Generating Functions

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April 5, 2014

Generating Functions

Definition

Assume we have a number serie $\{a_n\}_{n\geq 0}$, then its Ordinary Generating Function(OGF) is $a(x) = \sum_{n\geq 0} a_n x^n$; its Exponential Generating Function(EGF) is $\hat{a}(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

Now let's look at a simple example: let $n_n = a_{n+1}, n \geq 0$, first on the OGF, then

$$b(x) = \sum_{n\geq 0} b_n x^n = \frac{1}{x} \sum_{n\geq 0} a_{n+1} x^{n+1} = \frac{a(x) - a_0}{x}$$

Next on the EGF, then

$$\begin{aligned}
\hat{b}(x) &= \sum_n b_n \frac{x^n}{n!} \\
&= \sum_{n\geq 0} a_{n+1} \frac{x^n}{n!} \\
&= \sum_{n\geq 1} a_{n+1} \frac{(n+1)x^{n+1}}{x(n+1)!} \\
&= \sum_{n\geq 1} a_{n+1} \frac{(x^{n+1})'}{(n+1)!} \\
&= D_x \sum_{n\geq 0} a_{n+1} \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

Convolution

Assume we have two number series $a : \{a_n\}_{n\geq 0}, b : \{b_n\}_{n\geq 0}$
Define

$$c = a * b$$

then

$$\begin{aligned} c_n &= \sum_k a_k b_{n-k} \\ &= \sum_j b_j a_{n-j} \end{aligned}$$

$$\begin{aligned} c(x) &= \sum_{n\geq 0} c_n x^n \\ &= \sum_n \sum_k a_k b_{n-k} x^n \\ &= \sum_{k\geq 0} a_k \sum_{n\geq k} b_{n-k} x^n \\ &= \underbrace{\sum_{k\geq 0} a_k x^k}_{a(x)} \underbrace{\sum_{j\geq 0} b_j x^j}_{b(x)} \end{aligned}$$

So we get

$$c(x) = a(x)b(x)$$

and

$$\hat{c}(x) = \sum_n c_n \frac{x^n}{n!} = \sum_{n\geq 0} \sum_k a_k b_{n-k} \frac{x^n}{n!}$$

Next, let us define

$$r_n = \sum_k \binom{n}{k} a_k b_{n-k}$$

then we have

$$\hat{r}(x) = \sum_n \sum_k \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!} = \sum_k a_k \frac{x^k}{k!} \sum_{n\geq k} b_{n-k} \frac{x^{n-k}}{(n-k)!}$$

Let's look at an example based on this definition
Let $a_n = 1, b_n = f^n$, and $u_n = \sum_{k=0}^{n} 1 \cdot f^k \binom{n}{k}$, which matches the above definition

then we have the Exponential Generating Function of all the three terms

$$\hat{a}(x) = \sum 1 \cdot \frac{x^n}{n!} = e^x$$

$$\hat{b}(x) = \sum \frac{(fx)^n}{n!} = e^{fx}$$

$$\hat{u}(x) = e^x e^{fx} = e^{x(1+f)} = \sum u_n \frac{x^n}{n!} = \sum_n (1+f)^n \frac{x^n}{n!}$$

Then, from the formula of Binomial, we have

$$u_n = (1+f)^n$$

Next let's calculate the generating function of the Harmonic Number

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

Then the OGF of it is

$$\begin{aligned} H(x) &= \sum_{n\geq 1} H_n x^n \\ &= \sum_{n\geq 1} \sum_{j\geq 1} \frac{x^n}{j} \\ &= \sum_{j\geq 1} \frac{1}{j} \underbrace{\sum_{n\geq j} x^n}_{\frac{x^j}{1-x}} \\ &= \frac{1}{1-x} \sum_{j\geq 1} \frac{x^j}{j} \\ &= \frac{1}{1-x} ln(1-x) \end{aligned}$$