homework6

刘若涵 2020011126

1. $\varphi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3-43}{3x^2} = \frac{2}{3}x + \frac{43}{3x^2}$

$x_{n+1} = \varphi(x_n) = \frac{2}{3}x_n + \frac{43}{3x_n^2}$

$|\varphi'(x)| = \left|\frac{2}{3} - \frac{86}{3x^3}\right| < 1 \Rightarrow x^3 > \frac{86}{5}$

取 $x_0 = 3$

$x_1 = 3.5926$ $x_2 = 3.5056$ $x_3 = 3.5034$ $x_4 = 3.5034$

$|x_4 - x_3| < 0.001$ 停止迭代

$x^* = x_4 = 3.503$

$L = \max\limits_{x>3} |\varphi'(x)| = \frac{2}{3}$

舍入误差

$\delta_1 \leq \frac{1}{2} \times 10^{-4}$

$\delta_2 \leq L\delta_1 + \frac{1}{2} \times 10^{-4} = 0.833 \times 10^{-4}$

$\delta_3 \leq L\delta_2 + \frac{1}{2} \times 10^{-4} = 1.055 \times 10^{-4}$

$\delta_4 \leq L\delta_3 + \frac{1}{2} \times 10^{-4} = 1.204 \times 10^{-4}$

满足精度要求

2. 令 $f(x) = 6-2x+\cos x$

$f'(x) = -2-\sin x < 0$ $f(x)$ 单调递减.

$f(0) = 7 > 0$ $f(\pi) = 5-2\pi < 0$

$f(x)$ 有且仅有一个根，且该根位于 $(0,\pi)$ 区间内.

$\varphi(x) = 3+\frac{1}{2}\cos x$ 连续

$\forall x_0 \in \mathbb{R}$ $\varphi(x)$ 在区间 $[-|x_0|+2.5, |x_0|+3.5]$ 上满足

$\varphi(x) \in [2.5, 3.5] \subseteq [-|x_0|+2.5, |x_0|+3.5]$

$\forall x, \bar{x} \in [-|x_0|+2.5, |x_0|+3.5]$

$$\begin{aligned}
|\varphi(x)-\varphi(\bar{x})| &= \left|\frac{1}{2}\cos x-\frac{1}{2}\cos\bar{x}\right| \\
&= \left|-\sin\frac{x+\bar{x}}{2}\sin\frac{x-\bar{x}}{2}\right| \\
&\leq \left|\sin\frac{x-\bar{x}}{2}\right| \\
&\leq \frac{1}{2}|x-\bar{x}|
\end{aligned}$$

因此 $\forall x_0 \in \mathbb{R}$，迭代法 $x_{k+1} = 3+\frac{1}{2}\cos x_k$ 产生的序列 $\{x_k\}$ 收敛到方程的根.

3. 求解 $x=\dfrac{x(x^2+3a)}{3x^2+a}$ $\Rightarrow$ $x^*=0,\pm\sqrt{a}$.

$$\varphi(x)=\frac{x(x^2+3a)}{3x^2+a}$$

$$\varphi'(x)=\frac{3(x^2-a)^2}{(3x^2+a)^2}$$

$\varphi'(0)=3>1$ $\varphi'(\sqrt{a})=0<1$

且 $\varphi'(x)$ 在 $\sqrt{a}$ 处连续

$\therefore$ $\varphi(x)$ 产生的序列 $\{x_k\}$ 会收敛到 $\sqrt{a}$.（初值 $x_0>0$ 时）

$$\varphi''(x)=\frac{48ax(x^2-a)}{(3x^2+a)^3} \Rightarrow \varphi''(\sqrt{a})=0.$$

$$\varphi'''(x)=-\frac{48a(9x^4-18ax^2+a^2)}{(3x^2+a)^4} \Rightarrow \varphi'''(\sqrt{a})=\frac{3}{2a}\neq 0$$

$\therefore$ $\varphi(x)$ 产生的序列 $\{x_k\}$ 三阶收敛到 $\sqrt{a}$.

4. $f(x)=e^{-x}\ln x$

$f'(x)=e^{-x}\left(\frac{1}{x}-\ln x\right)$

$f''(x)=e^{-x}\left(\ln x-\frac{2}{x}-\frac{1}{x^2}\right)$

$f''(x)=0$ 的根即 $f(x)$ 的拐点

$\Leftrightarrow$ 求 $g(x)=\ln x-\frac{2}{x}-\frac{1}{x^2}=0$ 的根.

$g'(x)=\frac{1}{x}+\frac{2}{x^2}+\frac{2}{x^3}>0$ $g(x)$ 单调递增.

$g(2)<0$ $g(3)>0$

$g''(x)=-\frac{1}{x^2}-\frac{4}{x^3}-\frac{6}{x^4}<0$

$g(x)\in C^{(2)}[2,3]$ 满足:

① $g(2)\cdot g(3)<0$

② $g''(x)$ 在 $[2,3]$ 上不变号

③ $g'(x)\neq 0$

④ 选取初值 $x_0=2$ $g(x_0)g''(x_0)>0$

$\therefore$ 牛顿迭代法收敛

$\varphi(x)=x-\frac{g(x)}{g'(x)}=x-\frac{x^3\ln x-2x^2-x}{x^2+2x+2}$

设置迭代停止条件 $|x_{n+1}-x_n|<0.001$

$x_1 = \varphi(x_0) = 2.44548$

$x_2 = \varphi(x_1) = 2.54866$

$x_3 = \varphi(x_2) = 2.55244$

$x_4 = \varphi(x_3) = 2.55245$

迭代4次 $x^* = x_4 = 2.552$

$f(x_4) = 0.073$.

描点 $(2.552, 0.073)$