作业十

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5. $\overline{Y}_k = (a_{k1}, \cdots, a_{kn})\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$

$= AX$

则 $\varphi_{\overline{Y}_k}(\theta) = \varphi_{AX}(\theta) = \varphi_X(A^T\theta)$

$\varphi_X(\theta) = E(e^{i(\theta_1 X_1 + \cdots + \theta_n X_n)})$

$\underline{\underline{X_i \text{ iid}}} \ \prod_{i=1}^{n} e^{-\frac{\theta_i^2}{2}}$

$Cov(\overline{Y}_i, \overline{Y}_j) = Cov(\sum_{k=1}^{n} a_{ik} X_k, \sum_{l=1}^{n} a_{jl} X_l)$

$= \sum_{k=1}^{n}\sum_{l=1}^{n} a_{jk}a_{jl} Cov(X_k, X_l)$

$= \sum_{k=1}^{n} a_{ik}a_{jk} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

则 $\overline{Y}_k$ 的协方差矩阵为单位对角阵

$\overline{Y}_k \sim N(0,1)$

且当 $i \neq j$ 时 $Cov(\overline{Y}_i, \overline{Y}_j) = 0$ Y_i 之间相互独立

8 (a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{-1}^{1} \frac{1}{4}[1 + xy(x^2 - y^2)]\,dy$

$= \frac{1}{4}(y + \frac{x^3}{2}y^2 - \frac{x}{4}y^4)\Big|_{-1}^{1}$

$= \frac{1}{2} \quad -1 \le x \le 1$

$f_Y(y) = \frac{1}{2} \quad -1 \le y \le 1$

$$\varphi_X(\theta) = \int_{\mathbb{R}} e^{i\theta x} \cdot f_X(x)\,dx$$

$$= \int_{-1}^{1} \frac{1}{2} e^{i\theta x} dx$$

$$= \frac{\sin\theta}{\theta}$$

$$\varphi_Y(\theta) = \frac{\sin\theta}{\theta}$$

(b) $F_Z(z) = P(Z \le z)$

$$= P(V(X, Y) \le z)$$

$$= \iint_{X+Y\le z} (x+y) \cdot \frac{1}{4}(1 + xy(x^2 - y^2))\,dx\,dy$$

$$= \begin{cases} \frac{1}{4}(z+2) & -2 < z \le 0 \\ \frac{1}{4}(2-z) & 0 < z \le 2 \\ 0 & \text{else} \end{cases}$$

(c) $\varphi_Z(\theta) = \int_{-2}^{0} e^{i\theta z} \cdot \frac{1}{4}(z+2)\,dz + \int_0^2 e^{i\theta z} \cdot \frac{1}{4}(2-z)\,dz$

$= \left(\frac{\sin\theta}{\theta}\right)^2$ 不成立

12. $\varphi_Y(\theta) = E(e^{i\theta(X_1 + \cdots + X_V)})$

$E(e^{i\theta X_i}) = \varphi(\theta)$ $\quad V \sim P(\lambda)$

$$\varphi_Y(\theta) = \sum_{k=1}^{n} E(e^{i\theta(X_1 + \cdots + X_k)}) P(V=k)$$

$$= \sum_{k=1}^{n} \varphi^k(\theta) \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{\lambda(\varphi(\theta) - 1)}$$

$$E\bar{Y} = i^{-1}\varphi'_{\bar{Y}}(0)$$

$$= -i\lambda\,\varphi'(\theta)e^{\lambda(\varphi(\theta)-1)}\Big|_{\theta=0}$$

$$E\bar{X}_i = -i\varphi'_{\bar{X}}(0)$$

则 $E\bar{Y} = \lambda E\bar{X}_1$

16. 设 X 为 n 维 Gauss 随机向量

$X \sim N(\mu, \Sigma)$

设 $Y = AX + b$

$$EY = E(AX+b) = A\mu + b$$

$$\Sigma_Y = E[(Y-\mu_Y)(Y-\mu_Y)^T]$$

$$= E[(AX-A\mu)(AX-A\mu)^T]$$

$$= AE[(X-\mu)(X-\mu)^T]A^T$$

$$= A\Sigma A^T$$

$\therefore Y \sim N(A\mu+b,\ A\Sigma A^T)$

17. $\bar{X} \sim N(\mu, \Sigma)$

$\bar{X} - \mu \sim N(0, \Sigma)$

$\because \bar{X}$ 服从 N 维正态分布，则 Σ 正定

对正交矩阵 A

$A(\bar{X}-\mu) \sim N(0, A^T\Sigma A)$

$\Sigma' = A^T\Sigma A$ 为对角矩阵

$cov(Y_i, Y_j) = A\cdot A^T = I$ 当 $i \neq j$ 时 $A\cdot A^T = 0$

则 $\bar{Y}_1, \cdots, \bar{Y}_n$ 是独立随机变量

且 $\bar{Y}_i \sim N(0, \Sigma'_{ii}) = N(0, d_i)$

18 (a) $X \sim N(\mu, \Sigma)$

$$Y = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} X = BX \sim N(B\mu, B\Sigma B^T) = N(\mu', \Sigma')$$

$$\mu' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$\Sigma' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix}$$

协方差矩阵为 $\begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix}$

(b)
$$E(\bar{Y}_2 | \bar{Y}_1) = \mu_2' + \Sigma_{21}' \Sigma_{11}'^{-1} (\bar{Y}_1 - \mu_1')$$
$$= 8 + \frac{112}{5} \cdot \frac{5}{113} (\bar{Y}_1 - 7)$$
$$= 8 + \frac{112}{113} (\bar{Y}_1 - 7)$$

$$E(\bar{Y}_1 + \bar{Y}_2) = 15$$

(c)
$$E(X_2 | X_1) = \mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (X_1 - \mu_1)$$
$$= 2 + \frac{4}{5} \, 1 (X_1 - 1)$$
$$= 2 + \frac{4}{5} (X_1 - 1)$$

$$D(X_2 | X_1) = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$$
$$= 1 - \frac{4}{5} \cdot 1 \cdot \frac{4}{5}$$
$$= \frac{9}{25}$$

(d)

$$X_2 - E(X_2|X_1) = -\frac{4}{5}X_1 + X_2 - \frac{6}{5}$$

$$\begin{pmatrix} X_2 - E(X_2|X_1) \\ X_1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \frac{6}{5}$$

$$Cov(X_2 - E(X_2|X_1), X_1) = -\frac{4}{5} + \frac{4}{5} = 0$$

独立

19 必要性.

对 Brown 运动 $\{X_t : t \geq 0\}$ 有

$$E(|X_t - X_s|^2) = |t-s|$$

$$E(B) = t - 2s + s = t - s \qquad (s \leq t)$$

则为 Gauss 运动

充分性:

$$E(X_t - X_s) = 0$$

$$E(X_t - X_s)^2 = |t-s|$$

则 $E(X_{t_1} - X_{s_1})(X_{t_2} - X_{s_2}) = 0 \qquad \forall s_1 < t_1 \leq s_2 < t_2$

即 $X \sim N(0, t-s)$