Recurrence

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First order linear recurrence:

$$X_{n+1} = b_n X_n + a_n \qquad n \geq m$$

in which a_n, b_n and X_m are known.
Then we have the following general form to calculate X_{n+1}

$$\begin{aligned} X_{n+1} &= \sum_{j=m}^{n} a_j \prod_{i=j+1}^{n} b_i + x_m \prod_{i=m}^{n} b_i \\ &= a_n(\sum_{j=m}^{n-1} a_j \prod_{i=j+1}^{n-1} b_i + x_m \prod_{i=m}^{n-1} b_i) + b_n \end{aligned}$$

Using the above formulation, let's look at an interesting example:
Assuming a loan mortagage instance, which has the following definitions:

A - loan amount
m - monthly payment
ρ - monthly interest rate
R_n - principal at the end of month n
$R_0 = A$
$M = 360$ - loan duration

Each month the remaining money we have to pay is:

$$R_{n+1} = R_n(1 + \rho) - m$$

Then we have $R_M = 0$ because we have no money to pay when we have paid M months and paid off the loan. And now we want to calculate the monthly payment m.

$$
\begin{aligned}
R_M &= \sum_{j=0}^{M-1} (-m) \prod_{i=j+1}^{M-1} (1+\rho) + A(1+\rho)^M \\
&= \sum_{j=0}^{M-1} (-m)(1+\rho)^{M-j-1} + A(1+\rho)^M \\
&= (1+\rho)^M (A - m \sum_{j=0}^{M-1} \frac{1}{(1+\rho)^M} \\
&= (1+\rho)^M A - \frac{m}{\rho}((1+\rho)^M - 1) = 0
\end{aligned}
$$

So

$$m = \frac{A \cdot \rho(1+\rho)^M}{(1+\rho)^M - 1}$$

When we have the following instance:

$$
\begin{aligned}
A &= 100,000 \\
\rho &= \frac{0.06}{12} = 0.005
\end{aligned}
$$

Then we have $m = 598.98$.

Next, let's continue. First, give some calculation.

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} = \frac{n}{m}\binom{n-1}{m-1}$$

Proof:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{1}$$

$$\binom{n-1}{m-1} = \frac{(n-1)!}{(n-m)!(m-1)!} \tag{2}$$

$$\binom{n-1}{m} = \frac{(n-1)!}{(n-m-1)!m!} \tag{3}$$

(2)$\cdot \frac{n}{m}$ apparently equals to (1)
Define S_n the summation of the first n terms of $f(n)$, so

$$S_n = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n-1} f(k) + f(n)$$

Then if we are given:

$$a_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}$$

what can we derive from the above equation?
Using the above conclusion:

$$
\begin{aligned}
a_n &= \sum_{k=1}^{n}\left(\binom{n-1}{k-1}\frac{(-1)^{k+1}}{k}+\binom{n-1}{k}\frac{(-1)^{k+1}}{k}\right)\\
&= \underbrace{\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-1)^{k+1}}{k}}_{a_{n-1}}+\sum_{k=1}^{n}\frac{(-1)^{k+1}}{n}\binom{n}{k}\\
&= a_{n-1}-\frac{1}{n}\underbrace{\sum_{k=1}^{n}\binom{n}{k}\cdot(-1)^k}_{\text{binomial distribution}}\\
&= a_{n-1}+\frac{1}{n}
\end{aligned}
$$

Then we know that:

$$a_n=\sum_{k=1}^{n-1}b_k+b_n \qquad b_n=\frac{1}{n}$$

Then we get $a_n = H_n$.

Compositions
Look at the following problem:

$$n = a_1 + a_2 + ... + a_k \qquad a_k \geq 0, a_k \in N, n \in N$$

In how many ways can we obtain an exact above equation?
We define $C_{n,k}$ as the number of ways obtaining the above equation using k numbers.
Let's reformulate the above problem. It is same as the problem that we have n $1s$, using bars to partition the $1s$ into k segements. Obviously $k-1$ bars are needed.
Let's reformulate this problem again, it can be represented as we have $n+k-1$ element, choosing $k-1$ to change to bars.
Then we have:
I can't remember detail of the following part

$$
\begin{aligned}
C_{n.k} &= \sum_{i=0}^{n} C_{n-i,k-1} \\
&= \sum_{j=0}^{n} C_{j,k-1} \\
&= \sum_{j=0}^{n-1} C_{j,k-1} + C_{n,k-1} \\
&= C_{n-1,k-1} + C_{n,k-1}
\end{aligned}
$$

$$
\begin{aligned}
C_{n,k}^{(1)} &\stackrel{?}{=} \binom{n+k}{n} \stackrel{!}{=} \binom{n+k}{k} \\
C_{n,k}^{(2)} &\stackrel{?}{=} \binom{n+k-1}{k-1} = \binom{n+k-3}{k-2} + \binom{n+k-2}{k-2}
\end{aligned}
$$

General Solution of Recurrence
May contain parameters, that can be used to fit initial values.
$x_n^{(P)}$ - Particular Solution pays no attention to initial values
$x_n^{(0)}$ - Homogeneous Recurrence Replaces b_n by 0.

Theorem 1 *The general solution of a recurrence can be written as the sum of a general solution of the homogeneous recurrence and a particular solution of the fully recurrence.(Init values may not be satisfied), which can be formulated as the following representation:*

$$x_n = x_n^{(0)} + x_n^{(P)}$$

Example:

$$x_{n+1} = 2x_n + 1 \qquad n \geq 0, x_0 = 2$$

Hom.:

$$x_{n+1} = 2x_n = A \cdot 2^n$$

Par.:(Simply letting $x_{n+1} = x_n$)

$$x_n = -1$$

Then:

$$
\begin{aligned}
x_n &= A \cdot 2^n - 1 \\
2 &= A \cdot 2^0 - 1 \\
A &= 3
\end{aligned}
$$

Hence:

$$x_n = 3 \cdot 2^n - 1$$

A more complicated example:
we have n coins, define a independent random variable

$$C_i = \begin{cases} H & \text{with prob } \frac{1}{2i+1} \\ T & \text{with prob } 1 - \frac{1}{2i+1} = \frac{2i}{2i+1} \end{cases}$$

and

$$A_n\text{=tosses of all } n \text{ coins giving an odd number of } Hs$$

$$\begin{aligned} P(A_n) &= P_n \\ &= Pr(C_n(H)) \cdot (1 - P_{n-1}) + Pr(C_n(T)) \cdot P_{n-1} \\ &= \frac{1}{2n+1}(1 - P_{n-1} + \frac{2n}{2n+1}P_{n-1} \\ &= P_{n-1}(\frac{2n}{2n+1} - \frac{1}{2n+1}) + \frac{1}{2n+1} \qquad n \geq 1, P_0 = 0 \end{aligned}$$

Hom.:

$$P_n^{(0)} = \frac{2n-1}{2n+1}P_{n-1}^{(0)} = \frac{3}{2n+1}k$$

Par.:

$$g = g\frac{2n-1}{2n+1} + \frac{1}{2n+1}g = \frac{1}{2}$$

Hence:

$$\begin{aligned} P_n &= \frac{3k}{2n+1} + \frac{1}{2} \\ P_0 &= \frac{3k}{1} + \frac{1}{2} = 0 \\ k &= \frac{1}{6} \\ P_n &= \frac{1}{2} - \frac{1}{4n+2} \\ &= \frac{1}{2}(1 - \frac{1}{2n+1}) \\ &= \frac{n}{2n+1} \end{aligned}$$