# Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 5-2

## 3.2 Initial Value Problems for Higher Dimensional Wave Equations

### Key Ideas

- The approach starts by considering the 3D wave equation and then applying dimensional reduction to handle the 2D case.
- The main difficulty lies in the 3D wave equation, while the 2D case is relatively easier.
- The nature of solutions differs significantly between even and odd spatial dimensions for wave equations.
- Despite deriving 2D results from the 3D case, their solution properties remain fundamentally different.

### Physical Analogy: Gravity vs. Wave Propagation

- Gravity is governed by Poisson's equation, which does not evolve over time.
- A thought experiment: Suppose $\phi$ satisfies the 3D wave equation instead.
- General Relativity suggests that gravity behaves like waves, though in a much more complex manner.
- This leads to an intuitive approach: **body thinking**.
- If gravity were governed by a wave equation, it would exhibit oscillatory behavior.
- This would mean that gravitational strength at a point would fluctuate over time, akin to experiencing a roller-coaster effect.

### Simplest 3D Wave: Spherical Symmetry

- The simplest 3D wave solution is a **spherically symmetric wave**.
- Spherically symmetric waves simplify the problem by **reducing dependence on angular** coordinates.
- This allows for a transformation of the problem into an effectively 1D equation with a **radial coordinate**.

## Body Thinking: A Learning Strategy

- Body thinking involves associating abstract concepts with physical sensations.
- Example: A chemist mentally links chemical reactions to body movements, responding physically to conceptual changes.
- Applying this idea: Imagine being surrounded by a fluctuating gravitational field.
- In this model, gravity varies dynamically like a wave:
  - You feel gravitational strength increasing and decreasing.
  - The sensation is similar to riding a roller coaster while sitting still.

## Approach to the 3D Wave Equation

- Direct solution methods such as separation of variables are not feasible without boundary conditions.
- The only viable approach is to transform the 3D problem into a simpler 1D form.
- However, reducing 3D to 1D is challenging due to the greater degrees of freedom in 3D.
- To solve the 3D wave equation, we begin by identifying the simplest case: **spherical symmetry**.
- The problem-solving approach resembles detective work:
  - Collect key clues.
  - Formulate bold hypotheses.
  - Attempt to construct a viable solution.
- The first clue: consider the simplest wave solution, a spherically symmetric wave.
- Using the **method of dissecting complexity** (similar to *Pao Ding's Butchering the Ox* analogy), we rewrite the wave equation in spherical coordinates.
- This transformation simplifies the problem and helps in deriving an explicit solution.

In the previous section, we discussed the initial value problem of the **one-dimensional** wave equation and obtained **D'Alembert's formula**. For the **three-dimensional** wave equation, the solution can be expressed in a **spherical mean form**, which is commonly referred to as **Kirchhoff's formula**.

### 3.2.1 Kirchhoff's Formula for the Three-Dimensional Wave Equation

Now, let's consider the initial value problem of the three-dimensional wave equation

$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz})(-\infty < x, y, z < +\infty, t > 0), \tag{1}$$

$$u(x, y, z, 0) = \varphi(x, y, z), \quad u_t(x, y, z, 0) = \psi(x, y, z), \tag{2}$$

where $\varphi(x, y, z)$ and $\psi(x, y, z)$ are known functions.

## How to invent the method of spherical mean

**Clue 1:**

To explain how the sperical means method is conceived, let's first look at the **first clue**: the spherically symmetric solution of the three-dimensional wave equation.

Using spherical coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

The **wave equation in spherical coordinates** (by chain rules) is expressed as:

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

Since $u$ is **independent of $\theta$ and $\phi$** (due to the spherical symmetry)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

- We currently have the 1D wave equation describing string vibrations.
- The only available approach is the **traveling wave method**, which provides solutions in the form of moving waves.
- Our goal: Transform this equation into the **1D wave equation** to simplify analysis.

Recall a useful relation

$$\partial_r(r^2\partial_r u) = r\partial_r^2(ru) \tag{3}$$

Then we arrive at

$$\frac{\partial^2(ru)}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2(ru)}{\partial t^2}$$

The general solution gives:

$$ru = f(r-at) + g(r+at)$$

That is,

$$u = \frac{f(r-at)}{r} + \frac{g(r+at)}{r}.$$

This is the **spherically symmetric** solution of the three-dimensional wave equation.

**Clue 2:**

Using the D'Alembert's formula to **guess** the 3D formula,

- The goal is to rewrite the formula to unify two types of averages:
  - **Arithmetic Mean**
  - **Integral Mean**
- Mathematicians prefer **symmetry** and **consistency**, leading to the search for a **unified form** $\leftarrow$ **Aesthetic Criterion**.
- The **arithmetic mean** can be expressed as an **integral mean** by integration followed by differentiation.

$$
\begin{aligned}
u(x,t) =& \frac{1}{2}\left(\varphi(x-at)+\varphi(x+at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha \\
=& \frac{1}{2a}\partial_t\left(\int_{x-at}^{x+at}\varphi(\alpha)d\alpha\right)+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha \\
=& \frac{\partial}{\partial t}\left(\frac{t}{2at}\int_{x-at}^{x+at}\varphi(\alpha)\,d\alpha\right)+\frac{t}{2at}\int_{x-at}^{x+at}\psi(\alpha)\,d\alpha
\end{aligned}
$$

- The rewritten formula contains:
  - The integral mean of the initial displacement.
  - The integral mean of the initial velocity, scaled by time.
- Since the 1D and 3D cases are special cases of an $n$-dimensional wave equation, there must be an underlying relationship.
- Extending the 1D case to 3D leads to a natural question:
  - What type of averaging should be used in 3D?
  - Possibilities: **Spherical volume average** vs. **Spherical surface average**.
- Experimental verification (trial and errors) shows that **spherical surface averaging** is the appropriate choice for solving the 3D wave equation.

**Analogy** and **guess** (see Fig. 1):

$$
u(M,t)=\frac{\partial}{\partial t}\left(\frac{t}{4\pi(at)^2}\int_{S_{at}^M}\varphi(\xi,\eta,\zeta)\,dS\right)+\frac{t}{4\pi(at)^2}\int_{S_{at}^M}\psi(\xi,\eta,\zeta)\,dS \leftarrow \boxed{\text{Aesthetic Criteria}} \tag{4}
$$

This may be the solution to the three-dimensional wave equation, and

- It suggests that the solution **at a given point** can be expressed using the **spherical average around that point**.

- Conclusion:
  - **Spherical symmetry** is a crucial property in solving the 3D wave equation.
  - **Spherical surface averaging around every point** plays a fundamental role in the solution process.

## About (4)

- From the conjectured formula, in order to determine $u(t_0, x_0)$, we need to know the data on the sphere at any arbitrary time $t_1$. However, in reality, we only need the spherical average data at any given time $t_1$. This means that if we know the spherical average at any time $t_1$, it is sufficient to determine $u(t_0, x_0)$.
- However, the direct result of the spherical average data is the spherical average solution $\overline{u}$

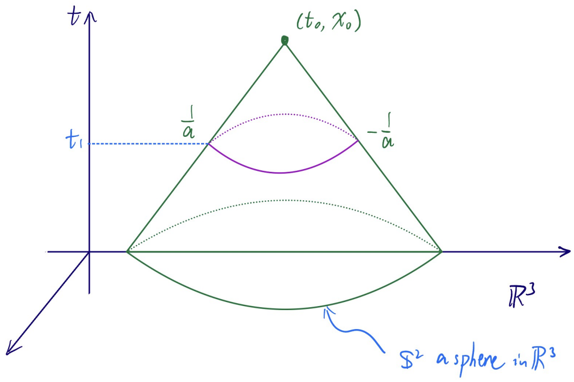


Figure 1: 3D wave analogy

instead of $u$. Therefore, we pose the following questions:

1. What is the relationship between $u$ and $\overline{u}$?
   - The information of $u$ and $\overline{u}$ should be equivalent.
     * If $u$ is known, $\overline{u}$ can be determined.
     * Conversely, knowing $\overline{u}$, intuition suggests that as the radius approaches zero, $\overline{u} \to u$.
     * This follows from the fact that as the concentric spheres around point $M$ shrink to a single point, the average over the sphere reduces to the value at $M$.
   - This equivalence allows us to study $u$ via $\overline{u}$.
     * $u$ is difficult to determine directly.
     * However, $\overline{u}$ depends only on a single spatial variable $r$, making it potentially solvable using a one-dimensional wave equation.
   - From the conjectured solution:
     * $u$ does not distinguish specific initial values but only their spherical averages.
     * This suggests redistributing values over the sphere such that every point takes the spherical average.
     * This transformation leads to a spherically symmetric problem, which may connect to **Clue 1**.
2. What equation does $\overline{u}$ satisfy?
   - The information of $u$ and $\overline{u}$ is equivalent (a conjecture that needs to be proven). Since the information of $u$ is equal to the information contained in the wave equation, it follows that the information of $\overline{u}$ should also be equal to the information of the wave equation. Therefore, the wave equation should be able to derive the equation for $\overline{u}$.

- The above considerations are purely based on the conjectures. Based on this idea, we can further refine and develop a mathematical method.

According to these clues, we conceive the ideas:

Clue 2: The solution may be related to the **spherical average** at each point.

$\downarrow$

Performing a spherical average at each point $\rightarrow$ **spherically symmetric** with respect to that point.

$\downarrow$

The spherical average **may** satisfy the **wave** equation with **spherical symmetry** (?1).

$\downarrow$

Use the **spherically symmetric wave** equation to find the **spherical average** (Clue 1).

$\downarrow$

Intuitively, the spherical average radius contracting may **degenerate** to the value at the **center of the sphere** (?2).

$\downarrow$

Form a closed loop to prove the conjecture of the 3D wave equation solution.

where (?1) and (?2) are conjectures that we have to prove. Next, we first prove these two questions.

## Key Questions and Solutions

**How does the spherical average relate to the original solution? (about (?2))**

- Consider a sphere of radius $r$ centered at any given point.
- Compute the spherical average over this sphere.
- As $r \to 0$, the spherical average approaches the value of the function at the center.
- This suggests that the spherical averaging method can be used to approximate the solution at any point.

**Does the spherical average satisfy a wave equation? (about (?1))**

- Two possible ways to check:
  1. **Assume the solution exists:** If the solution to the wave equation exists everywhere, we can explicitly compute its spherical average.
  2. **Use the wave equation itself:** The wave equation contains all necessary information about the solution, making it a valid tool to derive properties of the spherical average.

$$\boxed{\text{Wave Eqs} \xrightarrow{\text{directly derive}} \text{Eq. of Sperical Average (spherical symmetric wave eq.)}}$$

- Since the wave equation is **equivalent to the information** contained in the solution, if properly **transformed**, it should **hold for the spherical average** as well (like the energy method).

First, fix any point $M = (x, y, z)$, $S_r^M$ represents the sphere with center $M$ and radius $r$. Using spherical coordinates, a point on the sphere is given by:

$$P \equiv (\xi, \eta, \zeta) = (x + r\sin\theta\cos\phi, y + r\sin\theta\sin\phi, z + r\cos\theta).$$

Let $\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ represent the unit outward normal vector to the sphere $S_r^M$, then a point on the sphere $S_r^M$ can be simply written as $M + r\omega$. At the same time, $\omega$ can also be considered as a point on the unit sphere. Therefore, we also denote the surface element on the sphere as:

$$dS_r^M = r^2 \sin\theta d\theta d\phi \quad \text{and} \quad d\omega = \sin\theta d\theta d\phi.$$

Note the normalization relation ($d\omega$ is independent of $r$ and it leads to some convenience).

$$dS_r^M = r^2 d\omega.$$

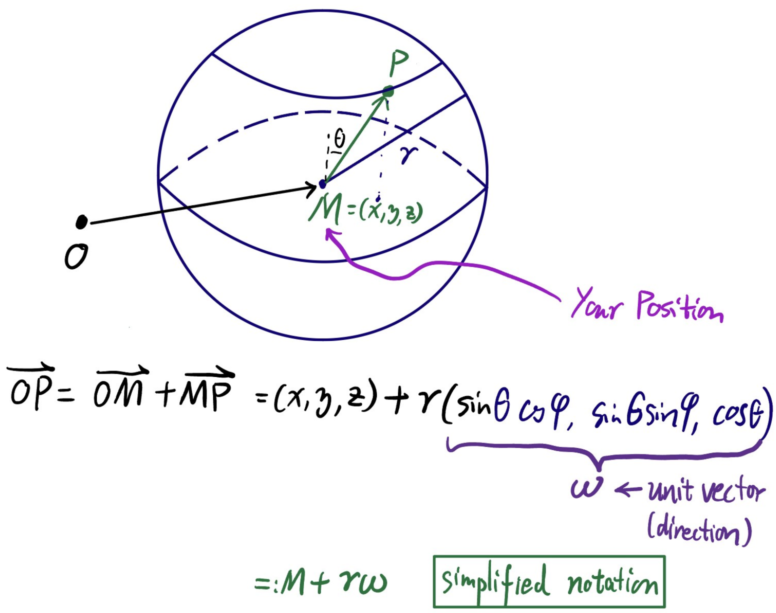


Figure 2: Notation on Sphere

Now introduce the **spherical average** of $u$ (note $dS_r^M = r^2 d\omega$):

$$\bar{u}(r,t) \equiv \frac{1}{4\pi r^2} \iint\limits_{S_r^M} u(P,t) dS_r^M = \frac{1}{4\pi} \iint\limits_{S_1^M} u(M + r\omega, t) d\omega.$$

**(Proof of** (?2)**)** Taking the limit as $r \to 0$ on both sides of the above equation, we get:

$$\lim_{r\to 0} \bar{u}(r,t) = \frac{1}{4\pi} \iint\limits_{S_1^M} u(M,t) d\omega = u(M,t) \leftarrow \boxed{\text{indep. of } \theta \text{ and } \phi}.$$

Furthermore, let $V_r^M$ denote the ball with center $M$ and radius $r$, then the volume integral on $V_r^M$ can be expressed in spherical coordinates as:

$$\iiint\limits_{V_r^M} f dV_r^M \overset{\text{Onion-peeling for integration}}{=} \int_0^r dr_1 \iint\limits_{S_{r_1}^M} f dS_{r_1}^M = \int_0^r dr_1 \iint\limits_{S_1^M} f(M + r_1\omega) r_1^2 d\omega.$$

**Claim (Proof of** (?1)**).** *$u$ solves $\partial_t^2 u = \Delta u \Rightarrow r\bar{u}$ solves $\partial_t^2 (r\bar{u}) = \partial_r^2 (r\bar{u})$. (For simplicity, take $a = 1$)*

*Proof.* First, we integrate the wave equation on $V_r^M$.

$$\underbrace{\int_{V_r^M} \partial_t^2 u \, dV}_{\text{LHS}} = \underbrace{\int_{V_r^M} \Delta u \, dV}_{\text{RHS}}$$

Then proceed the RHS by **Guass formula**, while the LHS by **onion-peeling for integration**:

$$\text{RHS} \overset{\text{Guass}}{=} \int_{\partial V_r^M} \mathbf{n} \cdot \nabla u dS = \int_{S_r^M} \partial_r u \, dS = \int_{S_1^M} \partial_r u \cdot r^2 d\omega = r^2 \partial_r \left( \int_{S_1^M} u(M + r\omega) d\omega \right) = 4\pi r^2 \partial_r \bar{u}(r,t);$$

$$\text{LHS} = \partial_t^2 \int_{V_r^M} u \, dV \overset{\text{Onion-peeling for integration}}{=} \partial_t^2 \left( \int_0^r \int_{S_1^M} u(M + r_1\omega) r_1^2 d\omega dr_1 \right) = 4\pi \partial_t^2 \int_0^r r_1^2 \bar{u}(r_1,t) dr_1.$$

where $\mathbf{n}$ is the outward normal. This impliles

$$4\pi \partial_t^2 \int_0^r r_1^2 \bar{u}(r_1,t) dr_1 = 4\pi r^2 \partial_r \bar{u}(r,t)$$

Taking the derivative on both sides with respect to $r$

$$\Rightarrow \underbrace{\partial_t^2 \partial_r \left( \int_0^r r_1^2 \bar{u}(r_1,t) dr_1 \right)}_{= r^2 \partial_t^2 \bar{u}} = \underbrace{\partial_r (r^2 \partial_r \bar{u}(r,t))}_{\overset{\text{by (3)}}{=} r \partial_r^2 (r\bar{u})}$$

Cancel one $r$

$$\Rightarrow \partial_t^2 (r\bar{u}) = \partial_r^2 (r\bar{u})$$

We finish the proof of this claim. $\square$

That is, we arrive at $(r\bar{u})_{tt} = a^2 (r\bar{u})_{rr}$. Therefore, the general solution for $r\bar{u}$ is

$$r\bar{u} = f(r + at) + g(r - at), \tag{5}$$

where $f$ and $g$ are twice continuously differentiable functions.

There are two approaches that can utilize initial values to determine $f$ and $g$ according to the following flowchart.

$$\begin{tikzcd}
\bar{u} \arrow[r, "r \to 0"] \arrow[d, "\text{Use initial value averaging}"'] & u \arrow[dl, "\text{Use initial values}"] \\
\begin{array}{c}\text{Determine} \\ f, g\end{array} &
\end{tikzcd}$$

(M1) **The blue line:** Feasible, use L'Hospital's rule to make preliminary assumptions about $f$ and $g$.

(M2) **The red line:** More straightforward.

**(Determine $f'$ and $g'$)** To apply Method (M2), we need to compute the **spherical average** of the **initial values**. First, we need to satisfy the initial conditions. From (5), i.e.,

$$r\bar{u} = f(r + at) + g(r - at),$$

we obtain

$$(r\bar{u})_t = r\bar{u}_t = af'(r + at) - ag'(r - at). \tag{6}$$

- In the form of $f + g$, $f' - g'$, to find $f$ and $g$, we only need to integrate $f' - g'$, or differentiate $f + g$. We use the method of differentiation for $f + g$, otherwise it's not easy to do.
- To differentiate $f + g$, we can only differentiate $f' + g'$ with respect to $r$.

Let use differentiate (5) with respect to $r$,

$$(r\bar{u})_r = \bar{u} + r\bar{u}_r = f'(r+at) + g'(r-at) \tag{7}$$

By (6) and (7), we calculate

$$\begin{aligned} 2f'(r+at) =& (r\bar{u})_r + \frac{1}{a}(r\bar{u})_t = \frac{\partial}{\partial r}\left(\frac{r}{4\pi}\int_{S_1^m} u\, d\omega\right) + \frac{1}{a}\frac{r}{4\pi}\int_{S_1^m} u_t\, d\omega \\ 2g'(r-at) =& (r\bar{u})_r - \frac{1}{a}(r\bar{u})_t = \frac{\partial}{\partial r}\left(\frac{r}{4\pi}\int_{S_1^m} u\, d\omega\right) - \frac{1}{a}\frac{r}{4\pi}\int_{S_1^m} u_t\, d\omega \end{aligned}$$

If $t = 0$, we obtain

$$\begin{aligned} 2f'(r) =& \frac{\partial}{\partial r}\left(\frac{r}{4\pi}\int_{S_1^m} \varphi(M+r\omega)\, d\omega\right) + \frac{1}{a}\frac{r}{4\pi}\int_{S_1^m} \psi(M+r\omega)\, d\omega \\ 2g'(r) =& \frac{\partial}{\partial r}\left(\frac{r}{4\pi}\int_{S_1^m} \varphi(M+r\omega)\, d\omega\right) - \frac{1}{a}\frac{r}{4\pi}\int_{S_1^m} \psi(M+r\omega)\, d\omega \end{aligned}$$

**(Obtain $u$)** On the other hand, using (7) (since we are considering the classical solution, $|\bar{u}_r| < \infty$ and $|\bar{u}_t| < \infty$),

$$\begin{aligned} u(M,t) = \lim_{r\to\infty} \bar{u}(r,t) = f'(at) + g'(-at) \leftarrow \boxed{\text{or L'Hospital law by (5)}} \\ =& \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{S_1^m} \varphi(M+at\omega)\, d\omega\right) + \frac{1}{2}\frac{t}{4\pi}\int_{S_1^m} \psi(M+at\omega)\, d\omega \\ &+ \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{S_1^m} \varphi(M+at\omega)\, d\omega\right) + \frac{1}{2}\frac{t}{4\pi}\int_{S_1^m} \psi(M+at\omega)\, d\omega \\ =& \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{S_1^m} \varphi(M+at\omega)\, d\omega\right) + \frac{t}{4\pi}\int_{S_1^m} \psi(M+at\omega)\, d\omega \\ =& \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2t^2}\int_{S_{at}^M} \varphi(\xi,\eta,\xi)dS\right) + \frac{t}{4\pi a^2t^2}\int_{S_{at}^M} \psi(\xi,\eta,\xi)dS. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} u(M,t) = & \underbrace{\frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2t^2}\int_{S_{at}^M} \varphi(\xi,\eta,\xi)dS\right)}_{\partial_t(t\times\text{the spherical average of the initial displacement on the } at \text{ sphere})} \\ & + \underbrace{\frac{t}{4\pi a^2t^2}\int_{S_{at}^M} \psi(\xi,\eta,\xi)dS}_{t\times\text{the spherical average of the initial velocity on the sphere of radius } at} \end{aligned}$$

## Another way which is not quite direct

In (6) and (7), let $r \to 0$, we obtain

$$f'(at) = g'(-at),$$

$$u(M,t) = \lim_{r\to 0} \bar{u}(r,t) = f'(at) + g'(-at) = 2f'(at).$$

In equations (6) and (7), taking $t = 0$ gives

$$(r\bar{u})_t \mid_{t=0} = af'(r) - ag'(r), \quad (r\bar{u})_r \mid_{t=0} = f'(r) + g'(r).$$

Then we arrive at

$$
\begin{aligned}
2f'(r) =&(r\bar{u})_r \mid_{t=0} + \frac{1}{a}(r\bar{u})_t \mid_{t=0} \\
=&\frac{\partial}{\partial r}\left(\frac{r}{4\pi r^2}\iint\limits_{S_r^M} u \mid_{t=0} dS_r^M\right) + \frac{r}{a}\left(\frac{1}{4\pi r^2}\iint\limits_{S_r^M} u_t \mid_{t=0} dS_r^M\right) \\
=&\frac{\partial}{\partial r}\left(\frac{r}{4\pi r^2}\iint\limits_{S_r^M} \varphi(P) dS_r^M\right) + \frac{r}{a}\left(\frac{1}{4\pi r^2}\iint\limits_{S_r^M} \psi(P) dS_r^M\right)
\end{aligned}
$$

Taking $r = at$ and substituting $u(M,t) = 2f'(at)$ gives

$$
\begin{aligned}
u(M,t) = 2f'(at) =&\frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2 t^2}\iint\limits_{S_{at}^M} \varphi(\xi,\eta,\xi) dS\right) + \frac{t}{4\pi a^2 t^2}\iint\limits_{S_{at}^M} \psi(\xi,\eta,\xi) dS \\
=&\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\iint\limits_{S_1^M} \varphi(M + at\omega) d\omega\right) + \frac{t}{4\pi}\iint\limits_{S_1^M} \psi(M + at\omega) d\omega.
\end{aligned}
$$

When the initial functions are sufficiently smooth, it is easy to verify that the function $u(x,y,z,t)$ represented by formula (8) is indeed the solution to problem (1)–(2).

**Ex 0.1.** *Solve the following initial value problem*

$$
\begin{cases}
u_{tt} = u_{xx} + u_{yy} + u_{zz} & (-\infty < x,y,z < +\infty, t > 0), \\
u(x,y,z,0) = 0, & u_t(x,y,z,0) = 2xy,
\end{cases}
$$

**Solution.** *From formula* (8)*, we get*

$$
\begin{aligned}
u(\underbrace{x,y,z,t}_{\textit{Your position}}) =&\frac{t}{2\pi}\int_0^{2\pi}\int_0^{\pi} \underbrace{(x + t\sin\theta\cos\varphi)(y + t\sin\theta\sin\varphi)\sin\theta}_{\textit{Sphere around you}} d\theta d\varphi \\
=&\frac{t}{2\pi}\int_0^{2\pi}\int_0^{\pi} (xy\sin\theta + xt\sin^2\theta\sin\varphi + yt\sin^2\theta\cos\varphi + t^2\sin^3\theta\cos\varphi\sin\varphi) d\theta d\varphi \\
=&\frac{t}{2\pi}\int_0^{2\pi}\int_0^{\pi} (x + t\sin\theta\cos\varphi)(y + t\sin\theta\sin\varphi)\sin\theta d\theta d\varphi \\
=&\frac{xyt}{2\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\varphi = 2xyt.
\end{aligned}
$$

## 3.2.2 Dimension Reduction Method

Using the dimension reduction method to solve the initial value problem of the two-dimensional wave equation.

$$
\begin{cases}
u_{tt} = a^2(u_{xx} + u_{yy}) & (-\infty < x,y < +\infty, t > 0), \\
u|_{t=0} = \varphi(x,y), \\
u_t|_{t=0} = \psi(x,y).
\end{cases} \tag{9}
$$

Since the initial value problem of the two-dimensional wave equation can be considered a special case of the initial value problem of the three-dimensional wave equation (see Fig. 3), the Poisson formula for the three-dimensional wave equation can be used to represent the solution of the initial value problem of the two-dimensional wave equation, and thus derive another form of the solution representation for the two-dimensional problem.

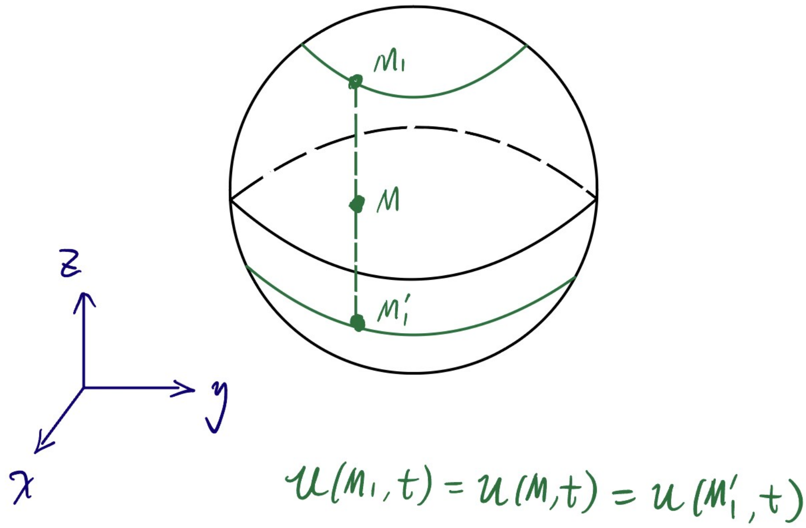


Figure 3: 2D wave equations

- The 2D problem is represented on a plane. Imagine a large trampoline with no boundaries, vibrating in two dimensions.
- To convert this 2D problem into a 3D case, we **extend the 2D surface along the $z$-axis**.
- This extension involves replicating the 2D surface infinitely along the $z$-axis, keeping the value of $u$ constant across different $z$ levels.
- After the surface is stacked infinitely, it forms a 3D structure, and we can apply 3D formulas such as Kirchhoff's equations.
- The **key idea** is that along the $z$-axis, $u_{zz} = 0$ because the value of $u$ does not change in the $z$-direction.
- This extension allows us to use the 3D Kirchhoff equation to solve the problem.
- To calculate any point in the stacked structure, we apply the second-order wave equation, averaging over a spherical surface surrounding the point.
- Finally, we need to project the result back onto the original 2D trampoline.

## Mathematical Interpretation

- The value of $u$ is constant along the $z$-axis, leading to $u_{zz} = 0$.

**Claim.**

*Initial data is independent of $z$, implies $u$ is independent of $z$ for all $t > 0$.*

*Proof.* Given $u_z|_{t=0} = 0$, $\partial_t u_z|_{t=0} = 0$, differentiating the wave equation with respect to $z$,

$$(u_z)_{tt} = a^2 \left((u_z)_{xx} + (u_z)_{yy}\right) \Rightarrow u_z \equiv 0.$$

□

- Once we reduce the problem to 3D, we can apply the 3D Kirchhoff equation directly to solve for $u(t)$.

Using Equation (8)

$$u(M,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\varphi(\xi,\eta,\zeta)dS\right) + \frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\psi(\xi,\eta,\zeta)dS$$

The solution to the initial value problem of the two-dimensional wave equation (9) can be obtained as

$$u(x,y,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\varphi dS\right) + \frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\psi dS,$$

where the integration is performed on the sphere $S_{at}^M$ in the three-dimensional space $(x,y,z)$.

- After extending the 2D surface to 3D, the task is to project the 3D spherical surface integral back onto the 2D trampoline surface.
- This involves using a projection formula for surface integrals that you learned in calculus.
- The formula for projecting a surface integral is:

$$\int_\Sigma f\,d\sigma = \int_S f\cos\gamma\,dS$$

  where $\gamma$ is the angle between the surface normals.
- This projection simplifies the surface integral from 3D to 2D.
- Alternatively, using the **standard method** of **calculating surface integrals** in calculus. If $z = \varphi(x,y)$, the integral can be transformed to:

$$\int_S f(x,y)\,dS = \int_\Sigma f(x,y)\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}\,dx\,dy$$

  where $f(x,y)$ represents the integrand function and the square root term adjusts for the curvature of the surface. For a sphere, $z=\varphi(x,y)=\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}$, then

$$(\varphi_x)^2 = \frac{(\xi-x)^2}{(at)^2-(\xi-x)^2-(\eta-y)^2} \quad \text{and} \quad (\varphi_y)^2 = \frac{(\eta-y)^2}{(at)^2-(\xi-x)^2-(\eta-y)^2}$$

  Then (see Fig. 3)

$$dS = \sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}d\sigma = \frac{at}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}d\sigma$$

- This formula is derived from standard techniques in multivariable calculus and is applicable for transforming integrals from a curved surface to a flat 2D plane.

## Key Concepts

- Projection of 3D integrals onto 2D involves transforming the integrand using standard surface integration formulas.
- The formula for surface integrals in calculus allows us to compute the projected integral over a 2D region from a 3D surface.
- A key aspect is recognizing how surface curvature and the normal vector impact the projection of the integral.

Since $\varphi$ and $\psi$ are functions independent of $z$, the integral on the spherical surface can be transformed into the integral on its projection on the plane $z =$ constant: $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \le a^2t^2$. Since the area element $dS$ on the spherical surface and its projection area element $d\sigma$ satisfy the following relationship (see Fig. 3):

$$d\sigma = \cos\gamma \cdot dS,$$

where $\gamma$ is the angle between the normal directions of these two surface elements. Therefore, we have:

$$\cos\gamma = \frac{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}{at}.$$

Note that the integrals over the upper and lower hemispheres both transform into integrals over the same circle, hence the integral over the circle $\Sigma_{at}^M$ should be **taken as twice the integral over the upper hemisphere**.

Thus,

$$\begin{aligned} u(x, y, t) =& \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2t^2}\iint\limits_{S_{at}^M} \varphi dS\right) + \frac{t}{4\pi a^2t^2}\iint\limits_{S_{at}^M} \psi dS \\ =& \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2t^2}\iint\limits_{\Sigma_{at}^M} \frac{2\varphi}{\cos\gamma} d\sigma\right) + \frac{t}{4\pi a^2t^2}\iint\limits_{\Sigma_{at}^M} \frac{2\psi}{\cos\gamma} d\sigma, \end{aligned}$$

$$u(x, y, t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{\Sigma_{at}^M} \frac{\varphi(\xi, \eta)d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}\right] + \frac{1}{2\pi a}\iint\limits_{\Sigma_{at}^M} \frac{\psi(\xi, \eta)d\sigma}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}}. \tag{10}$$

The above equation is called the **Poisson formula** for the initial value problem of the two-dimensional wave equation. Since the integration region $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \le a^2t^2$ is a circular domain centered at $M$ with radius $at$, we usually use **polar coordinates** to calculate the integral in equation (10).

**Ex 0.2.** *Solve the following problem*

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & (-\infty < x, y < +\infty, t > 0), \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 2xy. \end{cases}$$

**Solution.** *From equation* (10)*, we get*

$$\begin{aligned} u(x, y, t) =& \frac{1}{2\pi}\int_0^t \int_0^{2\pi} \frac{2(x + \rho\cos\theta)(y + \rho\sin\theta)}{\sqrt{t^2 - \rho^2}} \rho d\rho d\theta \\ =& \frac{xy}{\pi}\int_0^t \int_0^{2\pi} \frac{\rho d\rho d\theta}{\sqrt{t^2 - \rho^2}} = 2xyt. \end{aligned}$$

## Key Differences Between 2D and 3D Solution Formulas

- In 3D, the integral is taken over a **hollow spherical surface**, meaning the integration surface is a sphere with an empty interior.
- In 2D, the integral is taken over a **solid circular region**, meaning the entire disk, including its interior, contributes to the integration.
- This **difference** is a fundamental characteristic to remember when working with wave equations in different dimensions.

**Modification of the Integrand**

- In 3D, the integrand function is simply $\varphi, \psi$.
- In 2D, the integrand includes an additional factor due to the transformation of coordinates.
- As time increases, the denominator of the modified integrand grows, leading to a decrease in the integral's value.
- This results in **attenuation over time** in 2D wave propagation.

### 3.2.3 Physical Significance of the Solution

**Physical Interpretation**

- Consider a region $\Omega$ where an initial perturbation is introduced.
- Outside $\Omega$, the initial state is zero.
- Observing from a point $M$, one will detect the perturbation after some time as the wave propagates.
- Wave propagation speed is finite, meaning there is a delay before the perturbation reaches $M$.
- At time $T_1$, the value at $M$ is determined by an **average over a spherical shell (3D) or a circular disk (2D)**.
- The integration method used depends on whether the problem is in 2D or 3D.

Recall Solution (8)

$$u(M,t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\varphi(\xi,\eta,\zeta)dS\right) + \frac{t}{4\pi a^2 t^2}\iint_{S_{at}^M}\psi(\xi,\eta,\zeta)dS.$$

Assume that the initial disturbance occurs only within a finite region $\Omega$ in space. Outside of region $\Omega$, consider any point $M$, and examine the situation of the influence of the initial disturbance on point $M$ at different times (see Fig. 4).

We know that the value of the solution $u$ at point $M$ and time $t$, $u(M,t)$, is determined by the values of the initial functions $\varphi$ and $\psi$ on the sphere $S_{at}^M$. Therefore, the integral in equation (8) is not zero **only** when the sphere $S_{at}^M$ **intersects** with region $\Omega$, hence $u(M,t) \neq 0$.

Let $d$ and $D$ represent the **closest** and **farthest** distances from point $M$ to region $\Omega$, respectively, as shown in the figure. When $at < d$, the sphere $S_{at}^M$ is still at a distance from region $\Omega$, so the values of $\varphi$ and $\psi$ on this sphere are 0, the integral is 0, and thus $u(M,t) = 0$. At this time, the disturbance has not yet reached point $M$ (see Fig. 5).

When $d \leq at \leq D$, the sphere $S_{at}^M$ continuously intersects with region $\Omega$, the value of the integral is generally not 0, and the value of $u(M,t)$ is generally not 0 either. At this time, point $M$ is in a disturbed state. The initial disturbance reaches point $M$ instantaneously at $t = d/a$ (see Fig. 6)..

When $at > D$, the sphere $S_{at}^M$ has already passed the initial disturbance region $\Omega$ and no longer intersects with it. Starting from $t = D/a$, $u(M,t)$ takes the zero value again, indicating that the disturbance has already passed point $M$, and point $M$ returns to its original state of rest (see Fig. 7).

In a bounded region $\Omega$, any disturbance caused by a point propagates outward at speed $a$. Therefore, at time $t$, the **region affected by the initial disturbance** in $\Omega$ is **all of the spheres** centered at $p \in \Omega$ with radius $at$. When $t$ is sufficiently large, these spherical surfaces have **two envelopes**. The **outer envelope** is called the **front wavefront**, and the **inner envelope** is called the **rear wavefront**. The middle part between these two wavefronts is the region affected by the initial disturbance.

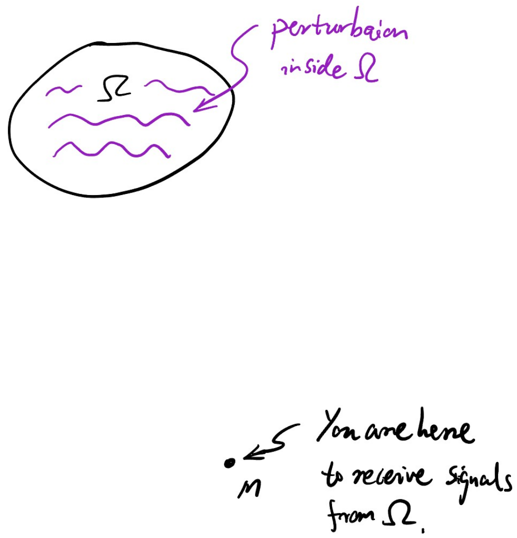


Figure 4: Physical meaning 1

From the above analysis, it can be seen that the wave generated by a point disturbance propagates as a spherical surface passing through $M$. A large number of point disturbances form the entirety of these spherical surfaces, which together constitute the disturbance region. Between the disturbance regions, there exist a leading envelope and a trailing envelope.

The part outside the front wavefront indicates the region where the wave has **not yet reached**, while the part inside the rear wavefront indicates the region where the wave has **passed and returned to its original state**. Therefore, when the initial disturbance is confined to a certain local area in space, the wave propagation has a clear front and rear wavefront. This phenomenon is known in physics as **Huygens' principle** or the **non-retroactive phenomenon**. Since the disturbance at point $M_0 \in \Omega$ at $t = t_0$ affects the sphere $S_{at_0}^{M_0}$ centered at $M_0$ with radius $at_0$, solution (8) is referred to as a **spherical wave**.

Recall Solution (10):

$$u(x,y,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{\Sigma_{at}^M} \frac{\varphi(\xi,\eta)d\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right] + \frac{1}{2\pi a}\iint\limits_{\Sigma_{at}^M} \frac{\psi(\xi,\eta)d\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}.$$

For the solution of the initial value problem of the two-dimensional wave equation (10), a similar discussion can be made. However, it is important to note that since the **integration is performed over the circular domain** $\Sigma_{at}^M : (\xi - x)^2 + (\eta - y)^2 \le a^2t^2$. Thus for any point $M$, once $u(M,t)$ changes from 0 to non-zero as time $t$ increases, it will **not gradually decrease back to** 0 as in the spatial case, but it will **gradually decrease** from a certain moment onward. Thus, there are **significant differences** between two-dimensional and three-dimensional situations (see Fig. 8).

For the two-dimensional case, the wave propagation **only has a front wavefront** and **no rear wavefront**, and **Huygens' principle no longer holds**. This phenomenon is called **wave diffusion**, or in other words, this type of wave exhibits a **retroactive effect**. For the two-dimensional problem, it can be considered that the initial disturbance occurs within an **infinitely long cylinder** and is **independent** of the $z$-coordinate. Thus, the initial disturbance at point $M_0$ should be considered as an initial disturbance along an infinitely long straight line passing through point $M_0$ and parallel to the $z$-axis. At $t = t_0$, its influence is within a cylindrical surface with this line as the axis and $at_0$ as the radius. Therefore, solution (10) is referred to as a **cylindrical wave**.

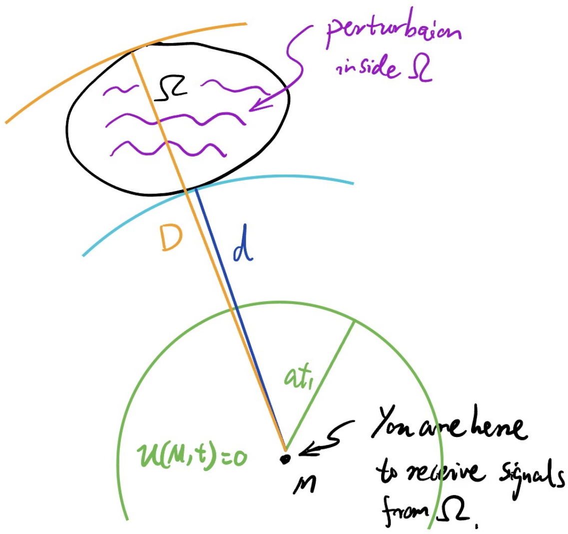


Figure 5: Physical meaning 2

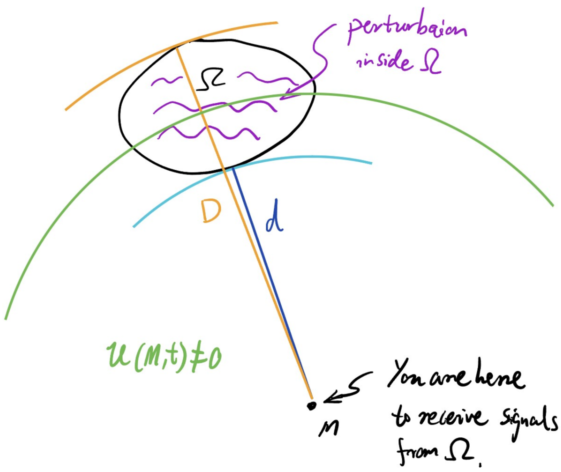


Figure 6: Physical meaning 3

### Summary

### Wave Propagation and Initial Disturbance (see Fig. 4)

- Consider a region $\Omega$ with an initial disturbance.
- Outside $\Omega$, the initial condition is zero.
- Observation point $M$ is used to analyze wave propagation.
- As time evolves, point $M$ will receive the wave signal after a certain delay due to finite propagation speed.

### Three Stages of Wave Reception

**First Stage: No Signal (see Fig. 5)**

- At time $t_1$, construct a spherical surface centered at $M$ with radius $at_1$.
- If this sphere does not intersect $\Omega$, the integral over the surface is zero.

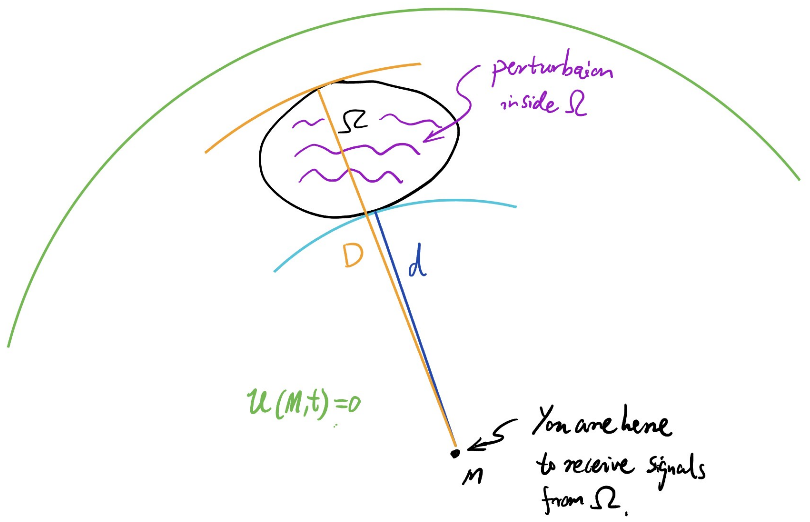


Figure 7: Physical meaning 4

- Thus, $u(M, t_1) = 0$, meaning no signal is received at $M$.

**Second Stage: Signal Reception (see Fig. 6)**

- At time $t_2$, the spherical surface intersects $\Omega$.
- Since the integral now includes nonzero contributions from $\Omega$, $u(M, t_2) \neq 0$.
- This indicates that the signal has reached $M$.

**Third Stage: Key Differences Between 2D and 3D**

**For 3D Waves (see Fig. 7):**

- At time $t_3$, the sphere extends beyond $\Omega$.
- The integral is again zero, implying that $u(M, t_3) = 0$.
- This demonstrates the principle of **Huygens' Principle** (no after-effect phenomenon).

**For 2D Waves (see Fig. 8):**

- In 2D, the integration region is a solid disk instead of a spherical shell.
- At time $t_3$, the integral remains nonzero since the solid region still overlaps with $\Omega$.
- However, the signal weakens over time due to a decay factor in the denominator.
- This results in a **persistence effect** and **wave dispersion**, unlike the 3D case.

## Key Differences Between 2D and 3D Wave Propagation

- **3D waves** exhibit **no after-effect**: once the signal passes, the region remains undisturbed ("Let bygones be bygones").
- **2D waves** exhibit **after-effect**: signals persist beyond initial interaction but decay over time ("What's done is done").
- The difference arises from the geometry of integration: spherical surface vs. solid disk.

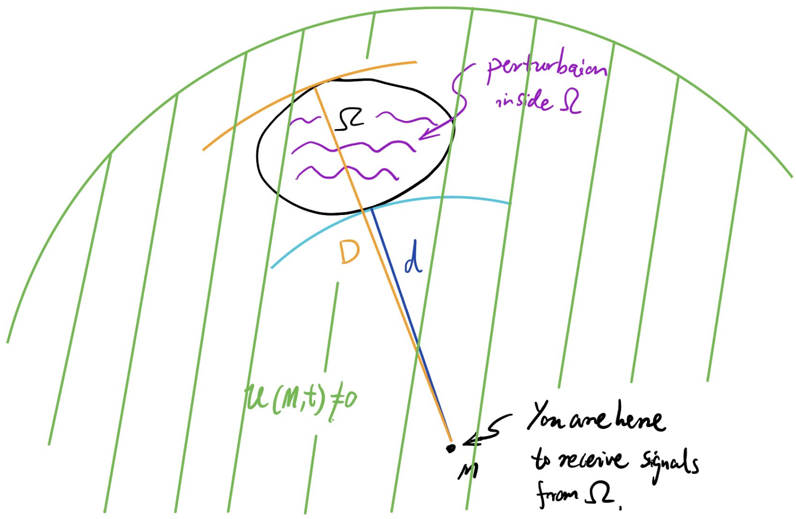


Figure 8: Physical meaning 5