1 Asymptotic Summation

Many analysis problems lead to sums for which we know no closed form. The problem of developing asymptotic estimates for such sums has been given much attention. In this section we present several approaches.

The first method assumes fast-varying summands, where the dominant contribution to the sum comes from a relatively small portion of the summation range; hence its name, "the critical range method." Then we described the Rice method, which is only useful for certain forms of summands involving binomial coefficients, and finally we come to the Euler summation formula, which is based on the rather obvious idea of replacing a sum by an integral. This is often a good choice for slowly changing functions (polynomials, rational or algebraic functions...), where the critical range method is inapplicable.

1.1 The Critical Range Method

Happily we find that many sums needed for analysis of algorithms have the following two properties that help in obtaining usable asymptotic values:

(a) The summands are all positive and unimodal in the index of summation (that is, they consist of one or two monotone sequences of terms).
(b) Since the terms are fast-changing in size, nearly the entire value of the sum comes from the contribution of a small fraction of the summation range.

Suppose we want to find the value of the sum that depends on a big parameter n:

$$S_n = \sum_{k \in D_n} a_n(k) \ ,$$

where D_n is a domain of summation (that may also depend on this parameter n). The second property above suggests the following procedure to handle such sums:

1. Look for the value of $k = k_0$ that maximizes $a_n(k)$ in the given domain:

$$a_n(k_0) = \max_{k \in D_n} a_n(k) \ . \tag{1}$$

2. Change the index of summation, so that the maximal term obtains at or near zero: $j = k_0 - k$, and rewrite the sum as

$$S_n = \sum_{j \in D'_n} a_n(k_0 - j) \ .$$

3. Evaluate the remaining sum. Sometimes this can be done directly, often various asymptotic approximation methods are naturally applied to estimate this value.

We demonstrate this method, called the **critical range method**, with an elaborate example, but it is appropriate to precede it with a suggestion (and a warning) that pertains to almost all work in asymptotics: *never trust a formula or a method 'blindly.' It is necessary that you get to understand the characteristics of the terms of the sum, to judge the suitability of the critical range used by the formula or method.*

We consider the following sum:

$$A_n = \sum_{k=0}^{\lambda n} \binom{n}{k}^r, \qquad \text{for some } 0 < \lambda \le 1,\ \ r > 0, \ \ \text{and large } \ n. \tag{2}$$

There seems to be no closed form for A_n except when $\lambda = 1$ or $1/2$ and r is a very small integer. The maximal value of the summand is obtained at

$$\max_{0\le k\le n} \binom{n}{k} = \begin{cases} \binom{n}{n/2} & n \text{ is even,} \\ \binom{n}{\lfloor n/2\rfloor} = \binom{n}{\lceil n/2\rceil} & n \text{ is odd.} \end{cases}$$

The derivation of the sum proceeds differently in the cases $\lambda = 1/2$, $\lambda < 1/2$, and $\lambda > 1/2$.

Let us start with $\lambda = 1/2$. If we assume that n is even, the sum goes up to the maximum value at $n/2$ when the sum goes to $\lfloor n/2\rfloor$). We shift the summation index: $j = \frac{n}{2} - k$, and then

$$A_n = \sum_{j=0}^{n/2} \binom{n}{\frac{n}{2} - j}^r. \tag{3}$$

This sum is as opaque as the original one, but is easier to manage, since the main contribution occurs when the index j is small, allowing to truncate expansions. We approximate the binomial coefficient $\binom{n}{k}$ using Stirling approximation, and this will suggest the continuation.

$$\binom{n}{k} = \frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \left[1 + O\left(\frac{1}{k} + \frac{1}{n-k}\right)\right]. \tag{4}$$

With our change of variables, $k=\dfrac{n}{2}-j, \quad n-k=\dfrac{n}{2}+j, \quad \dfrac{n}{k}=\dfrac{2n}{n-2j}, \quad \dfrac{n}{n-k}=\dfrac{2n}{n+2j}$. Then the square root term is

$$\frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}}=\frac{\sqrt{n}}{\sqrt{2\pi\left(\frac{n}{2}-j\right)\left(\frac{n}{2}+j\right)}}=\frac{\sqrt{n}}{\sqrt{2\pi\left(1-\frac{4j^2}{n^2}\right)\cdot\frac{n}{2}}}=\frac{\sqrt{2}}{\sqrt{\pi n}}\cdot\frac{1}{\sqrt{1-\frac{4j^2}{n^2}}}\,.$$

Since the contributing range of j is for small values only, we can use the binomial theorem to write for the last denominator

$$(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{3}{4}x^2+\cdots \quad\Longrightarrow\quad \left(1-\frac{4j^2}{n^2}\right)^{-1/2}=1+\frac{2j^2}{n^2}+O\left(\frac{j^4}{n^4}\right).$$

Therefore

$$\frac{\sqrt{n}}{\sqrt{2\pi k(n-k)}}=\frac{\sqrt{2}}{\sqrt{\pi n}}\cdot\left[1+\frac{2j^2}{n^2}+\cdots\right].$$

We shall neglect the second term in the brackets, since is below the relative error threshold we already established by selecting the Stirling approximation truncated as in Eq. (4). For the next two factors in the binomial coefficient we write:

$$\left(\frac{2n}{n-2j}\right)^{n/2-j}\left(\frac{2n}{n+2j}\right)^{n/2+j}=2^n\left(1-\frac{2j}{n}\right)^{-n/2+j}\left(1+\frac{2j}{n}\right)^{-n/2-j}\overset{\text{def}}{=}2^nT_n,$$

and then

$$\ln T_n=\left(-\frac{n}{2}+j\right)\ln\left(1-\frac{2j}{n}\right)-\left(\frac{n}{2}+j\right)\ln\left(1+\frac{2j}{n}\right). \tag{5}$$

Now use Maclaurin expansion of the logarithm function to second order, $\ln(1+x)=x-\frac{x^2}{2}$ which holds for $|x|<1$; here the role of x is played by j/n which is in $o(1)$. It follows that

$$\ln\left(1\pm\frac{2j}{n}\right)=\pm\frac{2j}{n}-\frac{1}{2}\left(\frac{2j}{n}\right)^2\pm\frac{1}{3}\left(\frac{2j}{n}\right)^3-\cdots.$$

Substituting these into Eq. (5) we find after extensive cancellations that

$$\ln T_n=\left(-\frac{n}{2}+j\right)\left[-\frac{2j}{n}-\frac{1}{2}\left(\frac{2j}{n}\right)^2-\cdots\right]-\left(\frac{n}{2}+j\right)\left[\frac{2j}{n}-\frac{1}{2}\left(\frac{2j}{n}\right)^2+\cdots\right]=$$

$$\frac{n}{2}\cdot\frac{2j}{n}-\frac{n}{2}\cdot\frac{2j}{n}-j\frac{2j}{n}-j\frac{2j}{n}+\frac{1}{2}\frac{n}{2}\cdot\left(\frac{2j}{n}\right)^2+\frac{1}{2}\frac{n}{2}\cdot\left(\frac{2j}{n}\right)^2-\frac{j}{2}\left(\frac{2j}{n}\right)^2+\frac{j}{2}\left(\frac{2j}{n}\right)^2+\cdots=$$

$$-\frac{4j^2}{n}+\frac{n}{2}\left(\frac{2j}{n}\right)^2+\cdots=-\frac{4j^2}{n}+\frac{2j^2}{n}+\cdots=-\frac{2j^2}{n}+O\left(\frac{j^4}{n^3}\right).$$

$\ln T_n=-\frac{2j^2}{n}+O\left(\frac{j^4}{n^3}\right)$, where the O term is supported by the Theorem shown in class about the remainder term in the expansion of a convergent function, and

$$T_n=\exp\left[-\frac{2j^2}{n}+O\left(\frac{j^4}{n^3}\right)\right]=\exp\left[-\frac{2j^2}{n}\right]\cdot\exp\left[O\left(\frac{j^4}{n^3}\right)\right]=\exp\left[-\frac{2j^2}{n}\right]\left[1+O\left(\frac{j^4}{n^3}\right)\right].$$

Since $T_n=1$ when $j=0$, the initial factor, $(2/(n\pi))^{1/2}\,2^n$ is our approximation of $\binom{n}{n/2}$.

Collecting all the pieces we find

$$\binom{n}{\frac{n}{2}-j}=\sqrt{\frac{2}{n\pi}}\,2^n\,e^{-2j^2/n}\left[1+O\left(\frac{j^4}{n^3}\right)\right]=\binom{n}{n/2}e^{-2j^2/n}\left[1+O\left(\frac{j^4}{n^3}\right)\right].\tag{6}$$

We have been justifying our retaining a single "error term" by citing our intention to sum over relatively small values of j only.

The underlying reason is our knowledge of the behavior of the binomial coefficients: $\binom{n}{k}$ increase very fast as k increases from 0, but slow down as k approaches $n/2$, and on the "shoulder" the rate of change is slow.

Since when $j=0$ we are at the middle of the shoulder, we need to sum until we effectively go off it; how far do we have to go? We need a quantitative estimate, and it is provided by the last relation above: Since $\binom{n}{\frac{n}{2}-j}\sim\binom{n}{n/2}e^{-2j^2/n}$, we clearly need to go further than $\sqrt{n}$, but in view of the largest remaining error term, j^4/n^3, we need not exceed $n^{3/4}$. If we pick, say, $n^{0.6}$ as the upper limit of the sum, then at that point we find the exponential factor is $\exp(-2n^{0.2})$, which even for moderate n is a very small fraction, and approaches zero exponentially fast as n increases.

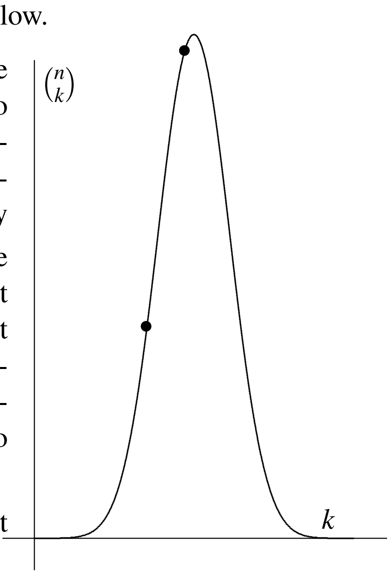

Figure 4:

We need not settle on any particular value, and just assume that we sum up to n^s, with $s\in(0.5,\,0.75)$.

Introducing this approximation into the sum (3), we need to evaluate:

$$\sum_{j=0}^{n^s}\binom{n}{n/2}^r e^{-2j^2r/n}=\binom{n}{n/2}^r\sum_{j=0}^{n^s}e^{-2j^2r/n}.\tag{7}$$

How can this sum be evaluated? we seem to escape one hurdle only to be faced with another, seemingly as high! Well, here we have a way. In the next section we go in some detail into the relationship between sums and integrals (and you may remember from the treatment of Riemann integrals in calculus the elements of this relationship), but at this stage we only observe the similarity of the above summand to the density function of the standard Normal random variable: That variable, denoted by $N(0,\sigma^2)$ has the density function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \quad \text{and} \quad \int_{-\infty}^{\infty} \varphi(x)\, dx = 1 \; .$$

Approximating the series by the corresponding Riemann integral we have, by comparison with the last relation,

$$\sum_{j=0}^{n^s} e^{-2j^2r/n} \approx \int_0^{n^s} e^{-\frac{2r}{n}x^2}\, dx \approx \int_0^{\infty} e^{-\frac{2r}{n}x^2}\, dx = \sqrt{\frac{n}{2r}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{n\pi}{8r}} \; . \tag{8}$$

Note the second transition, where we replaced the upper limit n^s by ∞: the reason is obviously that this integrand, behaving like the summand we examined above has vanishing contribution beyond n^s; hence changing the limit does not change the result, but allows us to use the fact that the density integrates to one.

The relative error introduced by extending the range of the sum is exponentially small in n, and is negligible in comparison with the relative error we have been carrying all along, $O(n^{-1})$, as seen in Eqs. (4) and (6). But what about replacing the sum by an integral? In the next subsection we present the tool, the Euler summation formula, which can answer just such questions. It can be shown (next HW?) that we need to add 1/2 to the value given in Eq. (8). Note that omitting this half would have contributed a relative error of $O(n^{-0.5})$, far larger than any we incurred so far! We finally obtain

$$\sum_{k=0}^{n/2} \binom{n}{k}^r = \binom{n}{n/2}^r \left[\frac{1}{2} + \left(\frac{\pi n}{8r}\right)^{1/2}\right] \left[1 + O\left(n^{-1}\right)\right]. \tag{9}$$

The relative error term is the one we used in the Stirling approximation: all the steps since then introduced much smaller errors.

$n \backslash r$	0.5	1	1.75	6.5	25	100
16	$-.00134470$	.01295039	.01754972	.01888521	.04874049	.24934293
64	$-.00006100$	.00354617	.00492026	.00572733	.00509321	.04279589
256	$-.00000374$	.00092976	.00130802	.00159774	.00153027	.00129960
1024	$-.00000024$	.00023817	.00033759	.00042365	.00042533	.00038876
4096	$-.00000001$	.00006028	.00008578	.00010920	.00011260	.00010799
16384	$-.00000000$	.00001516	.00002162	.00002773	.00002900	.00002858
65536	$-.00000000$	.00000380	.00000543	.00000699	.00000736	.00000736

A numerical illustration of the relative error in the calculation in Eq. (9)

The table above shows how well (and otherwise) this calculation performs, when viewed as an approximation. It shows the relative error of the estimate given in Eq. (9) in each of the indicated parameter pairs (n,r), and supports the claim that the error is in $O(n^{-1})$. The error generally increases with r; this is not reflected in the calculation. The dependence is probably complicated, but some information can be gleaned from the table: in most of the columns the error decreases proportionally to n^{-1}, except the first column! In that column, where $r = 1/2$, the relative error behaves like $O(n^{-2})$. This tells us that the coefficient of the term n^{-1} in the asymptotic expansion is proportional to $r-1/2$ (but the dependence almost disappears beyond the third column).

Figure 4 attempts to show the form of the function $\binom{n}{x}$, for $n = 100$, to illustrate the claims about the steep parts at the sides and the relatively flatter 'shoulder' near the center. In fact, in order for the diagram to show any detail it is substantially "squashed," and the actual values presented are $\binom{100}{x}^{0.2}$. In addition, to motivate further the selection of the value s above, two points are added, at $x = n/2 - 0.5\sqrt{n}$ and $x = n/2 - 1.5\sqrt{n}$. Note the ratio between the leading term in the result Eq. (9) and the leading term of the sum, at $j = 0$. It is large, having the value $(\pi n/8r)^{1/2}$. How to explain this, in view of the aimed 'steepness" of the function? The reason is that near this maximal term the function is not steep at all, and as our formula for the approximation of the change in the value of a binomial coefficient due to shifting the lower argument shows, changes *very slowly*. For $\lambda > 1/2$ the maximal term, at $k = n/2$, occurs well within the range of the summation. Everything proceeds as above, except that we have to use the integral on both sides of the origin $j = 0$. For the same reason the sum on one side above was completed for half the real line – now it would be allowed to cover the entire real line. The final result is then just double the value given in Eq. (9), independently of the precise value of λ (to see why such a blank statement makes sense, let $a = \lambda - 1/2$, and use the fact that for large n, $(1/2+\varepsilon)n > n/2 + an^s$, with the s used above, for arbitrarily small

$\varepsilon > 0$).

Now look at $\lambda < 1/2$. The maximum term falls again at the summation boundary, at the last term: $k_0 = \lfloor \lambda n \rfloor$. This, however, is not on the 'shoulder' of the curve of the function $\binom{n}{k}$, as before, but in the region where the function is steep indeed, and the maximal term, $\binom{n}{k_0}$, dominates the sum. As before, let $j = k_0 - k$, and again we have a partial sum of binomial coefficients, for which there is no closed form, so we need to develop an alternative. It will prove to be much simpler than before, when we needed to sum binomial coefficients close to $n/2$. Now, using the result which states that $\binom{n}{s+t} \sim \binom{n}{s} \left(\frac{n-s}{s}\right)^t$ when t^2 is very small compared to both s and $n-s$, we can write

$$\binom{n}{\alpha - j} \sim \binom{n}{\alpha} \left(\frac{\lambda}{1-\lambda}\right)^j.$$

The sum on j reduces to a geometric series:

$$\sum_{k=0}^{\lambda n} \binom{n}{k}^r \sim \sum_{j \geq 0} \binom{n}{\alpha}^r \left[\left(\frac{\lambda}{1-\lambda}\right)^r\right]^j = \binom{n}{\lfloor \lambda n \rfloor}^r \frac{1}{1-[\lambda/(1-\lambda)]^r}. \tag{10}$$