Lecture Notes: Mathematical Physics Equations and Special Functions

Chao Liu

Week 3-1

1 Heat Conduction in a Finite Rod (Continued)

Summary of Variable Separation Method

1. **Boundary Conditions Modification**
   - Four types of boundary conditions were discussed:
     - Both ends are of the first kind.
     - One end is of the first kind, the other is of the second kind.
     - One end is of the second kind, the other is of the first kind.
     - Both ends are of the second kind.
   - The modification mainly affects the solution of the SL problem in the variable separation method.
   - For these four types of boundary conditions, students are required to remember the corresponding eigenvalues, and eigenfunctions.
   - One type (left end is of the second kind, right end is of the first kind) was not discussed in class and needs to be calculated by students themselves.
2. **Equation Modification**
   - Two types of equations were discussed: wave equation and heat equation.
   - For the heat equation:
     - The time derivative term is reduced to one order.
     - The main modification in the variable separation method is the ODE for T, which becomes a first-order ODE.
     - The solution is an exponential decay function, $e^{-\lambda t}$, which reflects the physical phenomenon of heat transfer from high temperature to low temperature until equilibrium.
   - For the wave equation:
     - The time derivative term is of second order.
     - The ODE for T is a second-order ODE.
     - The solution is a combination of trigonometric functions (e.g., $T(t) = A\cos(\omega t) + B\sin(\omega t)$), which reflects the oscillatory nature of wave motion.
   - Physical Intuition
     - **Heat** equation solutions must exhibit **exponential decay** due to **thermal diffusion**.
     - **Wave** equation solutions must exhibit **oscillations** due to **periodic motion**.
     - **Damped oscillations** arise when additional terms introduce **decay** in wave equations.
3. **Current and Future Work**
   - This class will complete the discussion of the variable separation method.
   - Focus on three main examples:
     - Heat equation with both ends of the second kind boundary conditions.
     - Laplace equation in rectangular coordinates.

– Laplace equation in circular coordinates (main focus).

- The method for solving the heat equation with boundary conditions is similar to solving the Laplace equation in rectangular coordinates.
- The Laplace equation in circular coordinates will present more challenges and requires special attention.

1.1 Consider the Mixed Problem of the Homogeneous Heat Conduction Equation (Boundary Conditions are Both of the Second Type)

Problem 1.1. *The heat conduction problem on a finite rod with both ends $x = 0, x = l$ insulated, initial temperature distribution $\varphi(x)$, and no heat source.*

$$u_t = a^2 u_{xx} \quad (0 < x < l,\, t > 0), \tag{1}$$
$$u_x(0,t) = 0, \quad u_x(l,t) = 0, \tag{2}$$
$$u(x,0) = \varphi(x), \tag{3}$$

where $\varphi(x)$ is a given known function.

1. **Equation Type: Heat Equation**
   - When the equation becomes a heat equation, the function $T(t)$ is expected to be an exponential function.
   - This is because the time-dependent part of the heat equation typically results in exponential decay.
2. **Boundary Conditions: Second Kind**
   - When the boundary conditions are of the second kind, the eigenvalues and eigenfunctions will change accordingly.
   - This modification affects the form of the solution, requiring a re-evaluation of the eigenvalues and eigenfunctions.
3. **Approach to Solving the Problem**
   - Before solving, make initial predictions based on the type of equation and boundary conditions.
   - Use these predictions to guide the modification process.
   - The solution involves combining previously learned concepts and applying them to the specific problem at hand.

Solution. ***(1. Separation of Variables)*** *Let*

$$u(x,t) = X(x)T(t) \tag{4}$$

(2. PDE → ODEs) *Substituting (4) into equation (1) to separate variables yields two ordinary differential equations*

$$\begin{aligned} T'(t) + \lambda a^2 T(t) &= 0, \\ X''(x) + \lambda X(x) &= 0, \end{aligned}$$

From boundary conditions (2), we get $X'(0)T(t) = 0, X'(l)T(t) = 0$. Then

$$X'(0) = 0, \quad X'(l) = 0.$$

(3. Solving ODEs) *Solve the Boundary Value Problem of the Ordinary Differential Equation for* ***Non-zero Solutions****.*

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0. \tag{5}$$

1. *When $\lambda < 0$, the general solution of the equation is*

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$
$$X'(x) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x},$$

From boundary conditions, we get

$$\sqrt{-\lambda}(A - B) = 0,$$
$$\sqrt{-\lambda}(Ae^{\sqrt{-\lambda}l} - Be^{-\sqrt{-\lambda}l}) = 0.$$

This implies

$$A = B = 0, \implies X(x) \equiv 0. \text{ Therefore, (5) has no non-zero solutions.}$$

2. *When $\lambda = 0$, the general solution of the equation is $X_0(x) = A_0 x + B_0$, Then*

$$X_0'(x) = A_0.$$

From boundary conditions, we get

$$X_0'(0) = X_0'(l) = A_0 = 0 \quad \Rightarrow \quad X_0(x) = B_0.$$

Substituting $\lambda = 0$ into the equation $T'(t) + \lambda a^2 T(t) = 0$, we solve to get

$$T_0(t) = C_0.$$

*This gives a **non-trivial solution** to the conduction equation (1) satisfying boundary conditions (2)*

$$u_0(x,t) = \frac{1}{2}a_0,$$

where $a_0 = 2B_0C_0$ is an arbitrary constant.

3. *When $\lambda > 0$, the general solution of the equation has the following form*

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Then

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x,$$

From boundary conditions, we get

$$X'(0) = B\sqrt{\lambda} = 0 \implies B = 0 \quad \text{and} \quad X'(l) = -A\sqrt{\lambda}\sin\sqrt{\lambda}l = 0.$$

Assuming $X(x)$ is not identically zero, then $A \neq 0$, $\implies \sin\sqrt{\lambda}l = 0$,
Thus we get

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

Hence, a set of non-zero solutions is found

$$X_n(x) = A_n \cos\frac{n\pi x}{l} \quad (n = 1, 2, \ldots).$$

Now consider

$$T'(t) + \lambda a^2 T(t) = 0,$$

Substitute the eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \ldots).$$

into the equation to get

$$T'(t) + \left(\frac{n\pi a}{l}\right)^2 T(t) = 0, \quad (n = 1, 2, \ldots)$$

The general solution is

$$T_n(t) = D_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \quad (n = 1, 2, \ldots).$$

Summary of Solving Techniques and Course Insights

1. ***Solving the T Equation***

- *The T equation is relatively simple and can be solved using the separation of variables method for ODEs.*
- *The process involves writing the derivative of $\ln T$ and integrating it directly.*
- *For homogeneous equations, separation of variables is straightforward.*
- *For non-homogeneous equations, the integrating factor method is most effective.*

2. ***Importance of the Integrating Factor Method***

- *The ideas of integrating factor method is crucial for combining terms and simplifying the integration process.*
- *This technique will be frequently used in future problems and exercises.*
- *Understanding how to find and apply the integrating factor is essential.*

3. ***Characteristics of the Differential Equations Course***

- *Unlike other math courses that rely heavily on theorems and formulas, differential equations focus more on methods and underlying ideas.*
- *The course emphasizes the importance of mastering techniques rather than memorizing specific formulas.*
- *Methods like separation of variables and the idea of integrating factors are most fundamental and will be reinforced throughout the course.*

(4. Superposition of Series Solutions) *Thus, the non-zero solution satisfying equation (1) and boundary conditions (2) is*

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \cos\frac{n\pi x}{l} + \frac{1}{2}a_0, \tag{6}$$

where $a_n = A_n D_n$, $a_0 = 2B_0C_0$ are arbitrary constants.

(5. Initial Conditions Determine Coefficients) *In (6), let $t = 0$, and combine with the initial condition*

$$u(x,0) = \varphi(x),$$

we get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \varphi(x),$$

Then

$$a_0 = \frac{2}{l}\int_0^l \varphi(x)dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{n\pi x}{l}dx, \quad (n = 1, 2, \ldots). \tag{7}$$

Thus, the solution to the boundary value problem (1)-(3) is given by the series

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \cos\frac{n\pi x}{l} + \frac{1}{2}a_0.$$

where the coefficients a_n are determined by (7):

$$a_n = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{n\pi x}{l}dx, \quad (n = 0, 1, 2, \ldots).$$

Ex 1.1. *Solve the Following Problem*

$$\begin{cases} u_t = a^2 u_{xx} \quad (0 < x < \pi,\ t > 0), \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ u(x,0) = x^2(\pi - x)^2, \end{cases} \tag{8}$$

1. **Common Integrals in Exams and Homework**

- The most common integrals in this course involve:
  - Polynomials multiplied by trigonometric functions.
  - Polynomials multiplied by exponential functions.
- These integrals are typically solved using integration by parts.
- For higher powers (e.g., fourth power), multiple applications of integration by parts may be required. Each application reduces the power by one.

Solution. *Using the formula*

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \cos\frac{n\pi x}{l}.$$

where

$$a_n = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{n\pi x}{l}dx, \quad (n = 0, 1, 2, \dots).$$

Since $l = \pi$, we have

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2 \cos nx\,dx = -\frac{24[(-1)^n + 1]}{n^4} \quad (n \neq 0),$$

and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,dx = \frac{1}{15}\pi^4,$$

Substitute $a_0 = \frac{1}{15}\pi^4$, $a_n = -\frac{24[(-1)^n+1]}{n^4}$ $(n \neq 0)$, and $l = \pi$ into the formula

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \cos\frac{n\pi x}{l}.$$

The solution to the problem is

$$u(x,t) = \frac{1}{30}\pi^4 - \sum_{n=1}^{\infty} \frac{24[(-1)^n + 1]}{n^4} e^{-(na)^2 t} \cos nx.$$

2 Boundary Value Problems of the Two-Dimensional Laplace Equation

For certain special regions, the boundary value problems of the Laplace equation can also be solved using the method of separation of variables.

2.1 Boundary Value Problem of the Laplace Equation in a Rectangular Domain

Problem 2.1. *Consider the temperature distribution problem of a rectangular thin plate in a steady state. Assume the top and bottom surfaces of the plate are insulated (means 2D problem), and the two sides $(x = 0, x = a)$ are always kept at 0 degrees, while the other two sides $(y = 0, y = b)$ have temperatures $f(x)$ and $g(x)$ respectively. Find the temperature distribution pattern inside the plate in a steady state.*

Summary of Laplace Equation and Variable Separation Method

1. **Laplace Equation in Two Dimensions**

   - We focus on two-dimensional cases in this chapter because they involve two ODEs, which are simpler to handle.
   - Higher dimensions (e.g., three-dimensional) would involve more ODEs (two S-L problems) and more complex boundary conditions (see Chapter 5).

2. **Applicability of Variable Separation Method**

   - Variable separation is applicable to specific regions, such as rectangular and circular domains.
   - For rectangular regions:
     - The domain is defined by $u(x, y) = 0$ within a rectangle.
     - Separation of variables is straightforward because the boundaries are aligned with the coordinate axes.
   - For circular regions:
     - The domain is defined by $u(r, \theta) = 0$ within a circle.
     - Separation of variables is easier in polar coordinates due to the local orthogonality (i.e., variable separable) of r and θ.

3. **Other Related Regions**

   - Regions related to rectangles (e.g., infinite strips) can also be solved using variable separation.
   - Regions related to circles (e.g., annular and sector regions) can be handled similarly by modifying the circular case.
   - These regions will be covered in homework and exams.

4. **Limitations and Future Work**

   - More general regions are difficult to solve using variable separation.
   - Different methods may yield different forms of solutions, but they represent the same underlying function (see Chap. 4).

Solution. *Use $u(x, y)$ to represent the temperature at point (x, y) on the plate, i.e.,*

$$u_{xx} + u_{yy} = 0 \quad (0 < x < a,\ 0 < y < b), \tag{9}$$

$$u(x, 0) = f(x), \quad u(x, b) = g(x), \tag{10}$$

$$u(0, y) = 0, \quad u(a, y) = 0. \tag{11}$$

1. ***Boundary Conditions for Laplace Equation***

   - *The problem requires one pair of boundaries to have homogeneous (zero) boundary conditions.*
   - *The other pair of boundaries can have non-homogeneous boundary conditions.*

2. ***Approach to Solving the Problem***

   - *Given the homogeneous equation and boundary conditions, the separation of variables method is a suitable approach.*
   - *For Laplace's equation, there is no time derivative, so there are no initial conditions.*
   - *Instead, the Non-Homogeneous boundary conditions act as the "initial conditions" in the context of the separation of variables method.*

3. ***Role of Non-Homogeneous Boundaries***

- *The non-homogeneous boundary conditions are used to determine the coefficients in the solution.*
- *This is different from typical initial value problems, where initial conditions are used to determine coefficients.*
- *In this problem, the non-homogeneous boundaries play a crucial role in defining the solution.*

4. ***Steps in Separation of Variables***

- *Apply the separation of variables method to solve the Laplace equation.*
- *Use the homogeneous boundaries to solve the S-L problem.*
- *Use the non-homogeneous boundaries to determine the coefficients in the final solution.*

(1. Separation of Variables) *Applying the method of separation of variables, let*

$$u(x, y) = X(x)Y(y), \tag{12}$$

(2. PDE → ODEs) *Substitute (12) into equation (9), and separate variables to get*

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

where λ is a constant.

1. ***Importance of Homogeneity***

- *Homogeneous equations are required for the separation of variables method to work in this step.*
- *In non-homogeneous cases (to be discussed in the next class), the separation of variables method fails from the second step onward.*

2. ***Choosing the Sign of λ***

- *For homogeneous boundary conditions, especially when X is fixed at the boundaries, λ should be added a negative sign.*
- *This choice aligns with the form of the Sturm-Liouville problem for X, resulting in the ODE $X'' + \lambda X = 0$.*
- *Choosing a positive sign in front of λ may lead to an unfamiliar form, requiring additional work to derive the solution.*

Thus, we obtain two ordinary differential equations

$$X''(x) + \lambda X(x) = 0, \tag{13}$$
$$Y''(y) - \lambda Y(y) = 0. \tag{14}$$

From the homogeneous boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0,$$

we get

$$X(0) = X(a) = 0.$$

(3. Solving ODEs) *Now solve the boundary value problem of the ordinary differential equation*

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(a) = 0, \tag{15}$$

for non-zero solutions.

- *Students should be proficient in solving problems with all combinations of first and second kind boundary conditions.*

1. *When* $\lambda < 0$, *problem* (15) *has no non-trivial solutions.*
2. *When* $\lambda = 0$, *problem* (15) *also has no non-trivial solutions.*
3. *When* $\lambda > 0$, *problem* (15) *has non-trivial solutions.*
   At this time,
$$\lambda_n = \left(\frac{n\pi}{a}\right)^2,$$
   and the corresponding
$$X_n(x) = B_n \sin\frac{n\pi x}{a} \quad (n = 1, 2, \ldots).$$

Next, consider the equation

$$Y''(y) - \lambda Y(y) = 0,$$

Substitute λ_n *into equation* (14) *to get*

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0, \quad (n = 1, 2, \ldots).$$

The general solution is

$$Y_n(y) = C_n e^{\frac{n\pi}{a}y} + D_n e^{-\frac{n\pi}{a}y} \quad (n = 1, 2, \ldots).$$

Thus, we can obtain a series of particular solutions that satisfy the homogeneous boundary conditions (11) *for equation* (9)

$$u_n(x, y) = (a_n e^{\frac{n\pi}{a}y} + b_n e^{-\frac{n\pi}{a}y}) \sin\frac{n\pi}{a}x \quad (n = 1, 2, \ldots),$$

(4. Superposition of Series Solutions) *Since equation* (9) *and boundary conditions* (11) *are homogeneous, therefore*

$$u(x, y) = \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{a}y} + b_n e^{-\frac{n\pi}{a}y}) \sin\frac{n\pi}{a}x \tag{16}$$

still satisfies equation (9) *and the homogeneous boundary conditions* (11).

(5. Initial Conditions Determine Coefficients) *Applying the non-homogeneous boundary conditions*

$$u(x, 0) = f(x), \quad u(x, b) = g(x),$$

we have the relationships

$$\sum_{n=1}^{\infty} (a_n + b_n) \sin\frac{n\pi}{a}x = f(x),$$

and

$$\sum_{n=1}^{\infty} (a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}}) \sin\frac{n\pi}{a}x = g(x),$$

Using Fourier series coefficients, we get (first odd extension then periodic extension)

$$a_n + b_n = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi}{a}x dx,$$

$$a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} = \frac{2}{a}\int_0^a g(x) \sin\frac{n\pi}{a}x dx,$$

for $n = 1, 2, \ldots$.

Solving for a_n and b_n from the above equations and substituting back into (16) gives the solution to problem (9)-(11).

The solution to the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad (0 < x < a,\, 0 < y < b), \\ u(x,0) = f(x), \quad u(x,b) = g(x), \\ u(0,y) = 0, \quad u(a,y) = 0. \end{cases}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{a}y} + b_n e^{-\frac{n\pi}{a}y}) \sin \frac{n\pi}{a} x,$$

where

$$\begin{cases} a_n + b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x dx, \\ a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x dx, \end{cases}$$

for $n = 1, 2, \ldots$.

2. Boundary Value Problem of Laplace's Equation in a Circular Domain

- Regions related to circles (e.g., annular and sector regions) can be handled similarly by modifying the circular case.

Problem 2.2. *Consider the temperature distribution problem of a circular plate with radius r_0 under steady-state conditions. Assume the top and bottom surfaces of the plate are adiabatic, and the temperature on the circular boundary is known as $f(\theta)$ $(0 \le \theta \le 2\pi)$, and $f(0) = f(2\pi)$. Find the temperature distribution pattern under **steady-state conditions**.*

1. **Understanding the Physical Problem**
   - The term "steady-state" indicates that the problem involves Laplace's equation: $\Delta u = 0$.
2. **Dimensionality of the Problem**
   - The problem is two-dimensional because the temperature distribution is independent of the z-axis (insulated along z).
3. **Boundary Conditions**
   - The temperature on the circular boundary is given as $u(R, \theta) = f(\theta)$, where $\theta \in [0, 2\pi]$.
   - The periodicity condition $f(0) = f(2\pi)$ must be satisfied to ensure the temperature distribution is single-valued.
4. **Choice of Coordinate System**
   - Cartesian coordinates are not suitable for circular regions because they do not separate variables easily (due to the boundary conditions).
   - Polar coordinates (r, θ) are preferred due to their orthogonality and simplicity in handling circular boundaries (according to the boundary conditions).
   - Polar coordinates are often more suitable, especially for problems involving circular symmetry.
5. **Concept of 'Pao Ding Jie Niu'**

- 'Pao Ding Jie Niu' (the skill of a master butcher) is a Chinese idiom that emphasizes mastering a task through practice.
- This concept is widely applicable in mathematics and physics.
- In fluid dynamics, choosing the Lagrangian (comoving) coordinate system can simplify complex problems. For example, considering a fluid flow from the perspective of a moving reference frame (like a boat on a river) can make the problem easier to handle.

6. **Laplace's Equation in Polar Coordinates**
   - The Laplace operator in polar coordinates is given by:
   $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$
   - This can also be written as:
   $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$
7. **Mathematical Techniques**
   - The transformation from Cartesian to polar coordinates involves using the chain rule for partial derivatives.
   - Care must be taken to distinguish between functions defined in different coordinate systems to avoid confusion.
8. **Importance of Practice**
   - It is essential to practice deriving the Laplace operator in polar coordinates to understand its form and application.
   - Remembering the form of the Laplace operator in polar coordinates is crucial for solving problems involving circular regions.
9. **Two useful identities:**
   $$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r\frac{\partial^2}{\partial r^2}(ru) \quad \text{and} \quad r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r^2 u_{rr} + r u_r \quad \text{(Inspiring the Euler ODEs)}$$

Since the temperature under steady-state conditions satisfies Laplace's equation, and the region is circular, it is more convenient to use **polar coordinates** (the idea of "cutting up an ox like a master butcher"[1]) for the Laplace equation when applying the method of separation of variables.

Solution. *We use $u(r,\theta)$ to represent the temperature at point (r,θ) inside the circular plate. The problem can be formulated as the following boundary value problem:*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (0 < r < r_0), \quad (\textit{Not equivalent to } u_{xx} + u_{yy} = 0 \textit{ completely}) \tag{17}$$

$$u\big|_{r=r_0} = f(\theta). \tag{18}$$

Exercise: Verify that the Laplace equation $u_{xx}+u_{yy}=0$ *in polar coordinates takes the form* $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.
Hint: Perform polar coordinate transformation

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \quad \textit{and} \quad \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \arctan\frac{y}{x}. \end{cases}$$

[1] This is a classic Chinese idiom derived from the ancient text "Zhuangzi." It tells the tale of a skilled butcher who, through years of practice, develops exceptional expertise in his craft. The butcher is able to dismember an ox with such precision and ease that his movements appear to be a dance. His knife glides through the joints and along the natural lines of the ox's body, never needing to hack or force his way through bone or tissue.

For $u(x,y) := \bar{u}\big(r(x,y), \theta(x,y)\big)$ and $\bar{u}(r,\theta) := u\big(x(r,\theta), y(r,\theta)\big)$

$$u_x = \bar{u}_r \cdot r_x + \bar{u}_\theta \cdot \theta_x, \quad u_y = \bar{u}_r \cdot r_y + \bar{u}_\theta \cdot \theta_y$$

Further, we obtain

$$u_{xx} = (\bar{u}_{rr} r_x + \bar{u}_{r\theta}\theta_x) \cdot r_x + \bar{u}_r \cdot r_{xx} + (\bar{u}_{\theta r} r_x + \bar{u}_{\theta\theta}\theta_x) \cdot \theta_x + \bar{u}_\theta \cdot \theta_{xx}$$

and

$$u_{yy} = (\bar{u}_{rr} r_y + \bar{u}_{r\theta}\theta_y) \cdot r_y + \bar{u}_r \cdot r_{yy} + (\bar{u}_{\theta r} r_y + \bar{u}_{\theta\theta}\theta_y) \cdot \theta_y + \bar{u}_\theta \cdot \theta_{yy}$$

1. ***Coordinate Transformation Equivalence***
   - ***Question:*** *Is the transformation from Cartesian to polar coordinates equivalent?*
   - *This question is crucial as it impacts the solution process. Consider it if stuck.*
2. ***Characteristics of the Equation***
   - ***Question:*** *When solving equations, first examine their characteristics.*
   - *Homogeneous equations with homogeneous boundaries can use the method of separation of variables.*
   - *However, this problem does not satisfy homogeneous boundary conditions. What should be done?*
3. ***Solution Approach***
   - *Proceed with the method of separation of variables despite non-homogeneous boundaries.*
   - *Be prepared to modify and patch the method as necessary.*

(1. Separation of Variables) *Assume the solution to equation (17) is*

$$u(r,\theta) = R(r)\Phi(\theta),$$

(2. PDE → ODEs) *Substitute into equation (17) to get*

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$$

Separation of variables gives

$$\frac{r^2R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda$$

where λ is a constant.

1. ***Formulation of the Equation***
   - *We derive a form where one side is a function of R and the other side a function of θ, implying they must equal a constant.*
2. ***Choice of Constant Sign***
   - *The decision between a positive or negative constant is typically guided by homogeneous boundary conditions and the familiar SL problem.*
   - *In the absence of homogeneous boundary conditions, this guidance is not available, leading to a dilemma.*
3. ***Familiarity with Standard Forms***

- *It's beneficial to aim for a form we recognize, such as $X'' + \lambda X = 0$, which we are familiar with.*
- *This approach is driven by the **hope** of obtaining a solvable problem.*

4. ***Selection of the Constant***

- *We choose the constant with a positive sign to match a familiar form of the equation $X'' + \lambda X = 0$.*

5. ***Challenges in Separation of Variables***

- *The non-homogeneous boundary condition hinders the separation of variables, making the problem unsolvable in its current form.*

From this, two ordinary differential equations can be obtained

$$r^2 R'' + rR' - \lambda R = 0,$$
$$\Phi'' + \lambda\Phi = 0.$$

Question 2.1. *Which ODE should be solved first?*

- *Principle: First solve the S-L problem with homogeneous boundary conditions.*
- *However, now there are only non-homogeneous boundaries, and the boundary variables cannot be separated.*
- *The idea is to look for **implicit conditions**.*

1. ***Hidden Conditions in Boundary Value Problems:***
   - *A boundary value problem may have hidden conditions that are not explicitly given.*
   - *These hidden conditions can be derived from the given equation and boundary conditions through appropriate reasoning and physical considerations.*
   - *Hidden conditions may serve as **additional boundary conditions** or constraints for solving the problem.*
2. ***Coordinate Transformation and Information Loss:***
   - *When transforming coordinates from Cartesian to polar coordinates, **information is lost**.*
   - *The origin **loses its unique representation** in polar coordinates since $r = 0$ corresponds to an **undefined angle** θ.*
   - *To maintain a **one-to-one** correspondence, the origin is typically **excluded in polar** coordinates.*
   - ***Periodicity issues** arise because a point in Cartesian coordinates may correspond to **multiple angles in polar coordinates** (e.g., θ and $\theta + 2\pi$). For example, $(x, y) = (a, 0)$ becomes $(r, \theta) = (a, 0)$, $(a, 2\pi), \cdots$.*
3. ***Solving Periodicity Issues:***
   - *Method 1: Restrict the domain of θ to $[0, 2\pi]$ and enforce periodic boundary conditions such that*
   $$\begin{cases} u(r,\theta)\Big|_{\theta=0} = u(r,\theta)\Big|_{\theta=2\pi} \\ u_\theta(r,\theta)\Big|_{\theta=0} = u_\theta(r,\theta)\Big|_{\theta=2\pi} \end{cases}$$
   Since $\Phi'' + \lambda\Phi = 0$ is a second order ODE, the data upto u_θ is enough.

- *Method 2: Allow θ to take real values and enforce a periodic relationship $u(\theta) = u(\theta + 2\pi)$.*
- *Both methods ensure that the function values remain consistent across the transformation.*

4. ***Equivalence of Equations:***
   - *The transformed equation is* ***not completely equivalent*** *to the original Laplace equation in Cartesian coordinates.*
   - *The* ***loss of origin and introduction of periodicity*** *affect the equivalence of the equations.*
5. ***Supplementing Lost Information:***
   - *Lost information includes the* ***origin*** *and* ***periodicity****.*
     - *To address the* ***lost origin information****, impose a* ***boundedness condition*** *at the* ***origin****, requiring $|u(0,\theta)| < \infty$ (note $u(0,\theta) = \infty$ is a solution in polar coordinates, but not in Cartesian coordinates).*
     - *To address* ***periodicity****, introduce* ***periodic boundary conditions*** *such as $u(\theta + 2\pi) = u(\theta)$.*
6. ***Extending the Domain:***
   - *The original problem may define θ within $[0, 2\pi]$, but periodic boundary conditions require* ***extending*** *the domain to include all real values of θ periodically.*
   - *After extending the function to the entire real axis, it can be restricted back to the interval $[0, 2\pi]$.*
   - *Textbooks often implicitly assume analytic continuation without explicitly stating it.*

Since the temperature function $u(r,\theta)$ is single-valued, when θ changes from θ to $\theta+2\pi$, $u(r,\theta+2\pi) = u(r,\theta)$ holds, thus we have

$$\Phi(\theta + 2\pi) = \Phi(\theta).$$

At the same time, according to the physical meaning of the problem, the temperature at each point inside the circle should be bounded, hence $|u(0,\theta)| < +\infty$ holds, therefore $R(r)$ should satisfy the condition

$$|R(0)| < +\infty.$$

Thus, we obtain two ordinary differential equations with boundary conditions

$$\begin{cases} \Phi'' + \lambda\Phi = 0, \\ \Phi(\theta + 2\pi) = \Phi(\theta). \quad \textit{(The fifth SL problem: the periodic boundary.)} \end{cases} \tag{19}$$

and

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0, \\ |R(0)| < +\infty. \quad \textit{(Do not forget this new restriction!)} \end{cases} \tag{20}$$

(3. Solving ODEs) *We start with problem (19) and discuss λ in three cases:*
1. When $\lambda < 0$, the general solution of the equation is

$$\Phi(\theta) = Ae^{-\sqrt{-\lambda}\theta} + Be^{\sqrt{-\lambda}\theta}, \quad (\exp \textit{ monotonic functions, can be periodic})$$

where A, B are arbitrary constants. Since such functions do not satisfy periodic conditions, λ cannot take negative values.
2. When $\lambda = 0$, the general solution of the equation is

$$\Phi_0(\theta) = A_0\theta + B_0,$$

where A_0, B_0 are arbitrary constants. Only when $A_0 = 0$, the function Φ_0 satisfies periodic conditions. Therefore, when $\lambda = 0$, the solution to problem (19) is

$$\Phi_0(\theta) = B_0. \quad \textit{(horizontal line is periodic. Every number is its period by definition.)}$$

(Like the case with both sides second type boundary, there is 0 eigenvalue!)

Substituting $\lambda = 0$ into equation (20) gives

$$r^2 R'' + rR' - \lambda R = 0,$$

and its general solution is

$$R_0(r) = C_0 \ln r + D_0,$$

where C_0, D_0 are arbitrary constants.

General Solution to a Reducible Second-Order Differential Equation

Problem: *Find the general solution to the following reducible second-order differential equation for $R(r)$:*

$$r^2 R'' + rR' = 0$$

Solution. *Let*

$$R' = P(r) \Rightarrow R'' = P'(r)$$

Substitute into the original equation:

$$r^2 P'(r) + rP(r) = 0 \quad \Rightarrow \quad \frac{1}{P}dP = -\frac{1}{r}dr$$

Then

$$\Rightarrow \ln P = -\ln r + C \quad \Rightarrow \quad P(r) = \frac{C}{r}$$

Thus, we have

$$R'(r) = \frac{C}{r}$$

The general solution to the original equation is:

$$R(r) = C_0 \ln r + D_0,$$

where C_0, D_0 are arbitrary constants.

Only when $C_0 = 0$, the function R_0 satisfies the boundedness condition $|R(0)| < +\infty$. Therefore, when $\lambda = 0$, the solution to problem (20) is

$$R_0(r) = D_0.$$

Thus, a non-zero solution to the original equation (17) is obtained

$$u_0(r, \theta) = B_0 D_0 = \frac{1}{2}a_0.$$

3. When $\lambda > 0$, the general solution of the equation is

$$\Phi(\theta) = A \cos \sqrt{\lambda}\theta + B \sin \sqrt{\lambda}\theta,$$

where A, B are arbitrary constants. Since $\Phi(\theta + 2\pi) = \Phi(\theta)$,

$$\lambda = n^2 \quad (n = 1, 2, \ldots),$$

at this time, the solution to equation (19) can be expressed as

$$\Phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta.$$

The eigenfunction set corresponding to the Poisson equation in a circular domain (i.e., $\Phi'' + \lambda\Phi = 0$ and $\Phi(\theta + 2\pi) = \Phi(\theta)$) is:

$$\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cdots \cos n\theta, \sin n\theta, \cdots\}$$

Substituting $\lambda = n^2$ into equation (20) gives the ***Euler equation***

$$r^2R'' + rR' - n^2R = 0,$$

and its general solution is

$$R_n(r) = C_nr^n + D_nr^{-n}.$$

To ensure $|R(0)| < +\infty$, only $D_n = 0$ ($n = 1, 2, \ldots$), so

$$R_n(r) = C_nr^n \quad (n = 1, 2, \ldots).$$

General Form of Euler's Equation

The general form of Euler's equation is:

$$x^ny^{(n)} + P_1x^{n-1}y^{(n-1)} + \cdots + P_{n-1}xy' + P_ny = f(x).$$

Here, P_1 to P_n are constants, and $f(x)$ is a known function.

General Solution to a Second-Order Euler Equation

Problem: *Find the general solution to the following second-order Euler equation:*

$$r^2R'' + rR' - n^2R = 0, \quad (n = 1, 2, \ldots)$$

Solution. *Make the transformation $r = e^t \Rightarrow t = \ln r$.*

Thus, we have:

$$R_r = R_t \cdot \frac{1}{r}, \quad R_{rr} = \left(R_{tt} \cdot \frac{1}{r^2} + R_t \cdot \left(-\frac{1}{r^2}\right)\right) = \frac{1}{r^2}R_{tt} - \frac{1}{r^2}R_t.$$

Substitute into the original equation:

$$R_{tt} - R_t + R_t - n^2R = 0 \Rightarrow R_{tt} - n^2R = 0$$

$$\Rightarrow R_n = C_ne^{nt} + D_ne^{-nt}$$

Substitute $t = \ln r$ back to get the original form:

$$R_n(r) = C_nr^n + D_nr^{-n} \quad (n = 1, 2, \ldots)$$

The general solution to the original equation is:

$$R_n(r) = C_nr^n + D_nr^{-n} \quad (n = 1, 2, \ldots)$$

where C_n, D_n are arbitrary constants.

Another method–trial and error: *By observation ($r^2\partial_r^2, r\partial_r$), the solution should be r^α (since derivatives of polynomial reduce its order $r^2\partial_r^2 r^\alpha \sim r\partial_r r^\alpha \sim r^\alpha$) and substitute it in the equation to determine α.*

1. ***General Approach to Solving Equations:***
   - *Use both **intuitive guessing** and **inverse methods** to solve equations.*
   - *Transform **new equations** into **familiar** forms by **changing** either (1) the **function** or (2) the **variable**.*
2. ***Transformations in Differential Equations:***
   - *Since differential equations only involve two objects: (1) an **unknown function** and (2) its **variables**.*

- *Transformations can be made on (1) the **function itself** (e.g., the method of integration factor involves merging various functions to form a new function that simplifies the equation.) or (2) the **independent variable** (e.g., the Euler equation).*

3. ***Euler's Equation:***
   - *For Euler's equation, a variable transformation is applied.*
   - *Let $r = e^t$, then $t = \ln r$.*
   - *This leads to a simpler form of the differential equation.*
4. ***Guessing Solutions:***
   - *Assume solutions of the form $R = r^\alpha$.*
   - *Substitute into the equation to determine α.*
5. ***Exploratory Process:***
   - *The process of finding transformations is exploratory.*
   - *Use known forms and patterns to guide the search for new transformations.*
6. ***Chain Rule Application:***
   - *Use the chain rule to relate derivatives with respect to different variables.*
   - *This can simplify the process of finding suitable transformations.*
7. ***Physical Intuition:***
   - *Physicists often use intuitive methods that are not rigorous but efficient.*
   - *These methods can provide quick insights into the behavior of solutions.*
8. ***General Solution for Euler's Equation:***
   - *The general solution involves a linear combination of r^n and r^{-n}.*

The ideas for Euler ODE:

We start with the differential identity:

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} = r \frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right) \overset{Hope}{=} \frac{\partial}{\partial t}\frac{\partial}{\partial t} R$$

Transformation Goal:

$$\text{Hope a transformation: } r\frac{\partial}{\partial r} \to \frac{\partial}{\partial t} \quad (*)$$

Simplification:

$$\text{Then, } \frac{\partial^2 R}{\partial t^2} \text{ is easy since } \partial_t^2 R - n^2 R = 0 \Rightarrow R_n = Ce^{nt} + De^{-nt}$$

How to achieve (*)?

$$\frac{dR}{dt} \overset{chain\ rule\ (+)}{=} \frac{dr}{dt}\frac{dR}{dr} = r\frac{dR}{dr}$$

$$\left(\text{Physicist's approach: } \quad r\frac{\cancel{dR}}{dr} = \frac{\cancel{dR}}{dt} \Rightarrow dt = \frac{1}{r}dr = d\ln r\right)$$

***By chain rule** (+) **for** $R(r(t))$:*

$$\frac{dr}{dt} = r \Rightarrow d\ln r = dt \Rightarrow r = e^t$$

Then we find the transformation of variables.

Thus, when $\lambda = n^2 (n = 1, 2, \cdots)$*, we obtain a series of particular solutions for equation* (17).

$$u_n(r,\theta) = (a_n \cos n\theta + b_n \sin n\theta) r^n \quad (n = 1, 2, \cdots),$$

where $a_n = A_n C_n, b_n = B_n C_n$ *are arbitrary constants.*

(4. Superposition of Series Solutions) *Since equation* (17) *is linear and homogeneous, using the principle of superposition, we can obtain the series solution that satisfies the univalence and boundedness conditions for the equation as*

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n. \tag{21}$$

(5. Initial Conditions Determine Coefficients) *To determine the coefficients* a_n, b_n*, use the boundary condition* (18) *i.e.,* $u|_{r=r_0} = f(\theta)$.

$$u(r_0,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r_0^n = f(\theta), \quad (0 \le \theta \le 2\pi)$$

By Fourier series theory, we know

$$\begin{cases} a_n r_0^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \cdots), \\ b_n r_0^n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \cdots), \end{cases}$$

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \cdots), \end{cases}$$

Therefore, the solution to the boundary value problem (17)-(18) *is given by the series solution* (21).

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n.$$

where the coefficients a_n, b_n *are determined by equation* (22),

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta & (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta & (n = 1, 2, \cdots), \end{cases} \tag{22}$$

1. **Determine Coefficients:**
   - Use non-homogeneous boundary conditions (play the role of initial data) instead of initial values to determine coefficients.
2. **Domain Importance:**
   - The domain $[0, 2\pi]$ (θ restricts to $[0, 2\pi]$ from the extended $\mathbb{R}$) is crucial for defining the coefficients.
3. **Orthogonality and Integration:**
   - Use orthogonality of trigonometric functions over one period $[0, 2\pi]$ (one period ensures the orthogonality of trigonometric functions).
   - **Extend** the function to the entire real line **periodically** and then restrict it back to $[0, 2\pi]$.
4. **Integral Limits:**
   - Perform integration from 0 to 2π for determining coefficients.

Ex 2.1. *Solve the Following Problem*

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (0 < r < R), \\ u(R, \theta) = \theta \sin \theta. \end{cases}$$

Solution. *Using the formula*

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \quad (n = 1, 2, \cdots), \end{cases}$$

we obtain

$$\begin{cases} a_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin\theta \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin\theta \sin n\theta d\theta \quad (n = 1, 2, \cdots), \end{cases}$$

Since

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin\theta \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots),$$

we have

$$a_n = \frac{1}{2\pi R^n} \int_0^{2\pi} \theta[\sin(1+n)\theta + \sin(1-n)\theta] d\theta = \frac{1}{R^n} \cdot \frac{2}{n^2 - 1}, \quad (n \neq 1)$$

and

$$a_1 = \frac{1}{\pi R} \int_0^{2\pi} \theta \sin\theta \cos\theta d\theta = \frac{1}{2\pi R} \int_0^{2\pi} \theta \sin 2\theta d\theta = -\frac{1}{2R}$$

Especially,

$$a_0 = -2$$

And since

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} \theta \sin\theta \sin n\theta d\theta \quad (n = 1, 2, \cdots),$$

we have

$$b_n = \frac{1}{2\pi R^n} \int_0^{2\pi} \theta[\cos(1-n)\theta - \cos(1+n)\theta] d\theta = 0 \quad (n \neq 1)$$

and

$$b_1 = \frac{1}{\pi R} \int_0^{2\pi} \theta \sin^2\theta d\theta = \frac{1}{2\pi R} \int_0^{2\pi} \theta(1 - \cos 2\theta) d\theta = \frac{\pi}{R}.$$

Substituting the obtained coefficients into the series solution formula

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n.$$

The solution to the given problem is

$$u(r, \theta) = -1 - \frac{r}{2R}\cos\theta + \frac{\pi}{R} r \sin\theta + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \left(\frac{r}{R}\right)^n \cos n\theta.$$

Ex 2.2. *Solve the Following Problem*

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (0 < r < r_0), \\ u(r_0, \theta) = A \sin 2\theta. \end{cases} \tag{23}$$

Solution. *Using the formula*

$$\begin{cases} a_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi r_0^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \quad (n = 1, 2, \cdots), \end{cases}$$

we obtain

$$\begin{cases} a_n = \frac{1}{\pi R^n} \int_0^{2\pi} A \sin 2\theta \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots), \\ b_n = \frac{1}{\pi R^n} \int_0^{2\pi} A \sin 2\theta \sin n\theta d\theta \quad (n = 1, 2, \cdots), \end{cases}$$

Since

$$a_n = \frac{1}{\pi r_0^n}\int_0^{2\pi} A\sin 2\theta \cos n\theta d\theta \quad (n = 0, 1, 2, \cdots),$$

using the ***orthogonality*** *of trigonometric functions, we get*

$$a_n = 0 \quad (n = 0, 1, 2, \cdots),$$

and since

$$b_n = \frac{1}{\pi r_0^n}\int_0^{2\pi} A\sin 2\theta \sin n\theta d\theta \quad (n = 1, 2, \cdots),$$

using the ***orthogonality*** *of trigonometric functions again, we get*

$$b_n = 0 \quad (n \neq 2),$$

and

$$b_2 = \frac{A}{\pi r_0^2}\int_0^{2\pi} \sin^2 2\theta d\theta = \frac{A}{2\pi r_0^2}\int_0^{2\pi}(1-\cos 4\theta)d\theta = \frac{A}{r_0^2}.$$

Substituting the obtained coefficients into the series solution formula

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^n.$$

The solution to the given problem is

$$u(r,\theta) = \frac{A}{r_0^2}r^2 \sin 2\theta.$$

This example can also be solved by the **method of trial and error**.

The method of trial and error:
The trial method is very important. For certain practical problems, you can guess the shape of the solution based on the **physical meaning**, **algebraic structure** or **geometric characteristics** of the problem or the experience, then make an assumption of a specific form, and solve the equation to determine the parameters. (The method of separation of variables is essentially also a trial method, assuming a separable form.)

Ex 2.3. *Solve the Following Problem*

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (0 < r < r_0), \\ u(r_0,\theta) = A\sin 2\theta. \end{cases}$$

Solution. *Since the function $r^2 \sin 2\theta$ is harmonic, the function $C_1 r^2 \sin 2\theta + C_2$ is also harmonic, where C_1, C_2 are arbitrary constants. Let us assume (apriori assumption) the solution be*

$$u(r,\theta) = C_1 r^2 \sin 2\theta + C_2$$

From the boundary condition, we get

$$u(r_0,\theta) = C_1 r_0^2 \sin 2\theta + C_2 = A\sin 2\theta,$$

Comparing coefficients, we know

$$C_2 = 0, \quad C_1 r_0^2 = A \implies C_1 = \frac{A}{r_0^2}.$$

Thus, the solution to the given problem is

$$u(r,\theta) = \frac{A}{r_0^2}r^2 \sin 2\theta.$$

Summary of the forms of five common eigenfunction series

1. $u(0,t)=0,\ u(l,t)=0;\ \Rightarrow \left\{\sin\frac{n\pi x}{l}\right\}\ (n=1,2,\cdots);$
2. $u(0,t)=0,\ u_x(l,t)=0;\ \Rightarrow \left\{\sin\frac{(2n-1)\pi x}{2l}\right\}\ (n=1,2,\cdots);$
3. $u_x(0,t)=0,\ u(l,t)=0;\ \Rightarrow \left\{\cos\frac{(2n-1)\pi x}{2l}\right\}\ (n=1,2,\cdots);$
4. $u_x(0,t)=0,\ u_x(l,t)=0;\ \Rightarrow \left\{\cos\frac{n\pi x}{l}\right\}\ (n=0,1,2,\cdots);$

The above forms are applicable to one-dimensional vibration equations, heat conduction equations, and Poisson's equations on rectangular domains.

5. Eigenfunction series corresponding to Poisson's equation on a circular domain

$$\{1,\cos\theta,\sin\theta,\cos 2\theta,\sin 2\theta,\cdots\cos n\theta,\sin n\theta,\cdots\}$$