Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 7-1

1 Chapter 4 Green's Function Method

Why is PDE Courses Difficult?

- The lecture "Equations of Mathematical Physics (PDEs)" is difficult because:
  - Different sections introduce different problem-solving methods.
  - Each new method changes the perspective and approach to solving equations.
- Example of evolving techniques:
  - **Initial Simplicity in Chapter 2:** In Chapter 2, when learning the method of separation of variables, it seems manageable, and students feel they've grasped the main idea.
  - **Changing Approaches:** However, in Chapter 3, a new method is introduced—wave motion and averaging methods—requiring students to adjust their thinking.
  - **Continuous Shifts in Methodology:** After mastering one approach, such as the Green's function method, students find that previous methods no longer work, necessitating further adjustments.
  - **Adapting Learning Strategies:** The course continuously shifts its teaching methods, and students must constantly adapt their learning approach to keep up.

Structure of Chapter 4

- **Main Problem:** The chapter focuses on solving a boundary value problem represented by the equation

$$\begin{cases} \Delta u(x,y,z) = 0, & (x,y,z) \in \Omega \\ u\big|_{\Gamma} = f(x,y,z) \end{cases}$$

  which is the primary issue to be addressed in the first three sections.

- **First harmonic equations in §4.1:** In section 4.1, we focus only on the equation

$$\Delta u(x,y) = 0$$

  without considering boundary conditions, aiming to find a **general solution** in the **form of an integral**. This step is crucial as the integral form helps explain many aspects of the problem.

- **Adding Boundary Conditions §4.2:** Section 4.2 introduces boundary conditions and focus on

$$\begin{cases} \Delta u(x,y,z) = 0, & (x,y,z) \in \Omega \\ u\big|_{\Gamma} = f(x,y,z) \end{cases} \tag{1}$$

However, we do not solve the problem directly. Instead, we **find a transformation** that transform it into a simpler boundary value problem for a new variable v.

$$\begin{cases} \Delta v = 0, \quad (x, y, z) \in \Omega, \\ v\big|_{\Gamma} = \dfrac{1}{4\pi r_{MM_0}}\big|_{\Gamma} \end{cases} \tag{2}$$

- **Why the Problem (2) Seems Simpler:** The problem (2) appears simpler because the function f in (1) can take many forms, but we only need to solve one common problem in (2) if (1) and (2) share the same domain and boundary.
- **Key Approach:** The key to solving the problem is to find the solution for v (§4.3 on solving v), which is a transformed version of the original problem.
- **Role of Green's Function:** Once v is solved, we can convert it back to solve for u, effectively solving the original problem.
- **Solution Strategy (Section 4.2):** Section 4.2 **only** demonstrates and gives the **transformation** between u and v, but does **not solve** v.
- **Green's Function:** In section 4.2, v is identified as the Green's function, which helps solve the problem. The section also introduces a specific type of Green's function G, with another variation discussed later.

- **Solving for v §4.3:** In section 4.3, the goal is to **solve for** v, but this can **only** be done in **certain regions** (e.g., **half-space** or **spherical regions**). The solution uses the **method of images**, a physical approach, although the separation of variables method can also be applied.
- **Final Solution §4.3:** Once the Green's function v is solved, we can convert it back to find u and complete the solution to the original problem.
- **Trial methods in §4.4.**

Main Ideas

- **Interwoven Concepts:** Each section introduces additional related properties. For example, while studying Green's function, its properties must also be analyzed.
- **Harmonic Function:**
  - In Section 4.1, the function u is identified as a *harmonic function* and study the properties of harmonic functions.
  - A harmonic function satisfies $\Delta u = 0$ and must be C^2 to ensure the existence of second-order derivatives.
- **Understanding the Structure:** Without a clear grasp of the overall structure, students may struggle to distinguish between harmonic functions and Green's functions.

Why This Chapter Is Challenging

- The chapter focuses on solving the simplest partial differential equation: $\Delta u = 0$.
- **Paradox of Simplicity:** Despite its simplicity, it is difficult because:
  - The equation is well understood, leading to numerous useful properties.
- **Exam Expectations:**
  - Proofs are not tested directly.
  - However, students must understand the underlying ideas and apply them flexibly in problems $\leftarrow$ a defining feature of the study of PDEs.

In this chapter, we will introduce the key points and steps of using Green's function method to solve the boundary value problems of Laplace's equation. The solution of the first type boundary value problem of Laplace's equation will be expressed in the form of an integral through Green's function.

1.1 Green's Formula and Its Applications

Overview of the Approach

- The **goal** of Section 4.1 is to study the Laplace equation $\Delta u = 0$ and derive its general solution in itegral form.
- The process is divided into four steps:
  1. **Step 1: Understanding the Raw Material**
     - The first step provides fundamental Raw Material: **spherically and circularly symmetric** solutions.
     - This approach is similar to studying wave equations in three dimensions by first analyzing a spherical solution.
  2. **Step 2: Introducing a Mathematical Tool**
     - The second step presents an important mathematical tool: **the first and second Green's identities**←In fact, it is Gauss Formula.
     - Although it may seem unrelated at first, it is essential for the next step.
  3. **Step 3: Combining Raw Material and Mathematical Tool**
     - By embedding the raw material into the tool, the integral form of the general solution is obtained naturally.
  4. **Step 4: Studying Properties of the Solution**
     - Since u is a **harmonic function**, its **properties** are analyzed using the **integral representation**.
     - This step extends beyond the differential equation itself and focuses on the nature of harmonic functions.

Key Concepts

- **Green's Identity vs. Green's Function:**
  - The Green's identities used here should not be confused with Green's functions.
  - Green's identities are derived from Gauss's theorem, which relates surface integrals and volume integrals.
- **Main Thought Process:**
  1. Study fundamental solutions (raw material).
  2. Introduce Green's identities (mathematical tool).
  3. Combine both to derive the general solution.
  4. Analyze harmonic function properties based on this solution.

1.1.1 Spherically Symmetric Solutions (Raw Material)

Here, we first introduce the **circularly symmetric** solutions of the **two dimensional** Laplace's equation

$$u_{xx} + u_{yy} = 0$$

1. **Recognizing the Symmetry**: Since we are looking for a circularly symmetric solution, it is natural to consider a polar coordinate transformation.
2. **Coordinate Transformation**: Convert Cartesian coordinates to polar coordinates to exploit the symmetry.
3. **Laplace Operator in Polar Coordinates**: The Laplacian in two-dimensional polar coordinates is well known:
$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$
For a circularly symmetric function $u(r)$, the **dependence on θ vanishes**, simplifying to:
$$\Delta u = \frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0.$$
4. **Solving the ODE**: The above equation can be solved by standard techniques for ordinary differential equations.

The two dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ in polar coordinates is expressed as
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{3}$$
We seek the circularly symmetric solution $u = U(r)$ (i.e., the solution where $u = U(r)$ does not depend on θ) of equation (3). At this time, the above equation can be simplified to
$$\frac{d^2 U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = 0$$

- **Important note:** This equation is **NOT a Euler** equation for which we have a general solution formula, although it may look like it, but it does not meet the strict conditions for applying the standard Euler formula.
- **Recall:** The second order Euler equation
$$r^2 R_{rr} + rR_r - n^2 R = 0, \quad (n = 1, 2, \cdots)$$
- For the Euler equation, make the transformation $r = e^t \Leftrightarrow t = \ln r$ $(n \neq 0)$, then
$$R_{tt} - R_t + R_t - n^2 R = 0 \Leftrightarrow R_{tt} - n^2 R = 0$$
$$\Leftrightarrow R_n = C_n e^{nt} + D_n e^{-nt} \Leftrightarrow R_n(r) = C_n r^n + D_n r^{-n}, \quad (n = 1, 2, \cdots)$$
- If $n = 0$, we need use the different general solution for one real root.
- $\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = 0$ is an ODE, which can be solved. (There are many solution methods, such as the **method of separation of variables**). For example, let $F = \frac{dU}{dr}$, then $\frac{d\ln F}{dr} = -\frac{1}{r}$, integrating gives $\ln F = -\ln r + C$, so $F = \frac{C}{r} \Rightarrow \frac{dU}{dr} = \frac{C}{r} \Rightarrow U = C_1 \ln r + C_2$ Or transform it into $r^2 \frac{d^2U}{dr^2} + r\frac{dU}{dr} = 0$ (This is not an Euler equation, but it can be solved by $r = e^t$).

Its solution is
$$U = c_1 \ln r + c_2 \quad (r \neq 0)$$
where c_1 and c_2 are **arbitrary constants** (since there is no boundary condition). If we let $c_1 = -1$ and $c_2 = 0$ (choose **arbitrarily**), we can obtain
$$U_0 = \ln\frac{1}{r} \quad (r \neq 0)$$

It is usually called the **fundamental solution** of the **two dimensional** Laplace's equation.

Now we introduce the **spherically symmetric** solutions of the three - dimensional Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Make the spherical coordinate transformation

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases} \quad \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases}$$

By the chain rule of composite functions

$$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x + u_\varphi \cdot \varphi_x$$

$$u_{xx} = u_{rr}r_x^2 + u_{\theta\theta}\theta_x^2 + u_{\varphi\varphi}\varphi_x^2 + u_r \cdot r_{xx} + u_\theta \cdot \theta_{xx} + u_\varphi \cdot \varphi_{xx} + 2u_{r\theta}\theta_x r_x + 2u_{r\varphi}r_x\varphi_x + 2u_{\varphi\theta}\varphi_x\theta_x$$

We can transform the three dimensional Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ into the following form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{4}$$

We seek the spherically symmetric solution $u = U(r)$ (i.e., the solution where $u = U(r)$ does **not depend on** θ **and** φ) of equation (4). At this time, the above equation (4) can be simplified to

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 0 \quad \Rightarrow \quad \frac{dU}{dr} = \frac{c_1}{r^2}. \leftarrow \boxed{\text{Gravitational force}}$$

Its solution is

$$U = \frac{c_1}{r} + c_2 \quad (r \neq 0)$$

where c_1 and c_2 are arbitrary constants. If we let $c_1 = 1$ and $c_2 = 0$, we can obtain

$$U_0 = \frac{1}{r} \quad (r \neq 0)$$

It is usually called the **fundamental solution** of the three dimensional Laplace's equation.

Physical Meaning of Laplace's Equation

- The function u in Laplace's equation can represent a **gravitational potential** or an **electrostatic potential**.
- The **gradient** of u with respect to r corresponds to **field strength** (e.g., gravitational field or electric field).
- The force is given by the **negative gradient**:

$$\mathbf{F} = -\nabla u$$

- The solution follows an **inverse-square law**, consistent with **Newton's law of gravitation** and **Coulomb's law**.

Determining the Potential Function

- The potential function u is obtained by integrating the field strength.
- A general solution involves integration constants c_1 and c_2.
- The choice of these constants depends on **selecting a reference point** where **potential is zero**.
- Typically, the **potential is set to zero at infinity**:

$$u(\infty) = 0.$$

- This selection of zero potential is a form of **gauge choice**.

Is the Derived Solution a True Solution?

- The obtained solution satisfies Laplace's equation except at $r = 0$.
- At the origin $r = 0$, the solution diverges to infinity.
- Since Laplace's equation is originally defined in Cartesian coordinates, it should hold everywhere in the domain, including the origin.
- However, the transformation to spherical coordinates can lead to a loss of information at the origin.

Addressing the Singularity at the Origin

- The classical solution should not have singularities.
- To ensure well-posedness, we define the solution in the punctured domain:

$$\Omega' = \Omega \setminus \{M_0\},$$

  where M_0 is the origin.
- In this restricted domain, the solution satisfies Laplace's equation (or the Laplace's equation in polar coordinate).

Extending the Solution to Include the Origin

- To formally include M_0, the equation must be modified.
- The right-hand side of the equation should account for the singularity at M_0.
- A gravitational potential of this form suggests the presence of a **point mass**.
- Recall Poisson's equation in three dimensions:

$$\Delta u = \rho,$$

  where ρ represents the mass (or charge) density.
- A **point mass** located at M_0 corresponds to a singular mass distribution:

$$\rho(\mathbf{x}) = m\delta(\mathbf{x} - M_0),$$

  where $\delta(\mathbf{x} - M_0)$ is the Dirac delta function.
- This can be done by introducing the Dirac delta function $\delta(\mathbf{x} - M_0)$:

$$\Delta u = \kappa\delta(\mathbf{x} - M_0).$$

- This equation defines the **fundamental solution** of Laplace's equation.

Claim 1.1. *Suppose $u = \frac{1}{r}$ solve the Poission equation*

$$\Delta u = \kappa\delta(\mathbf{r} - \mathbf{r}_M)$$

then

$$\Delta u = -4\pi\delta$$

which implies $\kappa = -4\pi$.

Proof. Since κ is a constant, we can calculate it by taking integration on any region, for simplicity, we take a ball B_a,

$$\int_{B_a} \Delta u \, dV = \kappa \int_{B_a} \delta(\mathbf{r} - \mathbf{r}_M) \, dV = \kappa \leftarrow \boxed{\delta \text{ require integration to become meaningful.}}$$

Using the divergence theorem:

$$\int_{\partial B_a} \mathbf{n} \cdot \nabla u \, d\sigma = \int_{\partial B_a} \frac{\partial u}{\partial r} \, d\sigma = \kappa.$$

For $u = \frac{C}{r}$, we compute:

$$\frac{\partial u}{\partial r} = -\frac{C}{r^2}.$$

Thus, integrating over the sphere:

$$\int_{\partial B_a} \left(-\frac{C}{a^2} \right) d\sigma = -\frac{C}{a^2} \cdot 4\pi a^2 = -4\pi C.$$

Thus, we conclude: $\kappa = -4\pi C$. □

Summary

Fundamental Solution of the **Two Dimensional** Laplace Equation:

$$U_0 = \ln \frac{1}{r} \quad (r \neq 0)$$

Fundamental Solution of the **Three Dimensional** Laplace Equation:

$$U_0 = \frac{1}{r} \quad (r \neq 0)$$

Especially in the study of the three-dimensional Laplace equation, the **fundamental solution** plays an extremely important role.

It is easy to verify that when $r \neq 0$, the functions $\frac{1}{r}$ and $\ln \frac{1}{r}$ respectively satisfy the three-dimensional and two-dimensional Laplace equations.

In the rest of this Chapter, we focus on the problems in 3D case.

- **Parallel Extension to Two Dimensions (2D)**:
  - Every 3D derivation can be directly extended to 2D.
  - In 2D, replace 3D fundamental solutions with 2D fundamental solutions.
  - Corresponding modifications should be made accordingly.

1.2 Green's Formulas (Mathematical Tool)

Green's formulas are a direct **corollary** of the **Ostrogradsky-Gauss** formula.

Let Ω be a bounded region with a sufficiently smooth surface $\partial\Omega$ (also dennoted as $\Gamma := \partial\Omega$) as its boundary, and $u = u(x, y, z)$ (works for any n-dimensional as well) be an arbitrary function that is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ and has continuous partial derivatives in Ω. Then the following **Ostrogradsky-Gauss** formula holds

$$\int_{\Omega} \nabla \cdot \mathbf{F} dV = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{F} dS \tag{5}$$

where $F = (P, Q, R)$ is a vector, dV is the volume element, $\mathbf{n}$ is the **outer normal direction** of $\partial\Omega$, and dS is the surface element on $\partial\Omega$.

Let the functions u and v and all their first order partial derivatives be continuous on $\overline{\Omega}$, and have continuous second order partial derivatives in Ω. In **Gauss formula** (5), let $F = u\nabla v$, then we get:

$$\underbrace{\int_\Omega \nabla\cdot(u\nabla v)dV}_{\Downarrow\text{Leibniz}} \overset{\text{Gauss}}{=} \underbrace{\int_{\partial\Omega} u\nabla v\cdot\mathbf{n}dS}_{\Downarrow\text{Direction derivative}}$$

$$\int_\Omega u\Delta v dV + \int_\Omega \nabla u\cdot\nabla v dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dS$$

Then we get **Green's first formula**:

$$\int_\Omega u\Delta v dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n}dS - \int_\Omega \nabla u\cdot\nabla v dV \tag{6}$$

In formula (6), exchange the positions of the functions u and v, we get

$$\int_\Omega v\Delta u dV = \int_{\partial\Omega} v\frac{\partial u}{\partial n}dS - \int_\Omega \nabla v\cdot\nabla u dV \tag{7}$$

Subtract (7) from (6), then we get **Green's second formula**:

$$\int_\Omega (u\Delta v - v\Delta u)dV = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \tag{8}$$

Formula (8) holds for any functions u and v that have **first order continuous partial derivatives** on $\overline{\Omega}$ and **second order continuous partial derivatives** on Ω.

- **Generalization:** u and v can be Dirac δ finction.
- In application, if one need $\int_\Omega u\Delta v\, dV$, it can be calculated directly by

$$\begin{aligned}\int_\Omega u\Delta v\, dV &\overset{\text{Leibniz}}{=} -\int_\Omega \nabla\cdot(u\nabla v)\, dV - \int_\Omega \nabla u\cdot\nabla v\, dV\\ &\overset{\text{Gauss}}{=} \underbrace{\int_{\partial\Omega} u\nabla v\cdot\mathbf{n}\, dS}_{\int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS} - \int_\Omega \nabla u\cdot\nabla v\, dV.\end{aligned}$$

There is no need to memorize the formula.

1.3 Integral Expressions of Harmonic Functions

We use Green's formula to derive the integral expressions of harmonic functions.

First, note that the function

$$\frac{1}{r_{MM_0}} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}},$$

where $M_0(x_0, y_0, z_0)$ is a fixed point in the region Ω. Except at M_0, this function satisfies Laplace's equation everywhere. If the function u has first order continuous partial derivatives on $\Omega + \Gamma$ and is harmonic in Ω, then

$$u(M_0) = -\frac{1}{4\pi}\iint_\Gamma \left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) - \frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right]dS. \tag{9}$$

To derive the integral representation of harmonic functions, we utilize the **fundamental solution** obtained in the first part and apply **Green's second identity**. This section builds on the previous discussions, demonstrating how the solution structure naturally leads to the integral formula.

Choice of Functions

- Since u is **harmonic**, it satisfies $\Delta u = 0$ and belongs to $C^2(\Omega)$.
- We set v as the **fundamental solution**, given by:

$$v = \frac{1}{r_{MM_0}},$$

  where r_{MM_0} represents the Euclidean distance between a point M and a fixed point M_0.

Challenges and Resolution

While substituting into Green's second identity, we face an issue:

- v is singular at M_0, meaning $v \notin C^2(\Omega)$ and not even $C^1(\Omega)$.
- Direct substitution of v into Green's identity is not valid due to the singularity at M_0.

Resolution:

To address this, we adopt a standard idea from complex analysis:

1. **Remove the singularity** by excluding a **small ball** around M_0.
2. Consider a **modified domain** $\Omega \setminus B_\epsilon(M_0)$.
3. Apply Green's identity in the **punctured domain** and then take the **limit** as $\epsilon \to 0$.

This process allows for a well-defined integral representation of harmonic functions.

Proof. In formula (8), let u be a harmonic function and take $v = \frac{1}{r}$. Since the function v becomes infinite at the point M_0, Green's second formula (8) cannot be directly applied to the region Ω. However, if we remove a sphere $K_\varepsilon^{M_0}$ centered at M_0 with a sufficiently small positive radius ε from the region Ω, then the function v is continuously differentiable in the remaining region $\Omega - K_\varepsilon^{M_0}$ (as shown in Fig. 1).

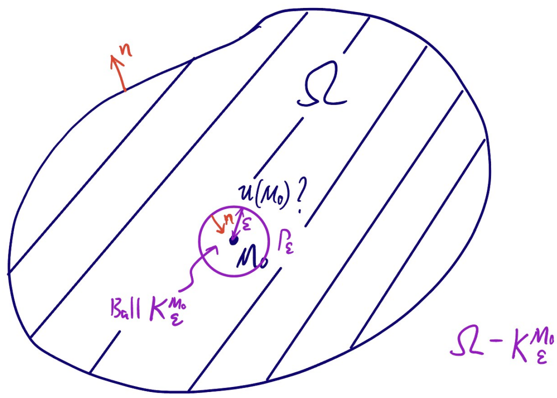

Figure 1: Domain of $\Omega - K_\varepsilon^{M_0}$

Apply formula (8) to the above mentioned **harmonic function** u and $v = \frac{1}{r}$ in the region $\Omega - K_\varepsilon^{M_0}$, we get

$$\cancel{\iiint\limits_{\Omega - K_\varepsilon^{M_0}} \left(u\Delta\frac{1}{r} - \frac{1}{r}\Delta u \right) d\Omega} = \iint\limits_{\Gamma + \Gamma_\varepsilon} \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) dS, \tag{10}$$

where Γ_ε is the spherical surface of the sphere $K_\varepsilon^{M_0}$.

Because in the region $\Omega - K_\varepsilon^{M_0}$, $\Delta u = 0$ and $\Delta\frac{1}{r} = 0$ (due to the **fundamental solution** $\Delta(\frac{1}{r}) = \delta(r - M_0)$), then formula (10) is transformed into

$$\iint_\Gamma \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) dS + \iint_{\Gamma_\varepsilon} \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) dS = 0. \tag{11}$$

On the spherical surface Γ_ε,

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{\varepsilon^2},$$

- The outward normal direction of $\Omega - K_\varepsilon^{M_0}$ points towards the center M_0 of the ball, which is opposite to the direction of the radial direction r. Thus $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$.

from which we can obtain

$$\iint_{\Gamma_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = \frac{1}{\varepsilon^2}\iint_{\Gamma_\varepsilon} udS = \frac{1}{\varepsilon^2}\cdot 4\pi\varepsilon^2\overline{u} = 4\pi\overline{u},$$

where $\overline{u}$ is the average value of the function u on the spherical surface Γ_ε.

On the other hand, since on the spherical surface Γ_ε

- Note $\mathbf{n}$ is the outward normal for $\Omega \setminus K_\varepsilon^{M_0}$, thus $\mathbf{n}$ is pointed to M_0 which is the inward normal for the ball $K_\varepsilon^{M_0}$;
- But for the Gauss formula, the outward normal should be $-\mathbf{n}$.

$$\iint_{\Gamma_\varepsilon} \frac{1}{r}\frac{\partial u}{\partial n}dS = \frac{1}{\varepsilon}\iint_{\Gamma_\varepsilon} \frac{\partial u}{\partial n}dS = \frac{1}{\varepsilon}\iint_{\Gamma_\varepsilon} \mathbf{n}\cdot\nabla u dS \overset{\text{Gauss}}{=} -\frac{1}{\varepsilon}\iint_{K_\varepsilon^{M_0}} \Delta u d\Omega = 0.$$

Then, substituting

$$\iint_{\Gamma_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = 4\pi\overline{u}, \quad \iint_{\Gamma_\varepsilon} \frac{1}{r}\frac{\partial u}{\partial n}dS = 0$$

into formula (11), we can get

$$\iint_\Gamma \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n} \right) dS + 4\pi\overline{u} = 0.$$

Now, let $\varepsilon \to 0$. Since $\lim_{\varepsilon\to 0}\overline{u} = u(M_0)$, from the above formula, we can obtain the integral expression (9) of the harmonic function u. □

This integral expression (9) shows that: for a harmonic function u with continuous first order partial derivatives on $\Omega+\Gamma$, its value at any point M_0 in the region Ω can be expressed by the integral expression (9) using the **values of this harmonic function** and its **normal derivative on the boundary** Γ of the region.

Key Insights from the Derivation

- The derivation of the integral representation involves a crucial technique: the **"removal of a small ball" around a singularity**.
- Understanding the idea of this method is essential since it appears frequently in problem-solving.

Interpretation of the Formula

- The formula is **slightly analogous to the Possion formula** in the "3D wave equation".
- To determine u at a point M_0, one must use:
  1. The values of u on the **boundary** Γ.
  2. The **normal derivative** $\frac{\partial u}{\partial n}$ on the **boundary**.
- The integral representation (9)

$$u(M_0) = -\frac{1}{4\pi} \iint\limits_{\Gamma} \left[u(M) \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) - \frac{1}{r_{MM_0}} \frac{\partial u(M)}{\partial n} \right] dS.$$

  shows that the function's **value** at any **interior point** "depends **only** on **boundary data**".

$$\boxed{\text{boundary data on } \Gamma \text{ surface}} \xrightarrow{\text{completely determined}} \boxed{\text{all interior information}}$$

Physical and Mathematical Implications

- The formula states that to determine the value of u at any point inside Ω, knowledge of boundary conditions is sufficient.
- This highlights a fundamental property of "harmonic functions": their internal values are entirely determined by boundary values.

When the point M_0 is taken outside the region Ω or on its boundary Γ, two other formulas can also be derived in the same way:

$$-\iint\limits_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS$$

$$= \begin{cases} 0, & \text{if } M_0 \text{ is outside } \Omega, \leftarrow \boxed{\text{No need for a ball of singularity, Green 2nd directly implies}} \\ 2\pi u(M_0), & \text{if } M_0 \text{ is on } \Gamma, \leftarrow \boxed{\text{Need for a half ball of singularity, Green 2nd directly implies}} \\ 4\pi u(M_0), & \text{if } M_0 \text{ is inside } \Omega. \end{cases}$$

Method 2: δ function method

Only two tools required:

- **Fundamental solution in 3D:** $\Delta \frac{1}{r} = -4\pi\delta(r - r_0)$ $(r = r_{MM_0}, r_0 = r_{M_0O})$
- **Green's second formula:**

$$\int_{\Omega} u\Delta v - v\Delta u \, dV = \int_{\partial\Omega} u\partial_n v - v\partial_n u \, dS$$

Proof. Let $v = \frac{1}{r}$ and u is **harmonic**, i.e., $\Delta u = 0$. Inserting these into the Green's second formula,

$$\Rightarrow \int_{\Omega}\Big(u \underbrace{\Delta \frac{1}{r}}_{=-4\pi\delta(r-r_0)} - \cancel{\frac{1}{r}\Delta u}\Big)\, dV = \int_{\partial\Omega} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\, dS.$$

$$\Rightarrow -4\pi u(r_0) = \int_{\partial\Omega} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u}{\partial n}\, dS.$$

Then we obtain

$$u(M_0) = -\frac{1}{4\pi}\iint\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) - \frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right] dS,$$

and complete the proof. □

If u is not a harmonic function, as long as it has first order continuous partial derivatives on $\Omega + \Gamma$ and $\Delta u = F$ in the region Ω, a formula similar to (9) can also be obtained

$$u(M_0) = -\frac{1}{4\pi}\iint_{\Gamma}\left[u(M)\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) - \frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right] dS - \frac{1}{4\pi}\iiint_{\Omega}\frac{F}{r_{MM_0}}\, d\Omega.$$

$\uparrow$ $\boxed{\text{It is easy to use the } \delta \text{ method to prove by replacing } \Delta u = F.}$

Supplementary 2: Integral expression of harmonic functions in two dimensional cases

First, consider the function

$$\ln\frac{1}{r_{MM_0}} = \ln\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

where $M_0(x_0, y_0)$ is a fixed point in the region D. Except at M_0, this function satisfies Laplace's equation everywhere. If the function u has continuous first order partial derivatives on $D + C$ and is **harmonic** in D, then

$$u(M_0) = -\underbrace{\frac{1}{2\pi}}_{\text{2D circumference is } 2\pi r}\int_C\left[u(M)\frac{\partial}{\partial n}\Big(\underbrace{\ln\frac{1}{r_{MM_0}}}_{\text{2D fund. solution}}\Big) - \ln\frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right] dS \tag{12}$$

Proof. In (8), let u be a harmonic function and take $v = \ln\frac{1}{r}$. Since the function v becomes infinite at the point M_0, Green's formula (8) cannot be directly applied to the region D. However, if a circle $K_\varepsilon^{M_0}$ with M_0 as the center and a sufficiently small positive number ε as the radius is removed from the region D, then the function v is continuously differentiable in the remaining region $D - K_\varepsilon^{M_0}$.

Apply formula (8) to the above mentioned harmonic function u and $v = \ln\frac{1}{r}$ in the region $D - K_\varepsilon^{M_0}$, we get

$$0 = \int_{C+C_\varepsilon}\left(u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) - \ln\frac{1}{r}\frac{\partial u}{\partial n}\right) dS$$

where C_ε is the circumference of the circle $K_\varepsilon^{M_0}$.

$$\int_C\left(u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) - \ln\frac{1}{r}\frac{\partial u}{\partial n}\right) dS + \int_{C_\varepsilon}\left(u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) - \ln\frac{1}{r}\frac{\partial u}{\partial n}\right) dS = 0. \tag{13}$$

On the circumference C_ε,

$$\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) = -\frac{\partial}{\partial r}\left(\ln\frac{1}{r}\right) = \frac{1}{r} = \frac{1}{\varepsilon}.$$

From this, we can obtain

$$\int_{C_\varepsilon} u\frac{\partial}{\partial n}\left(\ln\frac{1}{r}\right) dS = \frac{1}{\varepsilon}\int_{C_\varepsilon} u dS = \frac{1}{\varepsilon}\cdot 2\pi\varepsilon\overline{u} = 2\pi\overline{u}$$

where $\overline{u}$ is the average value of the function u on the circumference C_ε.

On the other hand, since on the circumference C_ε

$$\int_{C_\varepsilon} \ln \frac{1}{r} \frac{\partial u}{\partial n} dS = \ln \frac{1}{\varepsilon} \int_{C_\varepsilon} \frac{\partial u}{\partial n} dS = -\ln \frac{1}{\varepsilon} \int_{C_\varepsilon} \frac{\partial u}{\partial r} dS = -\ln \frac{1}{\varepsilon} \iint\limits_{K_\varepsilon^{M_0}} \Delta u d\sigma = 0$$

Then, substituting $\int_{C_\varepsilon} u \frac{\partial}{\partial n} \left(\ln \frac{1}{r}\right) dS = 2\pi \overline{u}$ and $\int_{C_\varepsilon} \ln \frac{1}{r} \frac{\partial u}{\partial n} dS = 0$ into formula (13), we can get

$$\int_C \left(u \frac{\partial}{\partial n} \left(\ln \frac{1}{r} \right) - \ln \frac{1}{r} \frac{\partial u}{\partial n} \right) dS + 2\pi \overline{u} = 0.$$

Now, let $\varepsilon \to 0$. Since $\lim_{\varepsilon \to 0} \overline{u} = u(M_0)$, from the above formula, we can obtain the integral expression (12) of the harmonic function u in two dimensional cases. $\square$