CS504 20140320 notes

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First let's look at some definitions:

Singletons, Matches, Coincidences Refer to items in the permutation which are in their original position. Actually, the term **Singletons, Matches,** and **Coincidences** are short for "*Cycle* of a single item."

Permutation Cycle is a subset of a permutation whose elements trade places with one another. For example, in the permutation group $\{4, 2, 1, 3\}$. (143) is a 3-cycle and (2) is a 1-cycle. Hence in the above definition, we only concern 1-cycle.

D_n -number of singletons in a permutation from S_n.

$d_{n,k}$ -number of permutations in S_n where $D_n = k$.

$\overline{D_n}$ -$d_{n,0}$, means the permutation is fully deranged.

Then from the above definitions, obviously, we have

$$d_{n,k} = \binom{n}{k} \cdot \overline{D_{n-k}}$$

which means we choose k positions for fixed singletons, and the other $n - k$ positions are all derangements.
We have recurrence

$$\overline{D_n} = (n-1)(\overline{D_{n-1}} + \overline{D_{n-2}}) \qquad n \geq 2, D_0 = 1, D_1 = 0$$

Then we have

$$\overline{D_{n+2}} = (n+1)(\overline{D_{n+1}} + \overline{D_n}) \qquad n \geq 0$$

The Exponential Generating Function of $\overline{D_n}$ is

$$\hat{D}(x) = \sum_{n \geq 0} \overline{D_n} \frac{X^n}{n!}$$

Then we have

$$
\begin{aligned}
\hat{D}''(x) &= \left(x\hat{D}'(x)\right)' + x\hat{D}'(x) + \hat{D}(x) \\
&= x\hat{D}''(x) + D'(x) + x\hat{D}'(x) + \hat{D}(x) \\
\hat{D}''(x)(1-x) &= \hat{D}(x) + (1+x)\hat{D}'(1+x) \\
\left((1-x)\hat{D}(x)\right)'' &= (-\hat{D}(x) + (1-x)\hat{D}'(x))' \\
&= -\hat{D}'(x) + (-\hat{D}'(x) + (1-x)\hat{D}''(x)) \\
\left((1-x)\hat{D}(x)\right)'' &= \hat{D}(x) + (1+x)\hat{D}'(x)
\end{aligned}
$$

Let $f(x) = (1-x)\hat{D}(x)$, then we have

$$
\begin{aligned}
f''(x) &= -f'(x) \\
\frac{f''}{f'} &= -1 \\
lnf' &= c - x \\
f' &= e^{c-x} = ke^{-x} \\
f(x) &= k_1 + k_2 e^{-x}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{D}(x) &= \frac{k_1 + k_2 e^{-x}}{1-x} \xrightarrow{D_0=1} k_1 + k_2 = 1 \\
D_1 = \hat{D}'(0) &\xrightarrow{D_1=0} \left.\frac{(-k_2 e^{-x}(1-x) + (k_1 + k_2 e^{-x})}{1}\right|_{x=0}
\end{aligned}
$$

Then we get $k_1 = 0, k_2 = 1$, then

$$
\begin{aligned}
\hat{D}(x) &= \frac{e^{-x}}{1-x} = \sum_{r\geq 0} \frac{(-x)^r}{r!} \\
D_n &= n! \sum_{k\geq 0} \frac{(-1)^k}{k!} \approx \frac{n!}{e}
\end{aligned}
$$

which is called **Incomplete Exponential Funcion**
Next define

$$
m_j = \begin{cases} 1 & \text{if } j \text{ is in position } j \\ 0 & \text{o.w.} \end{cases}
$$

Then we have

$$
E(M_n) = E\left(\sum_{j=1}^{n} m_j\right) = \sum_{j=1}^{n} \underbrace{E(m_j)}_{\frac{1}{n}} = 1
$$

2 2

Assume we have two number series $\{f_k\}_{k\geq 0}, \{g_n\}_{n\geq 0}$, and we have the relation between the two series

$$g_n = \sum_k \binom{n}{k} f_k$$

so, the generating function of g_n is

$$\begin{aligned}
g(x) &= \sum_n g_n x^n \\
&= \sum_n x^n \sum_k \binom{n}{k} f_k \\
&= \sum_k f_k \sum_{n\geq 0} \binom{n}{k} x^n \\
&= \sum_k f_k \frac{x^k}{(1-x)^{k+1}} \\
&= \frac{1}{1-x} \underbrace{\sum_k f_k \left(\frac{x}{1-x}\right)^k}_{f\left(\frac{x}{1-x}\right)} \\
g(x) &= \frac{1}{1-x} f\left(\frac{x}{1-x}\right)
\end{aligned}$$

then let $t = \frac{x}{1-x}$, we get

$$\begin{aligned}
x &= t(1-x) \\
x &= \frac{t}{1+x} \\
\frac{1}{1-x} &= 1+t \\
g\left(\frac{t}{1+t}\right) &= (1+t)f(t)
\end{aligned}$$

Now we have

$$
\begin{aligned}
f_n &= [t^n]f(t) \\
&= [t^n]\frac{1}{1+t}\sum_{k\geq 0} g_k \frac{t^k}{(1+t)^k} \\
&= \sum_{k\geq 0} g_k [t^n]\frac{t^k}{(1+t)^{k+1}} \\
&= \sum_{k\geq 0} g_k [t^{n-k}](1+t)^{-(k+1)}
\end{aligned}
$$

As $\binom{-k-1}{n-k} = \binom{k+1+n-k-1}{n-k}(-1)^{n-k}$ then

$$
f_n = \sum_{k\geq 0} g_k (-1)^{n-k}\binom{n}{k}
$$

As we have known $d_{n,k} = \binom{n}{k}D_{n-k}$, so

$$
\begin{aligned}
\underbrace{\sum_{k=0}^{n} d_{n,k}}_{n!} &= \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{n-j} \underbrace{D_{n-k}}_{j} \\
&= \sum_{j=0}^{n} \binom{n}{j} D_j \\
D_k &= \sum_{j} \binom{k}{j} j!(-1)^{k-j} \\
\frac{D_k}{k!} &= \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} = \sum_{r=0}^{k} \frac{(-1)^r}{r!}
\end{aligned}
$$