APR 07 2022 MATH 5A

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2.1 Limits of Sequence

Definition. *$\lim_{n\to\infty} a_n = L$ or $an \to L$ as $n \to \infty$ if we can get as close as we want to L by taking large enough n.*

$$\text{Limit exists} \begin{cases} \{a_n\} \text{ converges if } L \text{ is finite} \\ \{a_n\} \text{ diverges to } \infty \text{ if } L = \infty \\ \{a_n\} \text{ diverges to } -\infty \text{ if } L = -\infty \end{cases},$$

otherwise $\lim_{n\to\infty} a_n$ diverges and does not exist (D.N.E.).

Remark. *$\frac{\infty}{\infty}$ and $\frac{0}{0}$ are indeterminate forms, it doesn't say converges, diverges, or DNE. You try to simplify it more.*

Example. *Determine whether the sequence is convergent or divergent.*

$$a_n = \frac{n^3 - 7}{n}.$$

Solution. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3 - 7}{n} = \lim_{n\to\infty} \frac{(n^3 - 7)/n}{n/n} = \lim_{n\to\infty} \frac{n^2 - 7/n}{1} = \lim_{n\to\infty} \frac{n^2 - 0}{1} = \lim_{n\to\infty} n^2 = \infty.$$

So the limit diverges to infinity.

□

Example. *Find the limit. (Let q and r represent arbitrary real numbers.)*

$$\lim_{x\to\infty} \left(\sqrt{x^2 + qx} - \sqrt{x^2 + rx}\right).$$

Solution. First, we try to simplify the square root, the trick is to multiply it with its conjugate, we have

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2 + qx} - \sqrt{x^2 + rx}\right) &= \lim_{x\to\infty} \left(\sqrt{x^2 + qx} - \sqrt{x^2 + rx}\right) \times \frac{\left(\sqrt{x^2 + qx} + \sqrt{x^2 + rx}\right)}{\left(\sqrt{x^2 + qx} + \sqrt{x^2 + rx}\right)} \\
&= \lim_{x\to\infty} \frac{\left(\sqrt{x^2 + qx} - \sqrt{x^2 + rx}\right) \times \left(\sqrt{x^2 + qx} + \sqrt{x^2 + rx}\right)}{\sqrt{x^2 + qx} + \sqrt{x^2 + rx}} \\
&= \lim_{x\to\infty} \frac{(x^2 + qx) - (x^2 + rx)}{\sqrt{x^2 + qx} + \sqrt{x^2 + rx}} = \lim_{x\to\infty} \frac{(q - r)x}{\sqrt{x^2 + qx} + \sqrt{x^2 + rx}} \\
&= \lim_{x\to\infty} \frac{(q - r)x/x}{\sqrt{x^2 + qx}/x + \sqrt{x^2 + rx}/x} = \lim_{x\to\infty} \frac{q - r}{\sqrt{\frac{x^2 + qx}{x^2}} + \sqrt{\frac{x^2 + rx}{x^2}}} \\
&= \lim_{x\to\infty} \frac{q - r}{\sqrt{1 + \frac{q}{x}} + \sqrt{1 + \frac{r}{x}}} = \frac{q - r}{2}.
\end{aligned}$$

Notice that we use the following properties:

- $\dfrac{1}{x} = \dfrac{1}{\sqrt{x^2}}$
- $\dfrac{\sqrt{x^2+qx}}{\sqrt{x^2}} = \sqrt{\dfrac{x^2+qx}{x^2}}$

□

Definition. *Geometric sequence is a sequence of the form $a_n = ar^n$. A series is the total sum of the elements of a sequence, that is,*

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i.$$

A geometric series is the sum of a geometric sequence

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} ar^i = \frac{ar}{1-r}.$$

Example. *Find a geometric series expression of*

$$0.\overline{407} = 0.407407407407\cdots$$

Solution. We may write $a_1 = 0.407$, $a_2 = 0.000407$, $a_3 = 0.000000407$, and so on. So

$$0.\overline{407} = 0.407407407407\cdots = a_1 + a_2 + a_3 + \cdots$$

Also, we can observe that $\{a_n\}$ is a geometric sequence with a fixed ratio $r = 0.001$. The geometric sequence has the form $a_n = a \cdot r^n$, so

$$0.407 = a_1 = a \cdot r = a \cdot 0{,}001 \Rightarrow a = 407.$$

Thus, $a_n = a \cdot r^n = 407 \cdot 0.001^n$.

Thus the geometric series will be

$$0.\overline{407} = 0.407407407407\cdots = a_1 + a_2 + a_3 + \cdots = \frac{ar}{1-r} = \frac{407 \cdot 0.001}{1-0.001} = \frac{407}{999}.$$

□