Lecture Notes: Mathematical Physics Equations and Special Functions

Chao Liu

Week 8-1

0.1 Several important properties of Green's function

Theorem 0.1. *The Green's function $G(M, M_0)$ satisfies Laplace's equation everywhere except at the point $M = M_0$. When $M \to M_0$, $G(M, M_0)$ tends to infinity, and its order is the same as that of $\frac{1}{r_{MM_0}}$.*

Theorem 0.2. *The Green's function $G(M, M_0)$ is identically equal to 0 on the boundary Γ.*

- The theorem is simple but has two crucial points that are very useful in proofs of theorems.
- The **first key point** is that the function v is a harmonic function. This means v has the property of being C^2 inside the domain Ω, which is very useful for solving boundary value problems.
- The **second key point** is about the function G:
  - G is infinite at the point M_0, indicating a singularity at M_0, and has a growth order of $\frac{1}{r}$ near M_0.
  - G is zero on the boundary.
- v and G each have their own advantages:
  - v has good behavior inside Ω (**harmonic**), but boundary behavior is less ideal.
  - G has a **singularity** at M_0 but its boundary behavior is very good.

Proof. In fact, these theorems state the problem of G:

$$\begin{cases} \Delta G = -\delta(r - r_0) \\ G|_\Gamma = 0 \end{cases}$$

where r and r_0 give the position of M and M_0, respectively. In addition, the second statement of Theorem 0.1 can be proven by

$$G(M, M_0) = \underbrace{\frac{1}{4\pi r_{MM_0}}}_{\text{dominate}} - \underbrace{v}_{\text{bounded}}$$

and v is harmonic (thus is bounded in Ω since $v \in C^2$).

Another way:

$$\lim_{M \to M_0} \left(\frac{G(M, M_0)}{\frac{1}{r_{MM_0}}} \right) = \lim_{M \to M_0} \left(\frac{1}{4\pi} - r_{MM_0} v \right) = \frac{1}{4\pi}.$$

It completes the proof. □

- **Green's Function Notation:**
  - Green's function $G(M, M_0)$ is written with two indices: M (the **true variable**) and M_0 (a parameter indicating the **position** of the **unit positive charge**).
  - M represents the point inside the **conductor shell** where you place a probe to measure the potential.
  - M_0 is the position where the unit positive charge is placed.
- **Integral Variable and Charge Position:**

$$u(M_0) = -\iint_{\Gamma} f(M)\frac{\partial G(M, M_0)}{\partial n}dS$$

  - The integral variable in the expression for G is M, which is taken over the **boundary of** Ω.
  - M_0 is the **position of the unit positive charge** and is the **independent variable** you are solving for in $u(M_0)$.
- **Summary:**
  - The key is to understand that M_0 is the point where the **charge is placed** and the **independent variable of** u of the left hand side, and M is the **variable of integration** over the boundary and the **independent variable of** $G(M, M_0)$.

Theorem 0.3. *Inside the region* Ω*, the following inequality holds*

$$0 < G(M, M_0) < \frac{1}{4\pi r_{MM_0}}.$$

- Each function, v and G, has distinct advantages.
- Utilize the individual strengths of v and G.
- Perform transformations back and forth between v and G.

Proof. 1. Note

$$G < \frac{1}{4\pi r} \quad \Leftrightarrow \quad v > 0.$$

Let us focus on the system of v. Since

$$\begin{cases} \Delta v = 0 \\ v|_\Gamma = \frac{1}{4\pi r_{MM_0}}|_\Gamma > 0 \end{cases}, \leftarrow \boxed{\text{Using } v \text{ is harmonic!}}$$

by the **maximum principle**, we get $v > 0$.

$$\Rightarrow G(M, M_0) = \frac{1}{4\pi r_{MM_0}} - v < \frac{1}{4\pi r_{MM_0}}.$$

2. Prove $G(M, M_0) > 0$, that is, $v(M) < \frac{1}{4\pi r_{MM_0}}$. Note $G \sim \frac{1}{r_{MM_0}}$, when $M \to M_0$ there is a singularity.

We cannot directly use the maximum principle to get this conclusion (see Fig. 1). Applying the maximum principle to v, we get

$$\min_{M_1\in\Gamma}\frac{1}{4\pi r_{M_1M_0}} < v(M) < \max_{M_1\in\Gamma}\frac{1}{4\pi r_{M_1M_0}} = \frac{1}{4\pi r_{\min}},$$

which cannot give the result we want since what we want is

$$v(M) < \frac{1}{4\pi r_{MM_0}} \leftarrow \boxed{\text{Here } M \text{ is not always on the boundary. In fact } M\in\Omega}$$

- Here M is not always on the boundary. In fact $M\in\Omega$;
- In fact, from the example of Fig. 1, since $r_{MM_0} > r_{\min}$,

$$v(M) < \frac{1}{4\pi r_{MM_0}} < \frac{1}{4\pi r_{\min}}.$$

  That is, what we want is stronger than the one given by the maximum principle.

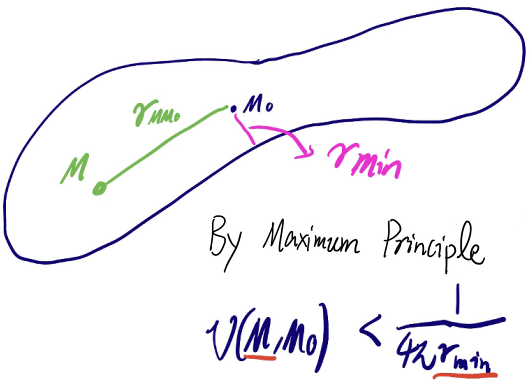

Figure 1: Theorem 0.3

- **Idea:** Excavate a small ball $K_\varepsilon^{M_0}$ (a small ball with M_0 as the center and ε as the radius). By taking $\varepsilon > 0$ small enough, $\frac{1}{4\pi\varepsilon} \to +\infty$ can be arbitrarily large.

Let $\Omega_\varepsilon = \Omega \setminus \overline{K_\varepsilon^{M_0}}$

Case 1: If $M\in\Omega_\varepsilon$, G is a harmonic function in Ω_ε and $G|_\Gamma = 0$. By the maximum principle, $G(M,M_0) > 0$. And we can take ε small enough ($\frac{1}{4\pi\varepsilon}$ can be large enough) such that $G|_{\partial K_\varepsilon^{M_0}} = \frac{1}{4\pi\varepsilon} - v > 0$ since $v\in C^2$ is harmonic and this implies v is bounded.

Case 2: If $M\notin\Omega_\varepsilon$, that is, $M\in K_\varepsilon^{M_0}$ (G has singularity, thus using v is better than G in this domain), by taking ε small enough, we can make

$$v < \frac{1}{4\pi\varepsilon} \leftarrow \boxed{v \text{ is harmonic and thus bounded by } C \text{ in } K_\varepsilon^{M_0}. \text{ Hence, there is } \varepsilon \text{ such that } \frac{1}{4\pi\varepsilon} > C > v}$$

Because $\frac{1}{4\pi\varepsilon}$ can be arbitrarily large, $v\in C^2 \Rightarrow v$ is bounded. Also, because $r_{MM_0} < \varepsilon \Rightarrow v < \frac{1}{4\pi\varepsilon} < \frac{1}{4\pi r_{MM_0}}$ (Question: Compare with Fig.1 and think about the differences between the case in a ball with the one in Ω). □

Theorem 0.4 (Symmetry). *The Green's function $G(M, M_0)$ has a symmetric property with respect to the independent variable M and the parameter variable M_0. That is, if $M_1, M_2 \in \Omega$, then*

$$G(M_1, M_2) = G(M_2, M_1).$$

- **Symmetry Property:**
  - Mathematically expressed as $G(M_1, M_2) = G(M_2, M_1)$.
  - Indicates that the Green's function remains unchanged when its arguments are swapped.
- **Physical Interpretation:**
  - Placing a positive charge at M_2 and measuring potential at M_1 gives the same result as placing a charge at M_1 and measuring at M_2.
  - Despite the positions of charges and measurement points being reversed, the potential remains the same.

The significance of this property in electrostatics can be described as follows: The electric potential generated at M_2 by a unit point charge at M_1 is equal to the electric potential generated at M_1 by a unit point charge at M_2. A principle similar to this is called the reciprocity principle in physics.

Method 1: (δ functions).

$$\begin{cases} \Delta G(M, M_i) = -\kappa\delta(\mathbf{x} - \mathbf{x}_i) \quad (\text{where } \mathbf{OM} = \mathbf{x}, \mathbf{OM_i} = \mathbf{x}_i, i = 1, 2) \\ G|_\Gamma = 0 \end{cases}$$

$$\begin{aligned} &\Rightarrow \int_\Omega [G(M, M_1)\Delta G(M, M_2) - G(M, M_2)\Delta G(M, M_1)]\, d\Omega \\ &= -\kappa \int_\Omega G(M, M_1)\delta(\mathbf{x} - \mathbf{x}_2) d\Omega + \kappa \int_\Omega G(M, M_2)\delta(\mathbf{x} - \mathbf{x}_1) d\Omega \\ &= -\kappa G(M_2, M_1) + \kappa G(M_1, M_2) \leftarrow \boxed{\text{Property of } \delta \text{ function}} \end{aligned} \tag{1}$$

Using the **2nd Green's identity** and the **boundary condition** $G|_\Gamma = 0$

$$(1) = \int_\Gamma \left[G(M, M_1)\frac{\partial G(M, M_2)}{\partial n} - G(M, M_2)\frac{\partial G(M, M_1)}{\partial n} \right] dS = 0 \tag{2}$$

Combining (1) and (2) gives $G(M_2, M_1) = G(M_1, M_2)$. □

Method 2: (Rigorous method. Excavate a small ball). Let's consider the region Ω'' obtained by removing two small balls $B_{\varepsilon_1}^{M_1}$, $B_{\varepsilon_2}^{M_2}$ centered at M_1 and M_2 from Ω (see Fig. 2). Applying the 2nd Green's identity on Ω'', we get:

$$\int_{\partial\Omega''} \left[G(M, M_1)\frac{\partial G(M, M_2)}{\partial n} - G(M, M_2)\frac{\partial G(M, M_1)}{\partial n} \right] dS = 0.$$

Here, $\partial\Omega''$ consists of $\partial B_{\varepsilon_1}^{M_1}$, $\partial B_{\varepsilon_2}^{M_2}$ and Γ. Because the boundary condition $G|_\Gamma = 0$, the integral along Γ is 0. Thus,

$$\int_{\partial B_{\varepsilon_1}^{M_1} + \partial B_{\varepsilon_2}^{M_2}} \left[G(M, M_1)\frac{\partial G(M, M_2)}{\partial n} - G(M, M_2)\frac{\partial G(M, M_1)}{\partial n} \right] dS = 0$$

By Theorem 0.1, $G(M, M_2) \sim \frac{1}{r_{MM_2}}$, thus $\lim_{M \to M_1} \frac{G(M, M_1)}{1/r_{MM_1}} = A$.

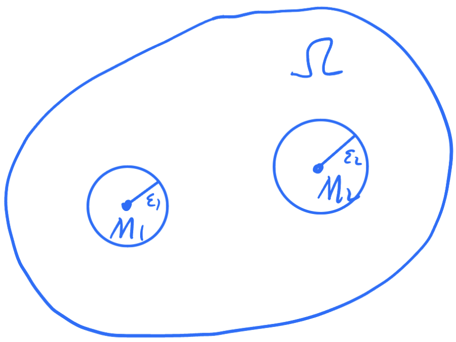

Figure 2: Theorem 0.4

$$
\begin{aligned}
&\Rightarrow \lim_{\varepsilon_1\to 0}\int_{\partial B^{M_1}_{\varepsilon_1}} G(M,M_1)\frac{\partial G(M,M_2)}{\partial n}dS \\
&= \lim_{\varepsilon_1\to 0}\int_{\partial B^{M_1}_{\varepsilon_1}} \frac{A}{r_{MM_1}}\frac{\partial G(M,M_2)}{\partial n}dS = \lim_{\varepsilon_1\to 0}\frac{A}{\varepsilon_1}\int_{\partial B^{M_1}_{\varepsilon_1}}\frac{\partial G(M,M_2)}{\partial n}dS \\
&\overset{\text{Gauss}}{=} \lim_{\varepsilon_1\to 0}\frac{A}{\varepsilon_1}\int_{B^{M_1}_{\varepsilon_1}} \Delta G(M,M_2)dS = 0
\end{aligned} \tag{3}
$$

In the last step, in $B^{M_1}_{\varepsilon_1}$, when ε_1 is small enough, such that $M_2 \notin B^{M_1}_{\varepsilon_1}$. Additionally,

$$
\begin{aligned}
&\lim_{\varepsilon_2\to 0}\int_{\partial B^{M_2}_{\varepsilon_2}} G(M,M_1)\frac{\partial G(M,M_2)}{\partial n}dS \\
=&A\lim_{\varepsilon_2\to 0}\int_{\partial B^{M_2}_{\varepsilon_2}} G(M,M_1)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_2}}\right)dS \qquad \left(n \text{ is inward normal, then } \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}\right) \\
=&A\lim_{\varepsilon_2\to 0}\int_{\partial B^{M_2}_{\varepsilon_2}} G(M,M_1)\frac{1}{r^2_{MM_2}}dS \\
=&A\lim_{\varepsilon_2\to 0}\frac{1}{\varepsilon_2^2}\int_{\partial B^{M_2}_{\varepsilon_2}} G(M,M_1)dS \\
=&A\lim_{\varepsilon_2\to 0}\frac{1}{\varepsilon_2^2}G(M_2,M_1)\cdot 4\pi\varepsilon_2^2 \quad \text{(using mean value theorem, or mean value theorem for integrals)} \\
=&4\pi A G(M_2,M_1).
\end{aligned} \tag{4}
$$

By a similar calculation of $\int_{\partial B^{M_1}_{\varepsilon_1}+\partial B^{M_2}_{\varepsilon_2}} G(M,M_2)\frac{\partial G(M,M_1)}{\partial n}dS$, and combining (3) and (4), we get:

$$
0 = \int_{\partial B^{M_1}_{\varepsilon_1}+\partial B^{M_2}_{\varepsilon_2}}\left[G(M,M_1)\frac{\partial G(M,M_2)}{\partial n} - G(M,M_2)\frac{\partial G(M,M_1)}{\partial n}\right]dS = 4\pi A G(M_2,M_1) - 4\pi A G(M_1,M_2)
$$

Then $G(M_2,M_1) = G(M_1,M_2)$ □

Theorem 0.5.

$$
\iint_\Gamma \frac{\partial G(M,M_0)}{\partial n}dS_M = -1.
$$

1. **(Method 1: Physical meaning)** This property seems familiar. When we talked about the first property of harmonic functions, it was similar, but with a harmonic function u instead of G. If we replace G with the harmonic function u, the value should be zero. This can be judged

directly from the **physical meaning**. For harmonic functions, the **flux is zero** because there are **no charges inside**.

2. **(Method 1: Physical meaning)** For the Green's function, the **physical meaning** is different. It represents a grounded conducting shell with a unit positive charge inside. Therefore, its **flux** should be equal to 1.

3. **(Method 1: Physical meaning)** The **negative sign** in the flux calculation comes from the fact that the electric field $\mathbf{E}$ is related to the potential G as a **negative gradient**. According to **Gauss's flux theorem**, the integral of $\mathbf{E} \cdot dS$ over a closed surface is equal to the charge enclosed, which is q.

4. **(Method 1: Physical meaning)** The relationship between $\mathbf{E}$ and G is given by $\mathbf{E} = -\nabla G$. The flux can be rewritten in terms of the outward normal vector $\mathbf{n}$ and the gradient of G:

$$1 = q = -\int_\Gamma \mathbf{n} \cdot \mathbf{E}\, dS = -\int_\Gamma \mathbf{n} \cdot \nabla G\, dS = -\int_\Gamma \frac{\partial G}{\partial \mathbf{n}}\, dS.$$

5. **(Method 2: Gauss's theorem)** Using **Gauss's theorem**, this surface integral can be converted into a volume integral involving ΔG. For a Green's function with a point charge, the Laplacian of G is related to the delta function:

$$\Delta G = -\delta(\mathbf{r}).$$

6. **(Method 3: Green's first identity)** The flux calculation using the Green's function can be related to the divergence theorem (**Gauss's theorem**) and the **Green's first identity**. These methods are **equivalent** and can be used to prove the same results.

7. **(Method 4: Transformation between u nd G–Nash's mountain-climbing-like metaphorical idea)**

   - Another interesting proof method mentioned is to transform the problem into solving a differential equation.
   - By comparing this relationship with the given equation, according to the form $\int \frac{\partial G}{\partial n}\, dS$, we make a connection to $\int f \frac{\partial G}{\partial n}\, dS = -u(M_0)$, that is, we want to prove

   $$-u(M_0) = \int f \frac{\partial G}{\partial n}\, dS = \int \frac{\partial G}{\partial n}\, dS = -1.$$

   providing $\Delta u = 0$ and $f = 1$ where $u|_\Gamma = f$.
   - Therefore, the problem is transformed to

   $$\begin{cases} \Delta u(x,y,z) = 0, & (x,y,z) \in \Omega \\ u|_\Gamma = f(x,y,z) = 1 \end{cases}$$

   and we have to solve u.

***Proof:* *(Method 4)*.** Consider the following Dirichlet problem

$$\begin{cases} \Delta u(x,y,z) = 0, & (x,y,z) \in \Omega \\ u|_\Gamma = f(x,y,z) = 1 \end{cases}$$

On the one hand, using the relation $u(M_0) = -\iint_\Gamma f(x,y,z) \frac{\partial G}{\partial n} dS$, we can obtain

$$u(M_0) = -\iint_\Gamma f(x,y,z) \frac{\partial G}{\partial n} dS = -\iint_\Gamma \frac{\partial G}{\partial n} dS,$$

On the other hand, by the extremum principle, the solution to this problem is $u \equiv 1$. According to the uniqueness of the solution to the Dirichlet problem, this theorem holds. □

0.2 Applications of Green's Function–Solve Green functions G and v

From the formula

$$u(M_0) = -\iint\limits_{\Gamma} f(M) \frac{\partial G(M, M_0)}{\partial n} dS, \tag{5}$$

it can be seen that for a region Ω bounded by a surface Γ, as long as its **Green's function is found**, then within this region, the solution to the **Dirichlet problem** of **Laplace's equation** can be expressed by this integral.

- In the previous section, we transformed the problem of finding u into the problem of finding the Green's functions G and v, but we did not solve for them. In this section, we will solve for the Green's functions G and v.
- The choice of the domain for the Green's function is crucial; arbitrary boundaries can make the Green's function difficult to determine.
- Two specific regions are discussed: half space and ball, using a physical approach combining physical thinking and mathematical derivation.
- The method used is known as the method of images or electrostatic analogy.

For some special regions, its Green's function can be obtained by the **method of electrical images** (**method of images–Physical Method**).

The so called **method of images** is to find the **image point** (symmetric point) M_1 of the point M_0 with respect to the boundary Γ **outside the region** Ω. Then, an **appropriate negative charge** is placed at this image point M_1. The **negative** electric potential generated by it and the **positive** electric potential generated by the unit positive charge at the point M_0 **cancel each other out** on the **surface** Γ. At this time, the **electric potential** of the electric field formed by the two inside Γ is **equivalent** to the required **Green's function**.

Remarks on the Method of Images

1. The fundamental **physical idea** of the **method of images**:
   - The essence of the method of images: **Equivalence** (a tool for physicists)
   - A fictitious negative **charge** is used to replace the **effect** of a **conducting shell**.
2. Determining the information of the **fictitious charge**:
   (a) **Charge amount**
   (b) **Position**
3. Principle of Determination:
   - The **electric potential** on the **conducting shell** is **0**. The fictitious charge has the **same effect** as the shell.
   - In other words, the criterion for finding the fictitious charge is that the electric fields produced by the fictitious charge and the conducting shell should cancel each other out at the boundary.
4. **Question:** Why must the **fictitious charge** be **equivalent** to the **induced charge** on the conducting **shell**? $\Leftrightarrow$ Just because the **electric potential** on the conducting **shell** Γ **is 0**, can it be guaranteed that the **internal electric potential** is also the **same**?

5. Since for the system of G:

$$\begin{cases} \Delta G = -\delta \\ G|_{\Gamma} = 0, \end{cases} \tag{6}$$

   the solution exists and is unique.

   $\Leftrightarrow$ For the electric field generated by a single electric charge, once the boundary electric potential is determined, there is only one distribution inside, which is also determined!

   - The uniqueness of the solution is ensured by the uniqueness of the Green's function, which is related to the maximum principle for the first Green function v.

6. Because from the problem (6) of G, it is not apparent why the potential is zero on the boundary. **Mathematically**, as long as there is such a boundary condition, the solution is unique. This offers a great deal of **freedom** in physics to employ **various manageable physical models** to simulate such mathematical boundary conditions.

0.2.1 Green's Function in the Half Space and the Dirichlet Problem

We aim to solve the Dirichlet problem in the upper half space $z > 0$. The problem is formulated as:

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad (z > 0), \tag{7}$$

$$u|_{z=0} = f(x, y), \quad -\infty < x, y < +\infty. \tag{8}$$

Three steps of the method of images:

1. Determine the image point (the charge quantity and position. Place M on the boundary for verification).

2. Write out the Green's function (Place M inside Ω).

3. Calculate $\frac{\partial G}{\partial n}\Big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again).

Step 1: Determine the image point (the charge quantity and position. Place M on the boundary for verification.)

First, we need to find the Green's function $G(M, M_0)$ (see Fig. 3). To this end, we place a unit positive charge at the point $M_0(x_0, y_0, z_0)$ in the upper half space $z > 0$, and a unit negative charge at the symmetric point $M_1(x_0, y_0, -z_0)$ of M_0 with respect to the plane $z = 0$ **(based on the symmetry, see the following discussions and other methods)**.

- The location of the unit positive charge can be **arbitrary**, but if one wants to know the **value of u at point** M_0, then place a unit positive **charge at** M_0.

$$u(M_0) = -\iint\limits_{\Gamma} f(M) \frac{\partial G(M, M_0)}{\partial n} dS.$$

- It is crucial to remember that the **independent variable of** u indicates the **location of the unit positive charge** in the Green's function.

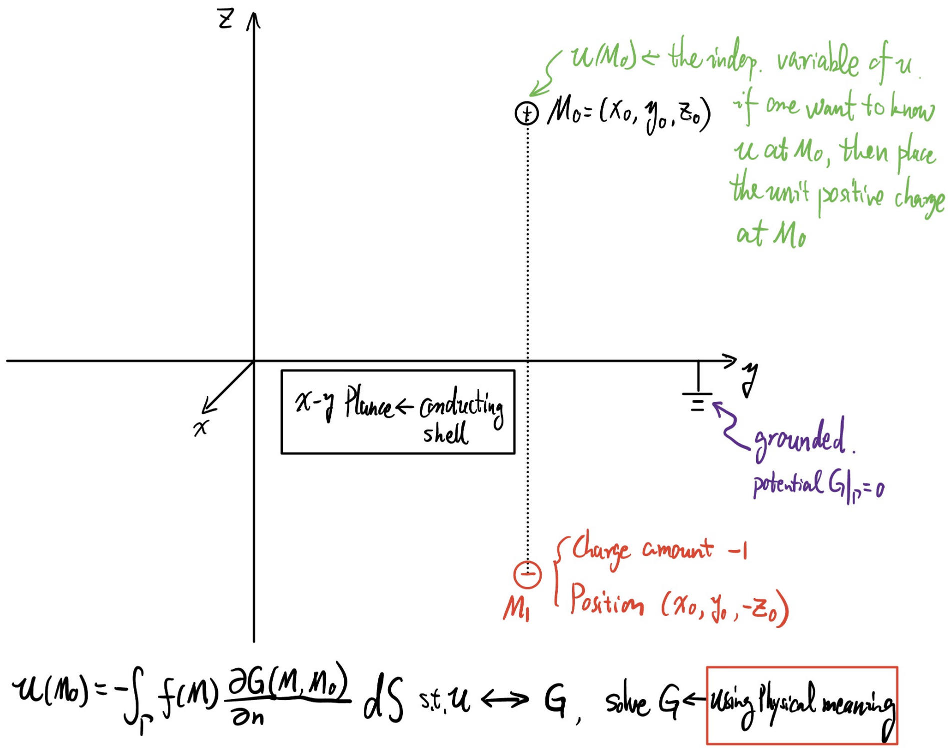

Figure 3: Green's function in the half space

- The goal is to find an **equivalent point**, also known as a **mirror point** or **fictitious charge**.
- On the boundary, the electric potential generated by the mirror point should **cancel out** the potential at the point of interest.
- This cancellation is the **criterion** for determining the mirror point.
- To find the mirror point, we need to determine its **position** and **charge**.
- **(Method 1: Symmetry)** The mirror point can often be **guessed** based on **symmetry**. The **symmetry** of this problem allows for a reasonable guess about the location of the mirror point.
- **(Method 2: A General method determining the mirror point–Not recommended)** Using the cancellation on the boundary, one can obtain **algebraic equations** to determine the **position** and **charge amount** of the mirror point. This is very difficult, the idea is given as follows (see Fig. 4):

$$\frac{1}{4\pi r} - \frac{q_1}{4\pi r_1} = 0 \Leftrightarrow \text{Criterion of cancellations.}$$

$$\Rightarrow \frac{r_1}{r} = q_1 \Rightarrow F(x,y) := \frac{(x_1-x)^2+(y_1-y)^2+z^2}{(x_0-x)^2+(y_0-y)^2+z_0^2} = q_1^2$$

Differentiate with respect to x and y respectively, because no matter how x and y move in the xy plane, the cancellation criterion is always satisfied. That is,

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0 \quad (\text{because} \quad q = \text{const.})$$

Additionally,

$$\left\{ \begin{array}{l} \frac{\partial x_1}{\partial x} = 0 \\ \frac{\partial x_1}{\partial y} = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \frac{\partial y_1}{\partial x} = 0 \\ \frac{\partial y_1}{\partial y} = 0 \end{array} \right.$$

- **(Method 3: Based on the specific (x, y) values)** Due to the criterion of cancellations,

$$\frac{(x_1 - x)^2 + (y_1 - y)^2 + z^2}{(x_0 - x)^2 + (y_0 - y)^2 + z^2} = q^2 \quad \forall x, y \in \mathbb{R}^2$$

  - There are 4 unknowns (x_1, y_1, z_1) and q, with only one equation.
  - To determine these 4 unknowns, we need 4 equations.
  - Since x, y can be chosen arbitrarily, we can obtain 4 equations by choosing 4 specific (x, y) values.
    1. $(x, y) = (x_0, y_0)$,
    $$\Rightarrow \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2 + z^2}{z_0^2} = q^2.$$
    2. $(x, y) = (x_1, y_1)$,
    $$\Rightarrow \frac{z_1^2}{(x_0 - x_1)^2 + (y_0 - y_1)^2 + z_0^2} = q^2$$
    3. $(x, y) = (0, 0)$,
    $$\Rightarrow \frac{x_1^2 + y_1^2 + z_1^2}{x_0^2 + y_0^2 + z_0^2} = q^2$$
    4. $(x, y) = (x_0, 0)$,
    $$\Rightarrow \frac{(x_1 - x_0)^2 + y_1^2 + z_1^2}{y_0^2 + z_0^2} = q^2$$

    With these 4 equations, the 4 unknowns can be solved.

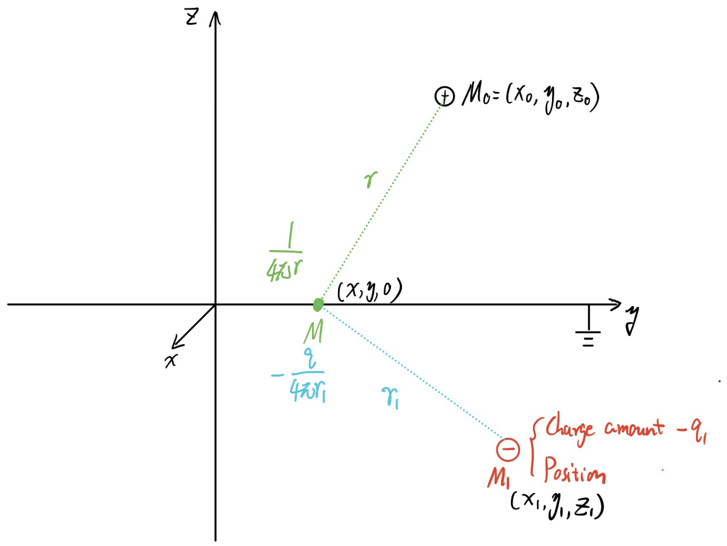

Figure 4: General idea for the mirror point

The electric potential of the electrostatic field formed by them is exactly 0 on the plane $z = 0$.

- **Verification of Mirror Point:** We need to verify if the electric potential at the boundary (see Fig. 5) .
- **Criterion for Position:** The sum of the potentials from the original charge and the mirror charge at the boundary should be zero.

- **Boundary Point Selection:** The point M **on the boundary** is selected for verification. That is, place M **on the boundary** for verification.
- **Dynamic Point M:** The selection of point M will change multiple times during the following calculation process (see Fig. 5, 6 and 7).
  - (Fig. 5) Take M on the boundary to **verify** the selected mirror point is correct. The criterion of this verification is the cancellation of potentials.

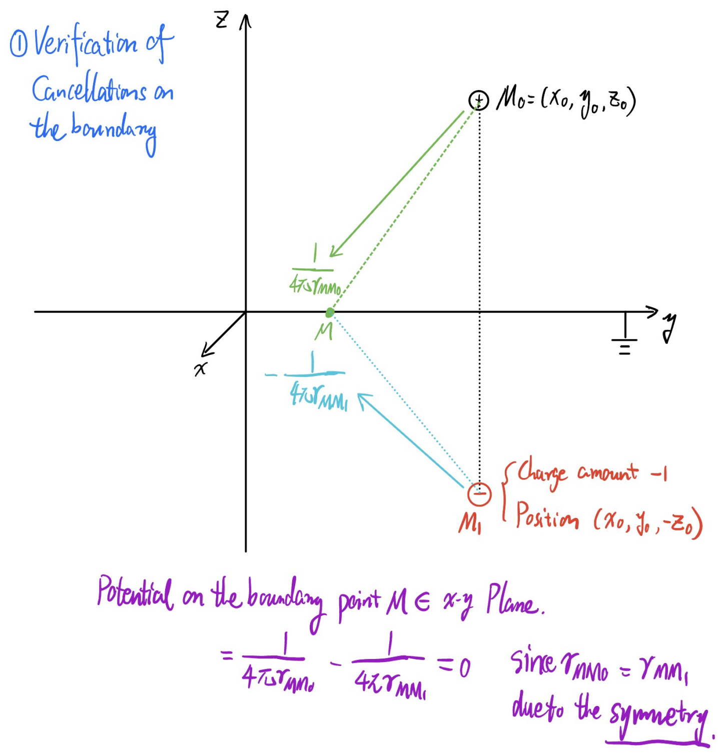

Figure 5: Step 1: Verification of cancellations

Step 2: Write out the Green's function (Place M inside Ω)

Therefore, **by placing M inside Ω**, the Green's function in the upper half space is

$$G(M, M_0) = \frac{1}{4\pi}\left(\frac{1}{r_{MM_0}} - \frac{1}{r_{MM_1}}\right),$$

where $r_{MM_0} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ and $r_{MM_1} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}$.

Step 3: Calculate $\frac{\partial G}{\partial n}\big|_{z=0}$ and use the formula to calculate $u(M_0)$ (Place M on the boundary again)

To solve problems (7) and (8), we need to calculate $\frac{\partial G}{\partial n}\big|_{z=0}$ (need to place M on the boundary again since the integration in (5) is on the boundary Γ, see Fig. 7). Since the outward normal direction on the plane $z = 0$ is the negative direction of the oz-axis, we have $\frac{\partial G}{\partial n}\big|_{z=0} = -\frac{\partial G}{\partial z}\big|_{z=0}$ (n is the direction of $-z$, the outward normal).

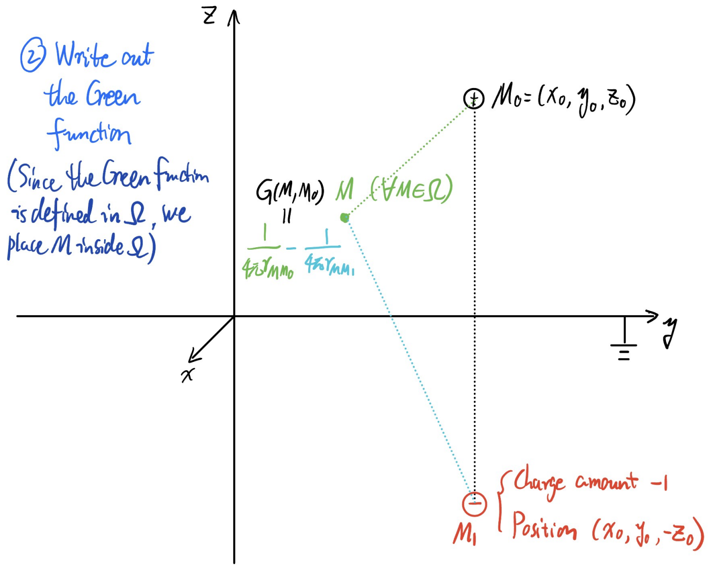

Figure 6: Step 2: Write out the Green function

We first calculate the partial derivative:

$$
\begin{aligned}
\left.\frac{\partial G}{\partial n}\right|_{z=0} &= -\left.\frac{\partial G}{\partial z}\right|_{z=0} \\
&= -\frac{1}{4\pi}\left\{\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right)\right. \\
&\quad \left.\left. -\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right)\right\}\right|_{z=0} \\
&= \frac{1}{4\pi}\left\{\frac{z-z_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right. \\
&\quad \left.\left. -\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right)\right\}\right|_{z=0} \\
&= \frac{1}{4\pi}\left.\left\{\frac{z-z_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} - \frac{z+z_0}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}}\right\}\right|_{z=0} \\
&= -\frac{1}{2\pi}\frac{z_0}{[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}},
\end{aligned} \tag{9}
$$

Substituting (9) into

$$u(M_0) = -\iint_{\Gamma} f(M)\frac{\partial G}{\partial n}dS,$$

we obtain the solution to the problems (7) and (8):

$$u(M_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x,y)z_0dxdy}{[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}}. \tag{10}$$

Ex 0.1. *Suppose that on the boundary of a homogeneous half space, the steady state temperature is maintained. It is equal to 1 inside the circle $K : x^2+y^2 < 1$ and 0 outside it. We want to find the steady state temperature distribution in the half space.*

Solution. *This problem can be reduced to the following well posed problem:*

$$
\begin{cases}
u_{xx}+u_{yy}+u_{zz}=0 & (z>0) \\
u|_{z=0} = f(x,y) = \begin{cases} 1, & x^2+y^2<1 \\ 0, & x^2+y^2 \geq 1 \end{cases}
\end{cases}
$$

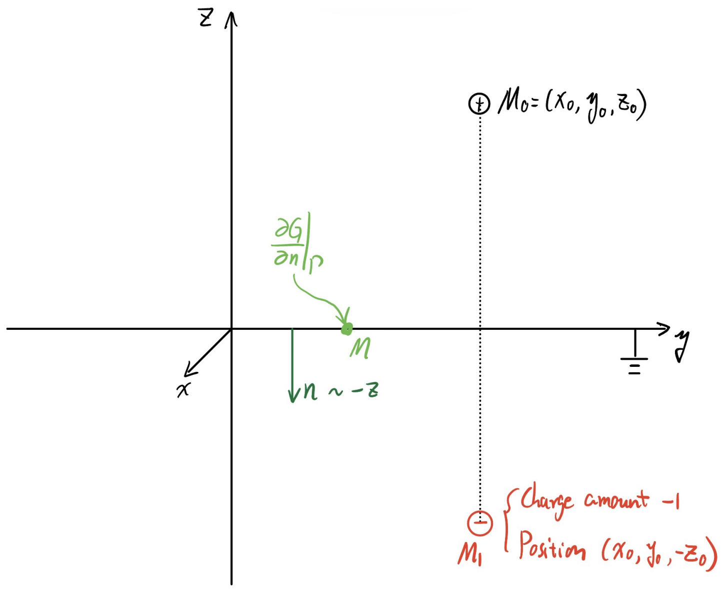

Figure 7: Step 3: Calculate $\frac{\partial G}{\partial n}|_{z=0}$

According to formula (10), we have

$$u(x_0,y_0,z_0)=\frac{z_0}{2\pi}\iint\limits_K \frac{dxdy}{[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}}$$

In particular, on the positive semi-axis of the oz-axis ($x_0=0, y_0=0$), we have

$$u(0,0,z_0)=\frac{z_0}{2\pi}\iint\limits_K \frac{dxdy}{(x^2+y^2+z_0^2)^{3/2}}$$

Since the integration region K is a circular region, we use polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$. Then

$$\begin{aligned}u(0,0,z_0)&=\frac{z_0}{2\pi}\int_0^1\int_0^{2\pi}\frac{rdrd\theta}{(r^2+z_0^2)^{3/2}}\\&=z_0\int_0^1\frac{rdr}{(r^2+z_0^2)^{3/2}}\\&=-z_0\frac{1}{(r^2+z_0^2)^{1/2}}\Big|_{r=0}^{r=1}\\&=1-\frac{z_0}{(1+z_0^2)^{1/2}}\end{aligned}$$

When $(0,0,z_0)$ approaches infinity along the positive semi-axis of the oz-axis, $u(0,0,z_0)\to 0$.

Supplementary: Green's function in the half plane and the Dirichlet problem

We aim to solve the Dirichlet problem in the upper half plane $y>0$. The problem is formulated as:

$$u_{xx}+u_{yy}=0 \quad (y>0), \tag{11}$$
$$u|_{y=0}=f(x), \quad -\infty<x<+\infty. \tag{12}$$

First, we find the Green's function $G(M,M_0)$. To this end, we place a unit positive charge at the point $M_0(x_0,y_0)$ in the upper half plane $y>0$, and a unit negative charge at the symmetric point $M_1(x_0,-y_0)$ of M_0 with respect to the boundary $y=0$. The electric potential of the electrostatic field formed by them is exactly 0 on the boundary $y=0$. Therefore, the Green's function in the upper half plane is

$$G(M,M_0)=\underbrace{\frac{1}{2\pi}}_{4\pi\to2\pi}\Big(\underbrace{\ln\frac{1}{r_{MM_0}}}_{\frac{1}{r_{MM_0}}\to\ln\frac{1}{r_{MM_0}}}-\ln\frac{1}{r_{MM_1}}\Big),$$

To solve problems (11) and (12), we need to calculate $\frac{\partial G}{\partial n}\big|_{y=0}$. Since the outward normal direction on the boundary $y = 0$ is the negative direction of the oy-axis, we have $\frac{\partial G}{\partial n}\big|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0}$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial n}\right|_{y=0} &= -\left.\frac{\partial G}{\partial y}\right|_{y=0} \\
&= -\frac{1}{2\pi}\left\{\frac{\partial}{\partial y}\left(\ln\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right) - \frac{\partial}{\partial y}\left(\ln\frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}}\right)\right\}\Bigg|_{y=0} \\
&= \frac{1}{2\pi}\left\{\frac{y-y_0}{[(x-x_0)^2+(y-y_0)^2]} - \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2}}\right)\right\}\Bigg|_{y=0} \\
&= \frac{1}{2\pi}\left\{\frac{y-y_0}{[(x-x_0)^2+(y-y_0)^2]} - \frac{y+y_0}{[(x-x_0)^2+(y+y_0)^2]}\right\}\Bigg|_{y=0} \\
&= -\frac{1}{\pi}\frac{y_0}{[(x-x_0)^2+y_0^2]},
\end{aligned} \tag{13}$$

Substituting (13) into

$$u(M_0) = -\int_C f(x,y)\frac{\partial G}{\partial n}dS,$$

we obtain the integral expression for the solution of the Dirichlet problem of the Laplace equation in the half plane (11) and (12):

$$u(M_0) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)y_0dx}{[(x-x_0)^2+y_0^2]}.$$