Lecture Notes: Mathematical Physics Equations and Special Functions

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Week 3-2

- **Method Discussed:** Separation of Variables
  - General format and steps are similar.
  - Differences lie in
    * The types of equations (determine the T-functions);
    * Boundary conditions (determine the SL problem and eigenfunctions, i.e., X-function).
- **Types of Equations:** Three main types
  - Affect the differentiation of the T (or Y)-function.
  - T-function equations can be oscillatory (u_{tt}) or exponentially decaying (u_t).
- **Boundary Conditions:** Five types (including periodic boundary conditions)
  - Five SL problems are presented under these boundary conditions.
  - These problems need to be memorized.
- **Current Requirement:**
  - Ability to immediately identify eigenvalues and eigenfunctions upon seeing the problem.
  - According to the PDE to guess the T-function and further guess the possible solution before solving the problem.

- **Trial Method Overview**
  - Also known as the "guessing method."
  - Separation of variables is a type of trial method.
  - Trial method allows guessing a more accurate form of the solution with some undetermined parameters.
  - Parameters are determined by substituting the guessed form back into the original equation and boundary conditions.
- **Challenges and Strategies for Guessing**
  - The difficulty lies in how to make an appropriate guess.
  - Common strategies for guessing:
    * **Physical Phenomena:** Use physical intuition (e.g., vibration of a string) to guide the guess.
    * **Geometric Properties:** Use symmetry or geometric constraints (e.g., spherical symmetry implies functions depend only on R).

* **Non-homogeneous Terms:** Guess based on the **form of boundary conditions** or **non-homogeneous terms** in the equation.

- **Specific Example**
  - Given boundary conditions (e.g., $u(r_0,\theta) = A\sin 2\theta$), guess the simplest form of the solution $C_1 r^\alpha \sin 2\theta + C_2$. That is, $\sin 2\theta \to \sin 2\theta$ and $A \to C_1 r^\alpha$. Since When r is fixed, start guessing from the simplest functions–polynomial functions r^α.

1 Transform the Series Solution into an Integral Form

The series solution (1) can be transformed into an integral form:

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n. \tag{1}$$

The coefficients a_n, b_n are determined by the following formulas (2):

$$\begin{cases} a_n = \frac{1}{\pi r_0^n}\int_0^{2\pi} f(\varphi)\cos n\varphi d\varphi \quad (n = 0,1,2,\cdots), \\ b_n = \frac{1}{\pi r_0^n}\int_0^{2\pi} f(\varphi)\sin n\varphi d\varphi \quad (n = 1,2,\cdots), \end{cases} \tag{2}$$

- **Objective:**
  - Convert the **series solution** of Laplace's equation into an **integral form**.
  - Preparation for Chapter 4, where a different method (Green's function method) will be used. The solution in Chapter 4 will be in integral form without summation symbols.
  - To ensure the solution from this chapter matches the form of the solution in Chapter 4.
- **Current Solution Form:**
  - Contains both integrals (a_n and b_n) and series summation.
- **Goal of the Transformation:**
  - Remove the summation symbol, implying finding the function to which the series converges.

Substituting (2) into (1) gives:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)d\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n \int_0^{2\pi} f(\varphi)\underbrace{(\cos n\theta\cos n\varphi + \sin n\theta \sin n\varphi)}_{\cos n(\theta-\varphi)} d\varphi.$$

Simplify to obtain

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2} + \underbrace{\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n \cos n(\theta-\varphi)}_{(*)}\right] f(\varphi)d\varphi \quad (r < r_0).$$

where $(*)$ means this form inspires us to use Geometric Series if $\cos(n\cdot)$ becomes $e^{n\cdot}$ and this can be done by Euler's formula

Using Euler's formula:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \quad (|q| < 1)$$

Perform the following identity transformation: Let $k = \frac{r}{r_0}$

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} k^n \cos n(\theta - \varphi) =& \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} k^n \left[e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}\right] \\
=& \frac{1}{2}\left[1 + \frac{ke^{i(\theta-\varphi)}}{1-ke^{i(\theta-\varphi)}} + \frac{ke^{-i(\theta-\varphi)}}{1-ke^{-i(\theta-\varphi)}}\right] \\
=& \frac{1}{2} \cdot \frac{1-k^2}{1-ke^{i(\theta-\varphi)} - ke^{-i(\theta-\varphi)} + k^2}
\end{aligned}$$

Then we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} k^n \cos n(\theta - \varphi) = \frac{1}{2} \cdot \frac{1-k^2}{1+k^2-2k\cos(\theta-\varphi)} \quad (|k| < 1),$$

So the series solution (1) can be expressed in integral form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\frac{r_0^2 - r^2}{r_0^2 + r^2 - 2r_0 r\cos n(\theta-\varphi)}d\varphi \quad (r < r_0),$$

This formula is called the **Poisson formula in a circular domain**.

2.4 Non-homogeneous Equations and Their Solving Problems

This section examines the boundary value problems of **Non-homogeneous equations** and introduces a commonly used method: **the method of eigenfunction expansion**.

We will use three types of boundary value problems as examples to illustrate the key points and steps of this method.

1. Forced vibration problems with boundary conditions.
2. Heat conduction problems in finite rods (with heat sources).
3. Poisson's equation (Non-homogeneous Laplace's equation).

I. Forced Vibration Problem with Boundary Conditions

Firstly, we consider the following problem

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) \quad (0 < x < l,\ t > 0), \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{cases} \qquad \text{(Multiple non-homogeneities)} \tag{3}$$

At this time, the vibration of the string is caused by two parts of interference:

One is the **external forcing force**, and the other is the **initial state** of the string. From the physical point of view, this vibration can be seen as a combination of the vibration caused solely by the forcing force and the vibration caused solely by the initial state.

General Ideas:

- **Isolate** and separate the **non-homogeneity** in the problem to multiple systems such that one system keeps only one non-homogeneity.
  - Express u as the sum of v and w, i.e., $u = v + w$ (**the principle of linear superposition**).
  - Each function v and w will handle different aspects of the non-homogeneity. That is,

* v solves

$$\begin{cases} \textbf{Non-homogeneous} \text{ equation} \\ \text{Homogeneous boundary} \\ \text{Homogeneous inital data} \end{cases} \qquad \to \text{using the eigenfunction method}$$

* w solves

$$\begin{cases} \text{Homogeneous equation} \\ \text{Homogeneous boundary} \\ \textbf{Non-homogeneous} \text{ inital data} \end{cases} \qquad \to \text{using separation of variables}$$

Thus, we can set the solution of problem (3) as

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ represents the displacement of the string caused **solely by the forcing force**, and $w(x,t)$ represents the displacement caused **solely by the initial state**, such that $v(x,t)$ and $w(x,t)$ satisfy the following boundary value problems respectively:

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x,t) \quad (0 < x < l,\ t > 0), \\ v(0,t) = 0, \quad v(l,t) = 0, \\ v(x,0) = v_t(x,0) = 0. \end{cases} \tag{4}$$

and

$$\begin{cases} w_{tt} = a^2 w_{xx} \quad (0 < x < l, t > 0), \\ w(0,t) = 0, \quad w(l,t) = 0, \\ w(x,0) = \varphi(x), \quad w_t(x,0) = \psi(x). \end{cases} \tag{5}$$

- Use the principle of linear superposition.

To solve the forced vibration problem with homogeneous boundary conditions and zero initial conditions:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) \quad (0 < x < l,\ t > 0), \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = u_t(x,0) = 0. \end{cases} \tag{6}$$

The above problem can be solved using a parameter variation method similar to that used for linear Non-homogeneous ordinary differential equations, and maintain the idea that:

The solution to this boundary value problem can be decomposed into an infinite number of standing waves, and the shape of each standing wave is still determined by the eigenfunctions of the vibrating body.

- **Non-homogeneous equations + homogeneous boundary + homogeneous initial conditions** → The eigenfunction method.
- Why these conditions? Consider this question in the following derivations.

How to invent the method of eigenfunctions? (A guessing method on every step)

- "A fair guessing" means one gives **a priori assumption**.

- We currently only have the method of separation of variables to solve PDEs.
- The method of separation of variables was discovered early through physical practice.
- When solving **non-homogeneous** equations, we still start with the separation of variables method to see if it can be solved.
- We need to understand **why a homogeneous equation** is required in the separation of variables method.
- When working on this problem, first **compare the differences**. By understanding these differences, we can identify where the difficulties will arise, and then we can figure out how to overcome them.
- In the five steps of the separation of variables method, the **second step** involves substituting back into the original equation.
- If the original equation contains a source function $f(t,x)$ that **cannot** be separated into a form like $T(t)X(x)$, the separation of variables method **fails**. **Non-homogeneous** terms **prevent** direct application of separation of variables.
- The function f is **generally not** separable into X and T components.
- When encountering this problem, we can consider using the **idea of superposition of solutions**.
- In the **fourth step** of the separation of variables method, we already use **superposition to satisfy initial conditions**.
- Initially, we assume that the solution can be separated into variables. However, in fact the final superposed solution cannot be separated by variables.
- Since the superposed solution already cannot be separated even in the method of separation of variables, we give a priori assumption (a weaker assumption) that directly superposing the solutions, i.e., assume the solution has the form that $u = \sum XT$ instead of $u = XT$.
- This approach allows us to handle cases where the original method fails due to non-separable terms.
- We can try to directly **superpose the solutions** even if they **cannot** be separated into variables. This is a weaker a priori assumption than the separation of variables since a lot of function can be expanded to a series based on some basis functions system.
- If the function f can be expressed in a separable form through summation as well (i.e., $f = \sum f_n(t)g_n(x)$, in fact, you will find that it is benefit to choose $g_n(x) = X_n(x)$ later), we can apply this idea to solve the problem.

Finding the Necessary Conditions: Introduce **a priori assumption**

$$u = \sum u_n(t)X_n(x) \to \text{to be determined } u_n \text{ and } X_n.$$

- **Goal:** We expect to **choose the appropriate functions $X_n(x)$ and coefficients $u_n(t)$, such that the equation and boundary, initial data are satisfied.**
- **Question:** How to choose or construct $X_n(x)$ and $u_n(t)$? $\leftarrow$ This is what we want to construct.

Assume (fair assumption) the expansion of f is denoted as (let $g_n(x) = X_n(x)$ already)

$$f = \sum f_n(t)X_n(x)$$

Let us see what the equation and data become:

- **The equation:**
$$\sum u_n'' X_n = \underbrace{\sum a^2 u_n X_n''}_{(+)} + \sum f_n X_n \tag{7}$$

- **The boundary:**
$$\begin{cases} u(0,t) = \sum u_n(t) X_n(0) = 0, \\ u(l,t) = \sum u_n(t) X_n(l) = 0. \end{cases}$$

- **The initial data:**
$$\begin{cases} u(x,0) = \sum u_n(0) X_n(x) = 0, \\ u_t(x,0) = \sum u_n'(0) X_n(x) = 0. \end{cases} \tag{8}$$

- By manipulating (7)–(8), try to find a way to construct X_n and u_n.

1. **(Using the equation)** By (7), we obtain
$$\sum u_n'' X_n = \underbrace{\sum a^2 u_n X_n''}_{(+)} + \sum f_n X_n \implies \sum (u_n'' - f_n) X_n - \underbrace{\sum a^2 u_n X_n''}_{(+)} = 0 \tag{9}$$
where $(+)$-term involving X_n'' cannot be combined with the other two terms, which involve X_n .

Remark 1.1. • *If each term in the summation (9) can be separated into a product of functions of X and T, we may be able to conclude that each term must be zero. That is,*
$$0 = \sum (u_n'' - f_n) X_n - \underbrace{\sum a^2 u_n X_n''}_{(+)} \overset{HOPE}{=} \sum T_n(t) X_n(x) \leftarrow \textit{Goal}$$

- *If successful, this* ***may*** *imply that $T_n(t) = 0$. In fact, this requires $\{X_n\}$ is a complete basis function system. However, in order to find an idea, we just roughly believe this is true currently and later reconsider if this can be satisfied (viewed as* ***a priori assumption****).*

In order to combine it with the other terms, we hope (to find the **simplest and familiar** way although there might be infinite ways and freedom).
$$\boxed{X_n'' = -\lambda X_n} \rightarrow \text{the key invent and a new a priori assumption} \tag{10}$$

- Since X_n is to be determined, we want to choose X_n to satisfy this ODE in order to determine X_n. However, this ODE can not determine X_n completely. We need boundary conditions. On the other hand, we observe this ODE is like the one in S-L problem, so we **hope** to construct **homogeneous boundary conditions**.

Remark 1.2. • *There are* ***multiple ways*** *to achieve (9), we only pick a simplest and efficient one, if it works against all odds, we finish. Otherwise, if it fails in the later steps, we choose another possible way to get through this step.*

- ***Motivation for the Choice:*** *The choice of a solution path is like navigating a maze with multiple possible routes. We choose one that works, but others might also be viable.*
- ***Simplicity and Familiarity:*** *We choose a path proportional to X_n because it is the simplest and most familiar. This choice implies $X_n'' = \text{constant} \times X_n$.*
- ***Requirement for Consistency:*** *To satisfy this choice, we require $X_n'' + \lambda X_n = 0$. This equation is* ***familiar*** *and aligns with our previous work:* ***SL problem****.*
- ***Extracting X_n:*** *By choosing this path, we can factor out X_n in the equation, simplifying the problem.*
- ***Hope and Consistency:*** *This choice is driven by our* ***hope*** *that it aligns with a* ***familiar form*** *and can be consistent with our previous results.*

2. **(Using the boundary)** We have been only using the equation so far, and now we need to incorporate the **boundary conditions**. Substituting the apriori assumption $u = \sum u_n(t)X_n(x)$ to the homogeneous boundary:

$$\begin{cases} u(0,t) = \sum u_n(t)X_n(0) = 0 \Leftarrow X_n(0) = 0 \\ u(l,t) = \sum u_n(t)X_n(l) = 0 \Leftarrow X_n(l) = 0 \end{cases} \quad \text{(a simplest condition and a priori assumption)}$$

   - This means if we choose $X_n(0) = 0$ and $X_n(l) = 0$, it can ensure the boundary condition satisfies the required boundary in (6). This gives a specific way on X_n to realize the boundary condition.
   - By choosing these boundaries and equation of X_n, we have used some freedom of this system and this leaves no other room for u_n but only one choice.

   Then we determine X_n by the S-L problem

$$\begin{cases} X_n'' + \lambda X_n = 0 \\ X_n(0) = 0, \quad X_n(l) = 0 \end{cases} \implies \boxed{\text{S-L problem leads to } X_n \text{ and } \lambda_n} \implies \text{determined } X_n(x) \tag{11}$$

   - This means we give a priori assumption on the form of u. $u = \sum u_n X_n$ where X_n satisfy this S-L problem (it is S-L problem for the corresponding homogeneous equation ans homogeneous boundary problem).

   Remark 1.3. • *For non-trivial solutions, the simplest condition is $X_n(0) = 0$, which can be viewed as a priori assumption.*

   - *We solve the equation for X_n with these boundary conditions, which leads to the eigenfunctions* $\sin\left(\frac{n\pi x}{l}\right)$.
   - *We design and identify a useful tool–S-L problem–to achieve the solution.*
   - *The X_n-ODE and its boundary conditions are simplest conditions for the system to hold, representing one possibility among many.*
   - *This approach is like choosing one path from multiple possible routes to reach the destination.*
   - *If the chosen path works, the task is completed; if not, the idea is deemed unfeasible.*
   - *We replace X_n with* $\sin\left(\frac{n\pi x}{l}\right)$ *and consider the **second hope**: finding u_n.*
   - *The coefficients f_n are found by taking the inner product of f with* $\sin\left(\frac{n\pi x}{l}\right)$ *and integrating.*

   If $X_n = \sin\frac{n\pi x}{l}$ solves the S-L problem, then (9) becomes:

$$\sum u_n'' X_n = -\sum a^2 \lambda u_n X_n + \sum f_n X_n \implies \sum (u_n'' - f_n + a^2\lambda u_n) X_n = 0$$

   - **Determining Coefficients:** Can we conclude that each coefficient is zero from the equation being zero? Yes, by using (1) a **Fourier series expansion** by solving the coefficient; (2) $\{X_n\}$ is a complete and linear independent (orthogonal in fact) basis. $\rightarrow$ we call this "compare the coefficient".

3. **(Using the initial data)** Comparing coefficients, we get:

$$u_n'' + a^2\lambda u_n - f_n = 0$$

   Using the initial data:

$$\begin{cases} u(x,0) = \sum u_n(0)X_n(x) = 0 \implies u_n(0) = 0 \\ u_t(x,0) = \sum u_n'(0)X_n(x) = 0 \implies u_n'(0) = 0 \end{cases}$$

$$\implies \begin{cases} u_n'' + a^2\lambda u_n - f_n = 0 \\ u_n(0) = 0, \quad u_n'(0) = 0 \end{cases} \implies \text{determined } u_n(t) \tag{12}$$

Considering (11) and (12), we arrive at

$$u(t,x) = \sum u_n(t)X_n(x) \rightarrow \text{determined}$$

- **Eigenfunction Method:** This process is essentially the eigenfunction method, which has been formalized by mathematicians to provide a clear and systematic approach. We list below.

Summary of the Method of Eigenfunction Expansion

1. Consider the eigenfunction system corresponding to the homogeneous problem (i.e., in (6), we set $f = 0$ and solve this homogeneous problem first, and we only need). **[Homogeneous Eigenfunction System]**
   - **Approach:** Solve the homogeneous problem by setting $f = 0$ in (6) and using the separation of variables method upto solving the S-L problem (referred to as the "2.5 step") to obtain the eigenfunctions X_n.
   - **Significance:** Once the eigenfunctions X_n are obtained, they form a **complete set** of eigenfunctions for the homogeneous problem.
2. Assume the solution of the non-homogeneous problem can be expanded using the eigenfunction series. **[Assumed Series Solution]**
3. Expand the free term using the eigenfunction series. **[Expansion of Free Term]**
   - Expand both u and f using the eigenfunctions X_n. This is analogous to expressing vectors in a linear space using a chosen basis, any function related to the equation can be expanded using the eigenfunctions.
4. Substitute the particular solution and the series form of the free term into the non-homogeneous equation, compare coefficients to obtain ODEs. **[Comparison of Coefficients]**
5. Use initial conditions to obtain the initial conditions for the ODEs. **[Initial Value (Boundary Value) Transformation]**
6. Solve the ODEs to obtain the coefficient functions. **[Solving ODEs]**

Basis decomposition method in algebra and PDEs

- In linear algebra and analytic geometry, to describe a linear space, it is essential to first establish a basis or a coordinate system. Only then can we effectively formulate and solve algebraic problems within this framework. Similarly, in our approach here, we first select a basis function system. Subsequently, we formulate all partial differential equation (PDE) problems within this basis or in terms of the corresponding eigenfunctions.

For this, we first discuss the **forced vibration problem** with **homogeneous boundary conditions** and **zero initial conditions**:

$$u_{tt} = a^2 u_{xx} + f(x,t) \quad (0 < x < l, \ t > 0), \tag{13}$$
$$u(0,t) = 0, \quad u(l,t) = 0, \tag{14}$$
$$u(x,0) = u_t(x,0) = 0. \tag{15}$$

Solution. ***[1. Homogeneous Eigenfunction System]*** *From the knowledge in Section* **??**, *the corresponding homogeneous equation to* (13) *is:*

$$u_{tt} = a^2 u_{xx},$$

which satisfies the homogeneous boundary conditions (14) *with the eigenfunctions satisfying:*

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

The eigenfunctions are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \quad (n = 1, 2, \ldots).$$

Thus, the eigenfunction series that satisfies the homogeneous boundary conditions (14) *for the corresponding homogeneous equation to* (13) *is* $\left\{\sin\left(\frac{n\pi x}{l}\right)\right\}$*.*

[2. Assumed Series Solution] *Assume the solution is*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right), \tag{16}$$

where $u_n(t)$ *is a function to be determined with respect to* t*.*

- ***Analogy to Linear Algebra:*** *Similar to linear algebra, where any vector can be decomposed using an orthogonal basis, functions in the equation can be decomposed using the eigenfunctions.*
- ***Key Idea:*** *The eigenfunction system provides a basis for decomposing all relevant functions in the problem.*

[3. Expansion of Free Term] *Expand the free term* $f(x,t)$ *in the equation into a Fourier series:*

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{l}\right), \tag{17}$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) dx \quad (n = 1, 2, \ldots).$$

[4. Comparison of Coefficients] *Substituting* (16) *and* (17) *into equation* (13) *gives:*

$$\sum_{n=1}^{\infty} \left[u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) - f_n(t)\right] \sin\left(\frac{n\pi x}{l}\right) = 0, \quad \textit{(wave equations in series version)}$$

from which we obtain[1]*:*

$$u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \quad (n = 1, 2, \ldots).$$

- ***Coefficient Comparison:*** *By "comparing coefficients", we can determine that the coefficients are zero. When mentioning "comparing coefficients", we mean*
  - ***Fourier Expansion Interpretation:*** *This process is analogous to a* ***Fourier expansion of zero****, where the coefficients are determined using inner products.*
  - *The definition of Linear Independence and the completeness of* $\{X_n\}$ *lead to that the coefficients are zero.*
- ***Orthogonality and Linear Independence:***
  - *Orthogonality implies linear independence.*
  - *For finite sums, linear independence can be used to conclude that coefficients are zero.*
  - *For infinite sums, roughly* ***completeness*** *is required to ensure that any functions in a function space can be represented by the basis function system.* ***Analogy to Linear Algebra:***
    - *In linear algebra, a complete set of linearly independent vectors can be used to represent any vector in the space.*
    - *If the set is incomplete (e.g., missing one vector), some vectors cannot be represented.*

[1] Completeness: refer to https://en.wikipedia.org/wiki/Orthonormal_basis#Incomplete_orthogonal_sets

[5. Initial Value Transformation] *Using the initial conditions (15) in expression (16) gives:*

$$u_n(0) = 0, \quad u_n'(0) = 0. \quad \textit{(Coefficient comparison–}\{X_n\}\textit{ orthogonality)}$$

- *Recall that the exchange between differentiation and summation can be treated as an a priori assumption.*

[6. Solving ODEs] *Thus, we obtain the following initial value problem for the ordinary differential equation:*

$$\begin{cases} u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \\ u_n(0) = u_n'(0) = 0, \quad (n = 1, 2, \ldots). \end{cases} \tag{52}$$

By applying the method of variation of parameters or Laplace transform in ordinary differential equations, the solution to problem (52) is

$$u_n(t) = \frac{l}{n\pi a} \int_0^t f_n(\tau) \sin\left(\frac{n\pi a}{l}(t-\tau)\right) d\tau \quad (n = 1, 2, \ldots).$$

Substituting u_n into

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right),$$

We obtain the solution to the mixed value problem (13)-(15).

Summary of Solving Non-Homogeneous ODEs

- **Solution Form:** The solution to the second order non-homogeneous ODE (52) involves a convolution of the **forcing term** f_n with a sine function, scaled by a coefficient.
- **Physical Interpretation:** This ODE describes vibrations, similar to the wave equation for a vibrating string. The **sine function** in the solution represents the **oscillatory nature** of the system.
- **Convolution with Sine Function:** The non-homogeneous term f_n (representing an external force) is convolved with a sine function. This convolution represents the response of the system to the external force.

Solution Using Laplace Transform

$$\begin{cases} u'(t) + k^2 u(t) = f(t), \\ u(0) = 0. \end{cases} \tag{18}$$

Solution. *Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get*

$$L[u'(t)] = sL[u(t)] - u(0)$$

$$sU(s) - u(0) + k^2 U(s) = F(s)$$

$$sU(s) + k^2 U(s) = F(s)$$

Thus

$$U(s) = \frac{1}{s+k^2} F(s).$$

$$L[e^{-at}] = \frac{1}{s+a}$$

Taking the inverse Laplace transform gives

$$u(t) = f(t) * e^{-k^2 t} = \int_0^t f(\tau) e^{-k^2(t-\tau)} d\tau.$$

Remark 1.4.

- *Advantages of Integral Transforms:*
  - *Derivatives and multipliers are interchanged.*
  - *Convolutions and products are interchanged.*
- *Integral transformations convert ordinary differential equations (ODEs) into algebraic equations.*
- *For partial differential equations (PDEs), they are similar and we will discuss this in Chap.* 3.

- **Analogy to Heat Equation:** For the ODE (18), it is similar to the heat equation, the solution involves an **exponential decay** term. The non-homogeneous term f would be convolved with an **exponential function** to describe the heat distribution.
- **General Form:** The solution to non-homogeneous equations can generally be expressed as the convolution of the forcing term f with a characteristic function (e.g., sine for vibrations, exponential for heat).

Another Method to Solve

$$\begin{cases} u'(t) + k^2 u(t) = f(t), \\ u(0) = 0. \end{cases}$$

Solution. *Directly use the general solution formula of the first-order linear differential equation to get*

$$u(t) = e^{-k^2 t} \left(\int_0^t e^{k^2 \tau} f(\tau) d\tau + C \right)$$

Using the condition $u(0) = 0$ *gives* $C = 0$ *. So the solution to the original problem can be expressed as*

$$u(t) = \int_0^t f(\tau) e^{-k^2(t-\tau)} d\tau.$$

Solution Using Laplace Transform

$$\begin{cases} u''(t) + k^2 u(t) = f(t), \\ u(0) = 0,\ u'(0) = 0. \end{cases} \tag{19}$$

Solution: Let $U(s) = L[u]$, $F(s) = L[f]$, take the Laplace transform of both sides of the equation to get

$$L[u''(t)] = s^2 L[u(t)] - su(0) - u'(0) \tag{20}$$

$$s^2 U(s) - su(0) - u'(0) + k^2 U(s) = F(s) \tag{21}$$

$$s^2 U(s) + k^2 U(s) = F(s) \tag{22}$$

Thus

$$U(s) = \frac{1}{k}\frac{k}{s^2+k^2}F(s). \tag{23}$$

$$L[\sin at] = \frac{a}{s^2+a^2} \tag{24}$$

Taking the inverse Laplace transform gives

$$u(t) = \frac{1}{k}f(t) * \sin kt = \frac{1}{k}\int_0^t f(\tau)\sin k(t-\tau)d\tau. \tag{25}$$

Recap on Variation of Parameters Method: Solving Second-Order Linear Non-Homogeneous ODEs

Given the differential equation:

$$y'' + p(x)y' + q(x)y = f(x) \tag{26}$$

Steps:

1. Write down the corresponding homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0$$

$$\Rightarrow y = c_1y_1(x) + c_2y_2(x), \text{ where } c_1, c_2 \text{ are constants}$$

2. **Assume** (a priori assumption) a particular solution (guess, trial method) for the non-homogeneous equation (26):

$$y = c_1(x)y_1(x) + c_2(x)y_2(x) \leftarrow \text{any function } y(x) \text{ can be written in this form} \tag{27}$$

- **Variation of Constants:** Treat constants c_1 and c_2 as functions.
- **Rationale:** Q: Why we have to write it as this form? A: This form and approach simplify the process of solving the differential equation. It is not the only way, but it is a efficient way. This form means y is completely unknown, which does not put any restriction on the solution y since by varying $c_1(x), c_2(x)$, y can be any function.
- **Simplification:** After setting up, you need to take the first and second derivatives of y and substitute them back into the original equation.
- **Advantages:** This method allows for significant simplification by eliminating many terms by using the homogeneous ODE since y_1 and y_2 solve it.
- **Non-uniqueness:** The form (27) of the solution does not represent a unique decomposition; different forms of $c_1(x)$ and $c_2(x)$ can lead to the same $y(x)$, since for some solution y and the given y_1, y_2, there are two functions c_1, c_2 unknown but only one equation. Any function can be expressed in such a form.
- **Note:** $y_1(x)$ and $y_2(x)$ are **known functions**, while $c_1(x)$ and $c_2(x)$ are **unknown functions**.

3. **Goal:** Substituting (27) into (26), we obtain an equation of $c_1(x)$ and $c_2(x)$.

- **Question:** One equation, two unknowns $c_1(x), c_2(x) \leftarrow$ an **indeterminate system**.
- **Ideas:** We can **arbitrarily add an equation** of $c_1(x)$ and $c_2(x)$ to (1) **simplify calculations** and (2) make the system **well-posed** (two hopes).
- **Note:** There's an opportunity to supplement the equation! It's important to grasp this well!!!

By calculations, we obtain

$$y' = c_1(x)y_1'(x) + c_2(x)y_2'(x) + \underbrace{[c_1'(x)y_1(x) + c_2'(x)y_2(x)]}_{(*)\text{ Let it vanish to obtain an extra eq.!}}$$

- Because if $(*) \neq 0$, y'' will introduce c_1'' and c_2'', making the processing more difficult. Why make things harder for ourselves?

- Recall that, with **two unknown functions** and **one equation**, there is **a degree of freedom**.

- **Choosing an Equation:** We can choose any equation for c_1 and c_2. We select $(*) = c_1'y_1 + c_2'y_2 = 0$ as our chosen equation.

Let:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{28}$$

- **Simplification Strategy:** By imposing this new equation, certain terms will cancel out, potentially simplifying the problem.

Thus:

$$y'' = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)$$

- **Differentiation:** After differentiating, the second derivative of y will only involve the first derivatives of c_1 and c_2, not the second derivatives.

Substituting y' and y'' into the original equation (26) yields

$$c_1y_1'' + c_2y_2'' + c_1'y_1' + c_2'y_2' + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) = f$$

Simplifying, we get:

$$\cancel{c_1(y_1'' + py_1' + qy_1)} + \cancel{c_2(y_2'' + py_2' + qy_2)} + c_1'y_1' + c_2'y_2' = f$$

- **Substitution Back:** Substitute y'' and y' back into the original equation and simplify. This will utilize the fact that y_1 and y_2 are solutions to the homogeneous equations.

$$\Rightarrow c_1'y_1' + c_2'y_2' = f \tag{29}$$

- **Resulting Simplicity:** The process simplifies the equation to just two terms.

From (26) and (29):

$$\begin{cases} c_1'y_1 + c_2'y_2 = 0 \\ c_1'y_1' + c_2'y_2' = f \end{cases} \xrightarrow{y_1, y_2 \text{ are known}} \begin{cases} c_1'(x) \\ c_2'(x) \end{cases} \xrightarrow{\text{Integration}} \begin{cases} c_1(x) \\ c_2(x) \end{cases}$$

- **Equation System:** We now have two equations (including the one we imposed) and two unknowns, allowing us to solve for c_1 and c_2.

Summary

- **Indeterminacy:** With two unknown functions and only one equation, the system is indeterminate, leading to potentially multiple solutions.
- **Freedom of Choice:** You can arbitrarily choose one equation for c_1 and c_2 to solve the system.
- **Utilization:** The freedom in choosing c_1 and c_2 should be used to facilitate the solution process.
- **Purpose of Imposing Equations:** Some students may **question why we impose certain equations**. The solutions c_1 and c_2 that we derive are only **one result among many possibilities**. As mentioned before, y can have many decomposition of the form (27), the **imposed equations** (28) **help us pick one specific** c_1, c_2.
- **Non-uniqueness of Solutions:** The decomposition c_1 and c_2 may not necessarily be the only solutions. There could be other valid solutions.
- **Validation of Solutions:** The solutions we obtain can indeed solve the equation, proving their applicability.
- Using the **degree of freedom** to impose an equation for solving is similar to the idea of **selecting an S-L problem** in the method of eigenfunctions. This is a path we artificially choose that is feasible, but it is not the only way. This represents the intrinsic conceptual connection between the method of eigenfunctions and the variation of parameters method.

Summary

For the initial value problem of ordinary differential equations, we have two cases:

1. Case 1: The differential equation is:

$$u_n''(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \tag{30}$$

with initial conditions:

$$u_n(0) = u_n'(0) = 0, \quad (n = 1, 2, \ldots).$$

The solution to this problem is given by:

$$\boxed{u_n(t) = \frac{l}{n\pi a}\int_0^t f_n(\tau)\sin\left(\frac{n\pi a}{l}(t-\tau)\right)d\tau \quad (n = 1, 2, \ldots).} \tag{31}$$

2. Case 2: The differential equation is:

$$u_n'(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t)$$

with initial condition:

$$u_n(0) = 0, \quad (n = 1, 2, \ldots).$$

The solution to this problem is given by:

$$\boxed{u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)} d\tau \quad (n = 1, 2, \ldots).} \tag{32}$$

These solutions are derived using the method of variation of parameters and Laplace transforms, which are powerful tools for solving linear differential equations with non-homogeneous terms and initial conditions.

Ex 1.1. *Solve the following problem*

$$\begin{cases} u_{tt} = a^2 u_{xx} + A\sin\omega t\cos\frac{\pi x}{l} \quad (0<x<l,\, t>0), \\ u_x(0,t)=0,\ u_x(l,t)=0, \\ u(x,0)=u_t(x,0)=0. \end{cases}$$

where A, ω are constants.

- **Similarity:** Non-homogeneous equation + Homogeneous boundary and initial data $\to$ the eigenfunction method.
- **Differences:** Both boundaries are second-type boundaries $\to$ S-L problem.

Solution. ***[1. Homogeneous Eigenfunction System]*** *From the knowledge of previous sections, it is known that the corresponding homogeneous equation of the original equation is*

$$u_{tt} = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous second type boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(l) = 0.$$

$$\Rightarrow \quad X_0(x) = B_0, \quad X_n(x) = A_n\cos\frac{n\pi x}{l} \quad (n=1,2,\ldots).$$

Therefore, it is known that the corresponding homogeneous equation of the original equation and the eigenfunction series that satisfy the homogeneous second type boundary conditions are $\{\cos\frac{n\pi x}{l}\}_{n=0}^{\infty}$.

[2. Assumed Series Solution] *Let the solution be*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos\frac{n\pi x}{l},$$

where $u_n(t)$ is a function to be determined with respect to t.

[3. Expansion of Free Term] $A\sin\omega t\cos\frac{\pi x}{l}$ *has already in the right form.*

[4. Comparison of Coefficients] *Substitute $u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos\frac{n\pi x}{l}$ into the original equation to simplify and obtain*

$$\sum_{n=0}^{\infty}\left[u_n'' + \left(\frac{n\pi a}{l}\right)^2 u_n\right]\cos\frac{n\pi x}{l} = A\sin\omega t\cos\frac{\pi x}{l},$$

Comparing the coefficients on both sides of the equation gives

$$u_n'' + \left(\frac{n\pi a}{l}\right)^2 u_n = 0 \quad (n\neq 1), \quad u_1'' + \left(\frac{\pi a}{l}\right)^2 u_1 = A\sin\omega t.$$

[5. Initial Value Transformation] *In $u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos\frac{n\pi x}{l}$, using the initial conditions gives*

$$\begin{cases} \sum_{n=0}^{\infty} u_n(0)\cos\frac{n\pi x}{l} = 0 \implies u_n(0)=0, \\ \sum_{n=0}^{\infty} u_n'(0)\cos\frac{n\pi x}{l} = 0 \implies u_n'(0)=0. \end{cases} \quad (n=0,1,2,\ldots).$$

Thus, we obtain two sets of initial value problems for ordinary differential equations

$$\begin{cases} u_n'' + \left(\frac{n\pi a}{l}\right)^2 u_n = 0 \quad (n\neq 1), \\ u_n(0) = u_n'(0) = 0. \end{cases} \quad \text{and} \quad \begin{cases} u_1'' + \left(\frac{\pi a}{l}\right)^2 u_1 = A\sin\omega t. \\ u_n(0) = u_n'(0) = 0. \end{cases}$$

1. *First, when $n \neq 1$, using the general solution formula we have*

$$u_n(t) = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t$$

and

$$u_n'(t) = -A_n \frac{n\pi a}{l} \sin \frac{n\pi a}{l} t + B_n \frac{n\pi a}{l} \cos \frac{n\pi a}{l} t$$

Using the conditions $u_n(0) = u_n'(0) = 0$, we get

$$u_n(t) = 0.$$

2. *when $n = 1$, by (31)*

$$u_1(t) = \frac{l}{\pi a} \int_0^t A \sin \omega\tau \sin \frac{\pi a(t-\tau)}{l} d\tau.$$

Since

$$\begin{aligned}
u_1(t) =& \frac{l}{\pi a} \int_0^t A \sin \omega\tau \sin \frac{\pi a(t-\tau)}{l} d\tau \\
=& \frac{Al}{2\pi a} \left\{ \int_0^t \cos\left[\left(\omega + \frac{\pi a}{l}\right)\tau - \frac{\pi a}{l} t\right] d\tau - \int_0^t \cos\left[\left(\omega - \frac{\pi a}{l}\right)\tau + \frac{\pi a}{l} t\right] d\tau \right\} \\
=& \frac{Al}{2\pi a} \left\{ \frac{1}{\omega + \frac{\pi a}{l}} \sin\left[\left(\omega + \frac{\pi a}{l}\right)\tau - \frac{\pi a}{l} t\right] \Bigg|_{\tau=0}^{\tau=t} - \frac{1}{\omega - \frac{\pi a}{l}} \sin\left[\left(\omega - \frac{\pi a}{l}\right)\tau + \frac{\pi a}{l} t\right] \Bigg|_{\tau=0}^{\tau=t} \right\} \\
=& \frac{Al}{2\pi a} \left(\frac{\sin \omega t + \sin \frac{\pi a}{l} t}{\omega + \frac{\pi a}{l}} - \frac{\sin \omega t - \sin \frac{\pi a}{l} t}{\omega - \frac{\pi a}{l}} \right) \\
=& \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - \left(\frac{\pi a}{l}\right)^2} \left(\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t \right).
\end{aligned}$$

Substituting

$$\begin{cases} u_n(t) = 0, \quad n \neq 1 \\ u_1(t) = \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - \left(\frac{\pi a}{l}\right)^2} \left(\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t\right) \end{cases}$$

into

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos \frac{n\pi x}{l},$$

we obtain the solution as

$$u(x,t) = \frac{Al}{\pi a} \cdot \frac{1}{\omega^2 - \left(\frac{\pi a}{l}\right)^2} \left(\omega \sin \frac{\pi a}{l} t - \frac{\pi a}{l} \sin \omega t\right) \cdot \cos \frac{\pi x}{l}.$$

2 Heat Conduction in a Finite Rod with a Heat Source

First, let's consider the following problem

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t) \qquad (0 < x < l,\ t > 0), \\ u(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = \varphi(x). \end{cases} \tag{33}$$

At this point, the heat conduction phenomenon is caused by two parts: one is the internal heat **source**, and the other is the **initial temperature** of the rod. Thus, this heat conduction phenomenon can be regarded as a **combination** of heat conduction caused solely by the internal heat source and heat conduction caused solely by the initial temperature.

Therefore, we can assume the solution to problem (33) to be

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ represents the temperature function caused solely by the **internal heat source**; and $w(x,t)$ represents the temperature function caused solely by the **initial temperature**; $v(x,t)$ and $w(x,t)$ satisfy the following mixed value problems respectively:

$$\begin{cases} v_t = a^2 v_{xx} + f(x,t) & (0 < x < l,\ t > 0), \\ v(0,t) = 0,\ v(l,t) = 0, \\ v(x,0) = 0. \end{cases}$$

$$\begin{cases} w_t = a^2 w_{xx} & (0 < x < l,\ t > 0), \\ w(0,t) = 0,\ w(l,t) = 0, \\ w(x,0) = \varphi(x). \end{cases}$$

- Isolating the non-homogeneity—utilizing the principle of linear superposition.

For this, we first discuss the case of **homogeneous boundary** conditions and **zero initial** conditions, taking the temperature at both ends maintained at 0 degrees as an example:

$$u_t = a^2 u_{xx} + f(x,t)(0 < x < l,\ t > 0), \tag{34}$$
$$u(0,t) = 0,\ u(l,t) = 0, \tag{35}$$
$$u(x,0) = 0. \tag{36}$$

We still use the **method of eigenfunctions** to solve this mixed value problem.

Solution. ***[1. Homogeneous Eigenfunction System]*** *From the knowledge of Section 2.2, the homogeneous equation corresponding to* (34)

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous first type boundary conditions satisfy (35)

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0.$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation (34) *that also satisfies the homogeneous first type boundary conditions* (35) *is* $\{\sin \frac{n\pi x}{l}\}$.

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t) & (0 < x < l,\ t > 0), \\ u(0,t) = 0,\ u(l,t) = 0, \\ u(x,0) = 0. \end{cases}$$

[2. Assumed Series Solution] *Expand the solution of the boundary value problem in terms of x using the eigenfunction series (Fourier sine series):*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}, \tag{37}$$

[3. Expansion of Free Term] *Expand the free term $f(x,t)$ in the equation using the same eigenfunction series:*

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \tag{38}$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi x}{l}\, dx \quad (n = 1, 2, \ldots).$$

[4. Comparison of Coefficients] *Substituting* (40)-(38) *into equation* (34) *yields*

$$\sum_{n=1}^{\infty} \left[u_n'(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) - f_n(t) \right] \sin \frac{n\pi x}{l} = 0,$$

from which we get

$$u_n'(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \quad (n = 1, 2, \ldots).$$

[5. Initial Value Transformation] *Using the initial condition* (36) *in expression* (40) *gives*

$$\sum_{n=1}^{\infty} u_n(0) \sin\frac{n\pi x}{l} = 0 \quad \Rightarrow \quad u_n(0) = 0, \quad (n = 1, 2, \ldots).$$

[6. Solving ODEs] *Thus, we obtain the following initial value problem for ordinary differential equations*

$$\begin{cases} u_n'(t) + \left(\frac{n\pi a}{l}\right)^2 u_n(t) = f_n(t) \\ u_n(0) = 0, \quad (n = 1, 2, \ldots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problem (30) *is*

$$u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)}\, d\tau \quad (n = 1, 2, \ldots).$$

Substituting

$$u_n(t) = \int_0^t f_n(\tau) e^{-\left(\frac{n\pi a}{l}\right)^2 (t-\tau)}\, d\tau \quad (n = 1, 2, \ldots).$$

into

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi x}{l},$$

we obtain the solution to the boundary value problem (34)-(36).

Ex 2.1. ***Solve the following problem***

$$\begin{cases} u_t = a^2 u_{xx} + A & (0 < x < l,\ t > 0), \\ u(0,t) = 0,\ u_x(l,t) = 0, \\ u(x,0) = 0. \end{cases} \tag{39}$$

where A is a constant.

Solution. ***[1. Homogeneous Eigenfunction System]*** *The corresponding homogeneous equation of the original equation*

$$u_t = a^2 u_{xx},$$

and the eigenfunctions that meet the homogeneous boundary conditions satisfy

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X'(l) = 0.$$

Then

$$X_n(x) = B_n \sin\frac{(2n+1)\pi x}{2l} \quad (n = 0, 1, 2, \ldots).$$

Thus, it is known that the eigenfunction series corresponding to the homogeneous equation and satisfying the homogeneous boundary conditions is $\left\{\sin\frac{(2n+1)\pi x}{2l}\right\}$.

[2. Assumed Series Solution] *Let the solution be*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin\frac{(2n+1)\pi x}{2l}. \tag{40}$$

[3. Expansion of Free Term] *Then expand A into the Fourier sine series according to the above eigenfunction series*

$$A = \sum_{n=0}^{\infty} A_n(t) \sin\frac{(2n+1)\pi x}{2l}, \tag{41}$$

where

$$A_n(t) = \frac{2}{l}\int_0^l A\sin\frac{(2n+1)\pi x}{2l}\,dx = \frac{4A}{(2n+1)\pi}.$$

[4. Comparison of Coefficients] *Substituting* (40)-(41) *into equation* (39) *yields*

$$\sum_{n=1}^{\infty}\left[u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) - \frac{4A}{(2n+1)\pi}\right]\sin\frac{(2n+1)\pi x}{2l} = 0,$$

from which we get

$$u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) = \frac{4A}{(2n+1)\pi}. \tag{42}$$

[5. Initial Value Transformation] *Using the initial condition in expression* (40) *gives*

$$\sum_{n=0}^{\infty} u_n(0)\sin\frac{(2n+1)\pi x}{2l} = 0 \quad \Rightarrow \quad u_n(0) = 0, \quad (n = 0, 1, 2, \ldots). \tag{43}$$

[6. Solving ODEs] *Thus, we obtain the following initial value problem for ordinary differential equations*

$$\begin{cases} u_n'(t) + \left(\frac{(2n+1)\pi a}{2l}\right)^2 u_n(t) = \frac{4A}{(2n+1)\pi} \\ u_n(0) = 0, \quad (n = 0, 1, 2, \ldots). \end{cases}$$

Applying the method of variation of parameters or Laplace transform for ordinary differential equations, the solution to problems (42)-(43) *is*

$$\begin{aligned} u_n(t) &= \int_0^t \frac{4A}{(2n+1)\pi} e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 (t-\tau)}\,d\tau \\ &= \frac{4A}{(2n+1)\pi}\int_0^t e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 (t-\tau)}\,d\tau \\ &= \frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}. \end{aligned}$$

Substituting

$$u_n(t) = \frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}$$

into

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\sin\frac{(2n+1)\pi}{2l}x,$$

we obtain the solution to the problem

$$u(x,t) = \sum_{n=0}^{\infty}\frac{16Al^2}{(2n+1)^3\pi^3 a^2}\left\{1 - e^{-\left[\frac{(2n+1)\pi a}{2l}\right]^2 t}\right\}\sin\frac{(2n+1)\pi}{2l}x.$$