CS504: Analysis of Computations and Systems — Spring 2014

Solution for Homework 6 Posted: 19:01, March 7, 2014

1. In the following $f()$ and $g()$ are the generating functions of some sequences, and therefore have power series developments. The symbols z and a there are an indeterminate and a constant, respectively. Explain the relations:

   $g(z) = f(z-a) \quad \Longrightarrow \quad (1)\ [(z+a)^n]\, f(z) = g_n; \quad (2)\ \ [z^n]\, f(z) = \sum_k \binom{k}{n} g_k\, a^{k-n}.$

   This can be used to connect the developments of the same function in two points.

   Solution: Evidently, if $g(z) = f(z-a)$ holds, presumably for any value of z, then also $g(z+a) = f(z)$. The left-hand side has the expansion $g(z+a) = \sum_{k\geqslant 0} g_k(z+a)^k$. Hence the first claim: $[(z+a)^n]g(z+a) = [(z+a)^n]\sum_{k\geqslant 0} g_k(z+a)^k = g_n$. And since $g(z+a) = f(z)$, the claim holds.

   The second one is more direct:

   $$[z^n]f(z) = [z^n]g(z+a) = [z^n]\sum_{k\geqslant 0} g_k(z+a)^k = \sum_{k\geqslant 0} g_k[z^n]\sum_j \binom{k}{j} z^j a^{k-j} = \sum_{k\geqslant 0} g_k \binom{k}{n} a^{k-n}$$

   where the extraction operator annihilates all the terms with powers of z which are not n.

   Now that I am done grading, two comments:

   1. Note the "annihilation" or selection action of the operator $[z^n]$. It is often useful in simplifying complex expressions, as it removes one summation level, by selecting a single term out of the summed over sequence.

   2. A note of different kind; a couple of students did more or less what I write above, but for the binomial expansion used the dummy index n (in lieu of my j), not noticing that it is already in use, in the extraction operator. This produced a meaningless mess. Dummy indices must be "fresh," and newly-minted.

2. Expand the OGF $d(x) = (4-2x+x^2)^{-1}$. Provide for d_n a closed form expression, without complex numbers.

   Solution: Complex numbers enter the calculation here since the given quadratic does not have real roots; instead, they are $1 \pm i\sqrt{3}$. We find

   $$d_n = [x^n]\,\frac{1}{4-2x+x^2} = [x^n]\frac{1}{(1-i\sqrt{3}-x)(1+i\sqrt{3}-x)} = [x^n]\frac{i}{2\sqrt{3}}\left(\frac{1}{1+i\sqrt{3}-x} - \frac{1}{1-i\sqrt{3}-x}\right),$$

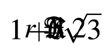
   following a partial fraction decomposition. This can be further developed,

   $$\begin{aligned} d_n &= \frac{i}{2\sqrt{3}}[x^n]\left(\frac{1}{1+i\sqrt{3}}\frac{1}{1-\frac{x}{1+i\sqrt{3}}} - \frac{1}{1-i\sqrt{3}}\frac{1}{1-\frac{x}{1-i\sqrt{3}}}\right) \\ &= \frac{i}{2\sqrt{3}}\left(\frac{1}{1+i\sqrt{3}}\left(\frac{1}{1+i\sqrt{3}}\right)^n - \frac{1}{1-i\sqrt{3}}\left(\frac{1}{1-i\sqrt{3}}\right)^n\right). \end{aligned}$$

While this involves complex numbers, they are complex conjugates, and the result of the subtraction is real. The expression can be somewhat simplified; however, you were asked for an explicitly real expression, not just real-valued, so we need to continue beyond a simplification.
Since $(1-i\sqrt{3})(1+i\sqrt{3}) = 4$, we can write $\dfrac{1}{1-i\sqrt{3}} = \dfrac{1}{4}(1+i\sqrt{3})$, and

$$d_n = \frac{i}{2\sqrt{3}}\frac{1}{4^{n+1}}\left((1-i\sqrt{3})^{n+1} - (1+i\sqrt{3})^{n+1}\right).$$

We continue with the polar representation of the complex numbers: $u+iv = r\exp\left(i\tan^{-1}(v/u)\right)$, where r is the absolute value of the number, $\sqrt{u^2+v^2}$, which is 2 for the numbers we have, and the tangent of the angle, θ in the diagram is $\sqrt{3}$, hence $\theta = \pi/3$.



Therefore $1 \pm i\sqrt{3} = 2e^{\pm i\theta}$, and $(1 \pm i\sqrt{3})^{n+1} = 2^{n+1}e^{\pm i(n+1)\pi/3}$. We use de Moivre formula, the facts that $\cos(-x) = \cos(x)$ while $\sin(-x) = -\sin(x)$, and substitute into d_n

$$d_n = \frac{i}{2\sqrt{3}}\frac{2^{n+1}}{4^{n+1}}\left(\cos(n+1)\frac{\pi}{3} - i\sin(n+1)\frac{\pi}{3} - \cos(n+1)\frac{\pi}{3} - i\sin(n+1)\frac{\pi}{3}\right)$$

The cosines cancel, the sines add, multiplying i by itself corrects the sign, hence,

$$d_n = \frac{1}{2^{n+1}\sqrt{3}}\sin(n+1)\frac{\pi}{3}, \qquad \text{while} \quad \sin k\frac{\pi}{3} = \begin{cases} 0 & k = 3r \\ \frac{\sqrt{3}}{2} & k = 6r+1, \quad \text{or } 6r+2 \\ -\frac{\sqrt{3}}{2} & k = 6r-1, \quad \text{or } 6r-2, \end{cases}$$

for any integer r. Therefore

$$d_n = \frac{V_n}{2^{n+2}}, \qquad \text{where} \quad V_n = \begin{cases} 0, & n = 3r-1 \\ 1, & n = 6r, \quad \text{or } \ 6r+1 \\ -1, & n = 6r-2, \quad \text{or } \ 6r-3. \end{cases}$$

And then it is always a smart idea to check, and the easiest way is to ask a computer with an algebra package (I use MAPLE) to do it. Here it is on MAPLE, enough to convince me I made no error:

```
t := 1/(4-2*x+x^2): series(t, x=0, 12);
1   1     1   3   1   4    1   6    1   7    1    9    1    10   / 12\
- + - x - -- x  - -- x  + --- x  + --- x  - ---- x  - ---- x   +O\x  /
4   8     32      64      256      512      2048      4096
```