Binomial Coefficients

Prof. Micha Hofri/Typeset: Chao Li

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Binomial Coefficients

First, let's look at some formulations

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Stirling Approximation

$$\begin{aligned} n! &= n^n\sqrt{2\pi n}e^{-n}\left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right) \\ &= \frac{n^n e^{-n}\sqrt{2\pi n}}{k^k e^{-k}\sqrt{2\pi k}(n-k)^{n-k}e^{-n+k}\sqrt{2\pi(n-k)}} \\ &= \left(\frac{n}{k}\right)^k\left(\frac{n}{n-k}\right)^{n-k}\sqrt{\frac{n}{k(n-k)2\pi}} \end{aligned}$$

Next, we have seen before

$$\binom{n}{k+1} = \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{k!(n-k)!}\cdot\frac{n-k}{k+1} = \binom{n}{k}\frac{n-k}{k+1}$$

$$\binom{a}{b} = \frac{a^{\underline{b}}}{b!} = \frac{a(a-1)\cdots(a-b+1)}{b!}$$

in which a is any number and $b \in N$, otherwise $\binom{a}{b} = 0$.
Assume we have $n, s \gg 1$ and $t \ll n, s$, saying $t^2 \in o(S, n-s)$ and $t \in Z$. Then

$$\begin{aligned}
\binom{n}{s+t} &= \frac{n^{\underline{s+t}}}{(s+t)!} \\
&= \frac{n^{\underline{s}}(n-s)^{\underline{t}}}{s!(s+t)^{\underline{t}}} \\
&= \binom{n}{s} \frac{(n-s)(n-s-1)(n-s-2)\cdots(n-s-t+1)}{(s+t)(s+t-1)\cdots(s+t-t+1)} \\
&= \binom{n}{s} \frac{(n-s)^t}{s^t} \underbrace{\frac{1\cdot(1-\frac{1}{n-s})(1-\frac{2}{n-s})\cdots(1-\frac{t-1}{n-s})}{(1+\frac{1}{s})(1+\frac{2}{s})\cdots(1+\frac{t}{s})}}_{D}
\end{aligned}$$

As $(1+\frac{1}{s})(1+\frac{2}{s})\cdots(1+\frac{t}{s}) = \frac{t(t+1)(3t^2-13t-2)}{24}$
So

$$\begin{aligned}
D &= \frac{1 - \frac{t(t-1)}{2(n-2)} + \frac{t(t-1)(t-2)(3t-1)}{24(n-s)^2}}{1 + \frac{t(t+1)}{2s} + \frac{t(t+1)(3t^2-13t-2)}{24s^2}} \\
&= \left(1 - \frac{t(t-1)}{2(n-s)}\right)\left(1 + \frac{t(t+1)}{2s}\right) + o\left(\frac{t^2}{s^2}\right) \\
&= 1 - \frac{t(t+1)}{2s} - \frac{t(t-1)}{2(n-s)} + o\left(t^2\left(\frac{1}{n-s} + \frac{1}{s}\right)\right)
\end{aligned}$$

Finally, we got

$$\binom{n}{s+t} = \binom{n}{s}\left(\frac{n-s}{s}\right)^t \left(1 - \frac{t(t+1)}{2s} - \frac{t(t-1)}{2(n-s)} + o\left(t^2\left(\frac{1}{s} + \frac{1}{n-s}\right)\right)\right)$$

Relative Error

Assume we have $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ then, we have

$$D_n = n!\left(\frac{1}{e} - \sum_{k>n} \frac{(-1)^k}{k!}\right) + O\left(\frac{1}{(n+1)!}\right) \qquad = \frac{n!}{e} - O(\frac{1}{n})$$

Then $O(1)$ is called the relative error, and $(1+O(1))$ is called relative error factor