An Analysis Of Field De-rotation For Alt-Az Mounted Telescopes

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Abstract

A field de-rotator is mounted on an alt-az telescope in order to compensate for the field rotation of the stars while the telescope is tracking the horizontal and vertical motions of one star (called the "track star") with the alt-az method. However, alt-az tracking of the track star is not perfect and thus corrections to these errors must be made with a guiding program. The only guiding choice that is compatible with field de-rotation is the off-axis guiding method. Unfortunately, off-axis guiding, like its name implies, can never use as the guide star, even if available, that is at the optical axis of the telescope, which also serves as the motor axis of the de-rotator. To the casual observer, it would seem that de-rotation would always introduce a guide error because of the mismatch between reference stars for the two systems. But, in fact, de-rotation coupled with alt-az tracking actually reduces the alt-az guide error! This reduction in guide error does not persist indefinitely because as the camera rotates, the initial guide axes used by the alt-az guiding program will rotate away from their initial directions and thus at some point, the guide axes will have rotated so far from their initial directions that off-axis guiding will fail. Thus, the loss of axes integrity that comes from field de-rotation naturally introduces a time integration limit. The goals of this paper are (a) to derive the formula for the rate of de-rotation when only alt-az tracking is used, (b) demonstrate the reduction in guide error when both de-rotation and alt-az tracking are used together, and (c) to calculate the integration time limit that arises from the loss of guide axes integrity.

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I. INTRODUCTION

It is well-known that an alt-az mounted telescope cannot be used for astrophotography because alt-az tracking only takes care of the horizontal and vertical positions of the track star as it moves in the sky but does not take care of the rotation of other stars about the track star. For example, Fig. 1 shows the view through a telescope of Orion when the middle star of Orion's belt, Alnilam, is alt-az tracked. And despite, Alnilam being pefectly tracked in all the frames, Orion rotates about Alnilam. This effect is more dramatically demonstrated in Fig. 2 where I have stacked all 12 frames shown in Fig. 1 on top of each other.

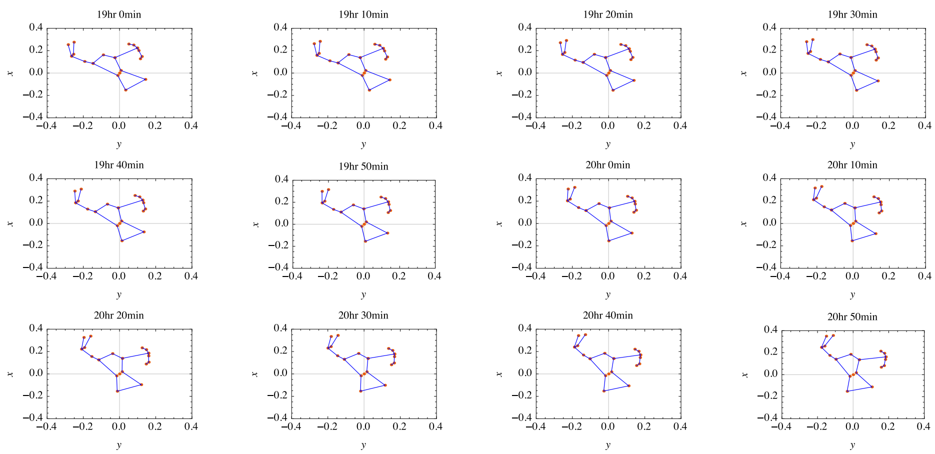

FIG. 1. These 12 frames represent the view of Orion seen through a telescope that is alt-az mounted in Chicago, on 24 Jan 2015 from 1900 hours to 2050 hours in 10 minute intervals. Alnilam, the middle star of Orion, is perfectly tracked in all these frames. And despite Alnilam being perfectly tracked, all the other stars of Orion rotate about it. Note: I have swapped the traditional directions of the x and y axis here because of the way I have named other vectors later in this paper.

Thus, there is a rotational degree of freedom that alt-az tracking does not correct. The traditional ways to take care of this effect are:

(a) Abandon the alt-az mount and use an equatorial mount.

(b) Add a field de-rotator to the alt-az mount to counter the field rotation.

Due to the limitations of both where I can mount my telescope to see Polaris in my yard and the obstruction of the telescope when my camera is mounted to do a proper Polar

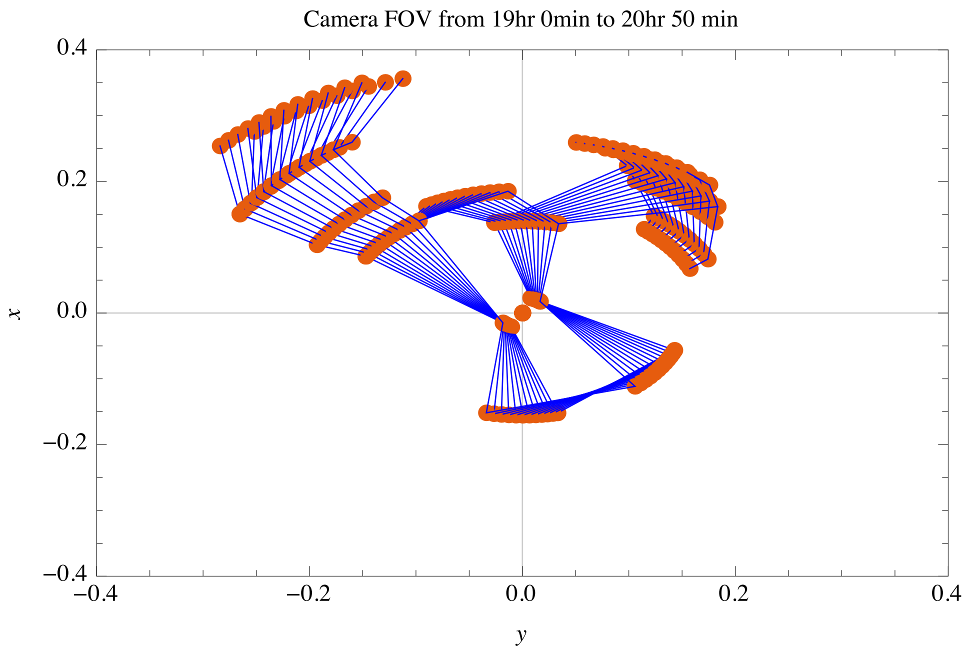

FIG. 2. The frames shown in Fig. 1 are stacked and the effect of rotating about Alnilam is more obvious.

alignment, the only method open to me is to use a field de-rotator in the alt-az mount configuration with an off-axis guider.[1] However, it is not clear to me that a field de-rotator would actually work properly because I suspect that there are limitations that have not been published widely. Therefore, the goal of this paper is to explore how well this technique works (or does not) for the astrophotography community and myself.

A. Nomenclature

I define both "track star" and "guide star" here in order to avoid any confusion later in the paper about which star I am talking about.

(a) track star The star that is being tracked by the telescope's alt-az motors and is at the optical axis of the telescope. This is usually the star that the user punches into the "goto" system of the telescope so that it shows up at the centre of the field of view. Tracking does not involve a feedback loop, i.e. it runs open loop.

(b) guide star The star that is used as a reference star by a software program that commands the alt-az motors to rotate the telescope in such a manner as to keep this star at a fixed location in the field of view of the telescope. Guiding involves a feedback loop. It is unnecessary for the guide star to be the same as the track star. In fact, for off-axis guiding, this is never the case.

II. ALT-AZ COORDINATES

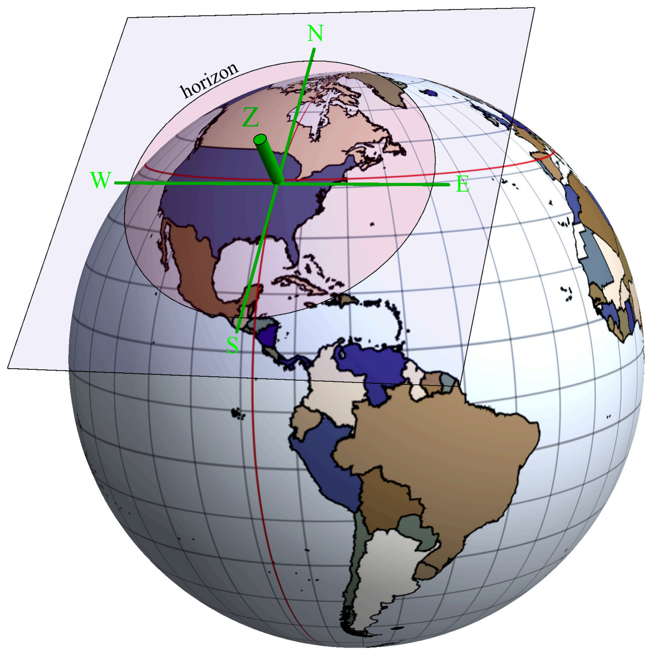

FIG. 3. The alt-az coordinate system starts with the construction of a tangential plane on the Earth's surface at the observer's location. A horizon is defined to be a circle that is in this plane centred at the observer. The North (N), South (S), East (E) and West (W) cardinal points are defined to be the the intersections of these directions with the horizon. The zenith (Z) is the vector that is normal to the horizon.

I have to explain how the alt-az system defines the coordinates of stars before we can continue with calculating the rate of field rotation of stars even when the telescope is guided.

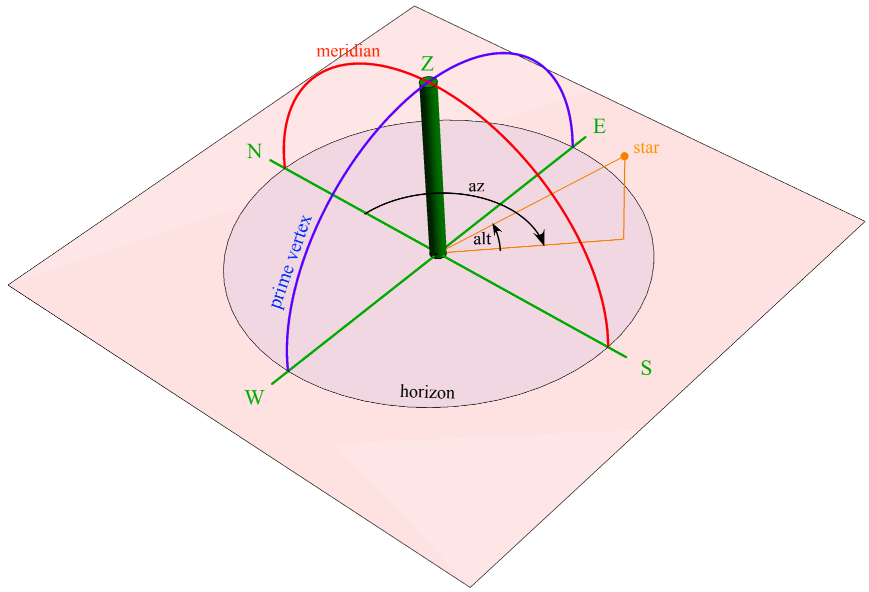

FIG. 4. The tangential plane from Fig. 3 is transferred to this figure here. The alt-az of a star is shown here. Its azimuth is measured clockwise from the North with North being defined as az $= 0°$. Its alt is measured w.r.t. the horizon with the horizon being defined as alt $= 0°$. The zenith is the point that is directly over the head of the observer. The meridian is the circle that passes through North and the zenith. The prime vertex is the circle that is orthogonal to the meridian.

The alt-az coordinate system is the system that is based on the observer's horizon.[2] The horizon is defined to be the circle that lies on the tangential plane of the Earth's sphere at the observer's location. This plane contains the cardinal directions north, south, east and west and the intersection of the horizon with these directions are the points North (N), South (S), East (E) and West (W). See Fig. 3. The meridian is defined to be the circle that passes through the zenith and N. The NS line that is formed by intersecting the meridian and the horizon points exactly in the north-south direction. The prime vertical is defined to be the circle that is orthogonal the meridian. In a similar fashion, the EW line that is formed by intersecting the prime vertical and the horizon points exactly in the east-west direction. The alt (altitude, denoted as "η") coordinate is the angle w.r.t. the horizon and az (azimuth, denoted as "ξ") is the angle w.r.t. to the NS line in the clockwise direction.

In this system, the stars do not have a fixed coordinate in alt-az because they are rotating about Polar the axis — the axis that extends from the centre of the Earth to Polaris — to first order. This shows up to the observer as stars rising in the East and setting in the West. Fig. 4 shows the alt-az system. Furthermore, the alt-az coordinate of any star is dependent on the location of the observer, i.e. their latitude and longitude, and the time the observation is made.

III. TRACKING AND FIELD ROTATION

When I point my telescope at the star at alt-az coordinates (η_0, ξ_0), the star will move away from where I'm pointing the telescope because the Earth is rotating. In order, to keep the telescope pointed at the star, the telescope will need to track the Earth's rotation. This is usually accomplished with the tracking electronics that comes with the telescope's "goto" system. Suppose I choose a star S_0 to observe, the tracking system should, in principle, keep this star at the centre of its field of view (FOV). Note that S_0 is not necessarily the star that is used for guiding. In fact, for off-axis guding, this star is never at the centre, it is at one edge of the FOV. And because S_0 is the track star, with perfect tracking, it will always remain at the centre of the FOV of the telescope.

As had been explained in the Introduction, stars that are not S_0 will rotate about it at some angular frequency. The goal of this section is to calculate this frequency.

A. Establishing the coordinate systems

The system that I want to solve for is shown in Fig. 5 and this system was constructed to fulfill the following criteria:

(i) The FOV of the camera that is attached to the telescope sees a very small angular patch of the sky. Therefore, I can approximate the camera's FOV as a tangential plane that is normal to the vector $\boldsymbol{S}_0$.

(ii) I can establish a Cartesion coordinate system, $\hat{x}\hat{y}\hat{z}$, that has its origin fixed on S_0. I will set up the $\hat{x}\hat{y}\hat{z}$ system so that as S_0 and all the stars move across the sky, the alt-az tracking electronics of the telescope keeps the origin of the $\hat{x}\hat{y}\hat{z}$ axes fixed on S_0.

(iii) Finally, let me choose another star S_1 that has alt-az coordinates (η_1, ξ_1) and is within the FOV of the camera. Since S_0 and S_1 are within the same small FOV of the camera, I can make the approximation that S_1 which lies on the celestial sphere, can be projected onto the tangential plane defined by $\boldsymbol{S}_0$ that has coordinate axes $\hat{x}\hat{y}$ shown in Fig. 5.

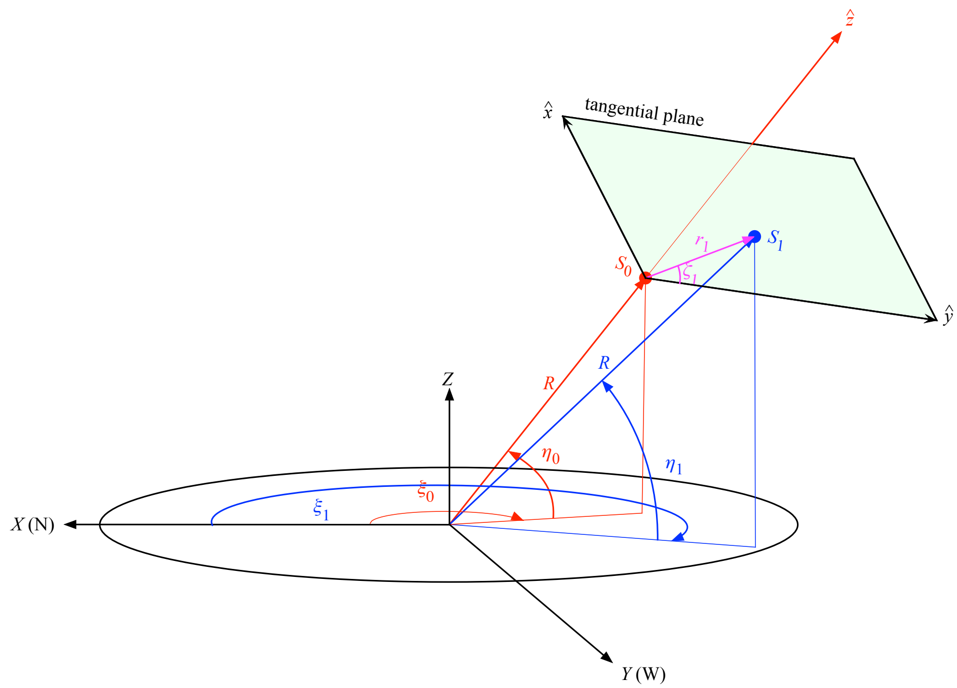

FIG. 5. The alt-az coordinates of the track star S_0 and some other star S_1 as seen on the tangential plane. A small patch on the tangential plane serves as the FOV of the camera. The angle ζ_1 is the angle of interest that I need to find in order to calculate the field rotation rate $\dot{\zeta}_1$.

1. The alt-az system as a Cartesian coordinate system

I can write down the coordinates of any star in terms of a Cartesian coordinate system XYZ that has the X axis pointing to the North, the Y axis pointing to the West and Z pointing overhead. This system is shown in Fig. 5. It is clear that the XYZ coordinate system that I have defined is consistent with the alt-az coordinate system.

The Cartesian coordinates of S_0 are

$$\boldsymbol{S}_0 = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} R\cos\eta_0\cos\xi_0 \\ -R\cos\eta_0\sin\xi_0 \\ R\sin\eta_0 \end{pmatrix} \tag{1}$$

where R is the radial distance to any star on the Celestial sphere. In general, all the quantities that interest us will eventually be independent of R because they deal with angles.

And similarly, I can write down the Cartesian coordinates of S_1 and they are

$$\boldsymbol{S}_1 = \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} R\cos\eta_1\cos\xi_1 \\ -R\cos\eta_1\sin\xi_1 \\ R\sin\eta_1 \end{pmatrix} \tag{2}$$

2. $\hat{x}\hat{y}\hat{z}$ system

Next, I want to construct the orthornormal vectors that define the $\hat{x}\hat{y}\hat{z}$ system. In this system, the origin of this system is always fixed on S_0 because the telescope is perfectly alt-az tracked. The tangential plane shown in Fig. 5 is defined by the vector $\boldsymbol{S}_0$ because it is normal to it. Let me write down the unit vector $\hat{\boldsymbol{z}}$ that is parallel to $\boldsymbol{S}_0$

$$\hat{\boldsymbol{z}} = \frac{1}{\sqrt{X_0^2 + Y_0^2 + Z_0^2}} \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \tag{3}$$

And without any loss of generality, I can create $\hat{\boldsymbol{y}}$ that is parallel to the horizontal plane, i.e. normal to $\boldsymbol{Z}$. In order to make sure that $\hat{\boldsymbol{y}}$ is in the tangential plane, I require that $\hat{\boldsymbol{y}}$ be normal to $\hat{\boldsymbol{z}}$. I can summarize these requirements mathematically below

$$\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = \begin{pmatrix} y_X \\ y_Y \\ y_Z \end{pmatrix} \cdot \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} = \boldsymbol{0} \tag{4}$$

$$\hat{\boldsymbol{y}} \cdot \boldsymbol{Z} = \begin{pmatrix} y_X \\ y_Y \\ y_Z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{0} \tag{5}$$

$$|\hat{\boldsymbol{y}}| = 1 \tag{6}$$

The above is easily solved and gives me

$$\hat{\boldsymbol{y}} = \begin{pmatrix} y_X \\ y_Y \\ y_Z \end{pmatrix} = \begin{pmatrix} \frac{Y_0}{\sqrt{X_0^2+Y_0^2}} \\ -\frac{X_0}{\sqrt{X_0^2+Y_0^2}} \\ 0 \end{pmatrix} \tag{7}$$

Finally $\hat{\boldsymbol{x}}$ is found by taking the cross product between $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \frac{1}{\sqrt{(X_0^2+Y_0^2)(X_0^2+Y_0^2+Z_0^2)}} \begin{pmatrix} -X_0 Z_0 \\ -Y_0 Z_0 \\ X_0^2+Y_0^2 \end{pmatrix} \tag{8}$$

B. The initial angle ζ_1

I want to calculate the angle ζ_1 that is between the star S_1 and the $\hat{\boldsymbol{y}}$ axis shown in Fig. 5. In order to find this angle, I have to find the coordinates of $\boldsymbol{r}_1$ in the $\hat{x}\hat{y}\hat{z}$ system. But first, in the XYZ coordinate system, $\boldsymbol{r}_1$ has the following coordinates

$$\boldsymbol{r}_1 = \boldsymbol{S}_1 - \boldsymbol{S}_0 = \begin{pmatrix} X_1 - X_0 \\ Y_1 - Y_0 \\ Z_1 - Z_0 \end{pmatrix} \tag{9}$$

The matrix $\tilde{\boldsymbol{R}}$ that transforms the coordinates any vector in the XYZ system to the $\hat{x}\hat{y}\hat{z}$ system is

$$\tilde{\boldsymbol{R}} = \begin{pmatrix} \hat{\boldsymbol{x}}^T \\ \hat{\boldsymbol{y}}^T \\ \hat{\boldsymbol{z}}^T \end{pmatrix} = \begin{pmatrix} -\frac{X_0 Z_0}{\sqrt{(X_0^2+Y_0^2)(X_0^2+Y_0^2+Z_0^2)}} & -\frac{Y_0 Z_0}{\sqrt{(X_0^2+Y_0^2)(X_0^2+Y_0^2+Z_0^2)}} & \frac{\sqrt{X_0^2+Y_0^2}}{\sqrt{X_0^2+Y_0^2+Z_0^2}} \\ \frac{Y_0}{\sqrt{X_0^2+Y_0^2}} & -\frac{X_0}{\sqrt{X_0^2+Y_0^2}} & 0 \\ \frac{X_0}{\sqrt{X_0^2+Y_0^2+Z_0^2}} & \frac{Y_0}{\sqrt{X_0^2+Y_0^2+Z_0^2}} & \frac{Z_0}{\sqrt{X_0^2+Y_0^2+Z_0^2}} \end{pmatrix} \tag{10}$$

Using the above matrix, I can calculate $\boldsymbol{r}_1$'s coordinates in the $\hat{x}\hat{y}\hat{z}$ system. I will denote the $\boldsymbol{r}_1$ vector in the $\hat{x}\hat{y}\hat{z}$ system as $\boldsymbol{r}'_1$

$$\boldsymbol{r}'_1 = \tilde{\boldsymbol{R}}\boldsymbol{r}_1 = \begin{pmatrix} \frac{-X_0X_1Z_0+X_0^2Z_1+Y_0(-Y_1Z_0+Y_0Z_1)}{\sqrt{(X_0^2+Y_0^2)(X_0^2+Y_0^2+Z_0^2)}} \\ \frac{X_1X_0-X_0Y_1}{\sqrt{X_0^2+Y_0^2}} \\ \frac{-X_0^2+X_0X_1-Y_0^2+Y_0Y_1+Z_0(-Z_0+Z_1)}{\sqrt{X_0^2+Y_0^2+Z_0^2}} \end{pmatrix} \equiv \begin{pmatrix} r'_{1\hat{x}} \\ r'_{1\hat{y}} \\ r'_{1\hat{z}} \end{pmatrix} \tag{11}$$

Therefore, the angle ζ_1 is given by

$$\tan\zeta_1 = \frac{r'_{1\hat{x}}}{r'_{1\hat{y}}} = \frac{-X_0X_1Z_0+X_0^2Z_1+Y_0(-Y_1Z_0+Y_0Z_1)}{(X_1Y_0-X_0Y_1)\sqrt{X_0^2+Y_0^2+Z_0^2}} \tag{12}$$

1. The effect from the rotation of the Earth

The stars in the sky do not stay still. They rotate about the Polar axis by an angular velocity Ω and the angular changes of the stars can be found by using a rotation matrix about the Polar axis. See Fig. 6. It is obvious from this figure that the Polar axis of rotation has the alt-az coordinates $(\eta_P, \xi_P) = (\phi_{\rm lat}, 0)$. I can write down its Cartesian coordinates as

$$\boldsymbol{S}_P = \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} = \begin{pmatrix} \cos\phi_{\rm lat} \\ 0 \\ \sin\phi_{\rm lat} \end{pmatrix} \tag{13}$$

The rotation matrix $\tilde{\boldsymbol{U}}_P$ about $\boldsymbol{S}_P$ is

$$\tilde{\boldsymbol{U}}_P(\theta) = \begin{pmatrix} \cos^2\phi_{\rm lat} + \sin^2\phi_{\rm lat}\cos\theta & -\sin\phi_{\rm lat}\sin\theta & (1-\cos\theta)\cos\phi_{\rm lat}\sin\phi_{\rm lat} \\ \sin\phi_{\rm lat}\sin\theta & \cos\theta & -\cos\phi_{\rm lat}\sin\theta \\ (1-\cos\theta)\cos\phi_{\rm lat}\sin\phi_{\rm lat} & \cos\phi_{\rm lat}\sin\theta & \sin^2\phi_{\rm lat} + \cos^2\phi_{\rm lat}\cos\theta \end{pmatrix} \tag{14}$$

where θ is the angle of rotation.

For us, $\theta = \Omega\delta t$ where δt is the time that has elapsed after observing the star at S_0 and Ω is the Earth's angular velocity. With the rotation matrix $\tilde{\boldsymbol{U}}_p$, I can calculate the position of any arbitrary star $\boldsymbol{S}$ at time δt. With no loss of generality, let me apply $\tilde{\boldsymbol{U}}_P$ to our track star S_0. At time δt, its position becomes

$$\begin{aligned} \boldsymbol{S}_0(\delta t) &= \tilde{\boldsymbol{U}}_P(\Omega\delta t)\boldsymbol{S}_0 \\ &= \begin{pmatrix} \cos\phi_{\rm lat}(X_0\cos\phi_{\rm lat} + Z_0\sin\phi_{\rm lat})(1-\cos\Omega\delta t) + X_0\cos\Omega\delta t - Y_0\sin\phi_{\rm lat}\sin\Omega\delta t \\ Y_0\cos\Omega\delta t + (X_0\sin\phi_{\rm lat} - Z_0\cos\phi_{\rm lat})\sin\Omega\delta t \\ \sin\phi_{\rm lat}(X_0\cos\phi_{\rm lat} + Z_0\sin\phi_{\rm lat})(1-\cos\Omega\delta t) + Z_0\cos\Omega\delta t + Y_0\cos\phi_{\rm lat}\sin\Omega\delta t \end{pmatrix} \\ &\approx \boldsymbol{S}_0(0) + \Omega\delta t \begin{pmatrix} -Y_0\sin\phi_{\rm lat} \\ X_0\sin\phi_{\rm lat} - Z_0\cos\phi_{\rm lat} \\ Y_0\cos\phi_{\rm lat} \end{pmatrix} \quad \text{if } \Omega\delta t \ll 1 \end{aligned} \tag{15}$$

And to obtain $\boldsymbol{S}_1(\delta t)$, I simply replace the subscript 0 with 1 in the above equation.

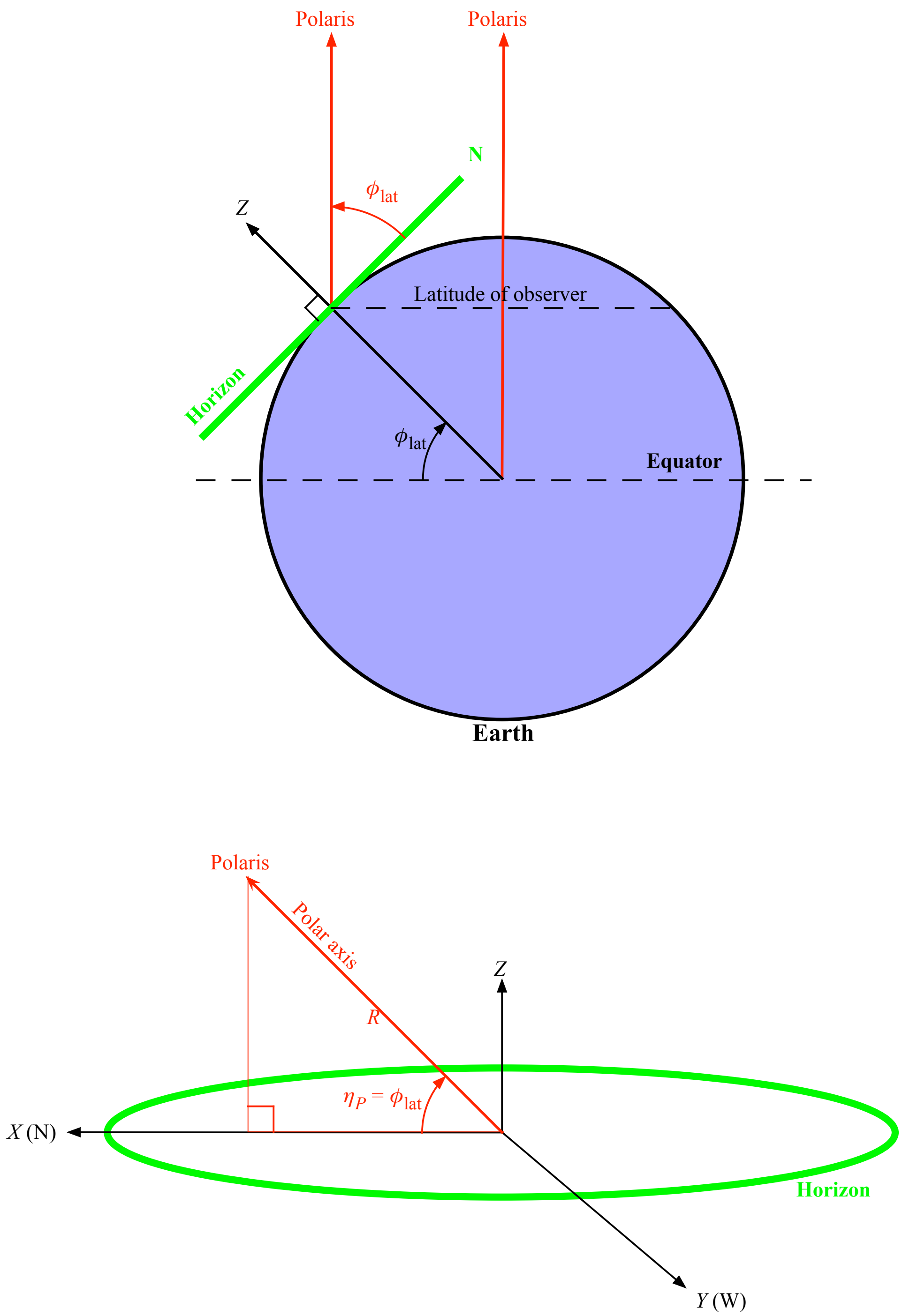

FIG. 6. For our purposes, the Earth's rotation axis essentially points at Polaris which I will call the Polar axis. The altitude angle of the Polar axis has the same angle as the observer's latitude, i.e. $\eta_P = \phi_{\text{lat}}$, and its azimuth angle $\xi_P = 0$ by definition.

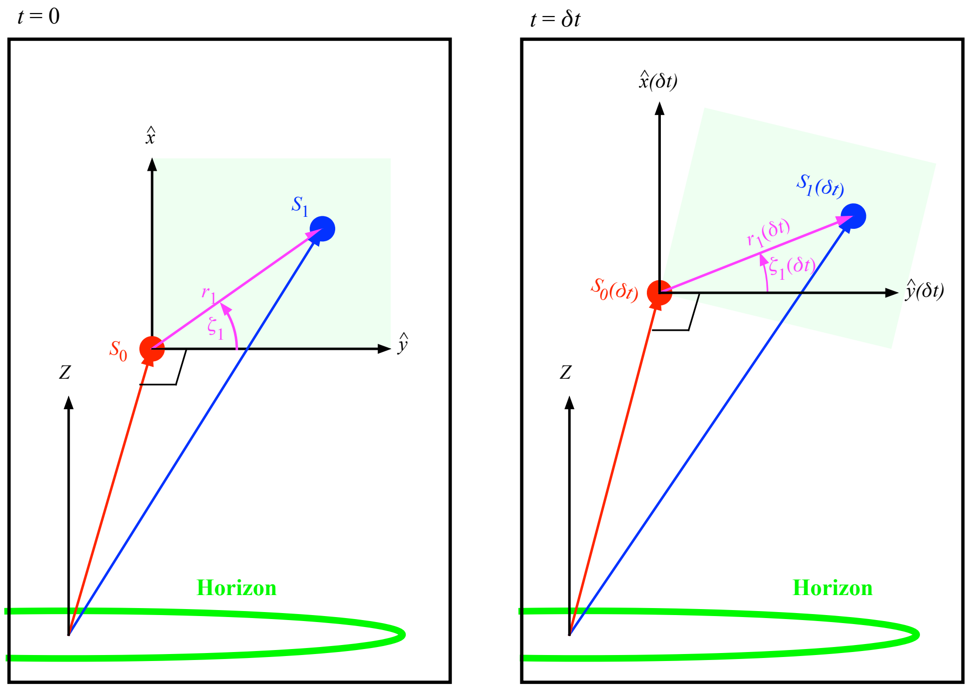

FIG. 7. At $t = 0$, the $\boldsymbol{r}_1$ vector and the $\hat{\boldsymbol{y}}$ vector encloses the angle of interest ζ_1. The light green rectangle represents the FOV at $t = 0$. At time $t = \delta t$, the stars have rotated and the original FOV (light green rectangle) has rotated by the same amount despite the alt-az tracking mechanism having moved the telescope so that S_0 is still at the same point in the new FOV defined by $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Notice that the mechanics of alt-az tracking always keeps $\hat{\boldsymbol{y}}$ parallel to the horizontal plane and normal to $\boldsymbol{S}_0$. The difference $\zeta_1(\delta t) - \zeta_1(0)$ tells us how much the stars have rotated w.r.t. S_0 during the time interval δt.

It is clear from Fig. 7 that the vector $\boldsymbol{r}_1(\delta t)$ is given by

$$\begin{aligned}
\boldsymbol{r}_1(\delta t) &= \boldsymbol{S}_1(\delta t) - \boldsymbol{S}_0(\delta t) \\
&= \boldsymbol{S}_1(0) - \boldsymbol{S}_0(0) + \Omega \delta t \begin{pmatrix} -(Y_1 - Y_0)\sin\phi_{\text{lat}} \\ (X_1 - X_0)\sin\phi_{\text{lat}} - (Z_1 - Z_0)\cos\phi_{\text{lat}} \\ (Y_1 - Y_0)\cos\phi_{\text{lat}} \end{pmatrix}
\end{aligned} \tag{16}$$

where I have applied Eq. 15.

In a similar vein, I must also rotate the tangential plane with coordinate system defined by $\boldsymbol{S}_0$ to $\boldsymbol{S}_0(\delta t)$. This is exactly what alt-az tracking does, it moves the telescope so that S_0 is always at the centre of its FOV. Since the tangential plane has moved, I have to create a new $\hat{x}\hat{y}\hat{z}(\delta t)$ coordinate system with the same method that I had used in section III A 2.

From these new vectors, I can construct the transformation $\tilde{\boldsymbol{R}}(\delta t)$ matrix that takes vectors from the XYZ coordinate system to the $\hat{x}\hat{y}\hat{z}(\delta t)$ system and it is

$$\tilde{\boldsymbol{R}}(\delta t) = \begin{pmatrix} \hat{\boldsymbol{x}}^T(\delta t) \\ \hat{\boldsymbol{y}}^T(\delta t) \\ \hat{\boldsymbol{z}}^T(\delta t) \end{pmatrix} \tag{17}$$

Unfortunately, writing out $\tilde{\boldsymbol{R}}(\delta t)$ is rather onerous and not very illuminating. But after a lot of algebra, once I have $\tilde{\boldsymbol{R}}(\delta t)$, I can transform $\boldsymbol{r}_1(\delta t)$ that is in XYZ coordinates to $\hat{x}\hat{y}\hat{z}(\delta t)$ coordinates

$$\begin{aligned} \boldsymbol{r}'_1(\delta t) &= \tilde{\boldsymbol{R}}(\delta t)\boldsymbol{r}_1(\delta t) \\ &= \tilde{\boldsymbol{R}}(\delta t)\boldsymbol{r}_1(0) + \Omega\delta t\tilde{\boldsymbol{R}} \begin{pmatrix} -(Y_1 - Y_0)\sin\phi_{\text{lat}} \\ (X_1 - X_0)\sin\phi_{\text{lat}} - (Z_1 - Z_0)\cos\phi_{\text{lat}} \\ (Y_1 - Y_0)\cos\phi_{\text{lat}} \end{pmatrix} \\ &\equiv \begin{pmatrix} r'_{1\hat{x}}(\delta t) \\ r'_{1\hat{y}}(\delta t) \\ r'_{1\hat{z}}(\delta t) \end{pmatrix} \end{aligned} \tag{18}$$

And by using Eq. 12 and the above equation for $\boldsymbol{r}'_1(\delta t)$, I can calculate the new angle $\zeta_1(\delta t)$ and the result is

$$\tan[\zeta_1(\delta t)] = \frac{r'_{1\hat{x}}(\delta t)}{r'_{1\hat{y}}(\delta t)} \tag{19}$$

C. Field rotation frequency $\dot{\zeta}_1$

Finally, I can calculate the field rotation frequency $\dot{\zeta}_1$. To do this, let me define $\delta\zeta_1$ to be the difference ζ_1 at $t = 0$ and $t = \delta t$, i.e.

$$\delta\zeta_1 = \zeta_1(\delta t) - \zeta_1(0) \tag{20}$$

In order to apply Eq. 12 and 19 to get the $\dot{\zeta}_1$, I have to take the tangent of $\delta\zeta_1$. This gives me

$$\begin{aligned} \tan(\delta\zeta_1) = \tan[\zeta_1(\delta t) - \zeta_1(0)] &= \frac{\tan[\zeta_1(\delta t)] - \tan[\zeta_1(0)]}{1 + \tan[\zeta_1(\delta t)]\tan[\zeta_1(0)]} \\ \approx \delta\zeta_1 \qquad &\text{if } \delta\zeta_1 \ll 1. \end{aligned} \tag{21}$$

where the last line in the above equation is a consequence of $\Omega\delta t \ll 1$.

Recall that Eq. 21 is the result in Cartesian XYZ coordinates. What I really want is the result in alt-az coordinates which I can get when I substitute Eq. 1 and 2 into Eq. 21. After I do this, I get a very simple solution for $\delta\zeta_1$ which is

$$\delta\zeta_1 = -\Omega\delta t\frac{\cos\xi_0\cos\phi_{\mathrm{lat}}}{\cos\eta_0} \tag{22}$$

It is important to notice that $\delta\zeta_1$ does not have any dependence on R or the choice of the non-tracked star S_1. This makes sense because otherwise the relative positions of the stars, for example constellations, as seen by the camera will become distorted as the stars rotate about the Polar axis.

The result that I have been hunting for, the field rotation frequency $\dot{\zeta}_1$, can be read off immediately from Eq. 22 and is

$$\boxed{\dot{\zeta}_1 = \lim_{\delta t\to 0}\frac{\delta\zeta_1}{\delta t} = -\Omega\frac{\cos\xi_0\cos\phi_{\mathrm{lat}}}{\cos\eta_0} \equiv \dot{\zeta}} \tag{23}$$

And from this equation, it is obvious that $\dot{\zeta}_1$ is independent of the choice of star S_1 and thus I have removed the subscript "1" from ζ_1 and will just call it ζ from now on.

Note: the Earth's angular rotation frequency is $\Omega = -[(4.178 \times 10^{-3}) \times \pi/180]$ rad/s. The negative sign comes from making sure that the stars rise from the East and sets in the West because when I apply the right hand rule about the Polar axis, a positive angular frequency means that the stars rise in the West and sets in the East.

1. *Speed Limit*

It is clear that $\dot{\zeta}$ has a singularity when the telescope points towards the zenith, i.e. $\eta_0 = 90°$. This also means that as the telescope gets closer and closer to the zenith, the stepper motor must rotate faster and faster to compensate for the Earth's rotation. This naturally sets a maximum altitude limit that the field de-rotator can be used. For example, if the maximum stepper motor revolution frequency, $\dot{\zeta}_{\mathrm{max}}$ rad/s, is known, then the maximum altitude η_{max} that the telecope can point to is

$$\eta_{\mathrm{max}} = \cos^{-1}\left(\frac{\Omega}{\dot{\zeta}_{\mathrm{max}}}\cos\xi_0\cos\phi_{\mathrm{lat}}\right) \tag{24}$$

For example, for where I live, Chicago's latitude is $\phi_{\mathrm{lat}} = 41.8369°$ and when I choose $\dot{\zeta}_{\mathrm{max}} = 1$ Hz $= 2\pi$ rad/s for maximum stepper motor torque, I obtain $\eta_{\mathrm{max}} = 89.9995°$

when $\xi_0 = 90^\circ$ (E) or 270° (W). Thus, in principle, as long as I do not point the telescope towards the zenith, the stepper motor revolution frequency is not the limiting factor. The only other integration time limiting factor comes from the loss of guide axes integrity which I will discuss in section IV B.

IV. OFF-AXIS GUIDING

Now, here's the section where I start discussing the limitations that field de-rotation introduces to off-axis guiding. I would like to remind the reader that "tracking" is distinct from "guiding" in this paper. Their definitions have been discussed in section I A. Here are some interesting consequences when de-rotation is added to tracking to off-axis guidng

(a) In off-axis guiding, a guide star S_g is chosen that is never at the optical axis of the telescope. (Note: In general, the rotation axis of the de-rotation motor is at the optical axis of the telescope.) It does seem, at first glance, that de-rotation adds to the guide error because the two axes are not coincident. However, I will demonstrate in the following section that, in fact, de-rotation coupled with tracking *reduces* guide errors. Unfortunately, this piece of good fortune does not persist indefinitely. See next item.

(b) As the field de-rotation mechanism rotates the camera, the up-down (traditionally called N-S) and left-right (traditionally called E-W) axes rotate away from their initial directions. At some point, the off-axis guiding software will fail because the axes have moved too far from their initial directions. This loss of axes integrity limits the integration time for photographing deep sky objects.

In the next subsections, I will analyze both (a) and (b).

A. Reduction of off-axis guiding error by de-rotation and tracking

I can demonstrate how the guide error is reduced to zero when a field de-rotator is added to alt-az tracking in the ideal world. I have drawn Fig. 8 to show the view through a telescope where there are two stars S_0 and S_g. The star S_0 is the track star that lies at the centre of the FOV and S_g is the guide star that lies at some position offset from S_0. Fig. 8(a) shows what happens at $t = 0$ and $t = \delta t$ later when only alt-az tracking is used to

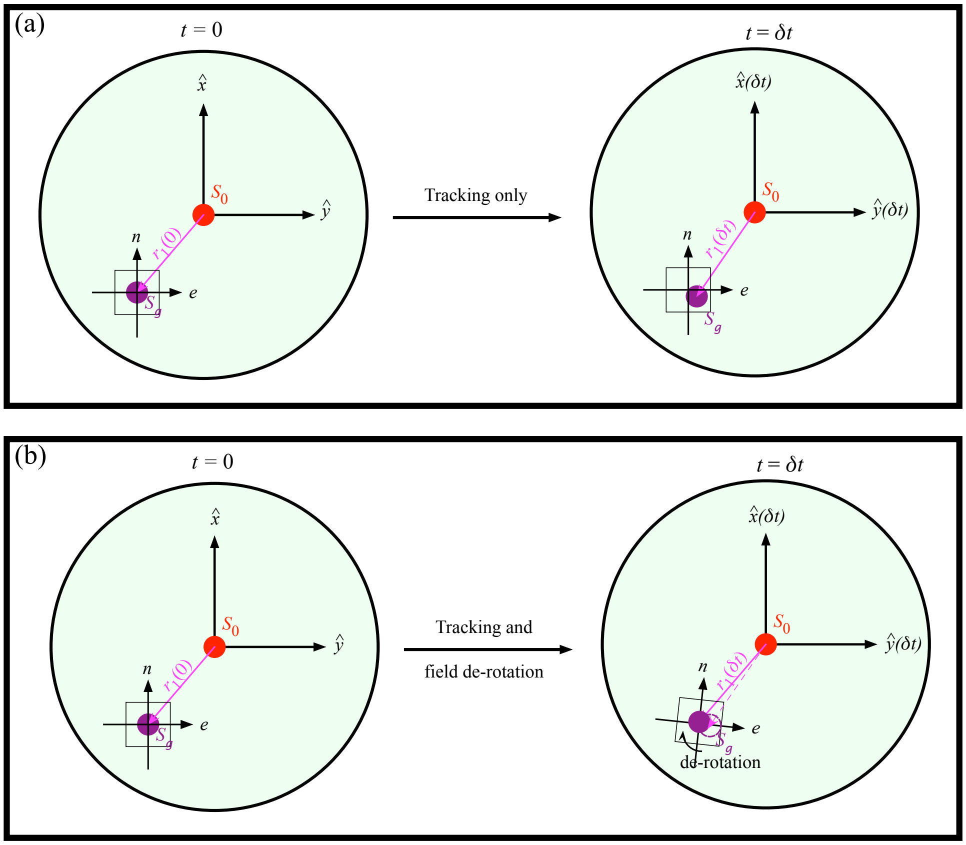

FIG. 8. These are the views through a telescope where its optical axis has been aligned to S_0 and a star S_g has been chosen to be the guide star. S_g lies in the cross-hairs of the off-axis guider. If alt-az tracking is the only method used to keep S_0 centred in the FOV, (a) shows that S_g rotates away from the off-axis cross-hairs due to field rotation that I had discussed earlier, while (b) is the case where both tracking and field-rotation is applied. In this case, S_g remains on the cross-hairs of the off-axis guider. Thus, de-rotation has *removed* any guide errors! However, notice that de-rotation has caused the guide reference axes $(\boldsymbol{n}, \boldsymbol{e})$ to rotate from their initial directions.

keep S_0 in the centre of the FOV. While S_0 remains fixed at the centre of the FOV, S_g has rotated away from the cross-hairs of the off-axis guider. However, when field de-rotation is added to alt-az tracking, S_g remains fixed in the cross-hairs of the off-axis guider. Thus, if both alt-az tracking and de-rotation are perfect, the off-axis guider has nothing to do! Unfortunately, neither mechanisms are perfect in real life and therefore there will be errors that the off-axis guider have to correct. Here is where the limitations to the integration time comes in: de-rotation rotates the guide reference axes $(\boldsymbol{e}, \boldsymbol{n})$ shown in Fig. 8(b) away from their original directions.

In the next subsection, I will complete the analysis of the loss of guide axes integrity. After that, I will do an analysis of how the errors from de-rotation and alt-az tracking affect off-axis guiding accuracy.

B. Loss of guide axes integrity

The loss of guide axes integrity is illustrated in Fig. 8(b) at $t = \delta t$ where due to field de-rotation, the formerly up-down ($\boldsymbol{n}$), left-right ($\boldsymbol{e}$) cross-hairs shown in the other FOV's have rotated so that they are no longer perfectly up-down or left-right. As time marches on, the $\boldsymbol{n}$ and $\boldsymbol{e}$ orthogonal vectors that make up the guide axes rotate further and further away from their initial directions and thus at some point in time, the off-axis guider cannot cope with the change in directions of the axes $\boldsymbol{n}$ and $\boldsymbol{e}$ because the alt-axis program no longer knows how to direct the alt-az motors to move the telescope up-down and left-right. This loss of guide axes integrity limits the integration time that can be used to photograph deep sky objects. In the next two section, I will calculate both the de-rotation angle and integration time limitations.

1. De-rotation angle limit

The first thing I need to do is to define the metric that tells me how well the alt-az tracking system performs after its original $(\boldsymbol{e}, \boldsymbol{n})$ axes rotates by an angle ζ to the new $(\boldsymbol{n}', \boldsymbol{e}')$ axes due to the actions of the field de-rotator.

Let me suppose that the guide star S_g is no longer at the centre of the cross hairs. See Fig. 9. The alt-az guider queries the camera to tell it where S_g is. The camera tells the alt-az guider that S_g is at polar coordinates (ε, γ) in the $(\boldsymbol{e}', \boldsymbol{n}')$ frame. The alt-az guider now knows that S_g is no longer at the centre of the cross hairs and thus applies the following correction to bring S_g back to the centre of the cross-hairs

$$\begin{aligned} \Delta e &= -\epsilon \cos \gamma \\ \Delta n &= -\epsilon \sin \gamma \end{aligned} \tag{25}$$

where Δe is the amount of correction in the $\boldsymbol{e}$ direction and Δn is the amount of correction in the $\boldsymbol{n}$ direction. Notice that these corrections are in the $(\boldsymbol{e}, \boldsymbol{n})$ frame because the off-axis

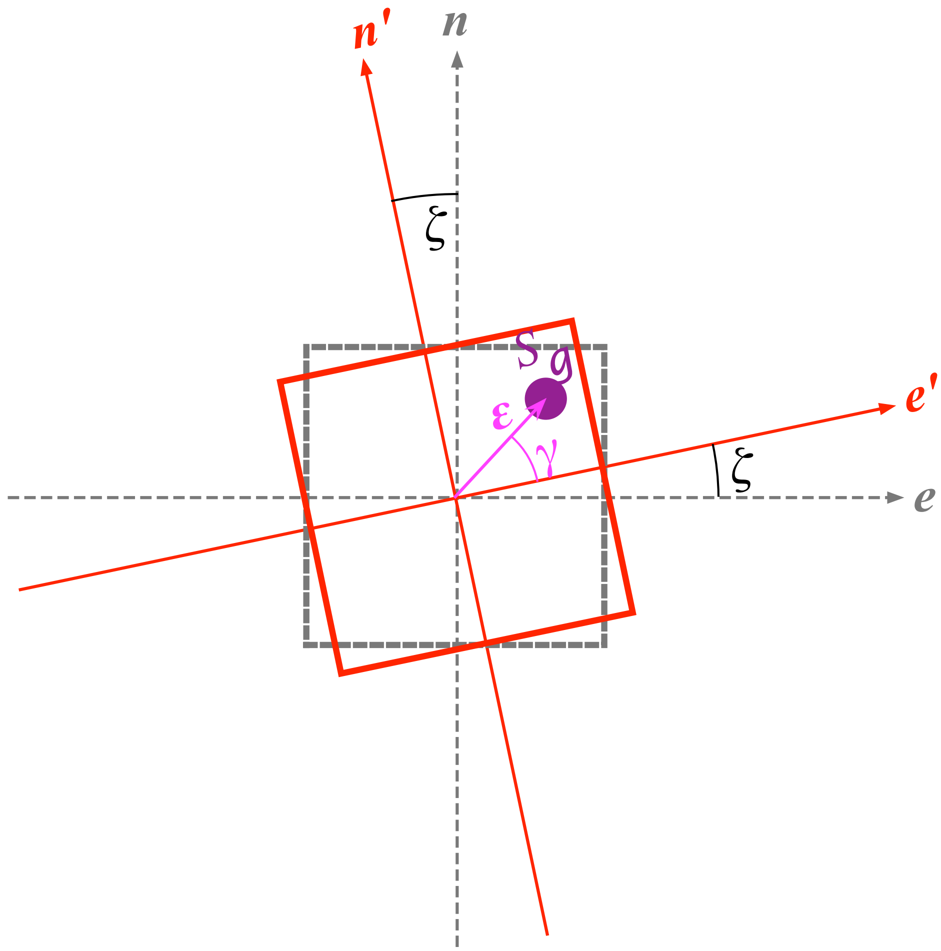

FIG. 9. The guide reference axes $(\boldsymbol{e}, \boldsymbol{n})$ is rotated by an angle ζ to $(\boldsymbol{n}', \boldsymbol{e}')$. The guide star S_g is offset from the centre of the cross hairs by the polar coordinates (ϵ, γ) w.r.t. the $(\boldsymbol{n}', \boldsymbol{e}')$ axes and $(\epsilon, \gamma + \zeta)$ w.r.t. the $(\boldsymbol{e}, \boldsymbol{n})$ axes.

guider program does not know that its guide axis has rotated. Thus the correction has not completely moved S_g back to the centre of the cross hairs. In fact, the residual errors are

$$\begin{aligned} \delta e &= \varepsilon[\cos(\gamma + \zeta) - \cos\gamma] \\ \delta n &= \varepsilon[\sin(\gamma + \zeta) - \sin\gamma] \end{aligned} \tag{26}$$

which tells me the added corrections, δe and δn, that are needed for the the alt-az guider to bring S_g back to the centre of the cross-hairs. These two numbers allow me to create an error radius $\delta\varepsilon$

$$\begin{aligned}\delta\varepsilon &= \sqrt{\delta e^2 + \delta n^2} \\ &= \varepsilon\sqrt{\left[\cos(\gamma+\zeta) - \cos\gamma\right]^2 + \left[\sin(\gamma+\zeta) - \sin\gamma\right]^2}\end{aligned} \tag{27}$$

Of course, the whole point of alt-az guiding is to require that the correction error be smaller than the error before correction, i.e. I require $\delta\varepsilon < \epsilon$ or else the guiding process becomes rather pointless! This inequality means that

$$\sqrt{\left[\cos(\gamma+\zeta) - \cos\gamma\right]^2 + \left[\sin(\gamma+\zeta) - \sin\gamma\right]^2} < 1 \tag{28}$$

If the initial angular error γ is uniformly distributed between 0 and 2π, the probability density function is

$$p(\gamma) = \begin{cases} \frac{1}{2\pi} & 0 \leq \gamma < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

then the mean value[3] of $\delta\varepsilon$ comes from integrating Eq. 28 and is

$$\begin{aligned}\int_{-\infty}^{\infty} d\gamma\, \sqrt{\left[\cos(\gamma+\zeta) - \cos\gamma\right]^2 + \left[\sin(\gamma+\zeta) - \sin\gamma\right]^2} p(\gamma) &< \int_{-\infty}^{\infty} d\gamma\, \left[1 \times p(\gamma)\right] \\ \sqrt{2(1-\cos\zeta)} &< 1 \\ \Rightarrow \quad \cos\zeta &> 1/2\end{aligned} \tag{30}$$

and thus $|\zeta| < \pi/3$ or 60°. This is a surprisingly large number that is independent of camera pixel size, focal length of the telescope etc. It only depends on how much the camera has rotated away from its initial position. How long the camera takes to rotate by 60° is in turn dependent on which star the telescope is pointing at. In the next section, I will calculate the maximum integration time for every point on the celestial sphere using this rotation angle limit.

C. Maximum integration time

The maximum integration time $t_{\max}$ is limited by the result from the previous section

$$|\zeta| = \left| \Omega \cos\phi_{\text{lat}} \int_0^{t_{\max}} dt\, \frac{\cos\xi(t)}{\cos\eta(t)} \right| < \pi/3 \tag{31}$$

and from Eq. 23 with $\xi(0) = \xi_0$ and $\eta(0) = \eta_0$.

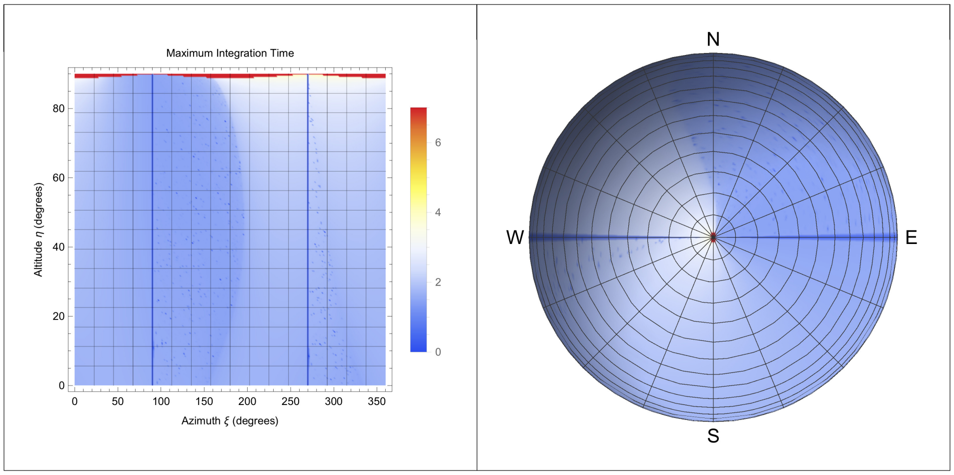

FIG. 10. The maximum integration time for each alt-az coordinate is plotted as a density plot as a projection onto a Cartesian coordinate system (left) and onto a hemisphere (right) as seen from the zenith. The integration times have been normalized by the function $-\log_{10}(t_{\max}/10^6)$ in these plots where 10^6 s has been used to represent infinite integration time. The longest integration time (value 0) is coloured blue while shortest is coloured red (value 7). As expected, as the telescope approaches the zenith $\eta = 90°$, the allowed integration time gets very short because the de-rotation frequency $\dot{\zeta} \to \infty$.

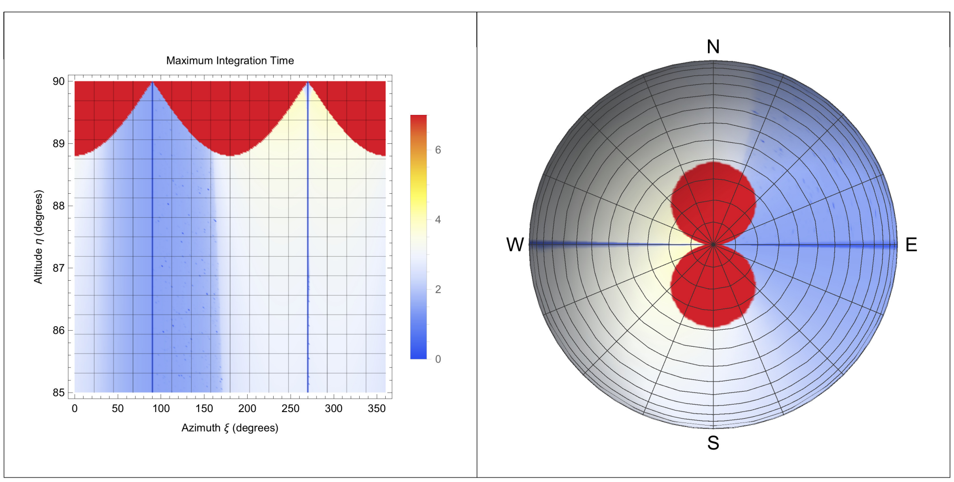

FIG. 11. Zoomed in view from $\eta = 85°$ to $90°$ of Fig. 10.

From Eq. 1, I can write $\cos\xi$ and $\cos\eta$ in Cartesian coordinates as

$$\begin{aligned}
\tan\xi &= -\frac{Y}{X} \quad \Rightarrow \quad \cos\xi = \frac{X}{\sqrt{X^2+Y^2}} \\
\tan\eta &= \frac{Z}{X}\cos\xi = \frac{Z}{\sqrt{X^2+Y^2}} \quad \Rightarrow \quad \cos\eta = \sqrt{\frac{X^2+Y^2}{X^2+Y^2+Z^2}}
\end{aligned} \tag{32}$$

Using the above results, I have

$$\frac{\cos\xi}{\cos\eta} = \frac{X\sqrt{X^2+Y^2+Z^2}}{X^2+Y^2} \tag{33}$$

and to see an explicit dependence on t on the rhs of the above, I can use Eq. 15 to get the equations to substitute into the rhs. These equations are:

$$\begin{aligned}
X(t) &= \cos\phi_{\mathrm{lat}}(X_0\cos\phi_{\mathrm{lat}} + Z_0\sin\phi_{\mathrm{lat}})(1-\cos\Omega t) + X_0\cos\Omega t - Y_0\sin\phi_{\mathrm{lat}}\sin\Omega t \\
Y(t) &= Y_0\cos\Omega t + (X_0\sin\phi_{\mathrm{lat}} - Z_0\cos\phi_{\mathrm{lat}})\sin\Omega t \\
Z(t)^2 &= R^2 - X(t)^2 - Y(t)^2
\end{aligned} \tag{34}$$

and Eq. 1 to write Eq. 33 in terms of the initial alt-az coordinates $(\eta_0, \xi_0,)$ of the star.

1. Numerical solution

I can obtain an analytic solution of ζ by integrating Eq. 31 after I have substituted in Eq. 33 and 34 into it. However, this solution is not very illuminating because it is quite complicated. Rather than wade through this algebraic morass, I will, instead, generate an example of the maximum integration time for every point on the celestial sphere at my Chicago location. The results are shown in Fig. 10 and 11.

In general, from these numerical results and those from section III C 1, I have found that as long as the telescope points between $0^\circ \le \eta < 85^\circ$, the minimum integration time is about 24 minutes and there are no limitations that arise from the stepper motor rotation speed.

Appendix: LX200 Alt-Az coordinate system

The LX200 defines its alt-az coordinate system differently than what I have done. The 0° of the azimuth for the LX200 is when it points South rather than North. My investigation shows that the azimuth relationship between the azimuth of the LX200, ξ_{LX200} and the azimuth ξ_0 discussed in this paper is:

$$\xi_{\mathrm{LX200}} = \xi_0 + 180^\circ \tag{A.1}$$

and if the result $\xi_{\mathrm{LX200}} > 360^\circ$, then $\xi_{\mathrm{LX200}} \to \xi_{\mathrm{LX200}} - 360^\circ$.

And going the other way, I have

$$\xi_0 = \xi_{\mathrm{LX200}} - 180^\circ \tag{A.2}$$

and if the result $\xi_0 < 0$ then $\xi_0 \to \xi_0 + 360^\circ$.

Thus, the derotation speed is related by a sign, i.e. $\dot{\zeta}_{\mathrm{LX200}} = -\dot{\zeta}_0$ between these two coordinate systems.

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[1] The off-axis guider is needed because the alt-az guider must see the guide star in the same field of view as the camera.

[2] The alt-az coordinate system is also called the horizonal coordinate system.

[3] A.A. Sveshnikov. *Problems in probability theory, mathematical statistics and theory of random functions*, chapter IV. Dover Publications, 1968.