Problem Set 7.3

(13)

$$10x + 4y - 2z = 14$$
$$-3w - 15x + y + 2z = 0$$
$$w + x + y = 6$$
$$8w - 5x + 5y - 10z = 26$$

$$\begin{bmatrix} 0 & 10 & 4 & -2 & 14 \\ -3 & -15 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 6 \\ 8 & -5 & 5 & -10 & 26 \end{bmatrix}$$

Pivot

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ -3 & -15 & 1 & 2 & 0 \\ 0 & 10 & 4 & -2 & 14 \\ 8 & -5 & 5 & -10 & 26 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & -12 & 4 & 2 & 18 \\ 0 & 10 & 4 & -2 & 14 \\ 0 & -13 & -3 & -10 & -22 \end{bmatrix}$$

Pivot

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & -12 & 4 & 2 & 18 \\ 0 & 0 & \frac{22}{3} & -\frac{1}{3} & 29 \\ 0 & 0 & -\frac{22}{3} & -\frac{73}{6} & -\frac{83}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & -12 & 4 & 2 & 18 \\ 0 & 0 & \frac{22}{3} & -\frac{1}{3} & 29 \\ 0 & 0 & 0 & -\frac{25}{2} & -\frac{21}{2} \end{bmatrix}$$

Pivot

* backsubstitution.

$$\begin{cases} w + x + y = 6 \\ -12x + 4y + 2z = 18 \\ \frac{22}{3}y - \frac{1}{3}z = 29 \\ -\frac{25}{2}z = -\frac{25}{2} \end{cases}$$

$\therefore z = 1$

$\frac{22}{3}y - \frac{1}{3} = 29$

$22y - 1 = 87$ $\quad$ $22y = 88$

$\therefore y = 4$

$-12x + 16 + 2 = 18$. $\quad$ $-12x = 0$ $\quad$ $\therefore x = 0$

$w + 0 + 4 = 6$ $\quad$ $\therefore w = 2.$

$\boxed{\therefore w = 2,\ x = 0,\ y = 4,\ z = 1}$

14)

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 1 & -1 & 3 & -3 & 3 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 10 & -4 & 10 & -8 & 10 \\ 2 & -2 & 6 & -6 & 6 \\ 6 & 12 & -21 & 6 & -21 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 0 & -19 & 5 & 49 & 5 \\ 0 & -5 & 5 & 5 & 5 \\ 0 & 3 & -24 & 39 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 0 & -5 & 5 & 5 & 5 \\ 0 & 3 & -24 & 39 & -24 \\ 0 & -19 & 5 & 49 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 0 & -5 & 5 & 5 & 5 \\ 0 & 0 & -21 & 42 & -21 \\ 0 & 0 & -14 & 28 & -14 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 0 & -5 & 5 & 5 & 5 \\ 0 & 0 & -21 & 42 & -21 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\tilde{A}$ $\tilde{A}^*$

$\boxed{\text{rank}(\tilde{A}) = \text{rank}(\tilde{A}^*) < n.}$

$\therefore$ infinitely many solution.

Problemset - 2.4

8)

pivot

$$\tilde{A} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 2 & 16 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \\ 0 & 12 & 0 & -12 \end{bmatrix}$$

pivot

$$= \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 12 & 0 & -12 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 12 & 0 & -12 \\ 0 & 0 & -60 & -150 \\ 0 & 0 & 0 & -30 \end{bmatrix}}$$

rank$(\tilde{A}) = 4$, 행공간 기저: $[2, 4, 8, 16]$, $[0, 12, 0, -12]$, $[0, 0, -60, -150]$, $[0, 0, 0, -30]$

pivot pivot

$$\tilde{A}^T = \begin{bmatrix} 2 & 16 & 4 & 2 \\ 4 & 8 & 8 & 16 \\ 8 & 4 & 16 & 8 \\ 16 & 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 16 & 4 & 2 \\ 0 & -24 & 0 & 12 \\ 0 & -60 & 0 & 0 \\ 0 & -126 & -30 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 16 & 4 & 2 \\ 0 & -24 & 0 & 12 \\ 0 & 0 & 0 & -30 \\ 0 & 0 & -30 & -15 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 16 & 4 & 2 \\ 0 & -24 & 0 & 12 \\ 0 & 0 & -30 & -15 \\ 0 & 0 & 0 & -30 \end{bmatrix}}$$

$\text{rank}(\tilde{A}) = \text{rank}(\tilde{A}^T) = 4$

열공간 기저 : $[2, 16, 4, 2]^T$, $[0\ \ -24\ \ 0\ \ 12]^T$, $[0\ \ 0\ \ -30\ \ -25]^T$, $[0\ \ 0\ \ 0\ \ -30]$

(b) Pivot

$$\tilde{A} = \begin{bmatrix} 5 & -2 & 1 & 0 \\ -2 & 0 & -4 & 1 \\ 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ -2 & 0 & -4 & 1 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Pivot

$$= \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & -8 & -26 & 5 \\ 0 & 18 & 56 & -10 \\ 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -8 & -26 & 5 \\ 0 & 18 & 56 & -10 \end{bmatrix}$$

Pivot

$$= \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 20 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 20 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(\tilde{A}) = 3$

행공간 기저 : $[1\ \ -4\ \ -11\ \ 2]$, $[0\ \ 1\ \ 2\ \ 0]$, $[0\ \ 0\ \ 20\ \ -10]$

$$\tilde{A}^T = \begin{bmatrix} 5 & -2 & 1 & 0 \\ -2 & 0 & -4 & 1 \\ 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -11 & 2 \\ -2 & 0 & -4 & 1 \\ 0 & 1 & 2 & 0 \\ 5 & -2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -11 & 2 \\ 0 & -8 & -26 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 18 & 56 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -8 & -26 & 5 \\ 0 & 18 & 56 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 20 & -10 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & -4 & 11 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 20 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$\text{rank}(\tilde{A}^T) = \text{rank}(\tilde{A}) = 3$

열공간 기저 : $[1\ -4\ 11\ 2]^T$, $[0\ 1\ 2\ 0]^T$, $[0\ 0\ 20\ -10]^T$

$\boxed{\text{Problem set 7.7}}$

(2) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (-1)^{1+1} a \begin{vmatrix} a & b \\ c & a \end{vmatrix} + (-1)^{1+2} b \begin{vmatrix} c & b \\ b & a \end{vmatrix}$

$+ (-1)^{1+3} c \begin{vmatrix} c & a \\ b & c \end{vmatrix}$

$= a(a^2 - bc) - b(ac - b^2)$

$+ c(c^2 - ab) = \boxed{a^3 + b^3 + c^3 - 3abc}$

(3)

$$\begin{vmatrix} 0 & 4 & -1 & 5 \\ -4 & 0 & 3 & -2 \\ 1 & -3 & 0 & 1 \\ -2 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \overset{\text{Pivot}}{1} & -3 & 0 & 1 \\ -4 & 0 & 3 & -2 \\ 0 & 4 & -1 & 5 \\ -2 & 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & -12 & 3 & 2 \\ 0 & 4 & -1 & 5 \\ 0 & -5 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & \overset{\text{Pivot}}{4} & -1 & 5 \\ 0 & -12 & 3 & 2 \\ 0 & -5 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & 0 & 17 \\ 0 & 0 & -\frac{1}{4} & \frac{29}{4} \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 & 1 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & -\frac{1}{4} & \frac{29}{4} \\ 0 & 0 & 0 & 17 \end{vmatrix}$$

$$= 1 \times 4 \times \left(-\frac{1}{4}\right) \times 17 = \boxed{-17}$$

Problem set 1.8

2) $\tilde{A} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$ $\quad \det(\tilde{A}) = \begin{vmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{vmatrix}$

$= \cos^2 2\theta + \sin^2\theta = 1 \quad \det(\tilde{A}) \neq 0 \Rightarrow \underline{\tilde{A}^{-1} \text{ exists.}}$

Gauss-Jordan elimination

pivot

$$\left[\begin{array}{cc|cc} \cos 2\theta & \sin 2\theta & 1 & 0 \\ -\sin 2\theta & \cos 2\theta & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \tan 2\theta & \sec 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 & 1 \end{array}\right]$$

pivot

$\underbrace{}_{\tilde{A}}$ $\quad \underbrace{}_{I}$

$$\left[\begin{array}{cc|cc} 1 & \tan 2\theta & \sec 2\theta & 0 \\ 0 & \sec 2\theta & \tan 2\theta & 1 \end{array}\right]$$

pivot

$$= \left[\begin{array}{cc|cc} 1 & 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sec 2\theta & \tan 2\theta & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \cos 2\theta & \sin 2\theta \\ 0 & 1 & -\sin 2\theta & \cos 2\theta \end{array}\right]$$

$\underbrace{}_{I}$ $\quad \underbrace{}_{\tilde{A}^{-1}}$

$$\boxed{\therefore \tilde{A}^{-1} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}}$$

6) $\tilde{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 3 \\ 0 & 3 & 5 \end{bmatrix}$ $\quad \det(\tilde{A}) = -4 \begin{vmatrix} 8 & 3 \\ 3 & 5 \end{vmatrix} = -4(40-9) \neq 0$

$\Rightarrow \underline{\tilde{A}^{-1} \text{ exists.}}$

$$\left[\begin{matrix} \underbrace{\begin{matrix} -4 & 0 & 0 \\ 0 & 8 & 3 \\ 0 & 3 & 5 \end{matrix}}_{\tilde{A}} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\tilde{I}} \end{matrix}\right] = \left[\begin{matrix} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{13}{8} & 0 & \frac{1}{8} & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{matrix}\right]$$

$$= \left[\begin{matrix} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{13}{8} & 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & 1 \end{matrix}\right] = \left[\begin{matrix} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & -13 \\ 0 & 0 & \frac{1}{8} & 0 & -\frac{3}{8} & 1 \end{matrix}\right]$$

$$= \left[\begin{matrix} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\tilde{I}} & \underbrace{\begin{matrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{matrix}}_{\tilde{A}^{-1}} \end{matrix}\right]$$

$$\boxed{\therefore \tilde{A}^{-1} = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix}}$$