EQ2341 Assignment 1

Yijie Li

May 20, 2025

1 Markov chain code verification

We consider the infinite-duration HMM defined by:

$$q = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}, \quad A = \begin{bmatrix} 0.99 & 0.01 \\ 0.03 & 0.97 \end{bmatrix}$$

Target: Calculate $P(S_t = j)$ theoretically and compare it with relative frequencies from codes. Let us denote the state distribution at time t as:

$$\pi_t = \begin{bmatrix} P(S_t = 1) \\ P(S_t = 2) \end{bmatrix}$$

At the initial time $t = 1$, the distribution is given by the initial probability vector:

$$\pi_1 = q = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

According to the Markov state transition equation:

$$\pi_{t+1} = A^T \cdot \pi_t$$

Let us compute the next few steps:

$$\pi_2 = A^T \cdot \pi_1 = \begin{bmatrix} 0.99 & 0.03 \\ 0.01 & 0.97 \end{bmatrix} \cdot \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

Here we find that $\pi_2 = \pi_1$. Further, it's easy to find that $\pi_t = \pi_{t-1} = ...\pi_2 = \pi_1$. The distribution is constont over time. This is the stationary distribution.

Through running the code, over the sequence of length $T = 10000$, relative frequency of $S_t = 1$ and $S_t = 2$ are 0.7369 and 0.2631, close to the result above.

2 HMM rand method verification

We define the output distributions as:

- $b_1(x) \sim \mathcal{N}(0, 1)$

- $b_2(x) \sim \mathcal{N}(3, 4)$

Following $\mu_X = E[X] = E_Z[E_X[X|Z]]$ and $var[X] = E_Z[var_X[X|Z]] + var_Z[E_X[X|Z]]$, we have

$$\mu_X = E_S[E_X[X|S]] = E_X[X|S=1]P(S=1) + E_X[X|S=2]P(S=2) = 0.75 \cdot 0 + 0.25 \cdot 3 = 0.75$$

Similarly,

$$var[X] = P(S=1)var_X[X|S=1] + P(S=2)var_X[X|S=2]+ \tag{1}$$

$$P(S=1)\left(E_X[X|S=1] - E[X]\right)^2 + P(S=2)\left(E_X[X|S=2] - E[X]\right)^2 \tag{2}$$

$$= 3.4375 \tag{3}$$

Measured results

When T=100000, the measured results are close enough to the result above.

- Sample mean of X_t: `0.7449`

- Sample variance of X_t: `3.4254`

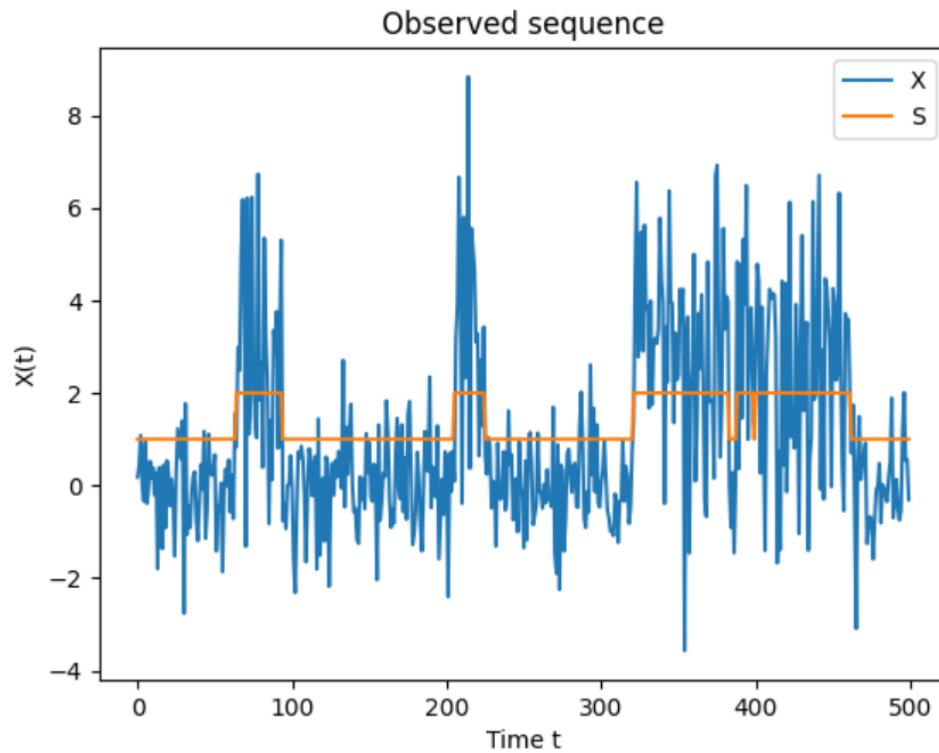# 3. Plot of 500-Element Output Sequence

Figure 1: Sample output X_t for $t = 1, \ldots, 500$ from the infinite-duration HMM

Observed behavior: The output values exhibit piecewise smooth segments around means 0 and 3, reflecting long stays in each state due to high self-transition probabilities.
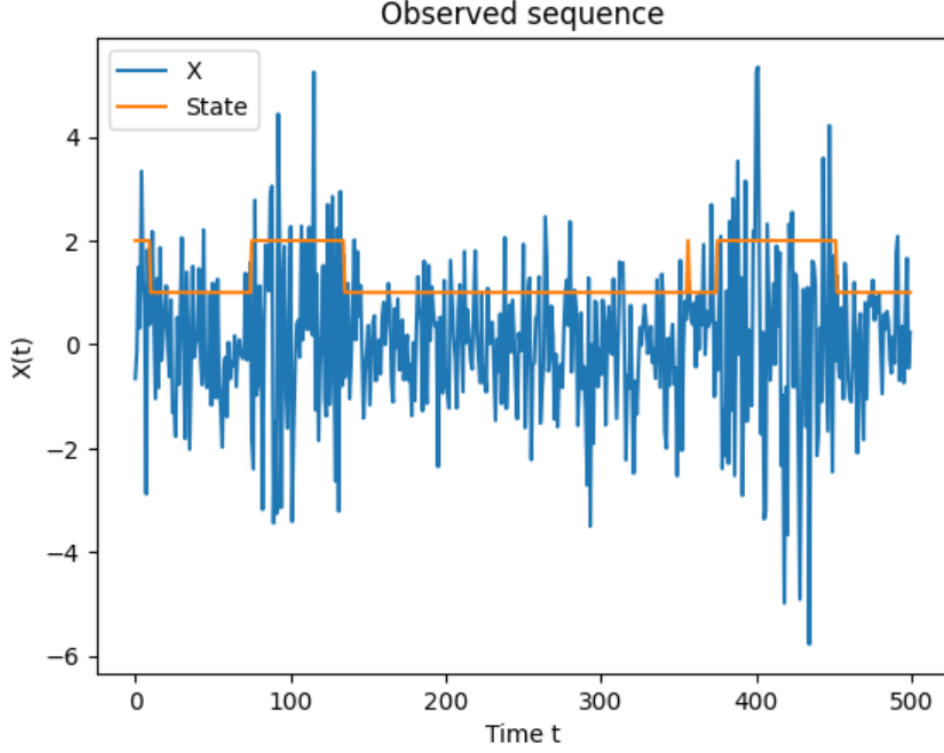
4. Second Infinite-Duration HMM Discussion

Figure 2: Caption

The hidden state sequence S_t is still generated by the same Markov chain.Therefore, the temporal structure remains present.The overall statistical behavior remains unchanged.

However,in the previous HMM the outputs from state 1 (mean 0) and state 2 (mean 3) were clearly separable while in this HMM, both states produce output centered at 0, but with different variances.

In the previous HMM, state estimation was relatively easy due to large mean differences, we can sample and estimate the mean value over a short period of time to estimate if it's state 1 or state 2. Since the variances in the new HMM differ, it is still possible (though less reliable) to infer likely states. For example, we can sample and estimate the variance(energy) over a short period of time to decide its state.

5. Finite-Duration HMM Definition and Discussion

Definition

$$q = [0.75, 0.25, 0], \quad A = \begin{bmatrix} 0.75 & 0.2 & 0.05 \\ 0.2 & 0.75 & 0.05 \end{bmatrix}$$

Test results

Observed sequence lengths: [36. 45. 17. 2. 24. 29. 18. 10. 3. 4.]

Discussion

The sequence generation terminates upon reaching state 3. The observed lengths are consistent with the termination probability.

6. Vector-Valued HMM Verification

Definition

The Markov chain is as follows:

$$q = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}, \quad A = \begin{bmatrix} 0.99 & 0.01 \\ 0.03 & 0.97 \end{bmatrix}, \quad b = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

We define a 2D Gaussian output for each state.

$$\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$
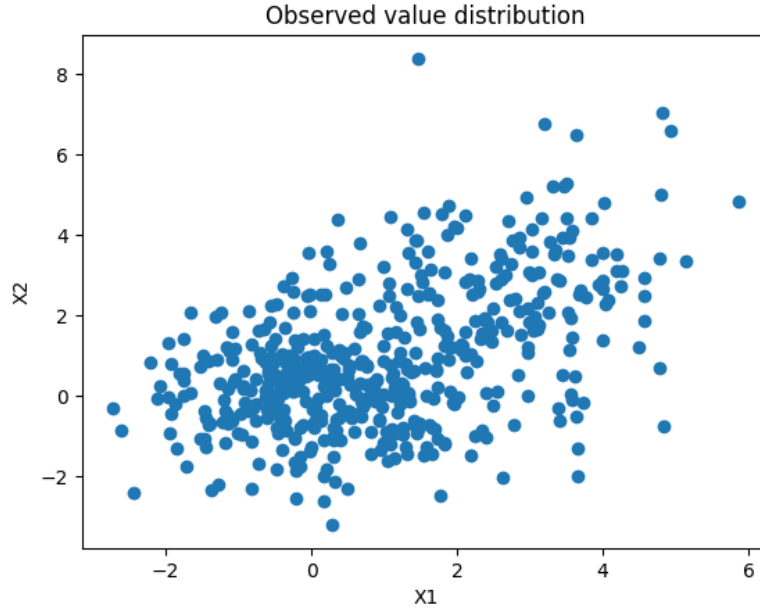

Figure 3: scatter plot of the 2D observations

The scatter plot of the 2D observations clearly demonstrates a tilted elliptical shape, with most points spreading along a positively sloped direction. This indicates that the two components, X_1 and X_2, are positively correlated.

This visual structure directly reflects the non-diagonal covariance matrix used in the HMM output distribution for state 1:

$$\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

The non-zero off-diagonal element (1) implies a positive linear relationship between the two dimensions.

Code Snippet

```
T = 500
# initialize Markov chain
q = np.array([0.75, 0.25])
```

```
A = np.array([[0.99, 0.01], [0.03, 0.97]])
mc = MarkovChain(q, A)


# initialize HMM
means1 = np.array([0, 0])
cov1 = np.array([[1, 0], [0, 1]])
means2 = np.array([2, 2])
cov2 = np.array([[2, 1], [1, 4]])
g1 = GaussD(means=means1, cov=cov1)  # Distribution for state = 1
g2 = GaussD(means=means2, cov=cov2)  # Distribution for state = 2

h = HMM(mc, [g1, g2])
X, S = h.rand(T)
t = np.arange(len(S))


cov_matrix = np.cov(X)  # shape = (2, 2)
print("Sample covariance matrix:\n", cov_matrix)


plt.plot(X[0], X[1], 'o', label='X')
plt.title("Observed value distribution")
plt.xlabel("X1")
plt.ylabel("X2")
```