공업수학2 6강

Chapter 7. Linear Algebra . Matrices vectors

linear transform $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

$$T(k\bar{u}) = kT(\bar{u})$$

$$\tilde{A}(\bar{u}+\bar{v}) = \tilde{A}\bar{u} + \tilde{A}\bar{v}$$

$$\tilde{A}(k\bar{u}) = k\tilde{A}\bar{u}$$

System of linear equations

$$\begin{aligned} x_1 + x_2 &= 5 \\ x_1 - x_2 &= 2 \end{aligned} \quad ; \quad \tilde{A}\cdot\bar{x} = \bar{b} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$\tilde{A}^*$; augmented matrix $[\tilde{A} ; \bar{b}]$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -2 & -3 \end{array}\right]$$

$$\tilde{A}^{(1)} ; \bar{b}^{(1)}$$

- GAUSS elimination and back substitution

$[m \times n]$: row echelon form ← elementary row operations

해는 3가지 안에 있음.

$$\left[\begin{array}{cccc|c} r_{11} & \cdots & & r_{1n} & f_1 \\ & r_{22} & & r_{2n} & f_2 \\ 0 & & \ddots & r_{rn} & f_r \\ & 0 & & & \vdots \\ & & & & f_n \end{array}\right]$$

1) no solution if $f_k \neq 0$ $(k>r)$

$\text{rank}(A) \neq \text{rank}(A^*)$

2) unique solution

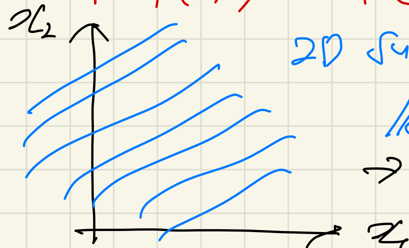
$\text{rank}(A^*) = \text{rank}(A) = n$.

3) Infinitely many solution.

$\rightarrow r < n$.

$\text{rank}(A) = \text{rank}(A^*) < n$.

(ex) (1) $x_1 + x_2 = 5$, $x_1 - x_2 = 2$



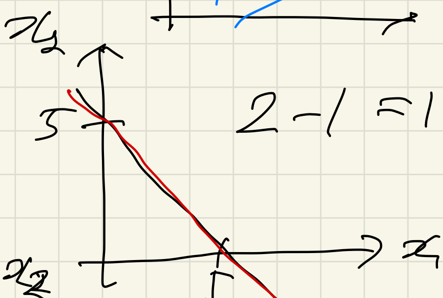
2D surface

$\mathbb{R}^2$

→ 해의 공간

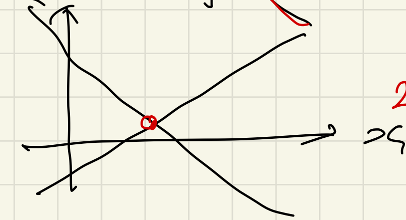
(2) $x_1 + x_2 = 5$

~~$x_1 - x_2 = 2$~~



$2 - 1 = 1$ $\mathbb{R}^1$ - 1D line

(3) $x_1 + x_2 = 5$, $x_1 - x_2 = 2$



$\mathbb{R}^0$

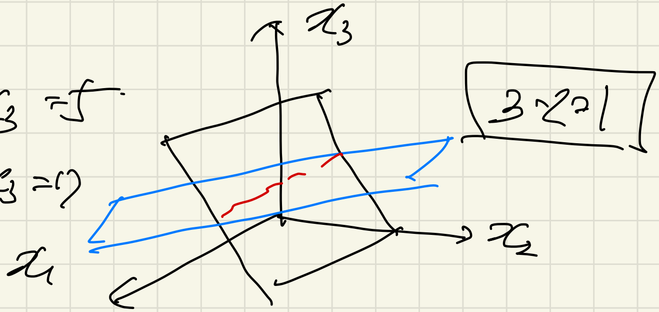
$2 - 2 = 0$.

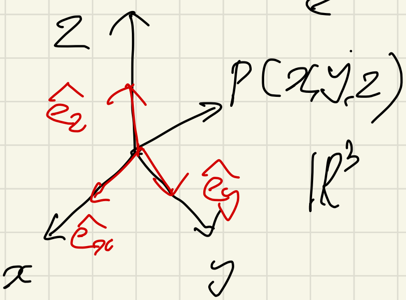
$$\begin{cases} x_1 - x_2 + x_3 = 0 & \text{— ①} \quad \mathbb{R}^2 : 3-1=2 \\ \cancel{-x_1 + x_2 - x_3 = 0} & \quad \mathbb{R}^3 ;\ 3-2 \\ \quad 10x_2 - 25x_3 = 90 & \text{— ②} \\ \cancel{20x_1 + 10x_2} \quad = 80 & \text{— ③} \quad \mathbb{R}^3 : 3-3=0 \end{cases}$$

ex2)

$x_1 + x_2 + x_3 = 5$

$x_1 - x_2 + 2x_3 = 17$





; vector space

$\mathbb{R}^3$

- $\mathrm{Rank}(A) = \mathrm{Rank}(A^T)$
- $[1, -1, 0]$ ✗
- $[-1, 2, 4]$
- $[1, 5, 7]$
- $[1, 1, 1]$

7.7) Determinant Cramer's Rule.

$$D = \det(\tilde{A}) = \begin{vmatrix} a_{11} & \cdots & a_{n1} \\ a_{21} & \ddots & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{vmatrix}$$

$$= \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{or.} \quad (-1)^{j+k} M_{jk}$$; Cofactor

M_{jk} ; minor

$$= \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$+ 0 \, (-1)^{1+3} \begin{vmatrix} 2 & 6 \\ -1 & 2 \end{vmatrix}$$

General properties of Determinants

(a) interchange → multiply

(b) $c\,\overline{a}_{(1)} + \overline{a}_{(2)}$ → does not alter

(c) $c\,\overline{a}_{(1)}$ → multiply by c

pivot

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

0

pivot

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix}$$

0 pivot 0

$$\frac{114}{24} = \frac{57}{12} = \frac{19}{4}$$

$$\overset{1.9}{\cancel{3.8}} \times \frac{19}{\cancel{4}_{2}} = \frac{36.1}{2} = 18.05$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix}$$

$$= 2 \times 5 \times 2.4 \times 47.25$$

$$= \boxed{1134}$$

⟺ Cramer's Rule. - ~

- a linear system of n equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

has a nonzero $D = \det(\tilde{A})$ the solution is given by $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, $\cdots$ $x_n = \frac{D_n}{D}$

D_k is obtained by replacing in D the kth column by the column with the entries $b_1, b_2 \cdots, b_n$

Ex) $a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$ $\Rightarrow x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

7.8) Inverse of a matrix

Gauss-Jordan Elimination

- $\tilde{A}\bar{x} = \bar{b} \rightarrow \tilde{A}^{-1}\tilde{A}\cdot\bar{x} = \tilde{A}^{-1}\cdot\bar{b},\ \bar{x} = \tilde{A}^{-1}\cdot\bar{b}$
- Inverse ; $\tilde{A}\cdot\tilde{A}^{-1} = \tilde{A}^{-1}\tilde{A} = \tilde{I}$

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

→ the inverse $\tilde{A}^{-1}$ of an $n \times n$ matrix $\tilde{A}$ exists if and only if (rank $\tilde{A} = n$) ; thus if and only if $\boxed{\det \tilde{A} \neq 0}$

→ nonsingular if rank $\tilde{A} = n \rightarrow \boxed{\tilde{A}^{-1} \text{ exists.}}$

singular if rank $\tilde{A} < n$

- Determination of the Inverse by the Gauss-Jordan - methode.

; guass elimination $\tilde{A}\cdot\bar{x} = \bar{b}$

$\Rightarrow$ augmented matrix $= [\tilde{A} ; \tilde{b}]$

$\Downarrow$ elementation. $\Rightarrow$ 변형

$[\tilde{A}^{(1)} ; \tilde{b}^{(1)}]$ $\Rightarrow$ Linear Combination of vectors.

$\Downarrow$

row echelon form.

$[\underbrace{\tilde{R}}_{r.} ; \tilde{b}^{*}] \rightarrow \boxed{\tilde{R}\tilde{X} = \tilde{b}^{*}}$

$\Rightarrow$ matrix Inverse.

$\tilde{A} \cdot \tilde{X} = \tilde{I}$, $\tilde{X} = \tilde{A}^{-1}$

$[\tilde{A} ; \tilde{I}] \rightarrow$ elementary operation.

$[\tilde{A}^{(1)} ; \tilde{I}^{(1)}] \rightarrow \tilde{A}^{(1)} \cdot \tilde{X} = \tilde{I}^{(1)}$

$\vdots$

$[\tilde{A}^{(n)} ; \tilde{I}^{(n)}] \Rightarrow \tilde{A}^{(n)} \cdot \tilde{X} = \tilde{I}^{(n)}$

$\Downarrow$

$\tilde{I}$; Gauss-Jordan-Elimination

$\tilde{I}^{(n)}$; $\tilde{I} \cdot \tilde{X} = \tilde{I}^{(n)}$

$\tilde{X} = \tilde{I}^{(n)} = \tilde{A}^{-1}$

Ex1) Determine inverse of

$$\tilde{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix},,$$

$\tilde{A}\tilde{X} = \tilde{I}$

$\tilde{A}^{-1}$

Guass - Jordon elimination

$$\tilde{A}^* = [\tilde{A} \,;\, \tilde{I}]$$

pivot

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ +3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Guass elimination

I elementary row operation.

pivot

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Pivot - Jordan elimination

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

pivot

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\underbrace{}_{\tilde{I}} \qquad \underbrace{}_{\boxed{\tilde{A}^{-1}}}$$