic function on a spherical surface;
- Tools for transforming spherical surface integrals into spherical volume integrals:
  - the onion peeling integration:

$$\int_{B_r^{M_0}} f\ dV = \int_{B_r^{M_0}} f\ r^2\sin\theta\ drd\theta d\varphi = \int_0^R\int_{S_r^{M_0}} f\ \underbrace{r^2\sin\theta\ d\theta d\varphi}_{dS\text{ (surface element)}} = \int_0^R\int_{S_r^{M_0}} f\ dS$$

  - Gauss's theorem.

Here, obviously only the onion peeling integration can be used

**Solution.** *By the spherical average formula, we have:*

$$\int_{B_\rho(a)} u(x,y,z)dxdydz = \int_0^\rho \left( \int_{\Gamma_r(a)} u(x,y,z)dS \right) dr = \int_0^\rho 4\pi r^2 u(a)dr = \frac{4}{3}\pi\rho^3 u(a).$$

*So* $u(a) = \frac{3}{4\pi\rho^3}\int_{B_\rho(a)} u(x,y,z)dxdydz$.

**Ex 2.5.** *Let* $u(x)$ *be harmonic in the disk* $B_R$ *centered at the origin with radius* $R$*, and continuous on* $\overline{B_R}$*. Denote* $M = \int_{B_R} u^2 d\Omega$*. Prove that:*

$$|u(0)| \le \frac{1}{R}\left(\frac{M}{\pi}\right)^{\frac{1}{2}}.$$

Note

$$\underbrace{|u(0)|}_{\text{value of the center of } B_R} \le \frac{1}{R} \underbrace{\left(\frac{M}{\pi}\right)^{\frac{1}{2}}}_{\text{the mean value of } u^2 \text{ on } B_R}.$$

- Related to the mean value theorem for harmonic functions;
- Need to build a relation $\int u^2 \to \int u$. The tools for this goal are:
  - The **Cauchy-Schwarz inequality**:

$$(\sum_k a_k b_k)^2 \le \sum_k a_k^2 \sum_k b_k^2 \quad \Leftrightarrow \quad \langle \mathbf{a}, \mathbf{b}\rangle \le \|\mathbf{a}\| \cdot \|\mathbf{b}\|.$$

  - The **Hölder inequality** (A continuous version of the **Cauchy-Schwarz inequality)**:

$$\left(\int fg\ dx\right)^2 \le \int f^2 dx \int g^2 dx \quad \Leftrightarrow \quad \langle f, g\rangle \le \|f\| \cdot \|g\|.$$

*Proof.* By the **mean value formula** for harmonic functions (the mean value theorem),

$$u(0) = \frac{1}{|B_R|}\int_{B_R} u\ d\Omega = \frac{1}{|\partial B_R|}\int_{\partial B_R} u\ ds$$

Using the **Hölder inequality** $\left(\int fg\ dx\right)^2 \le \int f^2 dx \int g^2 dx$, we have

$$u^2(0) = \frac{1}{|B_R|^2}\left(\int_{B_R} u\ d\Omega\right)^2 \le \frac{1}{|B_R|^2}\int_{B_R} u^2 d\Omega \int_{B_R} d\Omega = \frac{M}{|B_R|}$$

Since $|B_R| = \pi R^2$, we get

$$|u(0)| \le \left(\frac{M}{|B_R|}\right)^{\frac{1}{2}} = \frac{1}{R}\left(\frac{M}{\pi}\right)^{\frac{1}{2}}.$$

□

**Ex 2.6.** *Determine whether the following problem has a solution:*

$$\begin{cases} \Delta u = 1, & r < 1 \\ \frac{\partial u}{\partial n} = 0, & r = 1 \leftarrow \boxed{\textit{the second type boundary}} \end{cases}$$

- **Recall of Second Type Boundary Conditions:** For harmonic functions, $\int_\Gamma \frac{\partial u}{\partial n} dS = 0$ This is a necessary condition derived using the flux theorem.
- **Necessity of Understanding Methods:** Not all theorems learned may be directly applicable. It's crucial to understand the underlying methods. The flux theorem was used to prove a proposition previously; a similar approach is needed here.

**Solution.** *If $u$ is a solution of $\Delta u = 1$, then by Gauss's formula, we have*

$$\frac{4}{3}\pi = \int_{\Omega} \Delta u \; d\Omega = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0$$

*This is a contradiction. So the problem has no solution.*

**Ex 2.7.** *If $u = u(r, \theta)$ is a harmonic function, prove that $ru_r$ is also a harmonic function, and thereby prove the solution of the second kind boundary value problem:*

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, \quad 0 < r < R \\ \frac{\partial u}{\partial r}\big|_{r=R} = \varphi(\theta) \end{cases}$$

*is given by $u(r,\theta) = -\frac{R}{2\pi}\int_0^{2\pi} \varphi(\beta) \ln\left(R^2 + r^2 - 2Rr\cos(\beta - \theta)\right) d\beta + C$ when $\int_0^{2\pi} \varphi(\theta)d\theta = 0$, where $C$ is an arbitrary constant.*

*Proof.* Since $u = u(r, \theta)$ is a harmonic function, we have

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Let $v = ru_r$. Then

$$\begin{aligned} \Delta v &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(ru_r)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(ru_r) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(ru_r) + \frac{\partial^2 u}{\partial \theta^2}\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r^2\Delta u) = 0 \end{aligned}$$

So $v$ is also a harmonic function.

In this way, the second kind boundary value problem of $u$

$$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial r}\big|_{r=R} = \varphi(\theta) \end{cases}$$

is transformed into the first kind boundary value problem of $v$

$$\begin{cases} \Delta v = 0 \\ v|_{r=R} = R\varphi(\theta) \end{cases}$$

Using Poisson's formula, we have

$$v(r,\theta) = \frac{R}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\varphi(\beta)}{R^2 + r^2 - 2Rr\cos(\beta - \theta)} d\beta$$

Using the known condition $\int_0^{2\pi} \varphi(\theta)d\theta = 0$, we can rewrite $v$ as

$$\begin{aligned} v(r,\theta) &= \frac{R}{2\pi}\int_0^{2\pi} \left(\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\beta - \theta)} - 1\right)\varphi(\beta)d\beta \\ &= -\frac{R}{\pi}\int_0^{2\pi} \frac{r^2 - Rr\cos(\beta - \theta)}{R^2 + r^2 - 2Rr\cos(\beta - \theta)}\varphi(\beta)d\beta \end{aligned}$$

Then the solution of the original problem is:

$$\begin{aligned}
u(r,\theta) &= u(0,\theta) + \int_0^r \frac{v(\rho,\theta)}{\rho} d\rho \\
&= u(0,\theta) - \frac{R}{\pi}\int_0^{2\pi}\left(\int_0^r \frac{\rho - R\cos(\beta-\theta)}{R^2+\rho^2-2R\rho\cos(\beta-\theta)} d\rho\right)\varphi(\beta)d\beta \\
&= u(0,\theta) - \frac{R}{2\pi}\int_0^{2\pi} \varphi(\beta)\ln\left(R^2+r^2-2Rr\cos(\beta-\theta)\right)d\beta
\end{aligned}$$

where $u(0,\theta)$ is an arbitrary constant. □

# References


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