Lecture Notes: Mathematical Physics Equations and Special Functions

Chao Liu

Week 6-2

Ex 0.1. *Solve the Free Vibration Problem of a Semi-Infinite String*

$$u_{tt} = a^2 u_{xx}, \quad (x > 0, t > 0), \tag{1}$$
$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{2}$$
$$u(0,t) = f(t), \quad \lim_{x\to+\infty} u(x,t) = 0. \tag{3}$$

where $f(t)$ is a known function (satisfying the Laplace transform conditions), and $f(0) = 0$.

Choosing Between Laplace and Fourier Transforms

- If the domain is $x > 0, t > 0$, using the Laplace transform is often the simplest choice.
- Fourier transform can still be used, but it requires an extension and later restriction, which complicates the process.

Domain and Boundary Conditions

- If the given boundary conditions involve zero initial values, the Laplace transform is usually preferable.
- When applying the Laplace transform to a second-order equation, initial conditions appear naturally in the transformed equation. Given that the initial conditions are zero, these terms vanish, simplifying the equation.
- If transforming in the x variable instead of t, some boundary conditions may be missing, requiring additional conditions to be inferred.

Properties of the Laplace Transform

- The Laplace transform is an integral with respect to t, meaning it does not interfere with differentiation in x.
- Similarly, limits on x and integrals (Laplace integral for t) can be interchanged when taking limits at infinity.

Solution. *Take the Laplace transform of equations (1) and (3) with respect to t, denote:*

$$L[u(x,t)] = U(x,s), \quad L[f(t)] = F(s).$$

Equations (1)–(3) become:

$$a^2 \frac{d^2U}{dx^2} - s^2 U = 0, \tag{4}$$
$$U(0,s) = F(s), \quad \lim_{x\to+\infty} U(x,s) = 0. \tag{5}$$

The general solution to equation (4) is:

$$U(x,s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x},$$

Given a second-order constant coefficient ODE:

$$ay'' + by' + cy = 0,$$

where a, b, c are real or complex constants, the solution can be found using the following steps:

- *Assume a trial solution of the form $y = e^{\lambda x}$.*
- *Compute derivatives: $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$.*
- *Substituting into the ODE gives the characteristic equation:*

$$a\lambda^2 + b\lambda + c = 0.$$

Key Insights

- *The approach is based on the trial solution $e^{\lambda x}$, leveraging its property under differentiation.*
- *The characteristic equation arises naturally from substituting the assumed solution.*

From condition (5), we know $c_1 = 0$, $c_2 = F(s)$, thus we have:

$$U(x,s) = F(s)e^{-\frac{s}{a}x}.$$

Taking the inverse Laplace transform of the above equation, we get:

$$u(x,t) = L^{-1}[F(s)e^{-\frac{s}{a}x}]. \tag{6}$$

Using the inverse transform formula of the delay theorem of Laplace transform:

$$L^{-1}[F(s)e^{-sa}] = f(t-a) \quad (t > a).$$

Thus, equation (6) can be simplified to:

$$u(x,t) = \begin{cases} 0, & t < \frac{x}{a}, \\ f\left(t - \frac{x}{a}\right), & t > \frac{x}{a}. \end{cases}$$

This is the solution to the free vibration problem (1)–(3) of the semi-infinite string.

Ex 0.2. *Solve the Following Problem*

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < 1, t > 0 \leftarrow \boxed{\text{Laplace transf.}}), \\ u(x,0) = 4\sin \pi x, \\ u(0,t) = 0, \quad u(1,t) = 0. \end{cases} \tag{7}$$

Solution. *Obviously, take the Laplace transform with respect to t, denote:*

$$L[u(x,t)] = U(x,s),$$

Equation (7) can be transformed into:

$$a^2 \frac{d^2U}{dx^2} - sU = -4\sin \pi x, \tag{8}$$

with boundary conditions:

$$U(0,s) = 0, \quad U(1,s) = 0. \tag{9}$$

The general solution to equation (8) is:

$$U(x,s) = c_1 e^{\frac{\sqrt{s}}{a}x} + c_2 e^{-\frac{\sqrt{s}}{a}x} + \frac{4\sin \pi x}{s + a^2\pi^2},$$

- ***Type of Equation:*** *A second-order linear non-homogeneous ODE.*
- ***General Solution Method:***
  - *First, treat the ODE as homogeneous and find the general solution.*
  - *Then, add a particular solution for the non-homogeneous part.*
- ***Methods for Particular Solution:***
  - *Trial and Error Method: This is highly recommended for ODEs. Functions like exponential, sine, and cosine are useful due to their derivative invariance.*
  - *Variation of Parameters: Can be used but may be time-consuming.*
  - *Integral Transform: Not suitable for this ODE due to the domain issues (e.g., zero to one) and the need for extension and restriction.*
- ***Conclusion:*** *The trial and error method is the most efficient for solving this ODE. We calculate it below:*

Let

$$U = A\sin\pi x. \quad (A \text{ to be determined}).$$

Then

$$U'' = -A\pi^2 \sin\pi x$$

From

$$a^2U'' - sU = -4\sin\pi x \quad \Rightarrow \quad -Aa^2\pi^2\sin\pi x - sA\sin\pi x = -4\sin\pi x$$

From

$$\Rightarrow -A\cancel{\sin\pi x}(a^2\pi^2 + s) = -4\cancel{\sin\pi x} \Rightarrow A = \frac{4}{a^2\pi^2 + s}$$
$$\Rightarrow U = \frac{4}{a^2\pi^2 + s}\sin\pi x.$$

From conditions (9),

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\frac{\sqrt{3}}{2}} + c_2 e^{-\frac{\sqrt{3}}{2}} + \frac{4\sin\pi}{5+4\pi^2} = 0 \end{cases} \qquad \Rightarrow \qquad c_1 = c_2 = 0.$$

we know $c_1 = 0$, $c_2 = 0$, *thus we have:*

$$U(x, s) = \frac{4\sin\pi x}{s + a^2\pi^2}.$$

Taking the inverse Laplace transform of the above equation, we get the solution to problem (7)*:*

$$u(x, t) = 4\sin\pi x L^{-1}\left[\frac{1}{s + a^2\pi^2}\right] = 4e^{-a^2\pi^2 t}\sin\pi x.$$

This solution is exactly the same as the one obtained by separation of variables.

1 Exercise class

Ex 1.1. *Use the method of characteristics to solve the following initial-boundary value problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, x - at < 0, x > 0 \\ u\big|_{x-at=0} = \varphi(x), & u\big|_{x=0} = h(t). \end{cases}$$

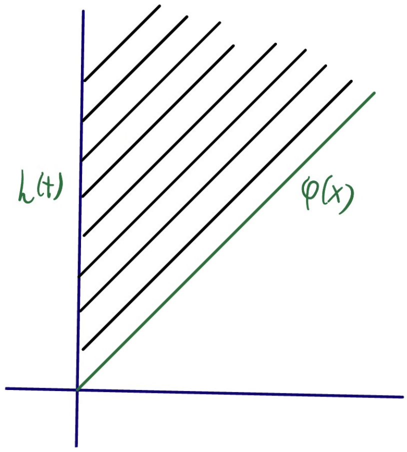

Figure 1: Goursat Problem

Solution. ***The general solution:*** *The general solution of the wave equation is:*

$$u(x,t) = f(x-at) + g(x+at). \leftarrow \boxed{\textit{Memorize the general solution of wave}}$$

where f, g are arbitrary functions with continuous second derivatives.

*Next, we use **boundary conditions** to determine the arbitrary functions f, g. First, from the condition:*

$$u\big|_{x-at=0} = \varphi(x) \implies f(0) + g(2x) = \varphi(x).$$

Let $\eta = 2x$,

$$\implies g(\eta) = \varphi\left(\frac{\eta}{2}\right) - f(0). \leftarrow \boxed{\textit{Know how to determine } f \textit{ and } g \textit{ by conditions}}$$

Using the condition:

$$u\big|_{x=0} = h(t) \implies f(-at) + g(at) = h(t).$$

Let $\xi = -at$

$$\implies f(\xi) = h\left(-\frac{\xi}{a}\right) - g(-\xi) = h\left(-\frac{\xi}{a}\right) - \varphi\left(-\frac{\xi}{2}\right) + f(0).$$

Substituting $f(\xi)$, $g(\eta)$ into the general solution formula gives the solution to the characteristic boundary problem as:

$$\begin{aligned} u(x,t) =& h\left(-\frac{x-at}{a}\right) - \varphi\left(-\frac{x-at}{a}\right) + f(0) + \varphi\left(\frac{x+at}{2}\right) - f(0) \\ =& \varphi\left(\frac{x+at}{2}\right) - \varphi\left(\frac{at-x}{2}\right) + h\left(\frac{at-x}{a}\right). \end{aligned}$$

Ex 1.2. *Goursat problem*

$$\begin{cases} u_{tt} = u_{xx} & (-\infty < x < +\infty, t > 0) \\ u|_{t=-x} = \varphi(x), & u|_{t=x} = \psi(x) \end{cases}$$

Solution. *General solution:*

$$u(x,t) = f_1(x+t) + f_2(x-t)$$

By initial data,

$$\begin{cases} f_1(0) + f_2(2x) = \varphi(x) \\ f_1(2x) + f_2(0) = \psi(x) \end{cases}$$

Let $y := 2x$, then

$$\begin{cases} f_1(0) + f_2(y) = \varphi\left(\frac{y}{2}\right) \\ f_1(y) + f_2(0) = \psi\left(\frac{y}{2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} f_2(y) = \varphi\left(\frac{y}{2}\right) - f_1(0) \\ f_1(y) = \psi\left(\frac{y}{2}\right) - f_2(0) \end{cases} \tag{10}$$

$$\Rightarrow u(x,t) = \psi\left(\frac{x+t}{2}\right) - f_2(0) + \varphi\left(\frac{x-t}{2}\right) - f_1(0) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - (f_1(0) + f_2(0))$$

Let $y = 0$ in (10) $\Rightarrow f_1(0) + f_2(0) = \varphi(0) = \psi(0) = \frac{1}{2}(\varphi(0) + \psi(0))$

$$\Rightarrow u(x,t) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) \underbrace{-\frac{1}{2}(\varphi(0) + \psi(0))}_{or\ -\varphi(0)\ or\ -\psi(0)}$$

Ex 1.3. *Solve the following problem*

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad (-\infty < x < \infty,\, t > 0) \tag{11}$$

$$u(x,0) = \varphi(x) \tag{12}$$

Solution (Method 1: like the derivations of the traveling wave method). *Let $\xi = x - at$, $\eta = x$, solve this problem in the (ξ, η) coordinate system. Let*

$$u(x,t) = \bar{u}(\xi(x,t), \eta(x,t)).$$

By the chain rules,

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial \bar{u}}{\partial \eta} = -a\frac{\partial \bar{u}}{\partial \xi}, \\ \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \bar{u}}{\partial \eta} = \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{u}}{\partial \eta}. \end{cases}$$

By (11), we obtain

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -a\frac{\partial \bar{u}}{\partial \xi} + a\frac{\partial \bar{u}}{\partial \eta} + a\frac{\partial \bar{u}}{\partial \eta} = 0.$$

That is,

$$\frac{\partial \bar{u}}{\partial \eta} = 0 \quad \Rightarrow \quad \bar{u}(\xi, \eta) = f(\xi) \quad \Rightarrow \quad u(x,t) = f(x - at).$$

Using the initial condition $u(x,0) = f(x) = \varphi(x)$,

$$\Rightarrow u(x,t) = \varphi(x - at).$$

Solution (Method 2: Characteristic methods). *Define a curve $x(t)$ in (x,t) plane by*

$$\frac{dx(t)}{dt} = a \quad \Leftrightarrow \quad x(t) = x(0) + at \leftarrow \boxed{\text{called the characteristic curves}}$$

Define a function $\bar{u}(t) := u(x(t), t)$, then by the chain rule and the equation (11), we obtain

$$\frac{d\bar{u}}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0.$$

Then integrating this equation, we arrive at

$$\bar{u}(t) = constant\ C \Leftrightarrow u(x(t), t) = C$$

By the initial data, we have $u(x(0), 0) = C = \varphi(x(0))$. Thus

$$u(x(t), t) = \varphi(x(0)) = \varphi(x(t) - at).$$

Then

$$u(x,t) = \varphi(x(0)) = \varphi(x - at).$$

Solution (**Method 3: Transforming to a traveling wave**). *Differentiate equation* (11) *with respect to* t *and* x *respectively, we get*

$$u_{tt} + au_{tx} = 0 \tag{13}$$

$$u_{tx} + au_{xx} = 0 \tag{14}$$

If u ***solves*** (11)***, then it solves*** (15)***:*** *Multiplying equation* (14) *by* a *and subtracting equation* (13) *from it, we obtain a one-dimensional wave equation*

$$u_{tt} - a^2 u_{xx} = 0 \tag{15}$$

The general solution of this equation is

$$u(x,t) = f_1(x+at) + f_2(x-at) \tag{16}$$

To find the particular solutions of equations (11) *and* (12)*, substitute equation* (16) *into equation* (11) *to get*

$$af_1'(x+at) - af_2'(x-at) + af_1'(x+at) + af_2'(x-at) = 0$$

which simplifies to

$$2af_1'(x+at) = 0 \tag{17}$$

Thus,

$$f_1(x+at) = C$$

Substituting equation (16) *into equation* (12)*, we get*

$$f_1(x) + f_2(x) = \varphi(x)$$

Solving, we find

$$f_2(x) = \varphi(x) - f_1(x)$$

Therefore,

$$f_2(x-at) = \varphi(x-at) - C. \tag{18}$$

Substituting equations (17) *and* (18) *into the general solution* (16)*, we obtain the solution to the initial value problem for the wave equation*

$$u(x,t) = \varphi(x-at)$$

Ex 1.4. *Vibration of a semi-infinite rod with one end free*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (0 < x < \infty, t > 0) \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = \psi(x) \quad (0 \le x < \infty) \\ u_x(0,t) = 0 & \textit{(second type; tension vanishes at } x = 0\textit{)} \end{cases}$$

Solution (Method 1: Extension methods)**.** *Extend the original problem to* $-\infty < x < +\infty$*, then the solution must satisfy*

$$\begin{cases} u_{tt} = a^2 u_{xx} & (-\infty < x < \infty, t > 0) \\ u(x,0) = \Phi(x) = \begin{cases} \varphi(x) & 0 \le x < \infty \\ f(x) & -\infty < x < 0 \end{cases} \\ u_t(x,0) = \Psi(x) = \begin{cases} \psi(x) & 0 \le x < \infty \\ g(x) & -\infty < x < 0 \end{cases} \end{cases}$$

where f *and* g *are to be determined.*

D'Alembert's formula implies

$$u(x,t) = \frac{1}{2}\left[\Phi(x+at) + \Phi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)\, d\xi. \tag{19}$$

Only if (19) *satisfies the original problem boundary conditions, it is the solution to the original problem by limiting to* $x \ge 0$.

The boundary condition verifies that the only extension is the even extension): *The problem is transformed into using the boundary conditions to determine f and g. By boundary conditions, we have*

$$u_x(0,t) = \frac{1}{2}\left[\Phi'(at) + \Phi'(-at)\right] + \frac{1}{2a}\left[\Psi(at) - \Psi(-at)\right] = 0$$

Since Φ and Ψ are independent,

$$\begin{cases} \Phi'(at) = -\Phi'(-at) \\ \Psi(at) = \Psi(-at) \end{cases}$$

Then

$$\begin{cases} -\Phi'(x) = \Phi'(-x) \Rightarrow \int_0^x \Phi'(\alpha)d\alpha = \int_0^x \big(-\Phi'(-\alpha)\big)d\alpha \Rightarrow \Phi(x) - \Phi(0) = \Phi(-x) - \Phi(0) \Rightarrow \Phi(x) = \Phi(-x) \\ \Psi(x) = \Psi(-x) \end{cases}$$

This means the Φ and Ψ satifying the boundary condition must both be even functions. Thus,

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0 \\ \varphi(-x), & x < 0 \end{cases} \quad \text{and} \quad \Psi(x) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x), & x < 0 \end{cases} \tag{20}$$

Note that $x + at$ is always greater than or equal to zero, thus from equation (9) we have:

$$\Phi(x + at) = \varphi(x + at)$$

$$\int_0^{x+at} \Psi(\xi)\, d\xi = \int_0^{x+at} \psi(\xi)\, d\xi$$

Since $x - at$ could be greater than, equal to, or less than zero:

1. *If $x - at \geq 0$, then from equation (20) we have:*

$$\Phi(x - at) = \varphi(x - at)$$

$$\int_{x-at}^{0} \Psi(\xi)\, d\xi = \int_{x-at}^{0} \psi(\xi)\, d\xi$$

2. *If $x - at < 0$, then from equation (20) we have:*

$$\Phi(x - at) = \varphi[-(x - at)] = \varphi(at - x)$$

$$\int_{x-at}^{0} \Psi(\xi)\, d\xi = \int_{x-at}^{0} \psi(-\xi)\, d\xi = -\int_{at-x}^{0} \psi(\eta)\, d\eta$$

Thus:

$$\int_{x-at}^{0} \Psi(\xi)\, d\xi = \int_0^{at-x} \psi(\xi)\, d\xi$$

By D'Alembert's formula, solve (19) to get. When $x \geq at$

$$u(x,t) = \frac{1}{2}\left[\varphi(x + at) + \varphi(x - at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi.$$

When $0 < x < at$

$$u(x,t) = \frac{1}{2}\left[\varphi(x + at) + \varphi(at - x)\right] + \frac{1}{2a}\int_0^{x+at} \psi(\xi)\, d\xi + \frac{1}{2a}\int_0^{at-x} \psi(\xi)\, d\xi.$$

- *From this example, it can be seen that for the one-dimensional wave problem in a semi-infinite domain, it can be extended to an infinite domain and then solved using d'Alembert's formula.*
- *If the endpoint has a **second-type homogeneous boundary condition**, then the initial conditions need to be extended as even extensions;*

- *If the endpoint has a* ***first-type homogeneous boundary condition****, a similar method can be used to deduce that the initial conditions need to be extended as odd extensions:*

$$\Phi(x) = \begin{cases} \varphi(x), & 0 \le x < \infty \\ -\varphi(-x), & -\infty < x < 0 \end{cases} \quad \text{and} \quad \Psi(x) = \begin{cases} \psi(x), & 0 \le x < \infty \\ -\psi(-x), & -\infty < x < 0 \end{cases}$$

Solution (Method 2)**.** *The general solution of equation $u_{tt} = a^2 u_{xx}$ is:*

$$u(x,t) = f_1(x+at) + f_2(x-at).$$

Thus, from initial data, we have:

$$f_1(x) + f_2(x) = \varphi(x) \quad \text{and} \quad af_1'(x) - af_2'(x) = \psi(x). \tag{21}$$

That is:

$$f_1(x) - f_2(x) = \frac{1}{a}\int_0^x \psi(\xi)\, d\xi + C \tag{22}$$

Where $C = f_1(0) - f_2(0)$. Solving equations (21) and (22) gives:

$$f_1(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_0^x \psi(\xi)\, d\xi + \frac{C}{2} \quad (0 \le x < \infty); \tag{23}$$

$$f_2(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_0^x \psi(\xi)\, d\xi - \frac{C}{2} \quad (0 \le x < \infty). \tag{24}$$

The above two equations are derived under the premise of $0 \le x < \infty$ since (21) holds for $0 \le x < \infty$ only by the inital data. Since $x + at$ is always greater than or equal to zero, from equation (23) we have:

$$f_1(x+at) = \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int_0^{x+at} \psi(\xi)\, d\xi + \frac{C}{2}. \tag{25}$$

As for $x - at$, it is not necessarily greater than zero.

(1) If $x - at \ge 0$, then from equation (24) we have:

$$f_2(x-at) = \frac{1}{2}\varphi(x-at) - \frac{1}{2a}\int_0^{x-at} \psi(\xi)\, d\xi - \frac{C}{2}. \tag{26}$$

(2) ***(Using the boundary to obtain the negative parts)*** *If $x - at < 0$, then (24) cannot be used. However, substituting the* ***boundary condition*** *into the general solution gives:*

$$f_1'(at) + f_2'(-at) = 0.$$

Let $x = at$, and integrate the above it from 0 to x to get:

$$f_1(x) - f_2(-x) = C$$

That is:

$$f_2(-x) = f_1(x) - C \quad (x \ge 0).$$

Thus:

$$f_2(x-at) = f_2[-(at-x)] = f_1(at-x) - C = \frac{1}{2}\varphi(at-x) + \frac{1}{2a}\int_0^{at-x} \psi(\xi)\, d\xi - C \quad (at - x \ge 0). \tag{27}$$

Substituting equations (25), (26), and (27) into the general solution, we get:

$$u(x,t) = \begin{cases} \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi, & x - at \ge 0 \\ \frac{1}{2}[\varphi(x+at) + \varphi(at-x)] + \frac{1}{2a}\left[\int_0^{x+at} \psi(\xi)\, d\xi + \int_0^{at-x} \psi(\xi)\, d\xi\right], & x - at < 0 \end{cases}$$

where $C = 0$ can be determined by data.

Ex 1.5.

$$\begin{cases} u_t - a^2 u_{xx} - bu_x - cu = f(x,t), & -\infty < x < \infty,\, t > 0 \\ u(x,0) = 0 \end{cases}$$

where a, b, c are constants. Solve using the method of integral transform.

Solution. *For x, use the Fourier transform:*

$$\hat{u}(\lambda,t) = F[u], \quad \hat{f}(\lambda,t) = F[f(x,t)]$$

Then

$$\frac{d\hat{u}}{dt} + a^2\lambda^2\hat{u} - ib\lambda\hat{u} - c\hat{u} = \hat{f} \quad \Rightarrow \quad \frac{d\hat{u}}{dt} + (a^2\lambda^2 - ib\lambda - c)\hat{u} = \hat{f}$$

$$\begin{cases} \frac{d\hat{u}}{dt} + (a^2\lambda^2 - ib\lambda - c)\hat{u} = \hat{f} \\ \hat{u}(\lambda,0) = 0 \end{cases}$$

There are multiple method to solve this ODE. I use the integrating factor method. $e^{(a^2\lambda^2 - ib\lambda - c)t}$ *is the integrating factor.*

$$\frac{d}{dt}\left(e^{(a^2\lambda^2 - ib\lambda - c)t}\hat{u}\right) = e^{(a^2\lambda^2 - ib\lambda - c)t}\hat{f}(\lambda,t).$$

Integrate both sides:

$$e^{(a^2\lambda^2 - ib\lambda - c)t}\hat{u}(\lambda,t) = \hat{u}(\lambda,0) + \int_0^t e^{(a^2\lambda^2 - ib\lambda - c)\tau}\hat{f}(\lambda,\tau)\,d\tau.$$

$$\Rightarrow \hat{u}(\lambda,t) = \int_0^t e^{(a^2\lambda^2 - ib\lambda - c)(\tau - t)}\hat{f}(\lambda,\tau)\,d\tau.$$

The inverse Fourier transform is used to obtain:

$$u(x,t) = \int_0^t e^{-c(\tau-t)}F^{-1}\left[e^{(a^2\lambda^2 - ib\lambda)(\tau-t)}\right] * f(x,\tau)\,d\tau$$

Using the Fourier transform table:

$$\begin{aligned} F^{-1}\left[e^{-a^2\lambda^2 t}\right] &= \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}} \\ F^{-1}\left[e^{(a^2\lambda^2 - ib\lambda)(\tau-t)}\right] &= \frac{1}{2a\sqrt{\pi(t-\tau)}}e^{\frac{(x-b(\tau-t))^2}{4a^2(\tau-t)}} \end{aligned}$$

$$\begin{aligned} \Rightarrow u(x,t) &= \int_0^t e^{-c(\tau-t)}\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{+\infty} e^{\frac{(x-y-b(\tau-t))^2}{4a^2(\tau-t)}} f(y,\tau)\,dy\,d\tau \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}}e^{\frac{(x-y-b(\tau-t))^2}{4a^2(\tau-t)} - c(\tau-t)} f(y,\tau)\,dy\,d\tau. \end{aligned}$$

Ex 1.6. *Solve the initial value problem:*

$$\begin{cases} u_{xx} - au_{tt} - bu_t - cu = 0 & (x > 0, t > 0) \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 & \textit{(Initial conditions)} \\ u(0,t) = \phi(t), \quad \lim_{x\to\infty} u(x,t) = 0 & \textit{(Boundary conditions)} \end{cases}$$

where a, b, c are constants, and $b^2 = 4ac$.

Solution. *Directly apply Laplace transform to t, let*

$$L[u(x,t)] = U(x,s), \quad L[\phi(t)] = \Phi(s)$$

Then the problem transforms into:

$$\begin{cases} U_{xx} - as^2U - bsU - cU = 0 \\ U(0,s) = \Phi(s), \quad \lim_{x\to\infty} U(x,s) = 0 \end{cases}$$

Since $b^2 = 4ac$, *we have* $as^2 + bs + c = \left(\sqrt{a}s + \frac{b}{2\sqrt{a}}\right)^2$

$$\Rightarrow \text{The solution is } U(x,s) = c_1 e^{\left(\sqrt{a}s + \frac{b}{2\sqrt{a}}\right)x} + c_2 e^{-\left(\sqrt{a}s + \frac{b}{2\sqrt{a}}\right)x}.$$

By the boundary conditions, $c_1 = 0$ *and* $c_2 = \Phi(s)$

$$\Rightarrow U(x,s) = \Phi(s) e^{-\left(\sqrt{a}s + \frac{b}{2\sqrt{a}}\right)x}.$$

Using the inverse Laplace transform:

$$\begin{aligned} u(x,t) = L^{-1}[U(x,s)] =& L^{-1}\left[\Phi(s)e^{-\sqrt{a}sx}\right] e^{-\frac{b}{2\sqrt{a}}x} \\ =& \begin{cases} \int_0^t \phi(t - \sqrt{a}x) e^{-\frac{b}{2\sqrt{a}}x} & t > \sqrt{a}x \\ 0 & t < \sqrt{a}x \end{cases}. \end{aligned}$$

Ex 1.7. *When the initial value* $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ *satisfy what conditions, does the solution of the one-dimensional wave equation consist only of a right-traveling wave?*

Solution (Method 1: D'Alembert solution)**.** *Use D'Alembert's formula, which is a special case of the general solution.*

$$\begin{aligned} u(x,t) =& \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi \leftarrow \boxed{\text{Newton-Leibniz formula}} \\ =& \underbrace{\frac{1}{2}\left[\varphi(x+at) + \frac{1}{2a}\Psi(x+at)\right]}_{\text{left-traveling wave}} + \underbrace{\frac{1}{2}\left[\varphi(x-at) - \frac{1}{2a}\Psi(x-at)\right]}_{\text{right-traveling wave}} \end{aligned}$$

The left-traveling wave: the term independent of x *and* t *is a constant.*

$$\Rightarrow a\varphi(x+at) + \Psi(x+at) = C \quad \text{Let } z = x + at \Rightarrow a\varphi(z) + \Psi(z) = C'$$

In other words, we find that $a\varphi'(z) + \psi(z) = 0$.

Solution (Method 2: Traveling wave solution)**.** *Suppose the solution of the initial value problem has only a right-traveling wave, i.e., the solution* $u(x,t) = f(x - at)$. *Then the initial conditions imply:*

$$\left.\begin{aligned} u(x,0) = f(x) = \varphi(x) \\ u_t(x,0) = -af'(x) = \psi(x) \end{aligned}\right\} \Rightarrow -a\varphi'(x) = \psi(x)$$

This is consistent with Method 1.

Ex 1.8. *Suppose* $f(x,y)$ *is a harmonic function (i.e.,* $\Delta f = 0$*),* $g(z) \in C^2(\mathbb{R})$, *solve the Cauchy problem*

$$\begin{cases} u_{tt} - a^2(u_{xx} + u_{yy} + u_{zz}) = 0 (x,y,z) \in \mathbb{R}^3,\ t > 0 \\ u|_{t=0} = f(x,y)g(z), \leftarrow \boxed{\text{inspiring the form of the solution}} \\ u_t|_{t=0} = 0 \end{cases}$$

Inspired by the form of the initial data, find a good variable transformations, which promotes this problem.

Solution (Ideas: Trial and Errors)**.** *Let* $u(x,y,z,t) = f(x,y)v(z,t) \leftarrow$ *guess this form of the solution according to its data.*

$$u_{xx} + u_{yy} + u_{zz} = f_{xx}v + f_{yy}v + fv_{zz} = \underbrace{(f_{xx} + f_{yy})}_{=0} v + fv_{zz} = fv_{zz}$$

This leads to

$$u_{tt} = a^2 f v_{tt}$$

Thus, the original problem transforms into

$$\begin{cases} v_{tt} - a^2 v_{zz} = 0 \\ v(z,0) = g(z), \quad v_t|_{t=0} = 0 \end{cases}$$

The solution can be obtained using the one-dimensional wave equation D'Alembert's formula.

$$v = \frac{1}{2}\left[g(z+at) + g(z-at)\right].$$

$$\Rightarrow u(x,y,z,t) = f(x,y)v(z,t) = \frac{1}{2}f(x,y)\left[g(z+at) + g(z-at)\right].$$

Ex 1.9. *Solve the initial value problem.*

$$\begin{cases} u_{tt} - 8(u_{xx} + u_{yy} + u_{zz}) = t^2x^2, \quad (x,y,z) \in \mathbb{R}^3,\, t > 0 \\ u|_{t=0} = y^2, \quad u_t|_{t=0} = z^2 \end{cases}$$

- Note these non-homogeneous terms are all just related to x, y and z, respectively. That is, they only involve one variable.
- Isolate the non-homogeneities.

Solution (Method of descent)**.** *Transform it into three one-dimensional problems, linearly superimpose them to solve the problem. By the principle of superposition, let*

$$u(x,y,z,t) = q(x,t) + p(y,t) + w(z,t)$$

Divide into three problems:

$$\begin{cases} q_{tt} - 8q_{xx} = t^2x^2 \\ q|_{t=0} = 0, \qquad q_t|_{t=0} = 0 \end{cases}$$

$$\begin{cases} p_{tt} - 8p_{yy} = 0 \\ p|_{t=0} = y^2, \qquad p_t|_{t=0} = 0 \end{cases}$$

$$\begin{cases} w_{tt} - 8w_{zz} = 0 \\ w|_{t=0} = 0, \qquad w_t|_{t=0} = z^3 \end{cases}$$

Using D'Alembert's formula, we get:

$$q(x,t) = \frac{1}{12a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \tau^2\xi^2\, d\xi\, d\tau = \frac{1}{12}t^4x^2 + \frac{2}{45}t^6$$

$$p(y,t) = \frac{1}{2}\left[(y+at)^2 + (y-at)^2\right] = y^2 + 8t^2$$

$$w(z,t) = \frac{1}{2a}\int_{z-at}^{z+at} \xi^2\, d\xi = tz^2 + \frac{8}{3}t^3$$

$$\Rightarrow u(x,y,z) = y^2 + 8t^2 + tz^2 + \frac{8}{3}t^3 + \frac{1}{12}t^4x^2 + \frac{2}{45}t^6$$

Ex 1.10. *Solve the initial value problem*

$$\begin{cases} u_{tt} - (u_{x_1x_1} + u_{x_2x_2}) = t\sin x_2, \quad (x_1,x_2) \in \mathbb{R}^2,\, t > 0. \\ u|_{t=0} = x_1^2, \quad u_t|_{t=0} = \sin x_2. \end{cases}$$

Solution. *Use the same method as the previous problem. Let* $u(x_1, x_2, t) = v(x_1, t) + w(x_2, t)$*, by superposition, the problem transforms into two one-dimensional problems. It can be verified that* $u_{x_1x_1} = v_{x_1x_1}$ *and* $u_{x_2x_2} = w_{x_2x_2}$*. (I)*

$$\begin{cases} v_{tt} - v_{x_1x_1} = 0 \\ v|_{t=0} = x_1^2, \quad v_t|_{t=0} = 0 \end{cases}$$

and (II)

$$\begin{cases} w_{tt} - w_{x_2x_2} = t \sin x_2 \\ w|_{t=0} = 0, \quad w_t|_{t=0} = \sin x_2 \end{cases}$$

Using D'Alembert's formula:

$$v = \frac{1}{2}\left[(x_1 - at)^2 + (x_1 + at)^2\right] = x_1^2 + t^2$$

$$w = \frac{1}{2}\int_{x_2-at}^{x_2+at} \sin\xi \, d\xi + \frac{1}{2}\int_0^t \int_{x_2-a(t-\tau)}^{x_2+a(t-\tau)} \tau \sin\xi \, d\xi \, d\tau = t \sin x_2$$

$$\Rightarrow u(x_1, x_2, t) = v + w = x_1^2 + t^2 + t \sin x_2$$

Ex 1.11. *Solve the initial value problem.*

$$\begin{cases} u_{tt} - a^2(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) = 0 \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \, t > 0 \\ u|_{t=0} = f(x_1) + g(x_2) \\ u_t|_{t=0} = \varphi(x_2) + \psi(x_3) \end{cases}$$

Solution. *Let* $u(x_1, x_2, x_2, t) = p(x_1, t) + q(x_2, t) + w(x_3, t)$*. The original problem can be transformed into three one-dimensional wave equations.*

(I)

$$\begin{cases} p_{tt} - a^2 p_{x_1x_1} = 0 \\ p|_{t=0} = f(x_1) \\ p_t|_{t=0} = 0 \end{cases}$$

(II)

$$\begin{cases} q_{tt} - a^2 q_{x_2x_2} = 0 \\ q|_{t=0} = g(x_2) \\ q_t|_{t=0} = \varphi(x_2) \end{cases}$$

(III)

$$\begin{cases} w_{tt} - a^2 w_{x_3x_3} = 0 \\ w|_{t=0} = 0 \\ w_t|_{t=0} = \psi(x_3) \end{cases}$$

Using D'Alembert's formula, we obtain:

$$\begin{aligned} p =& \frac{1}{2}\left[f(x_1 + at) + f(x_1 - at)\right] \\ q =& \frac{1}{2}\left[g(x_2 + at) + g(x_2 - at)\right] + \frac{1}{2a}\int_{x_2-at}^{x_2+at} \varphi(\xi) \, d\xi \\ w =& \frac{1}{2a}\int_{x_3-at}^{x_3+at} \psi(\xi) \, d\xi \end{aligned}$$

$$\begin{aligned} \Rightarrow u(x_1, x_2, x_3, t) =& p + q + w \\ =& \frac{1}{2}\left[f(x_1 + at) + f(x_1 - at) + g(x_2 + at) + g(x_2 - at)\right] \\ &+ \frac{1}{2a}\left[\int_{x_2-at}^{x_2+at} \varphi(\xi) \, d\xi + \int_{x_3-at}^{x_3+at} \psi(\xi) \, d\xi\right] \end{aligned}$$

Ex 1.12. *Solve the initial value problem.*

$$\begin{cases} u_{tt} - a^2(u_{x_1x_1} + u_{x_2x_2}) = \underbrace{c^2u}_{damping}, & (x_1, x_2) \in \mathbb{R}^2,\ t > 0 \\ u|_{t=0} = \varphi(x_1, x_2), \quad u_t|_{t=0} = \psi(x_1, x_2) \end{cases}$$

where c is a constant.

- **Ideas:** Introduce the third variable x_3, hoping that the damping c^2u is the second derivative of some new function with respect to x^3.
- **Tools:** Exponential functions help finish the above idea.
- **Strategy: Hope** find a function $w(x_1, x_2, x_3, t)$ such that:

  $$c^2u \sim a^2w_{x_3x_3}, \quad u_{tt} \sim w_{tt}$$

  $$u_{x_1x_1} \sim w_{x_1x_1}, \quad u_{x_2x_2} \sim w_{x_2x_2}$$

  Try a variable separable form $w(x_1, x_2, x_3, t) = f(x_3)u(x_1, x_2, t)$. Then we need $a^2w_{x_3x_3} = a^2f''(x_3)u = c^2uf$

  Thus, we can write:

  $$w_{tt} = fu_{tt}, \quad w_{x_1x_1} = fu_{x_1x_1}, \quad w_{x_2x_2} = fu_{x_2x_2}$$

  $$\Rightarrow \underbrace{fu_{tt}}_{w_{tt}} - a^2\underbrace{(fu_{x_1x_1} + fu_{x_2x_2})}_{w_{x_1x_1}+w_{x_2x_2}} = \underbrace{c^2uf}_{a^2w_{x_3x_3}=a^2f''(x_3)u}$$

  If $c^2f = a^2f''$, then the equation can be simplified as:

  $$w_{tt} - a^2(w_{x_1x_1} + w_{x_2x_2} + w_{x_3x_3}) = 0$$

  In order to find f, we solve this equation $c^2f = a^2f''$. Then $f = e^{\frac{c}{a}x_3}$ (any solution will work)
- In fact, If one have good intuition, and use the exponential function is invariant for derivations, it is easy to guess the for of w. Organize the above ideas to get the solution.

Solution (Method of Dimensional Lifting). *Let $w(x_1, x_2, x_3, t) = e^{\frac{c}{a}x_3}u(x_1, x_2, t)$, then w satisfies a three-dimensional wave equation. The initial value problem for w (where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$) is:*

$$\begin{cases} w_{tt} - a^2\Delta w = 0 \\ w|_{t=0} = e^{\frac{c}{a}x_3}\varphi(x_1, x_2), \quad w_t|_{t=0} = e^{\frac{c}{a}x_3}\psi(x_1, x_2) \end{cases}$$

By the three-dimensional Poisson formula (Kirchhoff formula):

$$\begin{aligned} w(x_1, x_2, x_3, t) = & \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{\frac{c}{a}(x_3+at\cos\theta)}\varphi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi \\ & + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{\frac{c}{a}(x_3+at\cos\theta)}\psi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi. \end{aligned}$$

$$\begin{aligned} \Rightarrow u(x_1, x_2, t) = & e^{-\frac{c}{a}x_3}w(x_1, x_2, x_3, t) \\ = & \frac{1}{4\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi} te^{ct\cos\theta}\varphi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi \\ & + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ct\cos\theta}\psi(x_1 + at\sin\theta\cos\phi, x_2 + at\sin\theta\sin\phi)\sin\theta\, d\theta\, d\phi. \end{aligned}$$

Ex 1.13. *Solve the following initial value problem using the method of characteristics:*

$$\begin{cases} u_{tt} + 2u_t = u_{xx} + 2u_x, & -\infty < x < +\infty, t > 0, \\ u(x,0) = e^{-x}\cos x, & u_t(x,0) = e^{-x}(\sin x - \cos x). \end{cases}$$

[Hint: Choose appropriate constants α and β as unknown function substitutions $v(x,t) = e^{\alpha t+\beta x}u(x,t)$ to transform the equation into a homogeneous string vibration equation.]

- When **derivatives of adjacent orders appear**, an integrating factor can be used to consolidate multiple terms into a single expression.
- The primary tools employed are the **integrating factor** and the **binomial theorem** for higher-order derivatives.
- (Method 1) By directly comparing with the binomial theorem, it is evident that we should multiply by $e^t e^x$.

$$\begin{array}{rccl} \text{The coefficients of } t\text{-derivatives terms} & 1 & 2 & 1 \leftarrow (e^t u)_{tt} \cdot e^x \\ \text{The coefficients of } x\text{-derivatives terms} & 1 & 2 & 1 \leftarrow (e^x u)_{xx} \cdot e^t \end{array}$$

$$\text{Let } (\underbrace{e^{t+x}u}_{v})_{tt} = (\underbrace{e^{x+t}u}_{v})_{xx}$$

- (Method 2) Assume $u = e^{\alpha t+\beta x}v$

$$u_{tt} = \alpha^2 e^{\alpha t+\beta x}v + 2\alpha e^{\alpha t+\beta x}u_t + e^{\alpha t+\beta x}u_{tt}$$

$$u_{xx} = \beta^2 e^{\alpha t+\beta x}v + 2\beta e^{\alpha t+\beta x}u_x + e^{\alpha t+\beta x}u_{xx}$$

$$2u_t = 2\alpha e^{\alpha t+\beta x}v + 2e^{\alpha t+\beta x}u_t$$

$$2u_x = 2\beta e^{\alpha t+\beta x}v + 2e^{\alpha t+\beta x}u_x$$

  Substitute into the original equation to get $u_t = u_x$, α and β should be determined.

Solution. *Let $v(x,t) = e^{t+x}u(x,t)$, then v satisfies*

$$\begin{cases} v_{tt} = v_{xx}, & -\infty < x < +\infty, t > 0, \\ v(x,0) = \cos x, & u_t(x,0) = \sin x. \end{cases}$$

Using D'Alembert's formula to solve, we get

$$v(x,t) = \frac{1}{2}[\cos(x-t) + \cos(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin\alpha\, d\alpha = \cos(x-t).$$

Ex 1.14. *Solve the following initial value problem using the method of traveling waves:*

$$\begin{cases} u_{tt} + 2u_t = u_{xx} + 4u_x + 3u, & -\infty < x < +\infty, t > 0, \\ u(x,0) = e^{-2x}\cos x, & u_t(x,0) = e^{-2x}(\sin x - \cos x). \end{cases}$$

[Hint: Choose appropriate constants α and β as unknown function substitutions $v(x,t) = e^{\alpha t+\beta x}u(x,t)$ to transform the equation into a homogeneous string vibration equation.]

$$\begin{array}{rcccl} \text{The coefficients of } t\text{-derivatives terms} & 1 & 2 & 1 & \leftarrow (e^t u)_{tt} \cdot e^{\alpha x} \\ \text{The coefficients of } x\text{-derivatives terms} & 1 & \underbrace{2\alpha}_{4} & \underbrace{\alpha^2}_{4} & \leftarrow (e^{\alpha x} u)_{xx} \cdot e^t \end{array}$$

$$\text{Then } \alpha = 2, \text{ Let } (\underbrace{e^{t+2x}u}_{v})_{tt} = (\underbrace{e^{2x+t}u}_{v})_{xx}$$

Solution. *Let $v(x,t) = e^{t+2x}u(x,t)$, then v satisfies*

$$\begin{cases} v_{tt} = v_{xx}, & -\infty < x < +\infty, t > 0, \\ v(x,0) = \cos x, & u_t(x,0) = \sin x. \end{cases}$$

Using D'Alembert's formula to solve, we get

$$v(x,t) = \frac{1}{2}[\cos(x-t) + \cos(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin\alpha \, d\alpha = \cos(x-t).$$

Thus,

$$u(x,t) = e^{-t-2x}\cos(x-t).$$

Ex 1.15. *Use the method of traveling waves to solve the following unbounded vibration problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx}, & -\infty < x < +\infty, \\ u(0,t) = t^2 + 3, & u_t|_{x=at} = 2x. \end{cases}$$

where a is a positive constant.

Solution. *The general solution of the wave equation is*

$$u(x,t) = f(x-at) + g(x+at)$$

Substitute the initial values

$$-af'(0) + ag'(2x) = 2x, \quad f(-at) + g(at) = t^2 + 3,$$

we get

$$ag'(x) = x + af'(0), \quad g(x) = \frac{x^2}{2a} + f'(0)x + c$$

Thus,

$$f(x) = \left(\frac{1}{a^2} - \frac{1}{2a}\right)x^2 + f'(0)x + 3 - c.$$

Therefore,

$$\begin{aligned} u(x,t) =& f(x-at) + g(x+at) \\ =& \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x-at)^2 + f'(0)(x-at) + 3 - c + \frac{(x+at)^2}{2a} + f'(0)(x+at) + c \\ =& \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x-at)^2 + \frac{(x+at)^2}{2a} + 2f'(0)x + 3. \end{aligned}$$

From $u_t(0,0) = 0$, we get $f'(0) = 0$. The solution to the equation is

$$u(x,t) = \left(\frac{1}{a^2} - \frac{1}{2a}\right)(x-at)^2 + \frac{(x+at)^2}{2a} + 3.$$

Ex 1.16. *Solve the following initial value problem using the method of traveling waves:*

$$\begin{cases} u_{tt} = u_{xx} - 2, & -\infty < x < +\infty, t > 0, \\ u(x,0) = x^2, & u_t(x,0) = \sin x, \quad -\infty < x < +\infty. \end{cases}$$

Solution. *From D'Alembert's formula, we know*

$$\begin{aligned} u(x,t) =& \frac{1}{2}\left((x+t)^2 + (x-t)^2\right) + \frac{1}{2}\int_{x-t}^{x+t} \sin\xi \, d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} (-2) \, d\xi \, d\tau \\ =& x^2 + t^2 - \frac{1}{2}(\cos(x+t) - \cos(x-t)) - t^2 \\ =& x^2 + \sin x \sin t. \end{aligned}$$

Ex 1.17. *Solve the following initial value problem using the method of integral transforms:*

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, & -\infty < x < +\infty, t > 0, \\ u(x,0) = e^{-x^2}, & -\infty < x < +\infty. \end{cases}$$

(Hint: $F^{-1}(e^{-\lambda^2 t}) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ *)*

Solution. *Perform the Fourier transform with respect to x, let $U(\lambda, t) = F(u(x,t))$, then we have*

$$\begin{aligned} U_t(\lambda, t) &= -\frac{1}{4}\lambda^2 U(\lambda, t), \\ U(\lambda, 0) &= F(e^{-x^2}). \end{aligned}$$

Thus,

$$U(\lambda, t) = e^{-\frac{1}{4}\lambda^2 t} F(e^{-x^2}).$$

Perform the inverse Fourier transform to obtain

$$\begin{aligned} u(x,t) =& F^{-1}(e^{-\frac{1}{4}\lambda^2 t}) * e^{-x^2} \\ =& \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}} * e^{-x^2} \\ =& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2} e^{-\frac{\xi^2}{t}} d\xi \\ =& \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{1+t}} \exp\left(\sqrt{\frac{t}{1+t}}\xi - \sqrt{\frac{1+t}{t}}x\right)^2 d\xi \\ =& \sqrt{1+t} e^{-\frac{x^2}{1+t}}. \end{aligned}$$

Ex 1.18. *Solve the following initial value problem using the method of integral transforms:*

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, t > 0, \\ u(x,0) = 0, u_t(x,0) = 0, & 0 \le x < +\infty, \\ u(0,t) = e^{-t}\sin(\omega t), & t \ge 0, \\ \lim_{x\to+\infty} |u(x,t)| < +\infty, & t \ge 0. \end{cases}$$

(Hint: If $F(s) = L[f(t)]$, then $L^{-1}[F(s)e^{-sa}] = \begin{cases} f(t-a), & t > a, \\ 0, & t < a. \end{cases}$ *)*

Solution. *Perform the Laplace transform with respect to t, let $U(x,s) = L(u(x,t))$, then we have*

$$s^2 U(x,s) = a^2 U_{xx}(x,s).$$

The general solution of this ODE is

$$U(x,s) = c_1(s) e^{-\frac{s}{a}x} + c_2(s) e^{\frac{s}{a}x}.$$

From the condition $u(0,t) = e^{-t}\sin(\omega t)$, we get

$$U(0,s) = L(e^{-t}\sin(\omega t)).$$

And since $\lim_{x\to+\infty}|u(x,t)| < +\infty$, for any $s > 0$, there exists $M < +\infty$ such that

$$\sup_{x\in(0,\infty)} |U(x,s)| < M.$$

Thus,

$$U(x,s) = L(e^{-t}\sin(at)) e^{-\frac{s}{a}x}.$$

Perform the inverse Laplace transform to obtain

$$u(x,t) = \begin{cases} e^{-(t-\frac{x}{a})}\sin(\omega(t-\frac{x}{a})), & t > \frac{x}{a}, \\ 0, & t < \frac{x}{a}. \end{cases}$$

Ex 1.19. *Solve the following initial value problem using the method of integral transforms:*

$$\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < +\infty, t > 0, \\ u(0,t) = e^{-t} - 1, \lim_{x\to+\infty} |u(x,t)| < +\infty, & t \geq 0, \\ u(x,0) = 0, u_t(x,0) = 0, & 0 \leq x < +\infty. \end{cases}$$

Solution. *Let $U(x,s) = L(u(x,t))$, perform the Laplace transform with respect to t to obtain*

$$s^2 U(x,s) = a^2 U_{xx}(x,s).$$

Solve to get

$$U(x,s) = A(s)e^{\frac{s}{a}x} + B(s)e^{-\frac{s}{a}x}.$$

From $\lim_{x\to+\infty} |u(x,t)| < +\infty$, we know $A(s) \equiv 0$. From $u(0,t) = e^{-t} - 1$, we know $B(s) = L(e^{-t} - 1)$. Thus,

$$U(x,s) = L(e^{-t} - 1)e^{-\frac{s}{a}x}.$$

Perform the inverse Laplace transform and use the delay theorem to obtain

$$u(x,t) = \begin{cases} e^{-(t-\frac{x}{a})} - 1, & x \leq at, \\ 0, & x > at. \end{cases}$$