homework10

刘若涵 2020011126

2.1(1)

$$A=\begin{bmatrix}1 & 1\\ -2 & 4\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s-1 & -1\\ 2 & s-4\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{1}{(s-2)(s-3)}\begin{bmatrix}s-4 & 1\\ -2 & s-1\end{bmatrix}=\begin{bmatrix}\frac{2}{s-2}-\frac{1}{s-3} & -\frac{1}{s-2}+\frac{1}{s-3}\\ \frac{2}{s-2}-\frac{2}{s-3} & -\frac{1}{s-2}+\frac{2}{s-3}\end{bmatrix}$$

$$L^{-1}\left((sI-A)^{-1}\right)=\begin{bmatrix}2e^{2t}-e^{3t} & -e^{2t}+e^{3t}\\ 2e^{2t}-2e^{3t} & -e^{2t}+2e^{3t}\end{bmatrix}$$

2.2

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -\theta\\ 0 & \theta & 0\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s & 0 & 0\\ 0 & s & \theta\\ 0 & -\theta & s\end{bmatrix}$$

$$(sI-A)^{-1}=\begin{bmatrix}\frac{1}{s} & 0 & 0\\ 0 & \frac{s}{s^2+\theta^2} & -\frac{\theta}{s^2+\theta^2}\\ 0 & \frac{\theta}{s^2+\theta^2} & \frac{s}{s^2+\theta^2}\end{bmatrix}$$

$$L^{-1}\left((sI-A)^{-1}\right)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta t & -\sin\theta t\\ 0 & \sin\theta t & \cos\theta t\end{bmatrix}$$

2.3 A为约当型矩阵

$$e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{6}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

推广为n阶

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\ 1 & t \\ 0 & 0 & 0 & \cdots\ 0 & 1 \end{bmatrix}$$

2.4 (2)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$SI-A = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & -1 & s \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{2(s+1)} - \frac{1}{2(s-1)} & -\frac{1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \\ 0 & \frac{1}{2(s+1)} + \frac{1}{2(s-1)} & \frac{1}{2(s+1)} - \frac{1}{2(s-1)} \\ 0 & \frac{1}{2(s+1)} - \frac{1}{2(s-1)} & \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \end{bmatrix}$$

$$L^{-1}((SI-A)^{-1}) = \begin{bmatrix} 1 & \frac{1}{2}(e^{t} - e^{-t}) & -1 + \frac{1}{2}(e^{t} + e^{-t}) \\ 0 & \frac{1}{2}(e^{t} + e^{-t}) & \frac{1}{2}(e^{t} - e^{-t}) \\ 0 & \frac{1}{2}(e^{t} - e^{-t}) & \frac{1}{2}(e^{t} + e^{-t}) \end{bmatrix}$$

2.5

$$A = \dot{\Phi}(0)\,\Phi^{-1}(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

2.6

$$X(t) = \Phi(t)\,X(0)$$

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^{-2t} & 2e^{-t} \\ -e^{-2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}+2e^{-t} & -2e^{-2t}+2e^{-t} \\ e^{-2t}-e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix}$$

$$A = \dot{\Phi}(0)\,\Phi^{-1}(0) = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

2.7

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} \qquad s+1 \quad s+3$$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{3}{2(s+1)} - \frac{1}{2(s+3)} & \frac{1}{2(s+1)} - \frac{1}{2(s+3)} \\ -\frac{3}{2(s+1)} + \frac{3}{2(s+3)} & -\frac{1}{2(s+1)} + \frac{3}{2(s+3)} \end{bmatrix}$$

$$\Phi(t) = L^{-1}\left((sI-A)^{-1}\right) = \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi(t)\,x(0) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \quad (t \geq 0)$$

2.8

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Phi(t) = e^{2t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{bmatrix}$$

$$x(t) = \Phi(t)\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ & e^{2t} & te^{2t} \\ & & e^{2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \quad (t \geq 0)$$

2.9

$$\Phi(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{\Phi}(t,0) = \begin{bmatrix} 2e^{2t}\cos t - e^{2t}\sin t & -2e^{2t}\sin t - e^{2t}\cos t \\ e^{t}\sin t + e^{t}\cos t & e^{t}\cos t - e^{t}\sin t \end{bmatrix}$$

$$= A(t)\,\Phi(t,0)$$

验证正确

$$\Phi(t,1) = \Phi(t,0)\,\Phi^{-1}(1,0) = \begin{bmatrix} e^{2t}\cos t & -e^{2t}\sin t \\ e^{t}\sin t & e^{t}\cos t \end{bmatrix}\begin{bmatrix} e^{-2}\cos 1 & e^{-1}\sin 1 \\ -e^{-2}\sin 1 & e^{-1}\cos 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t-2}\cos(t-1) & -e^{2t-1}\sin(t-1) \\ e^{t-2}\sin(t-1) & e^{t-1}\cos(t-1) \end{bmatrix}$$

2.11 $\dot{x}(t) = Ax(t) + B(t)u(t)$

$$e^{-At}(\dot{x}(t) - Ax(t)) = e^{-At}B(t)u(t)$$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}B(t)u(t)$$

$$e^{-At}x(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)d\tau$$

$$e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)d\tau$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau$$

2.12 (1)

$$\int_0^t A(\tau)d\tau = \begin{bmatrix} \frac{t^2}{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Phi(t,0) = \begin{bmatrix} e^{\frac{t^2}{2}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi^{-1}(t,0) = \begin{bmatrix} e^{-\frac{t^2}{2}} & 0 \\ 0 & 1 \end{bmatrix}$$

(2)

$$B = \int_0^t A(\tau)d\tau = \begin{bmatrix} 0 & -e^{-t} \\ e^{-t} & 0 \end{bmatrix}$$

$$SI - B = \begin{bmatrix} s & e^{-t} \\ -e^{-t} & s \end{bmatrix}$$

$$(sI-B)^{-1} = \begin{bmatrix} \dfrac{s}{s^2+e^{-2t}} & \dfrac{-e^{-t}}{s^2+e^{-2t}} \\ \dfrac{e^{-t}}{s^2+e^{-2t}} & \dfrac{s}{s^2+e^{-2t}} \end{bmatrix}$$

$$\Phi(t,0) = L^{-1}\left((sI-B)^{-1}\right) = \begin{bmatrix} \cos(te^{-t}) & -\sin(te^{-t}) \\ \sin(te^{-t}) & \cos(te^{-t}) \end{bmatrix}$$

$$\Phi^{-1}(t,0) = \begin{bmatrix} \cos(te^{-t}) & \sin(te^{-t}) \\ -\sin(te^{-t}) & \cos(te^{-t}) \end{bmatrix}$$

3.2

$$Q_k = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix} \qquad \mathrm{rank}(Q_k) = 3$$

1.15

2.1(1), 2.2, 2.3, 2.4(2), 2.5, 2.6, 2.7, 2.8, 2.9, 2.11(1), 2.12

3.2, 3.3, 3.4, 3.5, 3.6和3.7的能控部分

纸质版11月29日当堂交， 电子版11月29日晚23：59提交到网络学堂

说明：2.9题涉及时变系统的状态转移矩阵，其中符号$\Phi(t,t_0)$表示从t0时刻出发到t时刻的状态转移矩阵，满足$x(t)=\Phi(t,t_0)x(t_0)$

完全能控

3.3 (1)

$$\dot{x} = \left[\begin{array}{cc:c} -3 & 1 & \\ & -3 & \\ \hdashline & & -1 \end{array}\right] x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

不完全能控

(2)

$$Q_k = \begin{bmatrix} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{bmatrix} \qquad \mathrm{Rank}(Q_k) = 2 < 3$$

不完全能控

3.4 $c \neq 0$

3.5

$$Q_k = \begin{bmatrix} b & ab-1 \\ -1 & b \end{bmatrix}$$

$$\frac{ab-1}{b} \neq \frac{b}{-1}$$

$$ab-1 \neq -b^2$$

$$b^2+ab-1 \neq 0$$

3.6

$$Q_k = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -11 \\ -1 & 0 & -11 & 0 \end{bmatrix} \qquad \mathrm{Rank}(Q_k) = 4$$

完全能控

3.7

$$Q_k = \begin{bmatrix} 1 & a+b \\ 1 & c+d \end{bmatrix}$$

$$a+b \neq c+d$$