作业十一

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20.(a) 记 $X = AZ + \mu$ $X_a = BZ + \mu_a$

即 $X \sim N(\mu, AA^T)$ $X_a \sim N(\mu_a, BB^T)$

充分性. 若 $Cov(X_s, X_t)$ 只依赖于 $t-s$ 则 $Cov(X_{t_i}, X_{t_j}) = Cov(X_{t_i+a}, X_{t_j+a})$ $(j \geq i)$

故 $AA^T = BB^T$

又 EX_t 为常数 故 $EX = EX_a$ 即 $\mu = \mu_a$

故 X 与 X_a 同分布，即有相同的联合分布

又 $n, a, t_1, \cdots, t_n$ 有任意性 故 Gauss 过程为平稳的

必要性: X 与 X_a 有相同的联合分布 $\Rightarrow$ $\mu_a = \mu$，$BB^T = AA^T$，$B = A$

则 $EX_{t_i+a} = EX_{t_i}$ 由于 t_i 和 a 有任意性

故可得对 $\forall t \geq 0$，EX_t 恒定，即 EX_t 为常数

$BB^T = AA^T \Rightarrow Cov(X_{t_i}, X_{t_j}) = Cov(X_{t_i+a}, X_{t_j+a})$ $(j \geq i)$

即 $Cov(X_s, X_t) = Cov(X_{s+a}, X_{t+a})$ $(t \geq s)$

取 $a = -s$ 则 $Cov(X_t, X_s) = Cov(X_{t-s}, X_0)$ 只与 X_{t-s} 有关

故 $Cov(X_s, X_t)$ 只依赖于 $t-s$

(b) $B_t \sim N(0, t) \Rightarrow B_{e^{at}} \sim N(0, e^{at})$

$\Rightarrow U_t = e^{-\frac{at}{2}} B_{e^{at}} \sim N(0,1) \Rightarrow U_t$ 是 Gauss 过程

$$Cov(U_t, U_s) = e^{-\frac{a}{2}(t+s)} Cov(B_{e^{at}}, B_{e^{as}})$$

$$= e^{-\frac{a}{2}(t+s)} \cdot e^{at \wedge s} = e^{-a(\frac{t+s}{2} - t \wedge s)} = e^{\frac{-a|t-s|}{2}}$$

$EU_t = e^{-\frac{at}{2}} EB_{e^{at}} = 0$

故 $\{U_t : t \geq 0\}$ 是平稳的 Gauss 过程

21. (a) $B_1+\cdots+B_n = (1\ 1\ \cdots\ 1)\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} \sim N\left(0, (1\ 1\ 1\cdots 1)\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & \\ 1 & 2 & 3 & & \\ \vdots & & & \ddots & \\ 1 & & & & n \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right)$

$$= N\left(0, \frac{1}{6}n(n+1)(2n+1)\right)$$

(b) $Y_t = B_{t+1} - B_t \sim N(0,1)$ 即 Y_t 是 Gauss 过程

$EY_t = EB_{t+1} - EB_t = 0.$

$\forall t \geq s \quad cov(Y_s, Y_t) = cov(B_{t+1} - B_t, B_{s+1} - B_s)$

$$= cov(B_{t+1}, B_{s+1}) + cov(B_t, B_s) - cov(B_t, B_{s+1}) - cov(B_s, B_{t+1})$$

$$= s+1+s - t\wedge(s+1) - s = s+1 - t\wedge(s+1)$$

$$= -\frac{t-s-1}{2} - \frac{|t-s-1|}{2} \quad \text{只依赖于 } t-s$$

故 $\{Y_t : t\geq 0\}$ 是平稳过程

23. $B_t \sim N(0,t)$

$$E(B_s \mid B_t = x) = E(B_s) + \frac{cov(B_s, B_t)}{\sqrt{DB_s \cdot DB_t}}\sqrt{\frac{DB_s}{DB_t}}(x - EB_t)$$

$$= \frac{s}{\sqrt{st}}\sqrt{\frac{s}{t}} \cdot x = \frac{xs}{t}$$

24. $\because \{Y_t : t\geq 0\}$ 称为几何 Brown运动

$\therefore Y_t = e^{B_t}, \ t\geq 0$

$\therefore EY_t = E(e^{B_t}) = e^{\frac{1}{2}t}$

$DY_t = E(EY_t^2) - (EY_t)^2 = E(e^{2B_t}) - e^t$

$$= e^{2t} - e^t$$

3. $E\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)=\frac{1}{n}\sum_{k=1}^{n}E(X_k)$ 和 $D\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)=\frac{1}{n}\sum_{k=1}^{n}D(X_k)\le\frac{c}{n}$

利用切比雪夫不等式有：

$$1\geq P\left\{\left|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}E(X_k)\right|<\varepsilon\right\}$$

$$>1-\frac{D\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)}{\varepsilon^2}\geq 1-\frac{c}{n\varepsilon^2}$$

又∵ 对任意 $\varepsilon>0$，令 $n\to\infty$ 得

$$\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}E(X_k)\right|<\varepsilon\right\}=1$$

∴ $P\left\{\left|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}E(X_k)\right|<\varepsilon\right\}\to 0\ (n\to\infty)$.

4. X_i iid $\Rightarrow (X_i-\mu)^2$ iid $\forall i$

$E(X_k-\mu)^2=DX_k=\sigma^2$ 存在

则由 Khintchine LLN，$\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2\xrightarrow{P}\sigma^2$

5. 由 Chebyshev 不等式，$DX_n\le c\Rightarrow EX_n^2$ 存在 $(n\geq 1)$

则对 $\forall\varepsilon>0$，$P\left(\left|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}EX_k\right|\geq\varepsilon\right)\le\frac{D\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)}{\varepsilon^2}=\frac{D\left(\sum_{k=1}^{n}X_k\right)}{n^2\varepsilon^2}$

$D\left(\sum_{k=1}^{n}X_k\right)=\sum_{k=1}^{n}DX_k+2\sum_{i<j}Cov(X_i,X_j)\le nc+2\sum_{i<j}Cov(X_i,X_j)$

$$\lim_{|i-j|\to\infty}r(X_i,X_j)=\lim_{|i-j|\to+\infty}\frac{Cov(X_i,X_j)}{\sqrt{DX_iDX_j}}=0\Rightarrow\lim_{|i-j|\to+\infty}Cov(X_i,X_j)=0$$

且 $Cov(X_i,X_j)$ 对 $\forall i,j$ 有界，不妨设为 M

$$\lim_{n\to+\infty}2\sum_{i<j}Cov(X_i,X_j)=\lim_{N\to+\infty}\left(\sum_{|i-j|\geq N}Cov(X_i,X_j)+\sum_{|i-j|<N}Cov(X_i,X_j)\right)$$

$$=\sum_{i,j}\lim_{|i-j|\to+\infty}Cov(X_i,X_j)+\lim_{N\to+\infty}\sum_{|i-j|<N}Cov(X_i,X_j)$$

$$\le 0 + 2n(N-1)M$$

$$\therefore \frac{P(\sum_{k=1}^{n} X_k)}{n^2\varepsilon^2} \le \frac{nc + 2n(N-1)M}{n^2\varepsilon^2} = \frac{c+2(N-1)M}{n\varepsilon^2} \to 0 \quad (n\to+\infty)$$

故 对 $\forall \varepsilon>0$

$$P\left(\left|\frac{1}{n}\sum_{k=1}^{n}X_k - \frac{1}{n}\sum_{k=1}^{n}EX_k\right| \ge \varepsilon\right) \longrightarrow 0 \quad (n\to+\infty)$$

12. $\forall\ \varepsilon>0$

$$\lim_{n\to\infty} P(|M_n - a| \ge \varepsilon) = \lim_{n\to\infty} P(M_n \ge a+\varepsilon \text{ 或 } M_n \le a-\varepsilon)$$

$$= \lim_{n\to\infty} P(M_n \ge a+\varepsilon)$$

$$= \lim_{n\to\infty} P(X_1 \ge a+\varepsilon)P(X_2 \ge a+\varepsilon)\cdots P(X_n \ge a+\varepsilon)$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} [1 - P(X_i < a+\varepsilon)]$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \int_a^{a+\varepsilon} e^{-(x-a)} dx\right)$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} e^{-\varepsilon} = \lim_{n\to\infty} e^{-n\varepsilon} = 0$$

故 $M_n \xrightarrow{P} a$

13. 先证 $\ln Y_n \xrightarrow{P} \ln c$ 由于 $y=e^x$ 为连续函数 则有 $Y_n \xrightarrow{P} c$

$\ln Y_n = \frac{1}{n}\sum_{k=1}^{n} \ln X_k$ 记 $Z_k = \ln X_k$ 则 Z_k iid $(\forall k)$

$EZ_k = \int_0^1 \ln x_k\, dx_k = x(\ln x - 1)\Big|_0^1 = -1$

则有 $\frac{1}{n}\sum_{k=1}^{n} Z_k = \frac{1}{n}\sum_{k=1}^{n}\ln X_k \xrightarrow{P} -1$

故有 $Y_n \xrightarrow{P} e^{-1}$, $c = e^{-1}$

17 记 X_i 为每一次抛硬币的情况，则 X_i iid $X_i \sim \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

记 $Y = \sum_{i=1}^{n} X_i$ 即求 n st. $P(0.4 \leq \frac{Y}{n} \leq 0.6) \geq 0.9$ (*)

$EY = \sum_{i=1}^{n} EX_i = \frac{n}{2} \Rightarrow E(\frac{Y}{n}) = \frac{1}{2}$

$DY = \sum_{i=1}^{n} DX_i = \frac{n}{4} \Rightarrow D(\frac{Y}{n}) = \frac{1}{4n}$

则 (*) $\Leftrightarrow$ $P(-0.1 \leq \frac{Y}{n} - E(\frac{Y}{n}) \leq 0.1) \geq 0.9$

即 $P(|\frac{Y}{n} - E(\frac{Y}{n})| \leq 0.1) \geq 0.9$

由 chebyshev 不等式

$$P(|\frac{Y}{n} - E(\frac{Y}{n})| \leq 0.1) \geq 1 - \frac{D(\frac{Y}{n})}{0.1^2} = 1 - 100 \cdot \frac{1}{4n} = 1 - \frac{25}{n} = 0.9$$

$\Rightarrow n = 250$

故需抛 250 次.

正态近似：

$Y \sim B(n, \frac{1}{2}) \Rightarrow \frac{Y - \frac{1}{2}n}{\sqrt{\frac{1}{4}n}} \xrightarrow{D} N(0,1)$

$\Rightarrow P(\frac{Y - \frac{n}{2}}{\frac{1}{2}\sqrt{n}} \leq y) \xrightarrow{n \to +\infty} \Phi(y)$

$$P(0.4 \leq \frac{Y}{n} \leq 0.6) = P(-0.2\sqrt{n} \leq \frac{Y - \frac{n}{2}}{\frac{1}{2}\sqrt{n}} \leq 0.2\sqrt{n})$$

$$= 2\Phi(0.2\sqrt{n}) - 1 \geq 0.9$$

$\Rightarrow \Phi(0.2\sqrt{n}) \geq 0.95$

$\Rightarrow 0.2\sqrt{n} \geq 1.65$

$\Rightarrow n \geq 68.0625$ 即 $n \geq 69$

故需抛 69 次

19. X_i 为出售第i份报时路过人数.

X_i 服从几何分布

$P(X_i=k) = p(1-p)^k = \frac{1}{3}\times(\frac{2}{3})^k$

$EX_i = \frac{1}{p} = 3 \quad VarX_i = 6$

$EX = 3\times 100 = 300 \quad VarX = 6\times 100 = 600$

由中心极限定理得 $X \sim N(300, 600)$

则 $P(280 \le X \le 320) = 2\Phi(\frac{20}{\sqrt{600}}) - 1) = 0.5878$

21. 设X表示800间客房在某一时刻同时开动的空调数.

则 $X \sim B(800, 0.7)$

又设在任何时刻该宾馆至多有m台空调同时开动, 则只需供应 2x (千瓦) 的电力

$$P(0 \le X < m) \approx \Phi(\frac{m-800\times 0.7}{\sqrt{800\times 0.7\times 0.3}}) - \Phi(\frac{0-800\times 0.7}{\sqrt{800\times 0.7\times 0.3}})$$

$$\approx \Phi(\frac{m-560}{\sqrt{168}}) > 0.99$$

$m = 2.33\times\sqrt{168} + 560 \approx 590.2$

只需 $2\times 590.2 = 1180.4$ (千瓦) 电力