Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 2-2

1 Method of Separation of Variables

Recall:

Basic Steps of the Method of Separation of Variables:

1. **(Separation of Variables)** Assume separation of variables $u(x,t) = X(x)T(t)$
2. **(PDE $\rightarrow$ ODEs)** Substitute into the partial differential equation (PDE) to obtain two ordinary differential equations (ODEs) for $X(x)$ and $T(t)$ respectively
3. **(Solving ODEs)** Solve the ODEs:
   - $X(t)$-eq. $\overset{\text{by bdry condi.}}{\rightarrow}$ SL problem $\rightarrow$ Eigenvalues λ_n and eigenfunctions X_n (Standing waves appear here)
   - $T(t)$-eq. $\overset{\text{substituting } \lambda_n}{\rightarrow}$ Find the general solution of T_n
4. **(Superposition of Series Solutions)** Superpose the series solution $u = \sum^{\infty} u_n$
5. **(Initial Conditions Determine Coefficients)** Determine the Fourier coefficients using initial conditions

In this lecture:

1. We will discuss 4 to 5 major examples.
2. All examples follow a **structured five-step** separation of variables approach.
3. The class involves a high volume of information, requiring active participation and practice.

Learning Strategy:
To effectively follow along, students should:

1. Memorize the five-step framework.
2. Practice by solving problems alongside the instructor.
3. Compare each new example with the first one, identifying:
   (a) **Similarities**, which reinforce the core five-step-framework.
   (b) **Differences**, which determine **necessary modifications** in the steps.
4. **Understanding these differences** is crucial to avoid confusion.
5. You must study through comparison so that you can truly understand the underlying logic and changes.

Conclusion
By systematically comparing examples, students can develop a clear understanding of the separation of variables method and its applications.

1.1 Free Vibration of a Bounded String (Continued)

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \tag{1}$$

1.2 The physical meaning of the series solution for the boundary value problem

Take a general term of series (14) and transform it as follows:

$$\begin{aligned} u_n(x,t) &= \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \\ &= N_n \sin(\omega_n t + \theta_n) \sin \frac{n\pi x}{l}, \end{aligned} \tag{16}$$

where $N_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \arctan \frac{a_n}{b_n}$, $\omega_n = \frac{n\pi a}{l}$; θ_n is called the initial phase, and ω_n is called the frequency.

The physical meaning of equation (16) can be studied by first fixing the time t to observe the shape of the vibration wave at that moment; then fixing a point on the string to observe the vibration pattern at that point.

When $t = t_0$,

$$u_n(x,t_0) = N_n' \sin \frac{n\pi x}{l},$$

where $N_n' = N_n \sin(\omega_n t_0 + \theta_n)$ is a constant value. This indicates that at any time, the wave shape of $u_n(x,t_0)$ is a sine curve, and its amplitude is related to time t_0.

When $x = x_0$,

$$u_n(x_0,t) = N_n'' \sin(\omega_n t + \theta_n),$$

where $N_n'' = N_n \sin \frac{n\pi x_0}{l}$ is a constant value, indicating that each point x_0 on the string is undergoing simple harmonic motion, with an amplitude of $\left| N_n \sin \frac{n\pi x_0}{l} \right|$. The frequency is $\omega_n = \frac{n\pi a}{l}$, and the initial phase is θ_n. If another point is taken, the situation is the same, only the amplitude differs.

From the above, it is known that $u_n(x,t)$ represents a vibration wave where all points on the string vibrate with the same frequency in simple harmonic motion, with the same initial phase at each point, and the amplitude varies with the position of the point. The shape of this vibration wave at any moment is also a sine curve.

The solution is given by

$$u_n(x,t) = N_n \sin(\omega_n t + \theta_n) \sin \frac{n\pi x}{l}, \tag{16}$$

When $x_m = \frac{ml}{n} (m = 0, 1, 2, \dots, n)$, $u_n(x,t) = 0$, which indicates that these points remain stationary throughout the vibration process. Such points are referred to as nodes in the physical context of u_n. This implies that the vibration of $u_n(x,t)$ is a segmented vibration on the interval $[0,l]$, and such vibration waves containing nodes are called **standing waves**[1].

When $x_k = \frac{(2k-1)l}{2n} (k = 1, 2, \dots, n)$, the amplitude of the standing wave reaches its maximum value at these points, which are referred to as antinodes.

Thus, we know that $u_1, u_2, \dots, u_n, \dots$ are a series of standing waves, whose frequencies, initial phases, and amplitudes all vary with n. Therefore, it can be said that the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l} \tag{14}$$

is composed of a series of standing waves with different frequencies, initial phases, and amplitudes. Hence, people also call the method of separation of variables the method of standing waves.

Ex 1.1 (Examining the Free Vibration Problem of a String Fixed at Both Ends)**.** *Consider the free vibration problem of a string fixed at both ends:*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < 1,\, t > 0), \\ u(0,t) = 0,\ u(1,t) = 0, & \\ u(x,0) = \sin 2\pi x, \quad u_t(x,0) = x(1-x). & \end{cases}$$

[1] In Chap. 3, there is no boundary conditions constrain the wave, thus the wave is traveling instead of standing.

Solution. *Using the Formula*

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

Since $l=1$, the solution to this boundary value problem is

$$u(x,t)=\sum_{n=1}^{\infty}(a_n\cos n\pi at+b_n\sin n\pi at)\sin n\pi x.$$

Using formula again,

$$a_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}\,dx,\quad b_n=\frac{2}{n\pi a}\int_0^l\psi(x)\sin\frac{n\pi x}{l}\,dx,$$

we have

$$a_n=2\int_0^1\sin 2\pi x\sin n\pi x\,dx$$

$$b_n=\frac{2}{n\pi a}\int_0^1 x(1-x)\sin n\pi x\,dx$$

Given

$$a_n=2\int_0^1\sin 2\pi x\sin n\pi x\,dx=\begin{cases}0, & n\neq 2,\\ 1, & n=2.\end{cases}\qquad \textit{(orthogonality)}$$

$$\begin{aligned}b_n=&\frac{2}{n\pi a}\int_0^1 x(1-x)\sin n\pi x\,dx\\ =&\frac{2}{n\pi a}\left[-\frac{1}{n\pi}x(1-x)\cos n\pi x\Big|_0^1+\frac{1}{n\pi}\int_0^1(1-2x)\cos n\pi x\,dx\right]\\ =&\frac{2}{(n\pi)^2a}\left[\frac{1}{n\pi}(1-2x)\sin n\pi x\Big|_0^1+\frac{2}{n\pi}\int_0^1\sin n\pi x\,dx\right]\\ =&\frac{4}{(n\pi)^4a}\left[1-(-1)^n\right].\end{aligned}$$

(Integration by parts: Differentiation applied to a polynomial can continuously reduce its degree.)

Thus, we have

$$u(x,t)=\sum_{n=1}^{\infty}(a_n\cos n\pi at+b_n\sin n\pi at)\sin n\pi x,$$

where

$$a_n=2\int_0^1\sin 2\pi x\sin n\pi x\,dx=\begin{cases}0, & n\neq 2,\\ 1, & n=2.\end{cases}$$

$$b_n=\frac{4}{(n\pi)^4a}\left[1-(-1)^n\right]$$

Therefore, the solution to the boundary value problem is

$$u(x,t)=\cos 2\pi at\sin 2\pi x+\sum_{n=1}^{\infty}\frac{4}{(n\pi)^4a}\left[1-(-1)^n\right]\sin n\pi at\sin n\pi x.$$

To effectively learn the separation of variables method for differential equations, follow these suggestions:

1. Identify **similarities** with previous problems:

- Recognizing common structures (the homogeneous equation and boundary) helps determine if the method is applicable.

2. Focus on **differences**:
   - Differences dictate necessary modifications to the separation of variables method.
3. Active practice is essential:
   - Simply watching or reading is insufficient; hands-on problem-solving is required.
4. Fundamental skills in differentiation and integration are crucial:
   - These skills form the basis for successfully applying separation of variables.
5. The **eigenvalue problems** from these problems must be memorized.

Ex 1.2 (Free Vibration of a String with One End Fixed and the Other Free)**.**

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l,\, t > 0), \\ u(0,t) = 0,\, u_x(l,t) = 0, \\ u(x,0) = x^2 - 2lx,\, u_t(x,0) = 3\sin\frac{3\pi x}{2l}. \end{cases}$$

Similarities

- Both equations are $1+1$-dimensional (This chapter deals only with 2D; higher dimensions appear in Chapter 5.).
- The primary equation is the same: The wave equation determines that the ODE for T is a second-order oscillatory ODE.
- Two given functions: one is a sine function (trigonometric), and the other is a polynomial function. The coefficient determination follows the same approach:
  - Using orthogonality for integration.
  - Using integration by parts.
- Homogeneous equation + homogeneous boundary conditions $\rightarrow$ separation of variables, and five-step method should be followed for solution.

Differences

- Difference in boundary conditions:
  - First problem: both sides have Dirichlet (first-type) boundary conditions.
  - Second problem: left boundary is Dirichlet (first-type), right boundary is Neumann (second-type).
  - Third problem (as exercise): left boundary is Neumann (second-type), right boundary is Dirichlet (first-type), which is symmetric to the second problem.
  - Fourth problem: both boundaries are Neumann (second-type).
- The boundary conditions affect the eigenvalue problems. The major changes occur in:
  - Step 3: Solving the X-ODE (the eigenvalue problems).
  - Step 5: Determining coefficients due to different eigenfunctions.

2 Conclusion

Understanding the role of boundary conditions in separation of variables is crucial, as they determine how the problem is solved, particularly in the eigenvalue problems and coefficient determination.

Since the boundary conditions of this problem are different from (??) $u(0,t) = 0, u(l,t) = 0$, we cannot directly use the formula

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \sin \frac{n\pi x}{l}.$$

However, we can solve it using the method of separation of variables.

Solution. ***(1. Separation of Variables)*** *Let $u(x,t) = X(x)T(t)$.*

(2. PDE $\to$ ODEs) *Substituting into the equation and separating variables yields two ordinary differential equations*

$$T''(t) + \lambda a^2 T(t) = 0, \quad X''(x) + \lambda X(x) = 0,$$

(3. Solving ODEs)

(3a) *$X(t)$**-eq.*** $\overset{\textbf{\textit{by bdry condi.}}}{\to}$ ***SL problem:*** *From the boundary conditions, it is easy to obtain (Difference)*

$$X(0) = 0,\ X'(l) = 0.$$

Find the non-zero solution of the boundary value problem (Difference)

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

(First, solve the ODE corresponding to the homogeneous boundary conditions.)

(1) When $\lambda < 0$, there are no non-trivial solutions (using geometric thinking, the boundary requires $X(0) = 0$ and the slope at l also vanishes).

(2) When $\lambda = 0$, there are also no non-trivial solutions.

(3) When $\lambda > 0$, the general solution of the equation is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

From the boundary conditions, we get

$$X(0) = A = 0, \quad X'(l) = B\sqrt{\lambda}\cos\sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $B \neq 0 \implies \cos\sqrt{\lambda}l = 0$, thus

$$\lambda = \lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2 \quad (n = 0, 1, 2, \ldots). \quad \textit{eigenvaluess}$$

Thus, we find a set of non-zero solutions

$$X_n(x) = B_n \sin\frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \ldots). \quad \textit{eigenfunctions}$$

These eigenvalues and eigenfunctions can also be written as

$$\lambda = \lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2 \quad \textit{and} \quad X_n(x) = B_n \sin\frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \ldots).$$

(3b) *$T(t)$**-eq.*** $\overset{\textbf{\textit{substituting}}\ \lambda_n}{\to}$ ***Find the general solution of T_n:*** *Now consider*

$$T''(t) + \lambda a^2 T(t) = 0,$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2 \quad (n = 0, 1, 2, \ldots),$$

into the equation, we get (In the heat equation, T-ODE becomes a first-order ODE. Since the heat equation has only a first-order time derivative, it leads to exponential decay.)

$$T''(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 T(t) = 0,$$

and its general solution is

$$T_n(t) = C_n \cos\frac{(2n+1)\pi at}{2l} + D_n \sin\frac{(2n+1)\pi at}{2l} \quad (n = 0, 1, 2, \ldots).$$

(4. Superposition of Series Solutions) *Thus, the solution to the problem can be expressed as*

$$u(x,t) = \sum_{n=0}^{\infty} \left[a_n \cos\frac{(2n+1)\pi at}{2l} + b_n \sin\frac{(2n+1)\pi at}{2l}\right] \sin\frac{(2n+1)\pi x}{2l},$$

where $a_n = B_n C_n$, $b_n = B_n D_n$ are arbitrary constants.

(5. Initial Conditions Determine Coefficients) *In the above formula and its corresponding derivative, let $t = 0$, and combine the initial conditions*

$$u(x,0) = x^2 - 2lx, \quad u_t(x,0) = 3\sin\frac{3\pi x}{2l}.$$

We get[2]

$$\sum_{n=0}^{\infty} a_n \sin\frac{(2n+1)\pi x}{2l} = x^2 - 2lx,$$

$$\sum_{n=0}^{\infty} b_n \frac{(2n+1)\pi a}{2l} \sin\frac{(2n+1)\pi x}{2l} = 3\sin\frac{3\pi x}{2l}.$$

Thus,

$$a_n = \frac{2}{l}\int_0^l (x^2 - 2lx)\sin\frac{(2n+1)\pi x}{2l}\,dx = -\frac{32l^2}{(2n+1)^3\pi^3},$$

$$b_n \frac{(2n+1)\pi a}{2l} = \frac{2}{l}\int_0^l 3\sin\frac{3\pi x}{2l}\sin\frac{(2n+1)\pi x}{2l}\,dx = \begin{cases} 0, & n \neq 1, \\ \frac{2l}{\pi a}, & n = 1. \end{cases}$$

Therefore, the (formal) solution to the problem is

$$u(x,t) = \sum_{n=0}^{\infty} -\frac{32l^2}{(2n+1)^3\pi^3} \cos\frac{(2n+1)\pi at}{2l} \sin\frac{(2n+1)\pi x}{2l} + \frac{2l}{\pi a} \sin\frac{3\pi at}{2l} \sin\frac{3\pi x}{2l}.$$

Ex 2.1 (Free Vibration of a String with Both Ends Free)**.** *Consider the free vibration problem of a string with both ends free:*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < l,\ t > 0), \\ u_x(0,t) = 0,\ u_x(l,t) = 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

> **Similarities:** Homogeneous equations with homogeneous boundary — Try separation of variables.
> **Differences:** Both boundaries are the **second kind** of boundary conditions.

Solution. ***(1. Separation of Variables)*** *Let $u(x,t) = X(x)T(t)$,*

(2. PDE → ODEs) *Substituting into the equation and separating variables yields two ordinary differential equations*

$$T''(t) + \lambda a^2 T(t) = 0, \quad X''(x) + \lambda X(x) = 0,$$

(3. Solving ODEs) *From the boundary conditions, it is easy to obtain $X'(0) = 0$, $X'(l) = 0$. Solve the boundary value problem (Difference, New S-L prob.)*

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0.$$

for the non-zero solution.

[2] $\sin\frac{(2n+1)\pi x}{2l}$ still forms a orthogonal basis, see the last lecture and we need first extend the data functions oddly and then periodically.

(1) When $\lambda < 0$, there are no non-trivial solutions to this problem.

(2) When $\lambda = 0$, the general solution of the equation is $X_0(x) = A_0 x + B_0$, hence $X_0'(x) = A_0$. From the boundary conditions, we get $X_0'(0) = X_0'(l) = A_0 = 0$,

$$\implies X_0(x) = B_0.$$

Major Changes in the Solution

- *Previously, for $\lambda = 0$, only the trivial solution existed.*
- *Now, there **exists a nontrivial constant** solution.*
- *Reason:*
  - *Both boundary slopes are zero, forming a horizontal line.*
  - *This allows the solution to be a constant.*
  - *Previously, in addition to having zero slope, the solution also had to pass through the origin, restricting it to the trivial case.*

Substituting $\lambda = 0$ into the equation $T''(t) + \lambda a^2 T(t) = 0$, we solve to get

$$T_0(t) = C_0 t + D_0.$$

Thus, we obtain a non-trivial solution satisfying the second type of boundary conditions for the original vibration equation

$$u_0(x,t) = \frac{1}{2}(a_0 + b_0 t),$$

where $a_0 = 2B_0 D_0$, $b_0 = 2B_0 C_0$ are arbitrary constants (This notation is used to maintain consistency with the Fourier series later on).

(3) When $\lambda > 0$, the general solution of the equation is of the following form:

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

From the boundary conditions, we get

$$X'(0) = B\sqrt{\lambda} = 0 \implies B = 0,$$

$$X'(l) = -A\sqrt{\lambda}\sin\sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $A \neq 0$, $\sin\sqrt{\lambda}l = 0$, thus

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

Hence, we find a set of non-zero solutions

$$X_n(x) = A_n \cos\frac{n\pi x}{l} \quad (n = 1, 2, \ldots).$$

Now consider

$$T''(t) + \lambda a^2 T(t) = 0,$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots),$$

into the equation, we get

$$T''(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0, \quad (n = 1, 2, \ldots),$$

and its general solution is

$$T_n(t) = C_n \cos\frac{n\pi a t}{l} + D_n \sin\frac{n\pi a t}{l} \quad (n = 1, 2, \ldots).$$

*(**4. Superposition of Series Solutions**) Thus, the solution to the problem can be expressed as*

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right] \cos \frac{n\pi x}{l} + \frac{1}{2}(a_0 + b_0 t),$$

where $a_n = A_n C_n$, $b_n = A_n D_n$ are arbitrary constants.

- *When both boundaries are **second-type boundary** conditions, they only restrict the slope to zero, resulting in **nontrivial constant solutions** for $\lambda = 0$.*
- *For $\lambda = 0$, the corresponding eigenfunction is a constant $X_0(x) = A_0 = A_0 \cos \frac{0\pi x}{l}$. Thus, the complete eigenvalues and eigenfunctions are*

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = A_n \cos \frac{n\pi x}{l}, \qquad (n = \underbrace{0}_{\text{Constant solution}}, 1, 2, \ldots).$$

Conclusion

- *Attention to detail in counting indices is crucial as small errors can lead to incorrect series solutions.*
- *Ensure to clarify the starting point of indices when completing assignments.*

*(**5. Initial Conditions Determine Coefficients**) The derivative of the above formula with respect to t is*

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} \left(-a_n \sin \frac{n\pi at}{l} + b_n \cos \frac{n\pi at}{l} \right) \cos \frac{n\pi x}{l} + \frac{b_0}{2}.$$

Setting $t = 0$ in the above two equations and combining with the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),$$

we get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} = \varphi(x),$$

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \cos \frac{n\pi x}{l} = \psi(x).$$

Thus,

$$\begin{cases} a_n = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{n\pi x}{l} \, dx, \\ a_0 = \frac{2}{l} \int_0^l \varphi(x) \, dx, \\ b_n = \frac{2}{n\pi a} \int_0^l \psi(x) \sin \frac{n\pi x}{l} \, dx, \\ b_0 = \frac{2}{l} \int_0^l \psi(x) \, dx. \end{cases}$$

Substituting the determined a_n and b_n into the series solution expression, we obtain the solution to the original problem.

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{l} + b_n \sin \frac{n\pi at}{l} \right) \cos \frac{n\pi x}{l} + \frac{1}{2}(a_0 + b_0 t).$$

Summary of the forms of five common eigenfunction series

1. $u(0,t)=0,\ u(l,t)=0;\ \Rightarrow \left\{\sin\frac{n\pi x}{l}\right\}\ (n=1,2,\cdots);$
2. $u(0,t)=0,\ u_x(l,t)=0;\ \Rightarrow \left\{\sin\frac{(2n-1)\pi x}{2l}\right\}\ (n=1,2,\cdots);$
3. $u_x(0,t)=0,\ u(l,t)=0;\ \Rightarrow \left\{\cos\frac{(2n-1)\pi x}{2l}\right\}\ (n=1,2,\cdots);$
4. $u_x(0,t)=0,\ u_x(l,t)=0;\ \Rightarrow \left\{\cos\frac{n\pi x}{l}\right\}\ (n=0,1,2,\cdots);$

The above forms are applicable to one-dimensional vibration equations, heat conduction equations, and Poisson's equations on rectangular domains.

5. Eigenfunction series corresponding to Poisson's equation on a circular domain

$$\{1,\cos\theta,\sin\theta,\cos 2\theta,\sin 2\theta,\cdots\cos n\theta,\sin n\theta,\cdots\}$$

3 Heat Conduction in a Finite Rod

3.1 Consider the mixed problem of the homogeneous heat conduction equation (with boundary conditions of the first type)

$$\begin{cases} u_t=a^2u_{xx}\ (0<x<l,\ t>0), \\ u(0,t)=0,\ u(l,t)=0, \\ u(x,0)=\varphi(x), \end{cases} \tag{2}$$

where $\varphi(x)$ is a given known function.

Similarities

- Both involve homogeneous equations and boundary $\rightarrow$ separation of variables and fundamental five-step approach.
- Both end points satisfy the first-type (Dirichlet) boundary conditions.

Differences

- The equation transforms into the heat equation and the initial condition only needs to specify $u(x,0)$ due to the heat equation.

We will solve the problem (2) using the method of separation of variables (also known as the method of standing waves).

(1. Separation of Variables) First, let

$$u(x,t)=X(x)T(t),$$

(2. PDE $\rightarrow$ ODEs) Substitute it into the equation

$$u_t=a^2u_{xx}$$

and separate variables to obtain two ordinary differential equations

$$\begin{aligned} T'(t)+\lambda a^2T(t)=&0, \quad \text{(Difference: first order for heat; Normalization: } a^2 \text{ in } T\text{-ODE)} \\ X''(x)+\lambda X(x)=&0, \end{aligned}$$

(3. Solving ODEs)

(3a) $X(t)$**-eq.** $\overset{\textbf{by bdry condi.}}{\to}$ **SL problem:** From the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, we get

$$X(0) = 0,\ X(l) = 0.$$

Solve the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

for the non-zero solution.

1. When $\lambda < 0$, there are no non-trivial solutions to this problem.
2. When $\lambda = 0$, there are also no non-trivial solutions to this problem.
3. When $\lambda > 0$, there are non-trivial solutions to this problem.

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

$$X_n(x) = B_n \sin\frac{n\pi x}{l} \quad (n = 1, 2, \ldots).$$

(3b) $T(t)$**-eq.** $\overset{\textbf{substituting } \lambda_n}{\to}$ **Find the general solution of** T_n**:**
Now Consider

$$T'(t) + \lambda a^2 T(t) = 0, \tag{3}$$

substituting the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

into the equation, we get

$$T'(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0,$$

and its general solution is

$$T_n(t) = C_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \quad (n = 1, 2, \ldots).$$

- In the heat equation, T-ODE becomes a first-order ODE. Since the heat equation has only a first-order time derivative, it leads to exponential decay.
- In the wave equation, T-ODE becomes a second-order ODE. Since the wave equation has a second-order time derivative, it leads to oscillation (sin and cos function).

(4. Superposition of Series Solutions) Thus, the solution to the problem (2) that satisfies the homogeneous boundary conditions and has the form of variable separation is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin\frac{n\pi x}{l}, \tag{4}$$

where $a_n = B_n C_n$ are arbitrary constants.

(5. Initial Conditions Determine Coefficients) Using the initial condition $u(x,0) = \varphi(x)$, we get

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = \varphi(x),$$

$$a_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\, dx. \tag{5}$$

Equations (4) and (5) together give the particular solution to the problem (2).

$$
\begin{cases}
u_t = a^2 u_{xx}\,(0 < x < l,\, t > 0),\\
u(0,t) = 0,\; u(l,t) = 0,\\
u(x,0) = \varphi(x),
\end{cases}
$$

Note: If the boundary conditions in the problem are of the second or third kind, the method is similar. Please think for yourselves!

Prerequisite Knowledge: First-Order Linear Ordinary Differential Equation

$$
y' + a(x)y = b(x) \tag{6}
$$

The general solution formula. Here $a(x)$, $b(x)$ are known functions.

Let the antiderivative of $a(x)$ be $A(x)$, then the general solution formula for (6) is

$$
y = e^{-A(x)}\left(\int e^{A(x)}b(x)\,dx + C\right)
$$

where C is an arbitrary constant. Specifically, if $b(x) = 0$, then the general solution simplifies to

$$
y = Ce^{-A(x)}.
$$

Key Ideas

- The method of separation of variables works well for solving homogeneous ODE (3).
- When the ODE becomes non-homogeneous (6), we need a different approach.
- Recall that one **common method** is solving the homogeneous equation first and then finding a particular solution.
- Another method is the **variation of parameters**, which is useful for second-order equations.
- The simplest and most direct approach is the **integrating factor method**.

The basic Idea of Integrating Factor Method

- We seek to multiply the equation by a function $F(x)$, called the **integrating factor**, such that the left-hand side becomes the **derivative of a product** by the Leibniz rule.

Finding the Integrating Factor

Question 3.1. *When encountering forms like $y' + ay$ or $y'' + ay'$, which are **one order less**, the second term hinders direct integration. Is it possible to find a way to still integrate directly?*

A. *we can consider the **integrating factor**. Essentially, this is similar to the **Leibniz rule**:*

$$
(Fy)' = Fy' + F'y
$$

which is similar to and can be compared with $y' + ay$.

Thus, we can think, by multiplying F on the both sides of (6):

$$
\underbrace{Fy' + aFy}_{\text{Hope it is equal to } Fy'+F'y=(Fy)'} = Fb \tag{7}
$$

We expect the left hand of (7) is equal to $Fy' + F'y = (Fy)'$. Once feasible, what was originally not integrable on the left side (i.e., $y' + ay$) becomes an integrable form $(Fy)'$.

By comparison, we find that as long as $aFy = F'y$, that is, as long as $F' = aF$, Our expectation becomes true. Thus, we need

$$
(\ln F)' = a \quad \Rightarrow \quad F = e^{\int a dt}
$$

Why Use the Integrating Factor?

- One find the function $e^{\int a(x)dx}$ is a natural and direct choice as an integrating factor and you do not need the above general procedure because the **derivative of** exp **function remains proportional to itself**: $\frac{d}{dt}e^{\int a\,dt} = ae^{\int a\,dt}$, which is a "fixed point" for differentiation, <a useful property!>. sin, cos also have similar properties with respect to $\frac{d^2}{dt^2}$, hence they are useful in integration by parts.
- This idea can be generalized to other general forms, such as $y' + \frac{a}{t}y$. Following the same approach.
- When partial differential equations (PDEs) involve terms like $\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial x} + au$ and $\frac{\partial u}{\partial t} + au$, this idea of integrating factor can also be applied. For example,

$$\partial_t(e^{\int a\,dt}u) = e^{\int a\,dt}\partial_t u + ae^{\int a\,dt}u$$

$$\partial_t(e^{\int a\,dt}\partial_t u) = e^{\int a\,dt}\partial_t^2 u + ae^{\int a\,dt}\partial_t u.$$

- One idea to solve equations: Transform equations (PDEs or ODEs) that are unfamiliar into ones that are familiar. How to transform? there are two basic ideas:
  1. Function transformation
  2. Independent variable transformation

  The idea of the **integrating factor** mentioned above can be **used for function transformation** in order to achieve a **familiar** equation.

Examples

Ex 3.1. *For the PDE:* $u_{tt} + 2u_t = u_{xx} + 4u_x + 3u$

$$u_{tt} + 2u_t = u_{xx} + 4u_x + 3u \Rightarrow u_{tt} + 2u_t + u = u_{xx} + 4u_x + 4u$$

we let $v(x,t) = e^{t+2x}u(x,t)$*, then* v *satisfies the familiar wave equation* $v_{tt} = v_{xx}$.

Ex 3.2. *For the PDE:* $u_t = u_{xx} + 2u_x$

$$u_t = u_{xx} + 2u_x \Rightarrow u_t + u = u_{xx} + 2u_x + u$$
$$\Rightarrow e^x(e^t u_t + e^t u) = e^t(e^x u_{xx} + 2e^x u_x + e^x u) \Rightarrow e^x(e^t u)_t = e^t(e^x u)_{xx} \Rightarrow (e^{t+x}u)_t = (e^{t+x}u)_{xx}$$

We let $v(x,t) = e^{t+x}u$*, the* v *satisfy the familiar heat equation* $v_t = v_{xx}$.