Generating Functions - Part 2

Prof. Micha Hofri/Typeset: Chao Li

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Negating Upper Argument

For binomial coefficient, we have the following formula:

$$\binom{a}{b} = \binom{-a+b-1}{b}(-1)^b$$

Then let's consider the following example

let $a_n = \binom{n}{m}$, for $n \geq m$, then we have:

$$a_n = \binom{n}{m} = \binom{n}{n-m}$$

and its generating function:

$$a(x) = \sum_{n \geq m} x^n \binom{n}{m} = \sum_{n \geq m} x^n \binom{n}{n-m}$$

substitute our former formula into this equation, we then get

$$a(x) = \sum_{n \geq m} x^n \binom{-n+n-m-1}{n-m}(-1)^{n-m}$$

Let $n-m=j$ then we have

$$a(x) = \sum_{j \geq 0} x^{j-m} \binom{-n+n-m-1}{j}(-1)^j = x^m \sum_{j \geq 0} \binom{-m-1}{j}(-x)^j = \frac{x^m}{(1-x)^{m+1}}$$

An Example

Assume we have $C_n = 1 + (-1)^n, n \geq 0$ then its generating function is

$$\begin{aligned} C(x) &= \sum_{n\geq 0}(1+(-1)^n)x^n \\ &= \sum_{n\geq 0} x^n + \sum_{n\geq 0}(-x)^n \\ &= \frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2} \\ &= \sum_{k\geq 0} 2(x^2)^k \end{aligned}$$

Scaling

Assume we have two series $\{a_n\}$, $\{b_n\}$, and a constant c. We have $b_n \stackrel{\text{def}}{=} c^n a_n$ So the generating function of them could have the following feature that

$$b(x) = \sum (xc)^n a_n = a(cx)$$