Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 1-1

Course Arrangement

- **Office Hours**: Tentatively scheduled for Wednesday and Monday afternoons, 5th and 6th periods. Adjustments can be made based on feedback.
- **Homework**: Starting from the second week, about two exercises per week. Homework is due during the break between the two classes on Monday mornings.
- **Course Materials**: Slides, references, teaching plans, and homework problems (file name: "Mathematical Physics Equations Exercises, 9th Edition (Student Version) 2023") are available in the group files and course homepage.
- **Final Exam:** Final Grade = daily performance $\times 30\%$ + Final Exam $\times 70\%$.

Textbooks and References

- *Mathematical Physics Equations and Special Functions* (3rd Edition), Huazhong University of Science and Technology, Higher Education Press, 2019.
- *Mathematical Physics Equations and Special Functions*, Fang Ying, Huang Yi, Science Press, 2012.
- *Mathematical Physics Equations*, Jiang Yushan, Tsinghua University Press, 2014.
- *Partial Differential Equations and Boundary Value Problems with Fourier Series* (2nd Edition), Nakhle H. Asmar, Prentice Hall.
- *Methods of Mathematical Physics I, II*, Courant, Hilbert.
- *Mathematical Physics Methods*, Wu Chongshi.
- *Mathematical Physics Equations*, Gu Chaohao.
- *Lectures on Mathematical Physics Equations*, Jiang Lishang.
- *Mathematical Physics Methods*, Gu Qiao.

PDEs and ODEs

- Mathematical Physics Equations. A.K.A Partial Differential Eequation → PDE (several independent variable, e.g. $\partial_x^2\Phi(x,y)+\partial_y^2\Phi(x,y)=0$);
- Ordinary Differential Eequation → ODE (one independent variable, e.g. $y'(t)=f(t)$)
- PDEs "approximate", "translate" or even "precisely describe" the physical phenomenons.

Textbook

1. Chap 1: Derive the 3 types of PDEs (hyperbolic, parabolic and elliptic) and basic concepts;
2. Chap 2 – 4: Methods;
3. Chap 5: One special function–Bessel functions– tool for solving PDEs–methods can be applied to other special functions (see Chap 6).

Interesting Examples

The following two equations both describe the "gravity":

- General relativity–Einstein Equation ($G = T$)–(Quasilinear) Wave Equation ($\partial_t^2 g - \Delta g = T$)–gravitational wave ("like" water wave)!
- Newtonian gravity–Poisson Equation ($\Delta\Phi = \rho$)–Potential Equation–No wave (static distribution)!

Course Content Organization

- **By Equation Types**: Hyperbolic (wave equation), Elliptic (potential equation), Parabolic (heat equation).
- **By Methods**: Note the conditions under which each method applies.
- **Key feature of this Lecture:** Only Linear PDEs

Question 0.1. *Why is "linearity" required?*

A. *The linearity is extremely important in this course:*

- *The **derivations of equations** need this concept. Otherwise, the derivations are very confusing;*
- *The solving **methods** strongly **rely on the linear structures**.*
- *In many circumstances, **not far from the linear cases**, nonlinear problems can be **approximated** by the **linear** PDEs.*

Question 0.2. *What is the difference between the objects studied in Linear Algebra and Calculus?*

A.
- *Linear Algebra: **linear** mappings, **linear** transforms and **linear** functions (Represented by matrix...).*
- *Calculus: **nonlinear** (C^k-differntiable or integrable) functions and mappings.*

Question 0.3. *What is the relationship between the **linear objects** in Linear Algebra and the **nonlinear objects** in Calculus?*

A.
- *Core idea in calculus (LEGO idea): differetiation and integration;*
- *Differetiation (localization, disassemble): nonlinear objects locally are linear–local linearizations;*
- *Integrating (assemble) the local objects (differential elements) to return the nonlinear objects.*

Claim 0.1. *Once the problem is* ***locally linearized****, Linear Algebra steps in!*

Question 0.4. *What is the core theorem in Calculus?*

A.
- *Differentiation:* ***Taylor expansions****–disassemble a function to all order of small parts includiing the linear part. $f \in C^k(\mathbb{R})$, then the* ***nonlinear function***

$$f(x) = f(x_0) + \underbrace{f'(x_0)\Delta x}_{\text{linear part } (*)} + \underbrace{\frac{1}{2} f''(x_0)\Delta x^2}_{\text{Hessian}(**)} + \cdots$$

  - *The linear part $(*)$ can use linear algebra, and if $f : \mathbb{R}^n \to \mathbb{R}^m$, $f'(x_0)$ becomes an $m \times n$* ***gradient*** *matrix.*
  - *The second order $(**)$ is* ***Hessian*** *matrix–bilinear form and Quadratic form.*
  - *Under local approximations, Δx small enough, getting rid of the higher order terms, $f(x) - f(x_0) \sim f'(x_0)\Delta x$.*
- *Integration:* ***Newton-Leibniz theorem****–assemble the linear parts of a function to this function.*

- In the following derivation of the String Vibration Equation, we apply several approximations up to the linear terms (linearization);
- Alternatively, we could derive a fully nonlinear equation first, which is more complex, and then perform the linearization at the end.
- The derivation leads to some "ridiculous facts" due to the linear level. The true behavior is hidden in the higher-order terms.

1 Introduction (derive eqs, intro concepts, classify eqs and condi.– methods of solving eqs need this classifications)

1.1 String Vibration Equation and Definite Conditions

The string vibration equation was systematically studied by d'Alembert and others in the 18th century. It is a typical representative of a large class of partial differential equations.

1.1.1 Derivation of the String Vibration Equation

Question 1.1. *How to build a PDE?*

A 1.1. *Build a PDE $\Leftrightarrow$ Translate Physical phenomenon in terms of Math. language.*

1. ***Choice of Representation Quantities:*** *A representation quantity is a property you are interested in when describing a physical phenomenon. This could be a variable that captures how a certain property changes with respect to time and space.*

   Ex 1.1. *In the case of a vibrating string, the property you're interested in is the displacement from the equilibrium position. Thus, you choose the displacement as your representation quantity.*

   Ex 1.2. *In the case of heat conduction, the representation quantity would be temperature. For example, if you're considering a classroom where people emit heat, the temperature distribution in the room is what you want to study. In this case, the representation quantity is the temperature function, which describes how temperature varies spatially and temporally.*

2. ***Assumptions–Approximation and Simplification (Removing "Hair"):*** *In physics and mathematics, we often need to simplify models by removing extraneous details. This process of "removing the hair" refers to eliminating unnecessary or irrelevant properties to focus on the core elements of the system. The equations derived are thus approximations of real-world phenomena, rather than exact representations.*

   Ex 1.3. *When deriving the equations for a vibrating string, assumptions are made to simplify the model, such as neglecting air resistance or assuming a perfectly elastic string. These simplifications are part of the approximation process.*

3. ***Translation of Physical Laws into Mathematical Form*** *(math.-phys. bridge). The physical laws governing the system must be translated into mathematical equations.*

   Ex 1.4. *In the context of a vibrating string, the fundamental physical law we apply is* ***Newton's second law*** *of motion. For a vibrating string, you would apply Newton's second law to translate the physical phenomenon into a differential equation that describes the vibration.*

   Remark 1.1. *When applying Newton's second law in mechanics, remember it is primarily valid in the macroscopic world at low speeds. Furthermore, Newtonian mechanics applies to* ***point masses****, which is important to distinguish when studying different systems.*

   - *In mechanical systems, when dealing with* ***point masses****, you apply* ***Newton's second law****.*
   - *However, in other systems like* ***fields*** *(e.g., electromagnetic or gravitational fields), different types of mechanics apply.*

4. ***Local vs Global Viewpoint*** *(calculus involves, compatible with the choice of the physical laws). Since we are dealing with a differential equation, we need to apply differential thinking. The localization in this context refers to differentiation, and the integration corresponds to the process of globalizing the problem. Therefore, we should break the problem into small elements and differentiate them.*

   - *Localization (i.e., ideas of differentiation): differential equations reflect the* ***local properties****–require* ***localization****–take a local piece of the string, and study its properties.*
   - *Integration (easy global analysis for some cases, e.g., Gauss' flux theorem): since differential and integration are inverse operations, possible to use* ***global analysis*** *to derive* ***local equation*** *(i.e., PDE).*

   If you approach a problem from the perspective of fields, you're considering it as a global problem. However, when dealing with point mass mechanics, you must localize the problem. This localization is closely tied to differentiation. The reason Newton developed both classical mechanics and calculus was because they stemmed from the same idea. Thus, the mechanics of point masses naturally lead to differentiation.

   Ex 1.5. *For a vibrating string, you consider small segments of the string as point masses, and apply Newton's second law to each small piece to analyze the forces acting on it. This process involves localization.*

Remark 1.2. *Some further remarks on the "Local vs Global Viewpoint":*

- ***Physical Laws of Point Mechanics and Fields:*** *Some physical laws are inherently related to the mechanics of point masses, while others apply to the system as a whole. For instance, some physical laws describe the behavior of systems as a collective whole, rather than at individual points.*

- ***Energy and Global Laws:*** *Energy-based laws are examples of global physical laws. When analyzing such systems, you don't localize, but rather treat the system as a whole. For example, energy conservation can be analyzed globally without the need to focus on individual points.*

  Ex 1.6. *In the heat conduction problem, you consider the energy distribution across the entire system, which leads to the formulation of a global energy equation. The system is analyzed as a whole.*

- ***From Integral to Differential Equations:*** *When deriving equations such as the heat equation, the process starts with an integral formulation. Through integration, you obtain a global view. Then, by removing the integrals, you arrive at a differential equation. This process of going from integral to differential reflects a shift from a global to a local perspective.*

  Ex 1.7. *In deriving the heat equation, the heat transfer is initially considered as an integral over the system's volume. By simplifying, you eventually obtain a differential equation that describes the local temperature change over time and space.*

- ***Integral vs Differential Approaches:*** *When deriving equations, both integral and differential viewpoints can be used. You may start with a global perspective (integral form), and eventually convert to a local (differential) form. For example, in analyzing a vibrating string, you can either use the differential approach with Newton's second law, or the integral approach with the impulse theorem.*

- ***Approximation Process and Ideal Assumptions:*** *The equations we derive are always approximations of real-world phenomena. The process involves making idealized assumptions to simplify the system, which helps in deriving solvable equations.*

Consider a uniform, soft string of length L, fixed at both ends and stretched. Under external forces, the string undergoes small transverse vibrations near its equilibrium position. We **aim** to find the motion of each point on the string.

- **Choice of Representation Quantities:** The **displacement** from the equilibrium position.
- **Importance of Ideal Assumptions:** In order to simplify the equations and make them solvable, ideal assumptions are necessary. Without these assumptions, the number of variables and conditions will become too large, and the resulting equations would be highly complex, possibly unsolvable. Therefore, assumptions are made to reduce the complexity of the system, while still maintaining its most important characteristics.
- **Role of Simplifying Assumptions (ensure the linearity):** All assumptions made in the process are **aimed** at simplifying the problem into a linear equation with constant coefficients. While some assumptions may seem unreasonable or unrealistic, they help in achieving this simplified form, making the equations tractable.
- **Complex nonlinear equations are allowed:** If you don't simplify the problem and consider more factors, you can include more terms in the equation. However, as you reduce your assumptions, the resulting equation may become increasingly complex. Therefore, the equation we will derive here represents the simplest case. This does not mean it is the only possible formulation, but it is important to first understand this concept.

1.1.2 Assumptions

- The string is uniform, and its cross-sectional diameter is negligible compared to its length. The linear density is constant.
- The string is soft and does not resist bending[1]. The tension at each point is along the tangent direction, and the elongation follows **Hooke's law**[2].

[1] The biggest difference between a teaching stick and yarn, for example, lies in their resistance to deformation – one exerts a force that resists change in shape.

[2] That is, within the elastic limit, the deformation of an object is directly proportional to the external force that causes

- **Empirical formula**: Hooke's law is often treated as an empirical formula;
- Instead of considering it purely as empirical, we can interpret it as a fact derived from the first-order approximation (linear approximation), based on Taylor expansion. If the tension, T, is continuous and has k-derivatives, it can be expanded in a Taylor series as:

$$\Delta T = \frac{dT}{dx}(x_0)\Delta x + \frac{1}{2}\frac{d^2T}{dx^2}(x_0)(\Delta x)^2 + \cdots$$

  In Hooke's law, we discard higher-order terms and consider only the first-order term, assuming Δx is small. This gives us the linear approximation, which is essentially Hooke's law.

Since we are aiming linear equations in this course, we limit the expansion to the linear term. This assumption simplifies the problem to the linear framework, which is the essence of Hooke's law.

- The string undergoes **small transverse vibrations** in a plane: That is, the position of the string always remains close to a straight line (the equilibrium position), and each point on the string performs small vibrations in a direction perpendicular to the straight line, all within the same plane. ("Small" refers to the amplitude of the string's vibrations and the small tilt angle of the tangent at any point on the string). $u \ll 1$ and $u_x \ll 1$ (see Fig.3). The slope of the tangent line to the string with respect to x is equal to $\tan\alpha_i$, i.e., $u_x = \tan\alpha_i$. For small transverse vibrations, both u and u_x are very small. In other words, the angle α_i is also small.

Remark 1.3. *This is because if u_x is **small**, it **remains small regardless of the scale** at which the system is viewed, whereas the **vibration amplitude** u behaves **differently**. For example, in the "Ant-Man" scenario from the Marvel universe, an object appears small at a macroscopic scale, but when zoomed in, its size becomes more apparent. However, if the angle α remains small, the system will continue to appear small at any scale. Thus, we assume:*

$$u \ll 1 \quad \textit{and} \quad u_x \ll 1 \quad (\textit{see Fig. 3}).$$

Additionally, since the small angle approximation holds, we have:

$$\alpha_1, \alpha_2 \textit{ are small}, \quad \textit{and} \quad u_x \approx \tan(\alpha_i).$$

1.2 No external forces

First, let's discuss the case of string vibration without the influence of external forces during the vibration process. Keep in mind some key facts listed below before we peoceed.

- The problem of string vibration ultimately involves **translating Newton's Second Law** into a PDE.
- Since this is a **particle mechanics problem**, we need to **localize** the string by **analyzing a small segment**.
- Newton's Second Law is applied to this small segment (viewed as a particle and perform a force analysis), where forces are balanced by acceleration.
- After **force analysis**, the equation of motion can be derived.
- **Momentum or impulse principles** can also be used, as they are nearly equivalent to Newton's Second Law, though they are more general.
- We assume **no external forces** (e.g., gravity), so the only forces on both ends of the segment are the tension forces.
- The tension force is **tangential** to the string, and although the tangents at both ends are slightly different, we assume they are nearly the same due to the small size of the segment.

the deformation $\Delta T = -k\Delta x$. In the case of a string, the direction of the applied force is opposite to the direction of the deformation. This is because the force tries to restore the object to its original position, pulling it back towards its equilibrium. The negative sign in Hooke's law indicates this opposition between the force and the deformation.

1.2.1 Coordinate System

We choose a coordinate system (see Fig.1) where:

- The equilibrium position of the string aligns with the x-axis, fixed at endpoints $x = 0$ and $x = L$.
- $u(x,t)$ denotes the transverse displacement of the point at position x along the string at time t.

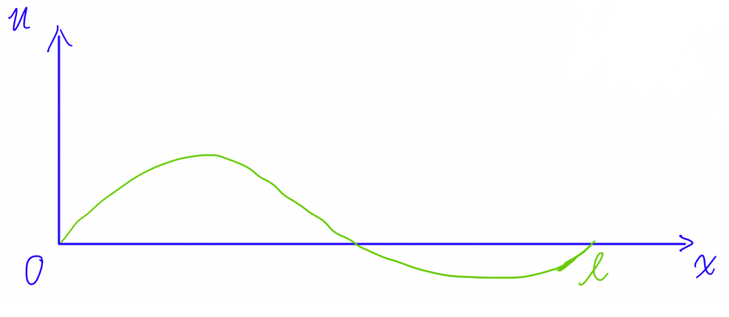

Figure 1: String 1

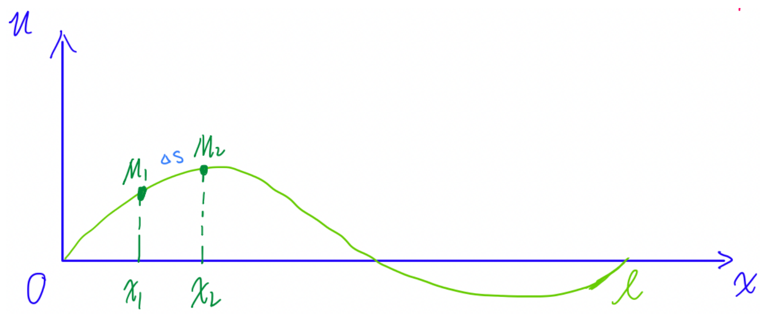

Figure 2: String 2

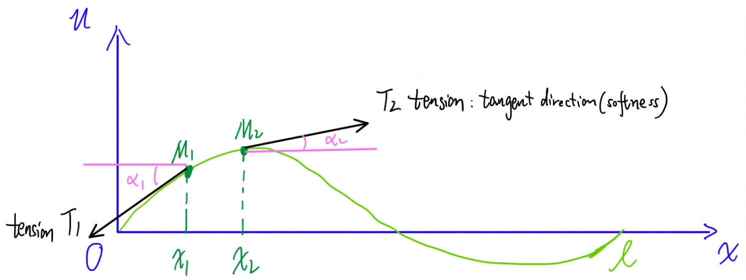

Figure 3: String 3

1.2.2 Deriving the free transverse vibration equation (wave equation)

Let $T_1(t) := T(t,x)|_{x_1}$ and $T_2(t) := T(t,x)|_{x_2}$ denote tensions at points x_1 and $x_2 := x_1 + \Delta x$, respectively, directed along the local tangents (see Fig. 3).

Performing a force analysis!

- **Horizontal components–balance:** For small vibrations, the horizontal components cancel: $T_1 \cos\alpha_1 = T_2 \cos\alpha_2$. Since

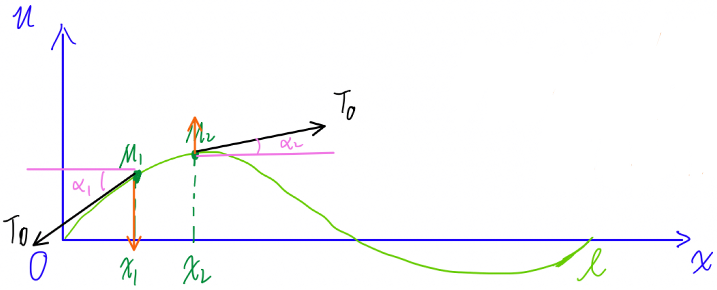

Figure 4: String 4

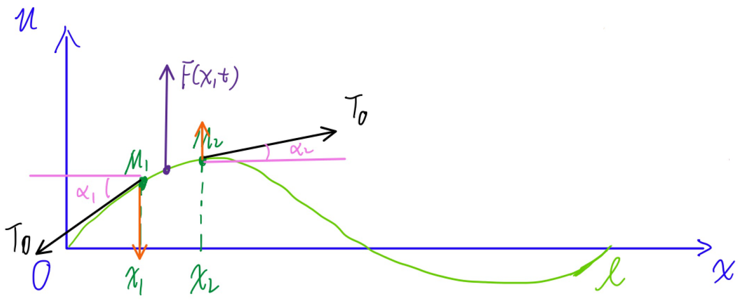

Figure 5: String 5

- For transverse vibration, movement occurs only in the vertical direction.
- In the horizontal direction, the two forces must balance each other.
- This balance results in no movement in the horizontal direction.

- **Vertical components–Newton's second law:** $T_2 \sin\alpha_2 - T_1 \sin\alpha_1 = \rho\Delta s \partial_t^2 u$. Since
  - The tension components in the vertical direction are given by: $T_2 \sin\alpha_2 - T_1 \sin\alpha_1$.
  - The net vertical force causes the string to move, so the forces must balance the mass times acceleration[3].
  - The **mass** of the string segment is the linear density multiplied by the length of the segment $m = \rho\Delta s$.
  - The **acceleration** is the rate of change in velocity for the segment[4] $\partial_t^2 u|_\eta$.

In fact, solving this equation is essentially an approximation process — a process of simplifying and approximating the equation.

$$\text{Horizontal:} \qquad T_2 \cos\alpha_2 - T_1 \cos\alpha_1 = 0, \tag{1}$$
$$\text{Vertical:} \qquad T_2 \sin\alpha_2 - T_1 \sin\alpha_1 = \rho\Delta s \partial_t^2 u|_\eta. \tag{2}$$

- **Eq. (1)⇒ T is constant in space:** Due to small angles α_1 and α_2, and since the goal is to linearize the equations, by approximating $\cos(\alpha) = 1 - \frac{1}{2}\alpha^2 + \cdots$, i.e, $\cos(\alpha) \approx 1$. Higher-order terms are neglected for simplicity, as we aim to obtain a linear equation. Then Eq. (1)⇒ $T_1(t) = T_2(t)$, which means tension does not depend on spatial position, i.e., $T_0(t) = T_1(t) = T_2(t)$ **is constant in space**[5].

[3]Applying Newton's Second Law: the net vertical force equals the mass times acceleration.

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

where mass is the linear density ρ times the string length L.

[4]Since the string segment Δx is very small, the difference in acceleration at each point on it will not be too large. Therefore, the acceleration at one point, denoted as η, can be used to approximate the acceleration at other points.

[5]This conclusion is valid under linear approximation. If higher-order terms in the Taylor expansion of $\cos(\alpha)$ were included, this would no longer hold.

- Eq. (2) becomes

$$T_0(t)\sin\alpha_2 - T_0(t)\sin\alpha_1 = \rho\Delta s\partial_t^2 u. \tag{3}$$

  - A similar approach (**aiming** the **linear** equation) is used with the sine function[6]:

$$\sin(\alpha) \approx \alpha \approx \tan\alpha \approx u_x$$

    for small angles. Eq. (3) becomes

$$T_0(t)\partial_x u|_{x_2} - T_0(t)\partial_x u|_{x_1} = T_0(t) \underbrace{\left(\partial_x u|_{x_2} - \partial_x u|_{x_1}\right)}_{\text{Mean Value Theorem}=\partial_x^2 u|_\xi\cdot\Delta x} = \rho\Delta s\partial_t^2 u|_\eta. \tag{4}$$

  - Using the Mean Value Theorem[7], We arrive at

$$T_0(t)\partial_x^2 u|_\xi \cdot \Delta x = \rho\Delta s\partial_t^2 u|_\eta. \tag{5}$$

Question 1.2. *There's also a tricky part, which is how to handle the arc length[8] ds?*

A. *Using the* ***arc length formula****, we know:*

$$\Delta s = \int_{x_1}^{x_2}\sqrt{1+u_x^2}\,dx, \quad u_x = \frac{\partial u}{\partial x} = \tan\alpha \ll 1.$$

Assuming the string only undergoes small vibrations, u_x^2 can be neglected compared to 1, thus The arc length $\Delta s \approx x_2 - x_1 = \Delta x$.

Remark 1.4.

- *Arc length ds is approximated using the* ***Pythagorean theorem*** *in the small displacement limit:*

$$ds = \sqrt{dx^2 + (du)^2}.$$

- *This leads to the approximation:*

$$ds \approx dx\left(1+u_x^2\right)^{1/2}.$$

- *For linearization, higher-order terms are discarded, leading to (using $(1+x)^\alpha = 1 + \alpha x + \cdots$):*

$$ds \approx dx.$$

- *This result implies that the arc length remains constant during vibration, which seems* ***counterintuitive***[9]*. This result holds* ***only under the assumption of linear*** *approximation.*

- *The increase in length includes a second-order small quantity, which means that when the value is small enough, it can be neglected.*

- *If the small quantity is* ***not negligible*** *(e.g.,* ***large perturbations****), this approximation no longer holds, and the equation would become* ***nonlinear or variable-coefficient****.*

Let us go back to Eq. (5):

- **T is constant in time:** Since ds does **not change with time**, that is, the deformation is 0 **with time**, by Hooke's law, the **change of tension** $T_0(t)$ is 0 with time. That is, $T_0(t)$ is independent time and thus a constant. The tension T becomes **independent of both space and time**, i.e., $T_0(t) = T_0 =$ Constant.

[6] $\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \cdots$ and $\tan\alpha = \alpha + \frac{\alpha^3}{3} + \cdots$.

[7] A mean value theorem (or Taylor expansion) is used to express the difference in function values at different points:

$$f(x_2) - f(x_1) = \left.\frac{df}{dx}\right|_\xi \cdot (x_2 - x_1).$$

[8] Since he arc length ds is neither u nor x. In this equation, the independent variables and functions we need are u and x, but how can we express s in terms of u and x?

[9] In a vibration process, the shortest distance between two points should increase if the path is curved, but the theory suggests that it does not, which seems **counterintuitive.**

- (5) becomes

$$T_0 \partial_x^2 u|_\xi \cdot \cancel{\Delta x} = \rho \cancel{\Delta x} \partial_t^2 u|_\eta \quad \Rightarrow \frac{T_0}{\rho} \partial_x^2 u|_\xi = \partial_t^2 u|_\eta \tag{6}$$

- Assumption: Let Δx be extremely small, tending to zero. As $\Delta x \to 0$, both terms involving Δx can be cancelled and ξ and $\eta \to x_1$. (6) becomes the **free transverse vibration equation** (or **wave equation**[10])

$$\partial_t^2 u = a^2 \partial_x^2 u \quad \text{where } a^2 := \frac{T_0}{\rho}, \quad (\rho = \text{Const. since "homogeous"}) \tag{7}$$

- The wave equation can be rewritten as

$$u_{tt} - a^2 u_{xx} = 0, \quad \text{"hyperbolic type"}.$$

The negative sign distinguishes wave equations from Laplace equations ("elliptic" type, $u_{xx} + u_{yy} = 0$), highlighting wave phenomena.

1.2.3 Forced Transverse Vibration

If an external force $F(x,t)$ (per unit length) acts vertically:

$$T \frac{\partial^2 u}{\partial x^2} \Delta x + F \Delta x = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

Simplifying gives the **forced vibration equation**:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad \text{where } a^2 := \frac{T_0}{\rho} \quad \text{and} \quad f := \frac{F}{\rho}$$

where $f(x,t)$ is the force density.

- If external forces are present, the equation becomes **non-homogeneous** (i.e., f exists).
- The homogeneous case ($f(x,t) = 0$) represents the **free vibration**.

1.3 General Wave Equations

The derived equations are **one-dimensional wave equations**. Analogously:

- **2D wave equation** (e.g., membrane vibration):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- **3D wave equation** (e.g., sound or electromagnetic waves):

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Remark 1.5. *The notation "(1+1)D", "(1+2)D", and "(1+3)D" is commonly used in relativity, where "1" refers to* ***time*** *and the number after "+" refers to* ***spatial dimensions****.*

[10] The vibration refers to observing the movement of a point after determining its spatial position, such as vibrating downward. However, if you also consider time and the spatial position simultaneously, the vibration essentially becomes a wave, representing the propagation of a wave. Therefore, this equation is referred to as the one-dimensional free vibration equation. It can also be called the one-dimensional wave equation.

1.4 Boundary and Initial Conditions

For a definite physical process, it is not enough to establish equations that describe the physical quantities involved. Additional conditions are required to fully describe the system's initial state and the physical conditions at the boundaries.

- The equation derived from physical laws (the equation is at the level of physical laws–universally applicable) may not have a unique solution.
- To ensure a unique solution, *boundary conditions* and *initial conditions* are necessary.
- These conditions help to select one case from the universal physical laws.
- We focus on initial and boundary conditions, though there are other types (e.g., characteristic problems).

Euqation+Condition=the determined solution!

Boundary and initial conditions include:

1.4.1 Initial Conditions

These describe the state of a process at its "initial" moment.

For the problem of string vibration, initial conditions refer to the displacement and velocity of the string at the "initial" moment.

$$\begin{cases} u\Big|_{t=0} = \varphi(x), & \text{Initial displacement} \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x). & \text{Initial velocity} \end{cases}$$

- Initial conditions specify the state at the initial time (e.g., at $t = 0$).
- For the wave equation, the two required initial conditions are:
  - The initial displacement $u(t = 0)$
  - The initial velocity $\frac{\partial u}{\partial t}(t = 0)$
- The **number of required initial conditions** depends on the **order of the equation**, and formally speaking, it should be **one order lower than the equation**.
  - For the wave equation, the time derivative has two derivatives, so both the initial position and initial velocity need to be specified.
  - For heat equations, which have only a first-order time derivative, only the initial position needs to be specified.

1.4.2 Boundary Conditions (Need to know its math expression, physical meaning, names)

These describe the physical conditions that the physical quantities of a process satisfy at the boundaries of the system.

For the string vibration problem, there are three basic types:

(1). Dirichlet Boundary Conditions (First Type)

If the motion law of one end of the string is known, represented as u, then the boundary condition can be expressed as:

- Non-homogeneous Boundary Condition $u|_{x=0} = \mu_1(t)$;
- In particular, if the end is **fixed**, the corresponding boundary condition is: $u|_{x=0} = 0$ ("Fixed boundary", "Homogeneous Boundary Condition").

Remark 1.6. • *The value (or behavior) of u itself at the boundary is given.*

- *In this case, u is a function of both space and time, but at the boundary, it is only a function of time (x is fixed due to the boundary).*
- *The physical meaning: The motion of the string at the boundary is specified, i.e., the displacement at the boundary follows a given function.*
- *Special cases: If $u|_{x=0} = 0$, it is called the* ***fixed boundary****, while if $u \neq 0$, it is a non-fixed boundary.*

(2). Neumann Boundary Conditions (Second Type)

If one end of the string (for example, $x = l$) slides freely along a line perpendicular to the x-axis and is not subject to any external force in the perpendicular direction, this boundary is called a **free boundary**.

- According to the component of the tension at the right end of the boundary element in the perpendicular direction $T_0 \partial_x u$, the condition at the free boundary is:

$$\partial_x u|_{x=0} = 0, \quad \text{Homogeneous Boundary Condition.}$$

- If the component of the boundary tension in the perpendicular direction is a known function $\mu_2(t)$, then the corresponding boundary condition is:

$$\partial_x u|_{x=0} = \mu_2(t), \quad \text{Non-homogeneous Boundary Condition.}$$

Remark 1.7. • *The first spatial derivative of u at the boundary is given.*

- *The physical meaning ($u_x = \tan\alpha \sim \sin\alpha$ perpendicular tension $T_0 \sin\alpha$): The value of the tension at the boundary is specified. If the tension is zero at the boundary, it is called a free boundary. If it is non-zero, it is a non-free boundary.*

(3). Robin Boundary Conditions (Third Type)

If one end of the string (for example, $x = 0$) is fixed on an elastic support, and the behavior of the support follows Hooke's Law.

If the position of the support is $u = 0$, then the value of u at the endpoint represents the behavior of the support at that point.

The vertical component $-T_0 \partial_x u$ of the string's force on the support is given by Hooke's Law, thus the boundary condition in the case of an elastic support is:

$$-T_0 \frac{\partial u}{\partial x}\bigg|_{x=l} = ku\big|_{x=l}.$$

- Homogeneous Boundary Condition:

$$\left(\frac{\partial u}{\partial x} + \alpha u \right)\bigg|_{x=l} = 0.$$

  where α is a known positive number.

- In mathematics, more general boundary conditions can also be considered:

$$\left(\frac{\partial u}{\partial x} + \alpha u \right)\bigg|_{x=l} = \mu_3(t), \quad \text{Non-homogeneous Boundary Condition}$$

  where $\mu_3(t)$ is a known function of t.

Remark 1.8. • *A linear combination of the value of u and its spatial derivative at the boundary is given.*

- *The physical meaning is the behavior of the elastic support on the boundary.*

Summary:

- The boundary conditions for partial differential equations (PDEs) specify how the solution behaves at the boundaries.
- The Dirichlet boundary condition specifies the value of u at the boundary.
- The Neumann boundary condition specifies the derivative of u at the boundary.
- The third type combines the value of u and its derivative in a linear fashion.

A problem consists of a partial differential equation and conditions. For instance, The initial-boundary value problem is given by:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u\Big|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \\ u\Big|_{x=0} = 0, \quad u\Big|_{x=l} = 0. \end{cases}$$

Depending on the type of conditions, the problems are further divided into:

- Initial Value Problem or Cauchy Problem
- Boundary Value Problem
- Mixed Problem or Initial-Boundary Value Problem

Homework: Exercise I