Computational Methods in QFT

Berends-Giele Recursion

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- Only deal with kinematic variables

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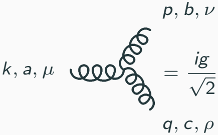
- QCD Lagrangian [Eidemueller et al., 2000]:

$$L = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi - \frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a}$$

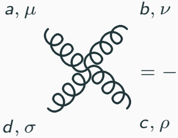
$$G^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu}$$

$$D_{\mu} = \partial_{\mu} - i g T_{a} A^{a}_{\mu}$$

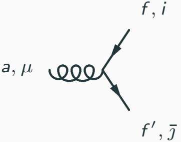
QCD Feynman rules [Mangano, 1999]



$$= \frac{ig}{\sqrt{2}} f^{abc} \left[\eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu} + \eta^{\mu\nu}(k-p)^{\rho} \right],$$

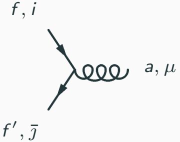


$$= -ig^2 \left[\begin{array}{r} f^{abe} f^{cde} (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ + f^{ace} f^{bde} (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\rho\nu}) \\ + f^{ade} f^{bce} (\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\rho}\eta^{\sigma\nu}) \end{array} \right],$$



$$= -\frac{ig\gamma^{\mu}\delta_{f}^{f'}}{\sqrt{2}} (T^{a})_{i}{}^{\bar{\jmath}},$$

$$= -\frac{i\delta^{ab}\eta^{\mu\nu}}{p^2},$$



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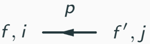

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$$f^{abc} = -\frac{i}{\sqrt{2}}\Big(Tr(T^a T^b T^c) - Tr(T^a T^c T^b)\Big) \tag{2}$$

Color Decopmposition of tree amplitudes

Treat color degrees of freedom by separating them from kinematical parts → partial amplitudes [Berends and Giele, 1987]

$$\begin{aligned}&\mathcal{A}_n^{tree}(\{p_i, h_i, a_i\})\\ &\quad = g^{n-2} \sum_{\sigma \in S_n/Z_n} Tr(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_n^{tree}(p_{\sigma(1)}, \ldots, p_{\sigma(n)}, h_{\sigma(1)}, \ldots, h_{\sigma(n)})\end{aligned} \tag{3}$$

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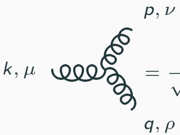
- Singularities and poles can only occure in adjacent momenta

Color ordered Feynman rules [Johansson and Ochirov, 2016]



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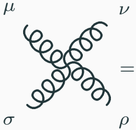

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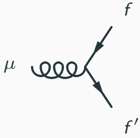

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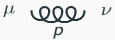

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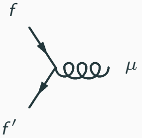

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Spinor helicity formalism for massless vector bosons

Spinor helicity formalism [Dixon, 2016], [Berends et al., 1981] - motivation

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Organize spin quantum numbers for external states
$\rightarrow$ helicity basis

$$p_i^\mu, h_i \rightarrow u_-(p_i) = \begin{pmatrix} \lambda_{i\alpha} \\ 0 \\ 0 \end{pmatrix} \leftrightarrow |i\rangle = \lambda_{i\alpha} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$u_+(p_i) = \begin{pmatrix} 0 \\ 0 \\ \tilde{\lambda}_i^{\dot{\alpha}} \end{pmatrix} \leftrightarrow |i] = \tilde{\lambda}_i^{\dot{\alpha}} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

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Define spinor products:

$$\langle ij\rangle \equiv \lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta$$

$$[ij] \equiv \tilde{\lambda}_{i\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{j\dot{\beta}}$$

with

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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- Anti-symmetry: $\langle ij\rangle = -\langle ji\rangle$, $[ij] = -[ji]$, $\langle ii\rangle = [ii] = 0$

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- Polarization vectors [Xu et al., 1987]:

$$\varepsilon_+^\mu(p,q) = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \quad \text{and} \quad \varepsilon_-^\mu(p,q) = -\frac{1}{\sqrt{2}}\frac{[q|\overline{\sigma}^\mu|p\rangle}{[qp]}$$

Parke-Taylor Formula / Berends-Giele Recursion

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General Parke-Taylor Formula [Berends and Giele, 1988]:

$$A_{jk}^{tree,MHV} = A_n^{tree}(1^+, \ldots, j^-, \ldots, k^-, \ldots, n^+) = i\frac{\langle jk\rangle^4}{\langle 12\rangle \ldots \langle n1\rangle}$$

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MHV: maximally helicity violating amplitudes

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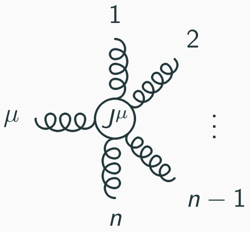
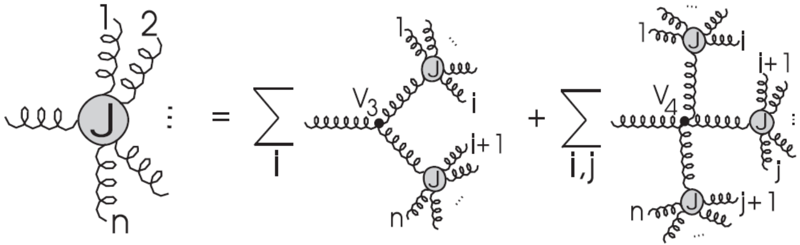


Recursion Idea 2/2 [Berends and Giele, 1988]



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$$\begin{aligned}\mathcal{A}_n^{tree}(\{p_i, h_i, a_i\}) \\ &= g^{n-2} \sum_{\sigma \in S_n / Z_n} Tr(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{tree}(p_{\sigma(1)}, \dots, p_{\sigma(n)}, h_{\sigma(1)}, \dots, h_{\sigma(n)})\end{aligned}$$

Special case 1/2

If only the helicity of the first gluon is negative and the rest are positive, then the equation for J^μ reduces to the following form [Berends and Giele, 1988]:

$$J^\mu(1^-, 2^+, \ldots, n^+) = \frac{\langle 1|\, \sigma^\mu \not{P}_{2,n}\, |1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n} \frac{\langle 1|\, \not{p}_m \not{P}_{1,m}\, |1\rangle}{P_{1,m-1}^2 P_{1,m}^2}, \qquad (4)$$

where the reference momenta are $q_1 = p_2$ and $q_2 = \cdots = q_n = p_1$.

Special case 2/2

If the helicities of all participating gluons are equal, then the equation for J^μ reduces to the following form [Berends and Giele, 1988]:

$$J^\mu(1^+, 2^+, \ldots, n^+) = \frac{\langle q|\, \sigma^\mu \not{P}_{1,n}\, |q\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle \cdots \langle n-1, n\rangle\langle nq\rangle}. \tag{5}$$

where the reference momentum q is the same for all gluons.

Proof of Parke-Taylor Formula [Berends and Giele, 1988]

$$A_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle \ldots \langle n1\rangle}$$

$$J^\mu(1^-, 2^+, \ldots, n^+) = \frac{\langle 1|\, \sigma^\mu \not{P}_{2,n}\, |1\rangle}{\sqrt{2}\langle 12\rangle \cdots \langle n1\rangle} \sum_{m=3}^{n} \frac{\langle 1|\, \not{p}_m \not{P}_{1,m}\, |1\rangle}{P_{1,m-1}^2 P_{1,m}^2}$$

Spinor helicity formalism - important identities

- Anti-symmetry: $\langle ij\rangle = -\langle ji\rangle$, $[ij] = -[ji]$, $\langle ii\rangle = [ii] = 0$
- Squaring: $\langle ij\rangle[ji] = 2p_i \cdot p_j = (p_i + p_j)^2 =: s_{ij}$
- 4-momentum: $p^\mu = \frac{1}{2}\langle p|\sigma^\mu|p]$
- Slash: $\not{p} = |p\rangle[p| + |p]\langle p|$
- Fierz: $\langle i|\sigma^\mu|j]\langle k|\sigma_\mu|\ell] = 2\langle ik\rangle[\ell j]$
- Charge conj.: $\langle i|\sigma^\mu|j] = [j|\overline{\sigma}^\mu|i\rangle$
- Shouten: $\langle ij\rangle\langle k| + \langle jk\rangle\langle i| + \langle ki\rangle\langle j| = 0$
- Polarization vectors [Xu et al., 1987]:

$$\varepsilon_+^\mu(p,q) = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle} \quad \text{and} \quad \varepsilon_-^\mu(p,q) = -\frac{1}{\sqrt{2}}\frac{[q|\overline{\sigma}^\mu|p\rangle}{[qp]}$$

Numerical demonstration

References I

Berends, F. A. and Giele, W. (1987).
The six-gluon process as an example of weyl-van der waerden spinor calculus.
Nuclear Physics B, 294:700–732.

Berends, F. A. and Giele, W. (1988).
Recursive calculations for processes with n gluons.
Nuclear Physics B, 306(4):759–808.

Berends, F. A., Kleiss, R., De Causmaecker, P., Gastmans, R., and Wu, T. T. (1981).
Single bremsstrahlung processes in gauge theories.
Physics Letters B, 103(2):124–128.

References II

Dixon, L. (1996).
Calculating scattering amplitudes efficiently.
arXiv preprint hep-ph/9601359.

Dixon, L. J. (2016).
A brief introduction to modern amplitude methods.
In *Journeys Through the Precision Frontier: Amplitudes for Colliders: TASI 2014 Proceedings of the 2014 Theoretical Advanced Study Institute in Elementary Particle Physics*, pages 39–97. World Scientific.

Eidemueller, M., Dosch, H. G., and Jamin, M. (2000).
The field strength correlator from qcd sum rules.
Nuclear Physics B-Proceedings Supplements, 86(1-3):421–425.

References III

Johansson, H. and Ochirov, A. (2016).
Color-kinematics duality for qcd amplitudes.
Journal of High Energy Physics, 2016(1):170.

Lehmann, H., Symanzik, K., and Zimmermann, W. (1955).
Zur formulierung quantisierter feldtheorien.
Il Nuovo Cimento (1955-1965), 1(1):205–225.

Mangano, M. L. (1999).
Introduction to qcd.
Technical report, CERN.

Parke, S. J. and Taylor, T. R. (1986).
Amplitude for n-gluon scattering.
Phys. Rev. Lett., 56:2459–2460.

References IV

Xu, Z., Zhang, D.-H., and Chang, L. (1987).
Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories.
Nucl. Phys., B291:392–428.

Questions?

Pauli Matrices

$$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$$
$$\overline{\sigma}^\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3)$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$