homework7

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1 ① $\|A\|_\infty = 6$

$A^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix} \Rightarrow \|A^{-1}\|_\infty = 1$

$Cond(A)_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 6$

② $B^TB = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$

$f(\lambda) = \det\begin{pmatrix} \lambda-5 & 4 & -1 \\ 4 & \lambda-6 & 4 \\ -1 & 4 & \lambda-5 \end{pmatrix} = (\lambda-4)(\lambda^2-12\lambda+4)$

$\Rightarrow \lambda_1 = 6+4\sqrt{2} \quad \lambda_2 = 6-4\sqrt{2} \quad \lambda_3 = 4$

$\|B\|_2 = \sqrt{\lambda_{max}(B^TB)} = \sqrt{6+4\sqrt{2}}$

$B^{-1} = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$

同理，求解 $(B^{-1})^T(B^{-1})$ 的特征值

$\lambda_1 = \frac{1}{6+4\sqrt{2}} \quad \lambda_2 = \frac{1}{4} \quad \lambda_3 = \frac{1}{6-4\sqrt{2}}$

$\lambda_{max}((B^{-1})^T(B^{-1})) = \frac{1}{6-4\sqrt{2}} = \frac{6+4\sqrt{2}}{4}$

$Cond(B)_2 = \|B\|_2 \|B^{-1}\|_2 = \sqrt{\frac{(6+4\sqrt{2})^2}{4}} = 3+2\sqrt{2}$

2. 由于 $A=LL^T$

设
$$L=\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L^T=\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & 0 & a_{33} \end{bmatrix}$$

由此得关系式：

$$a_{11}^2=2 \Rightarrow a_{11}=\sqrt{2}$$

$$a_{11}a_{21}=1 \Rightarrow a_{21}=\frac{\sqrt{2}}{2}$$

$$a_{11}a_{31}=0 \Rightarrow a_{31}=0$$

$$a_{21}^2+a_{22}^2=2 \Rightarrow a_{22}=\frac{\sqrt{6}}{2}$$

$$\left.\begin{array}{l} a_{21}a_{31}+a_{22}a_{32}=a \\ a_{31}^2+a_{32}^2+a_{33}^2=2 \end{array}\right. \Rightarrow \begin{array}{ll} \frac{\sqrt{6}}{2}a_{32}=a & ① \\ a_{32}^2+a_{33}^2=2 & ② \end{array}$$

根据条件知 $a_{33}>0$ 由①②知

$$a_{33}^2=2-\frac{2}{3}a^2>0 \Rightarrow -\sqrt{3}<a<\sqrt{3}$$

当 $a=1$ 时，代入得

$$L=\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 \\ 0 & \frac{\sqrt{6}}{3} & \frac{2\sqrt{3}}{3} \end{bmatrix}$$

3. 雅可比迭代法

$$B=-D^{-1}(L+U)=-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&a&0\\a&0&a\\0&a&0\end{bmatrix}=\begin{bmatrix}0&-a&0\\-a&0&-a\\0&-a&0\end{bmatrix}$$

求B特征值：

$|B-\lambda I|=0 \Rightarrow \lambda^3-2a^2\lambda=0$

$\lambda_1=0 \quad \lambda_2=-\sqrt{2}a \quad \lambda_3=\sqrt{2}a$

$\rho(B)=|\sqrt{2}a|<1 \Rightarrow$ 雅可比迭代法收敛时，$-\frac{\sqrt{2}}{2}<a<\frac{\sqrt{2}}{2}$

高斯-塞德尔迭代法：

$$B=-(D+L)^{-1}U=\begin{bmatrix}0&-a&0\\0&a^2&-a\\0&-a^3&a^2\end{bmatrix}$$

求B特征值： $|B-\lambda I|=0 \Rightarrow \lambda^2(\lambda-2a^2)=0$

$\lambda_1=0 \quad \lambda_2=0 \quad \lambda_3=2a^2$

$\rho(B)=2a^2<1 \Rightarrow$ 高斯-塞德尔迭代法收敛时，$-\frac{\sqrt{2}}{2}<a<\frac{\sqrt{2}}{2}$

4 $B=I+\alpha A=\begin{bmatrix}3\alpha+1&\alpha\\2\alpha&2\alpha+1\end{bmatrix}$

求B特征值： $|B-\lambda I|=0 \Rightarrow (\lambda-3\alpha-1)(\lambda-2\alpha-1)-2\alpha^2=0$

$\lambda_1=4\alpha+1 \quad \lambda_2=\alpha+1$

$\rho(B)<1 \Rightarrow -\frac{1}{2}<\alpha<0 \Rightarrow \alpha\in(-\frac{1}{2},0)$ 时，迭代收敛

5. $Ax = B \Leftrightarrow (I - TA)x + Tb = x$

取行变换矩阵 $T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 满足TA为上三角矩阵

则 $B = I - TA = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

由于 $\|B\|_\infty = \frac{1}{2} < 1$，故迭代法收敛

$$f = Tb = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} b$$

若b存在误差 δb 设 x^* 为精确解，则

$$x^* - x^{(k+1)} = B\left((x^* - x^{(k+1)}) + (x^{(k+1)} - x^{(k)})\right) + T\delta b$$

故 $(I - B)(x^* - x^{(k+1)}) = B(x^{(k+1)} - x^{(k)}) + T\delta b$

$$\Rightarrow \|x^* - x^{(k+1)}\|_\infty \le \frac{\|B\|_\infty}{1 - \|B\|_\infty}\|x^{(k+1)} - x^{(k)}\|_\infty + \frac{\|T\|_\infty \|\delta b\|_\infty}{1 - \|B\|_\infty}$$

由于 $\|B\|_\infty = \frac{1}{2}$ $\|T\|_\infty = 1$

故 $\|x^* - x^{(k+1)}\|_\infty \le \|x^{(k+1)} - x^{(k)}\|_\infty + 2\|\delta b\|_\infty$

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考虑取 $T = \frac{1}{2}\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ \frac{1}{3} & -\frac{2}{3} & \ddots & & \\ \vdots & \vdots & & \ddots & \\ \frac{(-1)^{n+1}}{n} & \frac{2\cdot(-1)^n}{n} & \cdots & -\frac{n-1}{n} & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & \frac{4}{3} & & \\ & & \ddots & \\ & & & \frac{2n}{n+1} \end{bmatrix}$

$$B = I - TA = \begin{bmatrix} 0 & -\frac{1}{2} & & & \\ & 0 & -\frac{2}{3} & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\frac{n-1}{n} \\ & & & & 0 \end{bmatrix}$$

$\rho(B) = 0 < 1 \Rightarrow$ 迭代法收敛

迭代法：$x^{(k+1)} = Bx^{(k)} + Tf$

$$Tf = \left(\frac{1}{2} \quad -\frac{1}{3} \quad \cdots \quad \frac{(-1)^{n+1} - n}{n+1}\right)^T$$

$$\|B\|_\infty = \frac{n-1}{n} \qquad \|Tf\|_\infty \le \frac{n-1}{n+1} \quad (n \geq 2)$$

① 使用事后估计分析误差.

$$\|x^{(k)} - x^*\| \le \frac{\|B\|}{1-\|B\|}\|x^{(k)} - x^*\|$$

$$\Rightarrow \|x^{(k)} - x^*\| \le \frac{\|B\|^k}{1-\|B\|}\|Tf\| = \frac{(n-1)^{k+1}}{n^{k-1}(n+1)}$$

② 舍入误差.

$$\|\delta_k\| \le \|B\| - \|\delta_{k-1}\| + \frac{1}{2}\times 10^{-m}$$

$$\Rightarrow \|\delta_k\| \le \frac{1-\|B\|^k}{1-\|B\|}\cdot\frac{1}{2}\times 10^{-m} = \frac{1}{2}\left(n - \frac{(n-1)^k}{n^{k-1}}\right)\times 10^{-m}$$

③ 分配误差:

$$\begin{cases} \|x^{(k)} - x^*\| = \dfrac{(n-1)^{k+1}}{n^{k-1}(n+1)} \le \frac{1}{2}\times 10^{-10} \\ \|\delta_k\| = \frac{1}{2}\left(n - \dfrac{(n-1)^k}{n^{k-1}}\right)\times 10^{-m} \le \frac{1}{2}\times 10^{-10} \end{cases}$$

通过上述方程给出符合要求的迭代次数k及存储位数m即可