UNIVERSITY OF CALIFORNIA,
IRVINE

# Class of Curvature Equations, Convexity, and Real Algebraic Geometry

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Chao-Ming Lin

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2023

# DEDICATION

To my family, friends, and teachers.

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# ACKNOWLEDGMENTS

First, I would like to thank my advisors Professor Zhiqin Lu and Professor Xiangwen Zhang. I was fortunate to have them advise me, they broadened my horizons in mathematics, taught me to appreciate theories in mathematics from different aspects, and boosted me to reach my full potential. I am very grateful to them for their constant guidance, patience, and support in mathematics at University of California, Irvine.

I also like to thank Professor Li-Sheng Tseng, Professor Jeffrey Streets, Professor Duong Hong Phong, Professor Po-Ning Chen, Professor Xin Fu, Professor Tsung-Ju Lee, and Professor Yu-Shen Lin for their academic help and support. I would like to thank Professor Bo Guan, Professor Richard Schoen, Professor Tristan Collins, Professor Siyuan Lu, and Professor Man-Chun Lee for their interest in my works and for their help in my career.

I would like to thank my friends at University of California, Irvine. I would like to thank Anthony Tsai for hanging out with me and sharing life as graduate students, we were foodies and we found all good Asian and Taiwanese food in this area. I would like to thank Hua Lin, Ellen Wang, Matthew Cheung, and Edward Martinez, I am lucky to have friends that regularly checked up, kept in contact, and chilled with me. I would like to thank Hsu Cheng and Yifan Guo for playing table tennis with me. I would like to thank Kuan-Hui Lee, Long-Sin Li, and Yi-Lin Tsai. I would like to thank Tin-Yau Tsang, his passion for mathematics and math society really impressed me a lot. I would like to thank Kai-Wei Zhao for his kindness and patience. I had a hard time, I appreciate conversations with him not only in mathematics but also in many fields. They are the ones who best understood the struggles of graduate school and the support I received was invaluable.

I would like to thank Heaton Christensen for his life experiences, knowledge, and financial advice. I learn a lot from him and our conversations always widen my field of vision. Heaton teaches me the importance of finding the core of things and helps me improve my resilience and confidence. I would like to give thanks to Heaton for everything.

I would like to thank the mathematics community I encountered prior to my graduate school at National Taiwan University, especially Professor Chung-Jun Tsai, Professor Chen-Yu Chi, Professor Mao-Pei Tsui, and Professor Yng-Ing Lee. I was not an outstanding student and failed several courses when I was in my bachelor degree. I am indebted for their tolerance and support at National Taiwan University. Chung-Jun is my master degree advisor at National Taiwan University, I am very grateful to him for showing his great patience and his guidance to me as a beginner in differential geometry. I also would like to thank my friends at National Taiwan University, I would like to thank Hsin-Po Wang, You-Cheng Chou, Chi-Shian Dai, Chung-Ru Lee, Chi-Yun Hsu, Chung-Ming Pan, Chen-Chih Lai, Yu-Ping Lin, and Christopher Kuo. I wish all of you the best when you are pursuing your future goal.

Last, I would like to thank my mother 何玉嬌 (Yu-Chiao Ho) and my father 林佑聰(Yu-Tsung Lin) for their unconditional love and support. Nothing would be possible without them and I hope my father will recover and see my achievement.

Portions of Chapter 2 of this dissertation are a reprint of the material currently in press at the Journal of Functional Analysis, available at https://doi.org/10.1016/j.jfa.2023.110038, used with permission from Elsevier.

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Mathematical Research Letters

# ABSTRACT OF THE DISSERTATION


Class of Curvature Equations, Convexity, and Real Algebraic Geometry

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Professor Zhiqin Lu, Co-Chair
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This dissertation works towards building a fundamental theory of general $\sigma_k$ equations and general inverse $\sigma_k$ equations, showing up in many different fields. For example, PDE, differential geometry, and complex geometry. Our primary goal is to construct nice algebra tools, especially related to real algebraic geometry, so that we can generalize previous classical equations to more complicated settings. Once the framework is settled, we aim to look for a priori estimates and further obtain the solvability of these equations. To be more precise, first, we introduce a special class of multilinear polynomials and a special class of univariate polynomials which are related to the convexity of these equations. Second, we study a priori estimates of these equations provided that the convexity and a $C$-subsolution are given. Last, by collecting these equations which have a priori estimates, we obtain a special algebraic set to apply the method of continuity and further look for the solvability. As an application, we apply our theory and results to the deformed Hermitian–Yang–Mills equation, an equation discovered around the same time by Mariño–Minasian–Moore–Strominger [54] and Leung–Yau–Zaslow [48] using different points of view when studying mirror symmetry in string theory. We confirm one of the conjectures by Collins–Jacob–Yau [18] of the deformed Hermitian–Yang–Mills equation when the complex dimension equals three or four.

# Chapter 1

# Introduction

In complex geometry, let $(M, \omega)$ be a compact connected Kähler manifold of complex dimension $n$ with a Kähler form $\omega$ and $[\chi_0] \in H^{1,1}(M; \mathbb{R})$, where $H^{1,1}(M; \mathbb{R})$ is the $(1,1)$-Dolbeault cohomology group. The study of the following equation is widely considered:

$$\chi^n = c_{n-1}\binom{n}{n-1}\chi^{n-1} \wedge \omega^1 + \cdots + c_1\binom{n}{1}\chi^1 \wedge \omega^{n-1} + c_0\binom{n}{0}\omega^n, \tag{1.1}$$

where $c_k$ are real functions on $M$ and $\chi \in [\chi_0]$ is a real smooth closed $(1,1)$-form. We call an equation having the same format as equation (1.1) a degree $n$ *general inverse $\sigma_k$ type equation*. A general inverse $\sigma_k$ type equation (1.1) is very likely to be ill-posed, but some special combinations of the coefficients raise some famous equations.

For example, by letting $[\chi_0]$ be a Kähler class, $c_k = 0$ for all $k \in \{1, \cdots, n-1\}$, and $c_0$ be a positive function, equation (1.1) becomes the complex Monge–Ampère equation in the Calabi conjecture [12, 13], which was solved by Yau [72]. Inspired by the study of the Hermitian–Yang–Mills connections by Donaldson [26] and Uhlenbeck–Yau [70], Donaldson [27] studied the J-equation using the moment map. The J-equation can be obtained by letting $[\chi_0]$ be a Kähler class, $c_k = 0$ for all $k \in \{0, \cdots, n-2\}$, and $c_{n-1}$ be a positive constant. The

J-equation was studied extensively by Chen [14], Collins–Székelyhidi [20], Lejmi–Székelyhidi [47], Song [65], Song–Weinkove [66], and the references therein.

There are also some examples with more non-zero terms. For example, the general inverse $\sigma_k$ equations with non-negative coefficients, which were studied by Fang–Lai–Ma [29], Collins–Székelyhidi [20], and Datar–Pingali [23]. Motivated by mirror symmetry in string theory, the deformed Hermitian–Yang–Mills equation, which will be abbreviated to the dHYM equation from now on, was discovered around the same time by Mariño–Minasian–Moore–Strominger [54] and Leung–Yau–Zaslow [48] using different points of view. The dHYM equation was initiated by Jacob–Yau [45] and can be formulated as follows:

$$\Im\big(\omega + \sqrt{-1}\chi\big)^n = \tan\big(\theta\big) \cdot \Re\big(\omega + \sqrt{-1}\chi\big)^n. \tag{1.2}$$

Here, $\Im$ and $\Re$ are the imaginary and real parts, respectively, and $\theta$ is a topological constant determined by the cohomology classes $[\omega]$ and $[\chi_0]$. The dHYM equation was studied extensively by Chen [14], Chu–Lee–Takahashi [17], Collins–Jacob–Yau [18], the author [50, 51] and the references therein. We should emphasize that there are many significant works that have been done recently. The interested reader is referred to [16, 19, 21, 22, 24, 43, 44, 46, 50, 53, 61, 63, 71] and the references therein.

If we write equation (1.1) in terms of the eigenvalues of the Hermitian endomorphism $\Lambda = \omega^{-1}\chi$ at a point, then we can rewrite equation (1.1) as

$$\lambda_1 \cdots \lambda_n = \sum_{k=0}^{n-1} c_k \sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{n-1} c_k \sigma_k(\lambda), \tag{1.3}$$

where $\lambda_i$ are the eigenvalues of $\Lambda$, $\sigma_k(\lambda_1, \cdots, \lambda_n)$ is the $k$-th elementary symmetric polynomial of $\{\lambda_1, \cdots, \lambda_n\}$, and we denote $\sigma_k(\lambda_1, \cdots, \lambda_n)$ by $\sigma_k(\lambda)$ for convenience. The following multivariate polynomial in $n$ variables $\{\lambda_1, \cdots, \lambda_n\}$

$$\lambda_1 \cdots \lambda_n - \sum_{k=0}^{n-1} c_k \sigma_k(\lambda_1, \cdots, \lambda_n) \tag{1.4}$$

is a special case of multilinear polynomials, that is, multivariate polynomials in which no variable occurs to a power of two or higher. We will call a multilinear polynomial having the same format as (1.4) a general inverse $\sigma_k$ type multilinear polynomial.

In convex geometry, on standard unit sphere $\mathbb{S}^n$, the following general Christoffel–Minkowski problem is studied extensively:

$$\sigma_s(W_u(x)) = \sum_{k=0}^{s-1} c_k(x) \sigma_k(W_u(x)) \tag{1.5}$$

where $c_k$ are real functions on $\mathbb{S}^n$ and $W_u(x) = u_{ij}(x) + u(x)\delta_{ij}$ is the spherical Hessian matrix of a function $u \colon \mathbb{S}^n \to \mathbb{R}$. Here, $u_{ij}$ are the second order covariant derivative with respect to any orthonormal frame on $\mathbb{S}^n$ and $\delta_{ij}$ is the standard Kronecker delta. We call an equation having the same format as equation (1.4) a degree $s$ *general $\sigma_k$ type equation* with $n$ variables.

For the case $s = 1$, equation (1.5) becomes the standard Christoffel problem which was solved by Firey [30, 31] and Berg [6]. For the case $s = n$ and $c_k = 0$ for $k \in \{1, \cdots, n-1\}$, equation (1.5) becomes the Minkowski problem which was studied extensively by Minkowski [55], Alexandrov [1], Lewy [49], Nirenberg [56], Pogorelov [62], and Cheng–Yau [15]. In [40], Guan–Zhang studied the solvability of a class of more general equations, they considered the case that $n \geq s \geq 2$ and $c_k \geq 0$ for $k \in \{0, \cdots, s-2\}$. There are also other general $\sigma_k$ type equations in different fields, but since we mainly focus on the general inverse $\sigma_k$ equations in this dissertation and because of the space limitations, the interested reader is referred to [10, 11, 20, 28, 29, 32, 33, 34, 35, 37, 38, 39, 42, 47, 57, 58, 59, 60, 64, 69] and the references therein.

Similarly, if we write equation (1.5) in terms of the eigenvalues of the Hessian matrix $W_u$ at a point, then we can rewrite equation (1.5) as

$$\sigma_s(\lambda) = \sum_{k=0}^{s-1} c_k \sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{s-1} c_k \sigma_k(\lambda), \tag{1.6}$$

where $\lambda_i$ are the eigenvalues of $W_u$, $\sigma_k(\lambda_1, \cdots, \lambda_n)$ is the $k$-th elementary symmetric polynomial of $\{\lambda_1, \cdots, \lambda_n\}$, and we denote $\sigma_k(\lambda_1, \cdots, \lambda_n)$ by $\sigma_k(\lambda)$ for convenience. The following multivariate polynomial in $n$ variables $\{\lambda_1, \cdots, \lambda_n\}$

$$\sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda_1, \cdots, \lambda_n) \tag{1.7}$$

is a special case of multilinear polynomials. We will call a multilinear polynomial having the same format as (1.7) a degree $s$ general $\sigma_k$ type multilinear polynomial with $n$ variables.

Throughout all these works, the convexity of either the equation itself or the level set plays a crucial role. To be more precise, to get a priori estimates, we highly rely on convexity.

Let us state some of our settings, definitions, and results now. This is a generalization of the author's works [51, 52]. First, we introduce the following stableness condition for general $\sigma_k$ type multilinear polynomials. For more details, see Section 2.2.

**Definition 1.1** ($\Upsilon$-stableness). Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. We say that this connected component $\Gamma_f^n$ of $f(\lambda)$ is $\Upsilon$-stable if

$$\Gamma_f^n \subseteq \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > q_{i_1, \cdots, i_{s-1}}\} \text{ for some } q = \left(q_{i_1, \cdots, i_{s-1}}\right)_{1 \le i_1 < \cdots < i_{s-1} \le n}.$$

Here, we treat $q$ as an element in the $\binom{n}{s-1}$-dimensional Euclidean space. We say that this connected component $\Gamma_f^n$ is strictly $\Upsilon$-stable if it is $\Upsilon$-stable and the boundary $\partial \Gamma_f^n$ is contained in the $\Upsilon_1$-cone.

The $\Upsilon_k$-cones will be defined later in Section 2.2 for $k \in \{1, \cdots, n-1\}$. In particular, the $\Upsilon_1$-cone is the $C$-subsolution cone introduced by Székelyhidi [68] and Guan [36]. With the $\Upsilon$-stableness condition, in Section 2.3, we prove that the boundary $\partial\Gamma_f^n$ of $\Gamma_f^n$ will be convex if $\Gamma_f^n$ is strictly $\Upsilon$-stable. In this case, this connected component equals $\partial\Gamma_f^n$. We have the following main result.

**Theorem 1.1** (Convexity of the general $\sigma_k$ equation)**.** *Consider the following general $\sigma_k$ equation $f(\lambda) := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda) = 0$, where $\sigma_k$ is the $k$-th elementary symmetric polynomial and $c_k$ are real numbers. Let $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. If $\Gamma_f^n$ is strictly $\Upsilon$-stable, then the boundary $\partial\Gamma_f^n$ is convex.*

The following general inverse $\sigma_k$ type equations are all strictly $\Upsilon$-stable, we will verify some of them in Section 2.4.

*Remark* 1.1. The following general inverse $\sigma_k$ type equations are all strictly $\Upsilon$-stable:

- Complex Monge–Ampère equation.
- J-equation.
- Hessian equation.
- Deformed Hermitian–Yang–Mills equation with supercritical phase.
- Special Lagrangian equation with supercritical phase.
- General inverse $\sigma_k$ equation with non-negative $c_k$ for $k \in \{0, \cdots, n-1\}$.

In practice, verifying the $\Upsilon$-stableness condition is not easy. Here, we introduce the following class of special univariate polynomials which plays an important role in determining the convexity of both general inverse $\sigma_k$ equations and general $\sigma_k$ equations. In Section 2.1, we will show more special properties of these special univariate polynomials. Now, we list some definitions and some interesting and important results.

**Definition 1.2** (Noetherian polynomial)**.** We say a degree $n$ real univariate polynomial $p(x)$ is right-Noetherian if for all $k \in \{0, \cdots, n-2\}$, there exists a real root of $p^{(k)}$ which is greater than or equal to the largest real root of $p^{(k+1)}$. Here $p^{(k)}$ is the $k$-th derivative of $p$. We say a right-Noetherian polynomial $p(x)$ is strictly right-Noetherian if the largest real root of $p(x)$ is strictly greater than the largest real root of $p'(x)$.

In Section 2.2, we will show that the right-Noetherianness condition is equivalent to the $\Upsilon$-stableness condition in the following sense. We get the following Positivstellensatz-type result generalizing the work in [51]. When the degree is small, we can explicitly write down the constraints using the resultant and the discriminant, see Section 2.4 for more examples when the degree equals three or four.

**Theorem 1.2** (Positivstellensatz)**.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. There exists a connected component $\Gamma_f^n$ of $\{f(\lambda) > 0\}$ which is $\Upsilon$-stable if and only if the diagonal restriction $r_f(x)$ of $f(\lambda)$, which is defined by the following*

$$r_f(x) \coloneqq f(x, \cdots, x) = \binom{n}{s} x^s - \sum_{k=0}^{s-1} c_k \binom{n}{k} x^k,$$

*is right-Noetherian. Moreover, $\Gamma_f^n$ is strictly $\Upsilon$-stable iff $r_f$ is strictly right-Noetherian.*

As an application of the Positivstellensatz Theorem, in Section 2.4, we will verify some general inverse $\sigma_k$ type equations and general $\sigma_k$ type equations. As a quick consequence of the Positivstellensatz Theorem, we can show that the level set of the following general inverse $\sigma_k$ equation is convex. This is also numerical checkable, which gives a large quantity of new convex sets.

**Example 1.1.** The following univariate polynomial $r_f(x) = x^5 - \sum_{k=0}^{3} c_k \binom{5}{k} x^k$ with $c_3 = 19, c_2 = -64, c_1 = 9$, and $c_0 = -20$ is strictly right-Noetherian. This is checkable using any computer. By rounding off to the third decimal place, we have

$$x_0 \sim 11.632,\ x_1 \sim 9.306,\ x_2 \sim 6.909,\ x_3 \sim 4.359,\ x_4 = 0.$$

Here, for $k \in \{0, \cdots, n-1\}$, we denote by $x_k$ the largest real root of the $k$-th derivative $r_f^{(k)}(x)$. This implies that the level set of the following general inverse $\sigma_k$ equation is convex

$$f(\lambda) = \lambda_1 \cdots \lambda_5 - \sum_{k=0}^{3} c_k \sigma_k(\lambda) = \lambda_1 \cdots \lambda_5 - 19\sigma_3(\lambda) + 64\sigma_2(\lambda) - 9\sigma_1(\lambda) + 20 = 0.$$

If a general $\sigma_k$ type multilinear polynomial has an $\Upsilon$-stable connected component, then, for convenience, we say this multilinear polynomial is $\Upsilon$-stable. In the following setting, we can also compare two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials.

**Definition 1.3** ($\Upsilon$-dominance)**.** Let $f(\lambda) \coloneqq \lambda_1 \cdots \lambda_n - \sum_{k=0}^{n-1} c_k \sigma_k(\lambda)$ and $g(\lambda) \coloneqq \lambda_1 \cdots \lambda_n - \sum_{k=0}^{n-1} d_k \sigma_k(\lambda)$ be two $\Upsilon$-stable general inverse $\sigma_k$ type multilinear polynomials. For $k \in \{0, \cdots, n-1\}$, we write $x_k$ the largest real root of the diagonal restriction $r_f^{(k)}$ of $f$ and $y_k$ the largest real root of the diagonal restriction $r_g^{(k)}$ of $g$. If $y_k \geq x_k$ for all $k \in \{0, \cdots, n-1\}$, then we say $g \gtrdot f$.

We get another Positivstellensatz-type result. This result implies that for $\Upsilon$-stable general inverse $\sigma_k$ type multilinear polynomials, the $\Upsilon$-dominance is equivalent to the set inclusion.

**Theorem 1.3** ($\Upsilon$-dominance)**.** *Let $f \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ and $g \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} d_k \sigma_k(\lambda)$ be two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials. Then $g \gtrdot f$ if and only if $\Gamma_g^n \subset \Gamma_f^n$.*

**Example 1.2.** The following univariate polynomial $r_g(x) = x^5 - \sum_{k=0}^{3} d_k \binom{5}{k} x^k$ with $d_3 = 19, d_2 = 65, d_1 = -2$, and $d_0 = -24$ is strictly right-Noetherian with roots:

$$y_0 \sim 15.250, \ y_1 \sim 11.673, \ y_2 \sim 8.066, \ y_3 \sim 4.359, \ y_4 = 0.$$

Here, for $k \in \{0, \cdots, 4\}$, we denote by $y_k$ the largest real root of the $k$-th derivative $r_g^{(k)}(x)$. We compare this $\Upsilon$-stable general inverse $\sigma_k$ type multilinear polynomial with the one in Example 1.1. Since $y_0 > x_0$, $y_1 > x_1$, $y_2 > x_2$, $y_3 = x_3$, and $y_4 = x_4$, we have $g \gtrdot f$. By

Theorem 1.3, we get

$$\begin{aligned}&\{\lambda_1\cdots\lambda_5 - 19\sigma_3(\lambda) - 65\sigma_2(\lambda) + 2\sigma_1(\lambda) + 24 > 0\}\\&\qquad\subset\{\lambda_1\cdots\lambda_5 - 19\sigma_3(\lambda) + 64\sigma_2(\lambda) - 9\sigma_1(\lambda) + 20 > 0\}.\end{aligned}$$

In this dissertation, we apply this framework to determine the solvability of a general inverse $\sigma_k$ equation on a compact connected Kähler manifold satisfying strictly $\Upsilon$-stableness condition at every point on the manifold. By Theorem 1.1, we know strictly $\Upsilon$-stableness condition will give us level set convexity, so we define the following set in Section 3.1.

**Definition 1.4.** For $\lambda = \{\lambda_1, \cdots, \lambda_n\}$, we define

$$\begin{aligned}\mathscr{C}_{n,s} &:= \Big\{(c_{s-1}, c_{s-2}, \cdots, c_0) \in \mathbb{R}^s\colon \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda) \text{ is strictly } \Upsilon\text{-stable}\Big\};\\ \tilde{\mathscr{C}}_{n,s} &:= \Big\{(c_{s-2}, c_{s-3}, \cdots, c_0) \in \mathbb{R}^{s-1}\colon \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda) \text{ is strictly } \Upsilon\text{-stable}\Big\}.\end{aligned}$$

For convenience, we denote $\mathscr{C}_{n,n}$ by $\mathscr{C}_n$ and $\tilde{\mathscr{C}}_{n,n}$ by $\tilde{\mathscr{C}}_n$.

So equation (1.1) can be viewed as a function $c$ from $M$ to $\mathscr{C}_n$, which is defined by

$$c\colon M \longrightarrow \mathscr{C}_n;\ c(p) \longmapsto (c_{n-1}(p), \cdots, c_1(p), c_0(p))$$

We can reformulate some classical general $\sigma_k$ equations or general inverse $\sigma_k$ equations into a function (or a constant map) from $M$ to $\mathscr{C}_{n,s}$. For example, let us reformulate the dHYM equation here and state one of the conjectures by Collins–Jacob–Yau [18].

*Conjecture* 1.1 (Reformulate deformed Hermitian–Yang–Mills equation). Let $(M, \omega)$ be a Kähler manifold with Kähler form $\omega$ and $[\chi_0]$ be a $(1,1)$-Dolbeault class. The deformed Hermitian–Yang–Mills equation with $\theta \in \big((n-2)\pi/2, n\pi/2\big)$ induces a point in $\tilde{\mathscr{C}}_n$ and we consider the following constant map $c_{\mathrm{dHYM}}\colon M \longrightarrow \tilde{\mathscr{C}}_n$. If there exists a $C$-subsolution to

$c_{\mathrm{dHYM}}$ in $[\chi_0]$, then there exists a $\chi \in [\chi_0]$ such that

$$\Im\big(\omega+\sqrt{-1}\chi\big)^n = \tan\big(\theta\big)\cdot \Re\big(\omega+\sqrt{-1}\chi\big)^n.$$

In Section 3.2, assuming we have a $C$-subsolution to an equation $d\colon M \to \mathscr{C}_{n,s}$, then we study the equations such that this subsolution is still a $C$-subsolution to them. By collecting these equations and by Theorem 1.3, we have an explicit expression of this set.

**Theorem 1.4.** *Given $d\colon M \to \mathscr{C}_{n,s}$, at any point $p \in M$, we write $x_k(p)$ the largest real root of the $k$-th derivative of $f_p(x) = \binom{n}{s}x^s - \sum_{k=0}^{s-1} d_k(p)\binom{n}{k}x^k$. Then the $C$-subsolution cone of $d$ at $p$ is contained in the $C$-subsolution cone of $c\colon M \to \mathscr{C}_{n,s}$ at $p$ if and only if for all $k \in \{1,\cdots,s-1\}$, we have $g_p^{(k)}(x_k(p)) \geq 0$. Here, $g_p(y) = \binom{n}{s}y^s - \sum_{k=0}^{s-1} c_k(p)\binom{n}{k}y^k$.*

It is still open whether the existence of a $C$-subsolution will provide a priori estimates, but the space of equations sharing same subsolution as $C$-subsolution is still worth considering. This space should be the space to find continuity path and apply a priori estimates. In Section 3.3, we show that the continuity path in Yau [72] and Collins–Székelyhidi [20] will lie in this space. In Chapter 4, we study a priori estimates of constant maps $d\colon M \to \tilde{\mathscr{C}_3}$ and $d\colon M \to \tilde{\mathscr{C}_4}$ provided the existence of a $C$-subsolution. We have the following result.

**Theorem 1.5** (A priori estimates)**.** *Let $S$ be a compact subset of the generic stratification of $\tilde{\mathscr{C}_4}$, $X$ be a $C$-subsolution to constant map $d\colon M \to \tilde{\mathscr{C}_4}$. If $X$ is again a $C$-subsolution to a constant map $c\colon M \to \tilde{\mathscr{C}_4}$ with $c \in S$ and $u\colon M \to \mathbb{R}$ is a solution to $c$. Then for every $\alpha \in (0,1)$, we have*

$$\|\partial\bar{\partial}u\|_{C^{2,\alpha}} \leq C\big(M, X, S, d, \omega, \alpha\big).$$

In Chapter 5, we apply our a priori estimates and find a continuity path connecting the original equation to a solvable one. In conclusion, we prove that if $d$ is in the generic open stratification of $\tilde{\mathscr{C}_4}$ and there exists a $C$-subsolution to $d$, then the degree four general inverse $\sigma_k$ equation $d\colon M \to \tilde{\mathscr{C}_4}$ with $d(p) = d$ is solvable. This result covers one of the author's work in [51].

**Theorem 1.6** (Solvability when $n = 4$)**.** *Suppose $d$ is in the generic open stratification of $\tilde{\mathscr{C}_4}$. If there exists a $C$-subsolution to $d$, then the degree four general inverse $\sigma_k$ equation*

$$X^4 - 6d_2\omega^2 \wedge X^2 - 4d_1\omega^3 \wedge X - d_0\omega^4 = 0.$$

*is solvable in the same cohomology class.*

**Theorem 1.7** (Deformed Hermitian–Yang–Mills equation, Lin [51])**.** *When the complex dimension equals three or four, Conjecture 1.1 is confirmed.*

The layout of this dissertation is as follows: in Chapter 2, we introduce some preliminary knowledge. We introduce a special class of univariate polynomials and a special class of multivariate polynomials. Moreover, we show that these special classes are related to the convexity of general $\sigma_k$ equation. By collecting these general $\sigma_k$ equations we get a special algebraic set, which is related to convexity. In Chapter 3, we reformulate general $\sigma_k$ equations into our framework. We may view these equations as functions mapping from the manifold $M$ to the special algebraic set introduced in Chapter 2. We also discuss a potential space to look for continuity paths and give some evidences showing that this potential space is compatible with some classical works. In Chapter 4, we study a priori estimates of general inverse $\sigma_k$ equations when the dimension is three or four. We show that the potential space is truly the right space because for any point in this space (which corresponds to a general inverse $\sigma_k$ equation) we have a priori estimates for it. In Chapter 5, we apply our results in Chapter 4 and conclude the solvability of degree three and degree four general inverse $\sigma_k$ equations with constant coefficients provided the existence of a $C$-subsolution.

# Chapter 2

# Preliminaries

In this chapter, we will outline some conventions, definitions, and results regarding algebra and complex geometry. Some ideas and details can be found in [51, 52] by the author. In Section 2.1, we introduce the class of right-Noetherian polynomials, which is related to the largest real roots of the derivatives of the polynomials. In Section 2.2, we consider some special semialgebraic sets in real algebraic geometry, which are defined by system of inequalities of polynomials with real coefficients. More precisely, we introduce the notion of $\Upsilon$-cones, which is a generalization of the $C$-subsolution cone introduced by Székelyhidi [68] and Guan [36]. In Section 2.3, we prove the convexity of the level sets introduced in Section 2.2. In Section 2.4, we apply our result in Section 2.3 to some classical examples. For example, when the degree is low, we can use the resultant and the discriminant to verify the convexity. In Section 2.5, we state some lemmas for symmetric functions.

## 2.1 Right-Noetherian Polynomials

In this section, we introduce the class of Noetherian polynomials, which will be used throughout this dissertation. The class of Noetherian polynomials has some special properties and will help us determine the convexity of the level set of any general inverse $\sigma_k$ type equation and any general $\sigma_k$ type equation. We will see this in the later sections.

**Definition 2.1** (Noetherian polynomial)**.** We say a degree $n$ real univariate polynomial $p(x)$ is right-Noetherian if for all $k \in \{0, \cdots, n-1\}$, there exists a real root of $p^{(k)}$ which is greater than or equal to the largest real root of $p^{(k+1)}$. Here, $p^{(k)}$ is the $k$-th derivative of $p$. We say a right-Noetherian polynomial $p(x)$ is strictly right-Noetherian if the largest real root of $p(x)$ is strictly greater than the largest real root of $p'(x)$.

As a consequence, we immediately have the following descending relation.

**Proposition 2.1.** *Let $p(x)$ be a real univariate polynomial of degree $n$ which is right-Noetherian. Then for any $k \in \{0, \cdots, n-1\}$, there exists a unique (ignoring multiplicity) real root of $p^{(k)}(x)$ which is greater than or equal to the largest real root of $p^{(k+1)}(x)$. Moreover, this real root is the largest real root of $p^{(k)}(x)$. In particular, if we denote $x_k$ to be the largest real root of $p^{(k)}(x)$, then $x_0 \geq x_1 \geq \cdots \geq x_{n-1}$.*

*Proof.* We prove this statement by mathematical induction on the degree $n$. When $n = 1$, there is nothing to prove. When $n = 2$, $p'(x)$ is a degree 1 polynomial and the only root will be the midpoint of the roots of $p(x)$. By the definition of right-Noetherianness, there exists a real root of $p(x)$ which is greater than or equal to the largest real root of $p'(x)$. Thus $p(x)$ is real rooted and if we ignore the multiplicity, then there exists a unique real root of $p(x)$ which is greater than or equal to the largest real root of $p'(x)$. Moreover, this real root will be the largest real root of $p(x)$. Suppose the statement is true when $n = m - 1$. When $n = m$, it suffices to check $p(x)$, the rest follows by mathematical induction. If there exists

$x_0$ and $\tilde{x}_0$ with $p(x_0) = 0 = p(\tilde{x}_0)$ and $\tilde{x}_0 > x_0 \geq x_1$, where $x_1$ is the largest real root of $p'(x)$. For convenience, we assume that the polynomial $p(x)$ is monic, we may write

$$p(x) = x^n + \sum_{k=0}^{n-1} c_k x^k.$$

Then $p'(x) = nx^{n-1} + \sum_{k=1}^{n-1} kc_k x^{k-1}$. Since $x_1$ is the largest real root of $p'(x)$, for any $x > x_1$, we have $p'(x) > 0$. By the fundamental theorem of calculus, we have

$$0 = p(\tilde{x}_0) = p(x_0) + \int_{x_0}^{\tilde{x}_0} p'(x)dx = 0 + \int_{x_0}^{\tilde{x}_0} p'(x)dx > 0,$$

which is a contradiction. This finishes the proof. $\square$

We give a quick example of right-Noetherian polynomial, the right-Noetherianness condition is checkable by any computer using long division algorithm and Sturm's theorem.

**Example 2.1.** The following univariate polynomial $r_f(x) = x^5 - \sum_{k=0}^{3} c_k \binom{5}{k} x^k$ with $c_3 = 19, c_2 = -64, c_1 = 9$, and $c_0 = -20$ is strictly right-Noetherian. This is checkable using any computer. By rounding off to the third decimal place, for $k \in \{0, \cdots, 4\}$, we have

$$x_0 \sim 11.632,\ x_1 \sim 9.306,\ x_2 \sim 6.909,\ x_3 \sim 4.359,\ x_4 = 0.$$

Here, we denote by $x_k$ the largest real root of the $k$-th derivative $r_f^{(k)}(x)$.

**Proposition 2.2.** *Let $p(x)$ be a real univariate polynomial of degree $n$ which is real rooted, that is, all roots are real numbers, then $p(x)$ is right-Noetherian.*

*Proof.* This follows immediately by the Gauss–Lucas theorem. If $p(x)$ is real rooted, then the roots of $p'(x)$ will be contained in the convex hull of the set of roots of $p(x)$. So $p'(x)$ will also be real rooted, the rest follows directly by mathematical induction. This finishes the proof. $\square$

*Remark* 2.1. A right-Noetherian polynomial might not be real rooted, a simple example will be $p(x) = x^3 - 1$. Then we have $p'(x) = 3x^2$ and $p''(x) = 6x$. If we denote by $x_i$ the largest real root of $p^{(i)}(x)$, then $x_0 = 1$, $x_1 = 0$, and $x_2 = 0$. So $p(x)$ will be right-Noetherian due to $x_0 \geq x_1 \geq x_2$. But the roots of $p(x) = x^3 - 1$ are: $1, (-1 + \sqrt{-3})/2$, and $(-1 - \sqrt{-3})/2$.

The log-concavity property of special univariate or multivariate polynomials were studied extensively by Brändén–Huh [7], Gurvits [41], Anari–Gharan–Vinzant [2], Anari–Liu–Gharan–Vinzant [3, 4], and Anari–Liu–Gharan–Vinzant–Vuong [5]. For the class of right-Noetherian polynomials, here, we not only show that any right-Noetherian polynomial will be strongly log-concave after translation, but we also show that the ratio will be monotone.

**Definition 2.2** (Log-concavity ratio)**.** Let $f\colon I \to \mathbb{R}$ be an analytic function, $I$ be an open interval in $\mathbb{R}$, and define $C_f := \{x \in I\colon f'(x) = 0\}$. For any point $x \in I$, we define the log-concavity ratio $\alpha_f(x)$ of $f(x)$ to be the following

$$\alpha_f(x) := \frac{f(x) \cdot f''(x)}{f'(x)^2} \tag{2.1}$$

if $x \notin C_f$. If $x \in C_f$ is a limit point of $C_f$, then we define $\alpha_f(x)$ to be 0, otherwise we define

$$\alpha_f(x) := \lim_{y \to x} \frac{f(y) \cdot f''(y)}{f'(y)^2},$$

where we allow $\alpha_f(x) = \infty$ or $\alpha_f(x) = -\infty$.

*Remark* 2.2. Let $f\colon I \to \mathbb{R}$ be an analytic function and $I$ be an open interval in $\mathbb{R}$, if $\alpha_f(x) \leq 1$ for all $x \in I$, then $f$ is logarithmically concave on $\{f > 0\}$.

The following Proposition 2.3 shows that for any real univariate polynomial $p(x)$, $p(x)$ (or $-p(x)$) will eventually be logarithmically concave when $x$ is sufficiently large.

**Proposition 2.3.** *Let $p$ be a real univariate polynomial of degree $n$, then* $\lim_{x\to\infty} \alpha_p(x) = 1 - 1/n$. *In particular, there exists a $N > 0$ sufficiently large such that $p$ (or $-p$ depends on* *the sign of the leading coefficient) is logarithmically concave on* $(N, \infty)$.

*Proof.* For $x$ sufficiently large, if we write $p(x) = \sum_{k=0}^{n} c_k x^k$, then by equation (2.1), we have

$$\alpha_p(x) = \frac{p(x)\cdot p''(x)}{p'(x)^2} = \frac{\sum_{k=0}^{n} c_k x^k \cdot \sum_{k=2}^{n} k(k-1)c_k x^{k-2}}{\left(\sum_{k=1}^{n} k c_k x^{k-1}\right)^2}.$$

Since $p'$ is a polynomial, by letting $x$ approach $\infty$, we can avoid critical points and get

$$\begin{aligned}\lim_{x\to\infty} \alpha_p(x) &= \lim_{x\to\infty} \frac{\sum_{k=0}^{n} c_k x^k \cdot \sum_{k=2}^{n} k(k-1)c_k x^{k-2}}{\left(\sum_{k=1}^{n} k c_k x^{k-1}\right)^2} = \lim_{x\to\infty} \frac{n(n-1)x^{2n-2} + O(x^{2n-3})}{n^2 x^{2n-2} + O(x^{2n-3})} \\ &= \lim_{x\to\infty} \frac{n-1+O(x^{-1})}{n+O(x^{-1})} = 1 - \frac{1}{n}.\end{aligned}$$

Here, we use the Big O notation for convenience. This finishes the proof. $\square$

The derivative of the log-concavity ratio in Definition 2.2 of any right-Noetherian polynomial will satisfy the following.

**Lemma 2.1.** *Let* $p(x)$ *be a right-Noetherian polynomial of degree* $n$. *For* $k \in \{1, \cdots, n-1\}$ *and* $x > x_k$, *where* $x_k$ *is the largest real root of* $p^{(k)}$, *then* $\alpha'_{p^{(k-1)}}(x) < 0$ *when* $2 > \alpha_{p^{(k)}}(x)$ *and* $\alpha_{p^{(k-1)}}(x) > 1/(2 - \alpha_{p^{(k)}}(x))$. *On the other hand,* $\alpha'_{p^{(k-1)}}(x) > 0$ *when* $\alpha_{p^{(k)}}(x) \geq 2$ *or* $2 > \alpha_{p^{(k)}}(x)$ *and* $1/(2 - \alpha_{p^{(k)}}(x)) > \alpha_{p^{(k-1)}}(x)$.

*Proof.* By taking the derivative of $\alpha_{p^{(k-1)}}(x)$ with respect to $x$, we get

$$\begin{aligned}\alpha'_{p^{(k-1)}}(x) &= \frac{d}{dx}\frac{p^{(k-1)}(x)p^{(k+1)}(x)}{p^{(k)}(x)^2} \\ &= \frac{p^{(k)}(x)^2 p^{(k+1)}(x) + p^{(k-1)}(x)p^{(k)}(x)p^{(k+2)}(x) - 2p^{(k-1)}(x)p^{(k+1)}(x)^2}{p^{(k)}(x)^3}.\end{aligned} \tag{2.2}$$

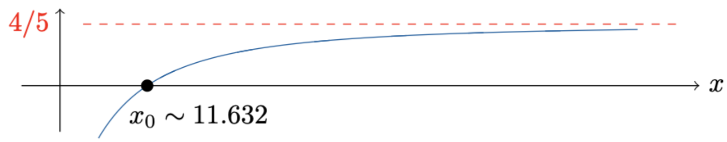


Figure 2.1: $\alpha_p(x)$ of $p(x) = x^5 - 19\binom{5}{3}x^3 + 64\binom{5}{2}x^2 - 9\binom{5}{1}x^1 + 20\binom{5}{0}x^0$.

Then the numerator of equation (2.2) will give us the following.

$$\begin{aligned} &p^{(k)}(x)^2 p^{(k+1)}(x) + p^{(k-1)}(x)p^{(k)}(x)p^{(k+2)}(x) - 2p^{(k-1)}(x)p^{(k+1)}(x)^2 \\ &\quad = p^{(k)}(x)^2 p^{(k+1)}(x)\Big(1 + \big(\alpha_{p^{(k)}}(x) - 2\big)\alpha_{p^{(k-1)}}(x)\Big). \end{aligned}$$

For convenience, we assume the leading coefficient of $p$ is positive. When $2 > \alpha_{p^{(k)}}(x)$ and $\alpha_{p^{(k-1)}}(x) > 1/(2 - \alpha_{p^{(k)}}(x))$, then since $x$ is greater than the largest real root of $p^{(k)}$, we get

$$\alpha'_{p^{(k-1)}}(x) = p^{(k+1)}(x)\frac{1 + \big(\alpha_{p^{(k)}}(x) - 2\big)\alpha_{p^{(k-1)}}(x)}{p^{(k)}(x)} < 0.$$

On the other hand, when $\alpha_{p^{(k)}}(x) \geq 2$ or $2 > \alpha_{p^{(k)}}(x)$ and $1/(2 - \alpha_{p^{(k)}}(x)) > \alpha_{p^{(k-1)}}(x)$,

$$\alpha'_{p^{(k-1)}}(x) = p^{(k+1)}(x)\frac{1 + \big(\alpha_{p^{(k)}}(x) - 2\big)\alpha_{p^{(k-1)}}(x)}{p^{(k)}(x)} > 0.$$

This finishes the proof. $\square$

Now, with all these preparations, we are able to prove the following important result for the class of right-Noetherian polynomials. We prove that this log-concavity ratio will be monotonic for right-Noetherian polynomials. In Figure 2.1, we plot the log-concavity ratio $\alpha_p(x)$ of $p(x) = x^5 - 19\binom{5}{3}x^3 + 64\binom{5}{2}x^2 - 9\binom{5}{1}x^1 + 20\binom{5}{0}x^0 = x^5 - 190x^3 + 640x^2 - 45x + 20$, which is right-Noetherian by Example 2.1.

**Theorem 2.1** (Monotonicity of log-concavity ratio)**.** *Let $p(x)$ be a right-Noetherian polynomial of degree $n$. Then the log-concavity ratio $\alpha_p(x)$ of $p(x)$ is monotonically increasing on $(x_1, \infty)$ with value from $-\infty$ to $1 - 1/n$ if $x_0 > x_1$ and on $[x_0, \infty)$ with value from $1 - 1/m$*

*to $1-1/n$ if $x_0 = x_1 = \cdots = x_{m-1}$. Here, $x_0$ is the largest real root of $p$, $x_k$ is the largest real root of the $k$-th derivative $p^{(k)}$ of $p$, and $m$ is the multiplicity of $p$ at $x_0$. In particular, if $p$ is right-Noetherian, then $p$ is always logarithmically concave when $x > x_0$.*

*Proof.* For convenience, we may assume that $p(x)$ is monic. By the definition of right-Noetherian, if we denote $x_i$ by the largest real root of $p^{(i)}(x)$, then by Proposition 2.1, we have $x_0 \geq x_1 \geq x_2 \geq \cdots \geq x_{n-1}$. We use mathematical induction on the degree of the polynomial. When the degree equals two, since $p$ has a real root $x_0$, we may write

$$p(x) = (x - x_0)(x - (2x_1 - x_0)) \tag{2.3}$$

with $x_0 \geq x_1$. If $x > x_1$, then by equation (2.3), we obtain

$$\alpha_p(x) = \frac{p(x)p''(x)}{p'(x)^2} = \frac{2(x-x_0)(x-(2x_1-x_0))}{4(x-x_1)^2} = \frac{1}{2} - \frac{(x_0-x_1)^2}{2(x-x_1)^2}.$$

So, for degree 2 right-Noetherian polynomials, $\alpha_p(x)$ is monotonically increasing from 0 to 1/2 from $x_0$ to $\infty$ if $x_0 = x_1$ and from $-\infty$ to 1/2 from $x_1$ to $\infty$ if $x_0 > x_1$. Suppose the statement holds when the degree equals $n-1$. When the degree equals $n$, say the multiplicity of the largest real root $x_0$ equals $m \geq 1$. We have

$$p(x) = (x-x_0)^m \cdot \sum_{k=m}^{n} \frac{p^{(k)}(x_0)}{k!}(x-x_0)^{k-m} = (x-x_0)^m \tilde{p}(x) \tag{2.4}$$

by using the Taylor series expansion of $p(x)$ at $x_0$ and set

$$\tilde{p}(x) \coloneqq \sum_{k=m}^{n} \frac{p^{(k)}(x_0)}{k!}(x-x_0)^{k-m}.$$

So the first and the second derivative of (2.4) with respect to $x$ can be written as

$$p'(x) = (x - x_0)^{m-1}\big(m\tilde{p}(x) + (x - x_0)\tilde{p}'(x)\big); \tag{2.5}$$

$$p''(x) = (x - x_0)^{m-2}\big(m(m-1)\tilde{p}(x) + 2m(x - x_0)\tilde{p}'(x) + (x - x_0)^2\tilde{p}''(x)\big). \tag{2.6}$$

There are two cases to consider: $m = 1$ or $m \geq 2$. For the case $m = 1$, since $p'$ is again right-Noetherian, $\alpha_{p'}$ is monotonically increasing on $[x_1, \infty)$ with value from $1 - 1/\tilde{m}$ to $1 - 1/(n-1)$, where $\tilde{m}$ is the multiplicity of $p'$ at $x_1$. When $x > x_1$, by Lemma 2.1, $\alpha_p'(x) > 0$ when $2 > \alpha_{p'}(x)$ and $1/(2 - \alpha_{p'}(x)) > \alpha_p(x)$. If we consider the set $I \coloneqq \big\{x \in (x_1, \infty) \colon 1/(2-\alpha_{p'}(x)) > \alpha_p(x)\big\}$, then the set $I$ is not empty because $x_0 \in I$. If we can show that $I = (x_1, \infty)$, then we are done. $I$ will be open by the continuity of functions $\alpha_p$ and $\alpha_{p'}$. If $I \neq (x_1, \infty)$, then we can find a smallest $\tilde{x} \in (x_1, \infty)$ such that $1/(2 - \alpha_{p'}(\tilde{x})) = \alpha_p(\tilde{x})$. This is ensured because $p$ is a polynomial and

$$\begin{aligned}\frac{1}{2 - \alpha_{p'}(x)} &- \alpha_p(x) \\ &= \frac{1}{(2 - \alpha_{p'}(x))p'(x)^2p''(x)}\Big(p'(x)^2p''(x) + p(x)p'(x)p'''(x) - 2p(x)p''(x)^2\Big).\end{aligned}$$

The second term $p'(x)^2p''(x) + p(x)p'(x)p'''(x) - 2p(x)p''(x)^2$ is just a polynomial so can only have finitely many zeros. Then by Lemma 2.1, at the point $\tilde{x}$, we have

$$\frac{d}{dx}\alpha_p(\tilde{x}) = \frac{p''(\tilde{x})\big(1 - (2 - \alpha_{p'}(\tilde{x}))\alpha_p(\tilde{x})\big)}{p'(\tilde{x})} = 0. \tag{2.7}$$

On the other hand, since $p'$ is right-Noetherian and by mathematical induction, we have

$$\frac{d}{dx}\Big|_{x=\tilde{x}} \frac{1}{2 - \alpha_{p'}(x)} = \frac{\alpha_{p'}'(\tilde{x})}{(2 - \alpha_{p'}(\tilde{x}))^2} > 0. \tag{2.8}$$

We get a contradiction. Otherwise, by standard calculus argument and equation (2.7), there exists a $\delta > 0$ sufficiently small such that if $|h| < \delta$, then we have

$$-\frac{1}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2} < \frac{\alpha_p(\tilde{x}) - \alpha_p(\tilde{x}-h)}{h} < \frac{1}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2}. \tag{2.9}$$

Also, since $\left(\frac{1}{2-\alpha_{p'}(x)}\right)' = \frac{\alpha'_{p'}(x)}{(2-\alpha_{p'}(x))^2}$, by inequality (2.8), for $\delta$ sufficiently small, we get

$$\frac{2}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2} < \frac{\frac{1}{2-\alpha_{p'}(\tilde{x})} - \frac{1}{2-\alpha_{p'}(\tilde{x}-h)}}{h} < \frac{4}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2}. \tag{2.10}$$

Since $\tilde{x}$ is the smallest value such that $1/(2-\alpha_{p'}(x)) = \alpha_p(x)$ and $x \in I$ for $x$ slightly larger than $x_1$, by the intermediate value theorem, we have $\tilde{x} - h \in I$ where $\delta > h > 0$. Hence, by inequalities (2.9) and (2.10), we obtain

$$\frac{1}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2} > \frac{\alpha_p(\tilde{x}) - \alpha_p(\tilde{x}-h)}{h} > \frac{\frac{1}{2-\alpha_{p'}(\tilde{x})} - \frac{1}{2-\alpha_{p'}(\tilde{x}-h)}}{h} > \frac{2}{3}\frac{\alpha'_{p'}(\tilde{x})}{(2-\alpha_{p'}(\tilde{x}))^2}.$$

This is a contradiction because $\alpha'_{p'}(x) > 0$ by mathematical induction. So $I = (x_1, \infty)$. If the multiplicity of $x_0$ equals 1, then $\alpha_p$ is increasing on $(x_1, \infty)$ with value from $-\infty$ to $1 - 1/n$.

For the second case, if the multiplicity of $x_0$ is greater than or equal to 2, then at $x_0$ we have $\alpha_p(x_0) = 1 - 1/m$ and $\alpha_{p'}(x_0) = 1 - 1/(m-1)$. This gives

$$\frac{1}{2-\alpha_{p'}(x_0)} = \frac{1}{2-(1-1/(m-1))} = 1 - \frac{1}{m} = \alpha_p(x_0).$$

We need to do some local analysis near $x_0$. Similar to before, we only need to consider the term $p'(x)^2p''(x) + p(x)p'(x)p'''(x) - 2p(x)p''(x)^2$. By equations (2.5) and (2.6), we get

$$\begin{aligned}
&p'(x)^2p''(x) + p(x)p'(x)p'''(x) - 2p(x)p''(x)^2 \\
&\quad = 2m(x-x_0)^{3m-3}\tilde{p}^2\tilde{p}' + 4m(x-x_0)^{3m-2}\big(\tilde{p}^2\tilde{p}'' - \tilde{p}\tilde{p}'^2\big)
\end{aligned}$$

$$+ m(x-x_0)^{3m-1}\Big(2\tilde{p}'^3 + \tilde{p}^2\tilde{p}''' - 3\tilde{p}\tilde{p}'\tilde{p}''\Big) + (x-x_0)^{3m}\Big(\tilde{p}\tilde{p}'\tilde{p}''' + \tilde{p}'^2\tilde{p}'' - 2\tilde{p}\tilde{p}''^2\Big).$$

When $x > x_0$ is sufficiently close to $x_0$, since $2m\tilde{p}(x)^2\tilde{p}'(x) > 0$, we get

$$p'(x)^2p''(x) + p(x)p'(x)p'''(x) - 2p(x)p''(x)^2 > 0.$$

Similarly, we define the set $I \coloneqq \Big\{x \in (x_0, \infty)\colon 1/(2-\alpha_{p'}(x)) > \alpha_p(x)\Big\}$ which is open and non-empty. Same as the previous argument, we get $I = (x_0, \infty)$, which implies that $\alpha_p$ is increasing on $[x_0, \infty)$ with value from $1 - 1/m$ to $1 - 1/n$. This finishes the proof. $\square$

As an application, we immediately obtain that for a right-Noetherian polynomial $p(x)$, the roots of $p(x)$, the roots of $p'(x)$, and the root of $p''(x)$ will satisfy the following relation.

**Proposition 2.4.** *Let $p(x)$ be a right-Noetherian polynomial of degree $n$. If we denote all the roots of $p(x)$ by $\alpha_1, \cdots, \alpha_n$, all the roots of $p'(x)$ by $\beta_1, \cdots, \beta_{n-1}$, and all the roots of $p''(x)$ by $\gamma_1, \cdots, \gamma_{n-2}$. If we write $x_k$ the largest real root of $p^{(k)}(x)$, then for $x > x_1$,*

$$\frac{\prod_{i=1}^{n}(x-\alpha_i)\cdot\prod_{i=1}^{n-2}(x-\gamma_i)}{\prod_{i=1}^{n-1}(x-\beta_i)^2}$$

*is monotonically increasing to 1 when $x$ approaches infinity.*

**Proposition 2.5.** *Let $p(x)$ be a right-Noetherian polynomial of degree $n$, then*

$$1 - \frac{2}{n} > \alpha_p(x)\alpha_{p'}(x)$$

*for $x \geq x_1$, where $x_1$ is the largest real root of $p'(x)$. In particular, for $x \geq x_1$,*

$$(n-2)p'(x)p''(x) \geq np(x)p'''(x).$$

*Proof.* By Theorem 2.1, for $x > x_1$, we have

$$1 - \frac{1}{n} > \alpha_p(x) > -\infty \quad \text{and} \quad 1 - \frac{1}{n-1} > \alpha_{p'}(x) \geq 0.$$

By multiplying them together, we always get the following upper bound: $1 - \frac{2}{n} > \alpha_p(x)\alpha_{p'}(x)$. Moreover, for $x = x_1$, since $p$ is right-Noetherian, we get

$$\frac{p'(x_1)p''(x_1)}{n} - \frac{p(x_1)p'''(x_1)}{n-2} = -\frac{p(x_1)p'''(x_1)}{n-2} \geq 0.$$

Also, for $x > x_1$, we obtain

$$\begin{aligned}\frac{p'(x)p''(x)}{n} - \frac{p(x)p'''(x)}{n-2} &= \frac{p'(x)p''(x)}{n-2}\Big(\frac{n-2}{n} - \frac{p(x)p''(x) \cdot p'(x)p'''(x)}{p'(x)^2 \cdot p''(x)^2}\Big) \\ &= \frac{p'(x)p''(x)}{n-2}\Big(\frac{n-2}{n} - \alpha_p(x)\alpha_{p'}(x)\Big) > 0.\end{aligned}$$

This finishes the proof. □

For the class of right-Noetherian polynomials, we show that this class will be strongly log-concave after translation. We state the definition here.

**Definition 2.3** (Strongly log-concave)**.** Let $p(x_1, \cdots, x_n)$ be a multivariate polynomial, we say $p$ is strongly log-concave if any order partial derivative is either identically zero or log-concave on $\mathbb{R}^n_{>0}$.

**Lemma 2.2.** *Let $p(x)$ be a real univariate polynomial of degree $n$ which is right-Noetherian, then $p$ is strongly log-concave after the translation $x \mapsto x - x_0$.*

*Proof.* This follows directly by Theorem 2.1. □

## 2.2 General $\sigma_k$ Equations and $\Upsilon$-Cones

In this section, we introduce the notion of $\Upsilon$-cones, which is an extension of the $C$-subsolution cone introduced by Székelyhidi [68] and Guan [36]. The arguments in this subsection might be tedious because any set in this section might have more than one connected components, we need to specify which connected component we are considering. After all the arguments in this subsection, there will be no ambiguity, so we may assume the set is the connected component we are interested in. We prove some Positivstellensatz type results and a Newton–Maclaurin type inequality on the $\Upsilon$-cones.

First, let us state some widely used notations, see Spruck [67] for more details. For an $n$-tuple numbers $\lambda = \{\lambda_1, \cdots, \lambda_n\}$, for $k \in \{1, \cdots, n\}$, the $k$-th elementary symmetric polynomial $\sigma_k(\lambda)$ of $\lambda$ will be

$$\sigma_k(\lambda) \coloneqq \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We also define $\sigma_0(\lambda) \coloneqq 1$ for convenience. For $l \in \{1, \cdots, n\}$ and pairwise distinct indices $i_1, \cdots, i_l$, where $i_j \in \{1, \cdots, n\}$ for all $j \in \{1, \cdots, l\}$, we denote the set $\lambda - \{\lambda_{i_1}, \cdots, \lambda_{i_l}\}$ by $\lambda_{;i_1, \cdots, i_l}$. In this section, we consider the following multilinear polynomial

$$\sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda). \tag{2.11}$$

We call a multilinear polynomial having the same format as polynomial (2.11) a general $\sigma_k$ type multilinear polynomial.

*Remark* 2.3. When $s = n$, we call this general $\sigma_k$ type multilinear polynomial a general inverse $\sigma_k$ type multilinear polynomial. In [52], the author has shown some results for general inverse $\sigma_k$ type multilinear polynomials. Here, we generalize and obtain more results for general $\sigma_k$ type multilinear polynomials.

**Lemma 2.3.** *By doing the substitution $\mu_i = \lambda_i - c_{s-1}/(n-s+1)$ for all $i \in \{1, \cdots, n\}$, then for all $k \in \{0, \cdots, n\}$, we have*

$$\sigma_k(\lambda) = \sum_{j=0}^{k} \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} \sigma_j(\mu)$$

*and the coefficients $d_j$ for $j \in \{1, \cdots, s-1\}$ will be*

$$d_j = \sum_{k=j}^{s-1} c_k \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} - \frac{c_{s-1}^{s-j}}{(n-s+1)^{s-j}} \binom{n-j}{s-j}.$$

*In addition, after substitution, the original general $\sigma_k$ type multilinear polynomial becomes*

$$\sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda) = \sigma_s(\mu) - \sum_{j=0}^{s-2} d_j \sigma_j(\mu)$$

*and for all positive integer $l$ and $i_a \in \{1, \cdots, n\}$ for all $a \in \{1, \cdots, l\}$, we have*

$$\frac{\partial^l}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_l}} \Big( \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda) \Big) = \frac{\partial^l}{\partial \mu_{i_1} \cdots \partial \mu_{i_l}} \Big( \sigma_s(\mu) - \sum_{j=0}^{s-2} d_j \sigma_j(\mu) \Big).$$

*Proof.* First, by doing the substitution, we have $\lambda = \mu + c_{s-1}/(n-s+1)$. Hence,

$$\sigma_k(\lambda) = \sigma_k\Big(\mu + \frac{c_{s-1}}{n-s+1}\Big) = \sum_{j=0}^{k} \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} \sigma_j(\mu).$$

Thus, after the substitution, we get

$$\begin{aligned}
&\sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda) \\
&\quad = \sum_{j=0}^{s} \frac{c_{s-1}^{s-j}}{(n-s+1)^{s-j}} \binom{n-j}{s-j} \sigma_j(\mu) - \sum_{k=0}^{s-1} c_k \sum_{j=0}^{k} \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} \sigma_j(\mu) \\
&\quad = \sigma_s(\mu) - \sum_{j=0}^{s-2} \Big( \sum_{k=j}^{s-1} c_k \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} - \frac{c_{s-1}^{s-j}}{(n-s+1)^{s-j}} \binom{n-j}{s-j} \Big) \sigma_j(\mu).
\end{aligned}$$

So, for any $j \in \{1, \cdots, s-2\}$, we have

$$d_j = \sum_{k=j}^{s-1} c_k \frac{c_{s-1}^{k-j}}{(n-s+1)^{k-j}} \binom{n-j}{k-j} - \frac{c_{s-1}^{s-j}}{(n-s+1)^{s-j}} \binom{n-j}{s-j}.$$

The rest follows by the change of variables formula. This finishes the proof. □

For convenience, by above Lemma 2.3, we may assume that $c_{s-1} = 0$ by doing this substitution. In most of the proofs in this section, we will do this substitution to simplify our proofs. We consider the following general $\sigma_k$ type multilinear polynomial instead.

$$f(\lambda) = f(\lambda_1, \cdots, \lambda_n) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda). \tag{2.12}$$

Now, we state the definition of $C$-subsolution here which was introduced by Székelyhidi [68] and Guan [36]. We will slightly adjust the settings in [68] to meet our settings.

**Definition 2.4** ($C$-Subsolution. Székelyhidi [68], Guan [36], and Trudinger [69])**.** Consider an equation $f(\lambda_1, \cdots, \lambda_n) = h$, where $f(\lambda_1, \cdots, \lambda_n)$ is a smooth symmetric function of variables $\{\lambda_1, \cdots, \lambda_n\}$. We assume that $f$ is defined in an open symmetric cone $\Gamma_f \subset \mathbb{R}^n$ satisfying $f > 0$, $\partial f / \partial \lambda_i > 0$ for all $i \in \{1, \cdots, n\}$ on $\Gamma_f$, and $\sup_{\partial \Gamma_f} f < h$. We say that $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{R}^n$ is a $C$-suboslution to the equation $f = h$ if the following set

$$F^h(\mu) \coloneqq \big\{ \lambda \colon f(\lambda) = h \ \text{ and } \ \lambda - \mu = (\lambda_1 - \mu_1, \cdots, \lambda_n - \mu_n) \in \Gamma_n \big\} \tag{2.13}$$

is bounded. By collecting all the $C$-subsolutions, we call this the $C$-subsolution cone.

**Definition 2.5** (Alternative definition of Definition 2.4. Székelyhidi [68] and Trudinger [69])**.** Suppose that $f$ is defined in an open symmetric cone $\Gamma_f \subset \mathbb{R}^n$ satisfying $f > 0$, $\partial f / \partial \lambda_i > 0$ for all $i \in \{1, \cdots, n\}$ on $\Gamma_f$, and $\sup_{\partial \Gamma_f} f < h$. Define

$$\Gamma_f^h \coloneqq \big\{ \lambda \in \Gamma_f \colon f(\lambda) > h \big\}. \tag{2.14}$$

For $\mu \in \mathbb{R}^n$, set (2.13) $F^h(\mu)$ is bounded if and only if $\lim_{t\to\infty} f(\mu + te_i) > h$ for all $i \in \{1, \cdots, n\}$, where $e_i$ is the $i$-th standard vector. We denote by $\Gamma_f^{n-1,h}$ the projection of $\Gamma_f^h$ onto $\mathbb{R}^{n-1}$ by dropping the last entry. We can show that for any $\mu \in \mathbb{R}^n$, $F^h(\mu)$ is bounded if and only if $\left(\mu_{a(1)}, \cdots, \mu_{a(n-1)}\right) \in \Gamma_f^{n-1,h}$ for every $a \in S_n$.

Let $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$, we are interested in whether there exists a nice connected component of $\{f(\lambda) > 0\}$. Inspired by the work of Trudinger [69] on the Dirichlet problem (over the reals) for equations of the eigenvalues of the Hessian, the results of Caffarelli–Nirenberg–Spruck [10], and the results of Collins–Székelyhidi [20]. In [51, 52], the author introduced the $\Upsilon$-cones to keep track of the information of the original equation as much as possible. We abstractly define the following sets.

**Definition 2.6** ($\Upsilon$-cones. Lin [51, 52])**.** Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$, we denote by $\Gamma_f^{n-1}$ the projection of $\Gamma_f^n$ onto $\mathbb{R}^{n-1}$ by dropping the last entry. We define

$$\Upsilon_1 \coloneqq \left\{\mu \in \mathbb{R}^n \colon \left(\mu_{a(1)}, \cdots, \mu_{a(n-1)}\right) \in \Gamma_f^{n-1}, \quad \forall a \in S_n\right\},$$

where $S_n$ is the symmetric group. For $s - 1 \geq k \geq 2$, we define the following $\Upsilon$-cones

$$\Upsilon_k \coloneqq \left\{\mu \in \mathbb{R}^n \colon \left(\mu_{a(1)}, \cdots, \mu_{a(n-k)}\right) \in \Gamma_f^{n-k}, \quad \forall a \in S_n\right\},$$

where we define $\Gamma_f^{n-k}$ inductively by the projection of $\Gamma_f^{n+1-k}$ onto $\mathbb{R}^{n-k}$ by dropping the last entry. For convenience, sometimes we write $\Upsilon_0 \coloneqq \Gamma_f^n$.

**Definition 2.7** ($\Upsilon$-stableness)**.** Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. We say that this

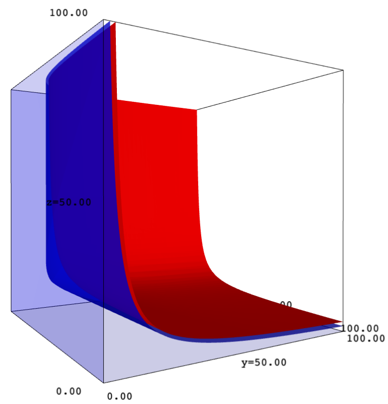


Figure 2.2: The $\Upsilon$-cones of the dHYM Equation when $n = 3$.

connected component $\Gamma_f^n$ of $f(\lambda)$ is $\Upsilon$-stable if

$$\Gamma_f^n \subseteq \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > q_{i_1,\cdots,i_{s-1}}\} \text{ for some } q = \left(q_{i_1,\cdots,i_{s-1}}\right)_{1 \le i_1 < \cdots < i_{s-1} \le n}.$$

Here, we treat $q$ as an element in the $\binom{n}{s-1}$-dimensional Euclidean space. We say that this connected component $\Gamma_f^n$ is strictly $\Upsilon$-stable if it is $\Upsilon$-stable and the boundary $\partial\Gamma_f^n$ is contained in the $\Upsilon_1$-cone.

*Remark* 2.4. Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. We will show that if $\Gamma_f^n$ is strictly $\Upsilon$-stable, then the symmetric cone $\Gamma_f$ in Definition 2.4 will always be contained in the $\Upsilon_1$-cone. Normally, we consider the largest possible $\Gamma_f$, which is in fact the $\Upsilon_1$-cone. So the $\Upsilon_1$-cone is the same as the $C$-subsolution cone introduced by Székelyhidi [68].

Above Figure 2.2 is an example of the $\Upsilon$-cones for the three-dimensional dHYM equation. The red hyperplane is the solution set $\{f(\lambda) = h\}$, and the darker blue cone in between is the boundary of the $\Upsilon_1$-cone. By Remark 2.4, the $\Upsilon_1$-cone is the $C$-subsolution cone introduced by Székelyhidi [68] and Guan [36]. Last, the outermost lighter blue cone is the boundary of the $\Upsilon_2$-cone. In fact, the $\Upsilon_2$-cone will be the positive orthant in this case.

**Lemma 2.4.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial* *and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. If $\Gamma_f^n$ is $\Upsilon$-stable, say*

$$\Gamma_f^n \subseteq \bigcap_{1\le i_1<\cdots<i_{s-1}\le n} \{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > q_{i_1,\cdots,i_{s-1}}\}$$

*with $q \in \mathbb{R}^{\binom{n}{s-1}}$, then we get*

$$\Gamma_f^n \subseteq \Upsilon_1 \subseteq \Upsilon_2 \subseteq \cdots \subseteq \Upsilon_{s-1} = \bigcap_{1\le i_1<\cdots<i_{s-1}\le n} \{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > c_{s-1}\},$$

*$c_{s-1} \geq q_{i_1,\cdots,i_{s-1}}$ for all $1 \le i_1 < \cdots < i_{s-1} \le n$. For any $l \in \{1, \cdots, s-1\}$, we define*

$$\Upsilon_l \coloneqq \bigcap_{1\le i_1<\cdots<i_l\le n} \Gamma^n_{f_{i_1\cdots i_l}} = \bigcap_{1\le i_1<\cdots<i_l\le n} \Big\{\sigma_{s-l}(\lambda_{;i_1,\cdots,i_l}) - \sum_{k=l}^{s-1} c_k \sigma_{k-l}(\lambda_{;i_1,\cdots,i_l}) > 0\Big\}.$$

*Moreover, we have $\Upsilon_l$ is open and connected. Here, we denote $\frac{\partial^l}{\partial i_1\cdots\partial i_l} f$ by $f_{i_1\cdots i_l}$.*

*Remark* 2.5. Notice that for the above Lemma 2.4, we need to specify each connected component inductively on the subindices to avoid ambiguity. For example, when $l = s-2$,

$$\begin{aligned} &\bigcap_{1\le i_1<\cdots<i_{s-2}\le n} \Big\{\sigma_2(\lambda_{;i_1,\cdots,i_{s-2}}) - \sum_{k=n-2}^{n-1} c_k \sigma_{k-(s-2)}(\lambda_{;i_1,\cdots,i_{s-2}}) > 0\Big\} \\ =& \bigcap_{1\le i_1<\cdots<i_{s-2}\le n} \{\sigma_2(\lambda_{;i_1,\cdots,i_{s-2}}) - c_{s-1}\sigma_1(\lambda_{;i_1,\cdots,i_{s-2}}) - c_{s-2} > 0\} \end{aligned}$$

will have two connected components. We specify the one which is contained in the next $\Upsilon$-cone $\Upsilon_{s-1} = \bigcap_{1\le i_1<\cdots<i_{s-1}\le n}\{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > c_{s-1}\}$. Similarly, we specify the connected component inductively til $\Upsilon_1$ by decreasing the subindices. But for notational convention, we abbreviate these expressions.

The $\Upsilon$-stableness in fact gives us some constraints on the coefficients $\{c_k\}_{k=0,\cdots,s-1}$. For example, if $\Gamma_f^n$ is $\Upsilon$-stable, then we get $\frac{n-s+2}{2(n-s+1)}c_{s-1}^2 + c_{s-2} \geq 0$. Otherwise, if $\frac{n-s+2}{2(n-s+1)}c_{s-1}^2 +$

$c_{s-2} < 0$, then

$$\bigcap_{1 \le i_1 < \cdots < i_{n-2} \le n} \left\{ \sigma_2(\lambda_{;i_1,\cdots,i_{s-2}}) - c_{s-1}\sigma_1(\lambda_{;i_1,\cdots,i_{s-2}}) - c_{s-2} > 0 \right\}$$

is not contained in $\Upsilon$-cone $\Upsilon_{s-1} = \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > c_{s-1}\}$, which violates the above Lemma 2.4. Later on, we will prove that the $\Upsilon$-stableness condition for the class of general $\sigma_k$ type multilinear polynomials is equivalent to the right-Noetherianness condition of its diagonal restriction. So the constraints can be derived using the resultants and can be written explicitly when the degree is low, see the author's work [51, 52] for more details. We will show some examples in Section 2.4 when the degree is less than or equal to four.

*Proof of Lemma 2.4.* There is nothing to prove when $s = 1$, so we may assume $s \ge 2$. We use mathematical induction on the number of variables $n$ to prove this. When $n = 1$, there is nothing to prove. When $n = 2$, we only need to consider the case $s = 2$ and this case can be done due to previous work in [52]. Suppose the statement is true when $n = m - 1$ and we assume $c_{s-1} = 0$ for convenience. When $n = m$, for the case $s = m$, this case can be done due to previous work in [52]. For $m - 1 \ge s \ge 2$, suppose that there exists $(\lambda_1, \cdots, \lambda_m) \in \Gamma_f^m$, such that

$$0 \ge \frac{\partial f}{\partial \lambda_i}(\lambda_1, \cdots, \lambda_m) = f_i(\lambda_1, \cdots, \lambda_m) = \sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i})$$

for some $i \in \{1, \cdots, m\}$. By fixing other entries, for $\tilde{\lambda}_i \le \lambda_i$, we get

$$\begin{aligned}
&\tilde{\lambda}_i\Big(\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i})\Big) + \sigma_s(\lambda_{;i}) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;i}) \\
&\quad \ge \lambda_i\Big(\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i})\Big) + \sigma_s(\lambda_{;i}) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;i}) = f(\lambda_1, \cdots, \lambda_m) > 0.
\end{aligned}$$

This implies that $(\lambda_1, \cdots, \lambda_{i-1}, \tilde{\lambda}_i, \lambda_{i+1}, \cdots, \lambda_m) \in \Gamma_f^m$ for all $\tilde{\lambda}_i \le \lambda_i$ due to the assumption that $\Gamma_f^m$ is connected. By letting $\tilde{\lambda}_i$ approach $-\infty$, we get a contradiction. Same as before, for any $i \in \{1, \cdots, m\}$, $\Gamma_f^m$ will be contained in the same connected component of

$$\Big\{f_i = \sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i}) > 0\Big\}.$$

Next, we prove that this connected component of $\big\{\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i}) > 0\big\}$ is contained in

$$\bigcap_{\{i_1 < \cdots < i_{s-2}\} \subset \{1, \cdots, i-1, i+1, \cdots n\}} \{\sigma_1(\lambda_{;i,i_1,\cdots,i_{s-2}}) > q_{i,i_1,\cdots,i_{s-2}}\}$$

by ignoring the Cartesian product $\mathbb{R}$ term. Without loss of generality, we only consider the case $i = 1$. Let $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_m) \in \Gamma_f^m$ and consider the following section

$$\lambda_1(\lambda_2, \cdots, \lambda_m) := \frac{-\sigma_s(\lambda_{;1}) + \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;1}) + f(\tilde{\lambda})}{\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1})},$$

where $f(\tilde{\lambda}) = \sigma_s(\tilde{\lambda}) - \sum_{k=0}^{s-2} c_k \sigma_k(\tilde{\lambda}) > 0$. Notice that this section $\lambda_1$ in $\Gamma_f^m$ is defined on this connected component of $\big\{\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1}) > 0\big\}$, continuous, and

$$\lambda_1(\tilde{\lambda}_2, \cdots, \tilde{\lambda}_m) = \frac{-\sigma_s(\tilde{\lambda}_{;1}) + \sum_{k=0}^{s-2} c_k \sigma_k(\tilde{\lambda}_{;1}) + f(\tilde{\lambda})}{\sigma_{s-1}(\tilde{\lambda}_{;1}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\tilde{\lambda}_{;1})} = \tilde{\lambda}_1.$$

Moreover, for any $(\lambda_2, \cdots, \lambda_m) \in \big\{\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1}) > 0\big\}$, we get

$$\lambda_1(\lambda_2, \cdots, \lambda_m)\sigma_{s-1}(\lambda_{;1}) - \lambda_1(\lambda_2, \cdots, \lambda_m) \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1}) + \sigma_s(\lambda_{;1}) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;1}) > 0.$$

Thus, if this connected component of $\big\{\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1}) > 0\big\}$ is not contained in $\bigcap_{\{i_1 < \cdots < i_{s-2}\} \subset \{2, \cdots, n\}} \{\sigma_1(\lambda_{;1,i_1,\cdots,i_{s-2}}) > q_{1,i_1,\cdots,i_{s-2}}\}$, then we get a contradiction. Hence, this connected component of $\big\{\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;i}) > 0\big\}$ is $\Upsilon$-stable and will be the unique connected component contained in the $\Upsilon_1$-cone of $\Gamma_{f_i}^{m-1}$ by mathematical induction.

We now show that the $\Upsilon_1$-cone of $\Gamma_f^m$ is exactly

$$\bigcap_{1\le i\le m}\Big\{\sigma_{s-1}(\lambda_{;i})-\sum_{k=1}^{s-2}c_k\sigma_{k-1}(\lambda_{;i})>0\Big\}.$$

Then, the rest follows from mathematical induction. By the previous arguments, we know

$$\Gamma_f^m\subseteq\bigcap_{1\le i\le m}\Big\{\sigma_{s-1}(\lambda_{;i})-\sum_{k=1}^{s-2}c_k\sigma_{k-1}(\lambda_{;i})>0\Big\}=:\Upsilon_1.$$

By mathematical induction, since $\big\{\sigma_{s-1}(\lambda_{;i})-\sum_{k=1}^{s-2}c_k\sigma_{k-1}(\lambda_{;i})>0\big\}$ is $\Upsilon$-stable, we get

$$\begin{aligned}\bigcap_{1\le i\le m}\Big\{\sigma_{s-1}(\lambda_{;i})-\sum_{k=1}^{s-2}c_k\sigma_{k-1}(\lambda_{;i})>0\Big\}&\subseteq\bigcap_{1\le i<j\le m}\Big\{\sigma_{s-2}(\lambda_{;i,j})-\sum_{k=2}^{s-2}c_k\sigma_{k-2}(\lambda_{;i,j})>0\Big\}\\&\subseteq\cdots\subseteq\bigcap_{1\le i_1<\cdots<i_{s-1}\le m}\{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}})>0\}.\end{aligned}$$

Last, similar to before, there exists a unique connected component of $\{f(\lambda)>0\}$ such that the intersection with $\Upsilon_1$ is not empty. This finishes the proof. $\square$

In the proof of Lemma 2.4, we also obtain the following result, let us list this result here.

**Lemma 2.5.** *Let $f(\lambda):=\sigma_s(\lambda)-\sum_{k=0}^{s-1}c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda)>0\}$. If $\Gamma_f^n$ is $\Upsilon$-stable, then for any $\lambda\in\Upsilon_l$, $\lambda+\overline{\Gamma}_n\subset\Upsilon_l$ for $l\in\{0,1,\cdots,s-1\}$. Here, we write $\Gamma_f^n=\Upsilon_0$ and $\overline{\Gamma}_n$ is the closure of $\Gamma_n$. As a consequence, for any $l\in\{0,1,\cdots,s-1\}$, as a set $\Upsilon_l$ will be*

$$\Upsilon_l=\bigcap_{1\le i_1<\cdots<i_l\le n}\Big\{\sigma_{s-l}(\lambda_{;i_1,\cdots,i_l})-\sum_{k=l}^{s-1}c_k\sigma_{k-l}(\lambda_{;i_1,\cdots,i_l})>0\Big\}\cap\Upsilon_{l+1}.$$

*In particular, for any $\lambda\in\big\{\sigma_s(\lambda)-\sum_{k=0}^{s-1}c_k\sigma_k(\lambda)>0\big\}\cap\Upsilon_1$, we have $\lambda\in\Gamma_f^n$.*

*Remark* 2.6. By Lemma 2.4, the $\Upsilon$-cones are defined by systems of inequalities of polynomials, so they are semialgebraic sets in real algebraic geometry.

The boundary of the $\Upsilon$-cones will have the following relations.

**Lemma 2.6.** *Let $f(\lambda) := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$. If $\Gamma_f^n$ is $\Upsilon$-stable and $\Upsilon_l \neq (c_{s-1}, \cdots, c_{s-1}) + \Gamma_n$ for some $l \in \{0, \cdots, s-2\}$, then either $\partial\Upsilon_l \cap \partial\Upsilon_{l+1} = \emptyset$ or $\{(x_l, \cdots, x_l)\}$. Here, $x_l$ will be the largest real value satisfies*

$$\binom{n-l}{s-l} x_l^{s-l} - \sum_{k=l}^{s-1} c_k \binom{n-l}{k-l} x_l^{k-l} = 0 = \binom{n-l-1}{s-l-1} x_l^{s-l-1} - \sum_{k=l+1}^{s-1} c_k \binom{n-l-1}{k-l-1} x_l^{k-l-1}.$$

*Moreover, if $\Gamma_f^n$ is strictly $\Upsilon$-stable, then $\partial\Upsilon_0 \cap \partial\Upsilon_1 = \partial\Gamma_f^n \cap \partial\Upsilon_1 = \emptyset$.*

*Proof.* There is nothing to prove when $s = 1$. In addition, when $s = n$, this can be done due to previous work in [52], so we may assume $n-1 \geq s \geq 2$. We use mathematical induction on the number of variables $n$ to prove this, for convenience, we assume $c_{s-1} = 0$. First, when $n = 1$ or $n = 2$, there is nothing to prove. Second, suppose the statement is true when $n = m-1$. Then, when $n = m$, we only need to prove the case that $\Upsilon_0 = \Gamma_f^m \neq \Gamma_m$, the rest follows directly by mathematical induction. If $\Upsilon_1 = \Gamma_m$, then $s = n$ and $f = \lambda_1 \cdots \lambda_m - c_0$ with $c_0 > 0$ by Lemma 2.4. This can be done due to [52]. We consider the case that $\Upsilon_1 \neq \Gamma_m$, for any $(\lambda_1, \cdots, \lambda_m) \in \partial\Upsilon_0$, we have

$$0 = \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda) = \lambda_1\Big(\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1})\Big) + \sigma_s(\lambda_{;1}) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}).$$

Due to Lemma 2.4, $\Upsilon_0$ is contained in $\Upsilon_1$. This implies that

$$\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1}) = \lambda_1\Big(\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1})\Big) \geq 0.$$

If $\partial\Upsilon_0 \cap \partial\Upsilon_1 \neq \emptyset$, we use the method of Lagrange multipliers to find the local extrema of

$\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})$ under the constraint $\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1}) = 0$. Let

$$\mathcal{F}(\lambda_2, \cdots, \lambda_m, \mu) := \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1}) - \mu\Big(\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1})\Big). \tag{2.15}$$

By taking the partial derivative of quantity (2.15) with respect to $\mu$ and $\lambda_i$, we have

$$\partial\mathcal{F}/\partial\mu = -\sigma_{s-1}(\lambda_{;1}) + \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1}); \tag{2.16}$$

$$\partial\mathcal{F}/\partial\lambda_i = \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1,i}) - \sigma_{s-1}(\lambda_{;1,i}) - \mu\Big(\sigma_{s-2}(\lambda_{;1,i}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;1,i})\Big), \tag{2.17}$$

for $i \in \{2, \cdots, m\}$. At $\nabla\mathcal{F} = 0$, we subtract equation (2.17) by (2.16) and get

$$0 = \partial\mathcal{F}/\partial\lambda_i - \partial\mathcal{F}/\partial\mu = (\lambda_i - \mu)\Big(\sigma_{s-2}(\lambda_{;1,i}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;1,i})\Big)$$

for all $i \in \{2, \cdots, m\}$. By mathematical induction, $\partial\Upsilon_1 \cap \partial\Upsilon_2 = \emptyset$ or $\{(x_1, \cdots, x_1)\}$. Here $x_1$ is the largest real value satisfies both

$$\binom{m-1}{s-1}x_1^{s-1} - \sum_{k=1}^{s-2} c_k\binom{n-1}{k-1}x_1^{k-1} = 0 = \binom{m-2}{s-2}x_1^{s-2} - \sum_{k=2}^{s-2} c_k\binom{m-2}{k-2}x_1^{k-2}.$$

No matter which case, there exists only one local minimum $(x_1, \cdots, x_1)$, where

$$\binom{m-1}{s-1}x_1^{s-1} - \sum_{k=1}^{s-2} c_k\binom{n-1}{k-1}x_1^{k-1} = 0.$$

Since we assume $\partial\Upsilon_0 \cap \partial\Upsilon_1 \neq \emptyset$ and by above, there exists only one critical point. It is a global minimum, $\partial\Upsilon_0 \cap \partial\Upsilon_1 = \{(x_1, \cdots, x_1)\}$, and $x_1$ also satisfies

$$\binom{m}{s}x_1^{s} - \sum_{k=0}^{s-2} c_k\binom{n}{k}x_1^{k} = 0.$$

This finishes the proof. $\square$

**Proposition 2.6.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. For $s > 1$ and any $q \in \mathbb{R}^n$, there exists at most one connected component of $\{f(\lambda) \neq 0\}$ which is contained in $\bigcap_{1\le i_1<\cdots<i_{s-1}\le n}\{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > q_{i_1,\cdots,i_{s-1}}\}$ and this connected component will be a connected component of $\{f(\lambda) > 0\}$.*

*Proof.* When $s = n$, this can be done due to previous work in [52], so we may assume $n - 1 \geq s \geq 2$. We use mathematical induction on the number of variables $n$ to prove this, for convenience, we assume $c_{s-1} = 0$. First, when $n = 1$ or $n = 2$, there is nothing to prove. Second, suppose the statement is true when $n = m-1$. Then, when $n = m$, for any connected component of $\{f(\lambda) \neq 0\}$ which is contained in $\bigcap_{1\le i_1<\cdots<i_{s-1}\le n}\{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > q_{i_1,\cdots,i_{s-1}}\}$ for some $q \in \mathbb{R}^{\binom{m}{s-1}}$. If there exists a point $(\lambda_1, \cdots, \lambda_m)$ such that $f_i(\lambda_1, \cdots, \lambda_m) = 0$ for some $i \in \{1, \cdots, m\}$. Then for any $\tilde{\lambda}_i \leq \lambda_i$, we always have

$$f(\lambda_1, \cdots, \lambda_{i-1}, \tilde{\lambda}_i, \lambda_{i+1}, \cdots, \lambda_m) = f(\lambda_1, \cdots, \lambda_m).$$

This gives a contradiction. By induction and similar to previous proofs, we see that this connected component of $\{f(\lambda) \neq 0\}$ will be contained in $\cap_{i\in\{1,\cdots,m\}}\{f_i > 0\}$. Here, by ignoring the Cartesian product $\mathbb{R}$ term, for convenience, we write $\{f_i > 0\}$ as the unique connected component of $\{f_i > 0\}$ which is contained in $\bigcap_{\{i_1<\cdots<i_{s-2}\}\subset\{1,\cdots,i-1,i+1,\cdots,m\}}\{\sigma_1(\lambda_{;i,i_1,\cdots,i_{s-2}}) > q_{i,i_1,\cdots,i_{s-2}}\}$. By the proof in Lemma 2.4, this connected component of $\{f \neq 0\}$ will be a connected component of $\{f > 0\}$ and is unique by Lemma 2.5. This finishes the proof. □

**Proposition 2.7.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$ which is $\Upsilon$-stable. Then for any $l \in \{0, \cdots, s-1\}$, the boundary $\partial\Upsilon_l$ of the $\Upsilon_l$-cone separates the ambient space $\mathbb{R}^n$ into two disjoint connected components.*

*Proof.* For convenience, we assume that $c_{s-1} = 0$. By Lemma 2.4, we have

$$\Gamma_f^n \subseteq \Upsilon_1 \subseteq \Upsilon_2 \subseteq \cdots \subseteq \Upsilon_{s-1} = \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1,\cdots,i_{s-1}}) > 0\}.$$

For any $l \in \{0, \cdots, s-1\}$, we consider the following two open sets $\Upsilon_l$ and $\mathbb{R}^n \backslash \overline{\Upsilon}_l$. By Lemma 2.4, $\Upsilon_l$ is connected. Similar to the proof in Lemma 2.4, we can also show that $\mathbb{R}^n \backslash \overline{\Upsilon}_l$ is connected. For any $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \backslash \overline{\Upsilon}_l$, we have a straight path connecting $(\lambda_1, \cdots, \lambda_n)$ to $(\lambda_{\min}, \lambda_2, \cdots, \lambda_n)$, where $\lambda_{\min} := \min\{\lambda_1, \cdots, \lambda_n\}$. Suppose that there exists $(\tilde{\lambda}, \lambda_2, \cdots, \lambda_n)$ on this path such that $(\tilde{\lambda}, \lambda_2, \cdots, \lambda_n) \in \Upsilon_l$. Then, by Lemma 2.5, we get $(\lambda_1, \cdots, \lambda_n) \in \Upsilon_l$, which gives a contradiction. We can find a piecewise linear path connecting $(\lambda_1, \cdots, \lambda_n)$ to $(\lambda_{\min}, \cdots, \lambda_{\min})$ and for any point on this path, this point is in $\mathbb{R}^n \backslash \overline{\Upsilon}_l$. Hence, $\mathbb{R}^n \backslash \overline{\Upsilon}_l$ is open and connected. By standard topology arguments, the boundary $\partial \Upsilon_l$ separates $\mathbb{R}^n$ into two disjoint connected components. This finishes the proof. □

**Proposition 2.8.** *Let $f(\lambda) := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial and $\Gamma_f^n$ be a connected component of $\{f(\lambda) > 0\}$ which is (strictly) $\Upsilon$-stable. For any $l \in \{1, \cdots, n-1\}$, suppose $(\mu_1, \cdots, \mu_l) \in \Gamma_f^l$. Then for any $a \in S_n$, by fixing the $a(k)$-th entry $\lambda_{a(k)}$ equals $\mu_k$ for all $k \in \{1, \cdots, l\}$, and treat the rest as variables. $f$ is a degree $\min\{s, n-l\}$ general $\sigma_k$ type multilinear polynomial with $n-l$ variables and the cross section will be (strictly) $\Upsilon$-stable.*

*Proof.* When $s = 1$, this is automatically true. When $s = n$, this can be done due to the work in [52]. So we may assume $2 \le s \le n-1$. We use mathematical induction on the number of variables $n$ to prove this, for convenience, we assume $c_{s-1} = 0$. First, when $n = 1$ or $n = 2$, the proof is straightforward. Second, suppose the statement is true when $n = m - 1$. Then, when $n = m$, it suffices to prove the statement by fixing a single entry, say $\mu_1 \in \Gamma_f^1$

for convenience, the rest follows by induction. By symmetry, we fix $\lambda_1 = \mu_1$, then we get

$$f(\mu_1, \lambda_2, \cdots, \lambda_m) = \sigma_s(\lambda_{;1}) + \mu_1 \sigma_{s-1}(\lambda_{;1}) - c_{s-2}\sigma_{s-2}(\lambda_{;1}) - \sum_{k=0}^{s-3} \Big( \mu_1 c_{k+1} + c_k \Big) \sigma_k(\lambda_{;1}).$$

By Lemma 2.4, since $\Gamma_f^m$ is $\Upsilon$-stable, $\Gamma_f^m \subset \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\}$. Thus, we can also verify that this connected component of

$$\begin{aligned} &\{f(\mu_1, \lambda_2, \cdots, \lambda_m) > 0\} \\ &\quad = \Big\{ \sigma_s(\lambda_{;1}) + \mu_1 \sigma_{s-1}(\lambda_{;1}) - c_{s-2}\sigma_{s-2}(\lambda_{;1}) - \sum_{k=0}^{s-3} \Big( \mu_1 c_{k+1} + c_k \Big) \sigma_k(\lambda_{;1}) > 0 \Big\} \end{aligned}$$

is contained in $\bigcap_{2 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\}$ by ignoring the first entry $\mu_1$. So the cross section $\lambda_1 = \mu_1$ is again $\Upsilon$-stable. For the strict $\Upsilon$-stableness, the proof is similar, so this finishes the proof. $\square$

**Theorem 2.2** (Positivstellensatz). *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. There exists a connected component $\Gamma_f^n$ of $\{f(\lambda) > 0\}$ which is $\Upsilon$-stable if and only if the diagonal restriction $r_f(x)$ of $f(\lambda)$, which is defined by the following*

$$r_f(x) \coloneqq f(x, \cdots, x) = \binom{n}{s} x^s - \sum_{k=0}^{s-1} c_k \binom{n}{k} x^k, \tag{2.18}$$

*is right-Noetherian. Moreover, $\Gamma_f^n$ is strictly $\Upsilon$-stable iff $r_f$ is strictly right-Noetherian.*

*Proof.* For convenience, we assume $c_{s-1} = 0$. If $\Gamma_f^n$ is $\Upsilon$-stable, by Lemma 2.4, then we have

$$\Gamma_f^n \subseteq \Upsilon_1 \subseteq \Upsilon_2 \subseteq \cdots \subseteq \Upsilon_{s-1} = \bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\}.$$

For $\lambda > 0$ sufficiently large, by Lemma 2.5, we have $(\lambda, \cdots, \lambda) \in \Gamma_f^n$. By decreasing the value of $\lambda$, since $\Gamma_f^n$ is contained in $\bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\}$, there exists a largest

$x_0 \geq 0$ such that

$$f(x_0) = f(x_0, \cdots, x_0) = r_f(x_0) = \binom{n}{s} x_0^s - \sum_{k=0}^{s-2} c_k \binom{n}{k} x_0^k = 0.$$

Otherwise we will get a contradiction. Similarly, there exists a largest $x_1 \geq 0$ such that

$$f_n(x_1) = f_n(x_1, \cdots, x_1) = \frac{1}{n} \frac{d}{dx}\Big|_{x=x_1} r_f(x) = \binom{n-1}{s-1} x_1^{s-1} - \sum_{k=1}^{s-2} c_k \binom{n-1}{k-1} x_1^{k-1} = 0.$$

If $x_1 > x_0$, then $f(x_1) = f(x_1, \cdots, x_1) > 0$. So, we have $(x_1, \cdots, x_1) \in \Gamma_f^n \subseteq \Upsilon_1$, which implies that $f_n(x_1) > 0$. This contradicts $f_n(x_1) = 0$. If we inductively let $x_i$ be the largest real root of $r_f^{(i)}(x)$ for $i \in \{0, \cdots, s-1\}$, then similarly we obtain the following right-Noetherianness of $r_f(x)$: $x_0 \geq x_1 \geq \cdots \geq x_{s-1} = 0$.

On the other hand, for convenience, we assume that $c_{s-1} = 0$. If the diagonal restriction $r_f(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-2} c_k \binom{n}{k} x^k$ is right-Noetherian, then we use mathematical induction on the number of variables $n$. When $n = 1$ or $n = 2$, this is true. Suppose the statement is true when $n = m-1$. When $n = m$, for any $i \in \{1, \cdots, m\}$ we get

$$r_{f_i}(x) = \binom{m-1}{s-1} x^{s-1} - \sum_{k=1}^{s-2} c_k \binom{m-1}{k-1} x^{k-1} = \frac{1}{m} \frac{d}{dx} r_f(x).$$

This implies that $r_{f_i}$ is still right-Noetherian. By mathematical induction, there exists a connected component $\Gamma_{f_i}^{m-1}$ of $\{f_i > 0\}$ which is $\Upsilon$-stable for all $i \in \{1, \cdots, m\}$. As a consequence, by Lemma 2.4, $\Gamma_{f_i}^{m-1}$ is contained in $\bigcap_{\{i_1 < \cdots < i_{s-2}\} \subset \{1, \cdots, i-1, i+1, \cdots, m\}} \{\sigma_1(\lambda_{;i, i_1, \cdots, i_{s-2}}) > 0\}$. By Lemma 2.6, if $\partial \Gamma_{f_i}^{m-1} \cap \partial \bigcap_{\{i_1 < \cdots < i_{s-2}\} \subset \{1, \cdots, i-1, i+1, \cdots, m\}} \{\sigma_1(\lambda_{;i, i_1, \cdots, i_{s-2}}) > 0\} \neq \emptyset$, then $f(\lambda) = \sigma_s(\lambda) - c_0$. We have $c_0 \geq 0$, which is guaranteed by the right-Noetherianness of $r_f$. Let $\Gamma_f^m$ be the connected component of $\{f(\lambda) > 0\}$ contained in $\bigcap_{1 \leq i_1 < \cdots < i_{s-1} \leq m} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\}$, then $\Gamma_f^m$ will be $\Upsilon$-stable.

Now, let $\Gamma_f^m$ be the open connected component of $\{f(\lambda) > 0\}$ containing the ray $\{(x, \cdots, x) \in$

$\mathbb{R}^m\colon x > x_0\}$. For the case $\partial\Gamma^{m-1}_{f_i} \cap \partial \bigcap_{\{i_1<\cdots<i_{s-2}\}\subset\{1,\cdots,i-1,i+1,\cdots,m\}}\{\sigma_1(\lambda_{;i,i_1,\cdots,i_{s-2}}) > 0\} = \emptyset$, suppose that $\Gamma^m_f$ is not contained in $\Gamma^m_{f_i}$ for some $i \in \{1,\cdots,m\}$. Here $\Gamma^m_{f_i} = \Gamma^{m-1}_{f_i} \times \mathbb{R}$. By Proposition 2.7, since $\Gamma^{m-1}_{f_i}$ is $\Upsilon$-stable, the boundary $\partial\Gamma^{m-1}_{f_i}$ separates $\mathbb{R}^{m-1}$ into two disjoint connected components. Thus, $\Gamma^m_{f_i}$ separates $\mathbb{R}^m$ into two disjoint connected components. Since any connected set in $\mathbb{R}^m$ is also path connected, there exists $(\tilde{\lambda}_1,\cdots,\tilde{\lambda}_m) \in \Gamma^m_f \cap \partial\Gamma^m_{f_i}$. That is,

$$f(\tilde{\lambda}) = \sigma_s(\tilde{\lambda}) - \sum_{k=0}^{s-2} c_k\sigma_k(\tilde{\lambda}) > 0 \quad \text{and} \quad \sigma_{s-1}(\tilde{\lambda}_{;i}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\tilde{\lambda}_{;i}) = 0$$

for some $i \in \{1,\cdots,m\}$. Say $i=1$ for convenience. At this point $(\tilde{\lambda}_1,\cdots,\tilde{\lambda}_m)$, we get

$$\sum_{k=0}^{s-2} c_k\sigma_k(\tilde{\lambda}_{;1}) - \sigma_s(\tilde{\lambda}_{;1}) = -f(\tilde{\lambda}) < 0. \tag{2.19}$$

Similar to the proof in previous Lemma 2.6, we use the method of Lagrange multipliers to find the local extrema of $\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})$ under the constraint $\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1}) = 0$. There exists only one local extremum at $(x_1,\cdots x_1)$, where $x_1$ is the largest real root of $r_{f_1}(x)$. Since $\Gamma^{m-1}_{f_i}$ is $\Upsilon$-stable, by Lemma 2.6, we can treat $\lambda_m$ as a function in terms of $\lambda_2,\cdots,\lambda_{m-1}$ and smooth when $(\lambda_2,\cdots,\lambda_m) \neq (x_1,\cdots,x_1)$. By taking the derivative of the quantity $\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})$ and the equation $f_1 = 0$ with respect to $i \in \{2,\cdots,m-1\}$, we get

$$\begin{aligned}
&\frac{\partial}{\partial\lambda_i}\Big(\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})\Big)\\
&\quad = \lambda_i\Big(\sigma_{s-2}(\lambda_{;1,i}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;1,i})\Big) + \lambda_m\frac{\partial\lambda_m}{\partial\lambda_i}\Big(\sigma_{s-2}(\lambda_{;1,m}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;1,m})\Big)\\
&\quad = \lambda_i f_{1i} + \lambda_m\frac{\partial\lambda_m}{\partial\lambda_i} f_{1m}.
\end{aligned} \tag{2.20}$$

Also, on $\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1}) = 0$, we have

$$0 = \frac{\partial}{\partial\lambda_i} f_1 = \frac{\partial}{\partial\lambda_i}\Big(\sigma_{s-1}(\lambda_{;1}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;1})\Big) = f_{1i} + \frac{\partial\lambda_m}{\partial\lambda_i} f_{1m}. \tag{2.21}$$

By combining equations (2.20) and (2.21), we get

$$\frac{\partial}{\partial\lambda_i}\Big(\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})\Big) = \lambda_i f_{1i} - \lambda_m f_{1i} = (\lambda_i - \lambda_m) f_{1i}. \tag{2.22}$$

By setting $\lambda_i \geq \lambda_m$ for all $i \in \{2, \cdots, m-1\}$ and since $\Gamma_{f_i}^{m-1}$ is $\Upsilon$-stable, quantity (2.22) is always positive on the level set $\{f_1 = 0\}$. By (2.22) and the fact that $(x_1, \cdots x_1)$ is the unique local extrema of $\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1})$, for any $(\lambda_2, \cdots, \lambda_m)$ on $\{f_1 = 0\}$, we have

$$\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;1}) - \sigma_s(\lambda_{;1}) \geq \sum_{k=0}^{s-2} c_k\binom{m-1}{k} x_1^k - \binom{m-1}{s} x_1^s. \tag{2.23}$$

Since $x_1$ is a root of $r_{f_1}$, we obtain,

$$\begin{aligned} r_f(x_1) &= x_1^m - \sum_{k=0}^{m-2} c_k\binom{m}{k} x_1^k \\ &= x_1\Big(\binom{m-1}{s-1} x_1^{s-1} - \sum_{k=1}^{s-2} c_k\binom{m-1}{k-1} x_1^{k-1}\Big) + \binom{m-1}{s} x_1^s - \sum_{k=0}^{s-2} c_k\binom{m-1}{k} x_1^k \\ &= \binom{m-1}{s} x_1^s - \sum_{k=0}^{s-2} c_k\binom{m-1}{k} x_1^k. \end{aligned} \tag{2.24}$$

By combining inequalities (2.19), (2.23), and (2.24), we get

$$\begin{aligned} 0 > -f(\tilde\lambda) &= \sum_{k=0}^{s-2} c_k\sigma_k(\tilde\lambda_{;1}) - \sigma_s(\tilde\lambda_{;1}) \geq \sum_{k=0}^{s-2} c_k\binom{m-1}{k} x_1^k - \binom{m-1}{s} x_1^s = -r_f(x_1) \\ &\geq -r_f(x_0) = 0. \end{aligned}$$

Here $x_0$ is the largest real root of $r_f$. This gives a contradiction, in conclusion, $\Gamma_f^m$ is contained in $\Gamma^m_{f_i}$ for all $i \in \{1, \cdots, m\}$. This implies that $\Gamma^m_f$ is $\Upsilon$-stable. Similarly, we can show that $\Gamma^n_f$ is strictly $\Upsilon$-stable if and only if $r_f$ is strictly right-Noetherian. This finishes the proof. $\square$

**Proposition 2.9.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. If one level set of $\{f(\lambda) = 0\}$ is contained in $\bigcap_{1 \le i_1 < \cdots < i_{s-1} \le n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > q_{i_1, \cdots, i_{s-1}}\}$ for some $q \in \mathbb{R}^{\binom{n}{s-1}}$, then this level set is unique and there exists a unique connected component of $\{f(\lambda) > 0\}$, which is strictly $\Upsilon$-stable and the boundary will be this level set.*

*Proof.* We only need to consider the case $s \ge 2$ and we use mathematical induction to prove this, for convenience, we assume $c_{s-1} = 0$. First, when $n = 1$, there is nothing to prove. Second, when $n = 2$, this can be done due to previous work in [52]. Suppose the statement is true when $n = m - 1$. Then, when $n = m$, if there exists a point $(\lambda_1, \cdots, \lambda_n)$ on this level set $\{f = 0\}$ such that $f_i(\lambda_1, \cdots, \lambda_n) = 0$ for some $i \in \{1, \cdots, n\}$, then for $\tilde{\lambda}_i \le \lambda_i$, we always have $f(\lambda_1, \cdots, \lambda_{i-1}, \tilde{\lambda}_i, \lambda_{i+1}, \cdots, \lambda_n) = f(\lambda_1, \cdots, \lambda_n) = 0$. By letting $\tilde{\lambda}_i$ approach $-\infty$, this gives a contradiction. By Proposition 2.6, this level set of $\{f = 0\}$ will be contained in $\cap_{i \in \{1, \cdots, m\}} \{f_i > 0\}$. Hence, this level set will be the following graph

$$\lambda_m = \frac{-\sigma_s(\lambda_{;m}) + \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;m})}{\sigma_{s-1}(\lambda_{;m}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;m})}$$

over $\{f_m > 0\}$. We define $\Gamma^m_f$ by

$$\Gamma^m_f \coloneqq \left\{ \lambda \colon (\lambda_1, \cdots, \lambda_{m-1}) \in \{f_m > 0\} \text{ and } \lambda_m > \frac{-\sigma_s(\lambda_{;m}) + \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;m})}{\sigma_{s-1}(\lambda_{;m}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;m})} \right\}.$$

We have $\Gamma^m_f$ is open and connected. If $\Gamma^m_f$ is not a connected component of $\{f(\lambda) > 0\}$, say there exists $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_m) \notin \Gamma^m_f$ in this connected component. Then it suffices to check the case that $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{m-1}) \notin \{f_m > 0\}$. Since connected set in $\mathbb{R}^m$ is also path connected and by induction, there exists $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m)$ such that

$$f(\hat{\lambda}_1, \cdots, \hat{\lambda}_{m-1}, \hat{\lambda}_m) = \sigma_s(\hat{\lambda}_{;m}) - \sum_{k=0}^{s-2} c_k \sigma_k(\hat{\lambda}_{;m}) > 0 \quad \text{and} \quad f_m(\hat{\lambda}_1, \cdots, \hat{\lambda}_{m-1}) = 0.$$

The rest follows by the proof in Theorem 2.2, we use the method of Lagrange multipliers to get a contradiction. So, $\Gamma_f^m$ is an open connected component of $\{f(\lambda) > 0\}$ which is strictly $\Upsilon$-stable and the boundary $\partial\Gamma_f^m$ will be this level set. Similarly, a connected component of $\{f(\lambda) > 0\}$ satisfies these properties will be unique. This finishes the proof. □

**Proposition 2.10.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a $\Upsilon$-stable general $\sigma_k$ type multi-linear polynomial. For any $k \in \{0, \cdots, n-1\}$ and $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \in \mathbb{R}^n$, there exists a unique $t_k \in \mathbb{R}$ such that $(\tilde{\lambda}_1 + t_k, \cdots, \tilde{\lambda}_n + t_k) \in \partial\Upsilon_k$. Moreover, $t_0 \geq t_1 \geq \cdots \geq t_{s-1}$.*

*Proof.* For convenience, we assume $c_{s-1} = 0$. By hypothesis and Lemma 2.4, since $f = \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda)$ is $\Upsilon$-stable, we get

$$\Gamma_f^n = \Upsilon_0 \subseteq \Upsilon_1 \subseteq \Upsilon_2 \subseteq \cdots \subseteq \Upsilon_{s-1} = \bigcap_{1 \leq i_1 < \cdots < i_{s-1} \leq n} \{\sigma_1(\lambda_{;i_1, \cdots, i_{s-1}}) > 0\},$$

and for any $l \in \{1, \cdots, s-1\}$

$$\Upsilon_l \coloneqq \bigcap_{1 \leq i_1 < \cdots < i_l \leq n} \Gamma^n_{f_{i_1 \cdots i_l}} = \bigcap_{1 \leq i_1 < \cdots < i_l \leq n} \Big\{\sigma_{s-l}(\lambda_{;i_1, \cdots, i_l}) - \sum_{k=l}^{s-2} c_k \sigma_{k-l}(\lambda_{;i_1, \cdots, i_l}) > 0\Big\}.$$

When $k = s-1$, by considering a system of linear equations, there exists a unique $t_{s-1} \in \mathbb{R}$ such that $(\tilde{\lambda}_1 + t_{s-1}, \cdots, \tilde{\lambda}_n + t_{s-1}) \in \partial\Upsilon_{s-1}$. Suppose the statement holds when $k = m$. When $k = m-1$, since $(\tilde{\lambda}_1 + t_m, \cdots, \tilde{\lambda}_n + t_m) \in \partial\Upsilon_m$, we have $(\tilde{\lambda}_1 + t_m, \cdots, \tilde{\lambda}_n + t_m) \notin \Upsilon_{m-1}$. If $(\tilde{\lambda}_1 + t_m, \cdots, \tilde{\lambda}_n + t_m) \in \partial\Upsilon_{m-1}$, then we have $t_{m-1} = t_m$. A quick observation gives $(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \in \Upsilon_{m-1}$ for $t > t_{m-1}$. This gives the uniqueness of $t_{m-1}$ and we justify the mathematical induction. If not, we try to show that there exists a unique $t_{m-1} > t_m$ such that $(\tilde{\lambda}_1 + t_{m-1}, \cdots, \tilde{\lambda}_n + t_{m-1}) \in \partial\Upsilon_{m-1}$. Again, we have $(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \in \Upsilon_m$ for $t > t_m$.

For $t > t_m$ sufficiently large, we have $(\tilde{\lambda}_1+t, \cdots, \tilde{\lambda}_n+t) \in \Upsilon_{m-1}$. Hence, by the intermediate value theorem, there exists a $t_{m-1} > t_m$ such that $(\tilde{\lambda}_1 + t_{m-1}, \cdots, \tilde{\lambda}_n + t_{m-1}) \in \Upsilon_{m-1}$. The uniqueness can be ensured because $(\tilde{\lambda}_1 + t_{m-1}, \cdots, \tilde{\lambda}_n + t_{m-1}) \in \Upsilon_m$, otherwise we will get a contradiction. This finishes the proof. □

Here, we define a first order differential operator $D$ on multilinear polynomials by

$$D = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i}.$$

We also define the following ratio for $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials.

**Definition 2.8.** Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a $\Upsilon$-stable general $\sigma_k$ type multilinear polynomial, we define the following ratio $\alpha_f$ of $f$ to be the following

$$\alpha_f \coloneqq \frac{f \cdot D^2 f}{(Df)^2}.$$

We have the following generalization of Theorem 2.1.

**Theorem 2.3** (General Newton–Maclaurin's Inequality)**.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a $\Upsilon$-stable general $\sigma_k$ type multilinear polynomial. For any $k \in \{0, \cdots, s-2\}$, the ratio $\alpha_{D^k f} = D^k f \cdot D^{k+2} f / (D^{k+1} f)^2$ is increasing on $\Upsilon_{k+1} \cap \{(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \colon t \in \mathbb{R}\}$ for any $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \in \Upsilon_{k+1}$. In particular, for any $k \in \{0, \cdots, s-2\}$, we have*

$$\alpha_{D^k f} = \frac{D^k f \cdot D^{k+2} f}{\left(D^{k+1} f\right)^2} \leq 1 - \frac{1}{s-k} \quad on \quad \Upsilon_{k+1}.$$

*Proof.* For any $k \in \{0, \cdots, s-2\}$, similar to Lemma 2.1, we have the following

$$D\alpha_{D^k f} = \frac{D^{k+2} f}{D^{k+1} f}\Big(1 + (\alpha_{D^{k+1} f} - 2)\alpha_{D^k f}\Big) \tag{2.25}$$

and

$$D^k f = k!\binom{n-s+k}{n-s}\sigma_{s-k}(\lambda) - \sum_{j=k}^{s-1} j!\binom{n-s+j}{n-s}c_j\sigma_{j-k}(\lambda). \tag{2.26}$$

We assume $c_{s-1} = 0$ for convenience and we use mathematical induction on the degree $s$ to prove this. When $s = 1$, then we have $f(\lambda) = \sigma_1(\lambda)$ and $\alpha_f = \frac{(\sigma_1(\lambda))\cdot 0}{n^2} = 0$. So the statement is automatically true. When $s = 2$, we have

$$\alpha_f = \frac{f\cdot D^2 f}{(Df)^2} = \frac{(\sigma_2(\lambda) - c_0)\cdot n(n-1)}{(n-1)^2\sigma_1^2(\lambda)} = \frac{n(\sigma_2(\lambda) - c_0)}{(n-1)\sigma_1^2(\lambda)}$$

and

$$D\alpha_f = \frac{n}{n-1}\frac{(n-1)\sigma_1^2(\lambda) - 2n\sigma_2(\lambda) + 2nc_0}{\sigma_1^3(\lambda)} = \frac{n}{n-1}\frac{\sum_{i<j}(\lambda_i - \lambda_j)^2 + 2nc_0}{\sigma_1^3(\lambda)} \geq 0$$

Notice that the equality happens only when $\lambda_1 = \cdots = \lambda_n$ and $c_0 = 0$. In addition, we have

$$\alpha_{Df} = \frac{Df\cdot D^3 f}{(D^2 f)^2} = \frac{(n-1)\sigma_1(\lambda)\cdot 0}{n^2(n-1)^2} = 0 \quad \text{and} \quad D\alpha_{Df} = 0.$$

So the statement holds when $s = 1$ or $s = 2$, suppose the statement holds when $s = m-1 < n$. When $s = m$, for $k \in \{1, \cdots, m-1\}$, by mathematical induction, the ratio $\alpha_{D^k f}$ will be increasing on $\Upsilon_{k+1} \cap \big\{(\tilde\lambda_1 + t, \cdots, \tilde\lambda_n + t) \colon t \in \mathbb{R}\big\}$ for any $(\tilde\lambda_1, \cdots, \tilde\lambda_n) \in \Upsilon_{k+1}$. In particular, by equation (2.25), we get

$$\begin{aligned}\alpha_{D^k f} &= \frac{D^k f\cdot D^{k+2} f}{\left(D^{k+1} f\right)^2} \leq \lim_{t\to\infty}\frac{D^k f(\lambda + t)\cdot D^{k+2} f(\lambda + t)}{(D^{k+1} f(\lambda + t))^2} \\ &= \lim_{t\to\infty}\frac{k!\binom{n-m+k}{n-m}\sigma_{m-k}(t)\cdot(k+2)!\binom{n-m+k+2}{n-m}\sigma_{m-k-2}(t)}{\left((k+1)!\binom{n-m+k+1}{n-m}\sigma_{m-k-1}(t)\right)^2} = \frac{m-k-1}{m-k}.\end{aligned}$$

We only need to prove the case when $k = 0$, by equation (2.25), we have

$$D\alpha_f = \frac{D^2 f}{Df}\Big(1 + (\alpha_{Df} - 2)\alpha_f\Big).$$

For any $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \in \Upsilon_1$, we define $I \coloneqq \big\{t \in \mathbb{R}\colon (\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \in \Upsilon_1$ and $\frac{1}{2-\alpha_{Df}}(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) > \alpha_f(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t)\big\}$. By Proposition 2.10, there exists $t_0$ such that $(\tilde{\lambda}_1 + t_0, \cdots, \tilde{\lambda}_n + t_0) \in \partial\Upsilon_0$. If $(\tilde{\lambda}_1 + t_0, \cdots, \tilde{\lambda}_n + t_0) \in \partial\Upsilon_1$, then by Lemma 2.6, either $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n$ or $\Upsilon_0 = \Gamma_n$. For the case $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n$, this can be done by Theorem 2.1. For the case $\Upsilon_0 = \Gamma_n$, we get $s = m = n$ and $f(\lambda) = \lambda_1 \cdots \lambda_n$. Hence, we have

$$D\alpha_f = \frac{2}{\sigma_{n-1}^3}\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big).$$

When $n = 1$ or $n = 2$, the proof is straight forward. When $n = 3$, we get

$$\sigma_2^2\sigma_1 - 4\sigma_3\sigma_1^2 + 3\sigma_3\sigma_2\sigma_0 = \lambda_1^3(\lambda_2 - \lambda_3)^2 + \lambda_2^3(\lambda_1 - \lambda_3)^2 + \lambda_3^3(\lambda_1 - \lambda_2)^2 \geq 0.$$

We consider the remaining case $n \geq 4$ and without loss of generality, we say $(\tilde{\lambda}_1 + t_0, \cdots, \tilde{\lambda}_n + t_0) = (\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_k, 0, \cdots, 0)$ and $n - 1 \geq k \geq 0$ with $\tilde{\epsilon}_j > 0$ for any $j \in \{1, \cdots, k\}$. The case $k = 0$ is the case $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n$, so we can ignore this case. Since we have

$$\begin{aligned}
&\lim_{t \to t_0^+}\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \\
&\quad = \lim_{t \to 0^+}\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_k + t, t, \cdots, t),
\end{aligned}$$

we consider the term $(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3})(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_k + t, t, \cdots, t)$ when $t$ is close to 0. If $n - 4 \geq k$, we get the following

$$\begin{aligned}
\sigma_n &= \tilde{\epsilon}_1 \cdots \tilde{\epsilon}_k t^{n-k} + \sigma_{k-1}(\tilde{\epsilon})t^{n-k+1} + O(t^{n-k+2}); \\
\sigma_{n-1} &= \binom{n-k}{1}\tilde{\epsilon}_1 \cdots \tilde{\epsilon}_k t^{n-k-1} + \binom{n-k+1}{1}\sigma_{k-1}(\tilde{\epsilon})t^{n-k} + O(t^{n-k+1});
\end{aligned}$$

$$\sigma_{n-2} = \binom{n-k}{2}\tilde{\epsilon}_1 \cdots \tilde{\epsilon}_k t^{n-k-2} + \binom{n-k+1}{2}\sigma_{k-1}(\tilde{\epsilon})t^{n-k-1} + O(t^{n-k});$$
$$\sigma_{n-3} = \binom{n-k}{3}\tilde{\epsilon}_1 \cdots \tilde{\epsilon}_k t^{n-k-3} + \binom{n-k+1}{3}\sigma_{k-1}(\tilde{\epsilon})t^{n-k-2} + O(t^{n-k-1}).$$

Here, we use the Big O notation. Thus, by some complicated computations, we have

$$\begin{aligned}&\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_k + t, t, \cdots, t)\\ &\quad = (n-k)\tilde{\epsilon}_1^2 \cdots \tilde{\epsilon}_k^2 \cdot \sigma_{k-1}(\tilde{\epsilon})t^{3n-3k-3} + O(t^{3n-3k-2}).\end{aligned}$$

For the case $n-1 \geq k \geq n-3$, if $k = n-1$, then we have

$$\begin{aligned}&\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_{n-1} + t, t)\\ &\quad = \tilde{\epsilon}_1^2 \cdots \tilde{\epsilon}_{n-1}^2 \cdot \sigma_{n-2}(\tilde{\epsilon}) + O(t).\end{aligned}$$

If $k = n-2$, then we get

$$\begin{aligned}&\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_{n-2} + t, t, t)\\ &\quad = 2\tilde{\epsilon}_1^2 \cdots \tilde{\epsilon}_{n-2}^2 \cdot \sigma_{n-3}(\tilde{\epsilon})t^3 + O(t^4).\end{aligned}$$

If $k = n-3$, then we get

$$\begin{aligned}&\Big(\sigma_{n-1}^2\sigma_{n-2} - 4\sigma_n\sigma_{n-2}^2 + 3\sigma_n\sigma_{n-1}\sigma_{n-3}\Big)(\tilde{\epsilon}_1 + t, \cdots, \tilde{\epsilon}_{n-3} + t, t, t, t)\\ &\quad = 3\tilde{\epsilon}_1^2 \cdots \tilde{\epsilon}_{n-3}^2 \cdot \sigma_{n-4}(\tilde{\epsilon})t^6 + O(t^7).\end{aligned}$$

No matter which case, when $t > t_0$ is sufficiently close to $t_0$, then $(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}_n + t) \in I$. So $I$ is not an empty set, moreover, $I$ is an open set by the continuity of $\alpha_f$ and $\alpha_{Df}$. If we can show that $I = (t_0, \infty)$, then we are done. If $I \neq (t_0, \infty)$, then we can find a smallest $\tilde{t} \in (t_0, \infty)$ such that $\frac{1}{2-\alpha_{Df}}(\tilde{\lambda}_1 + \tilde{t}, \cdots, \tilde{\lambda}_n + \tilde{t}) = \alpha_f(\tilde{\lambda}_1 + \tilde{t}, \cdots, \tilde{\lambda}_n + \tilde{t})$. This is ensured because we can treat $f$ as a polynomial in terms of $t$ and

$$\begin{aligned}
&\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+t)}-\alpha_f(\tilde{\lambda}+t)\\
&\quad=\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+t)}\big(1-(2-\alpha_{Df}(\tilde{\lambda}+t))\alpha_f(\tilde{\lambda}+t)\big)\\
&\quad=\frac{1}{2-\alpha_{Df}}\Big(1-\Big(2-\frac{DfD^3f}{(D^2f)^2}\Big)\frac{fD^2f}{(Df)^2}\Big)\\
&\quad=\frac{1}{(2-\alpha_{Df})(Df)^2D^2f}\Big((Df)^2D^2f+fDfD^3f-2f(D^2f)^2\Big).
\end{aligned}$$

The term $(Df)^2D^2f+fDfD^3f-2f(D^2f)^2$ is just a polynomial in terms of $t$, so can only have finitely many zeros. Then when $t=\tilde{t}$, we have

$$D\alpha_f(\tilde{\lambda}+\tilde{t})=\frac{D^2f(\tilde{\lambda}+\tilde{t})\big(1-(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))\alpha_f(\tilde{\lambda}+\tilde{t})\big)}{Df(\tilde{\lambda}+\tilde{t})}=0. \tag{2.27}$$

On the other hand, since $Df$ is $\Upsilon$-stable and by mathematical induction, we have

$$D\Big|_{t=\tilde{t}}\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+t)}=\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}>0. \tag{2.28}$$

We get a contradiction. Otherwise by standard calculus argument and equation (2.27), there exists a $\delta>0$ sufficiently small such that if $|h|<\delta$, then we have

$$-\frac{1}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}<\frac{\alpha_f(\tilde{\lambda}+\tilde{t})-\alpha_f(\tilde{\lambda}+\tilde{t}-h)}{h}<\frac{1}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}. \tag{2.29}$$

Since $D\big(\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+t)}\big)=\frac{D\alpha_{Df}(\tilde{\lambda}+t)}{(2-\alpha_{Df}(\tilde{\lambda}+t))^2}$, by inequality (2.28), for $\delta$ sufficiently small, we get

$$\frac{2}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}<\frac{\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+\tilde{t})}-\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}-h)}}{h}<\frac{4}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}. \tag{2.30}$$

Since $\tilde{t}$ is the smallest value such that $1/(2-\alpha_{Df}(\tilde{\lambda}+t))=\alpha_f(\tilde{\lambda}+t)$, by the intermediate value theorem, we have $\tilde{\lambda}+\tilde{t}-h\in I$ where $\delta>h>0$. Hence, by inequalities (2.29) and

(2.30), we obtain

$$\begin{aligned}\frac{1}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2} &> \frac{\alpha_f(\tilde{\lambda}+\tilde{t})-\alpha_f(\tilde{\lambda}+\tilde{t}-h)}{h} = \frac{\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+\tilde{t})}-\alpha_f(\tilde{\lambda}+\tilde{t}-h)}{h} \\ &> \frac{\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+\tilde{t})}-\frac{1}{2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}-h)}}{h} > \frac{2}{3}\frac{D\alpha_{Df}(\tilde{\lambda}+\tilde{t})}{(2-\alpha_{Df}(\tilde{\lambda}+\tilde{t}))^2}.\end{aligned}$$

This is a contradiction because $D\alpha_{Df} > 0$ by mathematical induction. Hence, $I = (t_0, \infty)$.

For the last case, if $(\tilde{\lambda}_1 + t_0, \cdots, \tilde{\lambda}_n + t_0) \notin \partial\Upsilon_1$, then $t_0 \in I$ due to the fact that

$$\alpha_f(\tilde{\lambda}+t_0) = \frac{f(\tilde{\lambda}+t_0)\cdot D^2 f(\tilde{\lambda}+t_0)}{(Df(\tilde{\lambda}+t_0))^2} = 0$$

and $2 - \alpha_{Df}(\tilde{\lambda}+t_0) \geq 2 - (1 - \frac{1}{m-1}) = \frac{s}{s-1} > 0$. Hence, $I$ is open and not an empty set. Similar to the previous argument, we get $I = (t_1, \infty)$, where $(\tilde{\lambda}_1 + t_1, \cdots, \tilde{\lambda}_n + t_1) \in \partial\Upsilon_1$. This finishes the proof. □

Even though the original Newton–Maclaurin's inequality holds as long as $\{\lambda_1, \cdots, \lambda_n\}$ are all real numbers, here, we mainly focus on the $\Upsilon$-cones and we obtain a monotonicity result due to Theorem 2.3.

**Proposition 2.11.** *The ratio* $\frac{\sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda)}{\sigma_k^2(\lambda)}$ *is increasing on* $\bigcap_{i=1}^n\big\{\sigma_k(\lambda_{;i}) > 0\big\} \cap \{(\tilde{\lambda}_1 + t, \cdots, \tilde{\lambda}+t) \colon t \in \mathbb{R}\}$ *for any* $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \in \bigcap_{i=1}^n\big\{\sigma_k(\lambda_{;i}) > 0\big\}$*. In particular, we have*

$$\frac{\sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda)}{\sigma_k^2(\lambda)} \leq \frac{k(n-k)}{(k+1)(n-k+1)} \qquad on \qquad \bigcap_{i=1}^n\big\{\sigma_k(\lambda_{;i}) > 0\big\},$$

*which is equivalent to*

$$\frac{\sigma_{k-1}(\lambda)}{\binom{n}{k-1}}\cdot\frac{\sigma_{k+1}(\lambda)}{\binom{n}{k+1}} \leq \Big(\frac{\sigma_k(\lambda)}{\binom{n}{k}}\Big)^2 \qquad on \qquad \bigcap_{i=1}^n\big\{\sigma_k(\lambda_{;i}) > 0\big\}.$$

*Proof.* By letting $f = \sigma_{k+1}(\lambda)$, the diagonal restriction will be $r_f(x) = \binom{n}{k+1}x^{k+1}$, which is right-Noetherian. By Theorem-2.3, this finishes the proof. $\square$

**Definition 2.9.** Let $f := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ and $g := \sigma_s(\lambda) - \sum_{k=0}^{s-1} d_k\sigma_k(\lambda)$ be two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials. For $i \in \{0, \cdots, s-1\}$, we write $x_i$ the largest real root of the $i$-th derivative of the diagonal restriction $r_f^{(i)}$ of $f$ and $y_i$ the largest real root of the $i$-th derivative of the diagonal restriction $r_g^{(i)}$ of $g$. If $y_i \geq x_i$ for all $i \in \{0, \cdots, s-1\}$, then we say $g \gtrdot f$.

**Theorem 2.4** ($\Upsilon$-dominance)**.** *Let $f := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ and $g := \sigma_s(\lambda) - \sum_{k=0}^{s-1} d_k\sigma_k(\lambda)$ be two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials. Then $g \gtrdot f$ if and only if $\Gamma_g^n \subset \Gamma_f^n$.*

*Proof.* First, if $s = 1$, then we can write $f(\lambda) = \lambda_1 + \cdots + \lambda_n - c_0$ and $g(\lambda) = \lambda_1 + \cdots + \lambda_n - d_0$. If $g \gtrdot f$, then we have $d_0/n \geq c_0/n$. This implies that $\Gamma_g^n \subset \Gamma_f^n$. On the other hand, if $\Gamma_g^n \subset \Gamma_f^n$, then we can also get $g \gtrdot f$.

Second, for the case $s = n$, this case can be done due to previous work in [52]. Then, we may assume that $2 \leq s \leq n-1$. We use mathematical induction on the variables $n$ to prove this, for convenience, we assume $d_{s-1} = 0$. When $n = 1$ or $n = 2$, the proof should be straightforward. Suppose the statement is true when $n = m-1$. Then, when $n = m$, if $\Gamma_g^m \subset \Gamma_f^m$, by denoting the largest real root of $r_f^{(k)}$ by $x_k$ and the largest real root of $r_g^{(k)}$ by $y_k$, we immediately get $y_0 \geq x_0$. The rest follows from mathematical induction, thus $g \gtrdot f$. On the other hand, if $g \gtrdot f$, suppose $\Gamma_g^m \not\subset \Gamma_f^m$, there exists $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_m)$ such that

$$g(\tilde{\lambda}) = \sigma_s(\tilde{\lambda}) - \sum_{k=0}^{s-2} d_k\sigma_k(\tilde{\lambda}) > 0 \quad \text{and} \quad f(\tilde{\lambda}) = \sigma_s(\tilde{\lambda}) - \sum_{k=0}^{s-1} c_k\sigma_k(\tilde{\lambda}) \leq 0.$$

Similar to the proof in previous Lemma 2.6, we use the method of Lagrange multipliers to find the local extrema of $f$ under the constraint $g = g(\tilde{\lambda})$. There exists only one local extremum at $(\tilde{y}_0, \cdots \tilde{y}_0)$, where $\tilde{y}_0$ is the largest real root of the diagonal restriction of $g - g(\tilde{\lambda})$. In addition, under the constraint $g = g(\tilde{\lambda}) > 0$, the partial derivative of $f$ with respect to $\lambda_i$

for $i \in \{1, \cdots, m-1\}$ will be

$$\frac{\partial}{\partial \lambda_i} f = f_i + \frac{\partial \lambda_m}{\partial \lambda_i} f_m = \frac{1}{g_m}\big(f_i g_m - g_i f_m\big) = \frac{\lambda_m - \lambda_i}{g_m}\big(g_m f_{im} - g_{im} f_m\big). \tag{2.31}$$

For the quantity $g_m f_{im} - f_{im} g_m$ in equation (2.31), we have

$$\frac{\partial}{\partial \lambda_i}\big(g_m f_{im} - g_{im} f_m\big) = g_{im} f_{im} - g_{im} f_{im} = 0.$$

So the quantity $g_m f_{im} - g_{im} f_m$ is independent of the value of $\lambda_i$ and $\lambda_m$. By Theorem 2.4 and Proposition 2.9, since $r_g$ is right-Noetherian, we have $g = g(\tilde{\lambda})$ is contained in the $\Upsilon_1$-cone $\Upsilon_1^g$ of $g - g(\tilde{\lambda})$. By fixing the values of $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{i-1}, \tilde{\lambda}_{i+1}, \cdots, \tilde{\lambda}_m$ and decreasing the value of the $i$-th entry, it will intersect with $\{g_m = 0\}$. At this intersection point, the quantity $g_m f_{im} - g_{im} f_m$ will be $g_m f_{im} - g_{im} f_m = -g_{im} f_m \leq 0$. The last inequality is due to mathematical induction, $g_m \gtrdot f_m$ if and only if $\Gamma^{m-1}_{g_m} \subset \Gamma^{m-1}_{f_m}$. For convenience, we suppose $\lambda_m$ is the smallest value between $\{\lambda_1, \cdots, \lambda_m\}$, then equation (2.31) will satisfy

$$\frac{\partial}{\partial \lambda_i} f = \frac{\lambda_m - \lambda_i}{g_m}(g_m f_{im} - g_{im} f_m) \geq 0.$$

By the fact that $(\tilde{y}_0, \cdots, \tilde{y}_0)$ is the unique local extrema of $f$ under the constraint $g = g(\tilde{\lambda})$,

$$0 \geq f(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_m) \geq f(\tilde{y}_0, \cdots, \tilde{y}_0) > f(y_0, \cdots, y_0) \geq f(x_0, \cdots, x_0) = 0.$$

This is a contradiction, hence we finish the proof. □

Here, we skip the proof of the following Lemma. By using mathematical induction, the proof should be straightforward.

**Lemma 2.7.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ and $g(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} d_k \sigma_k(\lambda)$ be two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials. If $g \gtrdot f$, then for any $(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \in \{g > 0\}$, we have $f(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n) \geq g(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$.*

By considering the difference of two $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials with one $\Upsilon$-dominant another, we get the following Positivstellensatz-type result.

**Lemma 2.8.** *Let* $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ *and* $g(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} d_k\sigma_k(\lambda)$ *be two* $\Upsilon$*-stable general* $\sigma_k$ *type multilinear polynomials. If* $g \gtrdot f$*, then*

$$\Gamma_g^n \subset \{f - g = \sum_{k=0}^{s-1}(d_k - c_k)\sigma_k(\lambda) \geq 0\}.$$

*Proof.* The proof follows from Theorem 2.4 and Lemma 2.7. □

Note that similar to before, we need to specify the connected component of $\{g - f > 0\}$. A simple application of Lemma 2.8 will be the inequality of arithmetic and geometric means.

## 2.3 Convexity of the Level Set

In this section, let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. If $\{f(\lambda) = 0\}$ is strictly $\Upsilon$-stable, then we use a classical way to prove the convexity of this level set $\{f(\lambda) = 0\}$. By doing the substitution in Lemma-2.3, we may assume $c_{s-1} = 0$ and consider the following general $\sigma_k$ equation

$$f(\lambda) = f(\lambda_1, \cdots, \lambda_n) = \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda) = 0.$$

There are two ways to compute the convexity, first, if we write

$$h = \frac{\sum_{k=0}^{s-2} c_k\sigma_k(\lambda)}{\sigma_s(\lambda)}, \tag{2.32}$$

then we have the following.

**Lemma 2.9.** *If the following $n-1\times n-1$ Hermitian matrix is positive semi-definite*

$$\Big(h_{ij}+h_{nn}\frac{h_ih_j}{h_n^2}-h_{in}\frac{h_j}{h_n}-h_{jn}\frac{h_i}{h_n}\Big)_{i,j\in\{1,\cdots,n-1\}}, \tag{2.33}$$

*then the level set $\{h=c\}$ is convex. Here, $h_i:=\partial h/\partial\lambda_i$ and $h_{ij}:=\partial^2 h/\partial\lambda_i\partial\lambda_j$*

*Proof.* Let $V=(V_1,\cdots,V_n)\in T_\lambda\big\{h=c\big\}$ be a tangent vector, which gives, $\sum_i h_iV_i=0$. Then, to get convexity, which is equivalent to the following quantity

$$\sum_{i,j}h_{ij}V_iV_{\bar{j}} \tag{2.34}$$

is non-negative. Since $V$ is a tangent vector, we can write $V_n=-\sum_{i=1}^{n-1}h_iV_i/h_n$. By plugging in quantity (2.34), we obtain

$$\begin{aligned}\sum_{i,j}h_{ij}V_iV_{\bar{j}}=&\sum_{i=1}^{n-1}\Big(h_{ii}+h_{nn}\frac{h_i^2}{h_n^2}-2h_{in}\frac{h_i}{h_n}\Big)|V_i|^2\\&+\sum_{1\leq i<j\leq n-1}\Big(h_{ij}+h_{nn}\frac{h_ih_j}{h_n^2}-h_{in}\frac{h_j}{h_n}-h_{jn}\frac{h_i}{h_n}\Big)(V_iV_{\bar{j}}+V_{\bar{i}}V_j).\end{aligned} \tag{2.35}$$

So, if the following $n-1\times n-1$ Hermitian matrix is positive semi-definite

$$\Big(h_{ij}+h_{nn}\frac{h_ih_j}{h_n^2}-h_{in}\frac{h_j}{h_n}-h_{jn}\frac{h_i}{h_n}\Big)_{i,j\in\{1,\cdots,n-1\}},$$

then the quantity (2.35) is non-negative. This implies that the level set $\{h=c\}$ is convex. $\square$

If we write

$$\lambda_n=-\frac{\sigma_s(\lambda_{;n})-\sum_{k=0}^{s-2}c_k\sigma_k(\lambda_{;n})}{\sigma_{s-1}(\lambda_{;n})-\sum_{k=1}^{s-2}c_k\sigma_{k-1}(\lambda_{;n})}, \tag{2.36}$$

then the Hessian of $\lambda_n$ is related to $n-1\times n-1$ Hermitian matrix (2.33) as follows.

**Lemma 2.10.** *Let $h = \sum_{k=0}^{s-2} c_k\sigma_k(\lambda)/\sigma_s(\lambda)$, then we have*

$$h_i = -\frac{H_{1;i}}{\sigma_s(\lambda)} \quad \textit{and} \quad h_{ij} = \frac{\sigma_{s-1}(\lambda_{;j})H_{1;i} + \sigma_{s-1}(\lambda_{;i})H_{1;j}}{\sigma_s^2(\lambda)} - (1-\delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)},$$

*where we denote by $h_i \coloneqq \partial h/\partial\lambda_i$ and $h_{ij} \coloneqq \partial^2 h/\partial\lambda_i\partial\lambda_j$. Moreover, we have*

$$\begin{aligned} &h_{ij} + h_{nn}\frac{h_i h_j}{h_n^2} - h_{in}\frac{h_j}{h_n} - h_{jn}\frac{h_i}{h_n} \\ &= \frac{H_{1;j}H_{2;i,n} + H_{1;i}H_{2;j,n}}{\sigma_s(\lambda)H_{1;n}} - (1-\delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)} = \frac{H_{1;n}}{\sigma_s(\lambda)}\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\lambda_n. \end{aligned} \tag{2.37}$$

*Here, for $i, j, k$ pairwise distinct, we denote*

$$\begin{aligned} H &\coloneqq h\sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda); & H_{0;i} &\coloneqq h\sigma_s(\lambda_{;i}) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;i}); \\ H_{1;i} &\coloneqq h\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i}); & H_{1;i,j} &\coloneqq h\sigma_{s-1}(\lambda_{;i,j}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i,j}); \\ H_{2;i,j} &\coloneqq h\sigma_{s-2}(\lambda_{;i,j}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;i,j}); & H_{2;i,j,k} &\coloneqq h\sigma_{s-2}(\lambda_{;i,j,k}) - \sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;i,j,k}). \end{aligned}$$

*Proof.* First, we have the following

$$0 = H = \lambda_i\Big(h\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i})\Big) + h\sigma_s(\lambda_{;i}) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;i}).$$

This implies that

$$H_{0;i} \coloneqq h\sigma_s(\lambda_{;i}) - \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;i}) = -\lambda_i H_{1;i}. \tag{2.38}$$

In addition, for $i, j, k$ pairwise distinct, we get

$$H_{1;i} = H_{1;i,j} + \lambda_j H_{2;i,j}; \tag{2.39}$$

$$H_{2;i,j} = H_{2;i,j,k} + \lambda_k H_{3;i,j,k}. \tag{2.40}$$

Second, for the first order partial derivatives of $h$, by equation (2.38), we get

$$\begin{aligned}
h_i = \frac{\partial h}{\partial \lambda_i} &= \frac{\partial}{\partial \lambda_i}\Big(\frac{\sum_{k=0}^{s-2} c_k \sigma_k(\lambda)}{\sigma_s(\lambda)}\Big) = \frac{\sigma_s(\lambda)\sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i}) - \sigma_{s-1}(\lambda_{;i})\sum_{k=0}^{s-2} c_k\sigma_k(\lambda)}{\sigma_s^2(\lambda)} \\
&= \frac{\sigma_s(\lambda_{;i})\sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i}) - \sigma_{s-1}(\lambda_{;i})\sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;i})}{\sigma_s^2(\lambda)} \\
&= \frac{\sigma_s(\lambda_{;i})\big(h\sigma_{s-1}(\lambda_{;i}) - H_{1;i}\big) - \sigma_{s-1}(\lambda_{;i})\big(h\sigma_s(\lambda_{;i}) - H_{0;i}\big)}{\sigma_s^2(\lambda)} = -\frac{H_{1;i}}{\sigma_s(\lambda)}.
\end{aligned} \tag{2.41}$$

Then, for the second order partial derivatives of $h$, by equation (2.38), we obtain

$$\begin{aligned}
h_{ij} = \frac{\partial^2 h}{\partial\lambda_i\partial\lambda_j} &= -\frac{\partial}{\partial\lambda_j}\Big(\frac{H_{1;i}}{\sigma_s(\lambda)}\Big) = -\frac{\partial}{\partial\lambda_j}\Big(\frac{h\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i})}{\sigma_s(\lambda)}\Big) \\
&= -\frac{h_j\sigma_{s-1}(\lambda_{;i}) + h(1-\delta_{ij})\sigma_{s-2}(\lambda_{;i,j}) - (1-\delta_{ij})\sum_{k=2}^{s-2} c_k\sigma_{k-2}(\lambda_{;i,j})}{\sigma_s(\lambda)} \\
&\quad\quad + \sigma_{s-1}(\lambda_{;j})\frac{h\sigma_{s-1}(\lambda_{;i}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;i})}{\sigma_s^2(\lambda)} \\
&= \frac{\sigma_{s-1}(\lambda_{;j})H_{1;i} + \sigma_{s-1}(\lambda_{;i})H_{1;j}}{\sigma_s^2(\lambda)} - (1-\delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)}.
\end{aligned} \tag{2.42}$$

Hence, by equation (2.41) and (2.42), we have

$$\begin{aligned}
&h_{ij} + h_{nn}\frac{h_ih_j}{h_n^2} - h_{in}\frac{h_j}{h_n} - h_{jn}\frac{h_i}{h_n} \\
&= \frac{\sigma_{s-1}(\lambda_{;j})H_{1;i} + \sigma_{s-1}(\lambda_{;i})H_{1;j}}{\sigma_s^2(\lambda)} - (1-\delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)} + 2\frac{\sigma_{s-1}(\lambda_{;n})H_{1;i}H_{1;j}}{\sigma_s^2(\lambda)H_{1;n}} \\
&\quad - \Big(\frac{\sigma_{s-1}(\lambda_{;i})H_{1;n} + \sigma_{s-1}(\lambda_{;n})H_{1;i}}{\sigma_s^2(\lambda)} - \frac{H_{2;i,n}}{\sigma_s(\lambda)}\Big)\frac{H_{1;j}}{H_{1;n}} \\
&\quad - \Big(\frac{\sigma_{s-1}(\lambda_{;j})H_{1;n} + \sigma_{s-1}(\lambda_{;n})H_{1;j}}{\sigma_s^2(\lambda)} - \frac{H_{2;j,n}}{\sigma_s(\lambda)}\Big)\frac{H_{1;i}}{H_{1;n}} \\
&= \frac{H_{1;j}H_{2;i,n} + H_{1;i}H_{2;j,n}}{\sigma_s(\lambda)H_{1;n}} - (1-\delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)}.
\end{aligned} \tag{2.43}$$

On the other hand, on $h = \sum_{k=0}^{s-2} c_k\sigma_k(\lambda)/\sigma_s(\lambda)$, we have

$$\lambda_n = \frac{-h\sigma_s(\lambda_{;n}) + \sum_{k=0}^{s-2} c_k\sigma_k(\lambda_{;n})}{h\sigma_{s-1}(\lambda_{;n}) - \sum_{k=1}^{s-2} c_k\sigma_{k-1}(\lambda_{;n})} = -\frac{H_{0;n}}{H_{1;n}}.$$

This implies that the first order partial derivatives of $\lambda_n$ will be

$$\frac{\partial \lambda_n}{\partial \lambda_i} = -\frac{\partial}{\partial \lambda_i}\Big(\frac{H_{0;n}}{H_{1;n}}\Big) = \frac{H_{1;i,n}H_{1;n} - H_{0;n}H_{2;i,n}}{H_{1;n}^2}.$$

So the second order partial derivatives of $\lambda_n$ will be

$$\begin{aligned}\frac{\partial^2}{\partial \lambda_i^2}\lambda_n &= -\frac{\partial}{\partial \lambda_i}\Big(\frac{H_{1;i,n}H_{1;n} - H_{0;n}H_{2;i,n}}{H_{1;n}^2}\Big) = 2H_{2;i,n}\frac{H_{1;i,n}H_{1;n} - H_{0;n}H_{2;i,n}}{H_{1;n}^3} \\ &= 2H_{2;i,n}\frac{(H_{1;i} - \lambda_n H_{2;i,n})H_{1;n} + \lambda_n H_{1;n}H_{2;i,n}}{H_{1;n}^3} = 2\frac{H_{1;i}H_{2;i,n}}{H_{1;n}^2} \qquad (2.44)\end{aligned}$$

and for $i \neq j$, similarly, we get

$$\begin{aligned}\frac{\partial^2}{\partial \lambda_j \partial \lambda_i}\lambda_n &= -\frac{\partial}{\partial \lambda_j}\Big(\frac{H_{1;i,n}H_{1;n} - H_{0;n}H_{2;i,n}}{H_{1;n}^2}\Big) \\ &= -\frac{H_{2;i,j,n}}{H_{1;n}} + \frac{H_{1;i,n}H_{2;j,n} + H_{1;j,n}H_{2;i,n} + H_{0;n}H_{3;i,j,n}}{H_{1;n}^2} - 2\frac{H_{0;n}H_{2;i,n}H_{2;j,n}}{H_{1;n}^3} \\ &= -\frac{H_{2;i,j}}{H_{1;n}} + \frac{H_{1;i}H_{2;j,n} + H_{1;j}H_{2;i,n}}{H_{1;n}^2}. \qquad (2.45)\end{aligned}$$

By comparing equations (2.43), (2.44), and (2.45), we con conclude that

$$h_{ij} + h_{nn}\frac{h_i h_j}{h_n^2} - h_{in}\frac{h_j}{h_n} - h_{jn}\frac{h_i}{h_n} = \frac{H_{1;n}}{\sigma_s(\lambda)}\frac{\partial^2}{\partial \lambda_i \partial \lambda_j}\lambda_n.$$

This finishes the proof. □

**Theorem 2.5** (Convexity of the general $\sigma_k$ equation)**.** *Let $f(\lambda) := \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k\sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. If the diagonal restriction $r_f(x) = \binom{n}{s}x^s - \sum_{k=0}^{s-1} c_k\binom{n}{k}x^k$ is strictly right-Noetherian, then $\{f = 0\}$ is convex.*

*Proof.* For the case $s = 1$, the level set is a hyperplane, so the level set will be convex. We assume $c_{s-1} = 0$ for convenience. We prove this by mathematical induction on the number of variables $n$. When $n = 2$, then we get $s = 2$ and this can be done due to previous work in

[52]. So for convenience, we may assume that $s < n$. Similar to the proof in [52], we **claim** that for any point in the enclosed region and every line passing through this point, the line will intersect with the level set of $\{f(\lambda) = 0\}$ at at least one point and at most two points.

Suppose the statement and the claim are true when $n = m - 1$. When $n = m \geq 3$, we first show that the claim is true. Let $(x_1, \cdots, x_m)$ be in the enclosed region and $(v_1, \cdots, v_m)$ be the tangent of the line. We define $\tilde{f}(t) \coloneqq f(x_1 + v_1 t, \cdots, x_m + v_m t)$, if $v_i = 0$ for some $i \in \{1, \cdots, m\}$. By fixing $x_i$ and Proposition 2.8, we may view $f(\bullet, x_i, \bullet)$ as a degree $s$ general $\sigma_k$ equation with $m-1$ variables. By mathematical induction, $\tilde{f}$ will intersect with the level set of $\{f(\bullet, x_i, \bullet) = 0\}$ at at least one point and at most two points. So we may assume $v_1 \cdots v_m \neq 0$, $\tilde{f}$ is a degree $s$ polynomial. If $s$ is an odd number, then by letting $t$ approach $\infty$ or $-\infty$, $\tilde{f}$ intersects with the level set of $\{f = 0\}$ at at least one point. If $s$ is an even number, then by letting $t$ approach $\infty$ or $-\infty$, $\tilde{f}_1 \coloneqq f_1(x_1 + v_1 t, \cdots, x_m + v_m t)$ intersects with the $\Upsilon_1$-cone at at least one point. Since $\{f = 0\}$ is contained in the $\Upsilon_1$-cone, so $\tilde{f}$ intersects with the level set of $\{f = 0\}$ at at least one point. Let $t_1 > 0$ be the smallest number (might not exist) such that $\tilde{f}(t_1) = 0$. If there exists $\epsilon > 0$ such that $\tilde{f}(t) > 0$ for all $\epsilon > t - t_1 > 0$, then similar to before, by doing a small perturbation on tangent, there exists new $t_1$ and $t_2$ with $t_2 > t_1$ such that $\tilde{f}(t_1) = 0 = \tilde{f}(t_2)$, $\tilde{f}(t) < 0$ for any $t \in (t_1, t_2)$, and $\tilde{f}(t) > 0$ for any $t \in (t_2, t_2 + \epsilon)$ for $\epsilon > 0$ small. For any $t \in (t_1, t_2)$, $0 > \tilde{f}(t) = f(x_1 + v_1 t, \cdots, x_m + v_m t)$. For the case $v_m > 0$, by considering $\frac{d}{dt}\tilde{f}_m$, we have

$$\frac{d}{dt}\tilde{f}_m = \frac{d}{dt} f_m(x_1 + v_1 t, \cdots, x_{m-1} + v_{m-1} t) = v_1 f_{1m} + \cdots + v_{m-1} f_{m-1\,m}.$$

If $\frac{d}{dt}\tilde{f}_m(t_2) = 0$, then for the case $s = 2$, we can perturb the tangent such that $\frac{d}{dt}\tilde{f}_m(t_2) \neq 0$. Moreover, for the case $s \geq 3$, again we can perturb the tangent such that $\frac{d}{dt}\tilde{f}_m$ is a polynomial of degree $s - 2$. Hence, there exists an $\epsilon_1 > 0$ sufficiently small such that for any $t \in (t_2 - \epsilon_1, t_2) \cup (t_2, t_2 + \epsilon_1)$, $\frac{d}{dt}\tilde{f}_m(t) \neq 0$ (actually the punctured disk with radius $\epsilon_1$ and center $t_2$ in the complex plane). Now, we may perturb $v_m$ slightly smaller so that $\tilde{f}(t_2) < 0$

and there exists $\tilde{t}_2 \in (t_2, t_2 + \epsilon_1)$ such that $\tilde{f}(\tilde{t}_2) = 0$ and $\frac{d}{dt}\tilde{f}_m(\tilde{t}_2) \neq 0$. This is ensured by considering $u_m > 0$ sufficiently small and we consider the following perturbation

$$f\big(x_1 + v_1 t, \cdots, x_{m-1} + v_{m-1}t, x_m + (v_m - u_m)t\big).$$

For $u_m > 0$ sufficiently small, we have

$$\begin{aligned} f\big(x_1 + v_1 t_2, \cdots, x_{m-1} + v_{m-1}t_2, x_m + (v_m - u_m)t_2\big) &< \tilde{f}(t_2) = 0; \\ f\big(x_1 + v_1(t_2 + \frac{\epsilon_1}{2}), \cdots, x_{m-1} + v_{m-1}(t_2 + \frac{\epsilon_1}{2}), x_m + (v_m - u_m)(t_2 + \frac{\epsilon_1}{2})\big) &> 0. \end{aligned}$$

So by the intermediate value theorem, there exists $\tilde{t}_2 \in (t_2, t_2 + \epsilon_1/2)$ such that

$$f\big(x_1 + v_1\tilde{t}_2, \cdots, x_{m-1} + v_{m-1}\tilde{t}_2, x_m + (v_m - u_m)\tilde{t}_2\big) = 0.$$

Moreover, $\frac{d}{dt}\tilde{f}_m(\tilde{t}_2) \neq 0$. By replacing $t_1$ and $t_2$, we may assume that there exists $t_1$ and $t_2$ with $t_2 > t_1$ such that $\tilde{f}(t_1) = 0 = \tilde{f}(t_2)$, $\tilde{f}(t) < 0$ for any $t \in (t_1, t_2)$, and $\frac{d}{dt}\tilde{f}_m(t_2) \neq 0$. For the case $\frac{d}{dt}\tilde{f}_m(t_2) > 0$, since $\tilde{f}(t_2) = 0$, we may pick $\tilde{\epsilon} > 0$ sufficiently small so that

$$f\big(x_1 + v_1(t_2 + \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 + \tilde{\epsilon}), x_m + v_m t_2\big) > 0.$$

For any $t \in (t_1, t_2)$, by fixing the last entry $x_m + v_m t$, since $v_m > 0$ and $t_1 > 0$, we have

$$\begin{aligned} \tilde{f}(t) = f(x_1 + v_1 t, \cdots, x_m + v_m t) &< 0; \\ f(x_1 + v_1 t_1, \cdots, x_{m-1} + v_{m-1}t_1, x_m + v_m t) &> \tilde{f}(t_1) = 0. \end{aligned}$$

By continuity, we have $\lim_{t \to t_2} f\big(x_1 + v_1(t_2 + \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 + \tilde{\epsilon}), x_m + v_m t\big) = f\big(x_1 + v_1(t_2 + \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 + \tilde{\epsilon}), x_m + v_m t_2\big) > 0$. By picking $\tilde{t}$ sufficiently close to

$t_2$, we have

$$f\big(x_1 + v_1(t_2 + \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 + \tilde{\epsilon}), x_m + v_m\tilde{t}\big) > 0.$$

Similar to before, by fixing the value $\tilde{t}$ and the last entry, we may view this as a line passing through a point in the enclosed region of a degree $s$ general $\sigma_k$ equation. By letting $g(t) \coloneqq f\big(x_1 + v_1 t, \cdots, x_{m-1} + v_{m-1}t, x_m + v_m\tilde{t}\big)$, we have

$$g(t_1) > 0,\ g(\tilde{t}) < 0,\ g(t_2 + \tilde{\epsilon}) > 0.$$

By letting $t$ approach $\infty$ or $-\infty$, we see that this line intersects with the level set at at least three points, which is a contradiction. For the case $\frac{d}{dt}\tilde{f}_m(t_2) < 0$, since $\tilde{f}(t_2) = f(x_1 + v_1t_2, \cdots, x_m + v_mt_2) = 0$, we may pick $\tilde{\epsilon} > 0$ sufficiently small so that

$$f\big(x_1 + v_1(t_2 - \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 - \tilde{\epsilon}), x_m + v_mt_2\big) > 0.$$

For any $t \in (t_1, t_2)$, by fixing the last entry $x_m + v_mt$, since $v_m > 0$ and $t_1 > 0$, we have

$$\begin{aligned} \tilde{f}(t) &= f(x_1 + v_1t, \cdots, x_m + v_mt) < 0; \\ f(x_1 + v_1t_1, \cdots, x_{m-1} + v_{m-1}t_1, x_m + v_mt) &> \tilde{f}(t_1) = 0. \end{aligned}$$

By continuity, we have $\lim_{t \to t_2} f\big(x_1 + v_1(t_2 - \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 - \tilde{\epsilon}), x_m + v_mt\big) = f\big(x_1 + v_1(t_2 - \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 - \tilde{\epsilon}), x_m + v_mt_2\big) > 0$. By picking $\tilde{t}$ sufficiently close to $t_2$, we have

$$f\big(x_1 + v_1(t_2 - \tilde{\epsilon}), \cdots, x_{m-1} + v_{m-1}(t_2 - \tilde{\epsilon}), x_m + v_m\tilde{t}\big) > 0.$$

The rest follows similarly, hence we get a contradiction. So $\tilde{f}(t) < 0$ for $\epsilon > t - t_1 > 0$ where $\epsilon > 0$ is small. For $t > t_1$, if this ray again intersects with the level set of $\{f = 0\}$, say $x_2$ is the smallest one. Thus $\tilde{f}(t_1) = 0 = \tilde{f}(t_2)$ and $\tilde{f}(t) < 0$ for any $t \in (t_1, t_2)$. The rest argument should be similar to above, which again gives us a contradiction. For the case $v_m < 0$, the proof is similar to above. In conclusion, for any ray passing through a point in the enclosed region, this ray intersects with the level set of $\{f = 0\}$ at at most one point. Combining this with the fact that the line will intersect with the level set of $\{f = 0\}$ at at least one point, we justify the claim. With this claim, using previous argument we can prove that the set $\{f = 0\}$ is convex because the enclosed region is convex. This finishes the proof. □

We prove the following Lemma to end this section, which shows that Theorem 2.1 is equivalent to the positive definiteness of the Hessian matrix of $\lambda_n$ on the curve $\{\lambda_1 = \cdots = \lambda_{n-1}\}$.

**Lemma 2.11.** *Let $f(\lambda) \coloneqq \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda)$ be a general $\sigma_k$ type multilinear polynomial. If the diagonal restriction $r_f(x)$ of $f$ is strictly right-Noetherian, then on the curve $\{\lambda_1 = \cdots = \lambda_{n-1}\}$ of the level set $\{f = 0\}$ with $\lambda_1 > x_1$, the positive definiteness of the following $n-1 \times n-1$ Hessian matrix*

$$\left(\frac{\partial^2 \lambda_n}{\partial \lambda_i \partial \lambda_j}\right)_{i,j\in\{1,\cdots,n-1\}}$$

*is equivalent to the monotonicity of log-concavity ratio of $r_f(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-1} c_k \binom{n}{k} x^k$. Here, $x_1$ is the largest real root of $r_f'$.*

*Proof.* For convenience, we assume $c_{s-1} = 0$. By Lemma 2.10, we show that the following matrix is positive-definite at every point on the curve $\{\lambda_1 = \cdots = \lambda_{n-1} = x\}$ with $x > x_1$:

$$\left(h_{ij} + h_{nn}\tfrac{h_i h_j}{h_n^2} - h_{in}\tfrac{h_j}{h_n} - h_{jn}\tfrac{h_i}{h_n}\right)_{i,j\in\{1,\cdots,n-1\}}.$$

By Lemma 2.10, we have

$$h_{ij} + h_{nn}\frac{h_i h_j}{h_n^2} - h_{in}\frac{h_j}{h_n} - h_{jn}\frac{h_i}{h_n} = \frac{H_{1;j}H_{2;i,n} + H_{1;i}H_{2;j,n}}{\sigma_s(\lambda) H_{1;n}} - (1 - \delta_{ij})\frac{H_{2;i,j}}{\sigma_s(\lambda)}.$$

Now, it suffices to show that the following $n-1 \times n-1$ matrix is positive semi-definite

$$
\begin{aligned}
C \coloneqq\ & 2H_{2;1,n}\big(H_{1;1,n} + \lambda_n H_{2;1,n}\big)\mathbb{1}_{n-1\times n-1} \\
& - H_{1;n}(H_{2;1,2,n} + \lambda_n H_{3;1,2,n})(\mathbb{1}_{n-1\times n-1} - \mathbb{I}_{n-1\times n-1}),
\end{aligned} \tag{2.46}
$$

where $\mathbb{1}_{n-1\times n-1}$ is the $n-1\times n-1$ all-ones matrix, $\mathbb{I}_{n-1\times n-1}$ is the $n-1\times n-1$ identity matrix, and we have

$$
\begin{aligned}
H_{1;n} &= \sigma_{s-1}(\lambda_{;n}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;n}) = \binom{n-1}{s-1} x^{s-1} - \sum_{k=1}^{s-2} c_k \binom{n-1}{k-1} x^{k-1};\\
H_{1;1,n} &= \sigma_{s-1}(\lambda_{;1,n}) - \sum_{k=1}^{s-2} c_k \sigma_{k-1}(\lambda_{;1,n}) = \binom{n-2}{s-1} x^{s-1} - \sum_{k=1}^{s-2} c_k \binom{n-2}{k-1} x^{k-1};\\
H_{2;1,n} &= \sigma_{s-2}(\lambda_{;1,n}) - \sum_{k=2}^{s-2} c_k \sigma_{k-2}(\lambda_{;1,n}) = \binom{n-2}{s-2} x^{s-2} - \sum_{k=2}^{s-2} c_k \binom{n-2}{k-2} x^{k-2};\\
H_{2;1,2,n} &= \sigma_{s-2}(\lambda_{;1,2,n}) - \sum_{k=2}^{s-2} c_k \sigma_{k-2}(\lambda_{;1,2,n}) = \binom{n-3}{s-2} x^{s-2} - \sum_{k=2}^{s-2} c_k \binom{n-3}{k-2} x^{k-2};\\
H_{3;1,2,n} &= \sigma_{s-3}(\lambda_{;1,2,n}) - \sum_{k=3}^{s-2} c_k \sigma_{k-3}(\lambda_{;1,2,n}) = \binom{n-3}{s-3} x^{s-3} - \sum_{k=3}^{s-2} c_k \binom{n-3}{k-3} x^{k-3}.
\end{aligned}
$$

By change of basis, to show $C$ is positive-definite, it is equivalent to showing that the matrix $O^*CO$ is positive-definite, where

$$
O \coloneqq \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1\\
-1 & 0 & 0 & \cdots & 0 & 1\\
0 & -1 & 0 & \cdots & 0 & 1\\
0 & 0 & -1 & \ddots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots\\
0 & 0 & \cdots & 0 & -1 & 1
\end{pmatrix}.
$$

The column vectors of matrix $O$ are in fact the eigenvectors of $\mathbb{1}_{n-1\times n-1}$, which makes the new matrix $O^*CO$ and the computations simpler. We have

$$
\begin{aligned}
&O^*CO \\
&= 2H_{2;1,n}\big(H_{1;1,n}+\lambda_n H_{2;1,n}\big)\begin{pmatrix} 0_{n-2\times n-2} & \vec{0} \\ \vec{0}^* & (n-1)^2 \end{pmatrix} \\
&\qquad - H_{1;n}(H_{2;1,2,n}+\lambda_n H_{3;1,2,n})\begin{pmatrix} -2 & -1 & \cdots & -1 & 0 \\ -1 & -2 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & -1 & \vdots \\ -1 & \cdots & -1 & -2 & 0 \\ 0 & 0 & \cdots & 0 & (n-1)(n-2) \end{pmatrix} \\
&= H_{1;n}H_{2;1,2}\left(\begin{pmatrix} 2 & 1 & \cdots & 1 & 0 \\ 1 & 2 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & \cdots & 1 & 2 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \begin{matrix} 2(n-1)^2 H_{2;1,n}(H_{1;1,n}+\lambda_n H_{2;1,n}) \\ -(n-1)(n-2)H_{1;n}(H_{2;1,2,n}+\lambda_n H_{3;1,2,n}) \end{matrix} \end{pmatrix}\right).
\end{aligned}
$$

We know $H_{1;n}$ is positive, $H_{2;1,2}$ is positive, and the following $n-2\times n-2$ matrix

$$
\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}
$$

is a positive-definite matrix with eigenvalues $\{1, \cdots, 1, n-1\}$. So to prove whether $O^*CO$ is positive-definite, it is equivalent to show that whether the following quantity is positive.

$$
2(n-1)H_{2;1,n}(H_{1;1,n}+\lambda_n H_{2;1,n}) - (n-2)H_{1;n}(H_{2;1,2,n}+\lambda_n H_{3;1,2,n}). \tag{2.47}
$$

Now, we use the equation itself, that is,

$$\lambda_n = -\frac{H_{0;n}}{H_{1;n}} = -\frac{\binom{n-1}{s}x^s - \sum_{k=0}^{s-2} c_k \binom{n-1}{k} x^k}{\binom{n-1}{s-1}x^{s-1} - \sum_{k=1}^{s-2} c_k \binom{n-1}{k-1} x^{k-1}}. \tag{2.48}$$

Here, we write

$$H_{0;n} = \sigma_s(\lambda_{;n}) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda_{;n}) = \binom{n-1}{s} x^s - \sum_{k=0}^{s-2} c_k \binom{n-1}{k} x^k.$$

If we consider the diagonal restriction $r_f(x) = \binom{n}{s}x^s - \sum_{k=0}^{s-2} c_k \binom{n}{k} x^k$ of $f$, then we have the following observations:

$$\begin{aligned}
&H = r_f; \quad H_{1;n} = \frac{r'_f}{n}; \quad H_{2;1,n} = \frac{r''_f}{n(n-1)}; \quad H_{3;1,2,n} = \frac{r'''_f}{n(n-1)(n-2)};\\
&H_{0;n} = \frac{n r_f - x r'_f}{n}; \quad H_{1;1,n} = \frac{(n-1)r'_f - x r''_f}{n(n-1)}; \quad H_{2;1,2,n} = \frac{(n-2)r''_f - x r'''_f}{n(n-1)(n-2)}.
\end{aligned}$$

Then by the above observations and (2.48), quantity (2.47) becomes

$$\begin{aligned}
&2(n-1)H_{2;1,n}(H_{1;1,n} + \lambda_n H_{2;1,n}) - (n-2)H_{1;n}(H_{2;1,2,n} + \lambda_n H_{3;1,2,n}) \qquad (2.49)\\
&= \frac{1}{H_{1;n}}\Big(2(n-1)H_{1;n}H_{1;1,n}H_{2;1,n} - 2(n-1)H_{0;n}H_{2;1,n}^2\\
&\qquad\qquad - (n-2)H_{1;n}^2 H_{2;1,2,n} + (n-2)H_{0;n}H_{1;n}H_{3;1,2,n}\Big)\\
&= \frac{1}{n^2(n-1)H_{1;n}}\Big((r'_f)^2 r''_f - 2r_f(r''_f)^2 + r_f r'_f r'''_f\Big) = \frac{r'_f(x)^3}{n^2(n-1)H_{1;n}}\frac{\partial}{\partial x}\Big(\frac{r_f(x)r''_f(x)}{r'_f(x)^2}\Big)\\
&= \frac{r'_f(x)^2}{n(n-1)}\frac{\partial}{\partial x}\alpha_f(x)
\end{aligned}$$

Here, for notation convention, we denote $\alpha_{r_f}(x)$ as $\alpha_f(x)$. In conclusion, we have shown that on the curve $\{\lambda_1 = \cdots = \lambda_{n-1}\}$, the positive definiteness of the Hessian matrix of $\lambda_n$ is equivalent to the monotonicity of log-concavity ratio $\alpha_f$. $\square$

## 2.4 Some Applications

In this section, we use our Convexity Theorem to verify some examples. First, when the degree is low, the Positivstellensatz Theorem can be verified using the resultants and the discriminant. Here, we give a different proof of the Positivstellensatz results in [51], the interested reader can also check [52] for general inverse $\sigma_k$ equations.

**Definition 2.10** (Resultant)**.** The **resultant** of two univariate polynomials $p_1(x)$ and $p_2(x)$ is defined as the determinant of their Sylvester matrix. To be more precise, if we write

$$
\begin{aligned}
&p_1(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0;\\
&p_2(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0,
\end{aligned}
$$

then the resultant of $p_1$ and $p_2$ is defined by the following.

$$
\operatorname{res}(p_1,p_2) \coloneqq \det \begin{pmatrix}
a_d & 0 \cdots & 0 & b_e & 0 \cdots & 0 \\
a_{d-1} & a_d \cdots & 0 & b_{e-1} & b_e \cdots & 0 \\
a_{d-2} & a_{d-1} & 0 & b_{e-2} & b_{e-1} & 0 \\
\vdots & \vdots & a_d & \vdots & \vdots & b_e \\
a_0 & a_1 \cdots & a_{d-1} & b_0 & b_1 \cdots & \vdots \\
0 & a_0 & \vdots & 0 & b_0 & \vdots \\
\vdots & \vdots & a_1 & \vdots & \vdots & b_1 \\
0 & 0 \cdots & a_0 & 0 & 0 \cdots & b_0
\end{pmatrix}. \tag{2.50}
$$

**Definition 2.11** (Discriminant)**.** Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree $n$ and the coefficient $a_0, \cdots, a_n$ are real numbers. The **discriminant** of $p$ is defined by

$$
\operatorname{discr}(p) \coloneqq \frac{(-1)^{n(n-1)/2}}{a_n} \operatorname{res}(p,p'). \tag{2.51}
$$

**Proposition 2.12.** *The level set of the following general inverse $\sigma_k$ equations are all convex.*

$$\lambda_1\lambda_2 - c_0 = 0, \qquad \textit{where } c_0 > 0;$$

$$\lambda_1\lambda_2\lambda_3 - c_1(\lambda_1+\lambda_2+\lambda_3) - c_0 = 0, \qquad \textit{where } c_1 \geq 0 \ \textit{ and } \ c_0 > -2c_1^{3/2};$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 - c_2\sigma_2(\lambda) - c_1\sigma_1(\lambda) - c_0 = 0,$$

*where $c_2 \geq 0$, $c_1 \geq -2c_2^{3/2}$, $c_0 > -3c_2x_1^2 - 3c_1x_1$, and*

$$x_1 = \begin{cases} \sqrt[3]{c_1} & , \textit{ when } c_2 = 0; \\ 2\sqrt{c_2}\cos\Big[\dfrac{1}{3}\arccos\Big(\dfrac{c_1}{2c_2^{3/2}}\Big)\Big] & , \textit{ when } c_2 > 0 \textit{ and } 4c_2^3 - c_1^2 \geq 0; \\ 2\sqrt{c_2}\cosh\Big[\dfrac{1}{3}\operatorname{arccosh}\Big(\dfrac{c_1}{2c_2^{3/2}}\Big)\Big] & , \textit{ when } c_1, c_2 > 0 \textit{ and } 4c_2^3 - c_1^2 \leq 0. \end{cases}$$

*Proof.* Here, we only prove the degree four case:

$$\lambda_1\lambda_2\lambda_3\lambda_4 - c_2\sigma_2(\lambda) - c_1\sigma_1(\lambda) - c_0 = 0.$$

First, the diagonal restriction and its derivatives (after normalizing) will be

$$\{x^4 - 6c_2x^2 - 4c_1x - c_0,\ x^3 - 3c_2x - c_1,\ x^2 - c_2,\ x\}.$$

Second, for the largest real roots, we have $x_2 = \sqrt{c_2}$, $x_3 = 0$. Then, for the depressed cubic polynomial $x^3 - 3c_2x - c_1$, we want

$$x_2^3 - 3c_2x_2 - c_1 = -2c_2^{3/2} - c_1 \leq 0.$$

That is, $c_1 \geq -2c_2^{3/2}$. We compute the discriminant of the cubic polynomial $x^3 - 3c_2x - c_1$, by (2.50) and (2.51), we have

$$\operatorname{discr}(x^3 - 3c_2x - c_1) = \operatorname{res}\big(x^3 - 3c_2x - c_1, 3x^2 - 3c_2\big)$$
$$= \det\begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ -3c_2 & 0 & -3c_2 & 0 & 3 \\ -c_1 & -3c_2 & 0 & -3c_2 & 0 \\ 0 & -c_1 & 0 & 0 & -3c_2 \end{pmatrix} = 27(4c_2^3 - c_1^2).$$

When $4c_2^3 - c_1^2 \geq 0$, then this is the case *casus irreducibilis*. When $4c_2^3 - c_1^2 \leq 0$, then the root can be represented using hyperbolic functions. So the largest real root $x_1$ will be

$$x_1 = \begin{cases} \sqrt[3]{c_1} & \text{, when } c_2 = 0; \\ 2\sqrt{c_2}\cos\Big[\dfrac{1}{3}\arccos\Big(\dfrac{c_1}{2c_2^{3/2}}\Big)\Big] & \text{, when } c_2 > 0 \text{ and } 4c_2^3 - c_1^2 \geq 0; \\ 2\sqrt{c_2}\cosh\Big[\dfrac{1}{3}\operatorname{arccosh}\Big(\dfrac{c_1}{2c_2^{3/2}}\Big)\Big] & \text{, when } c_1, c_2 > 0 \text{ and } 4c_2^3 - c_1^2 \leq 0. \end{cases}$$

Here, we take the branch $\arccos(\bullet) \in [0, \pi]$ and arccosh is the inverse hyperbolic cosine. Last, we plug $x_1$ in to the quartic polynomial $x^4 - 6c_2x^2 - 4c_1x - c_0$. Because we want $x_0 > x_1$, so we want the following to be true.

$$x_1^4 - 6c_2x_1^2 - 4c_1x_1 - c_0 = -3c_2x_1^2 - 3c_1x_1 - c_0 < 0.$$

That is, $c_0 > -3c_2x_1^2 - 3c_1x_1$. $\square$

In [40], Guan–Zhang studied the solvability of a general class of curvature equations. These curvature equations can be viewed as generalizations of the Christoffel–Minkowski problem in convex geometry. Guan–Zhang considered the following class of equations

$$\sigma_m(\lambda) + c_{m-1}\sigma_{m-1}(\lambda) = \sum_{k=0}^{m-2} c_k\sigma_k(\lambda),$$

where $n$ is the dimension of the space, $n \geq m \geq 2$, $c_k \geq 0$ for $k \in \{0, \cdots, m-2\}$, and $c_{m-1} \in \mathbb{R}$. They obtained a priori estimates for the admissible solutions. Here, we show that the level set is convex by our Convexity Theorem. When $m = n$, the following result can also be applied to the general inverse $\sigma_k$ equations with non-negative coefficients considered by Collins–Székelyhidi [20] and Fang–Lai–Ma [29].

**Lemma 2.12.** *The level set of the following general inverse $\sigma_k$ equation*

$$f(\lambda) := \sigma_m(\lambda) + c_{m-1}\sigma_{m-1}(\lambda) - \sum_{k=0}^{m-2} c_k\sigma_k(\lambda) = 0 \tag{2.52}$$

*is convex if $c_k \geq 0$ for $k \in \{0, \cdots, m-2\}$ with $\sum_{k=0}^{m-2} c_k > 0$ and $c_{m-1} \in \mathbb{R}$.*

*Proof.* Consider the following diagonal restriction $r_f(x)$ of equation (2.52), that is,

$$r_f(x) = \binom{n}{m}x^m + c_{m-1}\binom{n}{m-1}x^{m-1} - \sum_{k=0}^{m-2} c_k\binom{n}{k}x^k.$$

By Theorem 2.5, if $r_f$ is strictly right-Noetherian, then we are done. We prove this by mathematical induction on the degree $m$. We also **claim** that when $m \geq 2$, if the coefficients $c_k$ satisfy the hypothesis, then $x_0 > 0$. When $m = 1$, we have $r_f(x) = nx + c_0$, which is strictly right-Noetherian. When $m = 2$, we have $r_f(x) = \binom{n}{2}x^2 + nc_1x - c_0$; $r_f'(x) = n(n-1)x + nc_1$. If we write $x_1 = -\frac{c_1}{n-1}$ the largest real root of $r_f'$, then by the hypothesis, we get

$$r_f(-\frac{c_1}{n-1}) = -\frac{nc_1^2}{2(n-1)} - c_0 \leq -c_0 < 0.$$

This implies that $r_f$ is strictly right-Noetherian. Moreover, we have $r_f(0) = -c_0 < 0$, which implies that $x_0 > 0$. So the claim is true when $m = 2$. When $m = 3$, we obtain

$$r_f(x) = \binom{n}{3}x^3 + c_2\binom{n}{2}x^2 - \binom{n}{1}c_1x - c_0; \quad r'_f(x) = n\Big(\binom{n-1}{2}x^2 + \binom{n-1}{1}c_2x - c_1\Big);$$
$$r''_f(x) = n(n-1)\big((n-2)x + c_2\big).$$

We have $x_2 = -\frac{c_2}{n-2}$. If $c_1 > 0$, then $x_1 > \max\{x_2, 0\}$. In addition, we obtain

$$r_f(x_1) = \binom{n}{3}x^3 + c_2\binom{n}{2}x^2 - \binom{n}{1}c_1x - c_0 = -\frac{n(n-1)(n-2)}{12}x_1^3 - \frac{n}{2}c_1x_1 - c_0 < 0.$$

Thus, the largest real root $x_0$ of $r_f$ is greater than $x_1$, $r_f$ is strictly right-Noetherian. If $c_1 = 0$, then $x_1 = \max\{0, -\frac{2c_2}{n-2}\}$. If $c_2 < 0$, then similar to above, we get $r_f(x_1) < 0$. This implies that $x_0 > x_1$, $r_f$ is strictly right-Noetherian. Otherwise, if $c_2 \geq 0$, then $x_1 = 0$. For this case, by the hypothesis, we have $\sum_{k=0}^{3-2} c_k = c_0 + c_1 = c_0 > 0$. This implies that

$$r_f(x_1) = r_f(0) = -c_0 < 0.$$

Thus, $x_0 > x_1 = 0$, $r_f$ is again strictly right-Noetherian. No matter which case, the claim is true. Suppose the statement and the claim is true when $m = l - 1$. When $m = l$, we have

$$r_f(x) = \binom{n}{l}x^l + c_{l-1}\binom{n}{l-1}x^{l-1} - \sum_{k=0}^{l-2} c_k\binom{n}{k}x^k.$$

If we consider the first derivative of $r_f(x)$ with respect to $x$, then we obtain

$$r'_f(x) = n\Big(\binom{n-1}{l-1}x^{l-1} + c_{l-1}\binom{n-1}{l-2}x^{l-2} - \sum_{k=1}^{l-2} c_k\binom{n-1}{k-1}x^{k-1}\Big).$$

There are two cases to consider. First, if $\sum_{k=1}^{l-2} c_k > 0$, then $r'_f$ satisfies the hypothesis, so $r'_f$

is strictly right-Noetherian. Moreover, the largest real root $x_1$ of $r_f'$ will be positive. Also,

$$\begin{aligned}
r_f(x_1) &= \binom{n}{l}x_1^l + c_{l-1}\binom{n}{l-1}x_1^{l-1} - \sum_{k=0}^{l-2} c_k\binom{n}{k}x_1^k \\
&= \binom{n}{l}x_1^l - \sum_{k=0}^{l-2} c_k\binom{n}{k}x_1^k - \frac{x_1}{\binom{n-1}{l-2}}\binom{n}{l-1}\left(\binom{n-1}{l-1}x_1^{l-1} - \sum_{k=1}^{l-2} c_k\binom{n-1}{k-1}x_1^{k-1}\right) \\
&= -\frac{1}{l-1}x_1^l - \sum_{k=0}^{l-2} c_k\left(1-\frac{k}{l-1}\right)\binom{n}{k}x_1^k < 0.
\end{aligned}$$

In this case, $x_0 > x_1 > 0$, $r_f$ is strictly right-Noetherian. Second, if $\sum_{k=1}^{l-2} c_k = 0$, then $c_k = 0$ for all $k \in \{1, \cdots, l-2\}$. By hypothesis, we have $c_0 > 0$, so $r_f(x) = \binom{n}{l}x^l + c_{l-1}\binom{n}{l-1}x^{l-1} - c_0$. For $k \in \{1, \cdots, l-1\}$, we have $x_k = \max\{0, -\frac{l-k}{n-l+1}c_{m-1}\}$. We are done if $c_{m-1} < 0$. Otherwise, we have $x_1 = \cdots = x_{m-1} = 0$ and $x_0 > x_1 = 0$. Hence, no matter which case, $r_f$ is strictly right-Noetherian, and the claim is true. This finishes the proof. □

**Lemma 2.13.** *The level set of the deformed Hermitian–Yang–Mills equation*

$$\Im\big(\omega + \sqrt{-1}\chi\big)^n = \tan\big(\hat{\theta}\big) \cdot \Re\big(\omega + \sqrt{-1}\chi\big)^n$$

*is convex if $\theta$ is in the supercritical phase, that is, $\theta \in \big((n-2)\pi/2, n\pi/2\big)$. In addition, the level set is also convex if $\theta \in \big(-n\pi/2, -(n-2)\pi/2\big)$.*

*Proof.* First, it is well-known that the dHYM equation can be rewritten as $\sum_{i=1}^n \arctan(\lambda_i) = \theta$. Since $\theta \in \big((n-2)\pi/2, n\pi/2\big)$, we have

$$\frac{\theta - k\pi/2}{n-k} \in \left(\frac{(n-2-k)\pi}{2(n-k)}, \frac{(n-k)\pi}{2(n-k)}\right) = \left(\frac{\pi}{2} - \frac{\pi}{n-k}, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

for $k \in \{0, 1, \cdots, n-1\}$. Second, by Theorem 2.5, we consider the diagonal restriction, we get $(n-k)\arctan(x_k) = \theta - \frac{k\pi}{2}$ for $k \in \{0, 1, \cdots, n-1\}$, where $x_k$ is the largest real root of the $k$-th derivative of the diagonal restriction. We claim that:

$$x_0 = \tan\Big(\frac{\theta}{n}\Big) > \cdots > x_k = \tan\Big(\frac{\theta - k\pi/2}{n-k}\Big) > \cdots > x_{n-1} = \tan\Big(\theta - (n-1)\frac{\pi}{2}\Big).$$

Since $\tan(x)$ is increasing on $(-\pi/2, \pi/2)$, to check the above claim, it suffices to check

$$\frac{\theta}{n} > \frac{\theta - \pi/2}{n-1} > \cdots > \frac{\theta - k\pi/2}{n-k} > \cdots > \theta - (n-1)\frac{\pi}{2}.$$

This is true because the function $f(x) = (\theta - x\pi/2)/(n-x)$ is decreasing on $(-\infty, n)$. By Theorem 2.5, the level set is convex. $\square$

## 2.5 Basic Formulas of Symmetric Functions

In this section, we state some lemmas for symmetric functions first. One can also check the author's work [50] and the references there in for more details.

**Lemma 2.14.** *If $F(\Lambda) = f(\lambda_1, \ldots, \lambda_n)$ is a smooth function in the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of a Hermitian matrix $\Lambda$, then at a diagonal matrix $\Lambda$ with distinct eigenvalues $\lambda_i$, we get*

$$\frac{\partial F}{\partial \Lambda_i^j}(\Lambda) = \delta_{ij} f_i(\lambda); \quad \frac{\partial^2 F}{\partial \Lambda_i^j \Lambda_r^s}(\Lambda) = f_{ir}(\lambda)\delta_{ij}\delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j}(\lambda)(1-\delta_{ij})\delta_{is}\delta_{jr},$$

*where $f_i(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda)$ and $f_{ir} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_r}(\lambda)$.*

We denote $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ the eigenvalues of the Hermitian endomorphism $\omega^{i\bar{k}}\big(X + \sqrt{-1}\partial\bar{\partial}u\big)_{j\bar{k}}$. Since we are on a Kähler manifold, we can pick the following coordinates to simplify our computation.

**Lemma 2.15.** *At any point $p \in M$, there exist local holomorphic coordinates near $p$ such* *that*

$$\omega_{i\bar{j}}(p) = \delta_{ij}; \quad (X_u)_{i\bar{j}}(p) = \lambda_i \delta_{ij}; \quad \omega_{i\bar{j},k}(p) = 0,$$

*for all* $i, j, k \in \{1, \ldots, n\}$.

From now on, without further notice, we always use the above coordinates. We denote $\Lambda$ as the Hermitian endomorphism $\omega^{i\bar{k}}\big(X + \sqrt{-1}\partial\bar{\partial}u\big)_{j\bar{k}}$. Then the first and second derivatives of $\Lambda$ will be the following.

**Lemma 2.16.** *The first and second derivatives of* $\Lambda$ *are*

$$\begin{aligned}
\frac{\partial \Lambda_i^j}{\partial \bar{z}_k} &= \omega^{j\bar{p}}{}_{,\bar{k}}(X_u)_{i\bar{p}} + \omega^{j\bar{p}}(X_u)_{i\bar{p},\bar{k}} = -\omega^{j\bar{b}}\omega_{a\bar{b},\bar{k}}\omega^{a\bar{p}}(X_u)_{i\bar{p}} + \omega^{j\bar{p}}(X_u)_{i\bar{p},\bar{k}}, \\
\frac{\partial^2 \Lambda_i^j}{\partial z_l \partial \bar{z}_k} &= \omega^{j\bar{p}}{}_{,\bar{k}l}(X_u)_{i\bar{p}} + \omega^{j\bar{p}}{}_{,\bar{k}}(X_u)_{i\bar{p},l} + \omega^{j\bar{p}}{}_{,l}(X_u)_{i\bar{p},\bar{k}} + \omega^{j\bar{p}}(X_u)_{i\bar{p},\bar{k}l} \\
&= \omega^{j\bar{d}}\omega_{c\bar{d},l}\omega^{c\bar{b}}\omega_{a\bar{b},\bar{k}}\omega^{a\bar{p}}(X_u)_{i\bar{p}} - \omega^{j\bar{b}}\omega_{a\bar{b},\bar{k}l}\omega^{a\bar{p}}(X_u)_{i\bar{p}} + \omega^{j\bar{b}}\omega_{a\bar{b},\bar{k}}\omega^{a\bar{d}}\omega_{c\bar{d},l}\omega^{c\bar{p}}(X_u)_{i\bar{p}} \\
&\qquad - \omega^{j\bar{b}}\omega_{a\bar{b},\bar{k}}\omega^{a\bar{p}}(X_u)_{i\bar{p},l} - \omega^{j\bar{b}}\omega_{a\bar{b},l}\omega^{a\bar{p}}(X_u)_{i\bar{p},\bar{k}} + \omega^{j\bar{p}}(X_u)_{i\bar{p},\bar{k}l},
\end{aligned}$$

*where we denote* $(X_u)_{i\bar{j}} = X_{i\bar{j}} + u_{i\bar{j}}$ *and* $\Lambda$ *is the Hermitian endomorphism* $\omega^{-1}(X_u)$.

If we evaluate at any fixed point $p \in M$ and we use the coordinates in Lemma 2.15, we can simplify the first and second derivatives of $\Lambda$.

**Lemma 2.17.** *At any fixed point* $p$*, by picking the coordinates in Lemma 2.15, we get*

$$\frac{\partial \Lambda_i^j}{\partial \bar{z}_k}(p) = (X_u)_{i\bar{j},\bar{k}}; \quad \frac{\partial^2 \Lambda_i^j}{\partial z_l \partial \bar{z}_k}(p) = -\lambda_i \omega_{i\bar{j},\bar{k}l} + (X_u)_{i\bar{j},\bar{k}l}.$$

# Chapter 3

# Background

In this chapter, first, in Section 3.1, we collect all strictly $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials or strictly $\Upsilon$-stable general inverse $\sigma_k$ type multilinear polynomials. We may reformulate previous general inverse $\sigma_k$ type equations and general $\sigma_k$ type equations into our settings. Second, in Section 3.2, given a fixed strictly $\Upsilon$-stable general $\sigma_k$ equation or strictly $\Upsilon$-stable general inverse $\sigma_k$ equation and a $C$-subsolution to this equation. We study all strictly $\Upsilon$-stable general $\sigma_k$ equations or strictly $\Upsilon$-stable general inverse $\sigma_k$ equations such that the given one is still a $C$-subsolution to these equations. Even though it is still open whether a $C$-subsolution will provide a priori estimates, but it holds for all known examples. For example, the Monge–Ampère equation solved by Yau [72]; the J-equation studied extensively by Collins–Székelyhidi [20], Chen [14], Song–Weinkove [66]; and general inverse $\sigma_k$ equations with non-negative coefficients studied by Fang–Lai–Ma [29] and Collins–Székelyhidi [20]. So it is still worth considered based on these works and should be a potential space for finding continuity path. By Section 2.2, the $C$-subsolution cone introduced by Székelyhidi [68] and Guan [36] is the $\Upsilon_1$-cone provided that the original equation is strictly $\Upsilon$-stable. Moreover, with the aid of our $\Upsilon$-dominance Theorem, we can explicitly describe the space of all strictly $\Upsilon$-stable general $\sigma_k$ equations or strictly $\Upsilon$-stable general inverse $\sigma_k$

equations sharing the same function as a $C$-subsolution. Last, in Section 3.3, we show some examples such that the continuity paths will lie in our space defined in Section 3.2.

## 3.1 General $\sigma_k$ Equations and General Inverse $\sigma_k$ Equations

Let $(M, \omega)$ be a compact connected Kähler manifold of complex dimension $n$ and $[\chi_0] \in H^{1,1}(M; \mathbb{R})$, where $H^{1,1}(M; \mathbb{R})$ is the $(1,1)$-Dolbeault cohomology group. The study of the solvability of the following general inverse $\sigma_k$ equation is widely considered:

$$\chi^n = c_{n-1}\binom{n}{n-1}\chi^{n-1} \wedge \omega + \cdots + c_1\binom{n}{1}\chi \wedge \omega^{n-1} + c_0\binom{n}{0}\omega^n, \tag{3.1}$$

where $c_k$ are real functions on $M$ for $k \in \{0, \cdots, n-1\}$ and $\chi \in [\chi_0]$ is a real smooth, closed $(1,1)$-form. Or if possible, we can do a substitution $X \coloneqq \chi - c_{n-1}\omega$ and get

$$X^n = d_{n-2}\binom{n}{n-2}X^{n-2} \wedge \omega^2 + \cdots + d_1\binom{n}{1}X \wedge \omega^{n-1} + d_0\binom{n}{0}\omega^n, \tag{3.2}$$

where $d_k$ are real functions on $M$ for $k \in \{0, \cdots, n-2\}$. We can treat equation (3.1) or (3.2) as a function from the manifold $M$ to Euclidean space $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$, which is defined by

$$c\colon M \to \mathbb{R}^n;\ c(p) \coloneqq (c_{n-1}, \cdots, c_1, c_0) \ \text{ or } \ d\colon M \to \mathbb{R}^{n-1};\ d(p) \coloneqq (d_{n-2}, \cdots, d_1, d_0).$$

Similarly, we can view general $\sigma_k$ equation as a function from the manifold $M$ to Euclidean space. Based on the results in Section 2.2 and Section 2.3, to obtain convexity, we wish the coefficients of equation (3.1) or (3.2) satisfy some special properties. By collecting all strictly $\Upsilon$-stable general $\sigma_k$ type multilinear polynomials, we consider the following algebraic sets.

**Definition 3.1.** For $\lambda = \{\lambda_1, \cdots, \lambda_n\}$, we define

$$\mathscr{C}_{n,s} := \Big\{(c_{s-1}, c_{s-2}, \cdots, c_0) \in \mathbb{R}^s \colon \sigma_s(\lambda) - \sum_{k=0}^{s-1} c_k \sigma_k(\lambda) \text{ is strictly } \Upsilon\text{-stable}\Big\};$$

$$\tilde{\mathscr{C}}_{n,s} := \Big\{(c_{s-2}, c_{s-3}, \cdots, c_0) \in \mathbb{R}^{s-1} \colon \sigma_s(\lambda) - \sum_{k=0}^{s-2} c_k \sigma_k(\lambda) \text{ is strictly } \Upsilon\text{-stable}\Big\}.$$

For convenience, we denote $\mathscr{C}_{n,n}$ by $\mathscr{C}_n$ and $\tilde{\mathscr{C}}_{n,n}$ by $\tilde{\mathscr{C}}_n$.

When the number of variables $n$ is small, by Proposition 2.12, we get the following.

**Proposition 3.1.**

$$\tilde{\mathscr{C}}_2 = \{c_0 > 0\}; \quad \tilde{\mathscr{C}}_3 = \{(0, c_0) \colon c_0 > 0\} \cup \{(c_1, c_0) \colon c_1 > 0 \textit{ and } c_0 > -2c_1^{3/2}\};$$

$$\begin{aligned}\tilde{\mathscr{C}}_4 = &\Big\{(0,0,c_0) \colon c_0 > 0\Big\} \cup \Big\{(0, c_1, c_0) \colon c_1 > 0 \textit{ and } c_0 > -3c_1^{4/3}\Big\} \\ &\cup \Big\{(c_2, -2c_2^{3/2}, c_0) \colon c_2 > 0 \textit{ and } c_0 > 3c_2^2\Big\} \\ &\cup \Big\{(c_2, c_1, c_0) \colon c_2 > 0, c_1 > -2c_2^{3/2}, \textit{ and } c_0 > -3c_2 x_1^2 - 3c_1 x_1\Big\}.\end{aligned}$$

*Here, $x_1$ is the largest real root of $x^3 - 3c_2 x - c_1 = 0$.*

For the $\Upsilon$-cones defined in Section 2.2, if $\underline{\chi}$ is a $C$-subsolution to the equation $c \colon M \longrightarrow \mathscr{C}_{n,s}$, then we know that for any $p \in M$, $\binom{n-1}{s-1}\underline{\chi}^{s-1} \wedge \omega^{n-s} - \sum_{k=1}^{s-1} c_k \binom{n-1}{k-1}\underline{\chi}^{k-1} \wedge \omega^{n-k}$ is a positive $(n-1, n-1)$-form at $p$. If we consider the Hermitian endomorphism $\omega^{-1}\underline{\chi}$, we get the following.

*Remark* 3.1. $\underline{\chi} \in [\chi_0]$ is a $C$-subsolution to $c \colon M \to \mathscr{C}_{n,s}$ if and only if at each point $p \in M$, for any $l \in \{1, \cdots, s-1\}$ and any $a \in S_n$, we have

$$\sigma_l(\mu_{;a(1),\cdots,a(s-l)}) - \sum_{k=s-l}^{s-1} c_k \sigma_{k-s+l}(\mu_{;a(1),\cdots,a(s-l)}) > 0.$$

Here, $\mu = \{\mu_1, \cdots, \mu_n\}$ are the eigenvalues of the Hermitian endomorphism $\omega^{-1}\underline{\chi}$.

On the other hand, for any $p \in M$ and for any $(n-1)$-dimensional complex vector subspace $V^{n-1}$ of the $n$-dimensional complex vector space $T_p^{\mathbb{C}}(M)$, we have

$$\binom{n-1}{s-1}(\underline{\chi}|_{V^{n-1}})^{s-1} \wedge (\omega|_{V^{n-1}})^{n-s} - \sum_{k=1}^{s-1} c_k \binom{n-1}{k-1}(\underline{\chi}|_{V^{n-1}})^{k-1} \wedge (\omega|_{V^{n-1}})^{n-k} > 0.$$

By considering the complex Grassmannian space, for any $c \in \mathscr{C}_{n,s}$ and for any $\chi \in [\chi_0] \in H^{(1,1)}(M;\mathbb{R})$, we define the following function $\chi^{\omega,c} \colon M \to \mathbb{R}$ by

$$\chi^{\omega,c}(p) := \inf_{V \in \bigcup_{k=n-s+1}^{n-1} \mathrm{Gr}^{\mathbb{C}}(k, T_p^{\mathbb{C}}(M))} \left( \frac{(\chi|_V)^{s+\dim_{\mathbb{C}} V - n}(\omega|_V)^{n-s}}{(\omega|_V)^{\dim_{\mathbb{C}} V}} - \frac{\sum_{k=n-\dim_{\mathbb{C}} V}^{s-1} c_k \binom{\dim_{\mathbb{C}} V}{n-k} (\chi|_V)^{k+\dim_{\mathbb{C}} V - n}(\omega|_V)^{n-k}}{(\omega|_V)^{\dim_{\mathbb{C}} V}} \right).$$

Here, $\mathrm{Gr}^{\mathbb{C}}(k, T_p^{\mathbb{C}}(M))$ is the space of all complex $k$-dimensional vector subspaces of the complex $n$-dimensional vector space $T_p^{\mathbb{C}}(M)$.

*Remark* 3.2. $\underline{\chi} \in [\chi_0]$ is a $C$-subsolution to $c \colon M \to \mathscr{C}_{n,s}$ if and only if the function $\underline{\chi}^{\omega,c} \colon M \to \mathbb{R}$ is a positive function.

Here, similar to Remark 2.5, we need to consider all Grassmannians to avoid ambiguity. Most of the time, we consider the simpler case $c_{s-1} = 0$. We may reformulate the settings for classical general inverse $\sigma_k$ equations, for example, the complex Monge–Ampère equation.

**Theorem 3.1** (Reformulate Complex Monge–Ampère equation, Yau [72])**.** *Let $(M, \omega)$ be a Kähler manifold with Kähler form $\omega$ and $[\chi_0]$ be a $(1,1)$-Dolbeault class. Given a map $c \colon M \to \tilde{\mathscr{C}_n}$ satisfying the integrability condition, which is defined by*

$$c \colon M \longrightarrow \tilde{\mathscr{C}_n}; \quad p \longmapsto (\underbrace{0, \cdots, 0}_{n-2 \text{ copies}}, c_0(p)) \quad \text{and} \quad \int_M \chi_0^n = \int_M c_0 \omega^n.$$

*Suppose that there exists a $C$-subsolution to $c$ in $[\chi_0]$, that is, there exists a $\underline{\chi} \in [\chi_0]$ such* *that* $\inf_{V^k \in \mathrm{Gr}^{\mathbb{C}}(k, T_p^{\mathbb{C}}(M))} \frac{(\chi|_{V^k})^k}{(\omega|_{V^k})^k} > 0$ *for any* $p \in M$ *and* $k \in \{1, \cdots, n-1\}$*. Then there exists a unique* $\chi \in [\chi_0]$ *such that* $\chi^n = c_0 \omega^n$*.*

*Conjecture* 3.1 (Reformulate deformed Hermitian–Yang–Mills equation). Let $(M, \omega)$ be a Kähler manifold with Kähler form $\omega$ and $[\chi_0]$ be a $(1,1)$-Dolbeault class. The deformed Hermitian–Yang–Mills equation with $\theta \in \big((n-2)\pi/2, n\pi/2\big)$ induces a point in $\tilde{\mathscr{C}}_n$ and we consider the following constant map $c_{\mathrm{dHYM}} \colon M \longrightarrow \tilde{\mathscr{C}}_n$. If there exists a $C$-subsolution to $c_{\mathrm{dHYM}}$ in $[\chi_0]$, then there exists a $\chi \in [\chi_0]$ such that

$$\Im\big(\omega + \sqrt{-1}\chi\big)^n = \tan\big(\theta\big) \cdot \Re\big(\omega + \sqrt{-1}\chi\big)^n.$$

We state one of the author's work in [51].

**Theorem 3.2** (deformed Hermitian–Yang–Mills equation, Lin. [51])**.** *When the complex dimension equals three or four, Conjecture 3.1 is confirmed.*

## 3.2 Space for Continuity Path

By previous classical works, if $\underline{\chi} \in [\chi_0]$ is a $C$-subsolution to $d \colon M \to \mathscr{C}_{n,s}$ (or $\tilde{\mathscr{C}}_{n,s}$), then we are interested in whether $\underline{\chi}$ is again a $C$-subsolution to another function $c \colon M \to \mathscr{C}_{n,s}$ (or $\tilde{\mathscr{C}}_{n,s}$). Once we understand the space consisting such $c$, then it is a potential space to find a continuity path in this space connecting different equations and apply a priori estimates over this continuity path. In Chapter 4, for constant maps $d \colon M \to \mathscr{C}_n$, when the dimension $n$ equals three or four, it is justified that we obtain a priori estimates provided the existence of a $C$-subsolution. So the space consisting such $c$ is truly the space for continuity path.

**Theorem 3.3.** *Given* $d \colon M \to \mathscr{C}_{n,s}$*, at any point* $p \in M$*, we write* $x_k(p)$ *the largest real root of the* $k$*-th derivative of* $f_p(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-1} d_k(p) \binom{n}{k} x^k$*. Then the* $\Upsilon_1$*-cone of* $d$ *at* $p$ *is* *contained in the $\Upsilon_1$-cone of $c\colon M \to \mathscr{C}_{n,s}$ at $p$ if and only if for all $k \in \{1, \cdots, s-1\}$, we have $g_p^{(k)}(x_k(p)) \geq 0$. Here, $g_p(y) = \binom{n}{s} y^s - \sum_{k=0}^{s-1} c_k(p)\binom{n}{k} y^k$.*

*Proof.* To show that the $\Upsilon_1$-cone of $d\colon M \to \mathscr{C}_{n,s}$ at $p$ is contained in the $\Upsilon_1$-cone of $c\colon M \to \mathscr{C}_{n,s}$ at $p$. By Theorem 2.4, if we write $y_k(p)$ the largest real root of the $k$-th derivative of $\binom{n}{s} y^s - \sum_{k=0}^{s-1} c_k(p)\binom{n}{k} y^k$, then we are checking whether $x_k(p) \geq y_k(p)$ for all $k \in \{1, \cdots, s-1\}$. When $k = s-1$, $f_p^{(s-1)}(x) = \frac{n!}{(n-s)!}(x - \frac{d_{s-1}(p)}{n-s+1})$, $g_p^{(s-1)}(y) = \frac{n!}{(n-s)!}(y - \frac{c_{s-1}(p)}{n-s+1})$, $x_{s-1}(p) = \frac{d_{s-1}(p)}{n-s+1}$, and $y_{s-1}(p) = \frac{c_{s-1}(p)}{n-s+1}$. So, $x_{s-1}(p) \geq y_{s-1}(p)$ if and only if $g_p^{(s-1)}(x_{s-1}(p)) = \frac{n!}{(n-s)!}(x_{s-1}(p) - \frac{c_{s-1}(p)}{n-s+1}) \geq 0$. When $k = s-2$, we get $g_p^{(s-2)}(x_{s-2}(p)) = \frac{n!}{(n-s)!}(\frac{1}{2}x_{s-2}^2(p) - \frac{c_{s-1}}{n-s+1}x_{s-2} - \frac{c_{s-2}}{(n-s+1)(n-s+2)})$. If $x_{s-2}(p) \geq y_{s-2}(p)$, then since $y_{s-2}(p)$ is the largest real root of $g_p^{(s-2)}$, we get $g_p^{(s-2)}(x_{s-2}(p)) \geq 0$. On the other hand, if $g_p^{(s-2)}(x_{s-2}(p)) \geq 0$, then since $x_{s-2}(p) \geq x_{s-1}(p) \geq y_{s-1}(p)$, we get a contradiction by the proof in Proposition 2.1 when $y_{s-2}(p) > x_{s-2}(p)$. We use mathematical induction on $k$, suppose the statement is true when $k = l \geq 2$. When $k = l-1$, if $x_{l-1}(p) \geq y_{l-1}(p)$, then since $y_{l-1}(p)$ is the largest real root of $g_p^{(l-1)}$, we get $g_p^{(l-1)}(x_{l-1}(p)) \geq 0$. On the other hand, if $g_p^{(l-1)}(x_{l-1}(p)) \geq 0$, then since $x_{l-1}(p) \geq x_l(p) \geq y_l(p)$, we again have $x_{l-1}(p) \geq y_{l-1}(p)$. This finishes the proof. $\square$

As a consequence, we immediately get the following proposition.

**Proposition 3.2.** *Let $d\colon M \to \mathscr{C}_{n,s}$ and $\underline{\chi}$ be a $C$-subsolution to $d$. Then for any $p \in M$ and for any $c\colon M \to \mathscr{C}_{n,s}$ satisfies the following $s-1 \times s-1$ linear system:*

$$\begin{cases} \binom{n-s+1}{1} x_{s-1}(p) - c_{s-1}(p) \geq 0; \\ \binom{n-s+2}{2} x_{s-2}^2(p) - c_{s-1}(p)\binom{n-s+2}{1} x_{s-2}(p) - c_{s-2}(p) \geq 0; \\ \qquad\qquad\qquad \vdots \\ \binom{n-2}{s-2} x_2^{s-2}(p) - \sum_{k=2}^{s-1} c_k(p)\binom{n-2}{k-2} x_2^{k-2}(p) \geq 0; \\ \binom{n-1}{s-1} x_1^{s-1}(p) - \sum_{k=1}^{s-1} c_k(p)\binom{n-1}{k-1} x_1^{k-1}(p) \geq 0, \end{cases} \tag{3.3}$$

*we obtain that $\underline{\chi}$ is also a $C$-subsolution to $c$. That is to say, if $\underline{\chi}$ is a $C$-subsolution to $c$, then for any $p \in M$, $c(p) = (c_{s-1}(p), c_{s-2}(p), \cdots, c_0(p)) \in \mathscr{C}_{n,s} \subset \mathbb{R}^s$, $(c_{s-1}(p), c_{s-2}(p), \cdots, c_1(p))$ lies in one of the polyhedrons, containing $(-R, -R^2, \cdots, -R^{s-1})$ for $R > 0$ sufficiently large, defined by $s-1$ hypersurfaces passing through $(d_{s-1}(p), d_{s-2}(p), \cdots, d_1(p))$ with the following $s-1$ linearly independent vectors as normal vectors:*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \binom{n-s+2}{1} x_{s-2}(p) \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} \binom{n-2}{s-3} x_2^{s-3}(p) \\ \binom{n-2}{s-4} x_2^{s-4}(p) \\ \binom{n-2}{s-5} x_2^{s-5}(p) \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \binom{n-1}{s-2} x_1^{s-2}(p) \\ \binom{n-1}{s-3} x_1^{s-3}(p) \\ \binom{n-1}{s-4} x_1^{s-4}(p) \\ \vdots \\ \binom{n-1}{1} x_1(p) \\ 1 \end{pmatrix}.$$

Here, we also write a version for functions mapping to $\tilde{\mathscr{C}}_{n,s}$.

**Lemma 3.1.** *Given $d\colon M \to \tilde{\mathscr{C}}_{n,s}$, at any point $p \in M$, we write $x_k(p)$ the largest real root of the $k$-th derivative of $f_p(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-2} d_k(p) \binom{n}{k} x^k$. Then the $\Upsilon_1$-cone of $c$ at $p$ is contained in the $\Upsilon_1$-cone of $c\colon M \to \tilde{\mathscr{C}}_{n,s}$ at $p$ if and only if for all $k \in \{1, \cdots, s-2\}$, we have $g_p^{(k)}(x_k(p)) \geq 0$. Here, $g_p(y) = \binom{n}{s} y^s - \sum_{k=0}^{s-2} c_k(p) \binom{n}{k} y^k$.*

**Proposition 3.3.** *Let $d\colon M \to \tilde{\mathscr{C}}_{n,s}$ and $\underline{\chi}$ be a $C$-subsolution to $d$. Then for any $p \in M$ and for any $c\colon M \to \tilde{\mathscr{C}}_{n,s}$ satisfies the following $s-2 \times s-2$ linear system:*

$$\begin{cases} \binom{n-s+2}{2} x_{s-2}^2(p) - c_{s-2}(p) \geq 0; \\ \qquad\qquad \vdots \\ \binom{n-2}{s-2} x_2^{s-2}(p) - \sum_{k=2}^{s-2} c_k(p) \binom{n-2}{k-2} x_2^{k-2}(p) \geq 0; \\ \binom{n-1}{s-1} x_1^{s-1}(p) - \sum_{k=1}^{s-2} c_k(p) \binom{n-1}{k-1} x_1^{k-1}(p) \geq 0, \end{cases} \tag{3.4}$$

*we obtain that $\underline{\chi}$ is also a $C$-subsolution to $c$. That is to say, if $\underline{\chi}$ is a $C$-subsolution to* *c, then for any $p \in M$, $c(p) = (c_{s-2}(p), \cdots, c_0(p)) \in \tilde{\mathscr{C}}_{n,s} \subset \mathbb{R}^{s-1}$, $(c_{s-2}(p), \cdots, c_1(p))$ lies in one of the polyhedrons, containing $(-R, -R^2, \cdots, -R^{s-2})$ for $R > 0$ sufficiently large, defined by $s-2$ hypersurfaces passing through $(d_{s-2}(p), \cdots, d_1(p))$ with the following $s-2$ linearly independent vectors as normal vectors:*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \binom{n-s+3}{1} x_{s-3}(p) \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} \binom{n-2}{s-4} x_2^{s-4}(p) \\ \binom{n-2}{s-5} x_2^{s-5}(p) \\ \binom{n-2}{s-6} x_2^{s-6}(p) \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \binom{n-1}{s-3} x_1^{s-3}(p) \\ \binom{n-1}{s-4} x_1^{s-4}(p) \\ \binom{n-1}{s-5} x_1^{s-5}(p) \\ \vdots \\ \binom{n-1}{1} x_1(p) \\ 1 \end{pmatrix}.$$

*Remark* 3.3. In [51], the author found an explicit path in the space for continuity path, but without the aid of Theorem 2.4, it was not clear how the exact space looks like. Now, we have an explicit expression of this space for continuity path, this should provide us a more flexible way to find continuity paths. In Chapter 5, we will provide more details.

**Definition 3.2.** Let $d\colon M \to \mathscr{C}_{n,s}$, then for any $p \in M$, we may define the following polyhedron in $\mathscr{C}_{n,s}$ at $p$

$$\mathcal{P}^d(p) := \left\{ c \in \mathscr{C}_{n,s} \colon \binom{n-l}{s-l} x_l^{s-l}(p) - \sum_{k=l}^{s-1} c_k \binom{n-l}{k-l} x_l^{k-l}(p) \geq 0, \quad \forall l \in \{1, \cdots, s-1\} \right\}, \tag{3.5}$$

where $x_k(p)$ is the largest real root of the $k$-th derivative of $f_p(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-1} d_k(p) \binom{n}{k} x^k$. For any $d\colon M \to \mathscr{C}_{n,s}$, we write $c \in \mathcal{P}^d$ if for any $p \in M$, we have $c(p) \in \mathcal{P}^d(p)$.

Similarly, let $d\colon M \to \tilde{\mathscr{C}}_{n,s}$, then for any $p \in M$, we may define the following polyhedron in $\tilde{\mathscr{C}}_{n,s}$ at $p$

$$\tilde{\mathcal{P}}^d(p) := \left\{ c \in \tilde{\mathscr{C}}_{n,s} \colon \binom{n-l}{s-l} x_l^{s-l}(p) - \sum_{k=l}^{s-2} c_k \binom{n-l}{k-l} x_l^{k-l}(p) \geq 0, \quad \forall l \in \{1, \cdots, s-2\} \right\}, \tag{3.6}$$

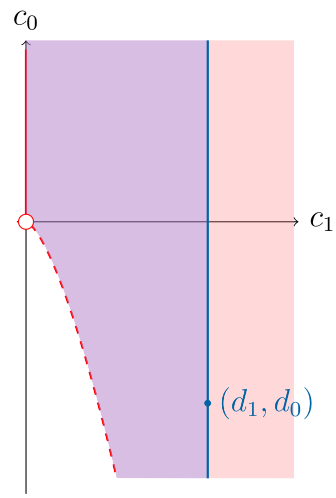


Figure 3.1: Polyhedron $\tilde{\mathcal{P}}^d$ of $\lambda_1\lambda_2\lambda_3 - d_1(\lambda_1 + \lambda_2 + \lambda_3) - d_0 = 0$.

where $x_k(p)$ is the largest real root of the $k$-th derivative of $f_p(x) = \binom{n}{s} x^s - \sum_{k=0}^{s-2} d_k(p)\binom{n}{k} x^k$. For any $d\colon M \to \tilde{\mathscr{C}}_{n,s}$, we write $c \in \tilde{\mathcal{P}}^d$ if for any $p \in M$, we have $c(p) \in \tilde{\mathcal{P}}^d(p)$.

Above Figure 3.1 is an example of the polyhedron $\tilde{\mathcal{P}}^d$ of $\lambda_1\lambda_2\lambda_3 - d_1(\lambda_1 + \lambda_2 + \lambda_3) - d_0 = 0$ with $d_1 = 4$ and $d_0 = -4$. The pink region is the set $\tilde{\mathscr{C}}_3$ and the purple region is the polyhedron $\tilde{\mathcal{P}}^d$.

## 3.3 Examples of Continuity Path

In 1978, by studying the complex Monge–Ampère equation, Shing-Tung Yau [72] resolved the Calabi conjecture [12, 13], which had been posed by Eugenio Calabi in 1954. This celebrated method by Yau is well-known nowadays, which is called the continuity method. The idea is to find a path connecting the unsolved equation to a well-understood solvable equation. In [72], Yau connected the unsolved equation to another complex Monge–Ampère equation having the given $C$-subsolution as the solution. Below, we will justify this continuity path lies in our polyhedron defined in Section 3.2.

**Proposition 3.4** (Complex Monge–Ampère equation, Yau [72])**.** *Suppose* $(M, \omega)$ *is a Kähler* *manifold and* $d\colon M \to \tilde{\mathscr{C}}_n$ *with* $d_i = 0$ *for all* $i \in \{1, \cdots, n-2\}$*. Let* $\underline{\chi} \in [\chi_0]$ *be a Kähler form, which is a* $C$*-subsolution to* $d$*. Then for the following continuity path*

$$d_t(p) \coloneqq \big(\underbrace{0, \cdots, 0}_{n-2 \text{ copies}}, td_0(p) + (1-t)\underline{\chi}^n/\omega^n(p)\big)$$

*with* $t \in [0,1]$*, we have* $d_t \in \tilde{\mathcal{P}}^d$ *for any* $t \in [0,1]$*.*

*Proof.* By Proposition 3.3, we first compute the largest real root of the $k$-th derivative of the diagonal restriction of $d$ for all $k \in \{1, \cdots, n-2\}$ at $p \in M$. The diagonal restriction at $p$ will be $x^n - d_0(p)$. Hence, $x_1(p) = 0, \cdots, x_{n-2}(p) = 0$ for all $p \in M$. By (3.6), we have

$$\tilde{\mathcal{P}}^d(p) \coloneqq \Big\{c \in \tilde{\mathscr{C}}_n\colon\ -c_l \geq 0, \quad \forall l \in \{1, \cdots, n-2\}\Big\} = \{(\underbrace{0, \cdots, 0}_{n-2 \text{ copies}}, c_0)\colon c_0 > 0\}. \quad (3.7)$$

So to check whether $d_t \in \tilde{\mathcal{P}}^d$ for any $t \in [0,1]$, by (3.7), that is to check whether $td_0(p) + (1-t)\underline{\chi}^n/\omega^n(p)$ is positive for any $t \in [0,1]$. This is true hence we finish the proof. □

**Proposition 3.5** (General inverse $\sigma_k$ equation with non-negative coefficients, Fang–Lai–Ma [29] and Collins–Székelyhidi [20])**.** *Suppose* $(M, \omega)$ *is a Kähler manifold and a constant map* $d\colon M \to \mathscr{C}_n$ *with* $d_i \geq 0$ *for all* $i \in \{0, \cdots, n-1\}$ *with* $\sum_{i=0}^{n-1} d_i > 0$*. Let* $\underline{\chi} \in [\chi_0]$ *be a* $C$*-subsolution to* $d$*. Then for the following continuity path*

$$d_t(p) \coloneqq \Big(td_{n-1}, td_{n-2}, \cdots, td_1, d_0 + (1-t)\frac{\binom{n}{n-1}d_{n-1}\Omega_1 + \cdots + \binom{n}{1}d_1\Omega_{n-1}}{\Omega_n}\Big),$$

*where* $\Omega_i \coloneqq \int_M \omega^i \wedge \chi_0^{n-i}$ *with* $t \in [0,1]$*, we have* $d_t \in \mathcal{P}^d$ *for any* $t \in [0,1]$*.*

*Proof.* First, by Lemma 2.12, we see $d_t$ is strictly $\Upsilon$-stable, that is, $d_t\colon M \to \mathscr{C}_n$. Second, by Proposition 3.2 and by (3.3), we want to verify whether the following $n-1 \times n-1$ linear system is always true for any $t \in [0,1]$:

$$\begin{cases} x_{n-1} - td_{n-1} \geq 0; \\ x_{n-2}^2 - 2td_{n-1}x_{n-2} - td_{n-2} \geq 0; \\ \vdots \\ x_2^{n-2} - t\sum_{k=2}^{n-1} d_k \binom{n-2}{k-2} x_2^{k-2} \geq 0; \\ x_1^{n-1} - t\sum_{k=1}^{n-1} d_k \binom{n-1}{k-1} x_1^{k-1} \geq 0, \end{cases} \tag{3.8}$$

where $x_k$ is the largest real root of the $k$-th derivative of $x^n - \sum_{k=0}^{n-1} d_k \binom{n}{k} x^k$ for $k \in \{1, \cdots, n-1\}$ and $x_k \geq 0$ by the proof of Lemma 2.12. Hence, for any $l \in \{1, \cdots, n-1\}$, we have

$$x_l^{n-l} - \sum_{k=l}^{n-1} d_k \binom{n-l}{k-l} x_l^{k-l} = 0. \tag{3.9}$$

Last, equation (3.9) implies that for any $l \in \{1, \cdots, n-1\}$, we get

$$x_l^{n-l} - t\sum_{k=l}^{n-1} d_k \binom{n-l}{k-l} x_l^{k-l} = (1-t)\sum_{k=l}^{n-1} d_k \binom{n-l}{k-l} x_l^{k-l} \geq 0.$$

This justifies $n-1 \times n-1$ linear system (3.8) is always true, this finishes the proof. $\square$

# Chapter 4

# A Priori Estimates

In this chapter, we study a priori estimates for functions $c\colon M^3 \to \tilde{\mathscr{C}}_3$ and $c\colon M^4 \to \tilde{\mathscr{C}}_4$ provided that a $C$-subsolution is given. We will study them individually in this dissertation and hope we will find a unified approach to handle all dimensions in the future. The ideas in this chapter come from the author's previous works [51, 52].

First, let us summarize the proof of our a priori estimates. Under the assumption of $C$-subsolution, we apply the Alexandroff–Bakelman–Pucci estimate to get a $C^0$ estimate. This $C^0$ estimate can be obtained following the proof in Székelyhidi [68], which is based on the method that Błocki [8, 9] used in the case of the complex Monge–Ampère equation. We will skip the proof of this $C^0$ estimate because it follows verbatim.

Second, we use the maximum principle to obtain that the $C^2$ norm can be bounded by the $C^1$ norm. The method is inspired by Hou–Ma–Wu [42] for the complex Hessian equations and used by Székelyhidi [68]. The interested reader is referred to [18, 68] and the references therein. Once we have the above type inequality, by a blow-up argument due to Dinew–Kołodziej [25], we can get an indirect $C^1$ estimate.

Last, to get $C^{2,\alpha}$ estimate, we follow the proof of the complex version of the Evans–Krylov theory in Siu [64], we can exploit the convexity of the solution sets to obtain a $C^{2,\alpha}$ estimates by a blow-up argument. Furthermore, for higher regularity, we apply the standard Schauder estimates and bootstrapping.

## 4.1 When Complex Dimension Equals Three

In this section, we are interested in the following equation on $M^3$:

$$X^3 = \binom{3}{1} c_1 X \wedge \omega^2 + \binom{3}{0} c_0 \omega^3 = 3c_1 X \wedge \omega^2 + c_0 \omega^3, \tag{4.1}$$

where $c_1$ is a constant and $c_0$ is a function on $M$. By Proposition 3.1, to have convexity, for any $z \in M$, we want $(c_1, c_0(z)) \in \tilde{\mathscr{C}_3}$ with

$$\tilde{\mathscr{C}_3} = \{(0, c_0)\colon c_0 > 0\} \cup \{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\}. \tag{4.2}$$

So equation (4.1) can be rewritten as a function $c\colon M^3 \to \tilde{\mathscr{C}_3}$. In this dissertation, we consider functions $c\colon M^3 \to \tilde{\mathscr{C}_3}$ with $c_1$ constant and range in a compact subset of the stratification $\{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\}$ of $\tilde{\mathscr{C}_3}$. For the case with the range in the stratification $\{(0, c_0)\colon c_0 > 0\}$, that is, $c\colon M^3 \to \{(0, c_0)\colon c_0 > 0\} \subset \tilde{\mathscr{C}_3}$. This is the three-dimensional complex Monge–Ampère equation, which can be done by Yau [72].

In this section, first, we always assume that there exists a $C$-subsolution $\underline{u}\colon M \to \mathbb{R}$ to a function $d\colon M^3 \to \tilde{\mathscr{C}_3}$ with $d_1$ constant. Then, we also call $X_{\underline{u}}$ this $C$-subsolution and by changing representative, we may assume $X$ is this $C$-subsolution. Because later on we want to use the method of continuity to obtain the solvability, we are interested in functions in the set $\tilde{\mathcal{P}}^d$. For any function $c \in \tilde{\mathcal{P}}^d$ with $c_1$ constant, we consider the following equation:

$$h_c(z, \lambda) = \frac{c_1\sigma_1(\lambda) + c_0(z)}{\lambda_1\lambda_2\lambda_3} = 1, \tag{4.3}$$

where $z \in M$ and $\lambda_i$ are the eigenvalues of $\omega^{-1}X_u$ at $z$. Note that we abbreviate $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and we always assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$ unless further notice. Most of the time, to save spaces, we will abbreviate $h = h_c(\lambda)$, $h_i = \partial h_c/\partial\lambda_i$, $h_{ij} = \partial^2 h_c/\partial\lambda_i\partial\lambda_j$ for $i, j \in \{1, 2, 3\}$ for notational convention. Unless specify otherwise, we always assume $S$ is a compact subset of $\{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\} \subset \tilde{\mathscr{C}}_3$ and we abbreviate $c_0 = c_0(z)$ for $z \in M^3$ with $(c_1, c_0) \in S \subset \tilde{\mathscr{C}}_3$.

### 4.1.1 The $C^2$ Estimate

Define a Hermitian endomorphism $\Lambda := \omega^{-1}X_u$, where $X_u = X + \sqrt{-1}\partial\bar{\partial}u$, and let $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the eigenvalues of $\Lambda$. We consider the following function $G(\Lambda) = \log(1+\lambda_1) = g(\lambda_1, \lambda_2, \lambda_3)$ and the following test function

$$U := -Au + G(\Lambda), \tag{4.4}$$

where $A \gg 0$ will be determined later. We want to apply the maximum principle to $U$, but since the eigenvalues of $\Lambda$ might not be distinct at the maximum point $q \in M$ of $U$, we do a perturbation here. The perturbation here, though not necessarily, is made to preserve the $\Upsilon$-cone structure for convenience. Assume $\lambda_1$ is large, otherwise, we are done, then

- we pick the constant matrix $B$ to be a diagonal matrix with real entries

$$B_{11} = \lambda_1; \quad B_{22} = \lambda_2/2; \quad B_{33} = 0.$$

By defining $\tilde{\Lambda} = \Lambda + B$, then $\tilde{\Lambda}$ has distinct eigenvalues near $q \in M$, which are $\{\tilde{\lambda}_1 = 2\lambda_1, \tilde{\lambda}_2 = 3\lambda_2/2, \tilde{\lambda}_3 = \lambda_3\}$. The eigenvalues of $\tilde{\Lambda}$ define smooth functions near the maximum point $q$. And we can check that the maximum point $q$ of $U$ in equation (4.4) is still the maximum point of the following locally defined test function

$$\tilde{U} := -Au + G(\tilde{\Lambda}). \tag{4.5}$$

Near the maximum point $q$ of $\tilde{U}$, we always use the coordinates in Lemma 2.15 unless otherwise noted. We instantly get the following lemma.

**Lemma 4.1.** *At the maximum point $q$ of $\tilde{U}$, by taking the first derivative of $\tilde{U}$ at $q$, we get*

$$0 = -Au_k(q) + \frac{1}{1+\tilde{\lambda}_1}(X_u)_{1\bar{1},k}, \tag{4.6}$$

*where we denote $u_k = \partial u/\partial z_k$ and $(X_u)_{1\bar{1},k} = \partial (X_u)_{1\bar{1}}/\partial z_k$.*

*Proof.* First, since $\tilde{U} = -Au + G(\tilde{\Lambda})$, if we take the first derivative, we obtain

$$\frac{\partial}{\partial z_k}\tilde{U} = -Au_k + \frac{\partial G}{\partial \Lambda_i^j}(\tilde{\Lambda})\frac{\partial \tilde{\Lambda}_i^j}{\partial z_k}.$$

At the maximum point $q$, we have $0 = -Au_k(p) + \frac{1}{1+\tilde{\lambda}_1}(X_u)_{1\bar{1},k}$, which finishes the proof. $\square$

For any $c \in \tilde{\mathcal{P}}^d$ with $c_1$ constant, we may define the following operator $\mathcal{L}_c$ by

$$\mathcal{L}_c := -\sum_{i,j,k}\frac{\partial H_c}{\partial \Lambda_i^k}(z,\Lambda)\omega^{k\bar{j}}(z)\frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{4.7}$$

where $\Lambda$ is the Hermitian endomorphism $\omega^{-1}X_u$ at $z$ and $H_c(z,\Lambda) = h_c(z,\lambda_1,\lambda_2,\lambda_3)$ is defined by $h_c(z,\lambda) = \big(c_1\sigma_1(\lambda) + c_0(z)\big)/\lambda_1\lambda_2\lambda_3$. We immediately have the following.

**Lemma 4.2.** *By taking* $h_c(\lambda) = \big(c_1\sigma_1(\lambda) + c_0\big)/\lambda_1\lambda_2\lambda_3$ *and* $g(\lambda) = \log(1+\lambda_1)$*, we have*

$$h_i = \frac{-c_1\sigma_1(\lambda_{;i}) - c_0}{\lambda_1\lambda_2\lambda_3\lambda_i}; \qquad h_{ij} = \frac{-c_1(\lambda_i+\lambda_j) + \big[c_1\sigma_1(\lambda)+c_0\big](1+\delta_{ij})}{\lambda_1\lambda_2\lambda_3\lambda_i\lambda_j};$$
$$g_i = \delta_{1i}\frac{1}{1+\lambda_1}; \qquad g_{ij} = -\delta_{1i}\delta_{1j}\frac{1}{(1+\lambda_1)^2}.$$

*Here, we denote* $h_i \coloneqq \partial h_c/\partial\lambda_i$, $g_i \coloneqq \partial g/\partial\lambda_i$, $h_{ij} \coloneqq \partial^2 h_c/\partial\lambda_i\partial\lambda_j$, *and* $g_{ij} \coloneqq \partial^2 g/\partial\lambda_i\partial\lambda_j$*. In particular, on* $\{\lambda_1\lambda_2\lambda_3 = c_1\sigma_1(\lambda) + c_0\}$*, we have*

$$h_i = \frac{c_1}{\lambda_1\lambda_2\lambda_3} - \frac{1}{\lambda_i} \quad \text{and} \quad h_{ij} = \frac{1+\delta_{ij}}{\lambda_i\lambda_j} - c_1\frac{\lambda_i+\lambda_j}{\lambda_1\lambda_2\lambda_3\lambda_i\lambda_j}.$$

**Lemma 4.3.** *For any point on the set* $\{h=1\}$*, we have*

$$-h_i = \frac{c_1\sigma_1(\lambda_{;i}) + c_0}{\lambda_1\lambda_2\lambda_3\lambda_i} = \frac{1}{\lambda_i} - \frac{c_1}{\lambda_1\lambda_2\lambda_3} > 0.$$

*Proof.* On $h = \frac{c_1\sigma_1(\lambda)+c_0}{\lambda_1\lambda_2\lambda_3} = 1$, this implies that $\lambda_i\big(\sigma_2(\lambda_{;i}) - c_1\big) = c_1\sigma_1(\lambda_{;i}) + c_0$. By Lemma 2.4, we have $\sigma_2(\lambda_{;i}) - c_1 > 0$ on $\sigma_3(\lambda) = c_1\sigma_1(\lambda) + c_0$. This implies that $c_1\sigma_1(\lambda_{;i}) + c_0 > 0$ and $-h_i > 0$. This finishes the proof. $\square$

To obtain a priori estimates, we focus on functions $c \in \tilde{\mathcal{P}}^d$ with $c_1$ constant and range in a compact subset $S$ of the stratification $\{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\}$ of $\tilde{\mathscr{C}}_3$. That is, besides $c_0, c_1$ are uniformly bounded from above and below, we also have $\inf_{c\in S} c_1 > 0$ and $\inf_{c\in S} c_0 + 2c_1^{3/2} > 0$. With these, we can get a priori estimates depending on this compact subset. With these, we can get a priori estimates depending on this compact subset. Also, to simplify estimates and compute asymptotic behavior, for the remainder of this subsection, we let $O_i$ be the Big $O$ notation that describes the limiting behavior when $\lambda_i$ approaches infinity. So $O_i(1)$ means the quantity will be bounded by a uniform constant if $\lambda_i$ is sufficiently large. we let $\Theta_i$ be the Big $\Theta$ notation that describes the limiting behavior when $\lambda_i$ approaches infinity. So $\Theta_i(1)$ means the quantity will be bounded from both above and below by a uniform constant if $\lambda_i$ is sufficiently large.

**Proposition 4.1.** *Let $h(\lambda) = \big(c_1\sigma_1(\lambda) + c_0\big)/\lambda_1\lambda_2\lambda_3$, on $\{\lambda_1\lambda_2\lambda_3 = c_1\sigma_1(\lambda) + c_0\}$, we have*

$$\sum_i h_i = \lambda_3^{-1}\Theta_1(1); \qquad \sum_i h_i^2 = \lambda_3^{-2}\Theta_1(1);$$
$$\sum_l h_{1l}h_l = \lambda_1^{-1}\lambda_3^{-2}O_1(1); \qquad \sum_l h_{2l}h_l = \lambda_2^{-1}\lambda_3^{-2}O_1(1);$$
$$\sum_l h_{3l}h_l = \lambda_3^{-3}O_1(1); \qquad \sum_{i,j} h_i h_{ij} h_j = \lambda_3^{-4}O_1(1).$$

*Proof.* The proof is by exhaustion. We consider two cases: the case $\lambda_3$ is uniformly bounded from below by a positive constant and the case $\lambda_3$ approaches 0. No matter which case, we always get these estimates. This finishes the proof. $\square$

Now, by taking the first and second derivatives of the equation $H_c(z,\Lambda) = 1$, we have the following Lemma. The proof should be straightforward; we apply Lemma 2.14, Lemma 2.15, Lemma 2.17, and Lemma 4.2. Or one can check the following reference [50] for more details.

**Lemma 4.4.** *Let $H_c(z,\Lambda) = 1$, then we have*

$$0 = \sum_{i,j} \frac{\partial H(\Lambda)}{\partial \Lambda_i^j}\frac{\partial \Lambda_i^j}{\partial z_k} + \frac{\frac{\partial c_0}{\partial z_k}}{\sigma_3(\Lambda)};$$
$$0 = \sum_{i,j}\Big(\frac{\partial^2 H(\Lambda)}{\partial \Lambda_i^j \partial \Lambda_r^s}\frac{\partial \Lambda_i^j}{\partial \bar z_k}\frac{\partial \Lambda_r^s}{\partial z_k} + \frac{\partial H(\Lambda)}{\partial \Lambda_i^j}\frac{\partial^2 \Lambda_i^j}{\partial z_k \partial \bar z_k} + \frac{\partial}{\partial \Lambda_i^j}\Big(\frac{2}{\sigma_3(\Lambda)}\Big)\Re\Big(\frac{\partial c_0}{\partial \bar z_k}\frac{\partial \Lambda_i^j}{\partial z_k}\Big)\Big) + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar z_k}}{\sigma_3(\Lambda)}.$$

*In particular, at the maximum point $q \in M$ of $\tilde U$, we have*

$$0 = \sum_i h_i (X_u)_{i\bar i,k} + \frac{\frac{\partial c_0}{\partial z_k}}{\lambda_1\lambda_2\lambda_3}; \tag{4.8}$$
$$0 = \sum_{i,j} h_{ij}(X_u)_{i\bar i,\bar k}(X_u)_{j\bar j,k} + \sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}|(X_u)_{j\bar i,k}|^2 + \sum_i h_i\Big((X_u)_{i\bar i,k\bar k} - \lambda_i\omega_{i\bar i,k\bar k}\Big)$$
$$+ \frac{\frac{\partial^2 c_0}{\partial z_k\partial \bar z_k}}{\lambda_1\lambda_2\lambda_3} - \sum_i \frac{2\Re\Big(\frac{\partial c_0}{\partial \bar z_k}(X_u)_{i\bar i,k}\Big)}{\lambda_1\lambda_2\lambda_3\lambda_i}. \tag{4.9}$$

*Proof.* The first and second derivatives should be straightforward. At the maximum point, suppose the eigenvalues are pairwise distinct satisfying $\lambda_1 > \lambda_2 > \lambda_3$. Since $\Lambda$ is a diagonal matrix, then

$$0 = \sum_i h_i (X_u)_{i\bar{i},k} + \frac{\frac{\partial c_0}{\partial z_k}}{\lambda_1\lambda_2\lambda_3}.$$

This is also true when the eigenvalues are not pairwise distinct. For the second derivative, if the eigenvalues at the maximum point $q$ are pairwise distinct, then

$$\begin{aligned}0 = &\sum_{i,j} h_{ij}(X_u)_{i\bar{i},\bar{k}}(X_u)_{j\bar{j},k} + \sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}|(X_u)_{j\bar{i},k}|^2 + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i\omega_{i\bar{i},k\bar{k}}\Big)\\ &+ \frac{\frac{\partial^2 c_0}{\partial z_k\partial\bar{z}_k}}{\lambda_1\lambda_2\lambda_3} - \frac{2}{\lambda_1\lambda_2\lambda_3\lambda_i}\Re\Big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\Big).\end{aligned}$$

This is also true when the eigenvalues are not pairwise distinct. $\square$

**Lemma 4.5.** *Let $d\colon M^3 \to \tilde{\mathscr{C}}_3$ and $X$ be a $C$-subsolution to $d$, then there exists uniform constants $N > 0$ and $\kappa > 0$, which are independent of $c \in \tilde{\mathcal{P}}^d$ with $c(z) \in S$ for any $z \in M$, such that if $\lambda_1 > N$, we have $\sum_i h_i u_{i\bar{i}} \geq -\kappa\sum_i h_i$.*

*Proof.* If $X$ is a $C$-subsolution to $d\colon M^3 \to \tilde{\mathscr{C}}_3$, then for all $z \in M$, we have $X^2 - d_1\omega^2 > 0$. Now, unless further notice, the following inequalities hold for any point $z \in M$. By choosing $\delta > 0$, $\kappa > 0$, and $\epsilon > 0$ sufficiently small, since $M$ is a compact manifold and $S$ is a compact subset, for any $c \in \tilde{\mathcal{P}}^d$ with $c \in S$, we get

$$(1-\delta)\big(X - \kappa\omega\big)^2 > (c_1 + \epsilon)\omega^2; \quad X - \kappa\omega > \epsilon\omega.$$

By the definition of $\tilde{\mathcal{P}}^d$, we also have $d_1 - c_1 = x_1^2 - c_1 \geq 0$. This implies that $d_1 \geq c_1$ for any $c \in \tilde{\mathcal{P}}^d$. Note that $u_{i\bar{i}} = \lambda_i - X_{i\bar{i}}$, so we can write

$$\begin{aligned}\sum_i h_i(u_{i\bar{i}} + \kappa) &= \sum_i h_i\big(\lambda_i - X_{i\bar{i}} + \kappa\big) = \sum_i \frac{-c_1\sigma_1(\lambda_{;i}) - c_0}{\lambda_1\lambda_2\lambda_3\lambda_i}(\lambda_i - X_{i\bar{i}} + \kappa) \\ &= -\frac{\sum_i A_i}{\lambda_1\lambda_2\lambda_3} + \sum_i (X_{i\bar{i}} - \kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_i},\end{aligned} \tag{4.10}$$

where we denote $A_i = c_1\sigma_1(\lambda_{;i}) + c_0$ for $i \in \{1, 2, 3\}$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3$, we have $A_3 \geq A_2 \geq A_1 > 0$. There are two cases to be considered:

- If $0 < \lambda_3 \leq \frac{X_{3\bar{3}} - \kappa}{3}$, then

$$(X_{3\bar{3}} - \kappa)\frac{A_3}{\lambda_1\lambda_2\lambda_3\lambda_3} \geq 3\frac{A_3}{\lambda_1\lambda_2\lambda_3}.$$

Hence, equation (4.10) gives $\sum_i h_i(u_{i\bar{i}} + \kappa) \geq -3\frac{A_3}{\lambda_1\lambda_2\lambda_3} + \sum_i (X_{i\bar{i}} - \kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_i} \geq 0$.

- If $\lambda_3 \geq \frac{X_{3\bar{3}} - \kappa}{3}$, we can show that $\lambda_2$ is bounded from above when $\lambda_1$ is large. To be more precise, suppose $\lambda_1 \geq \lambda_2 \geq N$ for $N > 0$ sufficiently large, then

$$\begin{aligned}\lambda_3 &= \frac{c_1(\lambda_1 + \lambda_2) + c_0}{\lambda_1\lambda_2 - c_1} \leq \frac{d_1(\lambda_1 + \lambda_2) + \sup_{c\in S}|c_0|}{\lambda_1\lambda_2 - d_1} \leq \frac{\sup_M d_1(\lambda_1 + \lambda_2) + \sup_{c\in S}|c_0|}{\lambda_1\lambda_2/2} \\ &\leq \frac{4\sup_M d_1 + 2\sup_{c\in S}|c_0|}{N}.\end{aligned}$$

We get a contradiction if $N$ is sufficiently large. So, we have $\frac{12\sup_M d_1 + 6\sup_{c\in S}|c_0|}{X_{3\bar{3}} - \kappa} \geq \lambda_2 \geq \lambda_3 \geq \frac{X_{3\bar{3}} - \kappa}{3} > 0$. With this, we can do a better estimate for $\lambda_2\lambda_3$, we have

$$\begin{aligned}\lambda_2\lambda_3 &= \frac{c_1(\lambda_1\lambda_2 + \lambda_2^2) + c_0\lambda_2}{\lambda_1\lambda_2 - c_1} = c_1 + \frac{c_1\lambda_2^2 + c_0\lambda_2 + c_1^2}{\lambda_1\lambda_2 - c_1} \leq c_1 + \frac{d_1\lambda_2^2 + \sup_{c\in S}|c_0|\lambda_2 + d_1^2}{\lambda_1\lambda_2 - c_1} \\ &\leq c_1 + \frac{d_1\lambda_2^2 + \sup_{c\in S}|c_0|\lambda_2 + d_1^2}{\lambda_1\lambda_2 - d_1}\end{aligned}$$

We see that when $\lambda_1$ is sufficiently large, for any $c \in \tilde{\mathcal{P}}^d$ satisfies the hypothesis, we have

$$\lambda_2\lambda_3 < c_1 + \epsilon. \tag{4.11}$$

In addition, by $(1-\delta)\big(X - \kappa\omega\big)^2 > (c_1+\epsilon)\omega^2$, we get the following,

$$(X_{2\bar{2}} - \kappa)\frac{1}{\lambda_2} + (X_{3\bar{3}} - \kappa)\frac{1}{\lambda_3} \geq 2\sqrt{\frac{(X_{2\bar{2}} - \kappa)(X_{3\bar{3}} - \kappa)}{\lambda_2\lambda_3}} \geq 2\sqrt{\frac{c_1+\epsilon}{(1-\delta)\lambda_2\lambda_3}} \tag{4.12}$$

By combining inequalities (4.10), (4.11), and (4.12), we may write

$$\begin{aligned}
\sum_i h_i(u_{i\bar{i}} + \kappa) &= \frac{-2c_1\sigma_1(\lambda) - 3c_0}{\lambda_1\lambda_2\lambda_3} + \sum_i (X_{i\bar{i}} - \kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_i} \\
&= -2 - \frac{c_0}{\lambda_1\lambda_2\lambda_3} + \sum_i (X_{i\bar{i}} - \kappa)\frac{\lambda_1\lambda_2\lambda_3 - c_1\lambda_i}{\lambda_1\lambda_2\lambda_3\lambda_i} \\
&\geq -2 - \frac{c_0}{\lambda_1\lambda_2\lambda_3} + (X_{2\bar{2}} - \kappa)\frac{\lambda_1\lambda_2\lambda_3 - c_1\lambda_2}{\lambda_1\lambda_2\lambda_3\lambda_2} + (X_{3\bar{3}} - \kappa)\frac{\lambda_1\lambda_2\lambda_3 - c_1\lambda_3}{\lambda_1\lambda_2\lambda_3\lambda_3} \\
&\geq -2 + (X_{2\bar{2}} - \kappa)\frac{1}{\lambda_2} + (X_{3\bar{3}} - \kappa)\frac{1}{\lambda_3} + \lambda_1^{-1}O_1(1) \\
&\geq -2 + 2\sqrt{\frac{(X_{2\bar{2}} - \kappa)(X_{3\bar{3}} - \kappa)}{\lambda_2\lambda_3}} + \lambda_1^{-1}O_1(1) \\
&\geq -2 + 2(1-\delta)^{-1/2} + \lambda_1^{-1}O_1(1) \geq \delta + \lambda_1^{-1}O_1(1).
\end{aligned}$$

Here, because in this case $\lambda_3$ has a lower bound, otherwise we will not get a lower order term $\lambda_1^{-1} \cdot O_1(1)$. In conclusion, we can find a uniform $N > 0$ such that if $\lambda_1 > N$, we have

$$\sum_i h_i(u_{i\bar{i}} + \kappa) \geq 0 \Longrightarrow \sum_i h_i u_{i\bar{i}} \geq -\sum_i h_i\kappa$$

for any $c \in \tilde{\mathcal{P}}^d$ with range in a compact subset of $\tilde{\mathscr{C}_3}$. This finishes the proof. □

**Lemma 4.6.** *With the same settings as in Lemma 4.5, there exists a uniform $N > 0$ and $\epsilon > 0$ such that if $\lambda_1 > N$, then $-h_2 - h_3 > \epsilon > 0$.*

*Proof.* If $\lambda_3$ is uniformly bounded from below by a positive constant, then by the proof in Lemma 4.5, we get $\lambda_2$ is uniformly bounded from above. We obtain

$$\begin{aligned}-h_2 - h_3 &= \frac{c_1\sigma_1(\lambda_{;2}) + c_0}{\lambda_1\lambda_2^2\lambda_3} + \frac{c_1\sigma_1(\lambda_{;3}) + c_0}{\lambda_1\lambda_2\lambda_3^2} = \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{2c_1}{\lambda_1\lambda_2\lambda_3} \\ &\geq \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{2d_1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \lambda_1^{-1}O_1(1) > \epsilon > 0.\end{aligned}$$

If $\lambda_3$ is sufficiently close to 0, and $\lambda_1$ is sufficiently large, then we have

$$-h_3 = \frac{1}{\lambda_3} - \frac{c_1}{\lambda_1\lambda_2\lambda_3} > \frac{1}{\lambda_3} - \frac{1}{\lambda_1} > \epsilon > 0.$$

This finishes the proof. $\square$

Now we let $C$ be a constant depending only on the stated data, but which may change from line to line. We can finish the proof of the following $C^2$ estimate.

**Theorem 4.1.** *Suppose $X$ is a $C$-subsolution to $d\colon M^3 \to \tilde{\mathscr{C}}_3$ with range in $S$ and $d_1$ constant. For any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1$ constant, if $u\colon M \to \mathbb{R}$ is a smooth function solving the equation $c\colon M^3 \to \tilde{\mathscr{C}}_3$, then there exists a constant $C$ such that*

$$|\partial\bar{\partial}u| \leq C\big(1 + \sup_M |\nabla u|^2\big).$$

*Here, $C = C(M, X, S, d, \omega, \operatorname{osc}_M u, \|c_0\|_{C^2})$ is a constant and $\nabla$ is the Levi-Civita connection with respect to $\omega$.*

*Proof.* We use the maximum principle to prove this, for any $c \in \tilde{\mathcal{P}}^d$ with $c \in S$, we can define the elliptic operator $\mathcal{L}_c$ in equation (4.7). First, by applying the operator $\mathcal{L}_c$ to $G(\tilde{\Lambda})$, at the maximum point $q$, we obtain

$$\mathcal{L}_c\big(G(\tilde{\Lambda})\big) = -\sum_{i,j,k} h_k g_{ij} \frac{\partial\tilde{\Lambda}_i^i}{\partial z_k}\frac{\partial\tilde{\Lambda}_j^j}{\partial\bar{z}_k} - \sum_k h_k \sum_{i\neq j} \frac{g_i - g_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}\frac{\partial\tilde{\Lambda}_j^i}{\partial z_k}\frac{\partial\tilde{\Lambda}_i^j}{\partial\bar{z}_k} - \sum_{i,k} h_k g_i \frac{\partial^2\tilde{\Lambda}_i^i}{\partial z_k\partial\bar{z}_k}$$

$$
\begin{aligned}
&= \sum_k h_k \frac{\left|(X_u)_{1\bar{1},k}\right|^2}{(1+\tilde{\lambda}_1)^2} + \sum_k h_k \frac{\lambda_1}{1+\tilde{\lambda}_1}\omega_{1\bar{1},k\bar{k}} - \sum_k h_k \frac{(X_u)_{1\bar{1},k\bar{k}}}{1+\tilde{\lambda}_1} \\
&\quad - \sum_k h_k \sum_{j\neq 1} \frac{\left|(X_u)_{j\bar{1},k}\right|^2 + \left|(X_u)_{1\bar{j},k}\right|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} \\
&\geq \sum_i h_i \frac{\left|(X_u)_{1\bar{1},i}\right|^2}{(1+\tilde{\lambda}_1)^2} - \sum_i h_i \frac{(X_u)_{1\bar{1},i\bar{i}}}{1+\tilde{\lambda}_1} - \sum_{j\neq 1} \frac{h_j\left|(X_u)_{1\bar{j},j}\right|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} + C\sum_i h_i. \qquad (4.13)
\end{aligned}
$$

Second, by equation (4.8) and equation (4.9), we have

$$
\begin{aligned}
0 &= \sum_{i,j} h_{ij}(X_u)_{i\bar{i},\bar{k}}(X_u)_{j\bar{j},k} + \sum_{i\neq j}\frac{|(X_u)_{j\bar{i},k}|^2}{\lambda_i\lambda_j} + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i\omega_{i\bar{i},k\bar{k}}\Big) \\
&\quad + \frac{\frac{\partial^2 c_0}{\partial z_k\partial\bar{z}_k}}{\lambda_1\lambda_2\lambda_3} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_i} \\
&= \sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l(X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2}h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l(X_u)_{l\bar{l},k}}{\sum_l h_l^2}h_j\Big) \\
&\quad + \frac{2\sum_{i,j,l} h_j h_{ij} h_l \Re\big((X_u)_{l\bar{l},k}(X_u)_{i\bar{i},\bar{k}}\big)}{\sum_l h_l^2} - \sum_{i,j} h_i h_j h_{ij}\frac{|\sum_l h_l(X_u)_{l\bar{l},k}|^2}{(\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\frac{|(X_u)_{j\bar{i},k}|^2}{\lambda_i\lambda_j} + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i\omega_{i\bar{i},k\bar{k}}\Big) + \frac{\frac{\partial^2 c_0}{\partial z_k\partial\bar{z}_k}}{\lambda_1\lambda_2\lambda_3} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_i} \\
&= \sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l(X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2}h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l(X_u)_{l\bar{l},k}}{\sum_l h_l^2}h_j\Big) \\
&\quad - \frac{2\sum_{i,j} h_j h_{ij}\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_k}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} + \sum_{i\neq j}\frac{|(X_u)_{j\bar{i},k}|^2}{\lambda_i\lambda_j} \\
&\quad + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i\omega_{i\bar{i},k\bar{k}}\Big) + \frac{\frac{\partial^2 c_0}{\partial z_k\partial\bar{z}_k}}{\lambda_1\lambda_2\lambda_3} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_i} \\
&\geq -\frac{2\sum_{i,j} h_j h_{ij}\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial\bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_k}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\frac{|(X_u)_{j\bar{i},k}|^2}{\lambda_i\lambda_j} + \sum_i h_i(X_u)_{i\bar{i},k\bar{k}} + C\sum_i h_i\lambda_i + \frac{\frac{\partial^2 c_0}{\partial z_k\partial\bar{z}_k}}{\lambda_1\lambda_2\lambda_3}, \qquad (4.14)
\end{aligned}
$$

where the inequality on the last line is due to the convexity of $h=1$ and $|\omega_{i\bar{i},k\bar{k}}| \leq C$. Since the solution set $\{\lambda_1\lambda_2\lambda_3 = c_1\sigma_1(\lambda) + c_0\}$ is convex and

$$(X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2} h_j = (X_u)_{j\bar{j},k} - \frac{\frac{\partial c_0}{\partial z_k} h_j}{\lambda_1\lambda_2\lambda_3 \sum_l h_l^2}$$

is a tangent vector on the solution set, we obtain that

$$\sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l (X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2} h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},k}}{\sum_l h_l^2} h_j\Big) \geq 0.$$

Hence, by setting $k = 1$, inequality (4.14) gives

$$\begin{aligned}
& -\sum_i h_i (X_u)_{1\bar{1},i\bar{i}} \\
& = -\sum_i h_i (X_u)_{i\bar{i},1\bar{1}} + \sum_i h_i\big((X_u)_{i\bar{i},1\bar{1}} - (X_u)_{1\bar{1},i\bar{i}}\big) \\
& = -\sum_i h_i (X_u)_{i\bar{i},1\bar{1}} + \sum_i h_i\big((X)_{i\bar{i},1\bar{1}} - (X)_{1\bar{1},i\bar{i}}\big) \\
& \geq -\frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{\lambda_1\lambda_2\lambda_3 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{\lambda_1\lambda_2\lambda_3\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_1}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} \\
& \quad + \sum_{i \neq j} \frac{|(X_u)_{j\bar{i},1}|^2}{\lambda_i\lambda_j} + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{\lambda_1\lambda_2\lambda_3} + C\sum_i h_i(1 + \lambda_i).
\end{aligned} \tag{4.15}$$

Combining Lemma 4.5, inequalities (4.13), (4.14), and (4.15), at the maximum point $q$, if $\lambda_1$ is sufficiently large, then we have

$$\begin{aligned}
\mathcal{L}_c(\tilde{U}) & \geq A\sum_i h_i u_{i\bar{i}} + \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + C\sum_i h_i - \sum_i h_i \frac{(X_u)_{1\bar{1},i\bar{i}}}{1+\tilde{\lambda}_1} - \sum_{j\neq 1} \frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} \\
& \geq A\sum_i h_i u_{i\bar{i}} + \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq i} \frac{|(X_u)_{j\bar{i},1}|^2}{(1+\tilde{\lambda}_1)\lambda_i\lambda_j} - \sum_{j\neq 1} \frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} \\
& \quad - \frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} \\
& \quad + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3} + C\sum_i h_i
\end{aligned}$$

$$
\begin{aligned}
&\geq \big(C - A\kappa\big)\sum_i h_i + \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} \frac{|(X_u)_{j\bar{1},1}|^2}{(1+\tilde{\lambda}_1)\lambda_i\lambda_j} - \sum_{j\neq 1} \frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} \\
&\quad - \frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} \\
&\quad + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3} \\
&\geq \big(C - A\kappa\big)\sum_i h_i + h_1 \frac{\big|(X_u)_{1\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} h_j \frac{\big|(X_u)_{1\bar{1},j}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} \frac{|(X_u)_{j\bar{1},1}|^2}{(1+\tilde{\lambda}_1)\lambda_i\lambda_j} \\
&\quad - \sum_{j\neq 1} \frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} - \frac{2\sum_l h_l h_{1l} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2\lambda_3} \\
&\quad - \frac{2\sum_{j\neq 1}\sum_l h_l h_{lj} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \sum_{j\neq 1} \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j} \\
&\quad - \frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2} + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3}. \qquad (4.16)
\end{aligned}
$$

We can also simplify some terms in inequality (4.16):

$$
\begin{aligned}
&\sum_{j\neq 1} h_j \frac{\big|(X_u)_{1\bar{1},j}\big|^2}{(1+\tilde{\lambda}_1)^2} + \frac{|(X_u)_{j\bar{1},1}|^2}{(1+\tilde{\lambda}_1)\lambda_1\lambda_j} = \sum_{j\neq 1} h_j \frac{\big|(X_u)_{j\bar{1},1} - T_j\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} \frac{|(X_u)_{j\bar{1},1}|^2}{(1+\tilde{\lambda}_1)\lambda_1\lambda_j} \\
&\quad \geq 2\sum_{j\neq 1} h_j \frac{\big|(X_u)_{j\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2} + 2\sum_{j\neq 1} h_j \frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} \frac{|(X_u)_{j\bar{1},1}|^2}{(1+\tilde{\lambda}_1)\lambda_1\lambda_j} \\
&\quad = \sum_{j\neq 1} \frac{2h_j\lambda_1\lambda_j + 1 + \tilde{\lambda}_1}{(1+\tilde{\lambda}_1)^2\lambda_1\lambda_j}\big|(X_u)_{j\bar{1},1}\big|^2 - 2\sum_{j\neq 1} \frac{c_1\sigma_1(\lambda_{;j}) + c_0}{\lambda_1\lambda_2\lambda_3\lambda_j}\frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2} \\
&\quad \geq \sum_{j\neq 1} \frac{-2\lambda_1 + 1 + \tilde{\lambda}_1}{(1+\tilde{\lambda}_1)^2\lambda_1\lambda_j}\big|(X_u)_{j\bar{1},1}\big|^2 - 2\sum_{j\neq 1} \frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2\lambda_j} \geq -2\sum_{j\neq 1} \frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2\lambda_j} \\
&\quad \geq -\frac{C}{\lambda_1^2\lambda_3} \geq -\frac{C}{\lambda_1\lambda_2\lambda_3} \geq -\frac{C}{\inf_{c\in S}\lambda_1\lambda_2\lambda_3} \geq -C. \qquad (4.17)
\end{aligned}
$$

Here, we denote $T_j \coloneqq (X_u)_{j\bar{1},1} - (X_u)_{1\bar{1},j} = X_{j\bar{1},1} - X_{1\bar{1},j}$. The last inequality is due to the fact that $S$ is a compact subset, so by our Theorem 2.4, $\lambda_1\lambda_2\lambda_3$ will have a uniform positive lower bound. In addition, for $j \neq 1$, we obtain

$$
\begin{aligned}
&-\frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} - \frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j}\\
&= -\frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)} - \frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{j},j}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{j},j}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j}\\
&\quad - \frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X_u)_{j\bar{j},1}-(X_u)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X_u)_{j\bar{j},1}-(X_u)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j}\\
&= -\frac{h_j}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)}\left|(X_u)_{1\bar{j},j} + \frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}(\tilde{\lambda}_1-\tilde{\lambda}_j)}{\lambda_1\lambda_2\lambda_3 h_j\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}(\tilde{\lambda}_1-\tilde{\lambda}_j)}{\lambda_1\lambda_2\lambda_3\lambda_j h_j}\right|^2\\
&\quad + \frac{(\tilde{\lambda}_1-\tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2 h_j}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2 - \frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2}\\
&\quad - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j}\\
&\geq \frac{(\tilde{\lambda}_1-\tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2 h_j}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2 - \frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2}\\
&\quad - \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j}. \qquad (4.18)
\end{aligned}
$$

We estimate some terms in inequality (4.16) and inequality (4.18). If $\lambda_3$ is uniformly bounded from below by a positive constant, then $\lambda_2$ is uniformly bounded from above. The estimates should be straightforward. Now, if $\lambda_3$ approaches 0, then we need to consider the limiting behavior of $h_2$ and $h_3$. For $h_3$, we have

$$\frac{1}{\lambda_3} > -h_3 = \frac{1}{\lambda_3} - \frac{c_1}{\lambda_1\lambda_2\lambda_3} > \frac{1}{\lambda_3} - \frac{1}{\lambda_1} > \frac{1}{2\lambda_3}.$$

So $h_3 = \lambda_3^{-1}\Theta_1(1)$. For $h_2$, the limiting behavior is slightly harder, we have

$$
\begin{aligned}
\frac{1}{\lambda_2} > -h_2 = \frac{1}{\lambda_2} - \frac{c_1}{\lambda_1\lambda_2\lambda_3} = \frac{c_1(\lambda_1+\lambda_3)+c_0}{\lambda_1\lambda_2^2\lambda_3} &\geq \frac{1}{2\lambda_2} + \frac{c_1\lambda_3+c_0}{2\lambda_1\lambda_2^2\lambda_3} \geq \frac{1}{2\lambda_2} + \frac{c_0}{2\lambda_1\lambda_2^2\lambda_3}\\
\geq \frac{1}{2\lambda_2} - \frac{\sup_{c\in S}|c_0|}{2\lambda_1\lambda_2^2\lambda_3} \geq \frac{1}{2\lambda_2} - \frac{\sup_{c\in S}|c_0|}{2c_1\lambda_1\lambda_2} &> \frac{1}{2\lambda_2} - \frac{\sup_{c\in S}|c_0|}{2c_1^{3/2}\lambda_1}.
\end{aligned}
$$

Hence, $h_2 = \lambda_2^{-1}\Theta_1(1)$. In conclusion, for $j \neq 1$, we have $h_j = \lambda_j^{-1}\Theta_1(1)$. Thus, by Proposition 4.1, for $j \neq 1$, we have

$$\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2 h_j^2}\Big|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\Big|^2 \leq \frac{C}{\lambda_j^{-2}}\Big|\frac{\sum_l h_l h_{lj}}{\sum_l h_l^2} + \frac{1}{\lambda_j}\Big|^2 \leq \frac{C}{\lambda_j^{-2}\lambda_j^2} \leq C; \tag{4.19}$$

$$\left|\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3 h_j \sum_l h_l^2}\right| \leq \frac{C\lambda_j^{-1}\lambda_3^{-2}}{\lambda_1\lambda_j^{-1}\lambda_3^{-2}} \leq \frac{C}{\lambda_1} \leq C; \tag{4.20}$$

$$\left|\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_j h_j}\right| \leq \frac{C}{\lambda_1\lambda_j\lambda_j^{-1}} \leq \frac{C}{\lambda_1} \leq C; \tag{4.21}$$

$$\left|\frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2(\sum_l h_l^2)^2}\right| \leq C\frac{\lambda_3^{-4}}{\lambda_1\lambda_3^{-4}} \leq \frac{C}{\lambda_1} \leq C; \tag{4.22}$$

$$\left|\frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3}\right| \leq \frac{C}{\lambda_1} \leq C. \tag{4.23}$$

Last, we have the following inequality

$$\begin{aligned}
&h_1\frac{\big|(X_u)_{1\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2} - \frac{2\sum_l h_l h_{1l}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\sum_l h_l^2} - \sum_{j\neq 1}\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2\lambda_3} \\
&\quad = h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1} - \Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1\sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2 \\
&\qquad - h_1\Big|\frac{\partial c_0}{\partial z_1}\Big|^2\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1\sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)^2 \\
&\quad \geq h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1} - \Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1\sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2.
\end{aligned} \tag{4.24}$$

Thus, by Lemma 4.5 and Lemma 4.6 and by inequalities (4.16), (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23), and (4.24), at the maximum point $q$ we obtain

$$\begin{aligned}
0 \geq \mathcal{L}_c(\tilde{U}) &\geq \big(C - A\kappa\big)\sum_i h_i + C\sum_{j\neq 1} h_j - C \\
&\qquad + h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1} - \Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1\sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2 \\
&\geq \frac{A\kappa\epsilon}{2} - \frac{1}{\lambda_1}\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1} - \Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1\sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2.
\end{aligned}$$

Here, we let $A$ sufficiently large to get the last inequality. So, we get

$$\begin{aligned}\sqrt{\frac{A\kappa\epsilon}{2}}\sqrt{\lambda_1} &\leq \Big|Au_1 - \Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1 \sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big| \\ &\leq A|u_1| + \Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1 \sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big|\Big|\frac{\partial c_0}{\partial z_1}\Big|. \end{aligned} \tag{4.25}$$

Similar to before, we estimate the quantity $\Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1 \sum_l h_l^2} + \frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big|$, we have

$$\begin{aligned}&\Big|\frac{1}{\lambda_1^2\lambda_2\lambda_3 h_1}\Big| = \frac{1}{c_1(\lambda_2+\lambda_3)+c_0} < \frac{1}{2c_1^{3/2}+c_0} \leq \frac{1}{\inf_{c\in S}(2c_1^{3/2}+c_0)} \leq C; \\ &\Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3 h_1 \sum_l h_l^2}\Big| \leq \frac{C\lambda_1^{-1}\lambda_3^{-2}}{(c_1(\lambda_2+\lambda_3)+c_0)\lambda_1^{-1}\lambda_3^{-2}} \leq C.\end{aligned}$$

In conclusion, we get $\sqrt{\frac{A\kappa\epsilon}{2}}\sqrt{\lambda_1} \leq A|u_1| + C$. This implies that

$$\lambda_1 \leq \frac{2}{A\kappa\epsilon}(A|u_1|+C)^2 \leq \frac{4}{A\kappa\epsilon}(A^2|u_1|^2 + C^2) \leq C(1+\sup_M |\nabla u|^2).$$

By plugging back to the original test function $U = -Au + G(\Lambda)$, we will obtain a $C^2$ estimate for any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1$ constant. This finishes the proof. □

### 4.1.2 The $C^1$ Estimate

Here, we use a blow-up argument proved by Collins–Jacob–Yau [18] to obtain a $C^1$ estimate. One can also check a more general setting considered by Székelyhidi [68], or the complex Hessian equation studied by Dinew–Kołodziej [25].

**Proposition 4.2** (Collins–Jacob–Yau [18]). *Suppose $u\colon M \to \mathbb{R}$ satisfies*

*(a)* $X + \sqrt{-1}\partial\bar{\partial}u \geq -K\omega$,

*(b)* $\|u\|_{L^\infty(M)} \leq K$,

*(c)* $\|\partial\bar\partial u\|_{L^\infty(M)} \le K\big(1+\sup_M |\nabla u|^2\big)$,

*for a uniform constant $K<\infty$. Then there exists a constant $C$, depending only on $M,\omega$, $X$, and $K$ such that* $\sup_M |\nabla u| \le C$.

### 4.1.3 Higher Order Estimates

The proof follows from Siu [64], here we use a standard blow-up argument inspired by Collins–Jacob–Yau [18]. The equation is elliptic and the solution set is convex, we can exploit the convexity of the solution set to obtain $C^{2,\alpha}$ estimates by a blow-up argument.

By shrinking the coordinate charts if necessary, we may assume that the manifold $M$ can be covered by finitely many coordinate charts $\bar U_a \subset V_a$ such that $X_u = \sqrt{-1}\partial\bar\partial u_a$ on $V_a$ for a smooth function $u_\alpha$ satisfying $\|u_a\|_{C^2(\bar U_a)} \le K$, where we use the standard Euclidean metric on $\mathbb{C}^3$ and $K$ is a uniform constant independent of $a$. For convenience, we focus on a fixed coordinate chart $V_a$, and we drop the subscript $a$. The function $u$ on $V$ satisfies

$$H_c(z,\partial\bar\partial u) = H_c\big(z,\Lambda(z)\big) = 1, \text{ for } z\in V,$$

where $\Lambda^j_i(z) = \omega^{j\bar k}(z)u_{i\bar k}(z)$ with eigenvalues in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3 - c_1\sigma_1(\lambda) - c_0(z) = 0$. Moreover, by fixing $\tilde z\in U$, we define the following operator which does not depend on $z\in V$,

$$\tilde H_{c,\tilde z}(\partial\bar\partial u) := H_c(\tilde z, \omega^{j\bar k}(\tilde z)u_{i\bar k}).$$

First, we prove a Hölder estimate for the second derivatives. We have the following.

**Lemma 4.7.** *Let $U\subset\mathbb{C}^3$ be a connected open set and fix $\tilde z\in U$. Suppose $u\colon U\subset\mathbb{C}^3\to\mathbb{R}$ is a $C^3$ function such that $\|\partial\bar\partial u\|_{L^\infty(U)}<\infty$ and the eigenvalues $\lambda\big(\omega^{j\bar k}(\tilde z)u_{i\bar k}(\tilde z)\big)$ of $\omega^{j\bar k}(\tilde z)u_{i\bar k}(\tilde z)$ in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3 - c_1\sigma_1(\lambda) - c_0(\tilde z) = 0$. If for all $z\in U$, $\tilde H_{c,\tilde z}(\partial\bar\partial u)(z) = 1$, then there* *exists a constant $\alpha \in (0,1)$ such that for any $R > 0$ with $\overline{B_{2R}} \subset U$, the function $u$ satisfies*

$$\|\partial\bar{\partial}u\|_{C^{\alpha}(B_R)} \leq C \cdot R^{-\alpha}.$$

*Here, $C = C\big(c, S, \|\partial\bar{\partial}u\|_{L^{\infty}(U)}\big)$.*

*Proof.* First, we may verify that for all $z \in U$, $\lambda\big(\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(z)\big)$ in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3 - c_1\sigma_1(\lambda) - c_0(\tilde{z}) = 0$. Second, let $\gamma$ be an arbitrary vector of $\mathbb{C}^n$, by differentiating $\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u) = 1$ with respect to $\gamma$ and then with respect to $\bar{\gamma}$ gives

$$\sum_{i,j} \frac{\partial \tilde{H}_{c,\tilde{z}}}{\partial u_{i\bar{j}}}(\partial\bar{\partial}u)u_{i\bar{j}\gamma} = 0; \quad \sum_{i,j,k,l} \frac{\partial^2 \tilde{H}_{c,\tilde{z}}}{\partial u_{i\bar{j}}\partial u_{k\bar{l}}}(\partial\bar{\partial}u)u_{k\bar{l}\bar{\gamma}}u_{i\bar{j}\gamma} + \sum_{i,j} \frac{\partial \tilde{H}_{c,\tilde{z}}}{\partial u_{i\bar{j}}}(\partial\bar{\partial}u)u_{i\bar{j}\gamma\bar{\gamma}} = 0.$$

By the convexity of the level set, we have

$$\sum_{i,j} \frac{\partial \tilde{H}_{c,\tilde{z}}}{\partial u_{i\bar{j}}}(\partial\bar{\partial}u)u_{i\bar{j}\gamma\bar{\gamma}} \leq 0.$$

Second, let $w = u_{\gamma\bar{\gamma}}$, then we may rewrite the equation as $-\sum_{i,j} \tilde{H}^{i\bar{j}}_{c,\tilde{z}}(\partial\bar{\partial}u)\partial_i\bar{\partial}_j w \geq 0$, where we denote $\tilde{H}^{i\bar{j}}_{c,\tilde{z}}(\partial\bar{\partial}u) := \partial\tilde{H}_{c,\tilde{z}}/\partial u_{i\bar{j}}(\partial\bar{\partial}u)$. By the hypothesis that $\|\partial\bar{\partial}u\|_{L^{\infty}(U)} < \infty$, the eigenvalues of $\sqrt{-1}\partial\bar{\partial}u$ have an upper bound and thus a positive lower bound by the compactness of $S$. Hence the operator $-\tilde{H}^{i\bar{j}}_{c,\tilde{z}}(\partial\bar{\partial}u)\frac{\partial^2}{\partial z_i\bar{\partial}z_j}$ is uniformly elliptic.

For $s = 1, 2$, let $M_s := \sup_{B_{sR}} w$, where $B_{sR}$ is a ball of radius $sR$ contained in $U$ having the same center. By the Krylov–Safanov's weak Harnack inequality [34], there is a constant $p > 0$ and $C > 0$ such that

$$\Big(\frac{1}{R^6}\int_{B_R}(M_2 - w)^p\Big)^{1/p} \leq C\big(M_2 - M_1 + R^{\frac{2(q-3)}{q}}\|h_{\gamma\bar{\gamma}}\|_{L^q(B_{2R})}\big),$$

where $q > 3$. Then, by the smoothness and convexity of the solution set, the tangent plane to the graph of $\tilde{H}_{c,\tilde{z}}$ at the point $\big(u_{i\bar{j}}(y)\big)$ is below the graph of $\tilde{H}_{c,\tilde{z}}$. Hence the tangent plane will be the supporting hyperplane, which implies

$$\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u(y)) - \tilde{H}^{i\bar{j}}_{c,\tilde{z}}(\partial\bar{\partial}u(y))\big(u_{i\bar{j}}(x) - u_{i\bar{j}}(y)\big) \geq \tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u(x)).$$

That is, $0 \geq -\tilde{H}^{i\bar{j}}_{c,\tilde{z}}(\partial\bar{\partial}u(y))\big(u_{i\bar{j}}(y) - u_{i\bar{j}}(x)\big)$. Last, the rest follows directly from the proof of the complex version of the Evans–Krylov theory in Siu [64]. □

Then, with the above Lemma 4.7, we can prove a Liouville-type result.

**Proposition 4.3.** *Let $\tilde{z} \in \mathbb{C}^3$. Suppose $u\colon \mathbb{C}^3 \to \mathbb{R}$ is a $C^3$ function such that $\|\partial\bar{\partial}u\|_{L^\infty(\mathbb{C}^3)} < \infty$ and the eigenvalues $\lambda\big(\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})\big)$ of $\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})$ in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3 - c_1\sigma_1(\lambda) - c_0(\tilde{z}) = 0$. If for all $z \in \mathbb{C}^3$, $\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u)(z) = 1$, then $u$ is a quadratic polynomial.*

*Proof.* The proof follows from Lemma 4.7 by letting $R \to \infty$. □

**Lemma 4.8.** *For $r > 0$, suppose $u\colon B_{2r} \subset \mathbb{C}^3 \to \mathbb{R}$ is a smooth function satisfying $H_c(z, \partial\bar{\partial}u) = 1$. Then, for every $\alpha \in (0,1)$, we have the estimate*

$$\|\partial\bar{\partial}u\|_{C^\alpha(B_{r/2})} \leq C(\alpha, c, S, \|\partial\bar{\partial}u\|_{L^\infty(B_{2r})}).$$

*Proof.* For each $z \in B_r$, we consider the following quantity

$$N_u \coloneqq \sup_{z\in B_r} d_z|\partial\partial\bar{\partial}u(z)|,$$

where $d_z \coloneqq \operatorname{dist}(z, \partial B_r)$. Suppose the supremum is achieved at $z_0 \in B_r$, then we consider the following smooth function $\tilde{u}\colon B_{N_u}(0) \to \mathbb{R}$ defined by

$$\tilde{u}(z) \coloneqq u\big(z_0 + d_{z_0}z/N_u\big)N_u^2/d_{z_0}^2 - A - A_i z_i,$$

where $A$, $A_i$ are chosen so that $\tilde{u}(0) = 0 = \partial\tilde{u}(0)$. Notice that

$$\partial\bar{\partial}\tilde{u}(z) = \partial\bar{\partial}u(z_0 + d_{z_0}z/N_u); \quad \|\partial\partial\bar{\partial}\tilde{u}\|_{L^\infty(B_{N_u}(0))} = 1 = |\partial\partial\bar{\partial}\tilde{u}(0)|.$$

In particular, we have $\|\partial\bar{\partial}\tilde{u}\|_{C^\alpha(B_r)} \leq r$ for every $\alpha \in (0,1)$ and $\tilde{u}$ solves

$$H_c\big(z_0 + d_{z_0}z/N_u, \partial\bar{\partial}\tilde{u}\big)(z) = 1, \quad z \in B_{N_u}(0).$$

By the hypothesis that $\|\partial\bar{\partial}u\|_{L^\infty(B_{2r})} < \infty$, the eigenvalues of $\sqrt{-1}\partial\bar{\partial}u$ have an upper bound and thus a positive lower bound, so $H_c(z,\cdot)$ is uniformly elliptic. The Schauder theory for fully nonlinear uniformly elliptic operators of the form $H_c(z,\partial\bar{\partial}u)$ implies that $\partial\tilde{u}$ is bounded in $C^{2,\alpha}(B_{N_u/2}(0))$, and so $\tilde{u}$ is controlled in $C^{3,\alpha}(B_{N_u/2}(0))$. Now, we prove this by contradiction. Suppose we have a sequence $\{u_n\}$ satisfying $H_c(z,\partial\bar{\partial}u_n) = 1$, where $u_n\colon B_{2r} \to \mathbb{R}$ such that $\|\partial\bar{\partial}u_n\|_{L^\infty(B_{2r})} \leq K$ but $N_{u_n} \geq n$. For each $n$, we let $z_n \in B_r$ be a point where $N_{u_n}$ is achieved. Since $\overline{B_r}$ is compact, by passing to a subsequence, we may assume that $z_n \to z_\infty \in \overline{B_r}$. Thus, we have functions $\tilde{u}_n\colon B_{N_{u_n}}(0) \to \mathbb{R}$ such that

$$\|\tilde{u}_n\|_{C^{3,\alpha}(B_{N_{u_n}}(0))} \leq C \text{ and } H_c\big(z_n + d_{z_n}z/N_{u_n}, \partial\bar{\partial}\tilde{u}_n\big)(z) = 1 \text{ for } z \in B_{N_{u_n}}(0).$$

Since $N_{u_n} \geq n$, by a diagonal argument, there exists a function $\tilde{u}_\infty\colon \mathbb{C}^3 \to \mathbb{R}$ and a subsequence such that $\{\tilde{u}_n\}_{n\geq k}$ converges uniformly to $\tilde{u}_\infty$ in $C^{3,\alpha'}(B_k(0))$ for some $\alpha' \in (0,1)$. In particular, we have $\tilde{H}_{c,z_\infty}(\partial\bar{\partial}\tilde{u}_\infty)(z) = 1$ and $|\partial\partial\bar{\partial}\tilde{u}_\infty(0)| = 1$. By Proposition 4.3, $\tilde{u}_\infty$ is a quadratic polynomial, which leads to a contradiction. □

By arguing locally, with Lemma 4.8 we have the following.

**Corollary 4.1.** *Suppose $S$ is a compact subset of the stratification $\{(c_1,c_0)\colon c_1 > 0$ and $c_0 > -2c_1^{3/2}\}$ of $\tilde{\mathscr{C}_3}$ and $X$ is a $C$-subsolution to $d\colon M^3 \to \tilde{\mathscr{C}_3}$ with range in $S$ and $d_1$ constant. For any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1$ constant, if $u\colon M \to \mathbb{R}$ is a smooth function solving* *the equation $c\colon M^3 \to \tilde{\mathscr{C}_3}$, then for every $\alpha \in (0,1)$, we have*

$$\|\partial\bar{\partial}u\|_{C^\alpha(M)} \leq C(M, X, S, d, \omega, \alpha, \|c_0\|_{C^2}, \|\partial\bar{\partial}u\|_{L^\infty(M)}).$$

## 4.2 When Complex Dimension Equals Four

In this section, we are interested in the following equation on $M^4$:

$$X^4 = \binom{4}{2} c_2 X^2 \wedge \omega^2 + \binom{4}{1} c_1 X \wedge \omega^3 + \binom{4}{0} c_0 \omega^4 = 6c_2 X^2 \wedge \omega^2 + 4c_1 X \wedge \omega^3 + c_0 \omega^4, \quad (4.26)$$

where $c_2, c_1$ are constants and $c_0$ is a function on $M$. By Proposition 3.1, to have convexity, for any $z \in M$, we want $(c_2, c_1, c_0(z)) \in \tilde{\mathscr{C}_4}$ with

$$\begin{aligned}\tilde{\mathscr{C}_4} = &\{(0,0,c_0)\colon c_0 > 0\} \cup \{(0, c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -3c_1^{4/3}\} \\ &\cup \{(c_2, -2c_2^{3/2}, c_0)\colon c_2 > 0 \text{ and } c_0 > 3c_2^2\} \\ &\cup \{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2}, \text{ and } c_0 > -3c_2x_1^2 - 3c_1x_1\}. \end{aligned} \quad (4.27)$$

Here, $x_1$ is the largest real root of $x^3 - 3c_2x - c_1$. So equation (4.26) can be rewritten as a function $c\colon M^4 \to \tilde{\mathscr{C}_4}$. In this dissertation, we consider functions $c\colon M^4 \to \tilde{\mathscr{C}_4}$ with $c_1$ and $c_2$ both constant and range in a compact subset $S$ of the stratification $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2}, \text{ and } c_0 > -3c_2x_1^2 - 3c_1x_1\}$ of $\tilde{\mathscr{C}_4}$. That is, besides $c_0, c_1, c_2$ are bounded from above and below, we also have $\inf_{c\in S} c_2 > 0$, $\inf_{c\in S} c_1 + 2c_2^{3/2} > 0$, and $\inf_{c\in S} c_0 + 3c_2x_1^2 + 3c_1x_1 > 0$.

In this section, first, we always assume that there exists a $C$-subsolution $\underline{u}\colon M \to \mathbb{R}$ to a function $d\colon M^4 \to \tilde{\mathscr{C}_4}$ with $d_1$ and $d_2$ constant. Then, we also call $X_{\underline{u}}$ this $C$-subsolution and by changing representative, we may assume $X$ is this $C$-subsolution. Because later on we want to use the method of continuity to obtain the solvability, we are interested in functions in $\tilde{\mathcal{P}}^d$. For any function $c \in \tilde{\mathcal{P}}^d$ with $c_1$ and $c_2$ constant, we consider the following equation:

$$h_c(z,\lambda) = \frac{c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0(z)}{\lambda_1\lambda_2\lambda_3\lambda_4} = 1, \tag{4.28}$$

where $z \in M$ and $\lambda_i$ are the eigenvalues of $\omega^{-1}X_u$ at $z$. Note that we abbreviate $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and we always assume $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ unless further notice. Most of the time, to save spaces, we will abbreviate $h = h_c(\lambda)$, $h_i = \partial h_c/\partial\lambda_i, h_{ij} = \partial^2 h_c/\partial\lambda_i\partial\lambda_j$ for $i, j \in \{1,2,3,4\}$ for notational convention. Unless specify otherwise, we always assume $S$ is a compact subset of $\big\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},$ and $c_0 > -3c_2x_1^2 - 3c_1x_1\big\} \subset \tilde{\mathscr{C}}_4$ and we abbreviate $c_0 = c_0(z)$ for $z \in M^4$ with $(c_2, c_1, c_0) \in S \subset \tilde{\mathscr{C}}_4$.

### 4.2.1 The $C^2$ Estimate

Define a Hermitian endomorphism $\Lambda \coloneqq \omega^{-1}X_u$, where $X_u = X + \sqrt{-1}\partial\bar{\partial}u$, and let $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the eigenvalues of $\Lambda$. We consider the following function $G(\Lambda) = \log(1+\lambda_1) = g(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and the following test function

$$U \coloneqq -Au + G(\Lambda), \tag{4.29}$$

where $A \gg 0$ will be determined later. We want to apply the maximum principle to $U$, but since the eigenvalues of $\Lambda$ might not be distinct at the maximum point $q \in M$ of $U$, we do a perturbation here. The perturbation here, though not necessarily, is made to preserve the $\Upsilon$-cone structure for convenience. Assume $\lambda_1$ is large, otherwise, we are done, then

- we pick the constant matrix $B$ to be a diagonal matrix with real entries

$$B_{11} = \lambda_1; \quad B_{22} = \lambda_2/2; \quad B_{33} = \lambda_3/3; \quad B_{44} = 0.$$

By defining $\tilde{\Lambda} = \Lambda + B$, then $\tilde{\Lambda}$ has distinct eigenvalues near $q \in M$, which are $\{\tilde{\lambda}_1 = 2\lambda_1, \tilde{\lambda}_2 = 3\lambda_2/2, \tilde{\lambda}_3 = 4\lambda_3/3, \tilde{\lambda}_4 = \lambda_4\}$. The eigenvalues of $\tilde{\Lambda}$ define smooth functions near the maximum point $q$. And we can check that the maximum point $q$ of $U$ in equation (4.29) is still the maximum point of the following locally defined test function

$$\tilde{U} \coloneqq -Au + G(\tilde{\Lambda}). \tag{4.30}$$

Near the maximum point $q$ of $\tilde{U}$, we always use the coordinates in Lemma 2.15 unless otherwise noted. Same as before, we instantly get the following lemma.

**Lemma 4.9.** *At the maximum point $q$ of $\tilde{U}$, by taking the first derivative of $\tilde{U}$ at $q$, we get*

$$0 = -Au_k(q) + \frac{1}{1+\tilde{\lambda}_1}(X_u)_{1\bar{1},k}, \tag{4.31}$$

*where we denote $u_k = \partial u/\partial z_k$ and $(X_u)_{1\bar{1},k} = \partial (X_u)_{1\bar{1}}/\partial z_k$.*

For any $c \in \tilde{\mathcal{P}}^d$ with $c_1$ and $c_2$ constant, we may define the following operator $\mathcal{L}_c$ by

$$\mathcal{L}_c \coloneqq -\sum_{i,j,k} \frac{\partial H_c}{\partial \Lambda_i^k}(z,\Lambda)\omega^{k\bar{j}}(z)\frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{4.32}$$

where $\Lambda$ is the Hermitian endomorphism $\omega^{-1}X_u$ at $z$ and $H_c(z,\Lambda) = h_c(z,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is defined by $h_c(z,\lambda) = \big(c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0(z)\big)/\lambda_1\lambda_2\lambda_3\lambda_4$. We have the following.

**Lemma 4.10.** *By taking $h_c(\lambda) = \big(c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0\big)/\lambda_1\lambda_2\lambda_3\lambda_4$ and $g(\lambda) = \log(1+\lambda_1)$, we have*

$$h_i = \frac{-c_2\sigma_2(\lambda_{;i}) - c_1\sigma_1(\lambda_{;i}) - c_0}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i}; \qquad h_{ij} = \frac{c_2\sigma_2(\lambda_{;i,j}) + c_1\sigma_1(\lambda_{;i,j}) + c_0}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i\lambda_j}(1+\delta_{ij});$$
$$g_i = \delta_{1i}\frac{1}{1+\lambda_1}; \qquad g_{ij} = -\delta_{1i}\delta_{1j}\frac{1}{(1+\lambda_1)^2}.$$

*Here, we denote $h_i \coloneqq \partial h_c/\partial\lambda_i$, $g_i \coloneqq \partial g/\partial\lambda_i$, $h_{ij} \coloneqq \partial^2 h_c/\partial\lambda_i\partial\lambda_j$, and $g_{ij} \coloneqq \partial^2 g/\partial\lambda_i\partial\lambda_j$. In* *particular, on* $\{\lambda_1\lambda_2\lambda_3\lambda_4 = c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0\}$*, we have*

$$h_i = \frac{c_2\sigma_1(\lambda_{;i}) + c_1}{\lambda_1\lambda_2\lambda_3\lambda_4} - \frac{1}{\lambda_i} \quad \text{and} \quad h_{ij} = \frac{1+\delta_{ij}}{\lambda_i\lambda_j} - \frac{c_2\big((\lambda_i+\lambda_j)\sigma_1(\lambda_{;i,j}) + \lambda_i\lambda_j\big) + c_1(\lambda_i+\lambda_j)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i\lambda_j}.$$

**Lemma 4.11.** *For any point on the set* $\{h = 1\}$*, we have*

$$-h_i = \frac{c_2\sigma_2(\lambda_{;i}) + c_1\sigma_1(\lambda_{;i}) + c_0}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} = \frac{c_2\sigma_1(\lambda_{;i}) + c_1}{\lambda_1\lambda_2\lambda_3\lambda_4} - \frac{1}{\lambda_i} > 0.$$

*Proof.* On $h = \frac{c_2\sigma_2(\lambda)+c_1\sigma_1(\lambda)+c_0}{\lambda_1\lambda_2\lambda_3} = 1$, this implies that $\lambda_i\big(\sigma_3(\lambda_{;i}) - c_2\sigma_1(\lambda_{;i}) - c_1\big) = c_2\sigma_2(\lambda_{;i}) + c_1\sigma_1(\lambda_{;i}) + c_0$. By Lemma 2.4, we have $\sigma_3(\lambda_{;i}) - c_2\sigma_1(\lambda_{;i}) - c_1 > 0$ on $\sigma_4(\lambda) = c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0$. Hence, $c_2\sigma_2(\lambda_{;i}) + c_1\sigma_1(\lambda_{;i}) + c_0 > 0$ and $-h_i > 0$. This finishes the proof. $\square$

To obtain a priori estimates, we focus on functions $c \in \tilde{\mathcal{P}}^d$ with $c_1$ and $c_2$ constant and range in a compact subset $S$ of the stratification $\big\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2}, \text{ and } c_0 > -3c_2x_1^2 - 3c_1x_1\big\}$ of $\tilde{\mathscr{C}}_4$. That is, besides $c_0, c_1, c_2$ are bounded from above and below, we also have $\inf_{c\in S} c_2 > 0$, $\inf_{c\in S} c_1 + 2c_2^{3/2} > 0$, and $\inf_{c\in S} c_0 + 3c_2x_1^2 + 3c_1x_1 > 0$. With these, we can get a priori estimates depending on this compact subset. Also, to simplify estimates and compute asymptotic behavior, for the remainder of this subsection, we let $O_i$ be the Big $O$ notation that describes the limiting behavior when $\lambda_i$ approaches infinity. So $O_i(1)$ means the quantity will be bounded by a uniform constant if $\lambda_i$ is sufficiently large. We let $\Theta_i$ be the Big $\Theta$ notation that describes the limiting behavior when $\lambda_i$ approaches infinity. So $\Theta_i(1)$ means the quantity will be bounded from both above and below by a uniform constant if $\lambda_i$ is sufficiently large.

**Proposition 4.4.** *Let* $h(\lambda) = \big(c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0\big)/\lambda_1\lambda_2\lambda_3\lambda_4$*, on* $\{\lambda_1\lambda_2\lambda_3\lambda_4 = c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0\}$*, we have*

$$\sum_i h_i = \lambda_4^{-1}\Theta_1(1); \qquad\qquad \sum_i h_i^2 = \lambda_4^{-2}\Theta_1(1);$$

$$\sum_l h_{1l}h_l = \lambda_1^{-1}\lambda_4^{-2}O_1(1); \qquad \sum_l h_{2l}h_l = \lambda_2^{-1}\lambda_4^{-2}O_1(1);$$
$$\sum_l h_{3l}h_l = \lambda_3^{-1}\lambda_4^{-2}O_1(1); \qquad \sum_l h_{4l}h_l = \lambda_4^{-3}O_1(1);$$
$$\sum_{i,j} h_i h_{ij} h_j = \lambda_4^{-4}O_1(1).$$

*Proof.* The proof is by exhaustion. We consider two cases: the case $\lambda_4$ is uniformly bounded from below by a positive constant and the case $\lambda_4$ approaches 0. No matter which case, we always get these estimates. This finishes the proof. $\square$

By taking the first and second derivatives of the equation $H_c(z, \Lambda) = 1$, we have the following.

**Lemma 4.12.** *Let $H_c(z, \Lambda) = 1$, then we have*

$$0 = \sum_{i,j} \frac{\partial H(\Lambda)}{\partial \Lambda_i^j}\frac{\partial \Lambda_i^j}{\partial z_k} + \frac{\frac{\partial c_0}{\partial z_k}}{\sigma_4(\Lambda)};$$
$$0 = \sum_{i,j}\Big(\frac{\partial^2 H(\Lambda)}{\partial \Lambda_i^j \partial \Lambda_r^s}\frac{\partial \Lambda_i^j}{\partial \bar{z}_k}\frac{\partial \Lambda_r^s}{\partial z_k} + \frac{\partial H(\Lambda)}{\partial \Lambda_i^j}\frac{\partial^2 \Lambda_i^j}{\partial z_k \partial \bar{z}_k} + \frac{\partial}{\partial \Lambda_i^j}\Big(\frac{2}{\sigma_4(\Lambda)}\Big)\Re\Big(\frac{\partial c_0}{\partial \bar{z}_k}\frac{\partial \Lambda_i^j}{\partial z_k}\Big)\Big) + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\sigma_4(\Lambda)}.$$

*In particular, at the maximum point $q \in M$ of $\tilde{U}$, we have*

$$0 = \sum_i h_i (X_u)_{i\bar{i},k} + \frac{\frac{\partial c_0}{\partial z_k}}{\lambda_1\lambda_2\lambda_3\lambda_4}; \tag{4.33}$$
$$\begin{aligned} 0 = &\sum_{i,j} h_{ij}(X_u)_{i\bar{i},\bar{k}}(X_u)_{j\bar{j},k} + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 \\ &+ \sum_i h_i\big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i \omega_{i\bar{i},k\bar{k}}\big) + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\lambda_1\lambda_2\lambda_3\lambda_4} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i}. \end{aligned} \tag{4.34}$$

*Proof.* The first and second derivatives should be straightforward. At the maximum point, suppose the eigenvalues are pairwise distinct satisfying $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Since $\Lambda$ is a diagonal matrix, then

$$0 = \sum_i h_i (X_u)_{i\bar{i},k} + \frac{\frac{\partial c_0}{\partial z_k}}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

This is also true when the eigenvalues are not pairwise distinct. For the second derivative, if the eigenvalues at the maximum point $q$ are pairwise distinct, then

$$\begin{aligned}0 = &\sum_{i,j} h_{ij}(X_u)_{i\bar{i},\bar{k}}(X_u)_{j\bar{j},k} + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 \\ &+ \sum_i h_i\big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i\omega_{i\bar{i},k\bar{k}}\big) + \frac{\frac{\partial^2 c_0}{\partial z_k\partial \bar{z}_k}}{\lambda_1\lambda_2\lambda_3} - \frac{2}{\lambda_1\lambda_2\lambda_3\lambda_i}\Re\Big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\Big).\end{aligned}$$

This is also true when the eigenvalues are not pairwise distinct. □

**Lemma 4.13.** *Let $d\colon M^4 \to \tilde{\mathscr{C}_4}$ and $X$ be a $C$-subsolution to $d$, then there exists uniform constants $N > 0$ and $\kappa > 0$, which are independent of $c \in \tilde{\mathcal{P}}^d$ with $c(z) \in S$ for any $z \in M$, such that if $\lambda_1 > N$, we have $\sum_i h_i u_{i\bar{i}} \geq -\kappa \sum_i h_i$.*

*Proof.* If $X$ is a $C$-subsolution to $d\colon M^4 \to \tilde{\mathscr{C}_4}$, then for all $z \in M$, we have $X^3 - 3d_2 X \wedge \omega^2 - d_1\omega^3 > 0$ and $X^2 - d_2\omega^2 > 0$. By choosing $\delta > 0$, $\kappa > 0$, and $\epsilon > 0$ sufficiently small, since $M$ is a compact manifold and $S$ is a compact subset, for any $c \in \tilde{\mathcal{P}}^d$ with $c \in S$, we get

$$\begin{aligned}&(1-\delta)(X-\kappa\omega)^3 - 3c_2(X-\kappa\omega)\wedge\omega^2 > (c_1+\epsilon)\omega^3;\\ &(1-\delta)\big(X-\kappa\omega\big)^2 > (c_2+\epsilon)\omega^2; \quad X - \kappa\omega > \epsilon\omega.\end{aligned}$$

By the definition of $\tilde{\mathcal{P}}^d$, we also have $d_2 - c_2 = x_2^2 - c_2 \geq 0$. This implies that $d_2 \geq c_2$ for any $c \in \tilde{\mathcal{P}}^d$. Note that $u_{i\bar{i}} = \lambda_i - X_{i\bar{i}}$, so we can write

$$\begin{aligned}\sum_i h_i(u_{i\bar{i}}+\kappa) = \sum_i h_i\big(\lambda_i - X_{i\bar{i}} + \kappa\big) &= -\sum_i \frac{c_2\sigma_2(\lambda_{;i}) + c_1\sigma_1(\lambda_{;i}) + c_0}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i}(\lambda_i - X_{i\bar{i}} + \kappa)\\ &= -\frac{\sum_i A_i}{\lambda_1\lambda_2\lambda_3\lambda_4} + \sum_i (X_{i\bar{i}} - \kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i}, \end{aligned} \tag{4.35}$$

where we denote $A_i = c_2\sigma_2(\lambda_{;i}) + c_1\sigma_1(\lambda_{;i}) + c_0$ for $i \in \{1,2,3,4\}$. Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, we have $A_4 \geq A_3 \geq A_2 \geq A_1 > 0$. There are two cases to be considered:

- If $0 < \lambda_4 \leq \frac{X_{4\bar{4}}-\kappa}{4}$, then

$$(X_{4\bar{4}} - \kappa)\frac{A_4}{\lambda_1\lambda_2\lambda_3\lambda_4^2} \geq 4\frac{A_4}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Hence, equation (4.35) gives $\sum_i h_i(u_{i\bar{i}} + \kappa) \geq -4\frac{A_4}{\lambda_1\lambda_2\lambda_3\lambda_4} + \sum_i (X_{i\bar{i}} - \kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \geq 0$.

- If $\lambda_4 \geq \frac{X_{4\bar{4}}-\kappa}{4}$, we can show that $\lambda_3$ is bounded from above when $\lambda_1$ is large. To be more precise, suppose $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq N$ for $N > 0$ sufficiently large, then

$$\begin{aligned}
\lambda_4 &= \frac{c_2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + c_1(\lambda_1+\lambda_2+\lambda_3) + c_0}{\lambda_1\lambda_2\lambda_3 - c_2(\lambda_1+\lambda_2+\lambda_3) - c_1} \\
&\leq \frac{d_2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + \sup_{c\in S}|c_1|(\lambda_1+\lambda_2+\lambda_3) + \sup_{c\in S}|c_0|}{\lambda_1\lambda_2\lambda_3 - d_2(\lambda_1+\lambda_2+\lambda_3) - d_1} \\
&\leq \frac{6\sup_M d_2 + 6\sup_{c\in S}|c_1| + 2\sup_{c\in S}|c_0|}{N}.
\end{aligned}$$

We get a contradiction if $N$ is sufficiently large. So, we have $8\frac{3\sup_M d_2 + 3\sup_{c\in S}|c_1| + \sup_{c\in S}|c_0|}{X_{4\bar{4}}-\kappa} \geq \lambda_3 \geq \lambda_4 \geq \frac{X_{4\bar{4}}-\kappa}{4} > 0$. If $\lambda_2$ is also sufficiently large, then for $\lambda_3\lambda_4$, we have

$$\begin{aligned}
\lambda_3\lambda_4 &= \lambda_3\frac{c_2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + c_1(\lambda_1+\lambda_2+\lambda_3) + c_0}{\lambda_1\lambda_2\lambda_3 - c_2(\lambda_1+\lambda_2+\lambda_3) - c_1} \\
&= c_2 + \frac{c_2\lambda_3^2(\lambda_1+\lambda_2) + (c_1 + c_2^2)(\lambda_1+\lambda_2+\lambda_3) + c_1c_2 + c_0}{\lambda_1\lambda_2\lambda_3 - c_2(\lambda_1+\lambda_2+\lambda_3) - c_1} \\
&\leq c_2 + \frac{d_2\lambda_3^2(\lambda_1+\lambda_2) + (\sup_{c\in S}|c_1| + d_2^2)(\lambda_1+\lambda_2+\lambda_3) + \sup_{c\in S}(d_2|c_1| + |c_0|)}{\lambda_1\lambda_2\lambda_3 - d_2(\lambda_1+\lambda_2+\lambda_3) - d_1}.
\end{aligned}$$

We see that when $\lambda_2$ is sufficiently large, for any $c \in \tilde{\mathcal{P}}^d$ satisfies the hypothesis, we have

$$\lambda_3\lambda_4 < c_2 + \epsilon. \tag{4.36}$$

In addition, by $(1-\delta)\big(X-\kappa\omega\big)^2 > (c_2+\epsilon)\omega^2$, we get the following,

$$(X_{3\bar{3}}-\kappa)\frac{1}{\lambda_3}+(X_{4\bar{4}}-\kappa)\frac{1}{\lambda_4} \geq 2\sqrt{\frac{(X_{3\bar{3}}-\kappa)(X_{4\bar{4}}-\kappa)}{\lambda_3\lambda_4}} \geq 2\sqrt{\frac{c_2+\epsilon}{(1-\delta)\lambda_3\lambda_4}} \tag{4.37}$$

By combining inequalities (4.35), (4.36), and (4.37), we may write

$$\begin{aligned}
\sum_i h_i(u_{i\bar{i}}+\kappa) &= \frac{-2c_2\sigma_2(\lambda)-3c_1\sigma_1(\lambda)-4c_0}{\lambda_1\lambda_2\lambda_3\lambda_4}+\sum_i(X_{i\bar{i}}-\kappa)\frac{A_i}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&= -2-\frac{c_1\sigma_1(\lambda)+2c_0}{\lambda_1\lambda_2\lambda_3\lambda_4}+\sum_i(X_{i\bar{i}}-\kappa)\frac{\lambda_1\lambda_2\lambda_3\lambda_4-c_2\lambda_i\sigma_1(\lambda_{;i})-c_1\lambda_i}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&\geq -2+(X_{3\bar{3}}-\kappa)\frac{1}{\lambda_3}+(X_{4\bar{4}}-\kappa)\frac{1}{\lambda_4}+\lambda_2^{-1}O_2(1) \\
&\geq -2+2\sqrt{\frac{(X_{3\bar{3}}-\kappa)(X_{4\bar{4}}-\kappa)}{\lambda_3\lambda_4}}+\lambda_2^{-1}O_2(1) \\
&\geq -2+2(1-\delta)^{-1/2}+\lambda_2^{-1}O_2(1) \geq \delta+\lambda_2^{-1}O_2(1).
\end{aligned}$$

If $\lambda_2$ is uniformly bounded from above, we get

$$\begin{aligned}
\sum_i h_i(u_{i\bar{i}}+\kappa) \geq (X_{2\bar{2}}-\kappa-\lambda_2)\frac{c_2(\lambda_3+\lambda_4)+c_1}{\lambda_2^2\lambda_3\lambda_4}+(X_{3\bar{3}}-\kappa-\lambda_3)\frac{c_2(\lambda_2+\lambda_4)+c_1}{\lambda_2\lambda_3^2\lambda_4} \\
+(X_{4\bar{4}}-\kappa-\lambda_4)\frac{c_2(\lambda_2+\lambda_3)+c_1}{\lambda_2\lambda_3\lambda_4^2}+\lambda_1^{-1}O_1(1).
\end{aligned}$$

We can treat the terms as inner product of the vector $(X_{2\bar{2}}-\kappa-\lambda_2, X_{3\bar{3}}-\kappa-\lambda_3, X_{4\bar{4}}-\kappa-\lambda_4)$ with the vector $(\frac{c_2(\lambda_3+\lambda_4)+c_1}{\lambda_2^2\lambda_3\lambda_4}, \frac{c_2(\lambda_2+\lambda_4)+c_1}{\lambda_2\lambda_3^2\lambda_4}, \frac{c_2(\lambda_2+\lambda_3)+c_1}{\lambda_2\lambda_3\lambda_4^2})$. If this inner product has a uniform positive lower bound, then we are done. Notice that when $\lambda_1$ is sufficiently large, since $\lambda_2, \lambda_3, \lambda_4$ are uniformly bounded from above and below, we have $\lambda_2\lambda_3\lambda_4-c_2(\lambda_2+\lambda_3+\lambda_4)-c_1$ is sufficiently close to 0. That is, when $\lambda_1$ is sufficiently large, we get

$$\delta\lambda_2\lambda_3\lambda_4 > \lambda_2\lambda_3\lambda_4-c_2(\lambda_2+\lambda_3+\lambda_4)-c_1 > 0.$$

Hence, for any triple $(\lambda_2, \lambda_3, \lambda_3)$ in this case, there exists a $\tilde{\delta} \in (0, \delta)$ such that $\lambda_2\lambda_3\lambda_4 -$

$c_2(\lambda_2+\lambda_3+\lambda_4)-c_1 = \tilde{\delta}\lambda_2\lambda_3\lambda_4$. In addition, we can check that for any $\tilde{\delta}\in[0,\delta]$, $(1-\tilde{\delta})\lambda_2\lambda_3\lambda_4 - c_2(\lambda_2+\lambda_3+\lambda_4)-c_1$ is strictly $\Upsilon$-stable hence convex by Theorem 2.5. Moreover, since $c\in\tilde{\mathcal{P}}^d$ with $c\in S$, we have $(1-\delta)(X-\kappa\omega)^3 - 3c_2(X-\kappa\omega)\wedge\omega^2 > (c_1+\epsilon)\omega^3$. By Hadamard's inequality, that is, the determinant of a positive-semidefinite Hermitian matrix is less than or equal to the product of its diagonal entries, we obtain

$$(1-\delta)(X_{2\bar{2}}-\kappa)(X_{3\bar{3}}-\kappa)(X_{4\bar{4}}-\kappa) - c_2\sum_{j=2}^{4}(X_{j\bar{j}}-\kappa) - c_1 > \epsilon.$$

Thus, for any $\tilde{\delta}\in[0,\delta]$, we get

$$(1-\tilde{\delta})(X_{2\bar{2}}-\kappa)(X_{3\bar{3}}-\kappa)(X_{4\bar{4}}-\kappa) - c_2\sum_{j=2}^{4}(X_{j\bar{j}}-\kappa) - c_1 > \epsilon.$$

This implies that the point $(X_{2\bar{2}}-\kappa, X_{3\bar{3}}-\kappa, X_{4\bar{4}}-\kappa)$ lies in the set $\{(1-\tilde{\delta})\lambda_2\lambda_3\lambda_4 - c_2(\lambda_2+\lambda_3+\lambda_4)-c_1>0\}$ for any $\tilde{\delta}\in[0,\delta]$. Moreover, the inner normal vector of $(1-\tilde{\delta})\lambda_2\lambda_3\lambda_4 - c_2(\lambda_2+\lambda_3+\lambda_4)-c_1=0$ at the point $(\lambda_2,\lambda_3,\lambda_4)$ will be $(\frac{c_2(\lambda_3+\lambda_4)+c_1}{\lambda_2^2\lambda_3\lambda_4}, \frac{c_2(\lambda_2+\lambda_4)+c_1}{\lambda_2\lambda_3^2\lambda_4}, \frac{c_2(\lambda_2+\lambda_3)+c_1}{\lambda_2\lambda_3\lambda_4^2})$. By the supporting hyperplane theorem, the inner product will be positive. Since $\tilde{\delta}\in[0,\delta]$ and $\lambda_2,\lambda_3,\lambda_4$ are uniformly bounded from above and below, so the inner product has a uniform positive lower bound. In conclusion, we can find a uniform $N>0$ such that if $\lambda_1>N$, we always have

$$\sum_i h_i(u_{i\bar{i}}+\kappa)\geq 0 \Longrightarrow \sum_i h_i u_{i\bar{i}} \geq -\sum_i h_i\kappa$$

for any $c\in\tilde{\mathcal{P}}^d$ with range in $S$. This finishes the proof. □

**Lemma 4.14.** *With the same settings as in Lemma 4.13, there exists a uniform $N>0$ and $\epsilon>0$ such that if $\lambda_1>N$, then $-h_2-h_3-h_4>\epsilon>0$.*

*Proof.* First, we have

$$-h_2 - h_3 - h_4 = \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} - \frac{c_2(3\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) + 3c_1}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Second, if $\lambda_3$ is sufficiently large, then $\lambda_4$ approaches 0 and

$$-h_2 - h_3 - h_4 > -h_4 = \frac{1}{\lambda_4} - \frac{c_2(\lambda_1 + \lambda_2 + \lambda_3) + c_1}{\lambda_1\lambda_2\lambda_3\lambda_4} \geq \frac{1}{\lambda_4} + \lambda_3^{-1}O_3(1).$$

So $-h_2 - h_3 - h_4$ is uniformly positive when $\lambda_3$ is sufficiently large. Then, if $\lambda_3$ is uniformly bounded from above and $\lambda_2$ is sufficiently large, then we get $2c_2 > \lambda_3\lambda_4 > c_2 > 0$ and

$$-h_2 - h_3 - h_4 \geq \frac{1}{\lambda_3} + \frac{1}{\lambda_4} + \lambda_2^{-1}O_2(1) \geq \frac{2}{\sqrt{\lambda_3\lambda_4}} + \lambda_2^{-1}O_2(1) > \frac{2}{\sqrt{2c_2}} + \lambda_2^{-1}O_2(1).$$

Last, if $\lambda_2$ is uniformly bounded from above and $\lambda_1$ is sufficiently large, then we can show that $\lambda_3\lambda_4 - c_2$ has a uniform positive lower bound. If not, for any $n \in \mathbb{N}$, there exists $\lambda_{2,n}$, $\lambda_{3,n}$ and $\lambda_{4,n}$ with $\lambda_{2,n} \geq \lambda_{3,n} \geq \lambda_{4,n}$ such that $\frac{1}{n} > \lambda_{3,n}\lambda_{4,n} - c_2 > 0$. This implies that

$$\begin{aligned}
&\lambda_{2,n}\lambda_{3,n}\lambda_{4,n} - c_2(\lambda_{2,n} + \lambda_{3,n} + \lambda_{4,n}) - c_1 \\
&\quad < \frac{\lambda_{2,n}}{n} - c_2(\lambda_{3,n} + \lambda_{4,n}) - c_1 < \frac{\lambda_{2,n}}{n} - 2c_2^{3/2} - c_1 < \frac{\lambda_{2,n}}{n} - \inf_{c\in S}(c_1 + 2c_2^{3/2}),
\end{aligned}$$

which gives a contradiction when $n$ is sufficiently large. Hence $\lambda_3\lambda_4 - c_2$ has a uniform positive lower bound, which implies that $\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} - \frac{3c_2}{\lambda_2\lambda_3\lambda_4}$ has a positive uniform lower bound. In addition, we obtain

$$\begin{aligned}
-h_2 - h_3 - h_4 &= \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} - \frac{c_2(3\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) + 3c_1}{\lambda_1\lambda_2\lambda_3\lambda_4} \\
&= \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} - \frac{3c_2}{\lambda_2\lambda_3\lambda_4} + \lambda_1^{-1}O_1(1).
\end{aligned}$$

Thus, we are done if $\lambda_1$ is sufficiently large. This finishes the proof. $\square$

Now we let $C$ be a constant depending only on the stated data, but which may change from line to line. We can finish the proof of the following $C^2$ estimate.

**Theorem 4.2.** *Suppose $X$ is a $C$-subsolution to $d\colon M^4 \to \tilde{\mathscr{C}_4}$ with range in $S$ and $d_1, d_2$ constant. For any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1, c_2$ constant, if $u\colon M \to \mathbb{R}$ is a smooth function solving the equation $c\colon M^4 \to \tilde{\mathscr{C}_4}$, then there exists a constant $C$ such that*

$$|\partial\bar{\partial}u| \leq C\big(1 + \sup_M |\nabla u|^2\big).$$

*Here, $C = C(M, X, S, d, \omega, \operatorname{osc}_M u, \|c_0\|_{C^2})$ is a constant and $\nabla$ is the Levi-Civita connection with respect to $\omega$.*

*Proof.* We use the maximum principle to prove this, for any $c \in \tilde{\mathcal{P}}^d$ with $c \in S$, we can define the elliptic operator $\mathcal{L}_c$ in equation (4.32). First, by applying the operator $\mathcal{L}_c$ to $G(\tilde{\Lambda})$, at the maximum point $q$, we obtain

$$\begin{aligned}
\mathcal{L}_c\big(G(\tilde{\Lambda})\big) &= -\sum_{i,j,k} h_k g_{ij} \frac{\partial \tilde{\Lambda}_i^i}{\partial z_k} \frac{\partial \tilde{\Lambda}_j^j}{\partial \bar{z}_k} - \sum_k h_k \sum_{i \neq j} \frac{g_i - g_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} \frac{\partial \tilde{\Lambda}_j^i}{\partial z_k} \frac{\partial \tilde{\Lambda}_i^j}{\partial \bar{z}_k} - \sum_{i,k} h_k g_i \frac{\partial^2 \tilde{\Lambda}_i^i}{\partial z_k \partial \bar{z}_k} \\
&= \sum_k h_k \frac{\big|(X_u)_{1\bar{1},k}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_k h_k \frac{\lambda_1}{1+\tilde{\lambda}_1} \omega_{1\bar{1},k\bar{k}} - \sum_k h_k \frac{(X_u)_{1\bar{1},k\bar{k}}}{1+\tilde{\lambda}_1} \\
&\quad - \sum_k h_k \sum_{j \neq 1} \frac{\big|(X_u)_{j\bar{1},k}\big|^2 + \big|(X_u)_{1\bar{j},k}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} \\
&\geq \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} - \sum_i h_i \frac{(X_u)_{1\bar{1},i\bar{i}}}{1+\tilde{\lambda}_1} - \sum_{j \neq 1} \frac{h_j \big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} + C \sum_i h_i. \qquad (4.38)
\end{aligned}$$

Second, by equation (4.33) and equation (4.34), we have

$$\begin{aligned}
0 &= \sum_{i,j} h_{ij} (X_u)_{i\bar{i},\bar{k}} (X_u)_{j\bar{j},k} + \sum_{i \neq j} \Big( \frac{1}{\lambda_i \lambda_j} - \frac{c_2}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \Big) |(X_u)_{j\bar{i},k}|^2 \\
&\quad + \sum_i h_i \big( (X_u)_{i\bar{i},k\bar{k}} - \lambda_i \omega_{i\bar{i},k\bar{k}} \big) + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_k} (X_u)_{i\bar{i},k}\big)}{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_i}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l (X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2} h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},k}}{\sum_l h_l^2} h_j\Big) \\
&\quad + \frac{2\sum_{i,j,l} h_j h_{ij} h_l \Re\big((X_u)_{l\bar{l},k}(X_u)_{i\bar{i},\bar{k}}\big)}{\sum_l h_l^2} - \sum_{i,j} h_i h_j h_{ij} \frac{|\sum_l h_l (X_u)_{l\bar{l},k}|^2}{(\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i \omega_{i\bar{i},k\bar{k}}\Big) \\
&\quad + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\lambda_1\lambda_2\lambda_3\lambda_4} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&= \sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l (X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2} h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},k}}{\sum_l h_l^2} h_j\Big) \\
&\quad - \frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2} - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_k}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 (\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 + \sum_i h_i\Big((X_u)_{i\bar{i},k\bar{k}} - \lambda_i \omega_{i\bar{i},k\bar{k}}\Big) \\
&\quad + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\lambda_1\lambda_2\lambda_3\lambda_4} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&\geq -\frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_k}(X_u)_{i\bar{i},k}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_k}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 (\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 + \sum_i h_i (X_u)_{i\bar{i},k\bar{k}} + C\sum_i h_i\lambda_i \\
&\quad + \frac{\frac{\partial^2 c_0}{\partial z_k \partial \bar{z}_k}}{\lambda_1\lambda_2\lambda_3\lambda_4}, \qquad (4.39)
\end{aligned}
$$

where the inequality on the last line is due to the convexity of $h = 1$ and $|\omega_{i\bar{i},k\bar{k}}| \leq C$. Since the solution set $\{\lambda_1\lambda_2\lambda_3\lambda_4 = c_2\sigma_2(\lambda) + c_1\sigma_1(\lambda) + c_0\}$ is convex and

$$
(X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},k}}{\sum_l h_l^2} h_j = (X_u)_{j\bar{j},k} - \frac{\frac{\partial c_0}{\partial z_k} h_j}{\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2}
$$

is a tangent vector on the solution set, we obtain that

$$
\sum_{i,j} h_{ij}\Big((X_u)_{i\bar{i},\bar{k}} - \frac{\sum_l h_l (X_u)_{l\bar{l},\bar{k}}}{\sum_l h_l^2} h_i\Big)\Big((X_u)_{j\bar{j},k} - \frac{\sum_l h_l (X_u)_{l\bar{l},k}}{\sum_l h_l^2} h_j\Big) \geq 0.
$$

Hence, by setting $k=1$, inequality (4.35) gives

$$
\begin{aligned}
&-\sum_i h_i (X_u)_{1\bar{1},i\bar{i}} \\
&= -\sum_i h_i (X_u)_{i\bar{i},1\bar{1}} + \sum_i h_i \big((X_u)_{i\bar{i},1\bar{1}} - (X_u)_{1\bar{1},i\bar{i}}\big) \\
&= -\sum_i h_i (X_u)_{i\bar{i},1\bar{1}} + \sum_i h_i \big((X)_{i\bar{i},1\bar{1}} - (X)_{1\bar{1},i\bar{i}}\big) \\
&\geq -\frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_1}|^2}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 (\sum_l h_l^2)^2} \\
&\quad + \sum_{i\neq j}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)|(X_u)_{j\bar{i},k}|^2 + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{\lambda_1\lambda_2\lambda_3\lambda_4} + C\sum_i h_i (1+\lambda_i). \qquad (4.40)
\end{aligned}
$$

Combining Lemma 4.13, inequalities (4.38), (4.39), and (4.40), at the maximum point $q$, if $\lambda_1$ is sufficiently large, then we have

$$
\begin{aligned}
\mathcal{L}_c(\tilde{U}) &\geq A\sum_i h_i u_{i\bar{i}} + \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + C\sum_i h_i - \sum_i h_i \frac{(X_u)_{1\bar{1},i\bar{i}}}{1+\tilde{\lambda}_1} - \sum_{j\neq 1} \frac{h_j \big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} \\
&\geq A\sum_i h_i u_{i\bar{i}} + \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq i}\Big(\frac{1}{\lambda_i\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{i},1}|^2}{1+\tilde{\lambda}_1} \\
&\quad - \sum_{j\neq 1} \frac{h_j \big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} - \frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&\quad - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 (\sum_l h_l^2)^2} + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4} + C\sum_i h_i \\
&\geq \sum_i h_i \frac{\big|(X_u)_{1\bar{1},i}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1}\Big(\frac{1}{\lambda_1\lambda_j} - \frac{c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{1},1}|^2}{1+\tilde{\lambda}_1} \\
&\quad - \sum_{j\neq 1} \frac{h_j \big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_j)} - \frac{2\sum_{i,j} h_j h_{ij} \Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4 \sum_l h_l^2} - \sum_i \frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{i\bar{i},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i} \\
&\quad - \frac{\sum_{i,j} h_i h_j h_{ij} |\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 (\sum_l h_l^2)^2} + \frac{\frac{\partial^2 c_0}{\partial z_1 \partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4} + \big(C - A\kappa\big)\sum_i h_i \\
&\geq h_1 \frac{\big|(X_u)_{1\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1} h_j \frac{\big|(X_u)_{1\bar{1},j}\big|^2}{(1+\tilde{\lambda}_1)^2} + \sum_{j\neq 1}\Big(\frac{\sigma_2(\lambda_{;1,j}) - c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{1},1}|^2}{1+\tilde{\lambda}_1}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j\neq 1}\frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)}-\frac{2\sum_l h_l h_{1l}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2\lambda_3\lambda_4}\\
&-\frac{2\sum_{j\neq 1}\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\sum_{j\neq 1}\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}\\
&-\frac{\sum_{i,j}h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2(\sum_l h_l^2)^2}+\frac{\frac{\partial^2 c_0}{\partial z_1\partial\bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4}+\big(C-A\kappa\big)\sum_i h_i. \qquad (4.41)
\end{aligned}
$$

We can also simplify some terms in inequality (4.41), for $j\neq 1$, we have:

$$
\begin{aligned}
&h_j\frac{\big|(X_u)_{1\bar{1},j}\big|^2}{(1+\tilde{\lambda}_1)^2}+\Big(\frac{\sigma_2(\lambda_{;1,j})-c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{1},1}|^2}{1+\tilde{\lambda}_1}\\
&\quad=h_j\frac{\big|(X_u)_{j\bar{1},1}-T_j\big|^2}{(1+\tilde{\lambda}_1)^2}+\Big(\frac{\sigma_2(\lambda_{;1,j})-c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{1},1}|^2}{1+\tilde{\lambda}_1}\\
&\quad\geq 2h_j\frac{\big|(X_u)_{j\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2}+2h_j\frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2}+\Big(\frac{\sigma_2(\lambda_{;1,j})-c_2}{\lambda_1\lambda_2\lambda_3\lambda_4}\Big)\frac{|(X_u)_{j\bar{1},1}|^2}{1+\tilde{\lambda}_1}\\
&\quad=\frac{2c_2\sigma_1(\lambda_{;1,j})+2c_1+\sigma_2(\lambda_{;1,j})-c_2}{(1+\tilde{\lambda}_1)^2\lambda_1\lambda_2\lambda_3\lambda_4}\big|(X_u)_{j\bar{1},1}\big|^2+2\Big(\frac{c_2\sigma_1(\lambda_{;j})+c_1}{\lambda_1\lambda_2\lambda_3\lambda_4}-\frac{1}{\lambda_j}\Big)\frac{\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2}\\
&\quad\geq-\frac{2\big|T_j\big|^2}{(1+\tilde{\lambda}_1)^2\lambda_j}\geq-\frac{C}{\lambda_1}\geq -C. \qquad (4.42)
\end{aligned}
$$

Here, we denote $T_j\coloneqq(X_u)_{j\bar{1},1}-(X_u)_{1\bar{1},j}=X_{j\bar{1},1}-X_{1\bar{1},j}$. In addition, for $j\neq 1$, we obtain

$$
\begin{aligned}
&-\frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)}-\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{j\bar{j},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}\\
&\quad=-\frac{h_j\big|(X_u)_{1\bar{j},j}\big|^2}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)}-\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{j},j}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{j},j}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}\\
&\qquad-\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X_u)_{j\bar{j},1}-(X_u)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X_u)_{j\bar{j},1}-(X_u)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}\\
&\quad=-\frac{h_j}{(1+\tilde{\lambda}_1)(\tilde{\lambda}_1-\tilde{\lambda}_j)}\left|(X_u)_{1\bar{j},j}+\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}(\tilde{\lambda}_1-\tilde{\lambda}_j)}{\lambda_1\lambda_2\lambda_3\lambda_4 h_j\sum_l h_l^2}+\frac{\frac{\partial c_0}{\partial z_1}(\tilde{\lambda}_1-\tilde{\lambda}_j)}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j h_j}\right|^2\\
&\qquad+\frac{(\tilde{\lambda}_1-\tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2}+\frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2-\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}\\
&\qquad-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{(\tilde{\lambda}_1-\tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2}+\frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2-\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}\\
&\quad-\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j}. \qquad (4.43)
\end{aligned}
$$

We estimate some terms in inequality (4.41) and inequality (4.43). If $\lambda_4$ is uniformly bounded from below by a positive constant, then $\lambda_3$ is uniformly bounded from above. If $\lambda_2$ is also uniformly bounded from above, the estimates should be straightforward. If $\lambda_2$ approaches infinity, then we get

$$
\begin{aligned}
-\lambda_2^2\lambda_3\lambda_4 h_2 &= c_2(\lambda_3+\lambda_4)+c_1+\lambda_1^{-1}O_1(1)=\Theta_1(1);\\
-h_3 &= \frac{1}{\lambda_3}-\lambda_2^{-1}O_1(1)=\lambda_3^{-1}\Theta_1(1);\\
-h_4 &= \frac{1}{\lambda_4}-\lambda_2^{-1}O_1(1)=\lambda_4^{-1}\Theta_1(1).
\end{aligned}
$$

In this case, by Proposition 4.4, for $j\neq 1$, we have

$$
\begin{aligned}
&\frac{(\tilde{\lambda}_1-\tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j^2}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2}+\frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2 \leq \frac{C}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j^2\lambda_j^2}\leq C;\\
&\left|\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4 h_j\sum_l h_l^2}\right| \leq \frac{C}{\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_j h_j}\leq C;\\
&\left|\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1}-(X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j h_j}\right| \leq \frac{C}{\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_j h_j}\leq C.
\end{aligned}
$$

Now, if $\lambda_4$ approaches 0, then we need to consider the limiting behavior of $h_2$, $h_3$ and $h_4$. For $h_4$, we have

$$
-h_4=\frac{1}{\lambda_4}-\frac{c_2(\lambda_1+\lambda_2+\lambda_3)+c_1}{\lambda_1\lambda_2\lambda_3\lambda_4}=\frac{1}{\lambda_4}+\lambda_2^{-1}O_1(1).
$$

So $h_4 = \lambda_4^{-1}\Theta_1(1)$. For $h_3$, the limiting behavior is slightly harder, we have

$$\frac{1}{\lambda_3} > -h_3 = \frac{c_2(\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4) + c_1(\lambda_1 + \lambda_2 + \lambda_4) + c_0}{\lambda_1\lambda_2\lambda_3^2\lambda_4} \geq \frac{c_2\lambda_1\lambda_2}{2\lambda_1\lambda_2\lambda_3^2\lambda_4} > \frac{1}{8\lambda_3}.$$

Hence, $h_3 = \lambda_3^{-1}\Theta_1(1)$. For $h_2$, the limiting behavior is slightly harder, we have

$$-\lambda_1\lambda_2^2\lambda_3\lambda_4 h_2 = c_2(\lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4) + c_1(\lambda_1 + \lambda_3 + \lambda_4) + c_0 = \lambda_1\lambda_3\Theta_1(1).$$

Hence, $-\lambda_1\lambda_2^2\lambda_3\lambda_4 h_2 = \lambda_1\lambda_3\Theta_1(1)$. Similar to before, by Proposition 4.1, for $j \neq 1$, we have

$$\begin{aligned}
&\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j^2}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2 \leq \frac{C}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2\lambda_j^2 h_j} \leq C;\\
&\left|\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4 h_j\sum_l h_l^2}\right| \leq \frac{C}{\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_j h_j} \leq C;\\
&\left|\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j h_j}\right| \leq \frac{C}{\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_j h_j} \leq C.
\end{aligned}$$

In conclusion, for $j \neq 1$, no matter which case, we always obtain

$$\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_j)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2 h_j^2}\left|\frac{\sum_l h_l h_{lj}\frac{\partial c_0}{\partial z_1}}{\sum_l h_l^2} + \frac{\frac{\partial c_0}{\partial z_1}}{\lambda_j}\right|^2 \leq C; \tag{4.44}$$

$$\left|\frac{2\sum_l h_l h_{lj}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4 h_j\sum_l h_l^2}\right| \leq C; \tag{4.45}$$

$$\left|\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}\big((X)_{j\bar{j},1} - (X)_{1\bar{j},j}\big)\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\lambda_j h_j}\right| \leq C; \tag{4.46}$$

$$\left|\frac{\sum_{i,j} h_i h_j h_{ij}|\frac{\partial c_0}{\partial z_1}|^2}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2(\sum_l h_l^2)^2}\right| \leq C; \tag{4.47}$$

$$\left|\frac{\frac{\partial^2 c_0}{\partial z_1\partial \bar{z}_1}}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4}\right| \leq C. \tag{4.48}$$

Last, we have the following inequality

$$
\begin{aligned}
&h_1\frac{\big|(X_u)_{1\bar{1},1}\big|^2}{(1+\tilde{\lambda}_1)^2}-\frac{2\sum_l h_l h_{1l}\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1\lambda_2\lambda_3\lambda_4\sum_l h_l^2}-\sum_{j\neq 1}\frac{2\Re\big(\frac{\partial c_0}{\partial \bar{z}_1}(X_u)_{1\bar{1},1}\big)}{(1+\tilde{\lambda}_1)\lambda_1^2\lambda_2\lambda_3\lambda_4}\\
&\quad= h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1}-\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2\\
&\qquad- h_1\Big|\frac{\partial c_0}{\partial z_1}\Big|^2\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)^2\\
&\quad\geq h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1}-\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2.
\end{aligned}
\tag{4.49}
$$

Thus, by Lemma 4.13 and Lemma 4.14 and by inequalities (4.41), (4.42), (4.43), (4.44), (4.45), (4.46), (4.47), (4.48), and (4.49), at the maximum point $q$ we obtain

$$
\begin{aligned}
0\geq \mathcal{L}_c(\tilde{U})&\geq \big(C-A\kappa\big)\sum_i h_i + C\sum_{j\neq 1}h_j - C\\
&\qquad+h_1\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1}-\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2\\
&\geq\frac{A\kappa\epsilon}{2}-\frac{1}{\lambda_1}\Big|\frac{(X_u)_{1\bar{1},1}}{1+\tilde{\lambda}_1}-\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|^2.
\end{aligned}
$$

Here, we let $A$ sufficiently large to get the last inequality. So, we get

$$
\begin{aligned}
\sqrt{\frac{A\kappa\epsilon}{2}}\sqrt{\lambda_1}&\leq\Big|Au_1-\Big(\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big)\frac{\partial c_0}{\partial z_1}\Big|\\
&\leq A|u_1|+\Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big|\Big|\frac{\partial c_0}{\partial z_1}\Big|.
\end{aligned}
\tag{4.50}
$$

Similar to before, we estimate the quantity $\Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}+\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big|$, we have

$$
\begin{aligned}
\Big|\frac{1}{\lambda_1^2\lambda_2\lambda_3\lambda_4 h_1}\Big|&=\frac{1}{c_2(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)+c_1(\lambda_2+\lambda_3+\lambda_4)+c_0}\\
&<\frac{1}{3c_2x_1^2+3c_1x_1+c_0}<\frac{1}{\inf_{c\in S}(3c_2x_1^2+3c_1x_1+c_0)}\leq C;\\
\Big|\frac{\sum_l h_l h_{1l}}{\lambda_1\lambda_2\lambda_3\lambda_4 h_1\sum_l h_l^2}\Big|&\leq\frac{C\lambda_1^{-1}\lambda_4^{-2}}{(c_2(\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)+c_1(\lambda_2+\lambda_3+\lambda_4)+c_0)\lambda_1^{-1}\lambda_4^{-2}}\leq C.
\end{aligned}
$$

Here, $x_1$ is the solution of the diagonal restriction of $\lambda_2\lambda_3\lambda_4 - c_2(\lambda_2+\lambda_3+\lambda_4) - c_1 = 0$. In conclusion, we get $\sqrt{\frac{A\kappa\epsilon}{2}}\sqrt{\lambda_1} \le A|u_1| + C$. This implies that

$$\lambda_1 \le \frac{2}{A\kappa\epsilon}(A|u_1|+C)^2 \le \frac{4}{A\kappa\epsilon}(A^2|u_1|^2+C^2) \le C(1+\sup_M |\nabla u|^2).$$

By plugging back to the original test function $U = -Au + G(\Lambda)$, we will obtain a $C^2$ estimate for any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1$, $c_2$ constant. This finishes the proof. $\square$

### 4.2.2 The $C^1$ Estimate

Here, same as Section 4.1.2, we use a blow-up argument proved by Collins–Jacob–Yau [18] to obtain the $C^1$ estimate. Since everything follows verbatim, so we do not state it here.

### 4.2.3 Higher Order Estimates

Here, the proofs are similar to the proofs in Section 4.1.3, so we just state the results here without writing down the proofs. The equation is elliptic and the solution set is convex, we can exploit the convexity of the solution set to obtain $C^{2,\alpha}$ estimates by a blow-up argument.

By shrinking the coordinate charts if necessary, we may assume that the manifold $M$ can be covered by finitely many coordinate charts $\bar{U}_a \subset V_a$ such that $X_u = \sqrt{-1}\partial\bar{\partial}u_a$ on $V_a$ for a smooth function $u_\alpha$ satisfying $\|u_a\|_{C^2(\bar{U}_a)} \le K$, where we use the standard Euclidean metric on $\mathbb{C}^4$ and $K$ is a uniform constant independent of $a$. For convenience, we focus on a fixed coordinate chart $V_a$, and we drop the subscript $a$. The function $u$ on $V$ satisfies

$$H_c(z, \partial\bar{\partial}u) = H_c\big(z, \Lambda(z)\big) = 1, \text{ for } z \in V,$$

where $\Lambda_i^j(z) = \omega^{j\bar{k}}(z)u_{i\bar{k}}(z)$ with eigenvalues in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3\lambda_4 - c_2\sigma_2(\lambda) - c_1\sigma_1(\lambda) -$

$c_0(z) = 0$. Moreover, by fixing $\tilde{z} \in U$, we define the following operator which does not depend on $z \in V$,

$$\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u) \coloneqq H_c(\tilde{z}, \omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}).$$

We have the following.

**Lemma 4.15.** *Let $U \subset \mathbb{C}^4$ be a connected open set and fix $\tilde{z} \in U$. Suppose $u\colon U \subset \mathbb{C}^4 \to \mathbb{R}$ is a $C^3$ function such that $\|\partial\bar{\partial}u\|_{L^\infty(U)} < \infty$ and the eigenvalues $\lambda\big(\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})\big)$ of $\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})$ in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3\lambda_4 - c_2\sigma_2(\lambda) - c_1\sigma_1(\lambda) - c_0(\tilde{z}) = 0$. If for all $z \in U$, $\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u)(z) = 1$, then there exists a constant $\alpha \in (0,1)$ such that for any $R > 0$ with $\overline{B_{2R}} \subset U$, the function $u$ satisfies*

$$\|\partial\bar{\partial}u\|_{C^\alpha(B_R)} \leq C \cdot R^{-\alpha}.$$

*Here, $C = C\big(c, S, \|\partial\bar{\partial}u\|_{L^\infty(U)}\big)$.*

Then, with the above Lemma 4.15, we can prove a Liouville-type result.

**Proposition 4.5.** *Let $\tilde{z} \in \mathbb{C}^4$. Suppose $u\colon \mathbb{C}^4 \to \mathbb{R}$ is a $C^3$ function such that $\|\partial\bar{\partial}u\|_{L^\infty(\mathbb{C}^3)} < \infty$ and the eigenvalues $\lambda\big(\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})\big)$ of $\omega^{j\bar{k}}(\tilde{z})u_{i\bar{k}}(\tilde{z})$ in the $\Upsilon_1$-cone of $\lambda_1\lambda_2\lambda_3\lambda_4 - c_2\sigma_2(\lambda) - c_1\sigma_1(\lambda) - c_0(\tilde{z}) = 0$. If for all $z \in \mathbb{C}^4$, $\tilde{H}_{c,\tilde{z}}(\partial\bar{\partial}u)(z) = 1$, then $u$ is a quadratic polynomial.*

**Lemma 4.16.** *For $r > 0$, suppose $u\colon B_{2r} \subset \mathbb{C}^4 \to \mathbb{R}$ is a smooth function satisfying $H_c(z, \partial\bar{\partial}u) = 1$. Then, for every $\alpha \in (0,1)$, we have the estimate*

$$\|\partial\bar{\partial}u\|_{C^\alpha(B_{r/2})} \leq C(\alpha, c, S, \|\partial\bar{\partial}u\|_{L^\infty(B_{2r})}).$$

By arguing locally, with Lemma 4.16 we have the following.

**Corollary 4.2.** *Suppose $S$ is a compact subset of the stratification $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 >$*

*$-2c_2^{3/2}$, and $c_0 > -3c_2x_1^2 - 3c_1x_1\}$ of $\tilde{\mathscr{C}_4}$ and $X$ is a $C$-subsolution to $d\colon M^4 \to \tilde{\mathscr{C}_4}$ with range in $S$ and $d_1$, $d_2$ constant, where $x_1$ is the largest real root of $x^3 - 3c_2x - c_1$. For any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ and $c_1$, $c_2$ constant, if $u\colon M \to \mathbb{R}$ is a smooth function solving the equation $c\colon M^4 \to \tilde{\mathscr{C}_4}$, then for every $\alpha \in (0,1)$, we have*

$$\|\partial\bar{\partial}u\|_{C^\alpha(M)} \leq C(M, X, S, d, \omega, \alpha, \|c_0\|_{C^2}, \|\partial\bar{\partial}u\|_{L^\infty(M)}).$$

# Chapter 5

# Existence Results

In this chapter, we study the solvability of the constant maps $d\colon M^3 \to \tilde{\mathscr{C}_3}$ and $d\colon M^4 \to \tilde{\mathscr{C}_4}$ provided that a $C$-subsolution is given. We will study them individually in this dissertation and hope we will find a unified approach to handle all dimensions in the future. The ideas in this chapter come from the author's previous works [51, 52].

We prove the following result in this chapter.

**Theorem 5.1.** *Suppose there exists a $C$-subsolution to constant map $d\colon M^3 \to \{(0, c_0)\colon c_0 > 0\} \subset \tilde{\mathscr{C}_3}$ or constant map $d\colon M^4 \to \{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},$ and $c_0 > -3c_2x_1^2 - 3c_1x_1\} \subset \tilde{\mathscr{C}_4}$, where $x_1$ is the largest real root of $x^3 - 3c_2x - c_1$. Then the general inverse $\sigma_k$ equation $d\colon M^3 \to \tilde{\mathscr{C}_3}$ or $d\colon M^4 \to \tilde{\mathscr{C}_4}$ is solvable.*

Theorem 5.1 confirms the following analytic conjecture by Collins–Jacob–Yau in [18] when the complex dimension equals three or four. In [18], with a slightly stronger $C$-subsolution assumption, Collins–Jacob–Yau were able to obtain the solvability. Here, in this chapter, we show that we can obtain the solvability from a usual $C$-subsolution.

*Conjecture* 5.1 (deformed Hermitian–Yang–Mills equation, Collins–Jacob–Yau [18]). If there exists a $C$-subsolution to the following equation,

$$\Im\big(\omega + \sqrt{-1}\chi\big)^n = \tan\big(\theta\big) \cdot \Re\big(\omega + \sqrt{-1}\chi\big)^n$$

with $\theta \in \big((n-2)\pi/2, n\pi/2\big)$, then the dHYM equation is solvable. Here, $\Im$ and $\Re$ are the imaginary and real parts, respectively, and $\theta$ is a topological constant determined by the cohomology classes $[\omega]$ and $[\chi]$.

## 5.1 When Complex Dimension Equals Three

In this section, we always assume that there exists a $C$-subsolution. By changing representative, we say $X$ is this $C$-subsolution. We are interested in the solvability of the following equation $d\colon M^3 \to \{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\} \subset \tilde{\mathscr{C}_3}$ with $d_1, d_0$ constant. That is,

$$X^3 - 3d_1\omega^2 \wedge X - d_0\omega^3 = 0. \tag{5.1}$$

By Section 4.1, we observe that if a $C$-subsolution to $d\colon M \to \tilde{\mathscr{C}_3}$ exists, then for any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ a compact subset of $\{(c_1, c_0)\colon c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\}$, we have a priori estimates. We try to find a continuity path in the space $\tilde{\mathcal{P}}^d$ connecting the original equation $d\colon M^3 \to \tilde{\mathscr{C}_3}$ to another solvable equation. When $d_0$ and $d_1$ are both non-negative with $d_1 + d_0 > 0$, this is solvable due to Collins–Székelyhidi [20]. Hence, we only focus on the case that $d_1 > 0$ and $d_0 < 0$. We will prove that this is solvable provided that a $C$-subsolution exists. In conclusion, for any constant map $d\colon M^3 \to \tilde{\mathscr{C}_3}$, it is solvable provided that a $C$-subsolution exists. We consider the following continuity path,

$$X^3 - 3d_1(t)\omega^2 \wedge X - d_0(t)\omega^3 = 0, \tag{5.2}$$

where $t \in [0, 1]$ and $d_0(t)$ and $d_1(t)$ are smooth functions in $t$ which satisfy all the following

$$\begin{aligned}
\text{Topological constraint: } & \Omega_0 - 3d_1(t)\Omega_2 - d_0(t)\Omega_3 = 0. \\
\text{Boundary constraints: } & d_1(1) = d_1; \quad d_0(1) = d_0; \quad d_1(0) > 0; \quad d_0(0) = 0. \\
\text{Positivstellensatz constraint: } & d(t) \in \tilde{\mathscr{C}_3}. \\
\Upsilon\text{-dominance constraint: } & d(t) \in \tilde{\mathcal{P}}^d.
\end{aligned}$$

Here, we denote $\Omega_i := \int_M \omega^i \wedge X^{3-i}$.

**Lemma 5.1.** *If* $(\Omega_2, \Omega_3) \in \Omega^{3,d}$*, then the following pair will satisfy all the 3-dimensional four constraints:*

$$d_1(t) := \frac{\Omega_0 - td_0\Omega_3}{3\Omega_2}; \quad d_0(t) := td_0.$$

*Here,* $\Omega_i = \int_M \omega^i \wedge X^{3-i}$ *and*

$$\Omega^{3,d} := \Big\{ \Omega_3 < \inf_{t\in[0,1)} -3\Omega_2 \frac{d_1 - (-d_0(t)/2)^{2/3}}{d_0 - d_0(t)} \Big\}.$$

*Proof.* First, the topological constraint is automatically satisfied. Second, we can check that they satisfy the boundary constraints

$$d_1(1) = \frac{\Omega_0 - d_0\Omega_3}{3\Omega_2} = d_1; \quad d_1(0) = \frac{\Omega_0}{3\Omega_2} > 0; \quad d_0(1) = d_0; \quad d_0(0) = 0.$$

Third, for the positivstellensatz constraint, when $t = 0$ or $t = 1$, the positivstellensatz constraint holds. We rewrite $d_1(t)$ as

$$d_1(t) = \frac{\Omega_0 - td_0\Omega_3}{3\Omega_2} = d_1 + \frac{d_0\Omega_3}{3\Omega_2}(1-t).$$

For $t \in (0,1)$, if $(\Omega_2, \Omega_3) \in \Omega^{3,d}$, then

$$d_1(t) = d_1 + \frac{d_0\Omega_3}{3\Omega_2}(1-t) > d_1 + \Big(-\frac{td_0}{2}\Big)^{2/3} - d_1 = \Big(-\frac{td_0}{2}\Big)^{2/3} = \Big(-\frac{d_0(t)}{2}\Big)^{2/3}.$$

This implies that $d_0(t) > -2d_1(t)^{3/2}$. Last, for the $\Upsilon$-dominance constraint, by Proposition 3.3, we have $\tilde{\mathcal{P}}^d = \{(c_1, c_0)\colon d_1 \geq c_1 > 0 \text{ and } c_0 > -2c_1^{3/2}\}$. We can verify that

$$d_1(t) = \frac{\Omega_0 - td_0\Omega_3}{3\Omega_2} \leq \frac{\Omega_0 - d_0\Omega_3}{3\Omega_2} = d_1.$$

This finishes the proof. □

**Theorem 5.2.** *If there exists a $C$-subsolution to equation (5.1), then the degree three general inverse $\sigma_k$ equation (5.1) is solvable.*

*Proof.* If a $C$-subsolution exists, say $X$, then pointwise we have $X^2 > d_1\omega^2$, this implies that

$$X^3 > d_1\omega^2 \wedge X \Longrightarrow \Omega_0 > d_1\Omega_2.$$

By rewriting the topological constraint, we get

$$-d_0\frac{\Omega_3}{\Omega_2} = 3d_1 - \frac{\Omega_0}{\Omega_2} < 2d_1. \tag{5.3}$$

On the other hand, consider the following quantity $\frac{d_1-(-td_0/2)^{2/3}}{1-t}$. For $t \in (0,1)$, we have

$$\frac{d_1 - (-\frac{td_0}{2})^{2/3}}{1-t} > d_1\frac{1-t^{2/3}}{1-t} \geq \frac{2d_1}{3}. \tag{5.4}$$

The last inequality is due to the fact that the function $(1-t^{2/3})/(1-t)$ is decreasing when $t \in (0,1)$ and by L'Hôpital's rule. Combining inequalities (5.3) and (5.4), we see that if there exists a $C$-subsolution, then we always have $(\Omega_2, \Omega_3) \in \Omega^{3,d}$. □

We state one of the author's work in [51].

**Corollary 5.1** (deformed Hermitian–Yang–Mills equation, L. [51])**.** *When the complex dimension equals three, Conjecture 5.1 is confirmed.*

*Proof.* When $n = 3$, the deformed Hermitian–Yang–Mills equation will be

$$\Im\big(\omega + \sqrt{-1}\chi\big)^3 = \tan(\theta) \cdot \Re\big(\omega + \sqrt{-1}\chi\big)^3.$$

By doing a substitution $X \coloneqq \chi - \tan(\theta)\omega$, the dHYM equation becomes

$$X^3 - 3\sec^2(\theta)\omega^2 \wedge X - 2\tan(\theta)\sec^2(\theta)\omega^3 = 0. \tag{5.5}$$

For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, we always have $2\tan(\theta)\sec^2(\theta) > -2\big(\sec^2(\theta)\big)^{3/2}$. By Theorem 5.2, Conjecture 5.1 is confirmed. This finishes the proof. $\square$

## 5.2 When Complex Dimension Equals Four

In this section, we always assume that there exists a $C$-subsolution. By changing representative, we say $X$ is this $C$-subsolution. We are interested in the solvability of the following equation $d\colon M^4 \to \{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2}, \text{ and } c_0 > -3c_2x_1^2 - 3c_1x_1\} \subset \tilde{\mathscr{C}_4}$ with $d_2, d_1, d_0$ constant, where $x_1$ is the largest real root of $x^3 - 3c_2x - c_1 = 0$. That is,

$$X^4 - 6d_2\omega^2 \wedge X^2 - 4d_1\omega^3 \wedge X - d_0\omega^4 = 0. \tag{5.6}$$

By Section 4.2, we observe that if a $C$-subsolution to $d\colon M \to \tilde{\mathscr{C}_4}$ exists, then for any $c \in \tilde{\mathcal{P}}^d$ with range in $S$ a compact subset of $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2}, \text{ and } c_0 > -3c_2x_1^2 - 3c_1x_1\}$, we have a priori estimates. We try to find a continuity path in the space $\tilde{\mathcal{P}}^d$ connecting the original equation $d\colon M^4 \to \tilde{\mathscr{C}_4}$ to another solvable equation. When $d_0, d_1$ and $d_2$ are all non-negative with $d_2 + d_1 + d_0 > 0$, this is solvable due to Collins–Székelyhidi [20]. Hence, we consider the other cases, we will prove that they are all solvable provided that a $C$-subsolution exists. We consider the following continuity path,

$$X^4 - 6d_2(t)\omega^2 \wedge X^2 - 4d_1(t)\omega^3 \wedge X - d_0(t)\omega^4 = 0, \tag{5.7}$$

where $t \in [0,1]$ and $d_0(t)$, $d_1(t)$, and $d_2(t)$ are smooth functions satisfy all the following

$$\begin{aligned}
\text{Topological constraint: } & \Omega_0 - 6d_2(t)\Omega_2 - 4d_1(t)\Omega_3 - d_0(t)\Omega_4 = 0. \\
\text{Boundary constraints: } & d_2(1) = d_2;\ d_1(1) = d_1;\ d_0(1) = d_0;\ d_2(0) > 0;\ d_1(0) \geq 0. \\
\text{Positivstellensatz constraint: } & d(t) \in \{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},\ \text{and } c_0 > -3c_2x_1^2 - \\
& 3c_1x_1\} \subset \tilde{\mathscr{C}_4}, \text{ where } x_1 \text{ is the largest real root of } x^3 = 3c_2x + c_1. \\
\Upsilon\text{-dominance constraint: } & d(t) \in \tilde{\mathcal{P}}^d.
\end{aligned}$$

Here, we denote $\Omega_i \coloneqq \int_M \omega^i \wedge X^{4-i}$. $(\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_4)$ will be a fixed value determined by the cohomology classes $\omega$ and $X$. If we have a priori estimates for compact subset of $\tilde{\mathscr{C}_4}$ or $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},\ \text{and } c_0 > -3c_2x_1^2 - 3c_1x_1\} \cup \{(0,0,c_0)\colon c_0 > 0\}$, then we can find a better path and do a uniform estimates. Here, since we only have a priori estimates for compact subset of $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},\ \text{and } c_0 > -3c_2x_1^2 - 3c_1x_1\}$, so we always required that the path will be in $\{(c_2, c_1, c_0)\colon c_2 > 0, c_1 > -2c_2^{3/2},\ \text{and } c_0 > -3c_2x_1^2 - 3c_1x_1\}$.

**Lemma 5.2.** *If $(\Omega_2, \Omega_3, \Omega_4) \in \Omega_\ell^{4,d}$, then there are two cases to consider: If $d_1 \geq 0$ and $d_0 < 0$, then the following triple will satisfy all the 4-dimensional four constraints:*

$$d_{2,\ell}(t) \coloneqq t^{2/3}d_2 + (1 - t^{2/3})\ell; \quad d_1(t) \coloneqq td_1; \quad d_{0,\ell}(t) \coloneqq \frac{\Omega_0 - 6d_{2,\ell}(t)\Omega_2 - 4d_1(t)\Omega_3}{\Omega_4}.$$

*Here, $\Omega_i \coloneqq \int_M \omega^i \wedge X^{4-i}$, $d_2/100 > \ell > 0$ sufficiently small, and*

$$\begin{aligned}
\Omega_\ell^{4,d} \coloneqq \Big\{0 < \inf_{t \in [0,1)} & 6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 \\
& + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4\Big\},
\end{aligned}$$

*where $x_{1,\ell}(t)$ is the largest real root of $x^3 - 3d_{2,\ell}(t)x - d_1(t) = 0$. If $d_1 < 0$, then the following triple will satisfy all the 4-dimensional four constraints:*

$$d_{2,\ell}(t) := \Big(d_2^{3/2} + \frac{(1-t)\ell d_1}{2}\Big)^{2/3}; \quad d_1(t) := td_1; \quad d_{0,\ell}(t) := \frac{\Omega_0 - 6d_{2,\ell}(t)\Omega_2 - 4d_1(t)\Omega_3}{\Omega_4}.$$

*Here, $\Omega_i := \int_M \omega^i \wedge X^{4-i}$, $\ell \in [1, -2d_2^{3/2}/d_1)$, and*

$$\Omega_\ell^{4,d} := \Big\{0 < \inf_{t\in[0,1)} 6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 \\ + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4\Big\},$$

*where $x_{1,\ell}(t)$ is the largest real root of $x^3 - 3d_{2,\ell}(t)x - d_1(t) = 0$.*

*Proof.* For the case $d_1 \geq 0$ and $d_0 < 0$, first, the topological constraint is automatically satisfied. Second, for the boundary constraints, it should be straightforward. Third, for the positivstellensatz constraint, when $t = 1$, the positivstellensatz constraint holds. For $t \in [0, 1)$, we have $d_{2,\ell}(t) > 0$, $d_1(t) + 2d_{2,\ell}^{3/2}(t) \geq 2\ell^{3/2} > 0$, and we can verify that the following quantity is always positive:

$$\begin{aligned} & d_{0,\ell}(t) + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t) \\ &\quad = \frac{\Omega_0 - 6d_{2,\ell}(t)\Omega_2 - 4d_1(t)\Omega_3}{\Omega_4} + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t) \\ &\quad = \frac{6(d_2 - d_{2,\ell}(t))\Omega_2 + 4d_1(1-t)\Omega_3}{\Omega_4} + d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3td_1x_{1,\ell}(t) > 0 \end{aligned}$$

due to the hypothesis that $(\Omega_2, \Omega_3, \Omega_4) \in \Omega_\ell^{4,d}$. Last, for the $\Upsilon$-dominance constraint, by Proposition 3.3, we have $\tilde{\mathcal{P}}^d = \{(c_2, c_1, c_0) \in \tilde{\mathscr{C}_4} \colon x_2^2 - c_2 = d_2 - c_2 \geq 0$ and $x_1^3 - 3c_2x_1 - c_1 \geq 0\}$. Here, $x_2$ is the largest real root of $x^2 - d_2 = 0$ and $x_1$ is the largest real root of

$x^3 - 3d_2x - d_1 = 0$. We can verify that $d_2 \geq d_{2,\ell}(t) = t^{2/3}d_2 + (1 - t^{2/3})\ell$ and

$$\frac{d}{dt}\big(x_1^3 - 3d_{2,\ell}(t)x_1 - d_1(t)\big) = -2t^{-1/3}(d_2 - \ell)x_1 - d_1 < 0 \text{ for } t \in (0, 1].$$

This implies that for $t \in [0, 1]$, $x_1^3 - 3d_{2,\ell}(t)x_1 - d_1(t) \geq x_1^3 - 3d_{2,\ell}(1)x_1 - d_1(1) = 0$.

For the case $d_1 < 0$, first, the topological constraint is automatically satisfied. Second, for the boundary constraints, we get $d_{2,\ell}(1) = d_2, d_1(1) = d_1, d_1(0) = 0, d_{,\ell}0(1) = d_0$, and

$$d_{2,\ell}(0) = \left(d_2^{3/2} + \frac{\ell d_1}{2}\right)^{2/3} > 0.$$

Third, for the positivstellensatz constraint, when $t = 1$, the positivstellensatz constraint holds. For $t \in [0, 1)$, we have $d_{2,\ell}(t) > 0$, $d_1(t) + 2d_{2,\ell}^{3/2}(t) = 2d_2^{3/2} + \ell d_1 + td_1(1 - \ell) > 0$, and we can verify that the following quantity is always positive:

$$\begin{aligned}
&d_{0,\ell}(t) + 3d_{2,\ell}(t)x_1^2(t) + 3d_1(t)x_1(t) \\
&\quad = \frac{\Omega_0 - 6d_{2,\ell}(t)\Omega_2 - 4d_1(t)\Omega_3}{\Omega_4} + 3d_{2,\ell}(t)x_1^2(t) + 3d_1(t)x_1(t) \\
&\quad = \frac{6(d_2 - d_{2,\ell}(t))\Omega_2 + 4d_1(1 - t)\Omega_3}{\Omega_4} + d_0 + 3d_{2,\ell}(t)x_1^2(t) + 3td_1x_1(t) > 0
\end{aligned}$$

due to the hypothesis that $(\Omega_2, \Omega_3, \Omega_4) \in \Omega_\ell^{4,d}$. Last, for the $\Upsilon$-dominance constraint, by Proposition 3.3, we have $\tilde{\mathcal{P}}^d = \{(c_2, c_1, c_0) \in \tilde{\mathscr{C}_4}\colon x_2^2 - c_2 = d_2 - c_2 \geq 0 \text{ and } x_1^3 - 3c_2x_1 - c_1 \geq 0\}$. Here, $x_2$ is the largest real root of $x^2 - d_2 = 0$ and $x_1$ is the largest real root of $x^3 - 3d_2x - d_1 = 0$. We can verify that $d_2 \geq d_{2,\ell}(t) = (d_2^{3/2} + (1 - t)\ell d_1/2)^{2/3}$ and

$$\frac{d}{dt}d_{2,\ell}(t) = \frac{-\ell d_1}{3}\left(d_2^{3/2} + \frac{\ell d_1}{2}\right)^{-1/3} > 0. \tag{5.8}$$

Inequality (5.8) gives

$$\begin{aligned}
&\frac{d}{dt}\big(x_1^3 - 3d_{2,\ell}(t)x_1 - d_1(t)\big) = -3d'_{2,\ell}(t)x_1 - d_1 \le -3d'_{2,\ell}(t)x_2 - d_1\\
&\quad \le -3d'_{2,\ell}(t)d_{2,\ell}^{1/2}(t) - d_1 = -2\frac{d}{dt}\big(d_{2,\ell}^{3/2}(t)\big) - d_1 = \ell d_1 - d_1 \le 0.
\end{aligned}$$

This implies that for $t \in [0,1]$, $x_1^3 - 3d_{2,\ell}(t)x_1 - d_1(t) \ge x_1^3 - 3d_{2,\ell}(1)x_1 - d_1(1) = 0$. This finishes the proof. $\square$

**Theorem 5.3.** *If there exists a $C$-subsolution to equation (5.6), then the degree four general inverse $\sigma_k$ equation (5.6) is solvable.*

*Proof.* If a $C$-subsolution exists, say $X$, then pointwise we have $X^3 > 3d_2\omega^2 \wedge X + d_1\omega^3$ and $X^2 > d_2\omega^2$, these imply that $\Omega_0 > 3d_2\Omega_2 + d_1\Omega_3$, $\Omega_0 > d_2\Omega_2$, and $\Omega_2 > d_2\Omega_4$. By the topological constraint $\Omega_0 = 6d_2\Omega_2 + 4d_1\Omega_3 + d_0\Omega_4$, we always get

$$3d_2\Omega_2 + 3d_1\Omega_3 + d_0\Omega_4 > 0; \tag{5.9}$$

$$5d_2\Omega_2 + 4d_1\Omega_3 + d_0\Omega_4 > 0. \tag{5.10}$$

The goal here is to check whether the following quantity

$$6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4 \tag{5.11}$$

is positive for $t \in [0,1]$ for some $\ell$. We consider two cases: $d_1 \ge 0$ and $d_1 < 0$.

• For the case $d_1 \ge 0$ and $d_0 < 0$. By inequality (5.9), quantity (5.11) becomes

$$\begin{aligned}
&6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4\\
&\quad > \big(2d_2(1 + 2t - 3t^{2/3}) - 6\ell(1 - t^{2/3})\big)\Omega_2\\
&\qquad + \Big(\frac{d_0}{3}(4t - 1) + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\Big)\Omega_4.
\end{aligned} \tag{5.12}$$

If we can show that quantity (5.12) is positive on $[0,1]$ for some $\ell$ sufficiently close to 0, then quantity (5.11) will also be positive, thus we are done. On $[0,1/4]$, we have

$$\begin{aligned}&\big(2d_2(1+2t-3t^{2/3})-6\ell(1-t^{2/3})\big)\Omega_2+\Big(\frac{d_0}{3}(4t-1)+3d_{2,\ell}(t)x_{1,\ell}^2(t)+3d_1(t)x_{1,\ell}(t)\Big)\Omega_4\\&\quad\geq\big(2d_2(1.5-3(1/4)^{2/3})-6\ell(1-(1/4)^{2/3})\big)\Omega_2\geq\big(0.6d_2-4\ell\big)\Omega_2>0\end{aligned}\tag{5.13}$$

provided that $d_2/100>\ell>0$. On $[1/4,1]$, we compute the derivative of the coefficient of $\Omega_4$ of quantity (5.12), we first consider the derivative of $x_{1,\ell}(t)$ with respect to $t$. Since $x_{1,\ell}(t)$ satisfies $x_{1,\ell}^3(t)-3d_{2,\ell}(t)x_{1,\ell}(t)-d_1(t)=0$, using implicit differentiation, we get

$$3(x_{1,\ell}^2(t)-d_{2,\ell}(t))x_{1,\ell}'(t)=3d_{2,\ell}'(t)x_{1,\ell}(t)+d_1'(t)=2t^{-1/3}(d_2-\ell)x_{1,\ell}(t)+d_1.$$

By Proposition 2.12, we have

$$x_{1,\ell}(t)=\begin{cases}2\sqrt{d_{2,\ell}(t)}\cos\Big[\frac{1}{3}\arccos\Big(\frac{d_1(t)}{2d_{2,\ell}^{3/2}(t)}\Big)\Big] & \text{, when } 4d_{2,\ell}^3(t)\geq d_1^2(t);\\ 2\sqrt{d_{2,\ell}(t)}\cosh\Big[\frac{1}{3}\operatorname{arccosh}\Big(\frac{d_1(t)}{2d_{2,\ell}^{3/2}(t)}\Big)\Big] & \text{, when } d_1^2(t)\geq 4d_{2,\ell}^3(t).\end{cases}$$

In addition, we have

$$\frac{d}{dt}\Big(\frac{d_1(t)}{2d_{2,\ell}^{3/2}(t)}\Big)=\frac{d}{dt}\Big(\frac{td_1}{2\big(t^{2/3}d_2+(1-t^{2/3})\ell\big)^{3/2}}\Big)=\frac{d_1\ell}{2\big(t^{2/3}d_2+(1-t^{2/3})\ell\big)^{5/2}}>0.$$

Hence, when $t\in[1/4,1]$, we obtain

$$\frac{d_1}{2d_2^{3/2}}\geq\frac{d_1(1)}{2d_{2,\ell}^{3/2}(1)}\geq\frac{d_1(t)}{2d_{2,\ell}^{3/2}(t)}\geq\frac{d_1(1/4)}{2d_{2,\ell}^{3/2}(1/4)}\geq 0.$$

This implies that $x_{1,\ell}(t)$ and

$$x_{1,\ell}'(t)=\frac{2t^{-1/3}(d_2-\ell)x_{1,\ell}(t)+d_1}{3(x_{1,\ell}^2(t)-d_{2,\ell}(t))}$$

both have a uniform lower bound and upper bound on $[1/4,1]$. The first derivative of the coefficient of $\Omega_4$ of quantity (5.12) will be

$$\begin{aligned}
&\frac{4d_0}{3}+3d_{2,\ell}'(t)x_{1,\ell}^2(t)+6d_{2,\ell}(t)x_{1,\ell}(t)x_{1,\ell}'(t)+3d_1'(t)x_{1,\ell}(t)+3d_1(t)x_{1,\ell}'(t)\\
&\quad=\frac{4d_0}{3}+2t^{-1/3}(d_2-\ell)x_{1,\ell}^2(t)+3d_1x_{1,\ell}(t)+3\big(2d_{2,\ell}(t)x_{1,\ell}(t)+td_1\big)x_{1,\ell}'(t)
\end{aligned}\tag{5.14}$$

and it has a uniform upper and lower bound for $\ell\in(0,d_2/100)$ and $t\in[1/4,1]$. The first derivative of the coefficient of $\Omega_2$ of quantity (5.12) will be

$$\frac{d}{dt}\big(2d_2(1+2t-3t^{2/3})-6\ell(1-t^{2/3})\big)=4d_2(1-t^{-1/3})+4\ell t^{-1/3}\tag{5.15}$$

and it has a uniform upper and lower bound on $[1/4,1]$. Let $\{\ell_i\}$ be a sequence such that $\ell_i\to 0$ as $i\to\infty$. We define the following sequence of functions on $[0,1]$:

$$\begin{aligned}
\mathscr{D}_i(t)&:=\big(2d_2(1+2t-3t^{2/3})-6\ell_i(1-t^{2/3})\big)\Omega_2\\
&\qquad+\Big(\frac{d_0}{3}(4t-1)+3d_{2,\ell_i}(t)x_{1,\ell_i}^2(t)+3d_1(t)x_{1,\ell_i}(t)\Big)\Omega_4.
\end{aligned}$$

Since derivatives (5.14) and (5.15) are uniformly bounded on $[1/4,1]$ when $\ell\in(0,d_2/100)$, so this sequence of functions $\{\mathscr{D}_i\}$ will be an equicontinuous sequence that converges uniformly to the following function on $[1/4,1]$:

$$\begin{aligned}
\mathscr{D}_\infty(t)&:=2d_2(1+2t-3t^{2/3})\Omega_2+\Big(\frac{d_0}{3}(4t-1)+3t^{4/3}(d_2x_1^2+d_1x_1)\Big)\Omega_4\\
&=2d_2(1+2t-3t^{2/3})\Omega_2\\
&\qquad+\Big(\frac{4t-1}{3}(d_0+3d_2x_1^2+3d_1x_1)+(1-4t+3t^{4/3})(d_2x_1^2+d_1x_1)\Big)\Omega_4.
\end{aligned}$$

We can check that $1+2t-3t^{2/3}$ and $1-4t+3t^{4/3}$ are decreasing on $[1/4,1]$, thus

$$\mathscr{D}_\infty(t)=2d_2(1+2t-3t^{2/3})\Omega_2$$

$$+\Big(\frac{4t-1}{3}(d_0+3d_2x_1^2+3d_1x_1)+(1-4t+3t^{4/3})(d_2x_1^2+d_1x_1)\Big)\Omega_4>0$$

on $[1/4,1]$. So, there exists $\ell_N>0$ sufficiently small such that

$$\begin{aligned}\mathscr{D}_N(t)&=\big(2d_2(1+2t-3t^{2/3})-6\ell_N(1-t^{2/3})\big)\Omega_2\\&\quad+\Big(\frac{d_0}{3}(4t-1)+3d_{2,\ell_N}(t)x_{1,\ell_N}^2(t)+3d_1(t)x_{1,\ell_N}(t)\Big)\Omega_4>0\end{aligned}\tag{5.16}$$

on $[1/4,1]$. By inequalities (5.13) and (5.16), we see that $\mathscr{D}_N>0$ on $[0,1]$. This implies that quantity (5.12) is positive, hence quantity (5.11) is also positive on $[0,1]$.

• For the case $0>d_1>-2d_2^{3/2}$, we consider two subcases: $d_0\le 0$ and $d_0>0$.

⋆ For the subcase $d_0\le 0$, by inequality (5.9), quantity (5.11) becomes

$$\begin{aligned}&6(d_2-d_{2,\ell}(t))\Omega_2+4(d_1-d_1(t))\Omega_3+\big(d_0+3d_{2,\ell}(t)x_{1,\ell}^2(t)+3d_1(t)x_{1,\ell}(t)\big)\Omega_4\\&\quad\ge 2(d_2+2td_2-3d_{2,\ell}(t))\Omega_2+\Big(\frac{4t-1}{3}d_0+3d_{2,\ell}(t)x_{1,\ell}^2(t)+3d_1(t)x_{1,\ell}(t)\Big)\Omega_4.\end{aligned}\tag{5.17}$$

So, instead of checking whether quantity (5.11) is positive, we check whether quantity (5.17) is positive. By taking the derivative of the coefficient of $\Omega_2$ of quantity (5.17), we get

$$\frac{d}{dt}(d_2+2td_2-3d_{2,\ell}(t))=2d_2+\ell d_1\Big(d_2^{3/2}+\frac{(1-t)\ell d_1}{2}\Big)^{-1/3}.\tag{5.18}$$

Above quantity (5.18) will be negative when $t<1+\frac{2d_2^{3/2}}{\ell d_1}+\frac{\ell^2d_1^2}{4d_2^3}$. When $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$, quantity will be negative when $t\le 1/4$. On the other hand, since $x_{1,\ell}(t)$ is the largest real root of $x^3-3d_{2,\ell}(t)x-d_1(t)=0$ and $0\ge d_1(t)>-2d_{2,\ell}^{3/2}(t)$, this is the case *casus irreducilis*, so we have

$$\begin{aligned}x_{1,\ell}(t)&=2d_{2,\ell}^{1/2}(t)\cos\Big[\frac{1}{3}\arccos\Big(\frac{d_1(t)}{2d_{2,\ell}^{3/2}(t)}\Big)\Big]\\&=2\Big(d_2^{3/2}+\frac{(1-t)\ell d_1}{2}\Big)^{1/3}\cos\Big[\frac{1}{3}\arccos\Big(\frac{td_1}{2d_2^{3/2}+(1-t)\ell d_1}\Big)\Big].\end{aligned}\tag{5.19}$$

Here, we specify the branch so that $\arccos\big(\bullet\big) \in [0,\pi]$. Thus, by (5.19), we have

$$
\begin{aligned}
&3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t) \\
&\quad = 3x_{1,\ell}^4(t) - 6d_{2,\ell}(t)x_{1,\ell}^2(t) \\
&\quad = 24d_{2,\ell}^2(t)\cos^2\Big[\frac{1}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\cos\Big[\frac{2}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big].
\end{aligned} \tag{5.20}
$$

In addition, we can verify that for $\ell \in [1, -2d_2^{3/2}/d_1)$, we get

$$
\frac{d}{dt}\Big(\frac{td_1}{2d_2^{3/2} + (1-t)\ell d_1}\Big) = \frac{d_1\big(2d_2^{3/2} + \ell d_1\big)}{\big(2d_2^{3/2} + (1-t)\ell d_1\big)^2} < 0.
$$

So on $[0,1]$ and for $\ell \in [1, -2d_2^{3/2}/d_1)$, we always have the following uniform bounds

$$
0 \geq \frac{td_1}{2d_2^{3/2} + (1-t)\ell d_1} \geq \frac{d_1}{2d_2^{3/2}} > -1.
$$

This implies that $\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)/3 \in [\pi/6, \pi/3)$ and

$$
\frac{3}{8} \geq \cos^2\Big[\frac{1}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\cos\Big[\frac{2}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big] > -\frac{1}{8}. \tag{5.21}
$$

Hence, on $[0,1/4]$, by inequalities (5.18), (5.20), and (5.21), quantity (5.17) becomes

$$
\begin{aligned}
&2(d_2 + 2td_2 - 3d_{2,\ell}(t))\Omega_2 + \Big(\frac{4t-1}{3}d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\Big)\Omega_4 \\
&\quad \geq 3(d_2 - 2d_{2,\ell}(1/4))\Omega_2 - 3d_{2,\ell}^2(t)\Omega_4 \geq 3\Big(d_2 - 2\Big(d_2^{3/2} + \frac{3\ell d_1}{8}\Big)^{2/3}\Big)\Omega_2 - 3d_{2,\ell}^2(1/4)\Omega_4 \\
&\quad \geq 0.6d_2\Omega_2 - 0.48d_2^2\Omega_4 > 0.12d_2^2\Omega_4 > 0
\end{aligned} \tag{5.22}
$$

provided that $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$, so $2^{-4/3}d_2 \leq d_{2,\ell}(1/4) \leq 0.4d_2$. For the subinterval $[1/4,1]$, we consider the derivative of the coefficient of $\Omega_4$ of quantity (5.17) with respect to $t$, by quality (5.20), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{4t-1}{3}d_0 + 3d_{2,\ell}(t)x_1^2(t) + 3d_1(t)x_1(t)\Big)\\
&\quad= \frac{4}{3}d_0 + 24\frac{d}{dt}\Big(d_{2,\ell}^2(t)\cos^2\Big[\frac{1}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\cos\Big[\frac{2}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\Big)\\
&\quad= \frac{4}{3}d_0 + 16d_{2,\ell}^{1/2}(t)\cos(\theta_{d_1,d_2,\ell}(t))\Big[\Big(d_{2,\ell}^{3/2}(t)\cos(3\theta_{d_1,d_2,\ell}(t))\Big)' + \Big(d_{2,\ell}^{3/2}(t)\Big)'\cos(\theta_{d_1,d_2,\ell}(t))\Big]\\
&\quad= \frac{4}{3}d_0 + 8d_1d_{2,\ell}^{1/2}(t)\cos(\theta_{d_1,d_2,\ell}(t))\Big[1-\ell\cos(\theta_{d_1,d_2,\ell}(t))\Big]. \qquad (5.23)
\end{aligned}$$

Here, $\theta_{d_1,d_2,\ell}(t) \coloneqq \frac{1}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big) \in (\pi/6, \pi/3)$. This is uniformly bounded on $[0,1]$. Let $\{\ell_i\}$ be a sequence such that $\ell_i \to -2d_2^{3/2}/d_1$ as $i \to \infty$. We define the following sequence of functions on $[0,1]$:

$$\mathscr{D}_i(t) \coloneqq 2(d_2 + 2td_2 - 3d_{2,\ell_i}(t))\Omega_2 + \Big(\frac{d_0}{3}(4t-1) + 3d_{2,\ell_i}(t)x_{1,\ell_i}^2(t) + 3d_1(t)x_{1,\ell_i}(t)\Big)\Omega_4.$$

Since derivatives (5.18) and (5.23) are uniformly bounded on $[1/4,1]$ when $\ell \in [1, -2d_2^{3/2}/d_1)$, so this sequence of functions $\{\mathscr{D}_i\}$ will be an equicontinuous sequence that converges uniformly to the following function on $[1/4,1]$:

$$\begin{aligned}
\mathscr{D}_\infty(t) &\coloneqq 2d_2(1+2t-3t^{2/3})\Omega_2\\
&\qquad + \Big(\frac{4t-1}{3}(d_0 + 3d_2x_1^2 + 3d_1x_1) + (1-4t+3t^{4/3})(d_2x_1^2 + d_1x_1)\Big)\Omega_4.
\end{aligned}$$

Since $1+2t-3t^{2/3}$ and $1-4t+3t^{4/3}$ are decreasing on $[1/4,1]$ and $d_2x_1^2 + d_1x_1 > -d_0 \geq 0$, $\mathscr{D}_\infty(t) > 0$ on $[1/4,1]$. So, there exists $\ell_N$ sufficiently close to $-2d_2^{3/2}/d_1$ such that

$$\begin{aligned}
\mathscr{D}_N(t) &= 2(d_2 + 2td_2 - 3d_{2,\ell_N}(t))\Omega_2\\
&\qquad + \Big(\frac{d_0}{3}(4t-1) + 3d_{2,\ell_N}(t)x_{1,\ell_N}^2(t) + 3d_1(t)x_{1,\ell_N}(t)\Big)\Omega_4 > 0 \qquad (5.24)
\end{aligned}$$

on $[1/4,1]$. By inequalities (5.22) and (5.24), we see that $\mathscr{D}_N > 0$ on $[0,1]$. This implies that quantity (5.17) is positive on $[0,1]$, hence quantity (5.11) is also positive on $[0,1]$.

$\star$ For the subcase $d_0 > 0$, we first consider the subinterval $[0, 1/4]$. On $[0, 1/4]$, we need to use both inequality (5.9) and inequality (5.10). We multiply inequality (5.9) by $12t$ and inequality (5.10) by $4 - 16t$, then quantity (5.11) becomes

$$\begin{aligned}
&6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4 \\
&\quad \geq (d_2 + 8td_2 - 6d_{2,\ell}(t))\Omega_2 + \big(3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4 \\
&\quad = (d_2 + 8td_2 - 6d_{2,\ell}(t))\Omega_2 \\
&\qquad + 24d_{2,\ell}^2(t)\cos^2\Big[\frac{1}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\cos\Big[\frac{2}{3}\arccos\big(d_1(t)/2d_{2,\ell}^{3/2}(t)\big)\Big]\Omega_4 \\
&\quad > (d_2 + 8td_2 - 6d_{2,\ell}(t))\Omega_2 - 3d_{2,\ell}^2(t)\Omega_4. \qquad (5.25)
\end{aligned}$$

The derivative of the coefficient of $\Omega_2$ of quantity (5.25) will be

$$\frac{d}{dt}(d_2 + 8td_2 - 6d_{2,\ell}(t)) = 8d_2 + 2\ell d_1\Big(d_2^{3/2} + \frac{(1-t)\ell d_1}{2}\Big)^{-1/3} \qquad (5.26)$$

and attains its minimum at $t = 1 + \frac{2d_2^{3/2}}{\ell d_1} + \frac{\ell^2 d_1^2}{23 d_2^3}$ with minimum value $9d_2 + \frac{16 d_2^{5/2}}{\ell d_1} - \frac{\ell^2 d_1^2}{8 d_2^2}$. When $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$, the minimum is attained close to $1/8$ and the minimum will be positive. This implies that when $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$, $d_2 + 8td_2 - 6d_{2,\ell}(t)$ will be positive on $[0, 1/4]$. Thus, by $\Omega_2 > d_2\Omega_4$ and (5.26), quantity (5.25) becomes

$$(d_2 + 8td_2 - 6d_{2,\ell}(t))\Omega_2 - 3d_{2,\ell}^2(t)\Omega_4 \geq \big(d_2^2 + 8td_2^2 - 6d_2 d_{2,\ell}(t) - 3d_{2,\ell}^2(t)\big)\Omega_4. \qquad (5.27)$$

The derivative of the coefficient of $\Omega_4$ of quantity (5.27) will be $8d_2^2 - 6d_2 d_{2,\ell}'(t) - 6d_{2,\ell}(t)d_{2,\ell}'(t)$ and is negative on $[0, 1/4]$ when $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$. This implies that $d_2^2 + 8td_2^2 - 6d_2 d_{2,\ell}(t) - 3d_{2,\ell}^2(t)$ will be decreasing. Hence, inequality (5.27) becomes

$$\begin{aligned}
&\big(d_2^2 + 8td_2^2 - 6d_2 d_{2,\ell}(t) - 3d_{2,\ell}^2(t)\big)\Omega_4 \\
&\quad \geq 3\big(d_2^2 - 2d_2 d_{2,\ell}(1/4) - d_{2,\ell}^2(1/4)\big)\Omega_4
\end{aligned}$$

$$= 3\Big(d_2^2 - 2d_2\Big(d_2^{3/2} + \frac{3\ell d_1}{8}\Big)^{2/3} - \Big(d_2^{3/2} + \frac{3\ell d_1}{8}\Big)^{4/3}\Big)\Omega_4 > 0 \tag{5.28}$$

provided that $2^{-4/3}d_2 \leq d_{2,\ell}(1/4) \leq 0.4d_2$ when $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$. So quantity (5.27) is positive on $[0, 1/4]$, which implies that quantity (5.11) is positive on $[0, 1/4]$. For $[1/4, 1]$, we use inequality (5.9), quantity (5.11) becomes

$$\begin{aligned}
&6(d_2 - d_{2,\ell}(t))\Omega_2 + 4(d_1 - d_1(t))\Omega_3 + \big(d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\big)\Omega_4 \\
&\quad \geq 2(d_2 + 2td_2 - 3d_{2,\ell}(t))\Omega_2 + \Big(\frac{4t-1}{3}d_0 + 3d_{2,\ell}(t)x_{1,\ell}^2(t) + 3d_1(t)x_{1,\ell}(t)\Big)\Omega_4.
\end{aligned}$$

Let $\{\ell_i\}$ be a sequence such that $\ell_i \to -2d_2^{3/2}/d_1$ as $i \to \infty$. We define the following sequence of functions on $[0, 1]$:

$$\mathscr{D}_i(t) \coloneqq 2(d_2 + 2td_2 - 3d_{2,\ell_i}(t))\Omega_2 + \Big(\frac{d_0}{3}(4t-1) + 3d_{2,\ell_i}(t)x_{1,\ell_i}^2(t) + 3d_1(t)x_{1,\ell_i}(t)\Big)\Omega_4.$$

Since derivatives (5.18) and (5.23) are uniformly bounded on $[1/4, 1]$ when $\ell \in [1, -2d_2^{3/2}/d_1)$, so this sequence of functions $\{\mathscr{D}_i\}$ will be an equicontinuous sequence that converges uniformly to the following function on $[1/4, 1]$:

$$\begin{aligned}
\mathscr{D}_\infty(t) &\coloneqq 2d_2(1 + 2t - 3t^{2/3})\Omega_2 \\
&\qquad + \Big(\frac{4t-1}{3}(d_0 + 3d_2x_1^2 + 3d_1x_1) + (1 - 4t + 3t^{4/3})(d_2x_1^2 + d_1x_1)\Big)\Omega_4 \\
&\geq 2d_2(1 + 2t - 3t^{2/3})\Omega_2 + \big((1 - 4t + 3t^{4/3})(d_2x_1^2 + d_1x_1)\big)\Omega_4 \\
&\qquad + \frac{4t-1}{3}(d_0 + 3d_2x_1^2 + 3d_1x_1)\Omega_4 \\
&\geq \Big(d_2^2(1 + 8t - 6t^{2/3} - 3t^{4/3}) + \frac{4t-1}{3}(d_0 + 3d_2x_1^2 + 3d_1x_1)\Big)\Omega_4 > 0.
\end{aligned}$$

Notice that the function $1 + 8t - 6t^{2/3} - 3t^{4/3}$ is decreasing on $[0, 1]$. So, there exists $\ell_N$

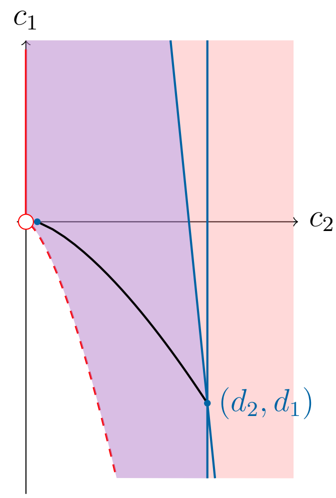


Figure 5.1: The continuity path in Lemma 5.2 when $d_1 < 0$.

sufficiently close to $-2d_2^{3/2}/d_1$ such that

$$\begin{aligned}\mathscr{D}_N(t) &= 2(d_2 + 2td_2 - 3d_{2,\ell_N}(t))\Omega_2 \\ &\quad + \Big(\frac{d_0}{3}(4t-1) + 3d_{2,\ell_N}(t)x_{1,\ell_N}^2(t) + 3d_1(t)x_{1,\ell_N}(t)\Big)\Omega_4 > 0 \end{aligned} \tag{5.29}$$

on $[1/4, 1]$. By inequalities (5.28) and (5.29), we see that quantity (5.11) is positive on $[0, 1]$.

In conclusion, if $d_1 \geq 0$ and $d_0 < 0$, then there exists $\ell$ sufficiently close to 0 such that $(\Omega_2, \Omega_3, \Omega_4) \in \Omega_\ell^{4,d}$. For the solvability, when $t = 0$, we have $d_{2,\ell}(0) = \ell > 0$ and $d_{0,\ell}(0) = (\Omega_0 - 6\ell\Omega_2)/\Omega_4 \geq 0$ provided that $\ell$ is sufficiently close to 0. This is solvable due to Collins–Székelyhidi [20], hence the original equation (5.6) is solvable due to the method of continuity. If $d_1 < 0$, then there exists $\ell$ sufficiently close to $-2d_2^{3/2}/d_1$ such that $(\Omega_2, \Omega_3, \Omega_4) \in \Omega_\ell^{4,d}$. In addition, when $t = 0$, we have $d_{2,\ell}(0) > 0$ and $d_{0,\ell}(0) = (\Omega_0 - 6d_{2,\ell}(0)\Omega_2)/\Omega_4 > 0$ provided that $\ell$ is sufficiently close to $-2d_2^{3/2}/d_1$. This finishes the proof. □

Figure 5.1 is an illustration of the continuity path in Lemma 5.2. Theorem 5.3 shows that we can meet all the 4-dimensional four constraints by letting the end point close to the origin. This continuity path was also considered in the author's work [51] when proving

Conjecture 5.1 by Collins–Jacob–Yau. We state the author's work here.

**Corollary 5.2** (deformed Hermitian–Yang–Mills equation, L. [51]). *When the complex dimension equals four, Conjecture 5.1 is confirmed.*

*Proof.* A clever way is by Lemma 2.13, the diagonal restriction of the deformed Hermitian–Yang–Mills equation is strictly right-Noetherian when the phase is supercritical. Hence, by above Theorem 5.2, we know that the dHYM equation is solvable when complex dimension equals four and there exists a $C$-subsolution. Or we can just check it directly, when $n = 4$, the dHYM equation will be

$$\cot(\theta) \cdot \Im\big(\omega + \sqrt{-1}\chi\big)^4 = \Re\big(\omega + \sqrt{-1}\chi\big)^4.$$

By doing a substitution $X \coloneqq \chi + \cot(\theta)\omega$, the dHYM equation becomes

$$X^4 - 6\csc^2(\theta)\omega^2 \wedge X^2 + 8\cot(\theta)\csc^2(\theta)\omega^3 \wedge X + \csc^2(\theta)(4 - 3\csc^2(\theta))\omega^4 = 0. \quad (5.30)$$

For $\theta \in (\pi, 2\pi)$, we always have $c_2 = \csc^2(\theta) > 0$, $c_1 = -2\cot(\theta)\csc^2(\theta) > -2c_2^{3/2} = -2(\csc^2(\theta))^{3/2} = 2\csc^3(\theta)$, and

$$\begin{aligned}
&c_0 + 3c_2x_1^2 + 3c_1x_1 \\
&\quad = -\csc^2(\theta)(4 - 3\csc^2(\theta)) + 24\csc^4(\theta)\cos^2\Big(\frac{\theta + 4\pi}{3}\Big)\cos\Big(\frac{2\theta + 8\pi}{3}\Big) \\
&\quad = -4\csc^4(\theta)\Big(\sin^2(\theta) - 3\Big[\cos\Big(\frac{2\theta + 2\pi}{3}\Big) + \frac{1}{2}\Big]^2\Big) \\
&\quad = -4\csc^4(\theta)\Big(\sin(\theta) + \sqrt{3}\cos\Big(\frac{2\theta + 2\pi}{3}\Big) + \frac{\sqrt{3}}{2}\Big)\Big(\sin(\theta) - \sqrt{3}\cos\Big(\frac{2\theta + 2\pi}{3}\Big) - \frac{\sqrt{3}}{2}\Big) \\
&\quad = -64\csc^4(\theta)\Big(\sin\Big(\frac{\theta + \pi}{3}\Big) - \frac{\sqrt{3}}{2}\Big)^3\Big(\sin\Big(\frac{\theta + \pi}{3}\Big) + \frac{\sqrt{3}}{2}\Big)^3 > 0
\end{aligned}$$

when $\theta \in (\pi, 2\pi)$. Here, $x_1 = -2\csc(\theta)\cos((\theta + 4\pi)/3)$ is the largest root of $x^3 - 3\csc^2(\theta)x + 2\cot(\theta)\csc^2(\theta)$. By Theorem 5.3, Conjecture 5.1 is confirmed. This finishes the proof. □

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