Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 10-1

1 5.3 Expansion into Series in Terms of Bessel Functions

- This section is about modifying the **fifth step** in the method of separation of variables, which is **determining the coefficients** using initial values.
- Previously, a Fourier series was written, with unknown coefficients determined by taking the inner product of the Fourier series with sine or cosine functions, utilizing their **orthogonality and norms**.
- Now, the method of separation of variables is modified by **replacing the trigonometric functions with Bessel functions**.
- This section focuses on studying the **orthogonality** and **norms** of **Bessel functions**.
- Similar to **Fourier series expansion** using trigonometric functions, many functions can be **expanded** using the **Bessel function system** under certain conditions.
- The resulting series after expansion is called the **Fourier-Bessel series**, which is a **generalization of the Fourier series**, replacing the basis function system with the **Bessel function system**.

When using Bessel functions to solve the definite solution problems of mathematical physics equations, we ultimately need to expand the known function into a series in terms of the Bessel function system. In this section, we will discuss this problem.

- We draw an analogy with the S-L problem solved in Chapter 2 to help understand the current problem (see Table 1).
- In the upcoming Section 5.4, we will solve the PDEs and see that the S-L problem is replaced by this Bessel equation.
- In Section 5.4, the primary focus will be on the method of separation of variables.
- It is essential to learn to use analogies to transfer the methods of eigenfunction and auxiliary function methods that you have previously learned to PDE problems involving Bessel equations.

At the beginning of this chapter, from the definite solution problem of the temperature distribution in a thin circular disk, we derived the **eigenvalue problem of the Bessel equation**:

$$r^2F'' + rF' + (\lambda r^2 - n^2)F = 0, \tag{1}$$

$$F(R) = 0, \quad |F(0)| < +\infty. \tag{2}$$

The **general solution** of equation (1) is

$$F(r) = CJ_n(\sqrt{\lambda}r) + DY_n(\sqrt{\lambda}r)$$

Since the general solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

is

$$y = CJ_n(x) + DY_n(x)$$

By the coordinate transformation $x = \sqrt{\lambda}r$ and $y(x) = F(r)$, we have

$$F(r) = CJ_n(\sqrt{\lambda}r) + DY_n(\sqrt{\lambda}r)$$

Since $Y_n(0)$ is infinite, according to the boundedness condition in the boundary condition (2), we know that $D = 0$. Thus

$$F(r) = CJ_n(\sqrt{\lambda}r)$$

In addition, by using the condition $F(R) = 0$ in (2), we obtain

$$J_n(\sqrt{\lambda}R) = 0. \tag{3}$$

1.0.1 5.3.1 Zeros of Bessel Functions

Equation (3) indicates that in order to find the eigenvalue λ of the eigenvalue problem (1)-(2), we need to determine whether the zeros of $J_n(x)$ exist. The so called **zeros of the Bessel function** refer to the values of x that make $J_n(x) = 0$.

There is a series of theorems about the **zeros of Bessel functions**.

1.0.2 5.3.1 Distribution of Zeros of Bessel Functions (Important Conclusions):

1. $J_n(x)$ has infinitely many simple real zeros. These zeros are symmetrically distributed about the origin on the x-axis. Therefore, $J_n(x)$ has **infinitely many positive zeros** (**countable**).
2. The zeros of $J_n(x)$ and $J_{n+1}(x)$ are **interlaced** with each other. Moreover, the zero with the smallest absolute value of $J_n(x)$ is closer to 0 than the zero with the smallest absolute value of $J_{n+1}(x)$. Naturally, $J_n(x)$ and $J_{n+1}(x)$ have **no common zeros**.
3. When x is large enough, the **distance between two adjacent zeros** of $J_n(x)$ is **close to** π.

- Bessel functions exhibit oscillatory behavior similar to trigonometric functions.
- Only J_0 reaches a maximum value of 1 at its peak, while others do not reach 1.
- Trigonometric functions always peak at 1, unlike higher-order Bessel functions.
- Bessel functions exhibit parity properties:
  - J_n is an **odd function** if n is **odd**.
  - J_n is an **even function** if n is **even**.
- J_0, being even, **resembles the cosine function** due to its symmetry about the y-axis.
- The shape of Bessel functions can be **visualized** by **distorting trigonometric functions**:
  - Imagine holding a trigonometric function and pulling the sides while keeping the middle high.
  - This distortion results in the characteristic shape of Bessel functions.

- Bessel functions can be thought of as a **twisted version** of trigonometric functions.
- Trigonometric functions like sine and cosine have a period of 2π, with **zeros spaced** π **apart**.
- As Bessel functions **approach infinity**, their behavior and **shape increasingly resemble trigonometric functions**.
- **Qualitative descriptions** of zeros provide insight into their **distribution**.
- **Specific values of zeros** can be **approximated** using **computational methods or numerical algorithms**.
- For nonlinear PDEs, exact solutions are often intractable; instead, we **describe their properties in this fashion**.
- We can **estimate** maximum and minimum values, frequency of oscillations, and asymptotic behavior.
- **Asymptotically**, solutions may **resemble known functions**, such as trigonometric functions.
- These **descriptive methods** are fundamental in PDE theory when **exact forms** are **not obtainable**.

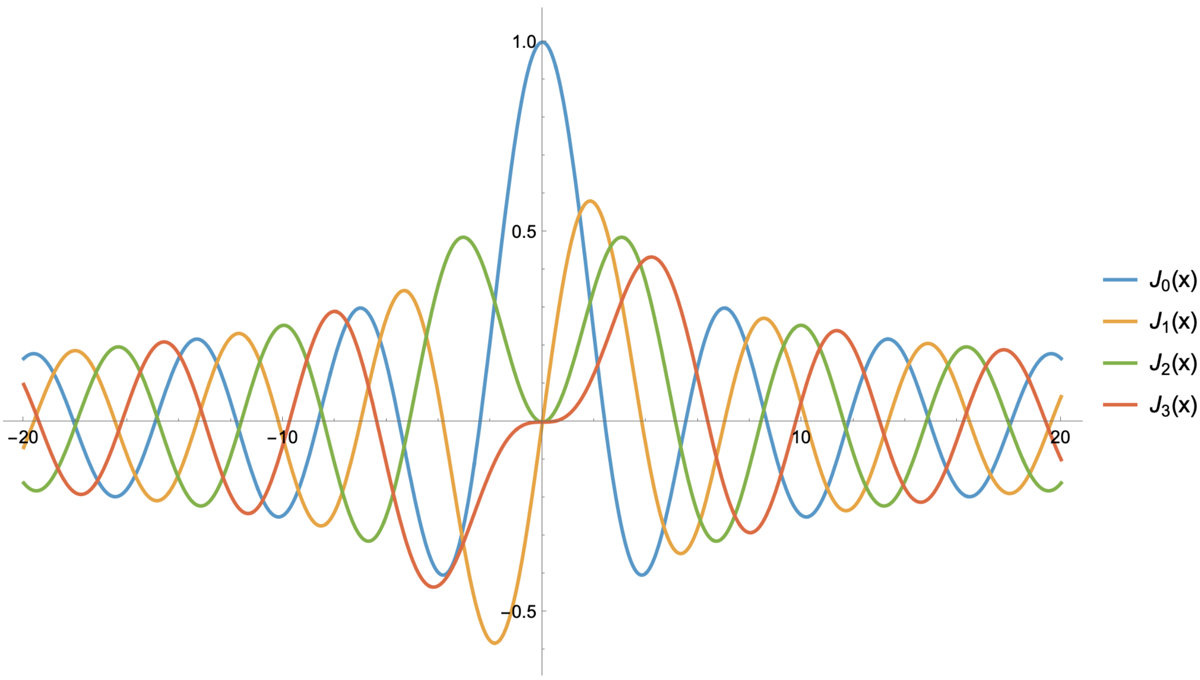

Figure 1: Zeros of Bessel functions 1

Integer order Bessel functions are more widely used, especially $J_0(x)$ and $J_1(x)$.

Applying the above conclusions about the zeros of Bessel functions, let $\mu_m^{(n)}$ $(m = 1, 2, \cdots)$ be the positive zeros of $J_n(x)$. Then from equation (3), we have

$$\sqrt{\lambda}R = \mu_m^{(n)} = \mu^{(n)\leftarrow\boxed{n \text{ in the Bessel equation ("}n\text{-th order")}}}_{m\leftarrow\boxed{\text{the } m\text{-th positive zero}}} \qquad (m = 1, 2, \cdots)$$

$$\lambda_m^{(n)} = \left(\frac{\mu_m^{(n)}}{R}\right)^2 \qquad (m = 1, 2, \cdots),$$

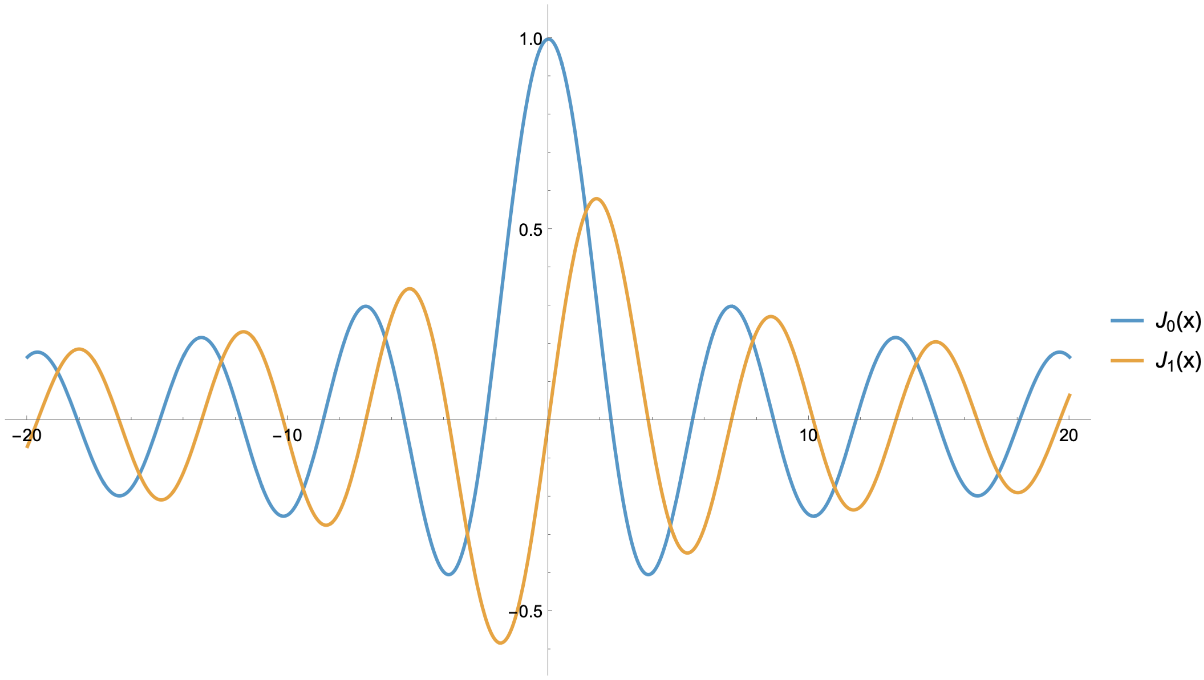

Figure 2: Zeros of Bessel functions 2

The eigenfunctions corresponding to these eigenvalues are (this form has a very strong correspondence with $\sin(\frac{n\pi}{l}x)$)

$$F_m(r) = J_n\left(\sqrt{\lambda_m^{(n)}}r\right) = \underbrace{J_n}_{\sin}\left(\underbrace{\frac{\mu_m^{(n)}}{R}}_{\frac{n\pi}{\ell}}\underbrace{r}_{x}\right) \qquad (m = 1, 2, \cdots). \tag{4}$$

Note the relations

$$J_n(\mu_m^{(n)}) = 0 \quad \leftarrow \boxed{\text{Definition of zeros}}$$

$$\left.\begin{aligned} J_{n+1}(\mu_m^{(n)}) \neq 0 \\ J_{n-1}(\mu_m^{(n)}) \neq 0 \end{aligned}\right\} \leftarrow \boxed{\text{Zeros must occur alternately, and there are no common zeros}}$$

1.0.3 5.3.2 Orthogonality of the Bessel Function System

Recall the Bessel equation and §2.6

$$\underbrace{r^2F'' + rF'}_{=r\partial_r(r\partial_r F)} + (\lambda r^2 - n^2)F = 0 \implies \partial_r(r\partial_r F) - \frac{n^2}{r}F + \lambda\underbrace{r}_{\text{Weight}}F = 0 \quad \text{(S-L problem)}$$

$\uparrow$ $\boxed{\text{Memorize this useful operator as mentioned before}}$

- Recall a general form of a S-L problem discussed in Section 2.6, which provides three key conclusions:
  1. There are **countably infinite** eigenvalues and eigenfunctions.
     - Transform it into the problem of **finding the zeros** of special functions (such as J_n, sin).

2. Eigenfunctions exhibit **orthogonality**.
   - If the eigenfunctions corresponding to the eigenvalue λ_n are denoted as $y_n(x)$, then all $y_n(x)$ form an orthogonal set with respect to **the weight function** $\rho(x)$, that is,

$$\int_a^b \rho(x) y_m(x) y_n(x) dx = 0 \quad (m \neq n).$$

   - Below we will use Bessel functions as an example to prove this conclusion.
   - The inner product must be replaced with the **weighted inner product**, that is, $\langle f, g\rangle_\rho = \int fg\rho dx$. For the Bessel equation, the **weight** function $\rho = r$.
3. Any function can be **expanded using these eigenfunctions**.

Theorem 1.1. *The sequence of n-**th order Bessel functions** (4) is **orthogonal** with weight r on the interval $(0, R)$, that is*

$$\left\langle J_n\left(\frac{\mu_m^{(n)}}{R} r\right), J_n\left(\frac{\mu_k^{(n)}}{R} r\right)\right\rangle_r = \int_0^R r \underbrace{J_n}_{n\text{-th order Bessel}} \left(\frac{\mu_m^{(n)}}{R} r\right) J_n\left(\frac{\mu_k^{(n)}}{R} r\right) dr = 0, \quad m \neq k. \tag{5}$$

- The following proof applies to general special functions, as long as the Bessel functions are replaced with other eigenfunction systems.
- Recalling the Exercise class Ex 2.10.2 in Chapter 2, the proof is the same.
- **Energy Method**:
  - **Multiply** the equation by the unknown function.
  - **Integrate** over the domain.
  - Apply **integration by parts** (Green's first identity in higher dimensions).
  - The **boundary terms** imply that λ must be positive for non-trivial solutions.

Proof. Rewrite the Bessel equation (1) as follows (to prove the orthogonality, we must use the standard form of S-L problem)

$$\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \left(\lambda r - \frac{n^2}{r}\right) F = 0 \quad \text{and} \quad F(R) = 0.$$

For the convenience of writing, let

$$F_1(r) = J_n(\alpha_1 r), \quad F_2(r) = J_n(\alpha_2 r),$$

where α_1 and α_2 are arbitrary parameters. Note, by taking $\alpha_1 = \mu_m^{(n)}/R$, $\alpha_2 = \mu_k^{(n)}/R$,

$$F_1(R) = J_n(\mu_m^{(n)}) = 0, \quad F_2(R) = J_n(\mu_k^{(n)}) = 0, \quad \mu_m^{(n)} \neq \mu_k^{(n)}. \tag{6}$$

That is, the Bessel solutions satisfy the boundary condition.

Then

$$\frac{d}{dr}\left(r\frac{dF_1}{dr}\right) + \left(\alpha_1^2 r - \frac{n^2}{r}\right) F_1 = 0,$$
$$\frac{d}{dr}\left(r\frac{dF_2}{dr}\right) + \left(\alpha_2^2 r - \frac{n^2}{r}\right) F_2 = 0.$$

Step 1: Multiply the equation by the unknown function. Multiply the above two equations by F_2 and F_1 respectively. Then

$$F_2\frac{d}{dr}\left(r\frac{dF_1}{dr}\right) + \left(\alpha_1^2 r - \frac{n^2}{r}\right) F_1F_2 = 0 \Rightarrow \int_0^R F_2\frac{d}{dr}\left(r\frac{dF_1}{dr}\right) + \left(\alpha_1^2 r - \frac{n^2}{r}\right) F_1F_2 dr = 0,$$

$$F_1 \frac{d}{dr}\left(r\frac{dF_2}{dr}\right) + \left(\alpha_2^2 r - \frac{n^2}{r}\right) F_2 F_1 = 0 \Rightarrow \int_0^R F_1 \frac{d}{dr}\left(r\frac{dF_2}{dr}\right) + \left(\alpha_2^2 r - \frac{n^2}{r}\right) F_2 F_1 dr = 0.$$

Step 2 and 3: Integrate, Apply integration by parts and obtain the symmetric forms, then subtract the symmetric forms: Integrate both sides of the above equation with respect to r from 0 to R

$$(\alpha_1^2 - \alpha_2^2)\int_0^R rF_1(r)F_2(r)dr + \left[rF_2\frac{dF_1}{dr} - rF_1\frac{dF_2}{dr}\right]_0^R = 0. \tag{7}$$

By taking $\alpha_1 = \mu_m^{(n)}/R$, $\alpha_2 = \mu_k^{(n)}/R$ and using (6), we can immediately obtain that equation (5) holds. The sequence of n-th order Bessel functions (4) is **orthogonal** with weight r on the interval $(0, R)$. □

1.1 5.3.3 The Norm of Bessel Functions

Theorem 1.2. *The square root of the definite integral*

$$\left\langle J_n\left(\frac{\mu_m^{(n)}}{R}r\right), J_n\left(\frac{\mu_m^{(n)}}{R}r\right)\right\rangle_r = \int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)dr = \frac{R^2}{2}J_{n-1}^2(\mu_m^{(n)}) = \frac{R^2}{2}J_{n+1}^2(\mu_m^{(n)})$$

is called the ***norm of the Bessel function*** $J_n\left(\frac{\mu_m^{(n)}}{R}r\right)$.

Proof. When $\alpha_1 \neq \alpha_2$, from equation (7), we have

$$\int_0^R rF_1(r)F_2(r)dr = -\frac{\left[rF_2\frac{dF_1}{dr} - rF_1\frac{dF_2}{dr}\right]_0^R}{\alpha_1^2 - \alpha_2^2}. \tag{8}$$

In the above formula, let $\alpha_1 = \mu_m^{(n)}/R$ fixed, and α_2 remain an arbitrary parameter (α_2 is not a eigenvalue, but it is close to α_1). Since

$$F_1(R) = J_n(\mu_m^{(n)}) = 0 \quad \text{and} \quad \left.\frac{dF_1}{dr}\right|_R = \alpha_1 J_n'(\alpha_1 R) = \frac{\mu_m^{(n)}}{R}J_n'(\mu_m^{(n)}),$$

the above formula (8) is transformed into

$$\mathcal{G}(\alpha_2) := \underbrace{\int_0^R rJ_n\left(\frac{\mu_m^{(n)}}{R}r\right)J_n(\alpha_2 r)dr}_{\textbf{Continuous}\text{ function of }\alpha_2\text{ since }J_n\text{ is continuous}} = -\frac{\mu_m^{(n)}J_n(\alpha_2 R)J_n'(\mu_m^{(n)})}{(\mu_m^{(n)}/R)^2 - \alpha_2^2}.$$

Since $\mathcal{G}(\alpha_2)$ is a continuous function and thus

$$\lim_{\alpha_2\to\mu_m^{(n)}/R}\mathcal{G}(\alpha_2) = \mathcal{G}(\mu_m^{(n)}/R) = \int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)dr.$$

However, when $\alpha_2 \to \mu_m^{(n)}/R$, the right-hand side of the above formula is an indeterminate form of the type $\frac{0}{0}$ (since $\lim_{\alpha_2\to\mu_m^{(n)}/R} J_n(\alpha_2 R) = J_n(\mu_m^{(n)}) = 0$). Applying L'Hopital's rule, we get

$$\begin{aligned}\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)dr &= \lim_{\alpha_2\to\mu_m^{(n)}/R} -\frac{\mu_m^{(n)}J_n'(\mu_m^{(n)})J_n'(\alpha_2 R)R}{-2\alpha_2} \\ &= \frac{\mu_m^{(n)}J_n'(\mu_m^{(n)})J_n'(\mu_m^{(n)})R}{2\mu_m^{(n)}/R} = \frac{R^2}{2}[J_n'(\mu_m^{(n)})]^2. \end{aligned} \tag{9}$$

From the recurrence formulas

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x), \quad xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x),$$

and

$$J_n(\mu_m^{(n)}) = 0,$$

we obtain

$$J_n'(\mu_m^{(n)}) = J_{n-1}(\mu_m^{(n)}), \quad J_n'(\mu_m^{(n)}) = -J_{n+1}(\mu_m^{(n)}).$$

Then equation (9) becomes

$$\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)dr = \frac{R^2}{2}J_{n-1}^2(\mu_m^{(n)}) = \frac{R^2}{2}J_{n+1}^2(\mu_m^{(n)}) \tag{10}$$

Since the Bessel functions $J_n(x)$ and $J_{n+1}(x)$ have **no common zeros**, from equation (10), we know that the norm of the Bessel function is not zero. □

1.2 5.3.4 Fourier-Bessel Series

- **Definition:** The Fourier-Bessel series is a type of series expansion **similar to the Fourier series**.
- **Function System Replacement:** It **replaces the trigonometric function system** used in Fourier series with a **new system of Bessel functions**.

When using Bessel functions to solve the definite solution problems of mathematical physics equations, it is often necessary to **expand** the known function into **a series in terms of the Bessel function system**. It can be proved that: if $f(r)$ is a piecewise smooth function defined in the interval $(0, R)$, and the integral

$$\int_0^R r^{\frac{1}{2}}|f(r)|dr \leftarrow \boxed{\text{ensures the integrability of the coefficients below, using Hölder's inequality}}$$

has a **finite value**, then it can be expanded into a series of the following form:

$$f(r) = \sum_{m=1}^{\infty} C_m J_n\left(\frac{\mu_m^{(n)}}{R}r\right), \tag{11}$$

Moreover, at the continuous points of $f(r)$, the series (11) converges to $f(r)$.

- **Starting Index:** The series typically starts from $m = 1$ because λ (the eigenvalue) is greater than 0, excluding the possibility of including a constant term.

At the discontinuous point r_0 of $f(r)$, the series converges to the average of the left hand and right hand limits at the point r_0, that is, it converges to $[f(r_0+0) + f(r_0-0)]/2$. The coefficient C_m is determined by the following formula

$$C_m = \frac{\int_0^R rf(r)J_n\left(\frac{\mu_m^{(n)}}{R}r\right)dr}{\frac{R^2}{2}J_{n+1}^2(\mu_m^{(n)})}. \tag{12}$$

The C_m determined by formula (12) is called the Fourier-Bessel coefficient, and the series (11) is called the **Fourier-Bessel series**.

In fact, multiply both sides of equation (11) by $rJ_n\left(\frac{\mu_k^{(n)}}{R}r\right)$ and integrate with respect to r from 0 to R, we get

$$\int_0^R rf(r)J_n\left(\frac{\mu_k^{(n)}}{R}r\right)dr = \sum_{m=1}^{\infty} C_m \int_0^R rJ_n\left(\frac{\mu_k^{(n)}}{R}r\right)J_n\left(\frac{\mu_m^{(n)}}{R}r\right)dr.$$

By the orthogonality of the Bessel function system,

$$\int_0^R rf(r)J_n\left(\frac{\mu_m^{(n)}}{R}r\right)dr = C_m\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R}r\right)dr.$$

Therefore,

$$C_m = \frac{\int_0^R r f(r) J_n\left(\frac{\mu_m^{(n)}}{R} r\right) dr}{\int_0^R r J_n^2\left(\frac{\mu_m^{(n)}}{R} r\right) dr}.$$

Or simply,

$$\left\langle f(r), J_n\left(\frac{\mu_k^{(n)}}{R} r\right)\right\rangle_r = \sum_{m=1}^{\infty} C_m \left\langle J_n\left(\frac{\mu_m^{(n)}}{R} r\right), J_n\left(\frac{\mu_k^{(n)}}{R} r\right)\right\rangle_r = C_k \left\langle J_n\left(\frac{\mu_k^{(n)}}{R} r\right), J_n\left(\frac{\mu_k^{(n)}}{R} r\right)\right\rangle_r$$

Then

$$C_m = \frac{\left\langle f(r), J_n\left(\frac{\mu_k^{(n)}}{R} r\right)\right\rangle_r}{\left\langle J_n\left(\frac{\mu_m^{(n)}}{R} r\right), J_n\left(\frac{\mu_m^{(n)}}{R} r\right)\right\rangle_r}$$

Ex 1.1. *Let $\mu_m^{(0)}$ $(m = 1, 2, \cdots)$ be the positive zeros of the function $J_0(x)$. Try to expand the function $f(x) = 1$ into a Fourier-Bessel series in terms of $J_0(\mu_m^{(0)} x)$ on the interval $(0, 1)$.*

Solution. *According to equations (11) and (12), we have*

$$1 = \sum_{m=1}^{\infty} C_m J_0(\mu_m^{(0)} x), \quad C_m = \frac{\int_0^1 x J_0(\mu_m^{(0)} x) dx}{\frac{1}{2} J_1^2(\mu_m^{(0)})}.$$

First, calculate the numerator. Let $\mu_m^{(0)} x = r$, then

$$\int_0^1 x J_0(\mu_m^{(0)} x) dx = \frac{1}{(\mu_m^{(0)})^2} \int_0^{\mu_m^{(0)}} r J_0(r) dr = \frac{1}{(\mu_m^{(0)})^2} [r J_1(r)]_0^{\mu_m^{(0)}} = \frac{1}{\mu_m^{(0)}} J_1(\mu_m^{(0)}).$$

Substitute it into C_m, we get

$$C_m = \frac{2}{\mu_m^{(0)} J_1(\mu_m^{(0)})}.$$

Therefore,

$$1 = \sum_{m=1}^{\infty} \frac{2}{\mu_m^{(0)} J_1(\mu_m^{(0)})} J_0(\mu_m^{(0)} x).$$

References

	Chapter 2	**Chapter 5**
S-L problem:	$X'' + \lambda X = 0$ $X(0) = X(\ell) = 0$	$r^2 F'' + rF' + (\lambda r^2 - n^2)F = 0$ $F(R) = 0 \quad \lvert F(0)\rvert < \infty$
Zeros (Solve a transcendental eq. to find λ)	$\sin \sqrt{\lambda}\ell = 0$ $\Rightarrow \sqrt{\lambda}\ell = n\pi (n = 1, 2, \ldots)$	$J_n(\sqrt{\lambda}R) = 0$ $\Rightarrow \sqrt{\lambda}R = \mu_m^{(n)}$
Eigenvalues	$\lambda_n = \left(\frac{\mu_m^{(n)}}{R}\right)^2$	$\lambda_n = \left(\frac{\mu_m^{(n)}}{R}\right)^2$
Eigenfunctions	$\sin \frac{n\pi x}{\ell}$	$J_n\left(\frac{\mu_m^{(n)}}{R} r\right)$
Orthogonality	$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx = 0$ $(n \neq m)$	$\int_0^R rJ_n\left(\frac{\mu_m^{(n)}}{R} r\right) J_n\left(\frac{\mu_k^{(n)}}{R} r\right) dr = 0$ $(m \neq k)$
Norms	$\int_{-\ell}^{\ell} \left(\sin\left(\frac{n\pi x}{\ell}\right)\right)^2 dx = \ell$	$\int_0^R rJ_n^2\left(\frac{\mu_m^{(n)}}{R} r\right) dr = \frac{R^2}{2}\left(J_n'(\mu_m^{(n)})\right)^2$ $= \frac{R^2}{2} J_{n\pm1}^2(\mu_m^{(n)})$
Expansions	$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$ $b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$	$f(r) = \sum_{m=1}^{\infty} C_m J_n\left(\frac{\mu_m^{(n)}}{R} r\right)$ $C_m = \frac{\int_0^R rf(r)J_n\left(\frac{\mu_m^{(n)}}{R} r\right) dr}{\frac{R^2}{2} J_{n+1}^2\left(\mu_m^{(n)}\right)}$

Table 1: Summary of Orthogonality, Norms, and Expansions