CS504 3/24/2014 notes

Prof. Micha Hofri/Typeset: Chao Li, Xinyue Liu

April 5, 2014

1 Manipulation of asymptotic Expansions

1.1 Main asymptotic classes:

The following are the main classes we encounter in analysis of algorithms and data structures. Remember that asymptotic class is related to rate of growth, not actual size.

1. logarithmic: $\log n$
2. polynomial: $n^a(a \in \mathbb{R}_+$; that is, a strictly positive real number).
3. exponential: $b^n(b > 1)$
4. super-exponential (many types, *e.g.* $3^{n^2}, 2^{2^n}$

As usual when dealing with asymptotics, functions are assigned to a class based on the leading term — $2^{3n} + n^3$ is an exponential function. We also disregard numerical multiplying factors, n^2 and $1000n^2 + 70n$ are both quadratic polynomials.

1.2 Simple examples, absolute and relative error terms

Let $f_n = \frac{2}{n+2}$ and $g_n = \frac{n^2}{n^2-5}$ be functions we want to emphasize their change as n increases. That is the purpose of asymptotic expansions.

$$f_n = \frac{2}{n+2} = \frac{2}{n}\frac{1}{1+\frac{2}{n}} = \frac{2}{n}\left(1 - \frac{2}{n} + \frac{4}{n^2} \mp \cdots\right) = \frac{2}{n} - \frac{4}{n^2} + O\left(\frac{1}{n^3}\right)$$

The last term is an absolute error term, of order $O\left(\frac{1}{n^3}\right)$, and since the leading term is of order $1/n$, the error term is a relative error of order $O\left(n^{-2}\right)$.

$$g_n = \frac{n^2}{n^2-5} = \frac{1}{1-\frac{5}{n^2}} = 1+\frac{5}{n^2}+\frac{25}{n^4}+\cdots = 1+\frac{5}{n^2}\left(1+O\left(\frac{1}{n^2}\right)\right)$$

Here the last term appears to be a n^{-2} relative error term, relative to the 1 it shares a pair of parentheses with, but this is not the right point of view: it needs to be compared with the leading term in the expression, which is 1 as well, hence it is both an absolute error term of order n^{-4}, and also such a relative error term.

1.3 Polynomials are ordered by their degree

That means

$$f_n = n^a, \quad g_n = n^b, \text{ and } 0 < a < b \implies f_n \in o(g_n).$$

And the reason is is that $C_n = f_n/g_n = n^{a-b}$, converges to 0 when $b > a$ as n increases.

While the claim made above was for positive powers, the same algebra holds for negative powers: the functions decrease, and at a faster rate as the powers go down; thus $n^{-3} \in o\left(n^{-1.7}\right)$. The same ordering: smaller is "smaller."

Similarly, exponentials are ordered by their base:

$$u_n = c^n, \quad v_n = d^n, \text{ and } 0 < c < d \implies u_n \in o(v_n),$$

for the same reason, $C_n = u_n/v_n = c^n/d^n = (c/d)^n \to 0$, as n increases since $(c/d) < 1$.

1.4 Polynomials are slower than exponentials

Now we compare functions from different asymptotic classes. Proving that logarithms are slower to increase than polynomials was given as homework assignment.

Next, we show the claim in the title of the subsection. Let $f_n = n^a, \quad g_n = b^n$ and the numerical parameters satisfy $a > 0$ and $b > 1$. The proof method is the same: we form the ratio $C_n = f_n/g_n$, and show it converges to zero. We do this by creating one more ratio, C_{n+1}/C_n, and show that at some value of n (which can be very small or very large, it does not matter for the validity of the claim, which is very patient), the ratio becomes smaller than 1, and stays so indefinitely as n increases.

$$\frac{C_{n+1}}{C_n} = \frac{(n+1)^a}{n^a} \cdot \frac{b^n}{b^{n+1}} = \left(1+\frac{1}{n}\right)^a \frac{1}{b} = \frac{1}{b} + \frac{a}{bn}\left(1+o(1)\right).$$

The $o(1)$ appears since the next term is of lower order than $1/n$. The fraction $1/b$ is smaller than 1, and once $\frac{a}{bn} < 1 - \frac{1}{b}$ is satisfied, at $n_o = a/(b-1)$, we know that for all $n > n_1 = 2n_o$ the value of the last ratio will stay under 1, and therefore $C_n \to 0$. The doubling of n_o was made to clear away whatever debris the $o(1)$ above can bring.

2 Asymptotic Manipulations

2.1 Asymptotic scale

Assume we have serires: $g_0(n), g_1(n), g_2(n)...$, and we know $g_i(n) \in \Omega[g_{i+1}(n)]$
Define

$$f_n = \sum_{i=0}^{k} g_i n + O(g_{k+1}^n)$$

, which is *Poincar series*.
Then the generating function of *Poincaré Series* is

$$f(r) = f_0 + f_1 r + f_2 r^2 + ...$$

which coverges in some $r < A \qquad (A > 1)$.Thus,

$$\begin{aligned} f(r) &= \underbrace{f_0 + f_1 r + ... + f_k r^k}_{\Sigma_{k\geq 0} k!} + r^{k+1} (\underbrace{f_{k+1} + f_{k+2} + ...}_{\text{finite value}}) \\ &= f_0 + f_1 r + ... + f_k r^k + O(r^{k+1}) \end{aligned}$$

2.2

Now consider

$$f(n) = n^{\frac{a}{n}}$$

perform some derivation

$$f(n) = \frac{a}{n} = e^{lnn^{\frac{a}{n}}} = e^{\frac{a}{n}lnn}$$

$$n^{\frac{a}{n}} = 1 + a\frac{lnn}{n} + O\left((\frac{lnn}{n})^2\right)$$

$$\frac{n^{\frac{a}{n}} - 1 - a\frac{lnn}{n}}{(\frac{lnn}{n})^2} \to 2$$

3 Exponential Integral

Now let's look at some integral functions, first

$$f(x) = \int_x^\infty \frac{e^{x-t}}{t} dt$$

Perform some calculation

$$\begin{aligned} f(x) &= -\frac{e^{x-t}}{t}\Bigg|_x^\infty - \int_x^\infty \frac{e^{x-t}}{t^2} dt \\ &= \frac{1}{x} + \frac{e^{x-t}}{t^2}\Bigg|_x^\infty + 2\int_x^\infty \frac{e^{x-t}}{t^3} \\ &= \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} \pm ... + \frac{(-1)^{k-1}(k-1)!}{x^k} + (-1)^k k! \int_x^\infty \frac{e^{x-t}}{t^{k+1}} dt \end{aligned}$$

Let $S_k(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} \pm ... + \frac{(-1)^{k-1}(k-1)!}{x^k}$, then

$$f(x) = S_k(x) + (-1)^k k! \int_x^\infty \frac{e^{x-t}}{t^{k+1}} dt$$

Then

$$\begin{aligned} |f(x) - S_k(x)| &= k! \int_x^\infty \frac{e^{x-t}}{t^{k+1}} dt \\ &\ll k! \int_x^\infty \frac{1}{t^{k+1}} dt \\ &= \frac{k!}{k+2} \cdot \frac{1}{x^{k+2}} \end{aligned}$$

3.1

Next example, calculate the approximation of $\frac{n}{n-1}$

$$\frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}$$

As $\frac{1}{1-\frac{1}{n}} = \sum_{l\geq 0}(\frac{1}{n})^l$

$$\frac{n}{n-1} = 1+\frac{1}{n}+O(\frac{1}{n^2})$$

Next example

$$\frac{n}{n-1} ln \frac{n}{n-1}$$

calculate the approximation of the above function with a asymptotic of $O(\frac{1}{n^4})$
First, calculate part of the function

$$ln\frac{n}{n-1} = ln\frac{1}{1-\frac{1}{n}} = -ln(1-\frac{1}{n}) = \frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+O(\frac{1}{n^4})$$

Then the whole function is

$$\begin{aligned}\frac{n}{n-1}ln\frac{n}{n-1} &= \left(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+O\left(\frac{1}{n^4}\right)\right)\left(\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+O\left(\frac{1}{n^4}\right)\right)\\ &= \begin{array}{ccccccc}\frac{1}{n} & + & \frac{1}{2n^2} & + & \frac{1}{3n^3} & + & \ldots\\ & & \frac{1}{n^2} & + & \frac{1}{2n^3} & + & \ldots\\ & & & & \frac{1}{n^3} & + & \ldots\end{array}\\ &= \frac{1}{n}+\frac{3}{2n^2}+\frac{11}{6}\frac{1}{n^3}+O\left(\frac{1}{n^4}\right)\end{aligned}$$

Next example

$$e^{0.1}+cos(0.1)-ln(0.9)$$

calculate the above function with a asymptotic approximation of $O(10^4)$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$$

$$cos(x) = 1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots$$

$$ln(0.9) = ln(1-x)|_{0.1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$e^{0.1} + cos(0.1) - ln(0.9)$$

$$\begin{array}{ccccccccc} = & 1 & + & 0.1 & + & 1/2 \cdot 1/100 & + & 1/6 \cdot 1/1000 & +\cdots \\ & 1 & & & - & 1/2 \cdot 1/100 & + & \cdots & \\ & & - & 0.1 & - & 1/2 \cdot 1/100 & - & 1/3 \cdot 1/1000 & +\cdots \\ = & 2 & - & 1/2 \cdot 1/100 & - & 1/6 \cdot 1/1000 & & & \end{array}$$

Now comes our final example of this part
As it is known that

$$H_n = lnn + r + \frac{a}{n} + O(\frac{1}{n^2})$$
$$H_{n+1} = ln(n+1) + r + \frac{a}{n+1}$$

Then we have

$$H_{n+1} - H_n = \frac{1}{n+1} = ln\frac{n+1}{n} + a(\frac{1}{n+1} - \frac{1}{n}) + ?$$

what is a
From the above equation, and $ln\frac{n+1}{n} = ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2}$ we can get

$$(\frac{1}{n+1} - \frac{1}{n})(1-a) + \frac{1}{n} = \frac{1}{n} - \frac{1}{2n^2} + \cdots$$
$$(a-1)\frac{1}{n(n+1)} = -\frac{1}{2n^2}$$

Since $\frac{1}{n(n+1)} = \frac{1}{n^2}\left(\frac{1}{1+\frac{1}{n}}\right) \approx \frac{1}{n^2}$, then

$$a - 1 = -\frac{1}{2}$$
$$a = \frac{1}{2}$$

So

$$H_n = lnn + r + \frac{1}{2n} + ? \qquad ? \text{ is } \frac{1}{12n^2}$$