Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 9-2

Recall the Bessel equation

$$\underbrace{x^2y'' + xy'}_{=x\partial_x(x\partial_x y)} + (x^2 - n^2)y = 0 \implies \partial_x(x\partial_x y) - \frac{n^2}{x}y + xy = 0 \quad \text{(S-L problem)} \tag{1}$$

$\uparrow$ $\boxed{\text{Memorize this useful operator as mentioned before}}$

- The equation is closely related to the parameter n; if n changes, the equation changes.
- The value of n determines the form of the solution, specifically the Bessel function, making it a crucial parameter.
- After normalization, it is found that the Bessel equation only contains the parameter n.
- λ is not a parameter.

0.0.1 Bessel Functions:

According to the theory of solutions of differential equations, equation (1) has a **generalized power series solution** in the following form:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{s+k} \quad (a_0 \neq 0), \tag{2}$$

where s is a constant.

- The power series method is inspired by the resemblance of the Bessel equation to the Euler equation, but the presence of an x^2y term in the Bessel equation makes the Euler equation's solution methods inapplicable.
- The Euler equation has two solution methods:
  - Variable transformation: Replace x with e^t to convert derivatives with respect to x into derivatives with respect to t, making coefficients constant.
  - Trial method: Guess the solution as $y = x^s$ and determine s by substituting it into the equation.
- For the Bessel equation, the trial method fails due to the x^2y term, which cannot be simplified by the usual substitution.
- The idea of using a power series comes from the observation that the solution should be related to polynomials, but a simple polynomial form is insufficient.

- **(Ideas)** In a manner similar to the Euler equation, if we assume a solution of the form x^n, substituting it into the differential equation typically produces an extra term involving x^{n+2}. To eliminate this term, we introduce a new term proportional to x^{n+2} as part of the solution. However, this modification in turn generates an x^{n+4} term. Repeating this process iteratively, we are led to introduce successively higher powers of x. This recursive structure suggests that the solution should naturally take the form of a power series in x.
- The idea of assuming the solution as a power series is also analogous to the method of eigenfunctions. In this context, the set of monomials $\{x^n\}$ can be regarded as a system of basis functions.

- The power series method assumes the solution is a power series of the form $\sum_{n=0}^{\infty} a_n x^n$ and attempts to determine the coefficients a_n.
- However, for the Bessel equation, one would find that the power series method alone cannot determine the coefficients a_n.
- The **generalized power series method** is introduced:
  - It assumes the solution starts from $a_0 x^s$ instead of a_0, i.e., $\sum_{k=0}^{\infty} a_k x^{k+s}$.
- For an ordinary differential equation (ODE) whose coefficients are free of singularities and are analytic, the power series method can be used to solve it (see ODE references, e.g., [1]).
- If the coefficients have a second order singularity (a regular singularity), the generalized power series method can be used to solve it (see ODE references, e.g., [1]).
- The coefficients of the Bessel equation have a regular singularity, and the generalized power series method can be used to solve it.

Next, we will determine s and a_k $(k = 0, 1, 2, \cdots)$. To this end, substitute (2) and

$$y' = \sum_{k=0}^{\infty} a_k (s+k) x^{s+k-1}, \quad y'' = \sum_{k=0}^{\infty} a_k (s+k-1)(s+k) x^{s+k-2}$$

into equation (1). We can obtain

$$\sum_{k=0}^{\infty} a_k (s+k-1)(s+k) x^{s+k} + \sum_{k=0}^{\infty} a_k (s+k) x^{s+k} - n^2 \sum_{k=0}^{\infty} a_k x^{s+k} + \sum_{k=0}^{\infty} a_k x^{s+k+2} = 0,$$

Let $k+2 = l \Rightarrow k = l-2$. Then,

$$\sum_{k=0}^{\infty} a_k x^{s+k+2} = \sum_{l=2}^{\infty} a_{l-2} x^{s+l} = \sum_{k=2}^{\infty} a_{k-2} x^{s+k}$$

k is a dummy variable, similar to the variable in integration, which can be replaced.

$$\sum_{k=0}^{\infty} a_k \left[(s+k-1)(s+k) + (s+k) - n^2 \right] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0,$$

$$\sum_{k=0}^{\infty} a_k \left[(s+k)^2 - n^2\right] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0,$$

$$(s^2 - n^2) a_0 x^s + \left[(s+1)^2 - n^2\right] a_1 x^{s+1} + \sum_{k=2}^{\infty} a_k \left[(s+k)^2 - n^2\right] x^{s+k} + \sum_{k=2}^{\infty} a_{k-2} x^{s+k} = 0,$$

$$(s^2 - n^2) a_0 x^s + \left[(s+1)^2 - n^2\right] a_1 x^{s+1} + \sum_{k=2}^{\infty} \left\{ \left[(s+k)^2 - n^2\right] a_k + a_{k-2} \right\} x^{s+k} = 0$$

By comparing the coefficients (due to the basis functions $\{x^{s+k}\}$ are linearly independent) on both sides of the above formula, we have **indicial equations**

$$(s^2 - n^2) a_0 = 0, \tag{3}$$

$$\left[(s+1)^2 - n^2\right] a_1 = 0, \tag{4}$$

$$\left[(s+k)^2 - n^2\right] a_k + a_{k-2} = 0 \quad (k = 2, 3, \cdots) \tag{5}$$

Since $a_0 \neq 0$, from (3) we can get $s_1 = n, s_2 = -n$.

Question 1: The variable s has two values. Is this advantageous or problematic?–This is advantageous!

- The parameter s can take two values, $s = \pm n$, which implies the existence of two potential solutions.
- Having two values for s is **beneficial** because it suggests the **possibility of two linearly independent solutions**.
- For a second-order ordinary differential equation (ODE), the general solution is formed by **two linearly independent** solutions.
- If two solutions y_1 and y_2 are **linearly independent**, the general solution can be expressed as $C_1 y_1 + C_2 y_2$, where C_1 and C_2 are arbitrary constants.
- The **presence of two distinct values** for s (roots) is **advantageous** and aligns with the expected structure of solutions for a second-order ODE.
- Linear dependence between the two potential solutions ($s = \pm n$) would be problematic, as it is not enough for the structure of the general solution for a second-order ODE.
- If only one solution were found (i.e., if $s = \pm n$ is corresponding to two linearly dependent solutions), it would be necessary to seek additional solutions.

First, take $s_1 = n$, substituting it into (4), we get $a_1 = 0$. Substituting it into (5), we get

$$a_k = -\frac{a_{k-2}}{k(2n+k)}. \quad (k = 2, 3, \cdots). \tag{6}$$

From (6), we know that

$$a_1 = a_3 = a_5 = \cdots = 0$$

In addition,

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2 \cdot 1 \cdot (n+1)},$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{2 \cdot 4(2n+2)(2n+4)} = \frac{a_0}{2^4 \cdot 2 \cdot 1(n+1)(n+2)},$$

$$\cdots\cdots$$

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} \cdot m! \underbrace{(n+1)(n+2)\cdots(n+m)}_{\text{If } n, m \text{ are integers, then this term is } (n+m)!/n!}},$$

Question 2: The coefficient a_0 cannot be determined –is this advantageous or problematic? –It is advantageous.

- It is **advantageous** because the undetermined a_0 represents **a free parameter**, **allowing for a general solution**.
- The presence of a_0 in the solution is natural because if y is a solution to the differential equation (1), then $a_0 y$ is also a solution for any non-zero a_0.
- From the **structure of the general solution**, a_0 plays the role of an **arbitrary constant**, just like C_1 and C_2 in $C_1 y_1 + C_2 y_2$.
- The **flexibility** of a_0 can be used to **simplify the expression** of the solution, making it more manageable and memorable.
- The **complexity** of the solution can be **reduced** by **strategically choosing** the value of a_0 to simplify the expression, particularly when dealing with factorials and powers of 2.

Basic knowledge of the Γ function (Review)

1. **Definition:**
$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \quad (x > 0),$$
$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

2. **Recurrence formula of the Γ function:**
$$\Gamma(x+1) = x\Gamma(x).$$
In particular, when x is a positive integer n, we have
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\Gamma(1) = n!.$$
$\uparrow$ $\boxed{\text{The gamma function can be regarded as a } \textbf{generalization of the factorial}.}$

3. When $n = 0, -1, -2, \cdots$
$$\frac{1}{\Gamma(n)} = 0 \quad (\text{or } \Gamma(n) = \infty).$$

Since a_0 is an arbitrary constant, we can choose its value in the following way: make the power of 2 in the coefficient of the general term the same as that of x, and simplify the denominator at the same time. For this purpose, take
$$a_0 = \frac{1}{2^n \Gamma(n+1)}.$$

Consider one term (if n, m are integers)

$$\begin{aligned} a_{2m}x^{n+2m} =& (-1)^m \frac{a_0}{2^{2m} \cdot m!(n+1)(n+2)\cdots(n+m)} x^{n+2m} \\ =& (-1)^m \frac{a_0}{m! \frac{(n+m)!}{n!}} \frac{x^{n+2m}}{2^{2m}} \\ =& \underbrace{(-1)^m \frac{1}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m}}_{\text{This is a nice form}} \underbrace{(a_0 2^n n!)}_{\text{let this term to be 1}} . \end{aligned}$$

Thus, we take $a_0 = \frac{1}{2^n n!}$. If n and m are not the integer, this becomes $a_0 = \frac{1}{2^n \Gamma(n+1)}$ and

$$a_{2m}x^{n+2m} = (-1)^m \frac{1}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m} .$$

Using the recurrence formula $n\Gamma(n) = \Gamma(n+1)$, the coefficient of the general term becomes

$$a_{2m} = (-1)^m \frac{1}{2^{n+2m} \cdot m!\Gamma(n+m+1)}$$

Substitute this coefficient expression back into (2). We get a particular solution of equation (1), denoted as $J_n(x)$

$$J_n(x) = \sum_{m=0}^{\infty} a_{2m}x^{n+2m} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m!\Gamma(n+m+1)}, \tag{7}$$

$J_n(x)$ is called the **Bessel function of the first kind of order** n.

- The solution obtained is a **formal solution**, which is based on assumptions about the interchangeability of differentiation and summation.
- To confirm that the **formal solution** is **indeed a solution**, two verifications are necessary:
  - **Verification of Identity**: Substitute the solution back into the original differential equation to check if it satisfies the equation (i.e., turns it into an identity).
  - **Convergence Verification**: Ensure that the series solution converges, as a divergent series cannot represent a valid function, let alone a solution.
- The coefficients in the series are now specific, allowing for convergence verification.
- The simplest method to verify convergence is the **ratio test** (also known as **D'Alembert's ratio test**), which involves comparing successive terms:
  - Compute the limit of the ratio of successive terms.
  - If the limit is less than 1, the series is convergent.
- This method is essentially a **comparison test**, comparing the series to a **geometric series** where the common ratio q is less than 1, ensuring convergence.

Also, since

$$\lim_{m\to\infty} \frac{|u_{m+1}|}{|u_m|} = \lim_{m\to\infty} \frac{x^2}{4(m+1)(n+m+1)} = 0 < 1.$$

Then, by the **ratio test (D'Alembert's test)**, the series (7) is **absolutely convergent** on the entire real axis.

Then let $s_2 = -n$, substituting it into (4), we get $a_1 = 0$, then substituting it into (5), we get

$$a_k = -\frac{a_{k-2}}{k(-2n+k)}. \quad (k = 2, 3, \cdots).$$

From the above formula, we know that

$$a_1 = a_3 = a_5 = \cdots = 0$$

In addition

$$a_2 = -\frac{a_0}{2(-2n+2)} = -\frac{a_0}{2^2 \cdot 1 \cdot (-n+1)},$$

$$a_4 = -\frac{a_2}{4(-2n+4)} = \frac{a_0}{2 \cdot 4(-2n+2)(-2n+4)} = \frac{a_0}{2^4 \cdot 2 \cdot 1(-n+1)(-n+2)},$$

$$\cdots\cdots$$

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} \cdot m!(-n+1)(-n+2)\cdots(-n+m)},$$

Since a_0 is an arbitrary constant, we can choose its value in the following way: make the power of 2 in the coefficient of the general term the same as that of x, and simplify the denominator at the same time. For this purpose, take

$$a_0 = \frac{1}{2^{-n}\Gamma(-n+1)}.$$

Using the recurrence formula $\Gamma(-n+1) = -n\Gamma(-n)$, the coefficient of the general term becomes

$$a_{2m} = (-1)^m \frac{1}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}$$

Substitute this coefficient expression back into (2). We get another particular solution of equation (1), denoted as $J_{-n}(x)$

$$J_{-n}(x) = \sum_{m=0}^{\infty} a_{2m} x^{-n+2m} = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}, \tag{8}$$

$J_{-n}(x)$ is called the **Bessel function of the first kind of order** $-n$.

- The **goal** is to generate the general solution using the two particular solutions obtained.
- A **general solution** requires the **linear combination** of **two linearly independent particular solutions**.
- The **task** is to determine whether the solutions J_n and J_{-n} are **linearly dependent or independent**.
- The analysis depends on the nature of n (whether it is an integer or not, and whether it is a half-integer).
- For n that is neither an integer nor a half-integer, J_n and J_{-n} are proven to be linearly independent.

Next, we will discuss in **three cases.**

Case 1: If n is **neither an integer (including 0) nor a half-odd integer**, then $s_1 - s_2 = 2n$ is also **not an integer**. Since $n \neq -n$, $J_n(x)$ and $J_{-n}(x)$ are **linearly independent**.

- The **reason** for **linear independence** is the **presence of negative exponents** in J_{-n} (i.e., terms like x^{-n}) and **only positive exponents** in J_n (i.e., terms like x^{2m}).

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m!\Gamma(n+m+1)},$$
$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}.$$

- **Linear dependence** would imply a **proportional relationship** between the two functions, which is **impossible** given their different exponent characteristics of J_n and J_{-n}.
- It is crucial to ensure that the **coefficients** of the **negative exponent** terms in J_{-n} are **non-zero** to confirm **linear independence**; otherwise, there might be a possibility of linear dependence.

According to the structure theorem of solutions of homogeneous linear ordinary differential equations, the general solution of equation (1) is

$$y = AJ_n(x) + BJ_{-n}(x), \tag{9}$$

where A and B are two arbitrary constants.

In (9), take (because n is neither an integer nor a half-odd integer)

$$A = \cot n\pi, \quad B = -\csc n\pi,$$

Then we get another particular solution of equation (1) that is linearly independent of $J_n(x)$, denoted as

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}. \tag{10}$$
$$\uparrow \boxed{n \neq \text{integer, thus } \sin n\pi \neq 0}$$

Therefore, the general solution of equation (1) can be written as

$$y = CJ_n(x) + DY_n(x), \tag{11}$$

$Y_n(x)$ is called the **Bessel function of the second kind** or the **Neumann function**.

- Introduction of a second type of Bessel function is necessary due to the **linear dependence** of J_n and J_{-n} when n is an **integer** (see below).
- The second type of Bessel function provides a **linearly independent solution** when n is an integer, which is crucial for forming the general solution.
- The construction of a new solution involves a **linear combination of J_n and J_{-n}**, denoted as Y_n, which is linearly independent from J_n.
- The general solution can be expressed as a linear combination of J_n and Y_n, which is **valid for all n, including integers**.
- The approach of using J_n and Y_n simplifies the solution process and provides a unified form for the general solution of Bessel's equation.
- It is recommended to focus on the linear combination of J_n and Y_n as the general solution for Bessel's equation, as it covers all cases, including when n is an integer.

Case 2: If n is an **integer (including 0)**, then $s_1 - s_2 = 2n$ is also an integer. Following the previous approach, we can also obtain two particular solutions of equation (1)

$$s_1 = n, \quad J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m!\Gamma(n+m+1)},$$

$$s_2 = -n, \quad J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{-n+2m}}{2^{-n+2m} \cdot m!\Gamma(-n+m+1)}, \tag{12}$$

Note that when $n \geq 0$ is an **integer**, using the recurrence formula of the Γ function, we can get $\Gamma(n+m+1) = (n+m)!$. Thus, one of the particular solutions (7) can be transformed into

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{n+2m}}{2^{n+2m} \cdot m!(n+m)!}, \tag{13}$$

And at this time, the function $J_{-n}(x)$ and $J_n(x)$ are linearly dependent.

In fact, without loss of generality, let n be a positive integer N.

- If $-N+m+1 \leq 0$ (i.e., $0 \leq m \leq N-1$), then $\Gamma(-N+m+1) = \infty$ and the coefficients vanish.

When $m = 0, 1, 2, \cdots, (N-1)$, $-n+m+1 = -N+m+1$ will be a negative integer or 0. For these values, $\Gamma(-N+m+1)$ is infinite. So

$$J_{-N}(x) = \sum_{m=N}^{\infty} (-1)^m \frac{x^{-N+2m}}{2^{-N+2m} \cdot m!\Gamma(-N+m+1)},$$

Let $m = N+k$, $(k = 0, 1, 2, \cdots)$, we get

$$J_{-N}(x) = \sum_{k=0}^{\infty} (-1)^{N+k} \frac{x^{N+2k}}{2^{N+2k}(N+k)!\Gamma(k+1)}$$

Then, after simplification, we have

$$\begin{aligned} J_{-N}(x) &= \sum_{k=0}^{\infty} (-1)^{N+k} \frac{x^{N+2k}}{2^{N+2k} \cdot k!(N+k)!} \\ &= (-1)^N \sum_{k=0}^{\infty} (-1)^k \frac{x^{N+2k}}{2^{N+2k} \cdot k!(N+k)!} \\ &= (-1)^N \sum_{m=0}^{\infty} (-1)^m \frac{x^{N+2m}}{2^{N+2m} \cdot m!(N+m)!} = (-1)^N J_N(x). \end{aligned}$$

This shows that when $n(n \geq 0)$ is an integer, $J_{-n}(x)$ and $J_n(x)$ are **linearly dependent**. In order to find the general solution of the Bessel equation, we still **need to find a particular solution** that is **linearly independent of** $J_n(x)$.

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}. \tag{14}$$

From formula (14), when n is not an integer, $Y_n(x)$ and $J_n(x)$ are linearly independent. When n is an integer, since

$$J_{-n} = (-1)^n J_n(x), \quad \cos n\pi = (-1)^n,$$

Then the right hand side of formula (14) becomes an indeterminate form of the type "$\frac{0}{0}$". At this time, we naturally define

$$Y_n(x) = \lim_{\alpha \to n} Y_\alpha(x) = \lim_{\alpha \to n} \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi},$$

where n is an integer and α is not an integer.

$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x) = \lim_{\alpha\to n} \frac{J_\alpha(x)\cos\alpha\pi - J_{-\alpha}(x)}{\sin\alpha\pi},$$

Applying L'Hopital's rule and through lengthy derivations (refer to [2]), we get

$$Y_0(x) = \frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2} + C\right) - \frac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\sum_{k=0}^{m-1}\frac{1}{k+1},$$

$$\begin{aligned} Y_n(x) =& \frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2} + C\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m} \\ & - \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}\cdot\left(\sum_{k=0}^{n+m-1}\frac{1}{k+1} + \sum_{k=0}^{m-1}\frac{1}{k+1}\right) \\ & (n = 1, 2, 3, \cdots), \end{aligned}$$

$$Y_n(0) \text{ is } \textbf{infinite} \quad (n = 1, 2, 3, \cdots),$$

where $C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.5772\cdots$, which is called the **Euler constant**. Obviously, $Y_n(x)$ is a **particular solution** of the n-th order Bessel equation that is **linearly independent** of $J_n(x)$ (since $Y_n(x)$ includes $\ln\frac{x}{2}$).

Memorize that $\ln\frac{x}{2}$ appears in Y_n, then remember two key points:

- J_n and Y_n are linearly independent;
- $Y_n(0) = \infty$.

- (Intuitive ideas) Y_n is derived from a **linear combination** of two solutions, but after taking the limit, it becomes **linearly independent** from them.
- (Intuitive ideas) This phenomenon is peculiar and highlights a significant difference between **finite-dimensional and infinite-dimensional** linear spaces.
- (Intuitive ideas) In finite-dimensional spaces, any linear combination remains within the space, whereas in infinite-dimensional spaces, taking limits can result in outcomes that lie outside the original space.
- (Intuitive ideas) This can lead to the creation of linearly independent solutions that were not present in the original space.
- Multiple Roots and Logarithmic-Type Solutions
  - When the roots differ by an integer, the two power series solutions obtained by the direct substitution method usually yield only one pure power series solution, while the other one **often involves logarithmic terms**. This phenomenon is common in the theory of ordinary differential equations. For example, the second solution may take the form

    $$y_2(x) = y_1(x)\ln x + \text{other terms in the power series expansion.}$$

    For the Bessel equation, when we seek a second independent solution, a solution **containing a $\ln x$ term naturally emerges**, which is precisely the characteristic of $Y_n(x)$.
- The introduction of $Y_n(x)$ was initially motivated by the need for **completeness of the solution space** from a mathematical perspective. Since the Bessel equation for integer order produces only one "good" solution using conventional methods, it becomes necessary to obtain a **second solution with logarithmic terms** through **analytic continuation** and a **limiting process**. Although this definition may appear **non-intuitive**, it actually reflects two key aspects:

- **Theoretical Aspect:** By constructing a second independent solution via the **limiting definition**, the **completeness** of the solution space is ensured.
- **Practical Aspect:** The introduction of $Y_n(x)$ facilitates a better adjustment and matching of boundary conditions in physical problems, thereby capturing the finer details of actual phenomena.

This thought process, driven jointly by mathematical analysis and physical requirements, is a vivid illustration of the continual refinement and evolution of mathematical methods.

- It is necessary to verify whether Y_n is a solution to the original Bessel's equation.
- The verification process involves substituting Y_n back into the original differential equation.
- Since Y_n is expressed as a series, one can assess its convergence and the behavior of its coefficients.

$$\lim_{\alpha \to n} \underbrace{\left[x^2 Y_\alpha'' + x Y_\alpha' + (x^2 - n^2) Y_\alpha\right]}_{=0} \underbrace{=}_{\text{We omit the details}} x^2 Y_n'' + x Y_n' + \left(x^2 - n^2\right) Y_n.$$

- The conclusion is that Y_n is indeed a solution to Bessel's equation and is linearly independent from J_n.
- With these two conclusions, the general solution of Bessel's equation can be constructed as a linear combination of J_n and Y_n.

In conclusion, regardless of whether n is an integer or not, the general solution of the Bessel equation (1) can be expressed as

$$y(x) = CJ_n(x) + DY_n(x),$$

where C and D are arbitrary real numbers, and n is an arbitrary real number. In addition, when n is an **integer**, from $J_n(-x) = (-1)^n J_n(x)$, it can be deduced that

- when n is an **even** number, $J_n(x)$ is an **even function** $\leftarrow$ it resembles the **cosine** function.;
- when n is an **odd** number, $J_n(x)$ is an **odd function** $\leftarrow$ it resembles the **sine** function..

Case 3: When n is a **half-odd integer**, it will be discussed in the next section.

Summary

- The primary focus of this subsection was on understanding the **form of Bessel's equation solutions**.
- Students should be **able to identify the parameter** n in a **Bessel equation** (**Memorize this equation**) and directly write down the solutions J_n and Y_n without intermediate derivations.
- Similar to how one would directly apply the general solution formula for second-order linear differential equations with constant coefficients, the **goal** is to **recognize n-Bessel's equation** and its solutions J_n and Y_n immediately.
- It is crucial to remember that $Y_n(0) = \infty$ due to the presence of $\ln\left(\frac{x}{2}\right)$ in its series representation.
- This infinity at $x = 0$ **violates the boundedness condition** often required in physical problems, leading to the exclusion of Y_n from the solution set in such cases.

- The **main takeaway** is
  - to recognize n-Bessel's equation and its solutions $CJ_n + DY_n$
  - to understand the implications of $Y_n(0) = \infty$ on the solution's applicability.

1 5.2 Recurrence Formulas of Bessel Functions

- This section focuses on fixing the fifth step in the method of separation of variables, which involves determining coefficients.
- To calculate these coefficients, one needs to compute integrals, specifically inner products, which often involve integrals over functions.
- Calculating these integrals requires the use of recursive relationships of functions.

There are certain relationships among Bessel functions of different orders. In this section, we will establish **recurrence formulas** that reflect these relationships. From the expression (7) of $J_n(x)$, the following two basic recurrence formulas can be derived:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \tag{15}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x). \tag{16}$$

Proof. In fact, multiply both sides of formula (7) by x^{-n}, and then differentiate with respect to x. We get

$$\begin{aligned}\frac{d}{dx}\left[x^{-n} J_n(x)\right] &= \frac{d}{dx}\left[\sum_{m=0}^{\infty}(-1)^m \frac{x^{2m}}{2^{n+2m}\cdot m!\Gamma(n+m+1)}\right] \\ &= \sum_{m=1}^{\infty}(-1)^m \frac{x^{2m-1}}{2^{n+2m-1}\cdot (m-1)!\Gamma(n+m+1)}\end{aligned}$$

Let $m = 1+k$, $(k = 0, 1, 2, \cdots)$, then

$$\begin{aligned}\frac{d}{dx}\left[x^{-n} J_n(x)\right] &= \sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{2^{n+2k+1}\cdot k!\Gamma(n+1+k+1)} \\ &= -x^{-n}\sum_{k=0}^{\infty}(-1)^k \frac{x^{n+1+2k}}{2^{n+1+2k}\cdot k!\Gamma(n+1+k+1)} \\ &= -x^{-n} J_{n+1}(x)\end{aligned}$$

Similarly, formula (15) can be proved. □

- The key to using the recursive relationships is flexibility in application.
- **Derivatives** are provided because they are the **inverse operations of integrals**.
- The actual calculations used involve **integration**, which is the **reverse process of differentiation**.
- The first formula provided is for the derivative of $x^n J_n$, but the useful formula for integration is $\int x^n J_{n-1}\,dx = x^n J_n$.
- Similarly, for the second formula, the integral of $x^{-n} J_{n+1}$ is $x^{-n} J_n$.

If we express the derivatives on the left hand sides of the above two formulas and simplify, we obtain

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x),$$

$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x).$$

Eliminating $J_n'(x)$ and $J_n(x)$ successively, we get

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \tag{17}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x). \tag{18}$$

Obviously, formulas (15), (16) are equivalent to formulas (17), (18).

How to memorize?

- The first formula describes the relationship among J_{n-1}, J_n, and J_{n+1}.
- It involves an **arithmetic mean**: $\boxed{\frac{J_{n+1} + J_{n-1}}{2} \sim J_n}$ with a correction factor $\frac{n}{x}$.
- The second formula is more conveniently remembered as $\boxed{\frac{J_{n+1} - J_{n-1}}{2} = -J_n'}$.
- The left side represents a finite difference ("**difference quotient**"), and the right side is a derivative ("**differential quotient**" with a **sign change**).

If the values of $J_{n-1}(x)$ and $J_n(x)$ are known, the value of $J_{n+1}(x)$ can be calculated from formula (17). In this way, through formula (17), Bessel functions of any positive integer order can be expressed in terms of Bessel functions of order 0 and order 1. In particular, when $n = 0$, from formula (16), we have

$$\boxed{J_0'(x) = -J_1(x);}$$

when $n = 1$, from formula (15), we have

$$\boxed{\frac{d}{dx}[xJ_1(x)] = xJ_0(x).} \tag{19}$$

How to memorize?

$$\begin{tikzcd} xJ_1 \arrow[r, bend left, "\frac{d}{dx}"] & x\boxed{J_0} \arrow[l, bend left, "\int"] \arrow[d, bend left, "\frac{d}{dx}"] \\ & \boxed{-J_1} \arrow[u, bend left, "\int"] \end{tikzcd}$$

Ex 1.1. *Calculate* $\int xJ_2(x)dx$.

Solution (Method 1)**.** *From formula* (17)*, we know that* $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$. *Then*

$$xJ_2(x) = 2J_1(x) - xJ_0(x)$$

$$\begin{aligned}\int xJ_2(x)dx &= 2\int J_1(x)dx - \int xJ_0(x)dx \\ &= -2J_0(x) - xJ_1(x) + c.\end{aligned}$$

Solution (Method 2)**.** *By* (18)*,* $J_2(x) = J_0(x) - 2J_1'(x)$*, then*

$$\begin{aligned}\int xJ_2(x)\mathrm{d}x &= \int xJ_0(x)\mathrm{d}x - 2\int xJ_1'(x)\mathrm{d}x \\ &= xJ_1(x) - 2\left(xJ_1(x) - \int J_1(x)\mathrm{d}x\right) \\ &= -xJ_1(x) - 2J_0(x) + c.\end{aligned}$$

Remark

- The purpose of calculating the integral is threefold:
  1. To **eliminate the integral sign** as much as possible.
  2. To **reduce the order of** J_n to lower values, since J_0 and J_1 are the simplest and most readily available from tables and graphs.
  3. To **minimize the polynomial orders**, as many problems involve polynomials multiplied by Bessel functions.
- The actual goal is not just to compute the integral but to **simplify it** by **reducing the indices** of J_n and x^n, and **removing the integral sign**.

Highlight:

- Using **different recurrence formulas** may lead to **different expressions**, but the **results are the same**, just with **different forms of expression**.
- When should we stop integrating? Simplify to the point where **no further simplification** is possible. For example, $\int J_0 dx$ (called **generalized hypergeometric function**) is left as it is!
- Try to express the results **in terms of** J_0 **and** J_1**.** Sometimes, it can also contain J_2, J_3, etc. In short, **simplify as much as possible**.

For the second kind Bessel functions, the following recurrence formulas also hold:

$$\begin{cases} \frac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x), \\ \frac{d}{dx}\left[x^{-n} Y_n(x)\right] = -x^{-n} Y_{n+1}(x), \\ Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x), \\ Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x). \end{cases}$$

- The relations of Y_n is often not used in this course.
- The reason for not using Y_n is that $Y_n(0) = \infty$, which violates the boundedness condition typically required in physical problems.
- Due to the unbounded nature of Y_n at $x = 0$, it is usually discarded.
- The relationships involving Y_n are thus not frequently applied in this course.

An **important feature** of Bessel functions when n **is a half-odd integer** is that they can be **expressed in terms of elementary functions**. First, calculate $J_{\frac{1}{2}}(x)$. From formula (7), we have

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{\frac{1}{2}+2m} \cdot m!\Gamma(\frac{1}{2}+m+1)}$$

Using the properties of the Γ function, we get

$$\begin{aligned}
\Gamma(\frac{3}{2}+m) &= (\frac{1}{2}+m)\Gamma(\frac{1}{2}+m) \\
&= (\frac{1}{2}+m)(m-\frac{1}{2})\Gamma(m-\frac{1}{2}) \\
&= \cdots \\
&= (\frac{1}{2}+m)(m-\frac{1}{2})\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) \\
&= \frac{1\cdot 3\cdot 5\cdots(2m+1)}{2^{m+1}}\sqrt{\pi}
\end{aligned}$$

Calculations:

$$\begin{aligned}
J_{\frac{1}{2}} &= \sum_{m=0}^{\infty}(-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{\frac{1}{2}+2m}m!\frac{1\cdot 3\cdot 5\cdots(2m+1)}{2^{m+1}}\sqrt{\pi}} \\
&= \sum_{m=0}^{\infty}(-1)^m \frac{x^{\frac{1}{2}+2m}}{2^{-\frac{1}{2}+m}m!\cdot 1\cdot 3\cdot 5\cdots(2m+1)\sqrt{\pi}} \\
&= \sqrt{2}\sum_{m=0}^{\infty}(-1)^m \frac{x^{\frac{1}{2}+2m}}{\underbrace{(2^m m!)}_{(2m)!!}\cdot\underbrace{1\cdot 3\cdot 5\cdots(2m+1)}_{(2m+1)!!}\sqrt{\pi}} \\
&= \sqrt{\frac{2}{\pi}}\sum_{m=0}^{\infty}(-1)^m \frac{x^{\frac{1}{2}+2m}}{(2m+1)!} \\
&= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}(-1)^m \frac{x^{1+2m}}{(2m+1)!}
\end{aligned}$$

Thus

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x. \tag{20}$$

$\boxed{\text{A sine function with decreasing amplitude as } x \text{ increases}}\uparrow$

Similarly, we can obtain

$$J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\cos x. \tag{21}$$

Applying formula (17), we have

$$\begin{aligned}
J_{\frac{3}{2}}(x) &= \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \\
&= \sqrt{\frac{2}{\pi x}}(-\cos x + \frac{1}{x}\sin x) \\
&= -\sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\cdot\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\sin x}{x}\right).
\end{aligned}$$

Similarly, applying formula (17), we get

$$\begin{aligned}
J_{-\frac{3}{2}}(x) &= \frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) \\
&= \sqrt{\frac{2}{\pi x}}(-\sin x + \frac{1}{x}\cos x) \\
&= \sqrt{\frac{2}{\pi}}x^{\frac{3}{2}}\cdot\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\cos x}{x}\right).
\end{aligned}$$

In general, we have

$$J_{\frac{2m+1}{2}}(x) = (-1)^m \sqrt{\frac{2}{\pi}} x^{m+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{\sin x}{x}\right),$$

$$J_{-\frac{2m+1}{2}}(x) = \sqrt{\frac{2}{\pi}} x^{m+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{\cos x}{x}\right).$$

Here, for convenience, we use the differential operator $(\frac{1}{x}\frac{d}{dx})^m$, which is an abbreviation for the operator $\frac{1}{x}\frac{d}{dx}$ acting m times successively. For example,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^2 \left(\frac{\sin x}{x}\right) = \frac{1}{x}\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right].$$

References

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