Proseminar on Computational Methods in QFT
Berends Giele Recursion
Report

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ETH Zürich, October 2019

Abstract

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In calculating amplitudes in QCD one question that arises is: What are the right kinematic variables for scattering amplitudes? Typically one uses the 4-momenta of involved particles.

For the momenta, specially their Lorentz-invariant products s_{ij} should be considered. Also the helicity basis [Berends et al., 1981] is very convenient, where the spin is quantized along the axis of p. For positive helicity, the spin points in the same direction as p, for negtive helicity in the opposite direction. We have $u_\pm(p)$, which is a 4-spinor. We know that for massless vectors there is a smaller representation which is two dimensional. For a four momentum p and a helicity h, we get two 2-spinors:

$$p^\mu, h \to u_-(p) = \begin{pmatrix} \lambda_{p\alpha} \\ 0 \\ 0 \end{pmatrix} \leftrightarrow |p\rangle = \lambda_{p\alpha} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$
$$u_+(p) = \begin{pmatrix} 0 \\ 0 \\ \tilde{\lambda}_p^{\dot{\alpha}} \end{pmatrix} \leftrightarrow |p] = \tilde{\lambda}_p^{\dot{\alpha}} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}.$$

In this representation, $|p\rangle$ is a column vector with helicity $-1/2$, whereas $|p]$ is a column vector with helicity $+1/2$. The dotted and undotted indices correpsond to two different spinor representations of the Lorentz-group. Positive $u_\pm$ and negative $v_\pm$ energy solutions are not distinct, since $p^2 = 0$.

Using this formalism, the massless Dirac equation can then be written as

$$\not{p}|p\rangle = \not{p}|p] = 0.$$

Definition 2.1 (Short notation of spinors)**.** As a short notation, when we are dealing with n particles, we have n four momenta labelled p_i, helicities labelled ${}_hi$ and maybe adjoint colors labelled a_i, where $i \in \{1, ..., n\}$. The i-th spinor can then be written as

$$|p_i\rangle \equiv |i\rangle,$$
$$|p_i] \equiv |i].$$

Next, we want to build some Lorentz-invariant quantities and define the spinor product.

Definition 2.2 (Spinor product)**.** The spinor product can be performed between spinors of the same representation

$$\langle ij\rangle \equiv \lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta,$$
$$[ij] \equiv \tilde{\lambda}_{i\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{j\dot{\beta}},$$

where the ϵ matrices are anti-symmetric and given by

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Definition 2.3 (Momentum products and sums)**.** To deal with multiple momenta later, we define some intermediate quantities

$$P_{i,j} = p_i + \cdots + p_j, \tag{2.1}$$
$$s_{ij} = (p_i + p_j)^2. \tag{2.2}$$

Definition 2.4 (Pauli matrices)**.** The Pauli matrices are defined as

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3),$$
$$\overline{\sigma}^\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3).$$

2.1 Important identities and definitions

Using the formalism defined above, there are now some identities that follow.

Lemma 2.1 (Fierz identity)**.**

$$\langle i|\sigma^\mu|j]\langle k|\sigma_\mu|l] = 2\langle ik\rangle[lj] \tag{2.3}$$

Lemma 2.2 (Charge conjugation identity)**.**

$$\langle i|\sigma^\mu|j] = [j|\overline{\sigma}^\mu|i\rangle \tag{2.4}$$

Lemma 2.3 (Schouten identity)**.**

$$\langle ij\rangle\,\langle k| + \langle jk\rangle\,\langle i| + \langle ki\rangle\,\langle j| = 0 \tag{2.5}$$

Lemma 2.4 (Squaring identity)**.**

$$\langle ij\rangle[ji] = 2p_i \cdot p_j = s_{ij} \tag{2.6}$$

Definition 2.5 (Feynman slash notation)**.** Feynmans slash notation has also a representation in the new formalism

$$\not{p} = \sigma^\mu p_\mu = p^\mu \sigma_\mu = |p\rangle[p| + |p]\langle p|. \tag{2.7}$$

Using the spinor products, we are now able to build polarisation vectors for massless vectors bosons with helicity $h = \pm 1$ [Xu et al., 1987],

$$\varepsilon_+^\mu(p,q) = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \quad \text{and} \quad \varepsilon_-^\mu(p,q) = -\frac{1}{\sqrt{2}}\frac{[q|\overline{\sigma}^\mu|p\rangle}{[qp]},$$

where q is a reference vector with massless momentum $q^2 = 0$. ε_+ is a state with helicity $h = +1$, because $|p]$ in the numerator gives $+1/2$ and $|p\rangle$ in the denominator also gives $+1/2$. The choice of q is a gauge choice and absolutely free as long as $q^2 = 0$ and q is not proportional to p, because else the denominator would vanish. This choice originates from the gauge invariance of the gauge field A_μ^a in the Lagrangian. The results in the end should not depend on any reference momentum q, but intermediate results could. When choosing a different q' as reference momentum, the polarisation vectors are shifted by an amount proportional to p^μ.

3 Color Ordering

Color ordering is a concept to treat the color component of scattering amplitudes. We want to decompose the problem into subproblems. This can be achieved by color ordering. In the Feynman rules of QCD, there is a lot of color dependency.

3.1 QCD gauge group

The gauge group of QCD is $SU(N_c)$ - unitary $N_c \times N_c$ matrices with unit determinant - with $N_c = 3$ the number of colors. The Lie algebra of this group is $\mathfrak{su}(N_c)$ - traceless hermitian $N_c \times N_c$ matrcies. The unit determinant converts to tracelessness, when going to the Lie algebra. The dimension of both the gauge group and its Lie algebra is $N_c^2 - 1$ (the tracelessness condition removes one dimention). We therefore have a basis of the Lie algebra - the generators of the gauge group - denoted by T^a, where a is the adjoint color index going from 1 to $N_c^2 - 1 = 8$.

Proposition 3.1 (Fierz identity)**.** The generators T^a of the gauge group satisfy the following identity [Berends and Giele, 1987]:

$$(T^a)_{i_1}{}^{\bar{\jmath}_1}(T^a)_{i_2}{}^{\bar{\jmath}_2} = \delta_{i_1}{}^{\bar{\jmath}_2}\delta_{i_2}{}^{\bar{\jmath}_1} - \frac{1}{N_c}\delta_{i_1}{}^{\bar{\jmath}_1}\delta_{i_2}{}^{\bar{\jmath}_2} \tag{3.1}$$

Proof. Equation (3.1) is just the statement that the T^a together with the identity form a basis of the $N_c \times N_c$ hermitian matrices. The T^a are a basis of $N_c \times N_c$ traceless hermitian matrices, since the identity is linearly indepdenant of all T^a. An arbitrary hermitian matrix A can therefore be written as

$$A = \sum_{a=1}^{N_c^2-1} b_a T^a + b_n id,$$

where $n = N_c^2$. We now apply the trace on both sides of that equation:

$$tr\,(A) = b_n N_c,$$

where we have used the fact that the T^a are traceless and the trace of the identity is its dimension. We obtain the n-th coefficient $b_n = tr\,(A)\,/N_c$. We can also multiply both sides with T^c and then take the trace and obtain the remaining coefficients:

$$tr\,(AT^c) = b_c,$$

where we again used the tracelessness of the T^a and their normalisation. Therefore we can write the i,j-th entry of the matrix A as

$$A_i{}^j = \sum_{a=1}^{N_c^2-1} (T^a)_k{}^l A_l{}^k (T^a)_i{}^j + \frac{tr\,(A)}{N_c}\delta_i{}^j.$$

Using $tr\,(A) = tr\,(Aid) = A_l{}^k \delta_k{}^l$ and $A_i{}^j = A_l{}^k \delta_i{}^l \delta_j{}^k$, we find

$$A_l{}^k \left[\sum_{a=1}^{N_c^2-1} (T^a)_k{}^l (T^a)_i{}^j + \frac{1}{N_c}\delta_k{}^l \delta_i{}^j - \delta_i{}^l \delta_j{}^k \right] = 0,$$

that holds for all $A_l{}^k$, which implies that the term is the square backets must be 0.

$\square$

Definition 3.1 (Normalisation)**.** We define the normalisation of the generators as

$$tr(T^a T^b) = \delta^{ab}. \tag{3.2}$$

This normalisation, though uncommon in literature, prevents later expressions from unnecessary $\sqrt{2}$ apprearing.

On the other hand, the generators also satisfy the relation

$$[T^a, T^b] = i\sqrt{2} \sum_{c=1}^{N_c^2-1} f^{abc} T^c, \tag{3.3}$$

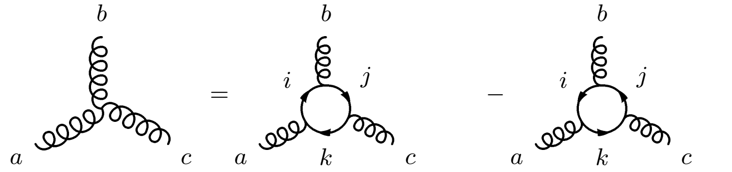
where the f^{abc} are anti-symmetric. The above relation can be solved for the f^{abc}: starting with equation (3.3) and multiplying from the right side with another T^d and taking the trace, we obtain

$$tr(T^a T^b T^d) - tr(T^b T^b T^d) = i\sqrt{2} \sum_{c=1}^{N_c^2-1} f^{abc} tr(T^c T^d). \tag{3.4}$$

Using the definition of the normalisation 3.1, the sum on the right hand side is over a Kronecker delta, therefore only one term is not vanishing

$$f^{abc} = -\frac{i}{\sqrt{2}} \Big(tr(T^a T^b T^c) - tr(T^b T^b T^c) \Big). \tag{3.5}$$

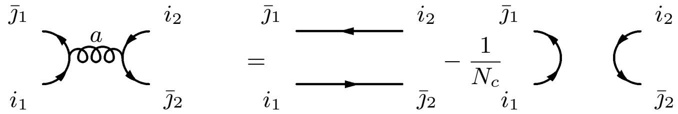
The expression for the structure constants (3.5) can be written pictorially in a very intuitive way taking only color into account:



(3.6)

$$
\begin{aligned}
i\sqrt{2}f^{abc} &= (T^a)_k{}^{\bar{\imath}}(T^b)_i{}^{\bar{\jmath}}(T^c)_j{}^{\bar{k}} - (T^a)_i{}^{\bar{k}}(T^b)_j{}^{\bar{\imath}}(T^c)_k{}^{\bar{\jmath}} \\
i\sqrt{2}f^{abc} &= tr(T^aT^bT^c) - tr(T^aT^cT^b).
\end{aligned}
$$

The structure constant f^{abc} is proportional to the 3-gluon vertex according to the Feynman rules (see appendix C), whereas the loop diagrams on the right hand side produce traces of all involved generators T^a using the Feynman rules for the quark-gloun vertices. The Fierz identity (3.1) can also be illustrated in a similar way:



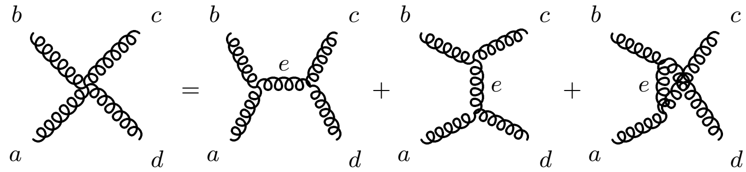
(3.7)

$$
(T^a)_{i_1}{}^{\bar{\jmath}_1}(T^a)_{i_2}{}^{\bar{\jmath}_2} = \delta_{i_1}{}^{\bar{\jmath}_2}\delta_{i_2}{}^{\bar{\jmath}_1} - \frac{1}{N_c}\,\delta_{i_1}{}^{\bar{\jmath}_1}\delta_{i_2}{}^{\bar{\jmath}_2}.
$$

Again the Feynman rules from appendix C where used to derive the pictures. The two equations can therfore be used to rewrite some guonic amplitudes. Let's see this in an example.

3.2 Color decomposition

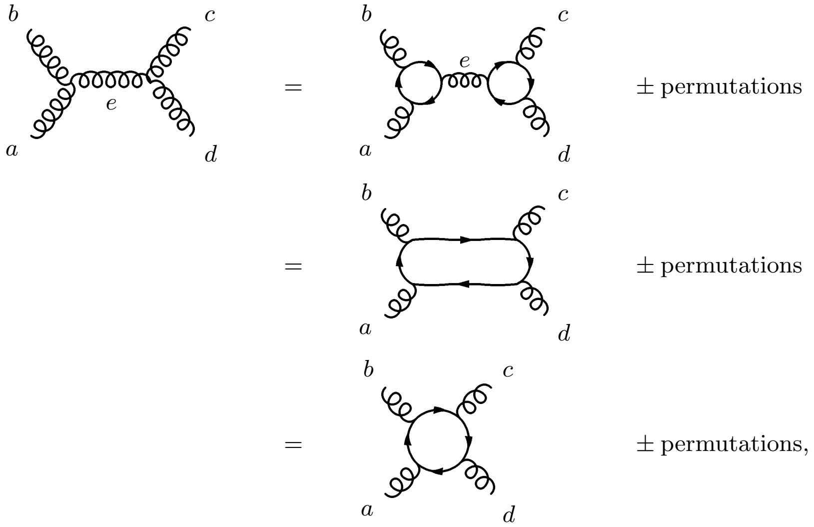
Example 3.1 (4-gluon vertex)**.** In this example, we decompose the 4-gluon vertex into multiple diagrams. We use the two pictorial representations (3.6) and (3.7). The 4-vertex can be decomposed into 3 components of each two 3-vertices by just looking at the Feynman rule:



(3.8)

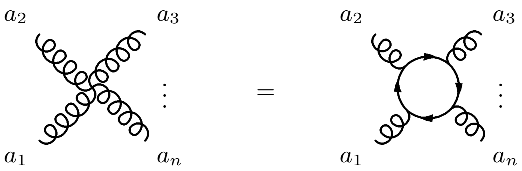
$$
f^{abe}f^{cde} \qquad f^{ade}f^{bce} \qquad f^{ace}f^{bde}.
$$

We decompose the vertex of the first diagram proportional to $f^{abe}f^{cde}$ (the others are analogue)



where we have used equation (3.6) in the first step and (3.7) in the second. It can be seen that the 4-gluon vertex decomposes into many permutations of traces of generators.

This also holds for n gluons,



$\pm$ permutations,

because the scattering amplitude of n gluons consists of 3- and 4-vertices. These vertices can be decomposed according to equation (3.6), equation (3.7) and the above example. In the end, one always ends up in many diagrams whose color part to the amplitude can be written in traces.

Definition 3.2 (Color decomposition)**.** This gives motivation to the following formula for a full pure gluonic tree amplitude:

$$\begin{aligned}
&\mathcal{A}_n^{tree}(\{p_i,h_i,a_i\}) \\
&\quad =g^{n-2} \sum_{\sigma \in S_n/Z_n} tr(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{tree}(p_{\sigma(1)}, \dots, p_{\sigma(n)}, h_{\sigma(1)}, \dots, h_{\sigma(n)}),
\end{aligned} \tag{3.9}$$

where n is the number of involved gluons, g is the strong coupling constant, S_n is the symmetric group, conststing of all permuatations of n objects, Z_n is the cyclic group, consisting of only cyclic permutations of n objects, S_n/Z_n sweeps out all distict cyclic permutations, T^a are the generators of the Lie algebra. As seen, the full amplutide depends on the momenta p_i, helicities h_i and color configurations a_i of all n gluons. The A_n^{tree} objects are not defined yet, these are the so called partial amplitudes and it has to be noticed that they do not depend on any color anymore. The whole color part of the amplitude is carried out into traces of generators and their permutations. Indeed the partial amplitudes still depend on the momenta and helicities, but in an ordered fashion. They recieve contributions only from Feynman diagrams with a particular ordering of gluons. This means that the partial amplitude are color ordered by construction, they need a fixed ordering of external legs. In literature they are also called color ordered amplitudes, dual amplitudes or primitive amplitudes (if one deos not consider loops).

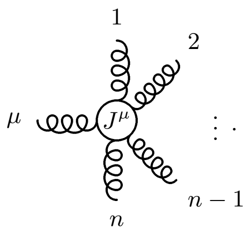
To construct the aprtial aplitudes, there is now a new set of Feynman rules for this: the color ordered Feynman rules (see section B). Notice that, the new Feynman rules now do not depend on any color anymore (no f^{abc} or T^a).

4 Berends-Giele current

We are now ready to formulate the problem. The goal is to calculate the partial amplitude of n gluons scattering at tree level. The recursion idea is the following.

4.1 Recursion idea

We define an auxiliary quantity called J^μ which we can contract with a approprate polarisation vector to obtain the partial amplitude. J^μ called the Berends-Giele-current of off-shell current and it consists of the sum of all color ordered Feynman graphs



The off-shell propagator is also included in J^μ. A pictorial representation of J^μ can be seen in figure 1.

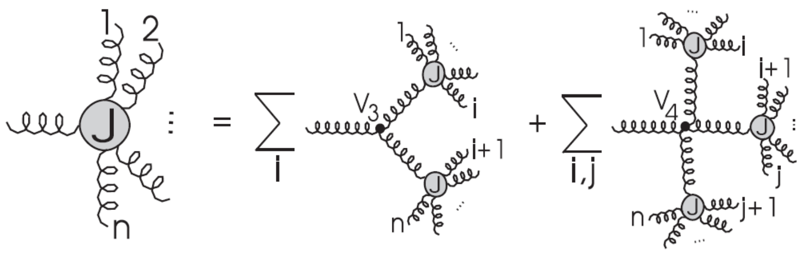


Figure 1: Graphical representation of the recursion relation for the off-shell current J^μ.

The image can be understood as follows. When we follow the off-shell leg μ into the current, we could approach a 3-vertex or a 4-vertex.

If it's a 3-vertex, then the remaining external legs from 1 to n must be split into two subsets to connect the next current to it. Since the diagrams are color ordered, we cannot split the set of n legs into two arbitrary sets. The sets must not be empty and must consist of only consequent numbers - this follows from the color ordering. So there are n possibilities to split them. If we therefore sum over these n possibilities we get the first sum in the term of figure 1.

If it's a 4-vertex, then the remaining external legs must be split into three subsets. The rules for the subsets are the same as for the 3-vertex. We therefore have $n(n-1)$ possitbilities. This can be written in terms of a double sum over two indices i and j and we obtain the second term in figure 1.

Therfore the current can be written as

$$\begin{aligned} J^\mu(1,\dots,n) = \frac{-i}{P_{1,n}^2}\Bigg[&\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1,\dots,i)J_\rho(i+1,\dots,n) \\ &+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} V_4^{\mu\nu\rho\sigma}J_\nu(1,\dots,i)J_\rho(i+1,\dots,j)J_\sigma(j+1,\dots,n)\Bigg] \end{aligned} \tag{4.1}$$

where the V_i are the colour ordered gluon self interaction vertices (see appendix B),

$$V_3^{\mu\nu\rho}(P,Q) = \frac{i}{\sqrt{2}}\left(\eta^{\nu\rho}\left(P-Q\right)^\mu + 2\eta^{\rho\mu}Q^\nu - 2\eta^{\mu\nu}P^\rho\right), \tag{4.2}$$

$$V_4^{\mu\nu\rho\sigma} = \frac{i}{2}\left(2\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right), \tag{4.3}$$

and the currents with only one gluon are the corresponding polarisation vectors

$$J^\mu(i^+) = \varepsilon_+^\mu(p_i,q_i) = \frac{1}{\sqrt{2}}\frac{\langle q_i|\sigma^\mu|p_i]}{\langle q_ip_i\rangle}, \qquad J_\mu(i^+) = \varepsilon_{+\mu}(p_i,q_i) = \frac{1}{\sqrt{2}}\frac{\langle q_i|\sigma_\mu|p_i]}{\langle q_ip_i\rangle}, \tag{4.4a}$$

$$J^\mu(i^-) = \varepsilon_-^\mu(p_i,q_i) = -\frac{1}{\sqrt{2}}\frac{[q_i|\overline{\sigma}^\mu|p_i\rangle}{[q_ip_i]}, \qquad J_\mu(i^-) = \varepsilon_{-\mu}(p_i,q_i) = -\frac{1}{\sqrt{2}}\frac{[q_i|\overline{\sigma}_\mu|p_i\rangle}{[q_ip_i]}. \tag{4.4b}$$

Notice that, the off-shell current J^μ is not gauge invariant - it depends on the reference momenta q_i. The question now is how to get the partial amplitude with this current? Partial amplitudes with $n+1$ external legs are obtained by calculating $J_\mu(1^\pm,\dots,n^\pm)$ for n external legs. Then the off-shell propagator is cut off. The remaining expression is contracted with the $n+1$-th polarisation vectors $\epsilon_\pm^\mu(p_{n+1},q_{n+1})$. This process is called LSZ-reduction [Lehmann et al., 1955]. the full amplitude can be obtained by applying equation (3.9).

Proposition 4.1. If the helicities of all participating gluons are equal, then equation (4.1) reduces to the following form [Berends and Giele, 1988]:

$$J^\mu(1^+,2^+,\dots,n^+) = \frac{\langle q|\,\sigma^\mu \not{P}_{1,n}\,|q\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}. \tag{4.5}$$

where the reference momentum q is the same for all gluons.

Proof. The proof is given by induction. We begin with the base case $n = 2$:

$$
\begin{aligned}
J^\mu(1^+, 2^+) &= \frac{-i}{P_{1,2}^2} V_3^{\mu\nu\rho}(P_{1,1}, P_{2,2}) J_\nu(1^+) J_\rho(2^+) \\
&= \frac{-i}{P_{1,2}^2} \frac{i}{\sqrt{2}} \left(\eta^{\nu\rho}\,(p_1 - p_2)^\mu + 2\eta^{\rho\mu} p_2^\nu - 2\eta^{\mu\nu} p_1^\rho\right) \frac{1}{\sqrt{2}} \frac{\langle q|\sigma_\nu|1]}{\langle q1\rangle} \frac{1}{\sqrt{2}} \frac{\langle q|\sigma_\rho|2]}{\langle q2\rangle} \\
&= \frac{-i}{P_{1,2}^2} \frac{i}{\sqrt{2}} \left(\eta^{\nu\rho}\,(p_1 - p_2)^\mu + 2\eta^{\rho\mu} p_2^\nu - 2\eta^{\mu\nu} p_1^\rho\right) \varepsilon_{+\nu}(p_1, q) \varepsilon_{+\rho}(p_2, q)
\end{aligned}
$$

We will obtain 3 terms. The first one will vanish, after applying the Fierz identity (2.3), since $\langle qq\rangle = [qq] = 0$,

$$
\begin{aligned}
\varepsilon_+(p_1, q) \cdot \varepsilon_+(p_2, q)\,(p_1 - p_2)^\mu &= \frac{1}{2} \frac{\langle q|\sigma^\rho|1]\langle q|\sigma_\rho|2]}{\langle q1\rangle\langle q2\rangle}\,(p_1 - p_2)^\mu \\
&= \frac{\langle qq\rangle[12]}{\langle q1\rangle\langle q2\rangle}\,(p_1 - p_2)^\mu \\
&= 0.
\end{aligned}
$$

The second term evaluates to

$$
\begin{aligned}
2\varepsilon_+^\mu(p_2, q)\; p_2 \cdot \varepsilon_+(p_1, q) &= 2 \frac{1}{\sqrt{2}} \frac{\langle q|\sigma^\mu|2]}{\langle q2\rangle} \frac{1}{\sqrt{2}} \frac{p_2^\nu \langle q|\sigma_\nu|1]}{\langle q1\rangle} \\
&= \frac{\langle q|\sigma^\mu|2]}{\langle q1\rangle\langle q2\rangle} \frac{1}{2} \langle 2|\sigma^\nu|2]\langle q|\sigma_\nu|1] \\
&= \frac{\langle q|\sigma^\mu|2]}{\langle q1\rangle\langle q2\rangle} \langle 2q\rangle[12] \\
&= \frac{\langle q|\sigma^\mu|2]\langle 2q\rangle}{\langle q1\rangle\langle q2\rangle} \frac{P_{1,2}^2}{\langle 21\rangle} \\
&= \frac{\langle q|\sigma^\mu \not{p}_2|q\rangle P_{1,2}^2}{\langle q1\rangle\langle 12\rangle\langle 2q\rangle}.
\end{aligned}
$$

Using this result for the third term gives (exchanging 1 and 2 and adding an overall minus)

$$
\begin{aligned}
-2\varepsilon_+^\mu(p_1, q)\; p_1 \cdot \varepsilon_+(p_2, q) &= -\frac{\langle q|\sigma^\mu \not{p}_1|q\rangle P_{1,2}^2}{\langle q2\rangle\langle 21\rangle\langle 1q\rangle} \\
&= \frac{\langle q|\sigma^\mu \not{p}_1|q\rangle P_{1,2}^2}{\langle q1\rangle\langle 12\rangle\langle 2q\rangle}.
\end{aligned}
$$

Therefore the whole term evaluates to

$$
J^\mu(1^+, 2^+) = \frac{\langle q|\sigma^\mu \not{P}_{1,2}|q\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle\langle 2q\rangle},
$$

which equals equation (4.5) for $n = 2$.

Now, we perform the induction step. Assuming that the proposition holds for $k \leq n-1$, we want to find $J^\mu(1^+, 2^+, \ldots, n^+)$. The V_4 term vanishes, because of

$$
\begin{aligned}
\eta^{\nu\sigma} J_\nu(1, \ldots, i) J_\sigma(j+1, \ldots, n) &\propto \eta^{\nu\sigma}\,\langle q|\,\sigma_\nu \not{P}_{1,i}\,|q\rangle\,\langle q|\,\sigma_\sigma \not{P}_{j+1,n}\,|q\rangle \\
&= \langle q|\,\sigma^\sigma \not{P}_{1,i}\,|q\rangle\,\langle q|\,\sigma_\sigma \not{P}_{j+1,n}\,|q\rangle \\
&\propto \langle qq\rangle \\
&= 0.
\end{aligned}
$$

The same argument also applies to $\eta^{\rho\sigma} J_\rho(i{+}1, \ldots, j) J_\sigma(j{+}1, \ldots, n) = 0$ and $\eta^{\nu\rho} J_\nu(1, \ldots, i) J_\rho(i{+}1, \ldots, j) = 0$. Thus the whole V_4 term evaluates to zero. The first term in V_3 also vanishes, because of the same argument, $\eta^{\nu\rho} J_\nu(1, \ldots, i) J_\rho(i+1, \ldots, n) = 0$. We are left with only the two remaining V_3 terms,

$$
\begin{aligned}
J^\mu(1^+,2^+,\dots,n^+) &= \frac{-i}{P_{1,n}^2}\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1,\dots,i)J_\rho(i+1,\dots,n)\\
&= \frac{\sqrt{2}}{P_{1,n}^2}\sum_{i=1}^{n-1}\left(\eta^{\rho\mu}P_{i+1,n}^\nu - \eta^{\mu\nu}P_{1,i}^\rho\right) J_\nu(1,\dots,i)J_\rho(i+1,\dots,n).
\end{aligned}
$$

By assumption, equation (4.5) holds for values smaller than or equal to $n-1$, hence we can substitute equation (4.5) for J_ν and J_ρ,

$$
\begin{aligned}
&\frac{\sqrt{2}}{P_{1,n}^2}\sum_{i=1}^{n-1}\left(\eta^{\rho\mu}P_{i+1,n}^\nu - \eta^{\mu\nu}P_{1,i}^\rho\right)\frac{\langle q|\,\sigma_\nu \not{P}_{1,i}\,|q\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle\cdots\langle i-1,i\rangle\langle iq\rangle}\frac{\langle q|\,\sigma_\rho \not{P}_{i+1,n}\,|q\rangle}{\sqrt{2}\langle q,i+1\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}\\
&=\frac{1}{\sqrt{2}P_{1,n}^2\langle q1\rangle\langle 12\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}\sum_{i=1}^{n-1}\frac{\langle i,i+1\rangle}{\langle iq\rangle\langle q,i+1\rangle}\\
&\quad\times\Big(\langle q|\,\not{P}_{i+1,n}\not{P}_{1,i}\,|q\rangle\,\langle q|\,\sigma^\mu\not{P}_{i+1,n}\,|q\rangle - \langle q|\,\sigma^\mu\not{P}_{1,i}\,|q\rangle\,\langle q|\,\not{P}_{1,i}\not{P}_{i+1,n}\,|q\rangle\Big)\\
&=\frac{\langle q|\,\sigma^\mu\not{P}_{1,n}\,|q\rangle}{\sqrt{2}P_{1,n}^2\langle q1\rangle\langle 12\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}\sum_{i=1}^{n-1}\frac{\langle i,i+1\rangle}{\langle iq\rangle\langle q,i+1\rangle}\\
&\quad\times\Big(\langle q|\,\sigma^\mu\not{P}_{i+1,n}\,|q\rangle\,\langle q|\,\not{P}_{1,i}\not{P}_{i+1,n}\,|q\rangle - \langle q|\,\not{P}_{i+1,n}\not{P}_{1,i}\,|q\rangle\,\langle q|\,\sigma^\mu\not{P}_{1,i}\,|q\rangle\Big)
\end{aligned}
$$

We use that $\not{a}\not{b} = 2ab\mathbb{I} - \not{b}\not{a}$ and therefore $\langle q|\,\not{P}_{1,i}\not{P}_{i+1,n}\,|q\rangle = -\langle q|\,\not{P}_{i+1,n}\not{P}_{1,i}\,|q\rangle$. We obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{2}P_{1,n}^2\langle q1\rangle\langle 12\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}\sum_{i=1}^{n-1}\frac{\langle i,i+1\rangle}{\langle iq\rangle\langle q,i+1\rangle}\langle q|\,\not{P}_{i+1,n}\not{P}_{1,i}\,|q\rangle\\
&\times\underbrace{\Big(\langle q|\,\sigma^\mu\not{P}_{i+1,n}\,|q\rangle + \langle q|\,\sigma^\mu\not{P}_{1,i}\,|q\rangle\Big)}_{=\langle q|\sigma^\mu\left(\not{P}_{i+1,n}+\not{P}_{1,i}\right)|q\rangle=\langle q|\sigma^\mu\not{P}_{1,n}|q\rangle}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\langle q|\,\not{P}_{i+1,n}\not{P}_{1,i}\,|q\rangle &= \langle q|\,\not{P}_{i+1,n}\left(\not{P}_{1,n}-\not{P}_{i+1,n}\right)|q\rangle\\
&= \langle q|\,\not{P}_{i+1,n}\not{P}_{1,n}\,|q\rangle - \langle q|\,\not{P}_{i+1,n}^2\,|q\rangle\\
&= \langle q|\,\not{P}_{i+1,n}\not{P}_{1,n}\,|q\rangle - \not{P}_{i+1,n}^2\langle qq\rangle\\
&= \langle q|\,\not{P}_{i+1,n}\not{P}_{1,n}\,|q\rangle\,,
\end{aligned}
$$

where we have used $\not{p}\not{p} = p^2\mathbb{I}$. The term then simplifies to

$$
\frac{\langle q|\,\sigma^\mu\not{P}_{1,n}\,|q\rangle}{\sqrt{2}P_{1,n}^2\langle q1\rangle\langle 12\rangle\cdots\langle n-1,n\rangle\langle nq\rangle}\left[\sum_{i=1}^{n-1}\frac{\langle i,i+1\rangle}{\langle iq\rangle\langle q,i+1\rangle}\langle q|\,\not{P}_{i+1,n}\right]\not{P}_{1,n}\,|q\rangle\,. \tag{4.6}
$$

The term in the square brackets can be simplified by the Shouten identity (2.5):

$$
\begin{aligned}
\sum_{i=1}^{n-1}\frac{\langle i,i+1\rangle}{\langle iq\rangle\langle q,i+1\rangle}\langle q|\,\not{P}_{i+1,n} &= \sum_{i=1}^{n-1}\left[\frac{-\langle i+1,q\rangle\,\langle i|\,\not{P}_{i+1,n}}{\langle iq\rangle\langle q,i+1\rangle}+\frac{-\langle qi\rangle\,\langle i+1|\,\not{P}_{i+1,n}}{\langle iq\rangle\langle q,i+1\rangle}\right] &(4.7)\\
&= \sum_{i=1}^{n-1}\left[\frac{\langle i|\,\not{P}_{i,n}}{\langle iq\rangle}-\frac{\langle i+1|\,\not{P}_{i+1,n}}{\langle i+1,q\rangle}\right] &(4.8)\\
&= \frac{\langle 1|\,\not{P}_{1,n}}{\langle 1q\rangle}-\frac{\langle n|\,\not{p}_n}{\langle nq\rangle} &(4.9)\\
&= \frac{\langle 1|\,\not{P}_{1,n}}{\langle 1q\rangle}. &(4.10)
\end{aligned}
$$

To remove the sum, we concluded that the second term in the sum always cancels with the first term of next summand. Thus, leaving only the first term with $i = 1$ and the second term with $i = n-1$. Using this on equation (4.6), we get the desired formula (4.5) for arbitrary n,

$$\begin{aligned} J^\mu(1^+, 2^+, \dots, n^+) &= \frac{\langle q|\, \sigma^\mu \not{P}_{1,n} \,|q\rangle}{\sqrt{2} P_{1,n}^2 \langle q1\rangle\langle 12\rangle \cdots \langle n-1, n\rangle\langle nq\rangle} \frac{\langle 1|\, \not{P}_{1,n}^2 \,|q\rangle}{\langle 1q\rangle} \\ &= \frac{\langle q|\, \sigma^\mu \not{P}_{1,n} \,|q\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle \cdots \langle n-1, n\rangle\langle nq\rangle}. \end{aligned}$$

□

Proposition 4.2. If only the helicity of the first gluon is negative and the rest are positive, then equation (4.1) reduces to the following form [Berends and Giele, 1988]:

$$J^\mu(1^-, 2^+, \dots, n^+) = \frac{\langle 1|\, \sigma^\mu \not{P}_{2,n} \,|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n} \frac{\langle 1|\, \not{p}_m \not{P}_{1,m} \,|1\rangle}{P_{1,m-1}^2 P_{1,m}^2}, \tag{4.11}$$

where the reference momenta are $q_1 = p_2$ and $q_2 = \dots = q_n = p_1$. So the reference momenta for the gluons with same helicities are equal.

Proof. The proof is given by induction. We begin with the case $n = 2$ which will be needed for the base case $n = 3$:

$$\begin{aligned} J^\mu(1^-, 2^+) &= \frac{-i}{P_{1,2}^2} V_3^{\mu\nu\rho}(P_{1,1}, P_{2,2}) J_\nu(1^-) J_\rho(2^+) \\ &= \frac{-i}{P_{1,2}^2} \frac{i}{\sqrt{2}} \left(\eta^{\nu\rho} \left(p_1 - p_2\right)^\mu + 2\eta^{\rho\mu} p_2^\nu - 2\eta^{\mu\nu} p_1^\rho\right) \varepsilon_{-\nu}(p_1, p_2) \varepsilon_{+\rho}(p_2, p_1). \end{aligned}$$

The first term evaluates to zero after applying the Fierz identity,

$$\begin{aligned} \varepsilon_-^\rho(p_1, p_2)\varepsilon_{+\rho}(p_2, p_1) &= \frac{1}{\sqrt{2}} \frac{[2|\overline{\sigma}^\rho|1\rangle}{[21]} \frac{1}{\sqrt{2}} \frac{\langle 1|\sigma_\rho|2]}{\langle 12\rangle} \\ &= \frac{1}{2} \frac{\langle 1|\sigma^\rho|2]\langle 1|\sigma_\rho|2]}{[21]\langle 12\rangle} \\ &= \frac{\langle 11\rangle[22]}{[21]\langle 12\rangle} = 0. \end{aligned}$$

The second and the third term also vanish,

$$\begin{aligned} p_2^\nu \varepsilon_{-\nu}(p_1, p_2) &= -\frac{1}{2} \langle 2|\sigma^\nu|2] \frac{1}{\sqrt{2}} \frac{[2|\overline{\sigma}_\nu|1\rangle}{[21]} \\ &= -\frac{\langle 21\rangle[22]}{\sqrt{2}[21]} = 0. \end{aligned}$$

Therefore, we get

$$J^\mu(1^-, 2^+) = 0. \tag{4.12}$$

Let's look at the $n = 3$ case,

$$\begin{aligned} J^\mu(1^-, 2^+, 3^+) = \frac{-i}{P_{1,3}^2} \Big[& V_3^{\mu\nu\rho}(P_{1,1}, P_{2,3}) J_\nu(1^-) J_\rho(2^+, 3^+) \\ & + V_3^{\mu\nu\rho}(P_{1,2}, P_{3,3}) J_\nu(1^-, 2^+) J_\rho(3^+) \\ & + V_4^{\mu\nu\rho\sigma} J_\nu(1^-) J_\rho(2^+) J_\sigma(3^+) \Big]. \end{aligned}$$

The second V_3 term vanishes, because it is the $n = 2$ case above. TODO: rest

Now, we perform the induction step:

$$J^\mu(1^-,2^+,\dots,n^+) = \frac{-i}{P_{1,n}^2}\left[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1^-,\dots,i^+)J_\rho((i+1)^+,\dots,n^+)\right.$$
$$\left.+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} V_4^{\mu\nu\rho\sigma} J_\nu(1^-,\dots,i^+)J_\rho((i+1)^+,\dots,j^+)J_\sigma((j+1)^+,\dots,n^+)\right]$$

The first V_3 term and all of the V_4 terms vanish, due to the choice of reference momenta and Fierzing. In each term in the double sum of the V_4's and in the first V_3 term, there appears a term proportional to

$$\eta^{\nu\sigma}\langle 1|\sigma_\nu \not{P}_{i,j}|1\rangle\langle 1|\sigma_\sigma \not{P}_{k,l}|1\rangle = -2\langle 11\rangle\langle 1|\not{P}_{k,l}\not{P}_{i,j}|1\rangle = 0,$$

where $i\le j, k\le l$ are some indices. The sum over i can actually be written as starting from 3 rather than 1, because the two cases $i=1$ and $i=2$, we will treat separately. Let's first examine $i=1$,

$$\begin{aligned}
I_1 &= \frac{-i}{P_{1,n}^2}V_3^{\mu\nu\rho}(p_1,P_{2,n})J_\nu(1^-)J_\rho(2^+,\dots,n^+)\\
&= -\frac{1}{\sqrt{2}P_{1,n}^2}\left(2\eta^{\rho\mu}P_{2,n}^\nu - 2\eta^{\mu\nu}p_1^\rho\right)\frac{1}{\sqrt{2}}\frac{[q_1|\overline{\sigma}_\nu|1\rangle}{[q_1,1]}\frac{\langle 1|\,\sigma_\rho\not{P}_{2,n}\,|1\rangle}{\sqrt{2}\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}\\
&= -\frac{\langle 1|\not{P}_{2,n}|q_1]\,\langle 1|\,\sigma^\mu\not{P}_{2,n}\,|1\rangle - \langle 1|\sigma^\mu|q_1]\,\langle 1|\,\not{p}_1\not{P}_{2,n}\,|1\rangle}{P_{1,n}^2\sqrt{2}[q_1,1]\langle 12\rangle\cdots\langle n1\rangle}\\
&= -\frac{\langle 1|\not{P}_{2,n}|q_1]}{[q_1,1]P_{1,n}^2}\frac{\langle 1|\,\sigma^\mu\not{P}_{2,n}\,|1\rangle}{\sqrt{2}\langle 12\rangle\cdots\langle n1\rangle}.
\end{aligned}$$

Using the spinor product (2.6) and the slash notation (2.7),

$$\begin{aligned}
-\frac{\langle 1|\not{P}_{2,n}|q_1]}{[q_1,1]P_{1,n}^2} &= \frac{\langle 1|\not{P}_{2,n}|q_1]\langle q_1,1\rangle}{(q_1+p_1)^2\,P_{1,n}^2}\\
&= \frac{\langle 1|\not{P}_{2,n}\not{q}_1|1\rangle}{(q_1+p_1)^2\,P_{1,n}^2}.
\end{aligned}$$

The reference momentum q_1 was not chosen until now. By staring at this, we see that, if the reference momentum choice is $q_1 = P_{2,n-1}$, then this looks very similar to the $m=n$ term in the sum of equation (4.11),

$$\begin{aligned}
\frac{\langle 1|\not{P}_{2,n}\not{q}_1|1\rangle}{(q_1+p_1)^2\,P_{1,n}^2} &= \frac{\langle 1|\not{P}_{2,n}\not{P}_{2,n-1}|1\rangle}{P_{1,n-1}^2P_{1,n}^2}\\
&= \frac{\langle 1|\not{P}_{1,n}\not{P}_{1,n-1}|1\rangle}{P_{1,n-1}^2P_{1,n}^2}\\
&= -\frac{\langle 1|\not{P}_{1,n}\not{p}_n|1\rangle}{P_{1,n-1}^2P_{1,n}^2}\\
&= \frac{\langle 1|\not{p}_n\not{P}_{1,n}|1\rangle}{P_{1,n-1}^2P_{1,n}^2}.
\end{aligned}$$

To summarize, the $i=1$ case:

$$I_1 = \frac{\langle 1|\,\sigma^\mu\not{P}_{1,n}\,|1\rangle}{\sqrt{2}\langle 12\rangle\cdots\langle n1\rangle}\frac{\langle 1|\not{p}_n\not{P}_{1,n}|1\rangle}{P_{1,n-1}^2P_{1,n}^2}. \tag{4.13}$$

The case for $i=2$ vanishes, because of equation (4.12),

$$I_2 = V_3^{\mu\nu\rho}(P_{1,2},P_{3,n})J_\nu(1^-,2^+)J_\rho(3^+,\dots,n^+) = 0.$$

Back to the full term, we are left with the two terms in V_3, where the reference momentum for J_σ is p_1 and the sum over i omitting the $i=2$ case (the $i=1$ case is taken out in front of the sum),

$$\begin{aligned} J^\mu(1^-,2^+,\ldots,n^+) &= I_1 + \frac{-i}{P_{1,n}^2}\left[\sum_{i=3}^{n-1}\frac{i}{\sqrt{2}}\left(2\eta^{\rho\mu}P^\nu_{i+1,n}-2\eta^{\mu\nu}P^\rho_{1,i}\right)\right.\\ &\quad\times\left.\left(\frac{\langle 1|\sigma_\nu \not{P}_{2,i}|1\rangle}{\sqrt{2}\langle 12\rangle\cdots\langle i1\rangle}\sum_{m=3}^{i}\frac{\langle 1|\not{p}_m\not{P}_{1,m}|1\rangle}{P^2_{1,m-1}P^2_{1,m}}\right)\left(\frac{\langle 1|\sigma_\rho\not{P}_{i+1,n}|1\rangle}{\sqrt{2}\langle 1,i+1\rangle\cdots\langle n1\rangle}\right)\right]\\ &= I_1+\frac{1}{P^2_{1,n}}\left[\sum_{i=3}^{n-1}\sum_{m=3}^{i}\frac{\langle 1|\not{p}_m\not{P}_{1,m}|1\rangle}{P^2_{1,m-1}P^2_{1,m}}\right.\\ &\quad\times\left.\frac{\langle 1|\not{P}_{i+1,n}\not{P}_{1,i}|1\rangle\langle 1|\sigma^\mu\not{P}_{i+1,n}|1\rangle-\langle 1|\sigma^\mu\not{P}_{1,i}|1\rangle\langle 1|\not{P}_{1,i}\not{P}_{i+1,n}|1\rangle}{\sqrt{2}\langle 12\rangle\cdots\langle i1\rangle\langle 1,i+1\rangle\cdots\langle n1\rangle}\right]. \end{aligned}$$

The two sums can be interchanged in the way illustrated by figure 2. We can therefore switch the two sums, according to

$$\sum_{i=3}^{n-1}\sum_{m=3}^{i}=\sum_{m=3}^{n-1}\sum_{i=m}^{n-1}.$$

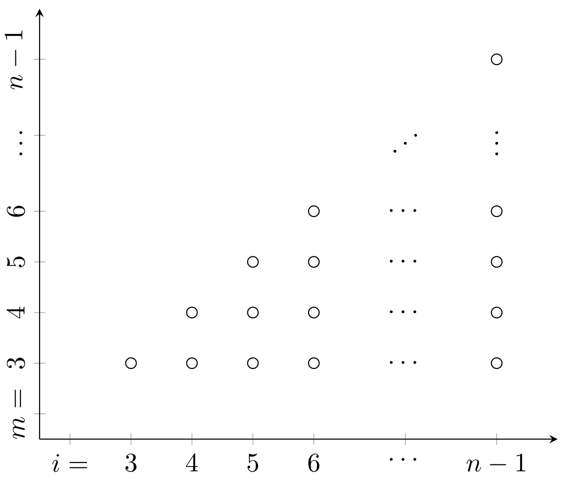


Figure 2: From the two sums, we can see that if $i=3$, then m takes values from 3 to i, so m can only be 3. When i increments to $i=4$, then m can take values 3 and 4 and so on. For $i=n-1$, m can take all values between 3 to $n-1$. If we sum over m before summing over i, we can read off the limits from the plot above. Namely m starts at 3 and go up to $n-1$, whereas i always starts from m and goes up to $n-1$.

The numerator can be simplyfied by using $\langle 1|\not{P}_{1,i}\not{P}_{i+1,n}|1\rangle = -\langle 1|\not{P}_{i+1,n}\not{P}_{1,i}|1\rangle$ and $\not{P}_{1,i} = \not{P}_{1,n}-\not{P}_{i+1,n}$, therefore $\langle 1|\not{P}_{i+1,n}\not{P}_{1,i}|1\rangle = \langle 1|\not{P}_{i+1,n}\not{P}_{1,n}|1\rangle$, where the right half is independent of i. The denominator can also be written as one part independent and one dependent of i,

$$\frac{1}{\langle 12\rangle\cdots\langle i1\rangle\langle 1,i+1\rangle\cdots\langle n1\rangle}=\frac{1}{\langle 12\rangle\cdots\langle n1\rangle}\frac{\langle i,i+1\rangle}{\langle i1\rangle\langle 1,i+1\rangle}.$$

Pulling all terms indepenent on i together out of the sum on i and switching the sums, results in

$$J^\mu(1^-,2^+,\ldots,n^+) = I_1+\frac{1}{P^2_{1,n}}\left[\frac{\langle 1|\sigma^\mu\not{P}_{1,n}|1\rangle}{\sqrt{2}\langle 12\rangle\cdots\langle n1\rangle}\sum_{m=3}^{n-1}\frac{\langle 1|\not{p}_m\not{P}_{1,m}|1\rangle}{P^2_{1,m-1}P^2_{1,m}}\right.$$

$$\times \sum_{i=m}^{n-1} \frac{\langle i, i+1\rangle}{\langle i1\rangle\langle 1, i+1\rangle} \langle 1|\not{P}_{i+1,n}\not{P}_{1,n}|1\rangle \Bigg].$$

We can evaluate the term in the second line that is dependent on i, which has appeared before in (4.10), and inserting equation (4.13) for I_1, concluding the proof,

$$\begin{aligned}
J^\mu(1^-, 2^+, \ldots, n^+) = I_1 + \frac{1}{P_{1,n}^2} &\left[\frac{\langle 1|\sigma^\mu \not{P}_{1,n}|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n-1} \frac{\langle 1|\not{p}_m \not{P}_{1,m}|1\rangle}{P_{1,m-1}^2 P_{1,m}^2} \right. \\
&\left. \times \frac{\langle m|\not{P}_{1,n}}{\langle m1\rangle} \not{P}_{1,n}|1\rangle \right] \\
&= I_1 + \frac{\langle 1|\sigma^\mu \not{P}_{1,n}|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n-1} \frac{\langle 1|\not{p}_m \not{P}_{1,m}|1\rangle}{P_{1,m-1}^2 P_{1,m}^2} \\
&= \frac{\langle 1|\,\sigma^\mu \not{P}_{1,n}\,|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \frac{\langle 1|\not{p}_n \not{P}_{1,n}|1\rangle}{P_{1,n-1}^2 P_{1,n}^2} + \frac{\langle 1|\sigma^\mu \not{P}_{1,n}|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n-1} \frac{\langle 1|\not{p}_m \not{P}_{1,m}|1\rangle}{P_{1,m-1}^2 P_{1,m}^2} \\
&= \frac{\langle 1|\sigma^\mu \not{P}_{1,n}|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n} \frac{\langle 1|\not{p}_m \not{P}_{1,m}|1\rangle}{P_{1,m-1}^2 P_{1,m}^2}.
\end{aligned}$$

□

4.2 Parke Taylor formula

Vector particles such as gluons do not have a conserved helicity. The most helicity violating amplitudes are zero at tree level. The first non-zero amplitudes are when two helicities have the opposite sign of the other $n-2$ gluons. These amplitudes are called MHV amplitudes (maximally helicity violating).

Theorem 4.3 (Parke Taylor [Parke and Taylor, 1986]). The tree-level amplitude for the first two gluons with negative helicity and the rest positive helicity can be written as

$$A_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 12\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}. \tag{4.14}$$

This formula was first speculated by Parke and Taylor and also numerically proven up to $n = 6$. It was first prooven by Berends and Giele [Berends and Giele, 1988].

Proof. To obtain amplitudes out of off-shell currents J_μ, we need to multiply the current by $iP_{1,n}^2$ and then contract it with an appropriate polarisation vector ε^μ which is on the mass-shell. Multiplication by $iP_{1,n}^2$ is needed, because the off-shell current is defined to include the propagator of external off-shell leg μ. In this way we could also achieve amplitudes with external quarks, instead of gluons. However, we then have $p_{n+1}^2 = P_{1,n}^2 = 0$. Therefore, we begin with

$$A_n^{tree}(1^-, 2^+, \ldots, n^+, (n+1)^-) \tag{4.15}$$

$$= iP_{1,n}^2 \varepsilon_-^\mu(p_{n+1}, q_{n+1}) J_\mu(1^-, 2^+, \ldots, n^+) \tag{4.16}$$

$$= -iP_{1,n}^2 \frac{1}{\sqrt{2}} \frac{[q_{n+1}|\overline{\sigma}^\mu|p_{n+1}\rangle}{[q_{n+1}, p_{n+1}]} \frac{\langle 1|\,\sigma_\mu \not{P}_{2,n}\,|1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n} \frac{\langle 1|\,\not{p}_m \not{P}_{1,m}\,|1\rangle}{P_{1,m-1}^2 P_{1,m}^2} \tag{4.17}$$

We used proposition 4.2 to express J_μ. The $\not{P}_{2,n}\,|1\rangle$ in the middle term, can actually the written as $\not{P}_{1,n}\,|1\rangle$ because the $|i\rangle$ are solutions to the massless Dirac equation, $\not{p}\,|p\rangle = 0$. We can also use (2.4) switch q and p in the numerator of the polarisation vector. Since $P_{1,n}^2 = 0$, all terms in the sum vanish except when $m = n$ - in this case $P_{1,n}^2$ is cancelled from the denominator.

Next, we evaluate

$$\not{P}_{1,n}\,|1\rangle = -\not{p}_{n+1}\,|1\rangle \tag{4.18}$$

$$= -\left(|n+1\rangle[n+1| + |n+1]\langle n+1|\right)|1\rangle \tag{4.19}$$

$$= -|n+1]\langle n+1, 1\rangle \tag{4.20}$$

$$= |n+1]\langle 1, n+1\rangle, \tag{4.21}$$

where we have used momentum conservation in the first step (4.18). From (4.18) to (4.19) definition (2.7) was inserted. In the last step, we used the antisymmetry of the spinor product. In the sum we have ($m = n$),

$$\begin{aligned} \not{p}_n \not{P}_{1,n} |1\rangle &= \not{p}_n |n+1]\langle 1, n+1\rangle & (4.22)\\ &= (|n\rangle[n| + |n]\langle n|)\, |n+1]\langle 1, n+1\rangle & (4.23)\\ &= |n\rangle[n, n+1]\langle 1, n+1\rangle, & (4.24) \end{aligned}$$

in an analogous way as before. Using these equations, we obtain (writing $|p_i\rangle$ as $|i\rangle$)

$$\begin{aligned} & A_n^{tree}(1^-, 2^+, \ldots, n^+, (n+1)^-) & (4.25)\\ &= -i\frac{1}{2}\frac{\langle n+1|\sigma^\mu|q_{n+1}]}{[q_{n+1}, n+1]}\frac{\langle 1|\sigma_\mu|n+1]\langle 1, n+1\rangle}{\langle 12\rangle\cdots\langle n1\rangle}\frac{\langle 1n\rangle[n, n+1]\langle 1, n+1\rangle}{P^2_{1,n-1}}. & (4.26) \end{aligned}$$

We have momentum conservation, therfore

$$\begin{aligned} P_{1,n-1} = -P_{n,n+1} \implies P^2_{1,n-1} &= P^2_{n,n+1} & (4.27)\\ &= (p_n + p_{n+1})^2 & (4.28)\\ &= s_{n,n+1} & (4.29)\\ &= \langle n, n+1\rangle[n+1, n], & (4.30) \end{aligned}$$

where we have used (2.6) in the last step. Applying the Fierz identity (2.3) to (4.26), we finally get

$$\begin{aligned} & A_n^{tree}(1^-, 2^+, \ldots, n^+, (n+1)^-) & (4.31)\\ &= -i\frac{\langle n+1, 1\rangle[n+1, q_{n+1}]\langle 1, n+1\rangle}{[q_{n+1}, n+1]\langle 12\rangle\cdots\langle n1\rangle}\frac{\langle 1n\rangle[n, n+1]\langle 1, n+1\rangle}{\langle n, n+1\rangle[n+1, n]} & (4.32)\\ &= -i\frac{-\langle 1, n+1\rangle[n+1, q_{n+1}]\langle 1, n+1\rangle}{-[n+1, q_{n+1}]\langle 12\rangle\cdots\langle n1\rangle}\frac{-\langle n1\rangle[n, n+1]\langle 1, n+1\rangle}{-\langle n, n+1\rangle[n, n+1]} & (4.33)\\ &= -i\frac{\langle 1, n+1\rangle^3}{\langle 12\rangle\cdots\langle n, n+1\rangle}\frac{-\langle 1, n+1\rangle}{\langle n+1, 1\rangle} & (4.34)\\ &= i\frac{\langle 1, n+1\rangle^4}{\langle 12\rangle\cdots\langle n+1, 1\rangle}. & (4.35) \end{aligned}$$

In the last step we multiplied by 1 to get the desired structure of the Parke Taylor formula. Notice that gluons with negative helicities, 1 and $n+1$, are adjacent. The formula for n gluons can therefore also be written as

$$A_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}, \tag{4.36}$$

by relabelling the momenta.

□

Notice that, the general Parke Taylor formula reads

$$A_n^{tree}(1^+, \ldots, j^-, \ldots k^-, \ldots, n^+) = i\frac{\langle jk\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}, \tag{4.37}$$

where j and k may not be adjacent.

5 Implementation

TODO

6 Summary

TODO

References

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[Berends et al., 1981] Berends, F. A., Kleiss, R., De Causmaecker, P., Gastmans, R., and Wu, T. T. (1981). Single bremsstrahlung processes in gauge theories. *Physics Letters B*, 103(2):124–128.

[Dixon, 1996] Dixon, L. (1996). Calculating scattering amplitudes efficiently. *arXiv preprint hep-ph/9601359*.

[Gruber, 2019] Gruber, R. (2019). Github repository: Source code of the implementation. `http://github.com/chaoos/report-proseminar-hs2019`. Accessed: 2019-12-03.

[Lehmann et al., 1955] Lehmann, H., Symanzik, K., and Zimmermann, W. (1955). Zur formulierung quantisierter feldtheorien. *Il Nuovo Cimento (1955-1965)*, 1(1):205–225.

[Mangano, 1999] Mangano, M. L. (1999). Introduction to qcd. Technical report, CERN.

[Parke and Taylor, 1986] Parke, S. J. and Taylor, T. R. (1986). Amplitude for n-gluon scattering. *Phys. Rev. Lett.*, 56:2459–2460.

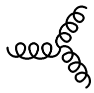
[Xu et al., 1987] Xu, Z., Zhang, D.-H., and Chang, L. (1987). Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories. *Nucl. Phys.*, B291:392–428.

Appendices

A QCD Feynman rules

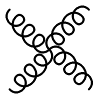
The Feynman rules for QCD in Feynman gauge are [Mangano, 1999]

p, b, ν

k, a, μ  $= \frac{ig}{\sqrt{2}} f^{abc} \left[\eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu} + \eta^{\mu\nu}(k-p)^{\rho}\right],$

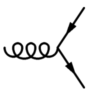
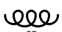
q, c, ρ

a, μ $\quad$ b, ν

 $= -ig^2 \begin{bmatrix} f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ +f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\rho\nu}) \\ +f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\sigma\nu}) \end{bmatrix},$

d, σ $\quad$ c, ρ

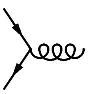
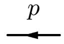
f, i

a, μ  $= -\frac{ig\gamma^{\mu}\delta_f^{f'}}{\sqrt{2}}(T^a)_i{}^{\bar{\jmath}},$ $\qquad$ a, μ  $b, \nu = -\frac{i\delta^{ab}\eta^{\mu\nu}}{p^2},$

p

$f', \bar{\jmath}$

f, i

 $a, \mu = \frac{ig\gamma^{\mu}\delta_f^{f'}}{\sqrt{2}}(T^a)_i{}^{\bar{\jmath}},$ $\qquad$ f, i  $f', j = \frac{i\delta_f^{f'}\delta_j^i}{p^2}.$

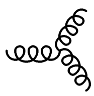
p

$f', \bar{\jmath}$

B Color ordered QCD Feynman rules

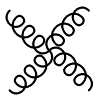
The color ordered Feynman rules for QCD in Feynman gauge are [Dixon, 1996]

p, ν

k, μ  $= \frac{i}{\sqrt{2}} \left[\eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu} + \eta^{\mu\nu}(k-p)^{\rho}\right],$

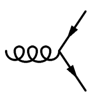
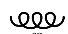
q, ρ

μ $\quad$ ν

 $= \frac{i}{2}\left[2\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right],$

σ $\quad$ ρ

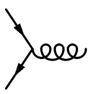
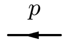
f

μ  $= -\frac{i\gamma^{\mu}\delta_f^{f'}}{\sqrt{2}},$ $\qquad$ μ  ν $= -\frac{i\eta^{\mu\nu}}{p^2},$

p

f'

f

 μ $= \frac{i\gamma^{\mu}\delta_f^{f'}}{\sqrt{2}},$ $\qquad$ f  f' $= \frac{i\delta_f^{f'}}{p^2}.$

p

f'

C Code

All code used in this report can be found in the GitHub repository [Gruber, 2019]