Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 4-2

New method in the Exercise Class:

- Once in standard S-L form, we will introduce an unfamiliar method called the **energy method**, not covered in textbooks.
- The energy method is essential for understanding certain aspects of Chapter 5.
- Without understanding the energy method, some parts of Chapter 5 will not be explained in textbooks.
- We will discuss this method in the exercise session.
- The three conclusions can be fully analogized to the properties of $X'' + \lambda X = 0$.

1 Exercise class

Ex 1.1. *Use the method of separation of variables to write out the following problem:*

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l,\, t > 0), \\ (u_x - \alpha u)\big|_{x=0} = 0, \quad \textit{(Third kind)} & \\ u_x\big|_{x=l} = 0, & \\ u(x,0) = \varphi(x) & \end{cases}$$

What is the eigenvalue problem? and write out

1. *when $\alpha = 0$ in the boundary conditions, the eigenvalues and eigenfunctions;*
2. *when $\alpha \to \infty$ in the boundary conditions, the eigenvalues and eigenfunctions;*

Solution. *For the following mixed value problem:*

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l,\, t > 0), \\ (u_x - \alpha u)\big|_{x=0} = 0, & \\ u_x\big|_{x=l} = 0, & \\ u(x,0) = \varphi(x) & \end{cases}$$

The eigenvalue problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ (X' - \alpha X)\big|_{x=0} = 0, \\ X'\big|_{x=l} = 0. \end{cases} \quad \textit{that is,} \quad \begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = \alpha X(0), \\ X'\big|_{x=l} = 0. \end{cases}$$

(1) When $\alpha = 0$ in the boundary conditions, the eigenvalue problem simplifies to

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'\big|_{x=0} = 0, \quad X'\big|_{x=l} = 0. \end{cases}$$

At this case, the corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{l} \quad (n = 0, 1, 2, \cdots).$$

(2) When $\alpha \to \infty$ in the boundary conditions, the eigenvalue problem simplifies to (since $X'(0) = \alpha X(0) \Leftrightarrow X(0) = \frac{1}{\alpha}X'(0)$)

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X\big|_{x=0} = 0, \quad X'\big|_{x=l} = 0. \end{cases}$$

At this case, the corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2, \quad X_n(x) = \sin\frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \cdots).$$

Ex 1.2. *Prove for the problem,*

$$\begin{cases} x^2y'' + 3xy' + \lambda y = 0, \quad (1 < x < e) \\ y(1) = y(e) = 0 \end{cases} \leftarrow \begin{cases} \boxed{\textit{in the class of the general S-L problem,}} \\ \boxed{\textit{play the role of } X'' + \lambda X = 0} \end{cases}$$

The eigenfunction series $\{y_n(x)\}$ is orthogonal on $[1, e]$ with respect to the weight function x. i.e.,

$$\boxed{\int_1^e x y_n(x) y_m(x) dx = \frac{1}{2}\delta_{mn}}$$

The first idea:

1. **Initial Step:** To prove the orthogonality of eigenfunctions, the first step is to find the eigenfunctions.
2. **Equation Substitution:** The current equation replaces the previous one involving $X''+\lambda X = 0$.
3. **Eigenvalue Problem Approach:** Similar to solving for X, we first find the general solution and then apply boundary conditions to determine parameters.
4. **General Solution for ODE:** The equation is a second-order linear ODE with variable coefficients, unlike the constant coefficient case.
5. **Euler's Equation Analogy:** The form of the equation **resembles Euler's equation**, which involves terms like x^2 and x times the second and first derivative, respectively.
6. **Transformation Method:** Use a variable transformation similar to solving Euler's equation to simplify the variable coefficients into constant coefficients.
7. **Transformation Outcome:** After transformation, the equation becomes a standard constant coefficient form ODE, which simplifies the process of finding solutions.
8. **Calculation of Transformation:** The process of transformation should be calculated, but it naturally follows if you aim for a constant coefficient form.

Solution (Method 1: Direct method). *(1) First, find the specific expression of the eigenfunction series $\{y_n(x)\}$.*

Make the transformation $x = e^t \iff t = \ln x$,

Then we have

$$y_x = y_t \cdot \frac{1}{x}, \quad y_{xx} = \left(y_{tt} \cdot \frac{1}{x}\right) \cdot \frac{1}{x} + y_t \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2}y_{tt} - \frac{1}{x^2}y_t,$$

Substitute into the original equation to get

$$y_{tt} - y_t + 3y_t + \lambda y = 0 \implies y_{tt} + 2y_t + \lambda y = 0$$

$$\implies \begin{cases} y_{tt} + 2y_t + \lambda y = 0 \\ y(1) = y(e) = 0. \end{cases} \quad \boxed{\leftrightarrow \textit{char. eq. } r^2 + 2r + \lambda = 0 \leftrightarrow r_\pm = -1 \pm \sqrt{1-\lambda}}$$

Methid 1: General method

Case (i) $\lambda < 1$

$$\begin{cases} y = Ce^{(1+\sqrt{\lambda})t} + De^{(1-\sqrt{\lambda})t} \\ \begin{cases} C + D = 0 \\ Ce^{-t\sqrt{\lambda}} + De^{t\sqrt{\lambda}} = 0 \end{cases} \end{cases}.$$

This leads to $C = D = 0$*, hence no non-trivial solutions.*

Case (ii) $\lambda = 1$

$$\begin{cases} y = (At + B)e^{-t} \\ \begin{cases} Be^{-t} = 0 \\ (A + B)e^{t} = 0 \end{cases} \end{cases}.$$

This leads to $B = 0$ *and* $A = 0$*, hence no non-trivial solutions.*

Case (iii) $\lambda > 1$

$$y = (A\cos(\sqrt{\lambda - 1}t) + B\sin(\sqrt{\lambda - 1}t))e^{-t}.$$

Using boundary conditions:

$$\begin{cases} A = 0, \\ (A\cos(\sqrt{\lambda - 1}) + B\sin(\sqrt{\lambda - 1}))e^{-1} = 0. \end{cases}$$

This leads to $A = 0$ *and* $\sin(\sqrt{\lambda-1}) = 0$*, hence* $\sqrt{\lambda - 1} = n\pi$*. Thus,* $\lambda = \lambda_n = 1 + (n\pi)^2$*, and* $y_n(t) = B_n \sin(n\pi t) \cdot e^{-t}$ *for* $n = 1, 2, \ldots$*.*

Summary of Transformation Techniques for Differential Equations

1. ***General Approach:*** *When encountering* ***unfamiliar*** *equations, transform them into* ***familiar*** *forms using technical skills.*
2. ***Two Main Techniques:***
   - ***Variable*** *transformation (e.g., for Euler's equation).*
   - ***Function*** *transformation to simplify equations.*
3. ***Goal:*** *Simplify the equation to a form where known solutions or conclusions can be directly applied.*
4. ***Desired Form: Aim*** *to transform it into a form similar to* $X'' + \lambda X = 0$*, which has known eigenvalues and eigenfunctions.*
5. ***Transformation Strategy:***
   - *Combine the first two derivative terms into a* ***single second-order derivative****.*
   - *Use the* ***binomial theorem*** *to handle terms involving* $(a + b)^{(n)}$ *and their derivatives.*
6. ***Application of Binomial Theorem:*** *Use the binomial theorem for derivatives to simplify higher-order terms.*

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \quad \textit{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Take $n=2$, $f=y$, what is g? Hope g is a fixed point for derivatives. Then

$$(yg)'' = y''g + 2y'g' + yg'' \overset{g \text{ a fixed point}}{=} (y'' + 2y' + y)g \tag{1}$$

7. ***Transformation Example:*** *Multiply both sides of the equation by e^t (a fixed point for derivation) to utilize the binomial theorem effectively.*
8. ***Resulting Equation:*** *After transformation, the equation becomes $X'' + (\lambda - 1)X = 0$, which is similar to the known form.*
9. ***Conclusion:*** *Once transformed, the equation can be solved using known conclusions without further case distinctions.*

Method 2: Integrating factor and binomial theorem method

$$\begin{aligned} 0 =& e^t \left(y_{tt} + 2y_t + y + (\lambda - 1)y \right) \\ =& \underbrace{(e^t y)_{tt}}_{X''} + \underbrace{(\lambda - 1)(e^t y)}_{(\lambda-1)X} \end{aligned}$$

According to th conclusion of $X'' + \lambda X = 0$, we conclude the solution.

$$X'' + (\lambda - 1)X = 0$$

$$\Rightarrow \lambda - 1 > 0 \Rightarrow \lambda - 1 = (n\pi)^2 \Rightarrow \lambda = 1 + (n\pi)^2, \quad (n = 1, 2, \dots)$$
$$\textit{and} \quad X_n = e^t y_n = \sin(n\pi t).$$

$$\Longrightarrow \lambda_n = (n\pi)^2 + 1, \quad y_n(t) = B_n e^{-t} \sin n\pi t \quad (n = 1, 2, \cdots).$$

Substitute $t = \ln x$ to get $y_n(x) = B_n \frac{1}{x} \sin(n\pi \ln x)$, $(n = 1, 2, \cdots)$. Then the eigenfunction series of the original problem is

$$\{y_n(x)\} = \left\{ \frac{1}{x} \sin(n\pi \ln x) \right\} \quad (n = 1, 2, \cdots)$$

(2) Now verify the orthogonality of the eigenfunction series $\{y_n(x)\}$.

$$\begin{aligned} &\int_1^e x y_n(x) y_m(x) dx \quad \textit{Make the transformation} \quad x = e^t \\ =& \int_0^1 e^{2t} y_n(t) y_m(t) dt = \int_0^1 \sin(n\pi t)\sin(m\pi t) dt = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n. \end{cases} \end{aligned}$$

Solution (Method 2: Energy method–Compatible with the nonlinear PDEs). *Let λ_n, y_n be the corresponding eigenvalues and eigenfunctions. (assume they exist!)*

Then λ_n, y_n satisfy:

$$x^2 y_n'' + 3x y_n' + \lambda_n y_n = 0 \Longrightarrow x^2 y_n'' + 3x^2 y_n' + \lambda_n x y_n = 0 \Longrightarrow (x^3 y_n')' + \lambda_n x y_n = 0$$

$$\left. \begin{array}{c} \underbrace{x^2 y'' + 3xy'}_{\Downarrow} + \lambda y = 0 \\ \underbrace{\frac{1}{x}(x^3 y'' + 3x^2 y')}_{\Downarrow} \\ \frac{1}{x}(x^3 y')' \end{array} \right\} \Rightarrow (x^3 y')' + \lambda x y = 0$$

- *The concept of* ***integrating factors*** *is crucial in transforming differential equations into a solvable form.*
- *To utilize the* ***energy method*** *for proving orthogonality, the equation must be converted into* ***Sturm-Liouville (S-L) standard form****.*
- *The orthogonality proof in* ***Chapter 5*** *also relies on this energy method.*

The physical origin of the concept of energy.

$$ma = \frac{d(mv)}{dt} = F.$$

$$\Rightarrow \quad v\frac{d(mv)}{dt} = F\frac{ds}{dt} \quad \Rightarrow \quad \int \frac{d\left(\frac{1}{2}mv^2\right)}{dt}dt = \int \frac{dW}{dt}dt \quad \Rightarrow \quad \frac{1}{2}mv^2 = W.$$

Mathematical formulation of this "energy concept".

Extract ideas:

- *Use unknown functions to multiply both sides of the differential equation;*
- *Integration (local to global);*
- *Symmetric energy obtained by integration by parts.*

Using y_m to multiply both sides and integrate from 1 to e, we get:

Step 1: eq. $\times$ **y**. $\quad \left(x^3y_n'\right)' y_m + \lambda_n x y_n y_m = 0$

Step 2: integration. $\quad \underbrace{\int_1^e \left(x^3y_n'\right)' y_m dx}_{\|\,(\text{Integration by Parts})} + \lambda_n \int_1^e x y_n y_m dx = 0$

Step 3: integration by parts. $\quad \underbrace{x^3y_n'y_m\Big|_1^e}_{\|\,\text{bdry. } y(1)=y(e)=0 \\ 0} - \int_1^e x^3 y_n' y_m' dx$

1. ***Advantage of Integration:*** *The* ***primary benefit*** *of integration is the ability to* ***move derivatives*** *(integration by parts).*
2. ***Utilizing Symmetry:*** *By moving derivatives, we can leverage specific symmetries to derive beneficial conclusions.*
3. ***Method Used:*** *Integration by parts is employed to shift derivatives.*
4. ***Result of Integration by Parts:*** *After applying integration by parts, a boundary term and an integral term remain.*
5. ***Boundary Term:*** *The boundary term involves the product of functions and their derivatives evaluated at the boundaries.*
6. ***Symmetry:*** *The resulting expression often exhibits* ***symmetry****, which is* ***desirable in mathematical and physical contexts****.*
7. ***Boundary Conditions:*** *The boundary conditions can be used to simplify the boundary term, often making it zero.*

8. ***Simplification:*** *If the boundary term is zero, the expression simplifies significantly.*

9. ***Final Form:*** *The simplified expression can be rearranged to a more recognizable form, facilitating further analysis.*

$$\Rightarrow \lambda_n \int_1^e x y_n y_m dx = \int_1^e x^3 y_n' y_m' dx. \tag{2}$$

By swapping m and n, we get:

$$\lambda_m \int_1^e x y_n y_m dx = \int_1^e x^3 y_n' y_m' dx. \tag{3}$$

Subtracting (3) from (2) gives:

$$(\lambda_n - \lambda_m) \int_1^e x y_n y_m dx = 0 \Rightarrow \text{if } n \neq m \text{ then } \int_1^e x y_n y_m dx = 0, \text{ orthogonality is proven.}$$

Idea: Use duality to achieve symmetries internally.

- If dual, there may be information sharing and exchange between dual systems. - The "inner product (dual form)" can be moved to reduce the order of the equation (tool: Integration by parts).

Techniques:

1. Both sides multiply the same unknown function.
2. Integration.
3. Integration by parts (eliminating boundary terms).

Comparison of Two Methods for Orthogonality of Eigenfunctions

1. **First Method:**
   - Requires finding the eigenfunction series explicitly.
   - After obtaining the specific form, substitute into the integral to verify if it equals zero.
2. **Second Method:**
   - Does not require finding the eigenfunctions Y_n explicitly.
   - Directly derives the orthogonality properties without explicit computation of Y_n.
3. **Key Difference:**
   - The first method involves **detailed computation of eigenfunctions**.
   - The second method focuses on orthogonality **without detailed computation of eigenfunctions**.
4. **Advantages of the Second Method:**
   - More efficient when detailed computation of eigenfunctions is unnecessary.
   - Provides a direct approach to proving orthogonality.
5. **Implementation of the Second Method:**
   - Utilize **properties of differential equations** and **boundary** conditions.
   - Apply integration techniques to show orthogonality without explicit eigenfunction computation.

Summary:

1. **Method Overview:** This method, initially seems strange, is known as the **energy method**, which simulates the **structure of kinetic energy**.
2. **Construction of Duality:** By **integrating** with an unknown function, we construct a **dual entity** that is **symmetric** with the original function.
3. **Symmetry and Simplification:** Through **integration by parts**, we achieve a **completely symmetric form**, which helps in simplifying the problem.

Ideas of Duality–Daoist Philosophy on Duality

The concept of duality reflects the idea of cooperation. When something is difficult or ineffective, it can be paired with something easier or more effective. By establishing an internal connection, their difficulty or efficiency can be balanced. The power of a collaborative team is greater than the sum of its individual members.

The Chinese classic *Tao Te Ching* expresses a **similar idea**:

万物负阴而抱阳
冲气以为和

道德经 · 第四十二章

This can be translated as

$$\begin{cases}\text{All things bear the shade on their backs}\\ \text{And the sun in their arms; } (\leftrightarrow \text{ multiplying the unknown and integrating it})\\ \text{By the blending of breath } (\leftrightarrow \text{ integration by parts})\\ \text{From the sun and the shade,}\\ \text{Equilibrium comes to the world. } (\leftrightarrow \text{ achieve the symmetric energy form})\end{cases}$$

or

"All things carry Yin and embrace Yang. Through their interaction, harmony is achieved."

This reflects the **natural principle of duality**, where seemingly opposing forces complement each other, achieving **balance through cooperation**.

In other words, **Contradiction** is the fundamental driving force of the development of things. Every entity contains two opposing aspects that are both in conflict and interdependent, and under certain conditions, they can transform into each other. Contradiction drives the development of things, causing them to continuously change, progress, and achieve balance.

1. **Basic Idea:** The underlying idea is one of **duality**, which is a **fundamental concept** in **functional analysis**.
2. **Benefits of Duality:** Duality helps **transfer complexity** from a difficult problem to a simpler one, reducing overall difficulty.
3. **Application in Differential Equations:** By multiplying by a simple function, integrating and shifting the derivatives, we **reduce the order** of the differential equation.
4. **Construction of Symmetry:** Lowering the order of derivatives through integration allows us to **construct symmetry**, a crucial tool in mathematical research.

Aesthetic approach

1. **Importance of Symmetry: Symmetry construction** is vital in mathematics, often inspired by physical or geometric intuition, or even **aesthetics**.
2. **Aesthetics in Mathematics:** The method reflects an **aesthetic approach** to mathematics, seeking **intrinsic symmetry** and **beauty** in mathematical forms.
3. **Artistic Approach:** This method embodies an **artistic perspective** in mathematics, aiming to construct **elegant structures** to derive solutions.
4. **Recurring Theme:** The concept of constructing symmetry and beauty will reappear in various contexts, emphasizing its importance in mathematical problem-solving.

Ex 1.3. *Eigenvalue Problem*

$$\begin{cases} u'' + \lambda u = 0; & 0 < x < l \\ u'(0) = u(l) = 0. \end{cases}$$

Solution. *Usually, the method is to exclude $\lambda \leq 0$, we can consider the general solutions separately and then use boundary conditions to determine the solutions.*

*We provide another method–the **energy method**:*

***Multiply** u on the both sides and **integrate** both sides with respect to x, and using **integration by parts**:*

$$\underbrace{\int_0^l uu''dx}_{\displaystyle \| \atop \displaystyle \underbrace{uu'\Big|_0^l}_{\| (*) \atop 0} - \int_0^l (u')^2 dx} + \lambda \int_0^l u^2 dx = 0$$

$(*)$ is due to the boundary condition $u(0) = u(l) = 0$, then we have:

$$\Rightarrow \quad \lambda \int_0^l u^2 dx = \int_0^l (u')^2 dx \quad \Rightarrow \quad \lambda = \frac{\int_0^l (u')^2 dx}{\int_0^l u^2 dx} \geq 0$$

When $\lambda = 0$, it implies $u'(x) = 0$, hence $u(x) = constant$. Using $u(l) = 0$, we get $u(x) \equiv 0$.

Thus, λ can only be positive $\lambda > 0$. In this case, the general solution is $u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. Using boundary conditions, we find $\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2$, and the corresponding eigenfunctions are $u_n(x) = \cos\left(\frac{(2n-1)\pi x}{2l}\right)$ for $n = 1, 2, \ldots$.

- The significance is that, although we cannot write the general solution in Chapter 5, this method still works and allows us to directly determine if the result is greater than or equal to zero. Otherwise, the previous method no longer applies.

Ex 1.4. *Let F and G be twice continuously differentiable functions.*

1. *Prove that $y(x,t) = F(2x+5t) + G(2x-5t)$ is the general solution of the equation $4y_{tt} = 25y_{xx}$.*
2. *Find the solution that satisfies the conditions $y(0,t) = y(\pi,t) = 0$, $y(x,0) = \sin 2x$, $y_t(x,0) = 0$.*

- Similar methods to those used in this example will be applied in Chapter 3.

Solution. *(2) From the conditions, we have:*

$$\begin{cases} y(0,t) = F(5t) + G(-5t) = 0 \\ y(\pi,t) = F(2\pi + 5t) + G(2\pi - 5t) = 0 \\ y(x,0) = F(2x) + G(2x) = \sin 2x \\ y_t(x,0) = 5F'(2x) - 5G'(2x) = 0 \end{cases}$$

- *There are four conditions. In essence, only two conditions are needed to determine the two functions. The other two conditions are naturally fulfilled.*

This implies:

$$F(y) = -G(-y) \tag{4}$$
$$F(2\pi + y) = -G(2\pi - y) \tag{5}$$
$$F(y) + G(y) = \sin y \tag{6}$$
$$F'(y) = G'(y) \quad \Rightarrow \quad F(y) = G(y) + C. \tag{7}$$

From equations (6) and (7), we get:

$$\begin{cases} 2F(y) = C + \sin y \Rightarrow F(y) = \frac{C+\sin y}{2} \\ 2G(y) = \sin y - C \Rightarrow G(y) = \frac{\sin y - C}{2} \end{cases} \tag{8}$$

From equation (4), we have:

$$F(y) = \frac{C + \sin y}{2} = -\frac{-\sin y - C}{2} = -G(-y) \quad \leftarrow \quad \textit{verifies } (4)$$

This implies:

$$C + \sin y = \sin y + C \quad \textit{(identity)}$$

Given:

$$F(2\pi + y) = \frac{C + \sin y}{2}$$
$$G(2\pi - y) = \frac{\sin(2\pi - y) - C}{2} = \frac{-\sin y - C}{2}$$

Thus:

$$F(2\pi + y) = -G(2\pi - y) = \frac{\sin y + C}{2} \quad \leftarrow \quad \textit{verifies } (5)$$

Using equation (8), we find:

$$\begin{aligned} y(x,t) =& \frac{\sin(2x + 5t) + C}{2} + \frac{\sin(2x - 5t) - C}{2} \\ =& \frac{1}{2}[\sin(2x + 5t) + \sin(2x - 5t)] \\ =& \sin 2x \cos 5t \end{aligned}$$

Ex 1.5. *Solve the Initial Boundary Value Problem*

$$\begin{cases} u_t - a^2 u_{xx} = 0 & 0 < x < l,\ t > 0 \\ u_x(0,t) - \sigma u(0,t) = 0 & t \geq 0 \\ u_x(l,t) + \sigma u(l,t) = 0 & t \geq 0 \\ u(x,0) = \varphi(x) & 0 \leq x \leq l \end{cases}$$

where $\sigma > 0$.

- Homogeneous equation and Homogeneous boundary $\rightarrow$ Try Separation of variables.

Solution. *Use separation of variables, let $u(x,t) = X(x)T(t)$, substitute into the system.*

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X'(0) - \sigma X(0) = 0, & X'(l) + \sigma X(l) = 0 \end{cases} \tag{9}$$

and

$$T' + a^2 \lambda T = 0 \tag{10}$$

From (9), we have:

$$\int_0^l XX''dx + \int_0^l \lambda X^2 dx = 0 \Rightarrow XX'\Big|_0^l - \int_0^l (X')^2 dx + \int_0^l \lambda X^2 dx = 0$$

$$\Rightarrow X(l)X'(l) - X(0)X'(0) + \lambda \int_0^l X^2 dx = \int_0^l (X')^2 dx \tag{11}$$

By boundary conditions:

$$X'(0) - \sigma X(0) = 0, \quad X'(l) + \sigma X(l) = 0$$

$$\Rightarrow -\sigma(X(l))^2 - \sigma(X(0))^2 + \lambda \int_0^l X^2 dx = \int_0^l (X')^2 dx$$

$$\Rightarrow \lambda \int_0^l X^2 dx = \sigma(X(l))^2 + \sigma(X(0))^2 + \int_0^l (X')^2 dx \geq 0$$

$$\Rightarrow \lambda \geq 0, \ \textit{if } \lambda = 0 \Rightarrow 0 \leq \int_0^l (X')^2 dx = -\sigma(X(l))^2 - \sigma(X(0))^2 \leq 0$$

$$\Rightarrow \quad X' \equiv 0 \quad \Rightarrow \quad X(x) = constant \quad \Rightarrow \quad \lambda > 0$$

For $\lambda > 0$, the general solution of (9) is:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \Rightarrow X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Using boundary conditions:

$$X'(0) - \sigma X(0) = B\sqrt{\lambda} - \sigma A = 0$$

$$X'(l) + \sigma X(l) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}l) + B\sqrt{\lambda}\cos(\sqrt{\lambda}l) + \sigma A\cos(\sqrt{\lambda}l) + \sigma B\sin(\sqrt{\lambda}l) = 0$$

This leads to:

$$\sqrt{\lambda}B - \sigma A = 0 \Rightarrow A = \frac{\sqrt{\lambda}}{\sigma}B$$

For a non-trivial solution, A and B must not both be zero.

$$(\sigma B - A\sqrt{\lambda})\sin(\sqrt{\lambda}l) + (\sqrt{\lambda}B + \sigma A)\cos(\sqrt{\lambda}l) = 0$$

$$\Rightarrow (\sigma - \frac{\lambda}{\sigma})\sin(\sqrt{\lambda}l) + 2\sqrt{\lambda}\cos(\sqrt{\lambda}l) = 0$$

$$\Rightarrow (\sigma^2 - \lambda)\sin(\sqrt{\lambda}l) + 2\sqrt{\lambda}\sigma\cos(\sqrt{\lambda}l) = 0 \tag{12}$$

(a) If *$\lambda = \sigma^2$, then $\cos\sqrt{\lambda}\ell = 0 \Rightarrow \sqrt{\lambda}\ell = \frac{2n-1}{2}\pi \Rightarrow \lambda = \left(\frac{2n-1}{2\ell}\pi\right)^2 = \sigma^2$ (for $n = 1, 2, \dots$) (a necessary condition!). If there exists an n_0 (where $n_0 = 1, 2, \dots$) such that $\sigma = \frac{2n_0-1}{2\ell}\pi$, then there is a unique $\lambda_0 = \left(\frac{2n_0-1}{2\ell}\pi\right)^2 = \sigma^2$.*

Since σ is given, there can only be one n_0 that makes $\sigma = \sqrt{\lambda_0}$. After σ is given, there is only one n_0, i.e., one λ_0.

The corresponding eigenfunction is:

$$\begin{aligned} X_0(x) =& \frac{\sqrt{\lambda_0}}{\sigma} \cos \sqrt{\lambda_0} x + \sin(\sqrt{\lambda_0} x) \\ =& \cos\left(\frac{2n-1}{2\ell}\pi x\right) + \sin\left(\frac{2n-1}{2\ell}\pi x\right) \\ =& \cos(\sigma x) + \sin(\sigma x) = \frac{2}{\sqrt{2}} \sin\left(\sigma x + \frac{\pi}{4}\right). \end{aligned}$$

(b) If *$\lambda \neq \sigma^2$, both sides can be divided by $\cos(\sqrt{\lambda} l)(\neq 0)$. (12) leads to*

$$\tan \gamma = \frac{2\sigma l \gamma}{\gamma^2 - (\sigma l)^2} \quad (\gamma = \sqrt{\lambda} l) \tag{13}$$

- *In this case, $\sigma^2 - \lambda \neq 0$, if $(\sigma^2 - \lambda)\sin(\sqrt{\lambda}\ell) = 0 \Rightarrow \sin(\sqrt{\lambda}\ell) = 0$ and $\cos(\sqrt{\lambda}\ell) = 0$, which ensures (12). However, they cannot both be zero at the same time.*

Summary Notes on Transcendental Equations

- ***Solving Transcendental Equations:***
  - *Generally, solving transcendental equations* ***graphically*** *or using* ***computational methods*** *to find approximate solutions.*
- ***Observation of Periodic Functions:***
  - *The curve of the function on the left hand side $\tan \gamma$ is* ***periodic*** *from $-\infty$ to $+\infty$.*
- ***Behavior of Rational Functions:***
  - *The rational function is defined from $-\infty$ to $+\infty$.*
  - *It consistently spans* ***from negative to positive infinity****.*
- ***Intersections and Eigenvalues:***
  - *The graph will have* ***infinitely many intersections*** *with the x-axis.*
  - *This implies the existence of* ***infinitely many, but countable, discrete eigenvalues****.*
- ***Existence and Distribution of Eigenvalues:***
  - *We can establish the existence of eigenvalues and understand their distribution properties.*
- ***Estimation of Eigenvalues:***
  - *Estimating eigenvalues is desirable but often challenging.*
- ***Sufficiency of Information:***
  - *With the knowledge of eigenvalue existence and distribution, we consider ourselves informed.*
- ***Challenges in Nonlinear Equations:***
  - *Achieving precise knowledge in the study of nonlinear equations is difficult.*

- ***Obtaining Eigenfunctions:***
  - *Once eigenvalues are determined, eigenfunctions can be found within the corresponding space.*
- ***Application in Chapter 5:***
  - *In the study of* ***Bessel functions****, eigenvalues are determined and recorded similarly.*

Thus, this equation has infinitely many positive roots, denoted as $\gamma_1, \gamma_2, \gamma_n, \ldots$*, and the corresponding eigenvalues* $\lambda_n = \left(\frac{\gamma_n}{l}\right)^2$*,* $n = 1, 2, \ldots$*. The corresponding eigenfunctions are:*

$$\begin{aligned} X_n(x) =& A_n \cos(\sqrt{\lambda_n} x) + B_n \sin(\sqrt{\lambda_n} x) \\ =& B_n \left(\sin \frac{\gamma_n x}{l} + \frac{\gamma_n}{\sigma l} \cos \frac{\gamma_n x}{l} \right) \\ =& K_n \sin \left(\frac{\gamma_n x}{l} + \theta_n \right) \end{aligned}$$

where $\theta_n = \arctan \frac{\sqrt{\lambda_n}}{\sigma}$ *and* $K_n = \frac{B_n}{\sigma l}\sqrt{(\sigma l)^2 + \gamma_n^2}$*. Therefore, all the eigenvalues are* $\lambda_0 = \sigma^2$*,* $\lambda_n = \left(\frac{\gamma_n}{l}\right)^2$*,* $n = 1, 2, \ldots$*. The corresponding eigenfunctions are* $X_0(x) = \sin\left(\sigma x + \frac{\pi}{4}\right)$*,* $X_n(x) = \sin\left(\frac{\gamma_n x}{l} + \theta_n\right)$*,* $n = 1, 2, \ldots$*.*

If $\lambda = \lambda_n$*, substitute into the* T*-equation to get* $T_n(t) = C_n e^{-a^2 \lambda_n t}$*,* $n = 0, 1, 2, \ldots$*.*

The superposition solution is:

$$u(x,t) = D_0 e^{-a^2 \sigma^2 t} \sin\left(\sigma x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} D_n e^{-a^2 \lambda_n t} \sin\left(\frac{\gamma_n x}{l} + \theta_n\right).$$

Initial conditions determines the coefficients:

$$D_0 = \frac{\int_0^l \sin\left(\sigma x + \frac{\pi}{4}\right) \varphi(x) dx}{\int_0^l \sin^2\left(\sigma x + \frac{\pi}{4}\right) dx}.$$

$$D_n = \frac{\int_0^l \sin\left(\frac{\gamma_n x}{l} + \theta_n\right) \varphi(x) dx}{\int_0^l \sin^2\left(\frac{\gamma_n x}{l} + \theta_n\right) dx} \quad (n = 1, 2, \ldots).$$

(b) *If for any positive integer* n*, there exists* $\sigma \neq \frac{(2n-1)\pi}{2l}$*, then* σ *is not a root of the transcendental equation (12). At this time, (12) and (13) have the same solution. Compared to case (a), there is one less* λ_0*.*

The eigenvalue is $\lambda_n = \left(\frac{\gamma_n}{l}\right)^2$*,* $n = 1, 2, \ldots$*.*

The corresponding eigenfunctions are $X_n(x) = \sin\left(\frac{\gamma_n x}{l} + \theta_n\right)$*,* $n = 1, 2, \ldots$*. where* γ_n *is the* n*-th positive root of the equation (13), and substitute* $\lambda = \lambda_n$ *into the equation, we obtain* $T_n(t) = C_n e^{-a^2 \lambda_n t}$*,* $n = 1, 2, \cdots$*.*

The superposition solution is: $u(x,t) = \sum_{n=1}^{\infty} D_n e^{-a^2 \lambda_n t} \sin\left(\frac{\gamma_n x}{l} + \theta_n\right)$*.*

Initial conditions determines the coefficients: $D_n = \frac{\int_0^l \sin\left(\frac{\gamma_n x}{l} + \theta_n\right) \varphi(x) dx}{\int_0^l \sin^2\left(\frac{\gamma_n x}{l} + \theta_n\right) dx} \quad (n = 1, 2, \ldots).$

Ex 1.6. *Solve the initial boundary value problem:*

$$\begin{cases} u_t = a^2 u_{xx} - b^2 u \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

where $0 < x < l$*,* $t > 0$ *and* b *is the known constant.*

- Question: Is the equation homogeneous or non-homogeneous?
- Conclusion: The equation is homogeneous.
- Reason: The term $-b^2u$ in question is related to u, not just the independent variable. Therefore, it is not a free term.
- The standard method is to directly use the separation of variables in five steps.
- The idea: This kind of adjacent order form can always be combined into one by introducing a new function $v = e^{bt}u$, thus transforming the equation into a simpler form.
- Note that

$$e^{b^2t}(u_t + b^2u) = e^{b^2t}u_t + b^2e^{b^2t}u = \underbrace{(e^{b^2t}u)_t}_{v_t} = a^2\underbrace{(e^{b^2t}u)_{xx}}_{v_{xx}}$$

Solution. *Let $u(x,t) = e^{-b^2t}v(x,t)$. The original problem can be transformed into:*

$$\begin{cases} v_t = a^2v_{xx} & 0 < x < l,\ t > 0 \\ v(0,t) = v(l,t) = 0 & t \geq 0 \\ v(x,0) = \varphi(x) & 0 \leq x \leq l \end{cases}$$

This becomes a classical heat conduction (mixed) problem.

Using the result from section 8.2, we get:

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2}{l^2}t} \sin\left(\frac{n\pi x}{l}\right)$$

where

$$C_n = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad n = 1, 2, \ldots$$

The solution to the original problem is:

$$u(x,t) = e^{-b^2t}\sum_{n=1}^{\infty}\left(\frac{2}{l}\int_0^l \varphi(x)\sin\left(\frac{n\pi x}{l}\right)dx\right)e^{-\frac{n^2\pi^2}{l^2}t}\sin\left(\frac{n\pi x}{l}\right).$$

Ex 1.7. *A rod of length l has an initial uniform temperature of 0°C. At $x = 0$, it maintains a* ***constant*** *temperature u_0, while at the ends $x = l$ and the lateral surface, it has heat exchange with the surrounding medium at 0°C. The temperature distribution $u(x,t)$ satisfies:*

$$\begin{cases} u_t = a^2u_{xx}\boxed{-b^2u} & \leftarrow \textit{(Integ. factor)} \\ u(0,t) = u_0 & \leftarrow \textit{(Non-homog. bdry)} \leftarrow \textit{(Simultaneous homogenization \& auxiliary func.)} \\ (u_x + \sigma u)|_{x=l} = 0 & \leftarrow \textit{(Third kind)} \\ u(x,0) = 0 \end{cases}$$

for $0 < x < l,\ t > 0$ and $\sigma > 0$. Solve for $u(x,t)$.

Solution.
- *Auxiliary function method. Let $u = v + w$. However, by transforming v, the equation can be simplified (simplified equation).*

Let $u(x,t) = e^{-b^2t}v(x,t) + w(x)$. We calculate

$$\begin{cases} u_t = e^{-b^2t}v_t - b^2e^{-b^2t}v \\ u_{xx} = e^{-b^2t}v_{xx} + w'' \end{cases}$$

$$\Rightarrow e^{-b^2t}v_t \cancel{-b^2e^{-b^2t}}v = a^2e^{-b^2t}v_{xx} + \underbrace{a^2w'' - b^2w}_{Let\ it\ =0} \cancel{-b^2e^{-b^2t}}v$$

Then we arrive at

$$\begin{cases} v_t - a^2v_{xx} = 0 & 0 < x < \ell,\, t > 0 \\ v(0,t) = (u_x + \sigma v)|_{x=\ell} = 0 & t \geq 0 \\ v(x,0) = -w \end{cases} \tag{14}$$

and

$$\begin{cases} a^2w'' - b^2w = 0, \quad 0 < x < \ell \\ w(0) = u_0 \\ w'(\ell) + \sigma w(\ell) = 0 \end{cases} \tag{15}$$

The general solution of (15) *is* $w(x) = Ae^{\frac{b}{a}x} + Be^{-\frac{b}{a}x}$ *and* $w'(x) = A\frac{b}{a}e^{\frac{b}{a}x} - \frac{b}{a}Be^{-\frac{b}{a}x}$*, using boundary conditions.*

$$\begin{cases} w(0) = A + B = u_0 \\ w'(\ell) + \sigma w(\ell) = A\frac{b}{a}e^{\frac{b}{a}\ell} - B\frac{b}{a}e^{-\frac{b}{a}\ell} + \sigma Ae^{\frac{b}{a}\ell} + \sigma Be^{-\frac{b}{a}\ell} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{u_0\left(\frac{b}{a}-\sigma\right)e^{-\frac{b}{a}\ell}}{2\left(\frac{b}{a}\cosh\frac{b}{a}\ell + \sigma\sinh\frac{b}{a}\ell\right)} \\ B = \frac{u_0\left(\frac{b}{a}+\sigma\right)e^{\frac{b}{a}\ell}}{2\left(\frac{b}{a}\cosh\frac{b}{a}\ell + \sigma\sinh\frac{b}{a}\ell\right)} \end{cases}$$

$$\Rightarrow w(x) = \frac{\frac{1}{a}\cosh\frac{b}{a}(x-\ell) - \sigma\sinh\frac{b}{a}(x-\ell)}{\frac{b}{a}\cosh\frac{b}{a}\ell + \sigma\sinh\frac{b}{a}\ell}u_0$$

The system (14) *can be solved by separation of variables. Let* $u(x,t) = X(x)T(t)$*, we get*

$$XT' - a^2X''T = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{a^2T} = -\lambda$$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + a^2\lambda T = 0 \end{cases}$$

Boundary conditions

$$\begin{cases} X(0) = 0 \\ X'(\ell) + \sigma X(\ell) = 0 \end{cases}$$

Then the Sturm-Liouville problem is

$$\begin{cases} X'' + \lambda X = 0 \quad (0 < x < \ell) \\ X(0) = X'(\ell) + \sigma X(\ell) = 0 \end{cases}$$

By the energy method,

$$\int_0^\ell XX''dx + \int_0^\ell \lambda X^2dx = 0 \quad \overset{Integration\ by\ parts}{\Rightarrow} \quad XX'\Big|_0^\ell - \int_0^\ell (X')^2dx + \int_0^\ell \lambda X^2dx = 0$$

$$\Rightarrow X(\ell)X'(\ell) + \lambda\int_0^\ell X^2dx = \int_0^\ell (X')^2dx$$

$$\Rightarrow -\sigma(X(\ell))^2 + \lambda\int_0^\ell x^2dx = \int_0^\ell (X')^2dx$$

$$\Rightarrow \lambda\int_0^\ell X^2dx = \sigma(X(\ell))^2 + \int_0^\ell (X')^2dx \geq 0$$

$$\Rightarrow \lambda \geq 0,\ \textit{if } \lambda = 0 \Rightarrow \int_0^\ell (X')^2 dx = -\sigma(X(\ell))^2 \leq 0$$

$$\Rightarrow \quad X' \equiv 0 \quad \Rightarrow \quad X(x) = const. \quad \Rightarrow \quad X(x) \equiv 0 \quad \Rightarrow \quad \lambda > 0.$$

If $\lambda > 0$, then the general solution is $X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Therefore $X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$.

Then $X(0) = A = 0$ and $X'(\ell) + \sigma X(\ell) = B\sqrt{\lambda}\cos\sqrt{\lambda}\ell + \sigma B\sin\sqrt{\lambda}\ell = 0$.

$$\Rightarrow \sqrt{\lambda}\cos\sqrt{\lambda}\ell = -\sigma\sin\sqrt{\lambda}\ell \quad (\neq 0 \textit{ unless } \cos \textit{ and } \sin \textit{ are zero at the same time, impossible}).$$

$$\Rightarrow \sqrt{\lambda} + \sigma\tan\sqrt{\lambda}\ell = 0. \tag{16}$$

The eigenvalues are the n-th positive roots of the transcendental equation (denoted as λ_n), and the eigenfunctions are $X_n(x) = \sin\sqrt{\lambda_n}x$. Putting λ_n into the T-equation leads to $T_n(t) = C_n e^{-a^2\lambda_n t}$.

Then the superposition series solution is

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-a^2\lambda_n t}\sin\sqrt{\lambda_n}x.$$

Initial data determine the coefficients

$$C_n = \frac{-\int_0^\ell w(x)\sin\sqrt{\lambda_n}x\,dx}{\int_0^\ell \sin^2\sqrt{\lambda_n}x\,dx} = \frac{-\int_0^\ell \frac{\frac{1}{a}\cosh\frac{b}{a}(x-\ell) - \sigma\sinh\frac{b}{a}(x-\ell)}{\frac{b}{a}\cosh\frac{b}{a}\ell + \sigma\sinh\frac{b}{a}\ell} u_0 \sin\sqrt{\lambda_n}x\,dx}{\int_0^\ell \sin^2\sqrt{\lambda_n}x\,dx}.$$

Ex 1.8. *The longitudinal vibration equation for a frustum with height l, top radius r, and bottom radius R is*

$$\left(1 - \frac{x}{l}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2}{\partial x}\left[\left(1 - \frac{x}{l}\right)^2 \frac{\partial u}{\partial x}\right]$$

where $a^2 = \frac{E}{\rho}$ is the modulus of elasticity, ρ is the density, and $h = \ell R/(R-r)$. If the ends of the frustum are rigidly fixed, with initial displacement $f(x)$ and initial velocity 0, find the longitudinal displacement $u(x,t)$.

Solution. *The corresponding spatial problem is*

$$\begin{cases} \left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right] & 0 < x < \ell,\ t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = 0 & 0 \leq x \leq \ell \\ u(0,t) = u(\ell,t) = 0 & t \geq 0 \end{cases}$$

Integrating factor:

Step-by-step analysis *of*

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{(h-x)^2}\frac{\partial}{\partial x}\left((h-x)^2\frac{\partial u}{\partial x}\right).$$

Focus on term

$$\frac{1}{(h-x)^2}\frac{\partial}{\partial x}\left((h-x)^2\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{(h-x)^2}{(h-x)^2}\frac{\partial u}{\partial x}\right) - \frac{2}{(h-x)^3}\cdot(h-x)^2\frac{\partial u}{\partial x} = u_{xx} - \frac{2}{h-x}u_x.$$

The equation becomes

$$\boxed{u_{tt} = a^2\left(u_{xx} - \frac{2}{h-x}u_x\right).}$$

Ideas:

- *$u_{xx} + Au_x$ can be combined into a derivative of a single term using the exponential function.*

- *The form $u_{xx} + \frac{A}{x} u_x$ also has a similar integrating factor.*

Beginners can use the method of the function to be determined to find the integrating factor.

Let a function $A(x)$, guess from the form that it is independent of t, and hope that the u_t term does not appear.

Multiply $A(x)$ to both sides of the above equation.

$$(A(x)u)_{tt} = a^2 \underbrace{\left(A(x)u_{xx} - \frac{2}{h-x} u_x A(x) \right)}_{Hope = (A(x)u)_{xx}}$$

First we calculate

$$(A(x)u)_x = A'u + A(x)u_x \quad \text{and} \quad (A(x)u)_{xx} = A''u + 2A'u_x + Au_{xx}.$$

From the hope $(A(x)u)_{xx} = A(x)u_{xx} - \frac{2}{h-x} u_x A(x)$, we get $A''u + 2A'u_x + Au_{xx} = Au_{xx} - \frac{2}{h-x} Au_x$,

$$\Rightarrow A''u + 2\left(A' + \frac{1}{h-x} \right) u_x = 0.$$

Try $A'' = 0$ (under this enforcing condition, we choose an integrating factor). We only need to require

$$A' + \frac{1}{h-x} A = 0.$$

$$\Rightarrow \quad (\ln A)' = -\frac{1}{h-x} \quad \Rightarrow \quad \ln A = \int \frac{1}{x-h} dx + C = \ln(x-h) + C.$$

$$\Rightarrow A = C(x-h) \quad \textit{while satisfying} \quad A'' = 0.$$

For simplicity, we take $A(x) = x - h$.

Then the equation can be written as $((x-h)u)_{tt} = a^2((x-h)u)_{xx}$ and let $v = (x-h)u$. Then the equation becomes $v_{tt} = a^2 v_{xx}$. The problem becomes:

$$\begin{cases} v_{tt} = a^2 v_{xx} \\ v(x,0) = (x-h)u(x,0) = (x-h)f(x),\ v_t(x,0) = (x-h)u_t(x,0) = 0 \\ v(0,t) = -hu(0,t) = 0,\ v(\ell,t) = (\ell-h)u(\ell,t) = 0 \end{cases}$$

That is

$$\begin{cases} v_{tt} = a^2 v_{xx} \\ v(x,0) = (x-h)f(x),\ v_t(x,0) = 0 \\ v(0,t) = v(\ell,t) = 0 \end{cases}$$

This is a form that everyone is familiar with, and can be found in §2.1.

$$v(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi at}{\ell} + b_n \sin \frac{n\pi at}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

where

$$\begin{cases} a_n = \frac{2}{\ell} \int_0^\ell (x-h)f(x) \sin \frac{n\pi x}{\ell}\, dx \\ b_n = 0 \end{cases}.$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{\ell} \int_0^\ell (x-h)f(x) \sin \frac{n\pi x}{\ell}\, dx \right) \cos \frac{n\pi at}{\ell} \sin \frac{n\pi x}{\ell}.$$

$$\Rightarrow u(x,t) = \frac{v(x,t)}{x-h} = \frac{2}{(x-h)\ell} \sum_{n=1}^{\infty} \left(\int_0^\ell (\xi-h)f(\xi) \sin \frac{n\pi \xi}{\ell}\, d\xi \right) \cos \frac{n\pi at}{\ell} \sin \frac{n\pi x}{\ell}.$$

Ex 1.9. *Consider a circular ring-shaped heat conductor with an inner radius r_1 and an outer radius r_2, insulated on the top and bottom. If the temperature of the inner circle is maintained at zero degrees, and the temperature of the outer circle is maintained at u_0 ($u_0 > 0$), find the temperature distribution $u(r,\theta)$ in the steady state. The problem is reduced to solving the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ in the steady state, which is a boundary value problem in polar coordinates:*

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & r_1 < r < r_2,\ 0 < \theta < 2\pi, \\ u(r_1,\theta) = 0,\ u(r_2,\theta) = u_0, & 0 < \theta < 2\pi, \\ u(r,\theta) = u(r,\theta+2\pi) & \text{(natural boundary condition)}. \end{cases}$$

- *This problem does **not include the origin**, so there is **no need** to add a **boundedness condition** at the origin. A **periodicity condition** is **sufficient**.*

Solution (Method 1: Ignore the shperical symmetry). *1. Assume the separated solution form is:*

$$u(r,\theta) = R(r)\Phi(\theta) \quad \text{(separation of variables)}$$

2. PDE → ODEs

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0 \quad \Rightarrow \quad -\frac{rR'' + rR'}{R} = \frac{\Phi''}{\Phi} = -\lambda$$

Thus, we have the following system of equations:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ r^2R'' + rR' - \lambda R = 0 \end{cases}$$

This implies:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(\theta) = \Phi(\theta+2\pi) \end{cases} \quad \text{(S-L problem)}$$

and

$$r^2R'' + rR' - \lambda R = 0$$

3. Solve ODEs

(a) When $\lambda < 0$, the general solution is $\Phi(\theta) = Ae^{\sqrt{-\lambda}\theta} + Be^{-\sqrt{-\lambda}\theta}$, which does not satisfy periodic boundary conditions, hence $A = B = 0$, no non-trivial solution.

(b) When $\lambda = 0$, the solution is $\Phi(\theta) = Ax + B$. To satisfy the periodic boundary conditions $\Phi(0) = \Phi(\theta + 2\pi)$,

$$\Rightarrow A = 0 \Rightarrow \Phi_0(\theta) = B_0$$

Solve ODE of R: $r^2R'' + rR' = 0$. The solution is $R(r) = C_0 \ln r + D_0$. Therefore, $u_0(r,\theta) = a_0 \ln r + b_0$.

(c) When $\lambda > 0$, the solution is

$$\Phi(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta).$$

If $\Phi(\theta)$ is periodic with period 2π, then

$$\sqrt{\lambda}(\theta + 2\pi) = \sqrt{\lambda}\theta + 2n\pi. \Rightarrow 2\sqrt{\lambda}\pi = 2n\pi \Rightarrow \lambda_n = n^2 \quad (n = 1,2,\ldots)$$

Thus,

$$\Phi_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta).$$

Solve ODE of R:

$$r^2R'' + rR' - n^2R = 0 \quad \text{(Euler equation)} \Rightarrow R_n(r) = C_nr^n + D_n\frac{1}{r^n} \quad (n = 1,2,\ldots)$$

$$\Rightarrow u_n(r,\theta) = (A_n\cos n\theta + B_n\sin n\theta)\left(C_nr^n + D_n\frac{1}{r^n}\right)$$

4. *Superposition of solutions*

$$u(r,\theta) = C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r^n + D_n \frac{1}{r^n} \right)$$

5. *Determine coefficients*

$$\begin{cases} u(r_1,\theta) = C_0 \ln r_1 + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \\ u(r_2,\theta) = C_0 \ln r_2 + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = u_0 \end{cases}$$

This leads to:

$$\begin{cases} C_0 \ln r_1 + D_0 = 0 \\ A_n \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \\ B_n \left(C_n r_1^n + D_n \frac{1}{r_1^n} \right) = 0 \end{cases} \quad \text{and} \quad \begin{cases} C_0 \ln r_2 + D_0 = u_0 \\ A_n \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = 0 \\ B_n \left(C_n r_2^n + D_n \frac{1}{r_2^n} \right) = 0 \end{cases}$$

Thus, we have the system of equations:

$$\begin{cases} C_0 \ln r_1 + D_0 = 0 \\ C_0 \ln r_1 + D_0 = u_0 \end{cases} \Rightarrow \begin{cases} C_0 = -\frac{u_0}{\ln \frac{r_1}{r_2}} = \frac{u_0}{\ln \frac{r_2}{r_1}} \\ D_0 = -\frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} \end{cases}$$

For the coefficients:

$$\begin{cases} A_n C_n r_1^n + A_n D_n \frac{1}{r_1^n} = 0 \\ A_n C_n r_2^n + A_n D_n \frac{1}{r_2^n} = 0 \end{cases} \Rightarrow \begin{cases} A_n C_n = 0 \\ A_n D_n = 0 \end{cases}$$

and similarly,

$$\begin{cases} B_n C_n = 0 \\ B_n D_n = 0 \end{cases}$$

Finally, the solution is:

$$u(r,\theta) = \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r - \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r_1 = u_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}$$

Solution (Method 2: Note the spherical symmetry). *From the boundary conditions, it is known that the circular ring has spherical symmetry, hence $u(r,\theta) = u(r)$, which is independent of θ.*

The equation becomes

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0 \Rightarrow \frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

$$\Rightarrow r \frac{du}{dr} = A \quad \Rightarrow \quad \frac{du}{dr} = \frac{A}{r} \quad \Rightarrow \quad du = \frac{A}{r} dr \quad \Rightarrow \quad u(r) = A \ln r + B$$

Using boundary conditions $u(r_1) = A \ln r_1 + B = 0$ and $u(r_2) = A \ln r_2 + B = u_0$, we get:

$$\begin{cases} A = \frac{u_0}{\ln \frac{r_2}{r_1}} \\ B = -\frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} \end{cases}.$$

Thus, the solution is:

$$u(r) = \frac{u_0}{\ln \frac{r_2}{r_1}} \ln r - \frac{u_0 \ln r_1}{\ln \frac{r_2}{r_1}} = u_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}.$$

Ex 1.10. *Find the solution to the following boundary value problem:*

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, & 0 < r < 1, \ 0 < \theta < \frac{\pi}{2}, \\ u(r,0) = 0, \ u\left(r, \frac{\pi}{2}\right) = 0, & 0 < r < 1, \\ u(1,\theta) = \theta \left(\frac{\pi}{2} - \theta \right), & 0 < \theta < \frac{\pi}{2}. \end{cases}$$

- Note that the problem has $0 < \theta < \frac{\pi}{2}$, hence it does **not have periodic boundary conditions** but has boundary conditions. However, the problem has **homogeneous boundary conditions** regarding θ, which are the conditions we need to establish a Sturm-Liouville (S-L) problem.

Solution. *1. Assume the separated solution form:*

$$u(r, \theta) = R(r)\Phi(\theta).$$

(separation of variables)

2. PDE transforms into ODEs:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ r^2 R'' + rR' - \lambda R = 0. \end{cases}$$

From the boundary conditions $u(r, 0) = 0$, $u(r, \frac{\pi}{2}) = 0$, we get:

$$R(r)\Phi(0) = 0, \quad R(r)\Phi(\frac{\pi}{2}) = 0$$

$$\Rightarrow \Phi(0) = 0, \quad \Phi(\frac{\pi}{2}) = 0$$

otherwise $R(r) = 0$ has no non-trivial solution.

3. Solving ODEs: (a) The S-L problem is given by:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = 0, \Phi\left(\frac{\pi}{2}\right) = 0 \end{cases} \leftarrow \boxed{\textit{Homog. bdry instead of periodic bdry!}}$$

The eigenvalues and corresponding eigenfunctions are (with $l = \frac{\pi}{2}$):

$$\lambda = \lambda_n = \left(\frac{n\pi}{\pi/2}\right)^2 = (2n)^2 = 4n^2 \quad (n = 1, 2, \ldots)$$

$$\Phi_n(\theta) = \sin(2n\theta) \quad (n = 1, 2, \ldots)$$

(b) The ODE for R becomes:

$$r^2 R'' + rR' - 4n^2 R = 0$$

The general solution is:

$$R_n(r) = C_n r^{2n} + D_n r^{-2n} \quad (n = 1, 2, \ldots)$$

Using the boundedness condition $|R_n(0)| < +\infty$ implies $D_n = 0$.

$$R_n(r) = C_n r^{2n}.$$

This implies

$$u_n(r, \theta) = C_n r^{2n} \sin(2n\theta) \quad (n = 1, 2, \ldots)$$

4. Superposition of solutions:

$$u(r, \theta) = \sum_{n=1}^{\infty} C_n r^{2n} \sin(2n\theta)$$

5. Determine coefficients:

$$u(1, \theta) = \sum_{n=1}^{\infty} C_n \sin(2n\theta) = \theta\left(\frac{\pi}{2} - \theta\right)$$

Using Fourier series to find C_n,

$$\begin{aligned}
C_n =& \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \theta\left(\frac{\pi}{2}-\theta\right)\sin(2n\theta)d\theta = -\frac{2}{n\pi}\int_0^{\frac{\pi}{2}} \theta(\frac{\pi}{2}-\theta)d\cos(2n\theta) \\
=& -\frac{2}{n\pi}\left[\cancel{\theta\left(\frac{\pi}{2}-\theta\right)\cos(2n\theta)\Big|_0^{\frac{\pi}{2}}} - \int_0^{\frac{\pi}{2}}\cos(2n\theta)\left(\frac{\pi}{2}-2\theta\right)d\theta\right] \\
=& \frac{2}{n\pi}\int_0^{\frac{\pi}{2}}\cos(2n\theta)\cdot\left(\frac{\pi}{2}-2\theta\right)d\theta \\
=& \frac{\pi}{n\pi}\int_0^{\frac{\pi}{2}}\cos(2n\theta)d\theta - \frac{2\cdot 2}{n\pi}\int_0^{\frac{\pi}{2}}\theta\cos(2n\theta)d\theta \\
=& \cancel{\frac{1}{2n\pi}\sin(2n\theta)\Big|_0^{\frac{\pi}{2}}} - \frac{2}{n^2\pi}\int_0^{\frac{\pi}{2}}\theta d\sin(2n\theta) \\
=& -\frac{2}{n^2\pi}\left[\cancel{\theta\sin(2n\theta)\Big|_0^{\frac{\pi}{2}}} - \int_0^{\frac{\pi}{2}}\sin(2n\theta)d\theta\right] \\
=& \frac{2}{n^2\pi}\int_0^{\frac{\pi}{2}}\sin(2n\theta)d\theta = -\frac{1}{n^3\pi}\cos(2n\theta)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{n^3\pi}(\cos(n\pi)-1) \\
=& -\frac{1}{n^3\pi}((-1)^n-1) = \frac{1}{n^3\pi}[1-(-1)^n].
\end{aligned}$$

Thus,

$$u(r,\theta) = \sum_{n=1}^{\infty}\frac{1}{n^3\pi}[1-(-1)^n]r^{2n}\sin(2n\theta).$$