Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 5-1

Structure of Chapter 3

- **Section 1:** Solution of the (1+1)D wave equation using the **traveling wave method**.
- **Section 2:** Solutions for (1+2)D and (1+3)D wave equations:

$$\left.\begin{cases}\textbf{1D: Traveling wave method.}\\ \textbf{2D: Dimensional reduction method.}\\ \textbf{3D: Spherical averaging method.}\end{cases}\right\}\text{Wave methods}$$

- These methods apply **only to wave equations** with **initial data**, unlike previous methods that work for various types of equations. i.e.,

$$\boxed{\text{wave equations + initial data (no boundary)} \to \text{wave methods}}$$

- **Section 3:** Introduces the **integral transform method**, which is independent of the previous sections.

Boundary Conditions and Initial Value Problems

- The problems addressed in Sections 1 and 2 are **initial value problems** (Cauchy problems).
- There are **no boundary conditions** since the spatial domain extends to infinity.
- This assumption is valid when boundaries are far from the region of interest.

Chapter 3: Method of Traveling Waves and Integral Transform Method

In this chapter, we will introduce two other methods for solving boundary value problems: one is the method of traveling waves (also known as d'Alembert's method), and the other is the integral transform method.

The method of traveling waves can only be used to solve the problems of wave equations within **unbounded regions**. Although it has significant limitations, it has special advantages for **wave problems**, making it one of the fundamental methods for solving differential equations.

The **integral transform method** is **not limited** by the type of equation and is mainly used for unbounded regions, but it can also be applied to bounded regions.

3.1 D'Alembert's Formula: Wave Propagation

3.1.1 D'Alembert's Solution for the String Vibration Equation

If the length of the string we are examining is very long, and what we need to know is only the vibration situation within a relatively short time and far from the boundary, then the influence of the boundary conditions can be ignored. We might as well consider the length of the string we are examining as infinite, and what we need to know is only the vibration situation within a finite range.

At this point, the boundary value problem is summarized in the following form:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) & (-\infty < x < +\infty, t > 0), \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x) \end{cases} \tag{1}$$

Solution Strategy:

- **Isolate non-homogeneous** terms and solve individually.
- Use **Homogenization** for non-homogeneous parts.

Isolate non-homogeneous terms: For the above initial value problem, since the differential equation and the inital conditions are both **linear**, the **superposition principle** also holds. That is, if $u_1(x,t)$ and $u_2(x,t)$ are respectively the solutions to the following initial value problems:

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < +\infty, t > 0), \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x) \end{cases} \tag{2}$$

and

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) & (-\infty < x < +\infty, t > 0), \\ u(x,0) = 0, & u_t(x,0) = 0 \end{cases} \tag{3}$$

then $u = u_1(x,t) + u_2(x,t)$ is the solution to the original problem (1).

This indicates that the comprehensive effect of the external factors represented by $f(x,t)$ and the initial vibration state represented by $\varphi(x), \psi(x)$ on the entire vibration process can be decomposed into the superposition of the effects produced by considering **only the external factors** and **only the initial vibration state** on the vibration process.

Consider the wave equation and initial conditions (2), that is,

$$u_{tt} = a^2 u_{xx} (-\infty < x < +\infty, t > 0), \tag{4}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \tag{5}$$

Recall: Two Basic Approaches to Solving Equations

- **Trial Method (Guessing)**: Chapter 2;
- **Inverse Method**: Chapter 3 and 4.

Core Idea

- A differential equation can be solved by finding its **inverse operation**, which is **integration**.
- Direct integration of the wave equation is difficult due to the presence of two independent variables $u_{tt} - a^2 u_{xx}$.
- The challenge: **which variable to integrate first** (time t or space x)? Can not get rid of second order derivatives in each variable.

Solution Strategy

- To simplify integration, a **variable transformation** is required. Since the principle

$$\text{unfamiliar eqs.} \quad \underset{\text{(2) variable transf}}{\overset{\text{(1) unkown transf.}}{\Longrightarrow}} \quad \text{familiar eqs.}$$

- The goal is to convert the equation into a form that allows **separable integration**, i.e., hope transform it to $u_{\xi\eta} = 0 \Rightarrow u_\xi = f_0(\xi) \Rightarrow u = f(\xi) + g(\eta)$.
- If successful, the equation can be integrated twice to obtain the solution, i.e., $u(\xi, \eta) = f(\xi) + g(\eta)$,.
- The presence of two arbitrary functions corresponds to the **second-order nature** of the wave equation.
- The question becomes:
  - How to transform $\boxed{u_{tt} = a^2 u_{xx} \to u_{\xi\eta} = 0}$?
- **Variable Transformation**:
  - Use coordinate transformations (e.g., $\xi = x - ct$, $\eta = x + ct$) to transform the wave equation (Why?).
  - **Method 1: Matrix Method**

$$(\partial_t, \partial_x)\begin{pmatrix} 1 & 0 \\ 0 & -a^2 \end{pmatrix}\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} = \underbrace{(\partial_t, \partial_x)\, Q^{-1}}_{=(\partial_\xi, \partial_\eta)} \underbrace{Q \begin{pmatrix} 1 & 0 \\ 0 & -a^2 \end{pmatrix} Q^T}_{\text{Hope}=\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}} \underbrace{(Q^T)^{-1}\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}}_{=:\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}}$$

By the knowledge of eigenvalues and eigenvectors, we conclude $\mu = -2a^2$, and

$$Q = \begin{pmatrix} a & 1 \\ -a & 1 \end{pmatrix}.$$

The inverse of Q can be computed as:

$$Q^{-1} = \begin{pmatrix} \frac{1}{2a} & \frac{1}{2} \\ -\frac{1}{2a} & \frac{1}{2} \end{pmatrix}.$$

Conclusion

- The key to solving the wave equation lies in finding an appropriate transformation.
- Once transformed, the equation can be integrated sequentially to determine the solution.

General solutions of waves: First, we examine the problem through a **variable transformation** (see Fig. 1, ξ =Constant and η =Constant, the coordinate lines):

$$\boxed{\xi = x - at, \quad \eta = x + at}, \quad \leftarrow \quad \boxed{\text{How to find it?}} \tag{6}$$

with the inverse transformation:

$$x = \frac{\xi + \eta}{2}, \quad t = \frac{\eta - \xi}{2a}.$$

Let $\bar{u} = \bar{u}(\xi, \eta)$ be the new unknown function, then:

$$u(x, t) = \bar{u}\left(\frac{\xi + \eta}{2}, \frac{\eta - \xi}{2a}\right) = \bar{u}(\xi, \eta).$$

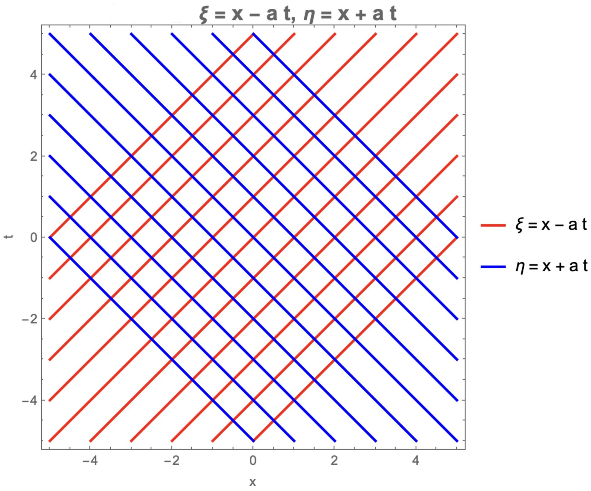

Figure 1: Coordinate $\xi = x - at$, $\eta = x + at$ for $a = 1$

Using the **chain rule** for composite functions, we obtain:

$$u_x = \bar{u}_\xi \cdot \xi_x + \bar{u}_\eta \cdot \eta_x = \bar{u}_\xi + \bar{u}_\eta,$$

$$u_{xx} = (\bar{u}_{\xi\xi} \cdot \xi_x + \bar{u}_{\xi\eta} \cdot \eta_x) + (\bar{u}_{\eta\xi} \cdot \xi_x + \bar{u}_{\eta\eta} \cdot \eta_x) = \bar{u}_{\xi\xi} + 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta}, \tag{7}$$

and similarly,

$$u_{tt} = a^2(\bar{u}_{\xi\xi} - 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta}). \tag{8}$$

Substituting (7) and (8) into equation (4) simplifies to:

$$\boxed{\bar{u}_{\xi\eta} = 0.} \leftarrow \boxed{\text{This is our expecting form!}} \tag{9}$$

Equation (9) can be solved directly by integration. Integrate once with respect to η, then integrate once with respect to ξ, to obtain the general solution of equation (9):

$$\bar{u}(\xi, \eta) = f(\xi) + g(\eta), \tag{10}$$

where f and g are arbitrary functions with continuous second derivatives.

Substituting the variable transformation (6) into (10) gives:

$$u(x,t) = f\,(x - at) + g\,(x + at)\,. \leftarrow \boxed{\text{general solutions, traveling wave solutions}} \tag{11}$$

- The main technique used is **variable substitution**.
- After substitution, differentiation is performed using the **chain rule**.

Why we select the variable transformation (6) and how to find it:

There are many methods:

Method 2: Chain rules The idea is to transform the wave equation $u_{tt} = a^2 u_{xx}$ using a change of variables. If there is **only one second-order term** instead of two second-order terms, direct integration is possible. Target: one second-order term.

$\partial_t^2 - a^2\partial_x^2 = (\partial_t + a\partial_x)(\partial_t - a\partial_x)$ since (analog to) $A^2 - B^2 = (A+B)(A-B)$, $(A = \partial_t, B + a\partial_x)$

Then

$$(\partial_t^2 - a^2\partial_x^2)u = \underbrace{(\partial_t + a\partial_x)}_{\text{Hope } \sim\partial_\eta}\underbrace{(\partial_t - a\partial_x)}_{\text{Hope } \sim\partial_\xi}u$$

This implies

$$\underbrace{\partial_\xi u}_{\overset{\text{chain rule}}{=}\frac{\partial t}{\partial \xi}\partial_t u + \frac{\partial x}{\partial \xi}\partial_x u} = l\partial_t u - al\partial_x u \quad \text{and} \quad \underbrace{\partial_\eta v}_{\overset{\text{chain rule}}{=}\frac{\partial t}{\partial \eta}\partial_t v + \frac{\partial x}{\partial \eta}\partial_x v} = k\partial_t v - ak\partial_x v$$

Then

$$\begin{cases} \frac{\partial t}{\partial \xi} = l & \Leftrightarrow t = l\xi + C_1(\eta) \\ \frac{\partial x}{\partial \xi} = -al & \Leftrightarrow x = -al\xi + C_3(\eta) \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial t}{\partial \eta} = k & \Leftrightarrow t = k\eta + C_2(\xi) \\ \frac{\partial x}{\partial \eta} = ak & \Leftrightarrow x = ak\eta + C_4(\xi) \end{cases}$$

Then

$$\begin{aligned} t = l\xi + C_1(\eta) = k\eta + C_2(\xi) \quad &\Rightarrow \quad t = l\xi + k\eta \\ x = -al\xi + C_3(\eta) = ak\eta + C_4(\xi) \quad &\Rightarrow \quad x = -al\xi + ak\eta \end{aligned}$$

If $l = k = 1$,

$$\begin{cases} x = -a\xi + a\eta \\ t = \xi + \eta \end{cases} \Rightarrow \begin{cases} \eta = \frac{1}{2}\left(t + \frac{x}{a}\right) \\ \xi = \frac{1}{2}\left(t - \frac{x}{a}\right) \end{cases}$$

If $k = \frac{1}{2a}, l = -\frac{1}{2a}$,

$$\begin{cases} x = \frac{1}{2}\xi + \frac{1}{2}\eta \\ t = -\frac{1}{2a}\xi + \frac{1}{2a}\eta \end{cases} \Rightarrow \begin{cases} \eta = x + at \\ \xi = x - at \end{cases}$$

These choices all work.

In the characteristic plane, the equation is an internal differential operator, which only restricts u and does not provide all derivative information.

Initial values determine f and g: Using the initial conditions (5) to determine the arbitrary functions f and g in the general solution (11), we substitute (5) into (11) to get:

$$f(x) + g(x) = \varphi(x), \tag{12}$$

$$-af'(x) + ag'(x) = \psi(x). \tag{13}$$

There to method to determine f and g:

- Integrating (13) (use this in the followings);
- Differentiating (12) (idea used in the next section).

Integrating (13) gives:

$$a(-f(x) + g(x)) + c = \int_{x_0}^{x} \psi(\alpha)d\alpha, \tag{14}$$

where x_0 is any point and c is the integration constant.

From (12) and (14), we derive:

$$\begin{cases} f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_{x_0}^{x}\psi(\alpha)d\alpha + \frac{c}{2a}, \\ g(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x}\psi(\alpha)d\alpha - \frac{c}{2a}. \end{cases} \tag{15}$$

Substituting (15) into the general solution (11) gives the solution to the initial value problem (4) and (5):

$$\boxed{u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha.} \tag{16}$$

This formula is known as **D'Alembert's formula** for the **free vibration** of an **infinitely long string**, or simply **D'Alembert's solution**. This method of solving is called **D'Alembert's method**.

Ex 0.1. *Use the method of characteristics to solve the following initial-boundary value problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, x - at < 0, x > 0 \\ u\big|_{x-at=0} = \varphi(x), & u\big|_{x=0} = h(t). \end{cases}$$

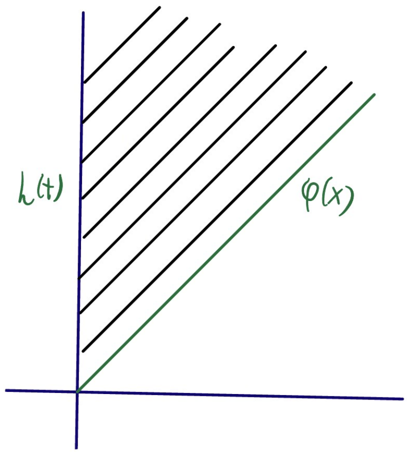

Figure 2: Goursat Problem

Solution. ***The general solution:*** *The general solution of the wave equation is:*

$$u(x,t) = f(x-at) + g(x+at). \leftarrow \boxed{\textit{Memorize the general solution of wave}}$$

where f, g are arbitrary functions with continuous second derivatives.

*Next, we use **boundary conditions** to determine the arbitrary functions f, g. First, from the condition:*

$$u\big|_{x-at=0} = \varphi(x) \implies f(0) + g(2x) = \varphi(x).$$

Let $\eta = 2x$,

$$\implies g(\eta) = \varphi\left(\frac{\eta}{2}\right) - f(0). \leftarrow \boxed{\textit{Know how to determine } f \textit{ and } g \textit{ by conditions}}$$

Using the condition:

$$u\big|_{x=0} = h(t) \implies f(-at) + g(at) = h(t).$$

Let $\xi = -at$

$$\implies f(\xi) = h\left(-\frac{\xi}{a}\right) - g(-\xi) = h\left(-\frac{\xi}{a}\right) - \varphi\left(-\frac{\xi}{2}\right) + f(0).$$

Substituting $f(\xi)$, $g(\eta)$ into the general solution formula gives the solution to the characteristic boundary problem as:

$$\begin{aligned} u(x,t) =& h\left(-\frac{x-at}{a}\right) - \varphi\left(-\frac{x-at}{a}\right) + f(0) + \varphi\left(\frac{x+at}{2}\right) - f(0) \\ =& \varphi\left(\frac{x+at}{2}\right) - \varphi\left(\frac{at-x}{2}\right) + h\left(\frac{at-x}{a}\right). \end{aligned}$$

Summary:

1. Write down the **general solution** of the differential equation.
2. Apply the **specific boundary conditions** to the general solution.
3. Solve the resulting system of equations, which typically involves **two unknowns** and **two equations.**

Note on D'Alembert's Formula

Consider the wave equation solution:

$$u(x,t) = \frac{1}{2}\left[\varphi(x-at) + \varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha$$

To understand $u(x_0, t_0)$, observe the geometric interpretation through characteristics (see Fig. 3):

$$u(x_0,t_0) = \underbrace{\frac{1}{2}\left[\varphi(A) + \varphi(B)\right]}_{\text{Arithmetic mean of the initial positions}} + \underbrace{t_0 \cdot \frac{1}{2at_0}\int_A^B \psi(\alpha)d\alpha}_{t_0 \times \text{Integral average of the initial velocity}}$$

where $A = x_0 - at_0$ and $B = x_0 + at_0$.

This formula represents the **average of** φ **at points** A **and** B, plus t_0 **times the average of** ψ **between** A **and** B.

- **Multiplication by** t_0**:** The multiplication by t_0 in the second term is to ensure **dimensional consistency** and to aid memory. It is crucial not to forget this multiplication.
- **Dimensional Analysis:** The term ψ represents initial velocity. When velocity is integrated, it results in displacement, which has a dimension of length. Multiplying by t_0 (time) ensures that the dimensions on both sides of the equation are consistent (length = speed × time).

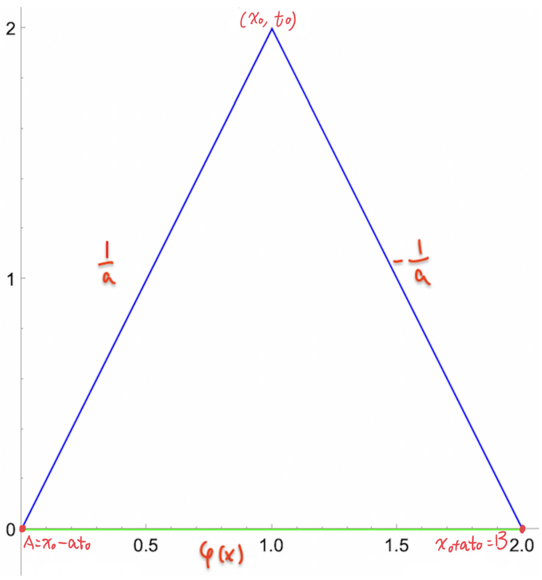

Figure 3: D'Alembert's formula

- **Formula Significance:** The formula indicates that the value of u **depends solely** on the information within the interval from $x - at$ to $x + at$. Information **outside** this interval has **no influence** on u.
- **Isolation of Information:** No matter what happens outside the interval $x - at$ to $x + at$, it does not affect the value of u at a specific point t. This demonstrates that u is isolated from external influences beyond this interval.
- **Importance of the Formula:** This formula is significant because it encapsulates the principle that the value of u is determined by local information within a specific segment, unaffected by external events.

3.2 Solving Non-Homogeneous Wave Equations

Consider the following wave equations and initial conditions (3), that is,

$$u_{tt} = a^2 u_{xx} + f(x,t)(-\infty < x < +\infty, t > 0), \tag{17}$$
$$u(x,0) = 0,\ u_t(x,0) = 0. \tag{18}$$

To solve problems (17) and (18), we use the **principle of homogenization** (or **Duhamel's principle**[1]), transforming the **non-homogeneous** equation into a corresponding **homogeneous equation**, thus directly **utilizing previous results** on homogeneous equations.

3.2.1 Homogenization Principle (Duhamel's principle)

We have solved part of the problem (homogeneous equation + non-homogeneous initial conditions) in the previous section. Next, we need to solve the non-homogeneous equation + homogeneous initial conditions part.

The previous section discussed the initial value problem for the homogeneous equation, which only involved Cauchy problems without boundary conditions. Here, we introduce another method called the homogenization principle, which transforms the non-homogeneous equation problem into a homogeneous equation problem, allowing us to directly use the results from the homogeneous equation solutions.

$$\boxed{\text{Non-Homogeneous Problem}} \xrightarrow{\text{Hope to transform (*)}} \boxed{\text{Homogeneous Problem}}$$

- **Question:** How to transform?
- **Ideas:** By **physical insight** + mathematical **proof**.
- **Answer:** $(*)$ is **Homogenization Principle**

- **Challenge without Boundary Conditions:** We currently lack the means to solve the problem as separation of variables and the method of eigenfunctions requires boundary conditions (due to S-L problem), which are not available.
- **Shift to Physical Insight:** Since mathematical methods are not directly applicable, we turn to physics. We interpret the physical meaning of the problem and use observations to guess what the solution might look like.

[1] See https://en.wikipedia.org/wiki/Duhamel%27s_principle

- **Verification with Mathematics:** Once a physical guess is made, we use mathematical methods to verify the solution. This approach combines physical intuition with mathematical rigor.
- **Importance of Physical Reasoning:** The physical reasoning, though **involving approximations**, provides a well-founded basis for guessing solutions. However, mathematical proof is still necessary to confirm the validity of these solutions.
- **Historical Use of This Method:** This method has been used before, similar to how separation of variables starts with physical insights followed by mathematical confirmation. Here, the role of physics is more prominent.
- **Restoring Physical Context:** To apply physical reasoning, we must first understand the physical meaning of this problem.

We use the principle of impulse and the concept of definite integral to derive the homogenization principle for the vibration of an infinitely long string.

Physical Intuition

Simply speaking, Divide the time interval $[0, t]$ into infinitely many small time intervals. In each small time interval, the external force f can be concentrated at one point, such as the initial velocity of the time interval (where $f dt$ is the impulse), and other parts of this time interval are considered to be free of external forces.

- The wave equation describes the vibration of a string.
- The source term f represents an external force density, meaning force per unit mass.
- Initial conditions:
  - Initial position is at equilibrium.
  - Initial velocity is zero (object is at rest).
- Without external force, the object remains at equilibrium without vibration.
- **Continuous external force** density **causes** vibrations.

Methodology: Discretization in Time

- Previous analysis used **spatial discretization** with Newton's Second Law.
- Now, we **discretize time instead of space**:
  - Imagine taking snapshots at discrete time intervals.
  - Divide time into segments: $t_0, t_1, t_2, \ldots, t_n$.
- External force f is decomposed into **discrete force impulses** f_i:
  - Defined as a **piecewise** function.
  - f_i is **nonzero** only within specific time intervals $[t_{i-1}, t_i]$.

$$f_i(x,t) = \begin{cases} f(x,t) & \text{if } t_{i-1} \le t < t_i \\ 0 & \text{otherwise} \end{cases}$$

- Summing all f_i reconstructs the original force f.

Equation Decomposition and Solution

- Decomposing force f into impulses f_i, the equation transforms into:

$$u_{i,tt} = u_{i,xx} + f_i$$
$$u_i|_{t=t_i} = 0, \; u_{i,t}|_{t=t_i} = 0.$$

- The solution u is obtained by summing solutions of each impulse component (the principle of linear superposition):

$$u = \sum u_i$$

Three Phases of Motion

1. **Before the impulse:** $t \le t_{i-1}$
   - **No external force**, system remains in equilibrium.
2. **During the impulse:** $t_{i-1} < t \le t_i$
   - **External force impulse** f_i is applied, causing vibration.
3. **After the impulse:** $t > t_i$
   - **External force returns to zero**, system evolves dynamically.

Phase 1: Initial Equilibrium

- Time interval: $t_0 \le t \le t_{i-1}$.
- The equation simplifies as the external force $f_\ell = 0$.
- The system **remains at rest**: $u_\ell = 0$.
- The **final state** of this phase is the **initial state** for the **next phase**:
  - $u_{i-1} = 0$.
  - Initial velocity $u'_{i-1} = 0$.

Phase 2: Impulse Effect

- Time interval: $t_{i-1} \le t \le t_i$.
- The equation now includes an external force: $f \neq 0$.
- Initial conditions:
  - Position: $u(t_{i-1}) = 0$.
  - Velocity: $u'(t_{i-1}) = 0$.
- Non-homogeneous equation + homogeneous data → No mathematical method to solve it since it is exactly (17)–(18). That is, the equation in Phase 2 is difficult to solve directly due to its non-homogeneous nature.
- The **differences** between Phase 2 and (17)–(18): The **time interval** is very **short** (t_{i-1} to t_i), thus an **approximation** can be applied.
- Instead of using mathematical methods, we **apply physical principles**.
- Use **impulse theorem**:
  - Impulse: $f \cdot dt$.
  - Resulting velocity: $u'(t_i) = f \cdot dt$.
  - Displacement approximation (linear assumption): $u(t_i) = f \cdot dt^2$ (second-order small term ignored, so $u(t_i) \approx 0$).

Phase 3: Solving the Wave Equation

- Time interval: $t_i \leq t$.
- The equation returns to a homogeneous wave equation ($f = 0$).
- Initial conditions:
  - $u(t_i) = 0$.
  - $u'(t_i) = f \cdot dt$.
- Solve using **d'Alembert's formula**.
- Introduce new variable transformation: $u_i = w_i \cdot dt$ to eliminate dt ($dt \approx \Delta t$ the difference of two constant time and treat Δt as a constant so that it can be moved out of the derivative).
- Reformulate the equation in terms of w_i and solve.

$$\begin{cases} w_{i,tt}\cancel{\Delta t} = w_{i,xx}\cancel{\Delta t} \\ w_i\cancel{\Delta t}|_{t=t_i} = 0, \; w_{i,t}\cancel{\Delta t}|_{t=t_i} = f\cancel{\Delta t}. \end{cases} \quad \leftarrow \text{denote } \tau = t_i.$$

- Use the **linear superposition principle**:

$$u(t) = \sum_{i=0}^{\infty} w_i \cdot dt \quad \Rightarrow \quad u(t) = \int_0^t w_i \, d\tau.$$

Conclusion

- The solution process follows three phases:
  1. **Initial equilibrium** (trivial solution).
  2. Application of **impulse theorem** to obtain **new initial conditions**.
  3. Solution of the wave equation using **d'Alembert's formula** and **integration**.
- This method is called the **Homogenization Principle (Duhamel's principle)**.

Approximation in the Physical Analysis

- The derivation was based on physical principles with several approximations.
- One key approximation was neglecting second-order small quantities like $(\Delta t)^2$.
- Thus, the obtained solution needs verification to confirm its correctness.

Verification by Substitution

- To verify whether the solution is correct, it must be substituted back into the original equation.
- This requires solving for w first.
- The function w is obtained by substituting u and solving the equation using the d'Alembert formula.

Challenges in the Verification Process

- The verification process involves derivatives of the variable limit integral.

Problem Overview

- The physical approach provides an approximate solution, but mathematical verification is required.
- The verification process involves:
  1. Solving for w.
  2. Substituting w to obtain u.
  3. Validating u as the solution to the original equation.

Mathematical verifications

Recall the initial value problem (17)–(18):

$$u_{tt} = a^2 u_{xx} + f(x,t)(-\infty < x < +\infty, t > 0),$$
$$u(x,0) = 0,\ u_t(x,0) = 0.$$

Theorem 0.1 (Homogenization Principle, Duhamel's Principle)**.** *If $w(x,t;\tau)$ is the solution of the initial value problem:*

$$\boxed{\textit{Eq. of Phase 3}} \rightarrow \begin{cases} w_{tt} = a^2 w_{xx} & (t > \tau), \\ w|_{t=\tau} = 0, & w_t|_{t=\tau} = f(x,\tau) \leftarrow \boxed{\textit{Impluse from Phase 2}} \end{cases} \tag{19}$$

where τ is a parameter, then the solution to the initial value problem (17)–(18) is:

$$\boxed{u(x,t) = \int_0^t w(x,t;\tau)d\tau.} \tag{20}$$

Transforming the Initial Condition

- The standard d'Alembert solution assumes initial conditions at $t = 0$.
- Our initial condition is at $t = \tau$.
- Introduce a **shifted time** variable:
$$t' = t - \tau.$$
- This transforms the equation into a form with **initial conditions at** $t' = 0$.

Proof. Let $t' = t - \tau$, and define $\bar{w}(x,t';\tau) = w(x,t'+\tau;\tau) = w(x,t;\tau)$, then problem (21) can be transformed into:

$$\begin{cases} \bar{w}_{t't'} = a^2 \bar{w}_{xx} & (t' > 0), \\ \bar{w}|_{t'=0} = 0, & \bar{w}_{t'}|_{t'=0} = f(x,\tau). \end{cases} \tag{21}$$

Using D'Alembert's formula (16), the solution to problem (21) is:

$$\bar{w}(x,t';\tau) = \frac{1}{2a}\int_{x-at'}^{x+at'} f(\xi,\tau)d\xi.$$

By letting $t' = t - \tau$, we transform the variables back to obtain:

$$w(x,t;\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi. \tag{22}$$

Substituting (22) into equation (20) gives the solution to the initial value problem (17)–(18):

$$\boxed{u(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.} \tag{23}$$

Verifying the Solution

- Verify by substitution:
  1. Ensure u satisfies the wave equation.
  2. Check initial conditions:
  $$u(0,x) = 0, \quad u_t(0,x) = 0.$$
- The verification requires differentiation under the integral sign due to the presence of variable limit integrals.

Two Derivative Formulas

(D1) The simplest:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(D2) m Derivative formula for integrals involving a parameter in a univariate function:

$$\frac{d}{dx}\underbrace{\left[\int_{u(x)}^{v(x)} f(t)dt\right]}_{=\int_0^{v(x)} f(t)dt - \int_0^{u(x)} f(t)dt} = f[v(x)]v'(x) - f[u(x)]u'(x)$$

(D3) Derivative formula for integrals involving a parameter in a bivariate function (Use this):

$$\frac{d}{dx}\underbrace{\left[\int_{\alpha(x)}^{\beta(x)} f(x,y)dy\right]}_{=:F(\alpha(x),\beta(x),x)} = f\big(x,\beta(x)\big)\beta'(x) - f\big(x,\alpha(x)\big)\alpha'(x) \underbrace{+ \int_{\alpha(x)}^{\beta(x)} f_x(x,y)dy}_{\text{extra term}} \leftarrow \boxed{\text{Replace } y!}$$

Proof. By **chain rules**,

$$\frac{d}{dx}F(\alpha(x),\beta(x),x) = \underbrace{\frac{\partial F}{\partial \alpha}}_{\text{Fix } \beta,x}\alpha' + \underbrace{\frac{\partial F}{\partial \beta}}_{\text{Fix } \alpha,x}\beta' + \underbrace{\frac{\partial F}{\partial x}}_{\text{Fix } \alpha,\beta} = -f(x,\alpha)\alpha' + f(x,\beta)\beta' + \int_\alpha^\beta f_x dy.$$

We complete the proof. □

In fact, the function determined by (23) is indeed the solution to problem (17)–(18). When f has continuous first derivatives, from (23) we have:

$$\begin{aligned} u_t =& \frac{1}{2a}\int_{x-a(t-t)}^{x+a(t-t)} f(\xi,t)d\xi + \frac{1}{2}\int_0^t f(x+a(t-\tau),\tau)d\tau + \frac{1}{2}\int_0^t f(x-a(t-\tau),\tau)d\tau \\ =& \frac{1}{2}\int_0^t \left[f(x+a(t-\tau),\tau) + f(x-a(t-\tau),\tau)\right] d\tau. \end{aligned}$$

Example on the computations

Note

$$u(x,t)=\frac{1}{2a}\int_0^t \underbrace{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi}_{=:G(x,t;\tau)}\,d\tau.$$

Then by (D2),

$$\begin{aligned}u_t(x,t)=&\frac{1}{2a}G(x,t;t)+\frac{1}{2a}\int_0^t G_t(x,t;\tau)\,d\tau\\=&\cancel{\frac{1}{2a}\int_{x-a(t-t)}^{x+a(t-t)} f(\xi,t)\,d\xi\,d\tau}+\frac{1}{2a}\int_0^t G_t(x,t;\tau)\,d\tau\end{aligned}$$

Then by (D3),

$$G_t(x,t;\tau)=f(x+a(t-\tau),\tau)\cdot a-f(x-a(t-\tau),\tau)\cdot(-a)$$

We then have:

$$\begin{aligned}u_{tt}=&f(x,t)+\frac{a}{2}\int_0^t [f'(x+a(t-\tau),\tau)-f'(x-a(t-\tau),\tau)]\,d\tau,\\u_x=&\frac{1}{2a}\int_0^t [f(x+a(t-\tau),\tau)-f(x-a(t-\tau),\tau)]\,d\tau,\\u_{xx}=&\frac{1}{2a}\int_0^t [f'(x+a(t-\tau),\tau)-f'(x-a(t-\tau),\tau)]\,d\tau.\end{aligned}$$

Thus, $u_{tt}=a^2u_{xx}+f(x,t)$, i.e., (23) satisfies equation (17). Verify the initial conditions (18). From (23) and the expression for u_t, we have:

$$u|_{t=0}=0,\quad u_t|_{t=0}=0.$$

This proves that the function determined by (23) is indeed the solution to the initial value problem (17)–(18). □

Equation (16) describes the general solution without external forces (2). That is,

$$u(x,t)=\frac{\varphi(x-at)+\varphi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha$$

solves

$$\begin{cases}u_{tt}=a^2u_{xx} & (-\infty<x<+\infty,t>0),\\u(x,0)=\varphi(x), & u_t(x,0)=\psi(x)\end{cases}$$

Equation (22) represents the effect of an external force integrated over time(3). That is,

$$u(x,t)=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau$$

solves

$$\begin{cases}u_{tt}=a^2u_{xx}+f(x,t) & (-\infty<x<+\infty,t>0),\\u(x,0)=0, & u_t(x,0)=0\end{cases}$$

By the **principle of superposition**, the solution to the problem (1),

$$\begin{cases}u_{tt}=a^2u_{xx}+f(x,t) & (-\infty<x<+\infty,t>0),\\u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x)\end{cases}$$

can be expressed as:

$$\boxed{u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau.} \tag{24}$$

and

$$u(x,t) = \underbrace{\frac{\varphi(x-at)+\varphi(x+at)}{2}}_{\text{Arithmetic mean of the initial positions}} + \underbrace{\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha}_{t\times\text{Integral average of the initial velocity}}$$
$$+ \underbrace{\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau}_{t^2\times\text{the average of the integral of the area of the triangle}} .$$

Ex 0.2. *Solve the following initial value problem*

$$\begin{cases} u_{tt} = u_{xx} + 2x & (-\infty < x < +\infty, t > 0), \\ u(x,0) = \sin x, & u_t(x,0) = x. \end{cases}$$

Solution. *Using formula* (24)*, we get*

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin(x+t)+\sin(x-t)] + \frac{1}{2}\int_{x-t}^{x+t}\alpha\, d\alpha + \frac{1}{2}\int_0^t\left(\int_{x-(t-\tau)}^{x+(t-\tau)} 2\xi\, d\xi\right)d\tau \\ &= \sin x\cos t + xt + xt^2. \end{aligned}$$

3.1.2 Physical Meaning of D'Alembert's Solution

$$\boxed{\text{Physical intuitions}} \to \boxed{\text{mathematical models}} \to \boxed{\text{Physical interpretations}}$$

From the general solution (11), the solution to the free vibration equation can be expressed as the sum of two functions $f(x-at)$ and $g(x+at)$. Through them, the nature of wave propagation can be clearly seen.

First, consider:

$$u_1 = f(x-at),$$

which is obviously a solution to equation (4). By giving t different values, one can see the vibration state of the string at each moment.

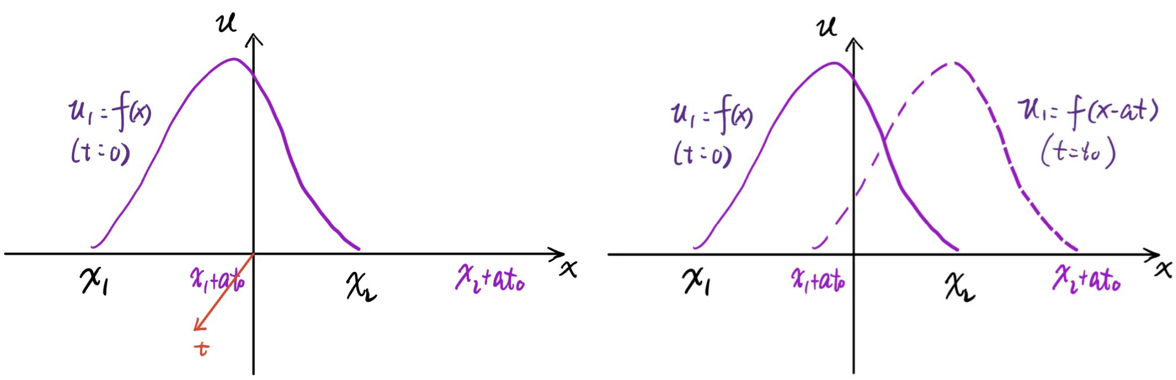

Figure 4: Wave propagation

- At $t = 0$, $u_1(x,0) = f(x)$, which corresponds to the vibration state at the initial moment (equivalent to the displacement state of the string at each point at the **initial moment**), as shown in Fig. 4.

- After time t_0, $u_1(x, t_0) = f(x - at_0)$, which is equivalent to the original shape $u_1 = f(x)$ **shifted to the right** by a distance at_0 on the (x, u) plane.
- As time progresses, this diagram will **continue** to **move to the right**, indicating that when the solution to equation (4) is expressed in the form $f(x - at)$, the wave shape of the vibration propagates **to the right** at a **constant speed** a.
- Thus, the vibration pattern described by the function $f(x - at)$ is called a **right-propagating wave**. Similarly, the solution in the form of $g(x + at)$ is called a **left-propagating wave**, which describes a wave pattern propagating to the **left** at a **constant speed** a.
- From this, it is evident that the general solution (11) represents any disturbance on the string propagating in the form of **traveling waves** in **both directions**, with the **propagation speed** being the constant a appearing in equation (4). **D'Alembert's solution method** is also known as the **method of traveling waves**.

3.1.3 Dependency Region, Determination Region, and Influence Region

Recall

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha.$$

solves

$$\begin{cases} u_{tt} = a^2 u_{xx} \quad (-\infty < x < +\infty, t > 0), \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

Question 0.1. *The solution of the initial value problem* (2) *at a point* (x, t) *is related to the values of the initial conditions on which points on the* x*-axis?*

- **(Dependency region of a point)** From D'Alembert's formula (16), it can be seen that the value of the solution at the point (x, t) **depends only** on the initial conditions on the interval $[x - at, x + at]$ on the x-axis, and is **independent of** the initial conditions at other points. This interval $[x - at, x + at]$ is called the **dependency region** of the **point** (x, t) (see Fig. 7).
- It is the interval intersected by the x-axis with the lines passing through the point (x, t) with slopes $\pm\frac{1}{a}$, as shown in Fig. 7.
- Consider an interval $[x_1, x_2]$ on the initial axis $t = 0$, draw a line through point x_1 with a slope of $\frac{1}{a}$, $x = x_1 + at$, and a line through point x_2 with a slope of $-\frac{1}{a}$, $x = x_2 - at$, together with the interval $[x_1, x_2]$ they form a **triangular region**, as shown in Fig. 7.
- In this triangular region, the dependency region of any point (x, t) falls within the interval $[x_1, x_2]$, therefore, the value of the solution in the triangular region is completely determined by the initial conditions on the interval $[x_1, x_2]$, and is independent of the initial conditions outside this interval.
- **(Determination region of an interval)** This triangular region is called the **determination region** of the **interval** $[x_1, x_2]$ (see Fig. 5 and 7). Given the initial conditions on the interval $[x_1, x_2]$, the solution to the initial value problem (2) can be determined within its determination region.

Question 0.2. *If at the initial moment* $t = 0$*, the disturbance exists only in a finite interval* $[x_1, x_2]$*, what is the range it affects after time* t*?*

- We know that the wave propagates in **both directions** at a certain **speed** a.
- Therefore, after time t, the range it propagates to (the range affected by the initial disturbance) is limited by the inequality
$$x_1 - at \le x \le x_2 + at \quad (t > 0) \tag{25}$$
and **outside** this range, it **remains** in a state of **rest**.

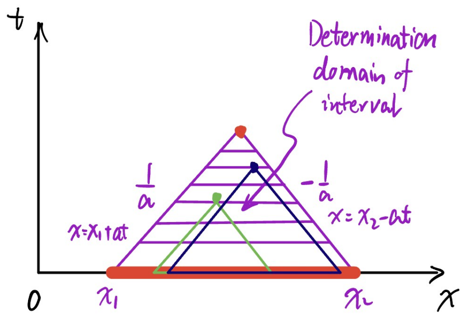

Figure 5: determination domains

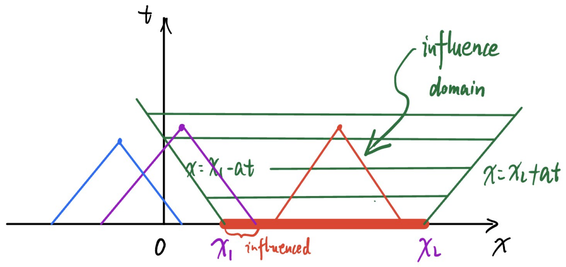

Figure 6: Influence domains

- **(Influence region of an interval)** On the (x, t) plane, the region represented by equation (25) is called the **influence region** of the **interval** $[x_1, x_2]$ (see Fig. 6 and 7). In this region, the solution $u(x, t)$ of the initial value problem is affected by the initial conditions on the interval $[x_1, x_2]$.
- Outside this region, the solution $u(x, t)$ of the initial value problem is not affected by the initial conditions on the interval $[x_1, x_2]$.
- In particular, shrinking the interval $[x_1, x_2]$ to a single point x_0, we can obtain the influence region of a single point x_0, as shown in Fig. 7. This influence region is the triangular area formed by the two lines passing through this point with slopes $\pm\frac{1}{a}$, $x = x_0 \pm at$.

Extended Materials:

Characteristics of Linear Wave Equations

- In a linear wave equation, the wave speed a is a constant.
- The solution is constructed using characteristic lines with slopes $\frac{1}{a}$ and $-\frac{1}{a}$, forming triangular influence regions.
- This structure results in straight-line characteristics, ensuring well-defined propagation.

Nonlinear Wave Equations and Curved Characteristics

- In nonlinear wave equations, the wave speed a may depend on t, x, or u.
- This leads to variable coefficients, altering the propagation behavior.
- Characteristic lines may become curved instead of straight.
- The influence region is no longer a simple triangle but a more complex shape.

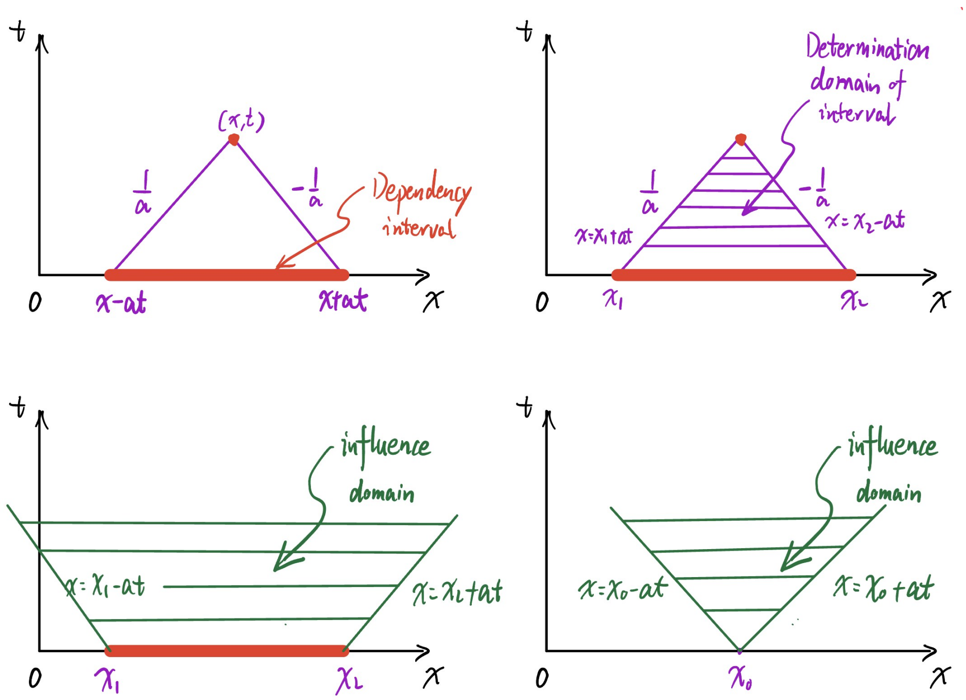

Figure 7: Domains

Finite Speed of Wave Propagation

- Waves propagate at a finite speed, meaning disturbances take time to reach different locations.
- If an event (e.g., an explosion) occurs at a point, the effects will not be felt instantaneously elsewhere.
- This principle ensures that causality is preserved:
  - Causes precede effects.
  - No information can propagate faster than the wave speed.

Causality in Wave Equations

- The finite speed of propagation enforces a strict order of events.
- No effect can precede its cause.
- This property is fundamental in physics and ensures consistency in signal transmission and interactions.