Lecture Notes: Mathematical Physics Equations and Special Functions

Chao Liu

Week 4-1

Recall:

- **Non-homogeneous equations + homogeneous boundary + homogeneous initial conditions** $\to$ The eigenfunction method.

1 Heat Conduction in a Finite Rod with a Heat Source (Conti.)

Ex 1.1. *Solve the following problem*

$$\begin{cases} u_t = a^2 u_{xx} + A & (0 < x < l,\, t > 0), \\ u(0,t) = 0,\, u_x(l,t) = 0, \\ u(x,0) = 0. \end{cases} \tag{1}$$

where A is a constant.

Solution. ***[1. Homogeneous Eigenfunction System]*** *The corresponding homogeneous equation of the original equation*

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

Then

$$X_n(x) = B_n \sin\frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \ldots).$$

(Note:) or written as

$$X_n(x) = B_n \sin\frac{(2n-1)\pi x}{2l} \quad (n = 1, 2, \ldots).$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation and satisfying the homogeneous boundary conditions is $\left\{\sin\frac{(2n+1)\pi x}{2l}\right\}$.

[2. Assumed Series Solution] *Let the solution be*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin\frac{(2n+1)\pi x}{2l}. \tag{2}$$

[3. Expansion of Free Term] *Then expand A into the Fourier sine series according to the above eigenfunction series*

$$A = \sum_{n=0}^{\infty} A_n(t) \sin\frac{(2n+1)\pi x}{2l}, \tag{3}$$

where

$$A_n(t) = \frac{2}{l}\int_0^l A\sin\frac{(2n+1)\pi x}{2l}\,dx = \frac{4A}{(2n+1)\pi}.$$

> *Recall:*
>
> - *The concept is analogous to expressing any vector in terms of a chosen set of basis vectors, similar to how coordinates are used in linear algebra and analytic geometry.*

[4. Comparison of Coefficients] *Substituting (2)-(3) into equation (1) yields*

$$\sum_{n=1}^{\infty}\left[u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) - \frac{4A}{(2n+1)\pi}\right]\sin\frac{(2n+1)\pi x}{2l} = 0,$$

from which we get (orthogonality of $\sin\frac{(2n+1)\pi x}{2l}$*)*

$$u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) = \frac{4A}{(2n+1)\pi}. \tag{4}$$

> *Memorizing:*
>
> - *The first order ODE* $\to$ *heat behavior* $\to$ *(Source term)*$*\exp(-coef. \times t)$*;*
> - *The second order ODE* $\to$ *oscillation behavior* $\to$ $\frac{1}{coef.}$*(Source term)*$*\sin(coef. \times t)$*.*

[5. Initial Value Transformation] *Using the initial condition in expression (2) gives*

$$\sum_{n=0}^{\infty} u_n(0)\sin\frac{(2n+1)\pi x}{2l} = 0 \quad\Rightarrow\quad u_n(0) = 0, \quad (n = 0, 1, 2, \ldots). \tag{5}$$

[6. Solving ODEs] *Thus, we obtain the following initial value problem for ordinary differential equations*

$$\begin{cases} u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) = \frac{4A}{(2n+1)\pi} \\ u_n(0) = 0, \quad (n = 0, 1, 2, \ldots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problems (4)-(5) is

$$\begin{aligned} u_n(t) &= \int_0^t \frac{4A}{(2n+1)\pi} e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 (t-\tau)}\,d\tau \\ &= \frac{4A}{(2n+1)\pi}\int_0^t e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 (t-\tau)}\,d\tau \\ &= \frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}. \end{aligned}$$

Substituting

$$u_n(t) = \frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}$$

into

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\sin\frac{(2n+1)\pi}{2l}x,$$

we obtain the solution to the problem

$$u(x,t) = \sum_{n=0}^{\infty} \frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}\sin\frac{(2n+1)\pi}{2l}x.$$

2 Poisson Equation (Non-homogeneous Laplace Equation)

Use the **method of eigenfunctions** to solve the boundary value problem of the non-homogeneous Laplace equation. We illustrate the key points and steps of solving such problems through examples.

Ex 2.1. *In a circle centered at the origin with a radius of 1, find the solution to the Poisson equation* $u_{xx} + u_{yy} = -2x$ *that satisfies the boundary condition* $u|_{x^2+y^2=1} = 0$.

> **Note:**
>
> - If the boundary condition is given by $u|_{x^2+y^2=1} = f$, our first step is to separate the non-homogeneous part into two systems. We then solve each system using the method of separation of variables and eigenfunction expansion, respectively.
> - The rectangular, strip, and sector domains can be analyzed and solved in a similar manner.

Solution. *Since the region is a circular domain, perform a polar coordinate transformation:*

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and denote $\bar{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$, *then the problem is reduced to:*

$$\bar{u}_{rr} + \frac{1}{r}\bar{u}_r + \frac{1}{r^2}\bar{u}_{\theta\theta} = -2r\cos\theta \quad (0 < r < 1), \tag{6}$$

$$\bar{u}|_{r=1} = 0. \tag{7}$$

> ***Recall:***
>
> - *For the problem discussed last time regarding solving the Laplace equation in a circular region, transforming to polar coordinates results in the* ***loss of two pieces of information****. To compensate, we add two conditions:* ***boundedness at the origin*** *and* ***periodicity****.*

[1. Homogeneous Eigenfunction System] *From the discussion in Section 2.3, the corresponding homogeneous equation to* (6) *that satisfies the single-valued condition has eigenfunctions satisfying:*

$$\Phi'' + \lambda\Phi = 0, \quad \Phi(\theta + 2\pi) = \Phi(\theta).$$

Thus, the eigenfunction series corresponding to (6) *that also satisfies the single-valued condition is:*

$$1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos n\theta, \sin n\theta, \ldots$$

[2. Assumed Series Solution] *By the method of eigenfunctions, assume the solution to equation* (6) *is:*

$$\bar{u}(r,\theta) = \sum_{n=0}^{\infty} \left[a_n(r)\cos n\theta + b_n(r)\sin n\theta\right]. \tag{8}$$

[4. Comparison of Coefficients] *Substituting* (8) *into equation* (6) *and simplifying, we get:*

$$\sum_{n=0}^{\infty}\left[\left(a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n\right)\cos n\theta + \left(b_n'' + \frac{1}{r}b_n' - \frac{n^2}{r^2}b_n\right)\sin n\theta\right] = -2r\cos\theta.$$

Comparing the coefficients of $\cos n\theta$ *and* $\sin n\theta$ *on both sides of the equation, we obtain*

$$a_1'' + \frac{1}{r}a_1' - \frac{1}{r^2}a_1 = -2r \quad (n = 1), \tag{9}$$

$$a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n = 0 \quad (n \neq 1), \quad \textit{(Euler)} \tag{10}$$

$$b_n'' + \frac{1}{r}b_n' - \frac{n^2}{r^2}b_n = 0. \quad \textit{(Euler)} \tag{11}$$

[5. Initial (boundary) Value Transformation] *Substitute boundary condition (7) into equation (8), then we have ("comparing the coefficient")*

$$a_n(1) = 0, \quad b_n(1) = 0. \tag{12}$$

According to the ***boundedness*** *of the function $\bar{u}(r, \theta)$, it follows that*

$$|a_n(0)| < +\infty, \quad |b_n(0)| < +\infty. \tag{13}$$

[6. Solving ODEs] *Since equations (10) and (11) are homogeneous Euler equations, their general solutions are:*

$$a_n(r) = A_n r^n + B_n r^{-n} \quad (n \neq 1),$$

and

$$b_n(r) = \bar{A}_n r^n + \bar{B}_n r^{-n}.$$

From condition (13), we get $B_n = 0$ and $\bar{B}_n = 0$. From condition (12), we get $A_n = 0$ and $\bar{A}_n = 0$. Therefore, $a_n(r) = 0$ (for $n \neq 1$), $b_n(r) = 0$.

Since equation (9) is a ***non-homogeneous Euler*** *equation, its general solution is:*

$$a_1(r) = c_1 r + c_2 r^{-1} - \frac{1}{4} r^3. \tag{14}$$

From condition (13), we get $c_2 = 0$. From condition (12), we get $c_1 = \frac{1}{4}$. Therefore,

$$a_1(r) = \frac{1}{4} r - \frac{1}{4} r^3.$$

Method of Variation of Parameters

We assume

$$a_1 = C_1(r) r + C_2(r) r^{-1}. \tag{15}$$

Then calculate

$$a_1' = C_1(r) - C_2(r)\frac{1}{r^2} \tag{16}$$

and

$$a_1'' = C_1'(r) - \frac{1}{r^2} C_2'(r) + \frac{2}{r^3} C_2(r) \tag{17}$$

Substituting (15)-(17) into (14), we have

$$\Rightarrow C_1'(r) - C_2'(r)\frac{1}{r^2} + C_2(r)\frac{2}{r^3} + \cancel{\frac{C_1}{r} - C_2(r)\frac{1}{r^3} - \frac{C_1}{r}} - C_2(r)\frac{1}{r^3} = -2r$$

$$\Rightarrow \begin{cases} C_1'(r) - C_2'(r)\frac{1}{r^2} = -2r \\ C_1'(r) + C_2'(r)\frac{1}{r^2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{C_2'(r)}{r^2} = r \Rightarrow C_2'(r) = r^3 \\ C_1'(r) = -r \end{cases}$$

$$\Rightarrow C_1(r) = -\frac{1}{2} r^2, \quad C_2(r) = \frac{1}{4} r^4.$$

$$\Rightarrow a_1(r) = -\frac{1}{2} r^3 + \frac{1}{4} r^3 = -\frac{1}{4} r^3.$$

Trial and errors

(similar to the second method for Euler ODE) According to the ***homogeneity*** *of every term, we know there is a special solution* $a_1 = Cr^3$ *and take it into* (9)*, we obtain*

$$\begin{cases} a_i' = 3Cr^2 \\ a_i'' = 6Cr \end{cases}$$
$$\Rightarrow 6Cr + \frac{1}{r} \cdot 3Cr^2 - \frac{1}{r^2} Cr^3 = -2r$$
$$\Rightarrow 6Cr + 3Cr - Cr = -2r$$
$$\Rightarrow 8C = -2 \Rightarrow C = -\frac{1}{4} \Rightarrow a_1 = -\frac{1}{4} r^3$$

The solution is then expressed as:

$$a_1(r) = \frac{1}{4} r - \frac{1}{4} r^3, \quad a_n(r) = 0 \quad (\text{ for } n \neq 1), \quad b_n(r) = 0.$$

Substituting these into the series solution:

$$\bar{u}(r, \theta) = \sum_{n=0}^{\infty} [a_n(r) \cos n\theta + b_n(r) \sin n\theta].$$

The solution to the problem is:

$$\bar{u}(r, \theta) = \frac{1}{4}(1 - r^2) r \cos \theta.$$

Converting to Cartesian coordinates:

$$u(x, y) = \frac{1}{4}[1 - (x^2 + y^2)]x.$$

Summary:

1. **Basic Six Steps of Eigenfunction Method.**
2. **Solution of Non-Homogeneous Euler ODE:**
   - Use the method of undetermined coefficients or variation of parameters.
   - Find a particular solution for the non-homogeneous part.
   - Combine with the solution of the corresponding homogeneous equation.
3. **Polar Coordinates System:**
   - Two conditions are typically missing in polar form expressions.
   - Additional implicit conditions are required to fully specify the system.

Alternative Solution Method (Trial and errors)

The Poisson equation in polar coordinates is given by:

$$\begin{cases} \bar{u}_{rr} + \frac{1}{r} \bar{u}_r + \frac{1}{r^2} \bar{u}_{\theta\theta} = -2r \cos \theta \quad (0 < r < 1), \\ \bar{u}|_{r=1} = 0. \end{cases}$$

Idea: If we know a particular solution w of the Poisson equation, then by making the function transformation $\bar{u} = v + w$, we can transform the Poisson equation into the Laplace equation. Then, by solving the boundary value problem of the Laplace equation, we can obtain the boundary value problem of the Poisson equation.

The **idea** is to use a known particular solution w to **transform the Poisson equation into a Laplace equation**

The particular solution carries the burden of all the non-homogeneity. Idea: find something to carry the burden of the non-homogeneity $\to$ will be used in next section for he non-homogeneity of the boundary.

$w = -\dfrac{1}{4}r^3\cos\theta$. (Similar to the previous and using the free term, there is a $\cos\theta \to Cr^3\cos\theta$)

Let $\bar{u}(r,\theta) = v(r,\theta) + w(r,\theta)$, then the problem can be transformed into:

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & (0 < r < 1), \\ v|_{r=1} = \frac{1}{4}\cos\theta. & \text{(Cost!)} \end{cases}$$

Assume (trial and errors, by observing $v|_{r=1} = \frac{1}{4}\cos\theta$) the solution to the Laplace equation is:

$$v(r,\theta) = Ar\cos\theta + B.$$

To satisfy the boundary condition:

$$v(1,\theta) = A\cos\theta + B = \frac{1}{4}\cos\theta.$$

Solving for A and B:

$$B = 0, \quad A = \frac{1}{4}, \quad \Rightarrow \quad v(r,\theta) = \frac{1}{4}r\cos\theta.$$

The final solution to the boundary value problem is:

$$\bar{u}(r,\theta) = v(r,\theta) + w(r,\theta) = \frac{1}{4}r\cos\theta - \frac{1}{4}r^3\cos\theta = \frac{1}{4}(1-r^2)r\cos\theta.$$

Supplementary Information

For the Poisson equation boundary value problem:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = F(r,\theta), & (0 < r < r_0), \\ u|_{r=r_0} = f(\theta). \end{cases}$$

Approach 1: Decompose the solution $u(r,\theta)$ into two parts:

$$u(r,\theta) = v(r,\theta) + w(r,\theta),$$

where $v(r,\theta)$ satisfy (use the method of eigenfunctions)

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = F(r,\theta), & (0 < r < r_0), \\ v|_{r=r_0} = 0. \end{cases}$$

and $w(r,\theta)$ satisfy (use separation of variables)

$$\begin{cases} w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = 0, & (0 < r < r_0), \\ w|_{r=r_0} = f(\theta). \end{cases}$$

Approach 2: (carry the burden of the non-homogeneity) 1. Find a particular solution $w(r,\theta)$ such that:

$$u(r,\theta) = v(r,\theta) + w(r,\theta),$$

2. Transform the Poisson equation into a Laplace equation:

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0, & (0 < r < r_0), \\ v|_{r=r_0} = f(\theta) - w(r_0,\theta). \end{cases}$$

Solve this problem using separation of variables or trial methods.

2.5 Problems with Non-homogeneous Boundary Conditions

In this section, we discuss the solution methods for problems with **non-homogeneous boundary** conditions.

Basic principle: Regardless of whether the equation is homogeneous or non-homogeneous, choose an auxiliary function $w(x,t)$, and through the function substitution $u(x,t) = v(x,t) + w(x,t)$, make the **boundary conditions** for the new unknown function $v(x,t)$ **homogeneous**. Using w carry the burden of the non-homogeneity of the boundary.

We will use the following problem as an example to illustrate the method of selecting the function substitution (also known as the **auxiliary function method**).

1. **Auxiliary Function Concept:**
   - The **idea** is straightforward: find a specific function to act as a **scapegoat**, attributing all the **faults** to it, in order to **save the other** terms.
2. **Purpose of the Scapegoat Function w:**
   - The given function w (we should find it) should satisfy boundary conditions u_1 and u_2.
   - It carries all the burdens of all **inhomogeneous** boundary conditions.
3. **Decomposition of u:**
   - Write u as $v + w$.
   - Use **linear superposition** to let w (given) handle the inhomogeneous boundaries.
4. **Properties of w:**
   - w is a given function (should find).
5. **Boundary Conditions for w:**
   - Hope to find a function w satisfying $w(0,t) = u_1$ and $w(l,t) = u_2$.
6. **Simplification for v:**
   - After subtracting w from u, v satisfies a **homogeneous boundary**.
7. **Finding w:**
   - w can be **any** function that **meets the boundary conditions** (this is the only requirement on w, no other conditions, so w can be chosen quite **freely**).
   - For example, w can be selected as **linear** functions or **parabolic curves** passing through the boundary points.
   - The auxiliary function is **not unique**.
8. **Principle:**
   - The **simplest** the auxiliary function.
9. **Standardization for Exams:**
   - To ensure consistency, students should use a specific method to choose w.
   - Suggested method: use a straight line between the two boundary points.

Summary:

- An auxiliary function as a scapegoat **carries all the non-homogeneous boundary**;

- It can be **chosen freely** (**not unique**) only require it **passes through the boundary point**. Thus the simplest choice is a **linear** function (**linear interpolation**) or **parabolic** curves passing through the boundary.
- Choosing different homogenization functions w, naturally leads to different boundary value problems for v, and thus the **solutions for v will also differ**. However, due to the **uniqueness of solutions** to mixed value problems, it ensures that the final u provided will be the **same**, even though the forms of the **expressions** might differ.

Consider the boundary value problem:

$$u_{tt} = a^2 u_{xx} + f(x,t) \quad (0 < x < l, t > 0), \tag{18}$$
$$u(0,t) = u_1(t), \quad u(l,t) = u_2(t), \tag{19}$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{20}$$

By making a function transformation to make the boundary conditions homogeneous, we let

$$u(x,t) = v(x,t) + w(x,t), \tag{21}$$

and choose the auxiliary function $w(x,t)$ such that the new unknown function $v(x,t)$ satisfies homogeneous boundary conditions, i.e.,

$$v(0,t) = 0, \quad v(l,t) = 0. \tag{22}$$

From (19) and (21), it is easy to see that to satisfy (22), it is only necessary that

$$w(0,t) = u_1(t), \quad w(l,t) = u_2(t). \tag{23}$$

In fact, there are **many functions** $w(x,t)$ that satisfy the two conditions in (23). For the convenience of future calculations, it is usually taken as a **linear function** of x, i.e., let

$$w(x,t) = A(t)x + B(t), \quad \leftarrow \quad \text{(Algebraic method)}$$

Determine $A(t)$ and $B(t)$ from condition (23) to get

$$B(t) = u_1(t), \quad A(t) = \frac{1}{l}[u_2(t) - u_1(t)],$$

Thus, we have

$$w(t,x) = \underbrace{\frac{u_2(t) - u_1(t)}{l}}_{\text{slope}} x + \underbrace{u_1(t)}_{\text{starting pt.}}.$$

Geometric methods for the linear auxiliary functions

- Given boundary values u_1 at the left endpoint and u_2 at the right endpoint.
- The auxiliary function is a **straight line passing through** $(0, u_1)$ and (l, u_2).
- The **slope** of the line is $(u_2 - u_1)/l$.
- The linear function can be expressed as $u_1 + (x/l) \cdot (u_2 - u_1)$.

Therefore, let

$$u(x,t) = v(x,t) + \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t).$$

Then the problem (18)-(20) can be transformed into a boundary value problem for $v(x,t)$:

$$\begin{cases} v_{tt} = a^2 v_{xx} + f_1(x,t) & (0 < x < l, \ t > 0), \\ v(0,t) = v(l,t) = 0, \\ v(x,0) = \varphi_1(x), \quad v_t(x,0) = \psi_1(x). \end{cases} \tag{24}$$

where the functions f_1, φ_1, and ψ_1 are defined as:

$$\begin{cases} f_1(x,t) = f(x,t) - \frac{x}{l}[u_2''(t) - u_1''(t)] - u_1''(t), \\ \varphi_1(x) = \varphi(x) - \frac{x}{l}[u_2(0) - u_1(0)] - u_1(0), \\ \psi_1(x) = \psi(x) - \frac{x}{l}[u_2'(0) - u_1'(0)] - u_1'(0). \end{cases} \leftarrow \begin{cases} \text{If you don't choose a linear function, there will} \\ \text{be more terms, causing unnecessary trouble.} \end{cases}$$

Recall: For the system (24),

1. **First Step:** Isolate the non-homogeneity.
2. **Solving Methods:**
   - Use separation of variables for one part (non-homogeneous initial data).
   - Use eigenfunction expansion for another part (non-homogeneous equation).
3. **General Approach:**
   - Transform the non-homogeneous boundary into a homogeneous one using an auxiliary function.
   - Isolate the non-homogeneous part and solve it using different methods.

Substitute the solution of equation (24) into:

$$u(x,t) = v(x,t) + \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t).$$

This gives the solution to the original problem (18)-(20).

If the boundary conditions are not all of the first kind, similar methods can be used to transform non-homogeneous boundary conditions into homogeneous ones. We provide the corresponding auxiliary function $w(x,t)$ expressions for the following cases of non-homogeneous boundary conditions:

1. $u(0,t) = u_1(t)$, $u(l,t) = u_2(t)$; $w(t,x) = \frac{x}{l}[u_2(t) - u_1(t)] + u_1(t)$.
2. $u(0,t) = u_1(t)$, $u_x(l,t) = u_2(t)$; $w(x,t) = u_2(t)x + u_1(t)$.
3. $u_x(0,t) = u_1(t)$, $u(l,t) = u_2(t)$; $w(x,t) = u_1(t)x + u_2(t) - lu_1(t)$.
4. $u_x(0,t) = u_1(t)$, $u_x(l,t) = u_2(t)$; $w(x,t) = \frac{u_2(t)-u_1(t)}{2l}x^2 + u_1(t)x$.

The above four auxiliary function cases are also applicable to the heat conduction equation.

Finding the Auxiliary Function:

- For the case (left end first kind, right end second kind), we want the auxiliary function to be a linear function.
- The left gives a point $(0, u_1(t))$ and the right gives the **slope** $u_2(t)$ (the derivative $u_x(l,t) = u_2(t)$)
- The linear function can be directly written as $u_1 + u_2 x$ according to this geometric translations.
- If $u(0,t) = u_1(t)$, $u_x(l,t) = u_2(t)$, then

$$w(x,t) = \underbrace{u_2(t)}_{\text{slope}}\, x + \underbrace{u_1(t)}_{\text{starting pt.}}$$

- If $u_x(0,t) = u_1(t)$, $u(l,t) = u_2(t)$, then

$$w(x,t) = \underbrace{u_1(t)}_{\text{slope}}\, x + \underbrace{u_2(t) - lu_1(t)}_{\text{starting pt., starting } u_2 \text{,using } u_1 \text{ slope to go back } l \text{ units}}$$

- If $u_x(0,t) = u_1(t)$, $u_x(l,t) = u_2(t)$, then it is **impossible** to find a **linear auxiliary** function if $u_1 \neq u_2$ because the **slopes** at the two endpoints are **different**.
- In this case, the simplest function is the parabolic function, but how to find it?
- It is not easy to find w, but it is **easy to find** w_x **to be linear** analog to the first case,
  - Since given the values of u_x at both endpoints, we start by finding the derivative of the auxiliary function, denoted as w_x instead of w.
  - Assume w_x is linear.

$$w_x(x,t) = \underbrace{\frac{u_2(t) - u_1(t)}{l}}_{\text{slope}} x + \underbrace{u_1(t)}_{\text{starting pt.}}.$$

Then **integrating** it yields the simplest auxiliary function

$$w(x,t) = \frac{u_2(t) - u_1(t)}{2l} x^2 + u_1(t)x.$$

- This can also be calculated by the method of coefficients to be determined.

Ex 2.2. *Solve the following problem*

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l,\ t > 0), \\ u(0,t) = t, \quad u(l,t) = 0, \\ u(x,0) = 0. \end{cases} \tag{25}$$

Solution. *Choose the* ***auxiliary function*** $w(x,t) = -\frac{t}{l}x + t$. *Let*

$$u(x,t) = v(x,t) - \frac{t}{l}x + t,$$

then problem (25) *is transformed into*

$$\begin{cases} v_t = a^2 v_{xx} + \frac{x}{l} - 1 & (0 < x < l,\ t > 0), \\ v(0,t) = 0, \quad v(l,t) = 0, \\ v(x,0) = 0. \end{cases} \tag{26}$$

To solve problem (26) *using the method of eigenfunctions, we set*

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{l} x, \tag{27}$$

Using the formula derived in Section 2.4.2 (formula (64)), we know

$$v_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)}\, d\tau,$$

Using the formula derived in Section 2.4.2 (formula 62), we know

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l} \int_0^l \left(\frac{x}{l} - 1\right) \sin \frac{n\pi x}{l}\, dx = -\frac{2}{n\pi}.$$

Substitute $f_n(t) = -\frac{2}{n\pi}$ *into* $v_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)}\, d\tau$,
we get

$$v_n(t) = -\frac{2}{n\pi} \int_0^t e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)}\, d\tau = \frac{2l^2}{(n\pi)^3 a^2} \left[e^{-\left(\frac{n\pi a}{l}\right)^2 t} - 1 \right], \tag{28}$$

Substitute (28) *into* (27) $v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi}{l}x$, *we obtain*

$$v(x,t)=\sum_{n=1}^{\infty}\frac{2l^2}{(n\pi)^3a^2}\left[e^{-(\frac{n\pi a}{l})^2 t}-1\right]\sin\frac{n\pi x}{l}.$$

Therefore, the solution to the original problem (25) *is*

$$u(x,t)=t\left(1-\frac{x}{l}\right)+\sum_{n=1}^{\infty}\frac{2l^2}{(n\pi)^3a^2}\left[e^{-(\frac{n\pi a}{l})^2 t}-1\right]\sin\frac{n\pi x}{l}.$$

Simultaneous homogenization

It is particularly important to note that for the given boundary value problem, for example:

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x,t), & (0<x<l,\ t>0),\\ u(0,t)=u_1(t), & u(l,t)=u_2(t),\\ u(x,0)=\varphi(x), & u_t(x,0)=\psi(x). \end{cases}$$

$$\downarrow \text{(If } f, u_1, u_2 \text{ indep. of } t\text{)}$$

$$\begin{cases} u_{tt}=a^2u_{xx}+f(x), & (0<x<l,\ t>0),\\ u(0,t)=u_1, & u(l,t)=u_2,\\ u(x,0)=\varphi(x), & u_t(x,0)=\psi(x). \end{cases} \qquad \to \text{Take } w(x) \text{ indep. of } t$$

If the **free term** f in the equation and the **boundary conditions** u_1, u_2 are **independent of the independent variable** t, in this case, we can choose an **auxiliary function** $w(x)$ (note it is independent of t), and then take the function substitution $u(x,t)=v(x,t)+w(x)$, to make the equation and the boundary conditions **homogeneous at the same time** (then the separation of variables applies).

Ex 2.3. *Solve the following problem*

$$\begin{cases} u_{tt}=a^2u_{xx}+\sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x & (0<x<l,\ t>0),\\ u(0,t)=3, \quad u(l,t)=6, & \\ u(x,0)=3\left(1+\frac{x}{l}\right), \quad u_t(x,0)=\sin\frac{4\pi}{l}x. & \end{cases} \tag{29}$$

Solution. *Assume the solution to the problem is*

$$u(x,t)=v(x,t)+w(x). \tag{30}$$

Substitute (30) *into the equation of problem* (31), *we get*

$$v_{tt}=a^2(v_{xx}+w''(x))+\sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x.$$

To make this equation homogeneous, naturally choose $w(x)$ *to satisfy*

$$a^2w''+\sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x=0.$$

Substituting (30) *into the boundary conditions of problem* (31), *we obtain*

$$\begin{cases} v(0,t)+w(0)=3, & v(l,t)+w(l)=6,\\ v(x,0)+w(x)=3\left(1+\frac{x}{l}\right), & v_t(x,0)=\sin\frac{4\pi}{l}x. \end{cases}$$

To make the boundary conditions of $v(x,t)$ *homogeneous as well,* $w(x)$ *must satisfy*

$$w(0,t)=3, \quad w(l,t)=6.$$

By making the substitution $u(x,t)=v(x,t)+w(x)$, *problem* (31) *is transformed into the following two problems:*

For $w(x)$:

$$\begin{cases} a^2 w'' + \sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x = 0, \\ w(0,t) = 3, \quad w(l,t) = 6. \end{cases} \tag{31}$$

For $v(x,t)$:

$$\begin{cases} v_{tt} = a^2 v_{xx} & (0 < x < l,\, t > 0), \\ v(0,t) = 0, \quad v(l,t) = 0, \\ v(x,0) = 3\left(1+\frac{x}{l}\right) - w(x), \quad v_t(x,0) = \sin\frac{4\pi}{l}x. \end{cases} \tag{32}$$

***Solving the ODE* (31):**

$$\frac{dw'}{dx} = -\frac{1}{a^2}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l}$$

By integration, we obtain

$$\begin{aligned} \Rightarrow w'(x) &= -\frac{1}{a^2}\int \sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l}\,dx + C_1 \\ &= -\frac{1}{2a^2}\int \sin\frac{4\pi x}{l}\,dx + C_1 \\ &= -\frac{l}{8\pi a^2}\cos\frac{4\pi x}{l} + C_1 \end{aligned}$$

Integrating it yields

$$\begin{aligned} \Rightarrow w(x) &= -\frac{l}{8\pi a^2}\int \cos\frac{4\pi x}{l}\,dx + C_1 x + C_2 \\ &= -\frac{l}{8\pi a^2}\cdot\frac{l}{4\pi}\sin\frac{4\pi x}{l} + C_1 x + C_2 \\ &= -\frac{l^2}{32\pi^2 a^2}\sin\frac{4\pi x}{l} + C_1 x + C_2 \end{aligned}$$

The boundary condition leads to $C_1, C_2 \Rightarrow C_2 = 3$.

$$w(l) = -\frac{l^2}{32\pi^2 a^2}\sin\frac{4\pi l}{l} + C_1 l + 3 = 6 \quad\Rightarrow\quad C_1 = \frac{3}{l}$$

Then,

$$\Rightarrow w(x) = -\frac{l^2}{32\pi^2 a^2}\sin\frac{4\pi x}{l} + \frac{3x}{l} + 3$$

Problem (31) is a boundary value problem for an ordinary differential equation, and its solution is:

$$w(x) = \frac{l^2}{32\pi^2 a^2}\sin\frac{4\pi}{l}x + 3\left(1+\frac{x}{l}\right).$$

Substitute the obtained $w(x)$ into problem (32):

$$\begin{cases} v_{tt} = a^2 v_{xx} & (0 < x < l,\, t > 0), \\ v(0,t) = 0, \quad v(l,t) = 0, \\ v(x,0) = -\frac{l^2}{32\pi^2 a^2}\sin\frac{4\pi}{l}x, \quad v_t(x,0) = \sin\frac{4\pi}{l}x. \end{cases}$$

Using the formula:

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi a t}{l} + b_n\sin\frac{n\pi a t}{l}\right)\sin\frac{n\pi x}{l},$$

where the coefficients a_n and b_n satisfy:

$$\begin{cases} a_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx, \\ b_n = \frac{2}{n\pi a}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx. \end{cases}$$

Thus,

$$v(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l},$$

where the coefficients a_n and b_n are calculated as follows:

$$\begin{cases} a_n=\frac{2}{l}\int_0^l -\frac{l^2}{32\pi^2a^2}\sin\frac{4\pi}{l}x\sin\frac{n\pi}{l}x\,dx=\begin{cases}0, & n\neq 4,\\ -\frac{l^2}{32\pi^2a^2}, & n=4.\end{cases}\\ b_n=\frac{2}{n\pi a}\int_0^l\sin\frac{4\pi x}{l}\sin\frac{n\pi x}{l}\,dx=\begin{cases}0, & n\neq 4,\\ \frac{l}{4\pi a}, & n=4.\end{cases}\end{cases}$$

Thus, the solution to problem (32) *is:*

$$v(x,t)=\left(-\frac{l^2}{32\pi^2a^2}\cos\frac{4\pi at}{l}+\frac{l}{4\pi a}\sin\frac{4\pi at}{l}\right)\sin\frac{4\pi}{l}x.$$

Therefore, the solution to the original problem (31) *is:*

$$u(x,t)=\left(-\frac{l^2}{32\pi^2a^2}\cos\frac{4\pi at}{l}+\frac{l}{4\pi a}\sin\frac{4\pi at}{l}\right)\sin\frac{4\pi}{l}x+\frac{l^2}{32\pi^2a^2}\sin\frac{4\pi}{l}x+3\left(1+\frac{x}{l}\right).$$

Solution (Alternative solution, a bad method if you use the linear auxiliary function for the simultaneous homogenization). *Choose the auxiliary function $w(x,t)=3\left(1+\frac{x}{l}\right)$, then*

$$u(x,t)=v(x,t)+3\left(1+\frac{x}{l}\right)$$

Substitute into problem (31) *to obtain*

$$\begin{cases} v_{tt}=a^2v_{xx}+\sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x,\\ v(0,t)=v(l,t)=0,\\ v(x,0)=0,\quad v_t(x,0)=\sin\frac{4\pi}{l}x.\end{cases}\quad\to\textit{(isolate non-homog. by linear superpositions)}\tag{33}$$

From the analysis in Section 2.4.1, we can set $v(x,t)=\bar{v}(x,t)+\bar{w}(x,t)$ and $\bar{v}(x,t)$ and $\bar{w}(x,t)$ satisfy the following boundary value problems respectively:

$$\begin{cases} \bar{v}_{tt}=a^2\bar{v}_{xx}+\sin\frac{2\pi}{l}x\cos\frac{2\pi}{l}x,\\ \bar{v}(0,t)=\bar{v}(l,t)=0,\\ \bar{v}(x,0)=0,\quad \bar{v}_t(x,0)=0.\end{cases}\quad\to\textit{(eigenfunction)}\tag{34}$$

and

$$\begin{cases} \bar{w}_{tt}=a^2\bar{w}_{xx},\\ \bar{w}(0,t)=\bar{w}(l,t)=0,\\ \bar{w}(x,0)=0,\quad \bar{w}_t(x,0)=\sin\frac{4\pi}{l}x.\end{cases}\quad\to\textit{(sep. of variables)}\tag{35}$$

Using formulas (14) and (15) from Section 2.1, we can calculate

$$\bar{w}(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

where the coefficients a_n and b_n are

$$\begin{cases} a_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx=0,\quad n\neq 4,\\ b_n=\frac{2}{n\pi a}\int_0^l\sin\frac{4\pi x}{l}\sin\frac{n\pi x}{l}\,dx=\begin{cases}0, & n\neq 4,\\ \frac{l}{4\pi a}, & n=4.\end{cases}\end{cases}$$

Then the solution to problem (35) *is*

$$\bar{w}(x,t)=\frac{l}{4\pi a}\sin\frac{4\pi a}{l}t\sin\frac{4\pi}{l}x.$$

Next, using the method of eigenfunctions to solve problem (34). Thus, let

$$\bar{v}(x,t) = \sum_{n=1}^{\infty} \bar{v}_n(t) \sin \frac{n\pi}{l} x$$

Using formula (53) derived in Section 2.4.1, we know

$$\bar{v}_n(t) = \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin \frac{n\pi a}{l}(t-\tau)\, d\tau,$$

Using formula (51) derived in Section 2.4.1, we know

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi x}{l}\, dx = \frac{1}{l} \int_0^l \sin \frac{4\pi x}{l} \sin \frac{n\pi x}{l}\, dx = \begin{cases} 0, & n \neq 4, \\ \frac{1}{2}, & n = 4. \end{cases}$$

1. *When $n \neq 4$, $\bar{v}_n(t) = 0$;*
2. *When $n = 4$, we obtain*

$$\bar{v}_n(t) = \frac{l}{8\pi a} \int_0^t \sin \frac{4\pi a}{l}(t-\tau)\, d\tau = \frac{l^2}{32\pi^2 a^2} \left(1 - \cos \frac{4\pi a}{l} t\right)$$

The solution to problem (34) is

$$\bar{v}(x,t) = \frac{l^2}{32\pi^2 a^2} \left(1 - \cos \frac{4\pi a}{l} t\right) \sin \frac{4\pi}{l} x.$$

Combine the solution to problem (35) $\bar{w}(x,t) = \frac{l}{4\pi a} \sin \frac{4\pi a}{l} t \sin \frac{4\pi}{l} x$ with the auxiliary function $w(x,t) = 3\left(1 + \frac{x}{l}\right)$ and the solution to problem (34) to obtain the solution to the original problem (31):

$$u(x,t) = v(x,t) + 3\left(1 + \frac{x}{l}\right) = \bar{v}(x,t) + \bar{w}(x,t) + 3\left(1 + \frac{x}{l}\right)$$

3 Eigenvalues and Eigenfunctions (Introduction)

In the first three sections of this chapter, when we applied the method of separation of variables to solve the boundary value problems related to the vibration equation, one-dimensional heat conduction equation, and two-dimensional Laplace equation, we needed to solve a boundary value problem of an ordinary differential equation containing the parameter λ:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0. \quad \leftarrow \quad \boxed{\text{The core of series solutions!}} \tag{36}$$

This type of problem is called an eigenvalue problem. It also belongs to the Sturm-Liouville problem.

This equation (36) is not enough for Chapter 5, we need to generalize it!

The general form of the Sturm-Liouville equation is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda\rho(x)y = 0 \rightarrow \boxed{\text{If } p = \rho = 1 \text{ and } q = 0, \text{ (37) becomes (36)}} \tag{37}$$

where

1. $p(x), p'(x) \in C[a,b], p(x) > 0$ for $a < x < b$;
2. $q(x) \in C[a,b]$, or $q(x) \in C(a,b)$, and **at most one endpoint has a first-order pole** (Bessel equation is in this form), and $q(x) \geq 0$;
3. $\rho(x) \in C[a,b], \rho(x) > 0$.

Equation (37) with boundary conditions is called the **Sturm-Liouville problem**. Those λ values that make the Sturm-Liouville problem have **non-zero solutions** are called the **eigenvalues** of the problem, and the corresponding **non-zero solutions** are called **eigenfunctions**.

Some conclusions about eigenvalues and eigenfunctions:

1. There are **(coutable) infinitely many** real eigenvalues:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \quad \leftarrow \quad \boxed{\text{countable infinite!}}$$

When $q(x) \ge 0$, $\lambda_n \ge 0$ $(n = 1, 2, 3, \cdots)$; corresponding to these eigenvalues, there are infinitely many eigenfunctions:

$$y_1(x), y_2(x), \cdots, y_n(x), \cdots.$$

> Generalize the case, for example, $\lambda_n = (\frac{n\pi}{l})^2, y_n(x) = X_n(x) = \sin\frac{n\pi x}{l}$, $n = 1, 2, \cdots$ for $X'' + \lambda X = 0$.

2. If the eigenfunction corresponding to the eigenvalue λ_n is denoted as $y_n(x)$, then all $y_n(x)$ form an **orthogonal** function system with **weight** function $\rho(x)$, that is

$$\langle y_m(x), y_n(x)\rangle_\rho := \int_a^b y_m(x)y_n(x)\underbrace{\rho(x)dx}_{\text{Measure}} = 0 \quad (m \ne n). \leftarrow \boxed{\text{Generalize the othogonality of } \sin, \cos}$$

> For $X'' + \lambda X = 0$, $\rho \equiv 1$, thus $\rho(x)dx = dx$. For general S-L problem, we can not ensure y_n are sin or cos, but we can ensure their **othogonality**.

3. (**Completeness**) Similar to Fourier series, the **expansion** in terms of **eigenfunctions** has the following convergence properties:

If the function $f(x)$ has continuous first-order derivatives and piecewise continuous second-order derivatives in (a, b), and satisfies the given boundary conditions, then $f(x)$ can be expanded in terms of eigenfunctions as an absolutely and uniformly convergent series in (a, b):

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad \leftarrow \quad \boxed{\text{Generalization of Fourier series}} \tag{38}$$

> Replace $\{1, \sin, \cos, \cdot\}$ to the eigenfunction system, and "any" good function can be **expanded by the eigenfunction system**. $\{y_n\}$ is complete.

where

$$c_n = \frac{\int_a^b \rho(x)f(x)y_n(x)dx}{\int_a^b \rho(x)y_n^2(x)dx} \quad (n = 1, 2, 3, \cdots); \leftarrow \boxed{\text{Calculated by inner product due to orthogonality}}$$

> It is easy to verify
>
> $$\langle y_m(x), f(x)\rangle_\rho = \sum_{n=1}^{\infty} c_n\langle y_m(x), y_n(x)\rangle_\rho = c_m\langle y_m(x), y_m(x)\rangle_\rho$$
>
> - In Chapter 5, when the trigonometric function system is replaced by the Bessel function system, there will also be a Bessel-Fourier expansion. This can be sought in the same way and also satisfies these properties.

If the function $f(x), f'(x)$ are **piecewise continuous** functions in (a, b), then the series (38) converges at the discontinuity point x_0 of $f(x)$ to

$$\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)],$$

and loses uniform convergence on (a, b).