

Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 9-1

1 Exercise Class (Continued)

Ex 1.1. 1. Let $r = \sqrt{x^2 + y^2}$, and $r_0 < 1$ be a positive constant. Denote $\Omega = \{(x, y) | r_0 < r < 1\}$, which is an annulus in the two dimensional plane. Given $a > 0$, if the function $u(x, y)$ satisfies:

$$\begin{cases} \Delta u = 0, & r_0 < r < 1 \\ u|_{r=1} = 0, & u|_{r=r_0} \leq a \ln \frac{1}{r_0} \end{cases}$$

Prove by the maximum principle or comparison principle that:

$$u(x, y) \leq a \ln \frac{1}{r} \quad \text{for any } (x, y) \in \Omega$$

- The extremum principle cannot be directly applied, as it yields $u \leq a \ln \frac{1}{r_0}$ (where r_0 is a constant), instead of the desired $u \leq a \ln \frac{1}{r}$ (where r is the variable).
- The problem involves comparing two functions.
- When discussing the comparison principle, it was mentioned that the extremum principle only applies to a single function.
- To handle two functions, construct a new function by taking their difference, and then apply the extremum principle.
- Note $a \ln \frac{1}{r}$ is a fundamental solution in the two-dimensional case satisfying $\Delta a \ln \frac{1}{r} = -2\pi a \delta(r - r_0)$.

2. If $u(x, y)$ is continuous in $\{(x, y) | 0 < r \leq 1\}$ and satisfies:

$$\begin{cases} \Delta u = 0, & 0 < r < 1 \\ u|_{r=1} = 0, & \lim_{r \rightarrow 0} \frac{u(x, y)}{\ln r} = 0 \end{cases}$$

Prove that:

$$u(x, y) \equiv 0, \quad 0 < r \leq 1$$

Proof of 1. Denote $w = a \ln \frac{1}{r} - u$. Then w satisfies:

$$\begin{cases} \Delta w = 0, & r_0 < r < 1 \\ w|_{r=1} = 0, & w|_{r=r_0} \geq 0 \end{cases}$$

By the maximum principle, $w \geq 0$, that is $u(x, y) \leq a \ln \frac{1}{r}$ when $r_0 \leq r \leq 1$. □

- **Comparison:** Both questions appear similar, but the difference lies in the inner boundary conditions.

- **Inner Boundary:**

- In the first scenario, the inner radius r_0 is fixed.
- In the second scenario, the inner boundary radius r approaches zero.

- **Behavior at Zero:**

- As $r \rightarrow 0$, $\ln r \rightarrow -\infty$.
- The condition $r \rightarrow 0$ implies u approaches zero slower than $\ln r$.

- **Theorem Implication:**

- If a function approaches zero slower than $\ln r$ on the boundary, it must be zero.
- This is a **strong** property of harmonic functions.

- **Proof Strategy:**

- Transform the problem into a form similar to Question 1.
- Use the limit definition to express boundary conditions.

- **Boundary Behavior:**

- The limit definition only applies to boundary points.

Proof of 2. (1) For any given $M = (x_1, y_1) \neq (0, 0)$, denote $r_1 = \sqrt{x_1^2 + y_1^2} \in (0, 1)$ (see Fig. 1).

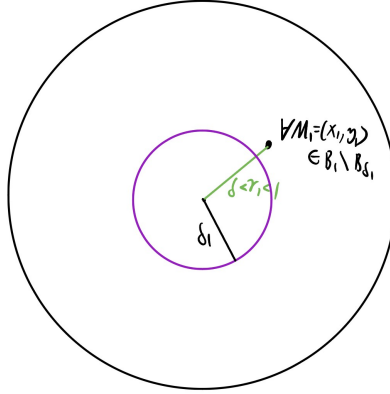


Figure 1: Example 1

- (2) Since $\lim_{r \rightarrow 0} \frac{u(x, y)}{\ln r} = 0$ (this condition can be viewed as the the inner boundary, i.e., r and (x, y) are on the boundary! a series boundary condition satisfies this limit), for any sufficient small $\epsilon > 0$, there exists a sufficient small constant $\delta \in (0, r_1)$ such that when

$$0 < r \text{ (the radius of the inner boundary)} < \delta < r_1,$$

we have $\left| \frac{u(x, y)}{\ln r} \right| \leq \epsilon$, which is equivalent to $|u(x, y)| \leq \epsilon |\ln r| = -\epsilon \ln r = \epsilon \ln \frac{1}{r}$ (since $\ln r < 0$ when $r < 1$). That is, there exists an inner boundary radius $r = \delta_1 < \delta$ such that $|u(x, y)| \leq \epsilon \ln \frac{1}{\delta_1}$ for $\sqrt{x^2 + y^2} = \delta_1$.

- (3) So we have

$$\begin{cases} \Delta u = 0, & \delta_1 < r < 1 \\ u|_{r=1} = 0, & u|_{r=\delta_1} \leq \epsilon \ln \frac{1}{\delta_1} \end{cases}$$

We can obtain, for any $r_1 > \delta_1$ (which means r_1 is in the annulus), $u(M) \leq \epsilon \ln \frac{1}{r_1}$ by 1. Similarly, we can get $u(M) \geq -\epsilon \ln \frac{1}{r_1}$, that is $|u(M)| \leq \epsilon \ln \frac{1}{r_1}$. Fix a r_1 in the annulus. Since ϵ can be

arbitrarily small and $\frac{1}{r_1}$ is fixed, by the squeeze theorem, letting $\epsilon \rightarrow 0$, we can get $u(M) = 0$ for any $M \neq (0, 0)$.

• **Transformation to Series of Problems:**

- The problem can be transformed into a series of boundary value problems: If u solves the original problem

$$\begin{cases} \Delta u = 0, & 0 < r < 1 \\ u|_{r=1} = 0, & \lim_{r \rightarrow 0} \frac{u(x,y)}{\ln r} = 0 \end{cases}.$$

It is equivalent to, for any sufficiently small $\epsilon > 0$, there is a constant $\delta_1 > 0$, such that u satisfies

$$\begin{cases} \Delta u = 0 \\ u|_{r=r_0} = 0, & |u(x,y)|_{r=\delta_1} \leq \epsilon \ln \frac{1}{\delta_1} \end{cases}$$

- Each problem in the series has a parameter δ_1 .
- Some students are confused about $-\epsilon \ln \frac{1}{r_1} < u(x_1, y_1) < \epsilon \ln \frac{1}{r_1}$, they misunderstand that $\epsilon \rightarrow 0$ leads to $r_1 \rightarrow 0$, thinking $\epsilon \ln \frac{1}{r_1}$ is an indeterminate form $0 \cdot \infty$. In fact, $\epsilon \rightarrow 0$ leads to $\delta_1 \rightarrow 0$, but r_1 is an arbitrary fixed value in the annulus greater than δ_1 , and r_1 does not necessarily change with δ_1 (it is fixed in the annulus).
- Therefore, we emphasize M is in the annulus and thus r_1 is fixed for ϵ .
- This is why we use the result of the first question. If we do not use the strong conclusion $u(x, y) < \epsilon \ln \frac{1}{r_1}$ from the first question and directly use the maximum principle, we can only get $u(x, y) < \epsilon \ln \frac{1}{\delta_1}$. This conclusion is obviously weaker than $u(x, y) < \epsilon \ln \frac{1}{r_1}$. We cannot deduce $u(x, y) = 0$ from $u(x, y) < \epsilon \ln \frac{1}{\delta_1}$ because $\epsilon \ln \frac{1}{\delta_1}$ is an indeterminate form as $\epsilon \rightarrow 0$ and $\delta_1 \rightarrow 0$. We need the strong result $u(x, y) < \epsilon \ln \frac{1}{r_1}$ to deduce $u(x, y) = 0$ since $\epsilon \ln \frac{1}{r_1} \rightarrow 0 \times (\text{a fixed and bounded value } \ln \frac{1}{r_1}) = 0$ since M ($r = r_1$) is fixed in the annulus.

□

Ex 1.2. It is known that a harmonic function on the plane satisfies the mean value formula:

$$u(M) = \frac{1}{2\pi r} \int_{S_r^M} u \, dS$$

where S_r^M is a circle centered at M with radius r .

1. Prove that the function $u(x, y) = e^y \cos x$ is a harmonic function.
2. Use the mean value formula to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\sin t} \cos(\cos t) dt = 1$$

3. Write the Green's function for the half plane $\{(x, y) \in \mathbb{R}^2 | x - y > 0\}$. (Hint: The symmetric point of (x, y) with respect to the line $y = x$ is (y, x))

Proof. Proof of 1: $u_{xx} + u_{yy} = -e^y \cos x + e^y \cos x = 0$

Proof of 2: By the mean value formula,

$$1 = u(0, 0) = \frac{1}{2\pi} \int_{S_1^0} u(x, y) dS = \frac{1}{2\pi} \int_{S_1^0} e^y \cos x dS$$

Let $x = r \cos t$, $y = r \sin t$, then

$$1 = \frac{1}{2\pi} \int_{S_1^0} e^y \cos x dS = \frac{1}{2\pi} \int_0^{2\pi} e^{\sin t} \cos(\cos t) dt$$

Solution of 3: Denote $M_0 = (x_0, y_0)$, $M_1 = (y_0, x_0)$, $M = (x, y)$, then

$$\begin{aligned} G(M, M_0) &= \frac{1}{2\pi} \left(\ln \frac{1}{r_{MM_0}} - \ln \frac{1}{r_{MM_1}} \right) \\ &= \frac{1}{2\pi} \left(\ln \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} - \ln \frac{1}{\sqrt{(x-y_0)^2 + (y-x_0)^2}} \right) \end{aligned}$$

□

Ex 1.3. 1. Let $u(x, y, z)$ be harmonic in the cube $\Omega = \{(x, y, z) : |x| < 1, |y| < 1, |z| < 1\}$, continuous on $\overline{\Omega}$, and satisfy $u(x, y, z)|_{\Gamma} = e^{-(x^2+y^2+z^2)}$ on the boundary Γ of Ω . Prove that $e^{-3} < u(x, y, z) < e^{-1}$ in Ω .

2. Let u be a smooth function in \mathbb{R}^3 . If $\Delta u \geq 0$, then u is called **subharmonic**. Prove that the following two propositions are equivalent:

(a) u is subharmonic in \mathbb{R}^3 .

(b) For any closed sphere Γ_r , $\iint_{\Gamma_r} \frac{\partial u}{\partial n} dS \geq 0$ holds, where n is the unit outward normal vector of Γ_r .

Proof. Proof of 1: The maximum value of u on the boundary is e^{-1} , and the minimum value is e^{-3} . Obviously, u is not a constant. According to the maximum principle, the maximum and minimum values of u can only be achieved on the boundary. Therefore, $e^{-3} < u(x, y, z) < e^{-1}$.

Proof of 2:

• “ \Rightarrow ” Let V_r be the sphere enclosed by Γ_r . In the first Green’s formula

$$\iiint_{V_r} v \Delta u \, d\Omega = \iint_{\Gamma_r} v \frac{\partial u}{\partial n} dS - \iiint_{V_r} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial v} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial v} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} \right) d\Omega$$

Take $v \equiv 1$, then

$$\iint_{\Gamma_r} \frac{\partial u}{\partial n} dS = \iiint_{V_r} \Delta u \, d\Omega \geq 0$$

• “ \Leftarrow ” Arbitrarily take a sphere V_r and denote its boundary as Γ_r . By the first Green’s formula,

$$\iiint_{V_r} \Delta u \, d\Omega = \iint_{\Gamma_r} \frac{\partial u}{\partial n} dS \geq 0$$

Due to the arbitrariness of V_r and the continuity of u , we know that $\Delta u \geq 0$.

Note: Gauss’s formula or the second Green’s formula can also be used. □

Ex 1.4. 1. State the mean value theorem for harmonic functions.

2. If u and v are harmonic in $\Omega = \{(x, y), x^2 + y^2 < 1\}$, and satisfy

$$u|_{\Gamma} = |xy|, \quad v|_{\Gamma} = x^3y - xy^3 \leftarrow \boxed{\text{One can change these two functions to obtain new exercises}}$$

on the boundary Γ of Ω . Prove that $u(x, y) > v(x, y)$ for any $(x, y) \in \Omega$.

Proof. Solution of 1: Let the function $u(M)$ be harmonic in the region Ω , and M_0 be any point in Ω . If Γ_a is a sphere centered at M_0 with radius a , and this sphere is completely inside the region Ω , then

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\Gamma_a} u \, dS.$$

Proof of 2: Let $w = u - v$, then w is harmonic in Ω and on the boundary

$$w|_{\Gamma} = |xy| - x^3y + xy^3 \geq 0.$$

And obviously w is not a constant. By the maximum principle, $w > 0$ in Ω , so $u > v$. □

Ex 1.5. Denote $B_R(M_0)$ as the sphere in \mathbb{R}^3 centered at M_0 with radius R , and its boundary is denoted as $\Gamma_R(M_0)$.

1. If u is harmonic in $B_R(M_0)$, continuous on the boundary $\Gamma_R(M_0)$ and $u \geq 0$, let $M \in B_R(M_0)$, and denote $r = |M - M_0| < R$. Using the Poisson formula for a spherical domain

$$u(M) = \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(P)(R^2 - r^2)}{(R^2 + r^2 - 2Rr \cos \gamma)^{3/2}} dS$$

prove the following Harnack's inequality:

$$\frac{R(R-r)}{(R+r)^2} u(M_0) \leq u(M) \leq \frac{R(R+r)}{(R-r)^2} u(M_0)$$

2. Prove Liouville's theorem: If $u \geq 0$ is a harmonic function in \mathbb{R}^3 , then u is a constant.

Proof. Proof of 1:

$$u(M_0) = \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS$$

Since $u \geq 0$ (**Ensure the direction of the inequality sign**) and $(R-r)^2 \leq R^2 + r^2 - 2Rr \cos \gamma \leq (R+r)^2$, we have

$$\begin{aligned} u(M) &\leq \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(y)(R^2 - r^2)}{(R-r)^3} dS = \frac{R(R+r)}{(R-r)^2} \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS = \frac{R(R+r)}{(R-r)^2} u(M_0) \\ u(M) &\geq \frac{1}{4\pi R} \iint_{\Gamma_R(M_0)} \frac{u(y)(R^2 - r^2)}{(R+r)^3} dS = \frac{R(R-r)}{(R+r)^2} \frac{1}{4\pi R^2} \iint_{\Gamma_R(M_0)} u(y) dS = \frac{R(R-r)}{(R+r)^2} u(M_0) \end{aligned}$$

Proof of 2: Without loss of generality, assume that u has a lower bound (if u has an upper bound, then consider the function $-u$), that is, there exists $M \in \mathbb{R}^3$ such that $u \geq M$. Let $v = u - M$, then $v \geq 0$, and v is a harmonic function in \mathbb{R}^3 . For any $x \in \mathbb{R}^3$, $R > r = |x|$, by Harnack's inequality,

$$\frac{R(R-r)}{(R+r)^2} v(O) \leq v(M) \leq \frac{R(R+r)}{(R-r)^2} v(O).$$

Let $R \rightarrow +\infty$, then $v(M) = v(O)$. Due to the arbitrariness of x , v is a constant, and thus u is a constant. \square

Ex 1.6. Use the method of images to find the Green's function for the first quadrant $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$.

Proof. Omitted. See, for instance, [1]. \square

Chapter 5 Bessel Functions

- **Bessel Functions (Chapter 5):** Also known as **cylindrical functions**, primarily used for studying the **two-dimensional Laplace operator**.
- **Importance of Bessel Functions:**
 - Common in electrical engineering, especially in designing circuit boards with cylindrical capacitors.
- **Special Functions and PDEs:**
 - **Special functions** are introduced to solve **partial differential equations (PDEs)**.
 - Bessel functions are directly linked to the **method of separation of variables**.

- **Separation of Variables Method (SVM for short):**

- Used in solving PDEs, involves solving ordinary differential equations (ODEs) that arise in the process.
- In higher dimensions, new types of ODEs appear, which are solved using **Bessel functions**.

- **Bessel Equations and Functions—Revise the third step (S-L problem) of SVM:**

- The solutions to these new ODEs (**Bessel equations**) are Bessel functions.
- **Bessel functions** play a role similar to **trigonometric functions** in Chapter 2.

- **Initial Value Problems and Coefficients—Revise the fifth step of SVM:**

- After introducing Bessel functions, further discussion is needed on how they are applied in solving initial value problems and determining coefficients.

Motivation and Problem Statement:

1. (Old) Due to limitations of the separation of variables method in Chapter 2 ($1 + 1 = 2$ dimensions), for higher dimensions, special functions need to be introduced.
2. (New) In practical applications, problems such as:
 - 3-dimensional Newtonian gravitational potential (Poisson's equation) or the distribution of electric potential
 - Electromagnetic waves in cylindrical waveguides
 - Vibration phenomena of drum membranes or speaker diaphragms in headphones
 - Heat conduction phenomena in cylindrical objects
3. **Nature of the problem:** Extension from low dimensional to high dimensional cases.

New Problems and Solutions:

- **Analogous extension:** Try to extend the separation of variables method for $1 + 1 = 2$ dimensions and see where problems occur, and attempt to remedy them.
- **Problems encountered in the analogy:**
 - The number of variables increases.
 - When separating variables successively, multiple ordinary differential equations (ODEs) will appear, and new Sturm-Liouville (S-L) problems will be encountered (the original separation of variables method has problems in solving S-L problems).
- **Solution:** Study and solve the new S-L problems for ODEs, replace the solution of S-L problems in the separation of variables method, obtain a new system of eigenfunctions, and study their properties to determine coefficients.

Review and outline of this Chapter:

- **Chapter Outline:** Understanding the structure of this chapter is crucial for grasping the content of Sections 5.1–5.3.

- **Chapter Basis:** Builds upon the five-step separation of variables method.
- **Unchanged Steps:** The overall steps of the separation of variables method remain the same.
- **Modification Goals (Sections 5.1-5.3):** Aim to adjust parts of the steps to facilitate solving higher-dimensional cases.
- **Specific Steps to Modify:** Only the third and fifth steps of the separation of variables method need alteration.
- **Basic steps of the separation of variables method**
 1. First, assume $u(x, t) = X(x)T(t)$.
 2. Transform the partial differential equation (PDE) into ordinary differential equations (ODEs).
 3. **(Revised) Solve the ODEs: Use the boundary conditions to solve the Sturm-Liouville (S-L) problem;** substitute the obtained eigenvalues λ_n into the $T(t)$ equation.
 - **Bessel Equation Emergence:** A Bessel equation, an ordinary differential equation (ODE), appears as a Sturm-Liouville (S-L) problem.
 - **Section 5.1:** Introduces Bessel equations and their solutions, known as Bessel functions, to solve the **new S-L Problem** (see Table 1).
 - **Solution Role:** The series of Bessel functions serve a role analogous to the trigonometric functions series, acting as eigenfunctions in the solution process.
 4. Superpose the series solution $u(x, t)$.
 5. **(Revised) Determine the coefficients a_n, b_n from the initial values.**
 - **Coefficient Determination (Step 5):** Involves using inner products to find coefficients, previously used with trigonometric orthogonality.
 - **Zeros, Orthogonality, Norms and completeness (leading to the formula of the coefficients, Section 5.3):** For Bessel functions, orthogonality and norms must be established, which is the focus of Section 5.3.
 - **Recurrence Relations (Section 5.2):** Teaches how to calculate integrals involving Bessel functions in the formula of the coefficients, essential for determining coefficients. That is, by employing recurrence relations, the complexity of integrating higher-order Bessel functions is systematically reduced to more manageable, lower-order integrals.
 - * **Integration and Differentiation Relationship:** Integration and differentiation are inverse processes.
 - * **Use of Trigonometric Derivatives in Integration:** When integrating trigonometric functions, we often use their derivatives, such as $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\sin x) = \cos x$, which are crucial for integration by parts.
 - * **Recurrence Relations in Trigonometry:** These derivatives are a form of recurrence relations, showing how derivatives of trigonometric functions cycle back to the original function with a sign change after two derivatives.
 - * **Recurrence Relations in Bessel Functions:** Bessel functions also exhibit recurrence relations, where the derivative of a Bessel function relates to other Bessel functions of different orders.
 - * **Application in Integration:** These recurrence relations are useful in integration, allowing us to transform integrals involving Bessel functions into more manageable forms.
 - * **Integral Calculation Using Recurrence Relations:** By understanding how derivatives of Bessel functions relate to each other, we can simplify the process of integrating these functions.
- **Connection Between Sections:**

- **Chapter Overview:** This chapter focuses on modifying steps in the separation of variables method for solving higher-dimensional PDEs.
 - **Sections 5.1-5.3:** These sections explain how to adjust steps 3 and 5 of the separation of variables method. Sections 5.1, 5.2, and 5.3 are interconnected, with 5.1 solving for Bessel functions, 5.2 for integrals, and 5.3 for orthogonality.
 - **Application to PDEs:** Section 5.4 integrates the knowledge from the first three sections to solve PDEs, using the modified five-step method, including non-homogeneous equations and non-homogeneous boundary conditions.
 - **Auxiliary Function Method:** Not covered in detail in lectures but included in exercises and homework; involves finding a function to handle boundary conditions, reducing the problem to separation of variables or eigenfunction methods.
 - **Eigenfunction Method:** Covered with an example; slightly more complex than previously studied due to non-homogeneous conditions.
- Review the process of the separation of variables method for Laplace’s equation in a circular domain and its system of eigenfunctions: $1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta, \dots, \cos n\theta, \sin n\theta, \dots$
 - **Methodology for Special Functions:**
 - **Analogous Approach:** The methodology for studying other special functions is similar to that used for Bessel functions.
 - **Applicability of Chapter’s Framework:** The approach outlined in this chapter can be similarly applied to the study of other special functions.

Chapter 2	Chapter 5
S-L problem: $X'' + \lambda X = 0$ with homogeneous boundary	S-L problem: Bessel equations with $F(R) = 0$ (Step 3; §5.1)
Eigenfunctions (basis): {trigonometric functions}	Eigenfunctions (basis): {Bessel functions} (Step 3; §5.1)
Infinite but countable eigenvalues; orthogonality; Fourier expansions	Infinite but countable eigenvalues; orthogonality; Fourier-Bessel expansions (Step 5; §5.3)
How to calculate the integral $\int f \sin$ to get the Fourier coefficient	How to calculate the integral $\int f \times \text{Bessel function}$ to get the Fourier-Bessel coefficient (Step 5; §5.2)

Table 1: Comparison of Bessel functions and trigonometric functions

When using the **separation of variables method** to solve the boundary value problems of other partial differential equations, boundary value problems of other forms of ordinary differential equations will also be derived, thus introducing various systems of **coordinate functions**. These systems of **coordinate functions** are what people usually call **special functions**. In this chapter, we will derive the Bessel equation through separation of variables for boundary value problems in the **cylindrical coordinate system**; then discuss the solution method of this equation and the properties of its solutions; finally, introduce some applications of Bessel functions in solving problems of mathematical physics equations.

2 Bessel Equation and Bessel Functions

2.1 Derivation of the Bessel Equation

When using the **separation of variables method** to solve the **vibration problem** of a **circular membrane** or the law of **instantaneous temperature distribution** on a thin **circular disk**, we will encounter the **Bessel equation**. Below, we will derive the Bessel equation taking the instantaneous temperature distribution of a circular disk as an example.

Consider a thin circular disk of radius R with adiabatic upper and lower surfaces. The temperature on the boundary of the disk is always kept at 0 degrees, and the initial temperature distribution is known. We want to find the law of instantaneous temperature distribution inside the disk. Let $u(x, y, t)$ represent the temperature at point (x, y) on the disk at time t .

This problem is reduced to solving the following boundary value problem:

$$u_t = a^2(u_{xx} + u_{yy}) \quad (0 < x^2 + y^2 < R^2) \quad (1)$$

$$u|_{x^2+y^2=R^2} = 0 \quad (2)$$

$$u|_{t=0} = \varphi(x, y) \quad (3)$$

Use the **separation of variables method** to solve this problem. Let $u(x, y, t) = V(x, y)T(t)$ and substitute it into equation (1), we get

$$VT' = a^2(V_{xx} + V_{yy})T.$$

Multiply both sides by $\frac{1}{a^2VT}$, we obtain

$$\frac{T'}{a^2T} = \frac{V_{xx} + V_{yy}}{V} = -\lambda, \quad (\lambda > 0) \leftarrow \boxed{\text{Why? Explain later.}}$$

Then we have

$$T' + \lambda a^2 T = 0 \quad (4)$$

$$V_{xx} + V_{yy} + \lambda V = 0 \leftarrow \boxed{\text{Helmholtz equation } \Delta V + \lambda V = 0} \quad (5)$$

Equation (4) is the **Helmholtz equation**.

- **General n -Dimensional Form:** The n -Dimensional Helmholtz equation is generally written as $\Delta V + \lambda V = 0$.
- **One-Dimensional Case:** For $n = 1$, the Helmholtz equation reduces to the familiar one-dimensional form $V'' + \lambda V = 0$, where $\lambda > 0$ has been proven using various methods. For instance, by **classification of λ** :
 - For $\lambda < 0$, the solution is a linear combination of exponentials.
 - For $\lambda = 0$, the solution is linear, which does not satisfy non-trivial boundary conditions.
 - Therefore, λ must be greater than zero.
- **Proof of $\lambda > 0$:** We have demonstrated that $\lambda > 0$ using various methods for one-dimensional case, with the **most robust method** being the **energy method**.
- **Proof Without General Solution:** In higher dimensions without a general solution formula, we use the energy method to prove $\lambda > 0$.
- **Energy Method:**
 - **Multiply** the equation by V
 - **Integrate** over the domain.
 - Apply **integration by parts** (Green's first identity in higher dimensions).
 - The **boundary terms** imply that λ must be positive for non-trivial solutions.

The solution of equation (4) is

$$T(t) = Ae^{-\lambda a^2 t}$$

From the boundary condition (2), we have

$$T(t)V(x, y)|_{x^2+y^2=R^2} = 0$$

That is

$$V|_{x^2+y^2=R^2} = 0. \quad (6)$$

To find the non zero solutions of equation (5) that satisfy condition (6),

$$\begin{cases} V_{xx} + V_{yy} + \lambda V = 0 \\ V|_{x^2+y^2=R^2} = 0 \end{cases}$$

We use the polar coordinate system in the plane, and then the boundary value problem becomes

$$\frac{\partial^2 \bar{V}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{V}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{V}}{\partial \theta^2} + \lambda \bar{V} = 0 \quad (0 < r < R), \quad (7)$$

$$\bar{V}|_{r=R} = 0. \quad (8)$$

Why $\lambda > 0$?

Multiply equation (5) **by** V on both sides, we get

$$VV_{xx} + VV_{yy} + \lambda V^2 = 0$$

that is $V\Delta V + \lambda V^2 = 0$. Then **integrating** it yields

$$\iint_D (V\Delta V + \lambda V^2) d\sigma = 0,$$

where $D = \{(x, y) | x^2 + y^2 \leq R^2\}$ and $\partial D = \{(x, y) | x^2 + y^2 = R^2\}$. By **Gauss's formula** or **Green's first formula**, we have

$$\underbrace{\oint_{\partial D} V \frac{\partial V}{\partial n} ds}_{\text{Due to the boundary condition}} - \iint_D [|\nabla V|^2 - \lambda V^2] d\sigma = 0.$$

Then

$$\iint_D [|\nabla V|^2 - \lambda V^2] d\sigma = 0. \quad (9)$$

Then

$$\lambda = \frac{\iint_D |\nabla V|^2 d\sigma}{\iint_D V^2 d\sigma} \geq 0.$$

Thus, $\lambda \geq 0$ or $V \equiv 0$.

If $\lambda = 0$, then

- (Method 1) by (9), $\nabla V \equiv 0 \implies V \equiv \text{Constant}$. With the help of the boundary condition, $V \equiv 0$.
- (Method 2) the boundary value problem becomes $\Delta V = 0$. From the boundary condition $V|_{\partial D} = 0$, according to the **extreme value theorem**, $V \equiv 0$. So there is only the trivial solution when $\lambda = 0$.

Let $\bar{V}(r, \theta) = F(r)G(\theta)$ and substitute it into equation (7), we get

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' + \lambda FG = 0 \leftarrow \boxed{\text{If } \lambda = 0, \text{ it reduces to the Laplace equation (see Chapter 2)}}$$

- The following derivation, if $\lambda = 0$, reduces to the separation of variables method for the Laplace equation on a circular domain as discussed in Chapter 2, involving **boundedness conditions** at the center and **periodicity conditions**. We have highlighted the changes in **red** font.

Multiply both sides by $\frac{r^2}{FG}$ and rearrange the terms, we obtain (it is similar to the procedures of the Laplace equation, see Chapter 2)

$$\frac{G''}{G} = -\frac{r^2 F'' + rF' + \lambda r^2 F}{F} = -\mu \leftarrow \boxed{\text{taking minus sign since it leads to the familiar } G'' + \mu G = 0}$$

Then we have

$$G'' + \mu G = 0, \quad (10)$$

$$r^2 F'' + rF' + (\lambda r^2 - \mu)F = 0. \leftarrow \boxed{\text{If } \lambda = 0, \text{ it becomes \textbf{Euler equation}}}$$

Since the temperature function $u(x, y, t)$ is single valued, $V(x, y)$ must also be a single valued function, that is, $\bar{V}(r, \theta) = \bar{V}(r, \theta + 2\pi)$.

$$\implies G(\theta) = G(\theta + 2\pi) \leftarrow \boxed{\text{Periodic condition}}$$

Solving the boundary value problem of the ordinary differential equation

$$G'' + \mu G = 0, \quad G(\theta) = G(\theta + 2\pi)$$

we can obtain

$$\mu = n^2 \quad (n = 0, 1, 2, \dots)$$

$$G_0(\theta) = \frac{1}{2}a_0 \quad G_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \quad (n = 1, 2, \dots)$$

Substitute $\mu = n^2$ into equation (11), we get

$$\boxed{r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0,} \quad (12)$$

This equation is called the **n -th order Bessel equation**.

- The method for solving Euler's equation cannot be applied because $\lambda r^2 F$ is not well-behaved for $r = e^t$, and it cannot be transformed into an ordinary differential equation (ODE) with constant coefficients.

From the boundary condition (8) $\bar{V}|_{r=R} = 0$, we know that $\bar{V}(R, \theta) = F(R)G(\theta) = 0$.

$$\implies F(R) = 0$$

In addition, since the temperature on the disk is finite, especially at the center of the disk, we can obtain

$$|F(0)| < +\infty \leftarrow \boxed{\text{Boundness condition}}$$

Therefore, the final solution of the original boundary value problem is reduced to finding the eigenvalues and eigenfunctions of the problem

$$\begin{cases} r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0, \\ F(R) = 0, \quad |F(0)| < +\infty \end{cases} \leftarrow \boxed{\text{ODEs}}$$

- **Next Section Focus:** On solving ODEs using Power Series Method and Generalized Power Series Method.
- **Forgetting Physical Context:** From now on, we will focus solely on analyzing the Bessel equation, disregarding the physical background.
- **General Study of Bessel Equation:** We will study the general Bessel equation for **general values of n , not just integers**.

- **Eliminating the Parameter λ :** We will perform a transformation to simplify the equation by eliminating the parameter λ . Perform a variable rescaling to eliminate the parameter λ .

$$\underbrace{(\sqrt{\lambda})^2 r^2}_{x^2} \frac{d^2 F}{d(\underbrace{(\sqrt{\lambda})^2 r^2}_{x^2})} + \underbrace{(\sqrt{\lambda}) r}_x \frac{dF}{d(\underbrace{(\sqrt{\lambda}) r}_x)} + \underbrace{((\sqrt{\lambda})^2 r^2 - n^2)}_{x^2} F = 0$$

- **Importance of n :** n determines the **order** of the Bessel equation, which is crucial for different solutions.

If we let $x = \sqrt{\lambda}r$, and denote $F(r) = F\left(\frac{x}{\sqrt{\lambda}}\right) = y(x)$, then

$$F_r = y_x \cdot \sqrt{\lambda}, \quad F_{rr} = (y_{xx} \cdot \sqrt{\lambda}) \cdot \sqrt{\lambda} = y_{xx} \cdot \lambda$$

Substitute the above formulas into equation (12), we can get

$$x^2 y'' + xy' + (x^2 - n^2)y = 0. \quad (13)$$

Equation (13) is a second order linear ordinary differential equation with variable coefficients, and its solutions are called **Bessel functions**. (Sometimes they are called **cylindrical functions**).

References

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