作业十四

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10. $D\left(\frac{\overline{X}}{\alpha}\right) = \frac{1}{\alpha^2}D\overline{X} = \frac{1}{n^2\alpha^2}D\left(\sum_{i=1}^{n}X_i\right) = \frac{n}{n^2\alpha^2}\cdot\frac{\alpha}{\lambda^2} = \frac{1}{n\alpha\lambda^2}$

$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda)\right] = -E\left(-\frac{\alpha}{\lambda^2}\right) = \frac{\alpha}{\lambda^2}$

$$\frac{(g'(\theta))^2}{nI(\lambda)} = \frac{\left(-\frac{1}{\lambda^2}\right)^2}{\frac{n\alpha}{\lambda^2}} = \frac{1}{n\alpha\lambda^2}$$

故 $\frac{\overline{X}}{\alpha}$ 是 $g(\lambda)=\frac{1}{\lambda}$ 的有效估计，也是其UMVUE

11 $(X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_n)$ 的联合密度函数为

$p(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n, a, \sigma^2)$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{m+n} 2^{-\frac{n}{2}} e^{-\sum_{i=1}^{m}\frac{(x_i-a)^2}{2\sigma^2} - \sum_{i=1}^{n}\frac{(y_i-a)^2}{4\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{m+n} 2^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{m}x_i^2 + \sum_{i=1}^{n}y_i^2\right) + \frac{m\bar{x}+\frac{1}{2}n\bar{y}}{\sigma^2}a - \frac{m+\frac{1}{2}n}{2\sigma^2}a^2}$$

设 $\varphi(x_1, x_2, \cdots, x_n)$ 为0的任一无偏估计

$$E(\varphi) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi\, p(x_1,\cdots,x_m,y_1,\cdots,y_n,a,\sigma^2)\,dx_1\cdots dx_m dy_1\cdots dy_n$$

$= 0$ (1)

式(1)两端对 a 求导得

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{(m\bar{x}+\frac{1}{2}n\bar{y})}{\sigma^2}\varphi\, p(x_1,\cdots,x_m,y_1,\cdots,y_n,a,\sigma^2)\,dx_1\cdots dx_m dy_1\cdots dy_n = 0 \quad (2)$$

即 $E((m\bar{x}+\frac{1}{2}n\bar{y})\varphi) = 0$

$Cov(m\bar{x}+\frac{1}{2}n\bar{y}, \varphi) = E\left((m\bar{x}+\frac{1}{2}n\bar{y})\varphi\right) - E(m\bar{x}+\frac{1}{2}n\bar{y})E(\varphi) = 0$

又 $E(m\bar{x}+\frac{1}{2}n\bar{y})=(m+\frac{1}{2}n)a$

$\Rightarrow \dfrac{m\bar{x}+\frac{1}{2}n\bar{y}}{m+\frac{1}{2}n}$ 为 a 的 UMVUE

式(2)两端对 a 求导得

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\frac{(m\bar{x}+\frac{1}{2}n\bar{y})^2}{\sigma^4}-\frac{m+\frac{1}{2}n}{\sigma^2}\,\frac{m\bar{x}+\frac{1}{2}n\bar{y}}{\sigma^2}a\right]\cdot$$

$$\varphi\, p(x_1,\cdots,x_m,y_1,\cdots,y_n,a,\sigma^2)\,dx_1\cdots dx_m\,dy_1\cdots dy_n=0$$

即 $E((m\bar{x}+\frac{1}{2}n\bar{y})^2\varphi)=0$

式(1)两端对 σ^2 求导得

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\sum_{i=1}^m x_i^2+\frac{1}{2}\sum_{i=1}^n y_i^2\right)\cdot\varphi\, p(x_1,\cdots,x_m,y_1,\cdots,y_n,a,\sigma^2)\,dx_1\cdots dx_m\,dy_1\cdots dy_n=0$$

即 $E\left(\left(\sum_{i=1}^m x_i^2+\frac{1}{2}\sum_{i=1}^n y_i^2\right)\varphi\right)=0$.

记 $T=\sum_{i=1}^m x_i^2+\frac{1}{2}\sum_{i=1}^n y_i^2-\dfrac{(m\bar{x}+\frac{1}{2}n\bar{y})^2}{m+\frac{1}{2}n}$

$E(T\varphi)=0$

$\Rightarrow Cov(T,\varphi)=0$.

$E\left(\sum_{i=1}^m x_i^2+\frac{1}{2}\sum_{i=1}^n y_i^2\right)=(m+\frac{1}{2}n)a^2+(m+n)\sigma^2$

$E(m\bar{x}+\frac{1}{2}n\bar{y})^2=(m+\frac{1}{2}n)^2a^2+(m+\frac{1}{2}n)\sigma^2$

故 $E(T)=(m+\frac{1}{2}n)a^2+(m+n)\sigma^2-(m+\frac{1}{2}n)a^2-\sigma^2$

$\qquad=(m+n-1)\sigma^2$

故 $\dfrac{1}{m+n-1}$ 为 T 的 UMVUE

12　设 $\varphi(x_1,x_2,\cdots,x_n)$ 为 0 的任一无偏估计

$$E(\varphi)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi\cdot\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x_i-\mu)^2}{2}}dx_1\cdots dx_n=0$$

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\varphi\cdot(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2+n\bar{x}\mu-\frac{n\mu^2}{2}}dx_1\cdots dx_n=0 \quad (1)$$

式(1)两端对 μ 求导得.

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}n\bar{x}\varphi\cdot(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2+n\bar{x}\mu-\frac{n\mu^2}{2}}dx_1\cdots dx_n=0 \quad (2)$$

即 $E(\bar{x}\varphi)=0$,

式(2)两端对 μ 求导得

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[(n\bar{x})^2-n\bar{x}\cdot n\mu\right]\varphi\cdot(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2+n\bar{x}\mu-\frac{n\mu^2}{2}}dx_1\cdots dx_n=0$$

即 $E(\bar{x}^2\varphi)=0$.

记 $T=\bar{x}^2-\frac{1}{n}$

则 $\mathrm{Cov}(T,\varphi)=0$,　$E(T)=\mu^2$

故 $T=\bar{x}^2-\frac{1}{n}$ 为 μ^2 的 UMVUE

$$\mathrm{Var}(T)=\mathrm{Var}(\bar{x}^2)=\frac{2}{n^2}+\frac{4}{n}\mu^2$$

C-R 下界为 $\dfrac{4\mu^2}{nI(\mu)}=\dfrac{4\mu^2}{n}$

故此UMVUE方差达不到 C-R不等式的下界.

14. (1)

$$p(x_i;\theta)=\left(\frac{1-\theta}{2}\right)^{\frac{x_i(x_i-1)}{2}}\cdot\left(\frac{1}{2}\right)^{(1-x_i)(1+x_i)}\cdot\left(\frac{\theta}{2}\right)^{\frac{x_i(1+x_i)}{2}}$$

$$L(\theta)=\prod_{i=1}^{n}p(x_i;\theta)=\left(\frac{1-\theta}{2}\right)^{\sum_{i=1}^{n}\frac{x_i(x_i-1)}{2}}\cdot\left(\frac{1}{2}\right)^{\sum_{i=1}^{n}(1-x_i^2)}\cdot\left(\frac{\theta}{2}\right)^{\sum_{i=1}^{n}\frac{x_i(x_i+1)}{2}}$$

$$\ln L(\theta)=\sum_{i=1}^{n}\frac{x_i(x_i-1)}{2}\ln\frac{1-\theta}{2}+\sum_{i=1}^{n}(1-x_i^2)\ln\frac{1}{2}+\sum_{i=1}^{n}\frac{x_i(x_i+1)}{2}\ln\frac{\theta}{2}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{x_i(x_i-1)}{2} \frac{-1}{1-\theta} + \sum_{i=1}^{n} \frac{x_i(x_i+1)}{2} \frac{1}{\theta} = 0$$

$$\Rightarrow \hat{\theta}_1 = \frac{\sum_{i=1}^{n} \frac{x_i^2+x_i}{2}}{\sum_{i=1}^{n} x_i^2} = \frac{1}{2} + \frac{\sum_{i=1}^{n} x_i}{2\sum_{i=1}^{n} x_i^2}$$

$$EX = \frac{\theta}{2} - \frac{1-\theta}{2} = \theta - \frac{1}{2}$$

$$\Rightarrow E\bar{X} = \theta - \frac{1}{2}$$

$$E\hat{\theta}_1 = \frac{1}{2} + \frac{n}{2} E\left(\frac{x_1}{\sum_{i=1}^{n} x_i^2}\right)$$

$$E\left(\frac{x_1}{\sum_{i=1}^{n} x_i^2}\right) = E\left(\frac{1}{1+\sum_{i=1}^{n} x_i^2}\right) \cdot \frac{\theta}{2} + E\left(\frac{-1}{1+\sum_{i=1}^{n} x_i^2}\right) \cdot \frac{1-\theta}{2}$$

$$= \left(\theta - \frac{1}{2}\right) E\left(\frac{1}{1+\sum_{i=1}^{n} x_i^2}\right)$$

$$E\left(\frac{1}{1+\sum_{i=1}^{n} x_i^2}\right) = \sum_{k=0}^{n-1} \frac{1}{k+1} C_{n-1}^{k} \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{n-1} \sum_{k=0}^{n-1} \frac{1}{n} C_n^{k+1}$$

$$= \frac{1}{n} \left(\frac{1}{2}\right)^{n-1} \sum_{k=1}^{n} C_n^k$$

$$= \frac{1}{n} \left(\frac{1}{2}\right)^{n-1} \cdot (2^n - 1)$$

$$E\hat{\theta}_1 = \frac{1}{2} + \frac{n}{2}\left(\theta - \frac{1}{2}\right) \cdot \frac{1}{n} \cdot \left(\frac{1}{2}\right)^{n-1} \cdot (2^n - 1)$$

$$= \frac{1}{2} + \left(\theta - \frac{1}{2}\right)\left(1 - \left(\frac{1}{2}\right)^n\right) \neq 0.$$

故 $\hat{\theta}_1$ 不是无偏估计

(2) $EX = \theta - \frac{1}{2}$

令 $EX = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$\Rightarrow \hat{\theta}_2 = \frac{1}{2} + \bar{X}$

(3) $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln p(x;\theta)\right]$

$= E\left[\frac{X(X-1)}{2}\frac{1}{(\theta-1)^2} + \frac{X(X+1)}{2}\frac{1}{\theta^2}\right]$

$= \frac{1-\theta}{2}\cdot\frac{1}{(\theta-1)^2} + \frac{\theta}{2}\cdot\frac{1}{\theta^2}$

$= \frac{1}{2}\left(\frac{1}{1-\theta} + \frac{1}{\theta}\right)$

$= \frac{1}{2\theta(1-\theta)}$

$\frac{(\theta')^2}{nI(\theta)} = \frac{2\theta(1-\theta)}{n}$ 为 θ 的无偏估计的方差的 C-R 下界.

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3 (1) $L(x_1,\cdots,x_n;\theta) = \prod_{i=1}^{n} P(x_i;\theta) = \theta^n(1-\theta)^{\sum_{i=1}^{n} x_i}$

则后验分布为 $\pi(\theta|x_1,\cdots,x_n) \propto \theta^n(1-\theta)^{\sum_{i=1}^{n} x_i}$ $(0<\theta<1)$

即 $\theta_{后验} \sim Beta(n+1, \sum_{i=1}^{n} x_i + 1)$

(2) $E[\pi(\theta|x_1,\cdots,x_n)] = \frac{n+1}{n+\sum_{i=1}^{n} x_i + 2} \xlongequal{n=4} \frac{5}{20} = \frac{1}{4}$

6. $L(x_1,\cdots,x_n;\theta) = \prod_{i=1}^{n} P(x_i|\theta) = \frac{2^n\prod_{i=1}^{n} x_i}{\theta^{2n}}$ $(0<x_i<\theta)$

(1) $\pi(\theta) = U(0,1) \Rightarrow \pi_1(\theta|x_1,\cdots,x_n) = \frac{2^n\prod_{i=1}^{n} x_i}{k_1\theta^{2n}}$

$$k_1 = \int_{x_{(n)}}^{1} \frac{2^n \prod_{i=1}^{n} x_i}{\theta^{2n}} d\theta$$

$$= 2^n \prod_{i=1}^{n} x_i \frac{1}{-2n+1} (1 - x_{(n)}^{1-2n})$$

$$\therefore \pi_1(\theta | x_1, \cdots, x_n) = \frac{1-2n}{(1-x_{(n)}^{1-2n})\theta^{2n}} = \frac{2n-1}{(x_{(n)}^{1-2n}-1)\theta^{2n}} \quad (x_{(n)} < \theta < 1)$$

(2) $\pi(\theta) = 3\theta^2 \quad (0<\theta<1)$

$$\Rightarrow \pi_2(\theta | x_1, \cdots, x_n) = \frac{3 \cdot 2^n \prod_{i=1}^{n} x_i}{k_2 \, \theta^{2n-2}}$$

$$k_2 = \int_{x_{(n)}}^{1} \frac{3 \cdot 2^n \prod_{i=1}^{n} x_i}{\theta^{2n-2}} d\theta$$

$$= 3 \cdot 2^n \prod_{i=1}^{n} x_i \frac{1}{3-2n} (1 - x_{(n)}^{3-2n})$$

$$\therefore \pi_2(\theta | x_1, \cdots, x_n) = \frac{3-2n}{(1-x_{(n)}^{3-2n})\theta^{2n-2}} = \frac{2n-3}{(x_{(n)}^{3-2n}-1)\theta^{2n-2}} \quad (x_{(n)} < \theta < 1).$$

11. $h(x_1, x_2 \cdots, x_n, \theta) = \theta^{-n} \frac{192}{\theta^4} \quad (0 < x_{(1)} < x_{(n)} < \theta, \ \theta \geq 4)$

$\because x_{(1)} = 3, \quad x_{(3)} = 8$

则 $h(x_1, x_2, \cdots, x_n, \theta) = 192 \theta^{-7} \quad (\theta > 8)$

$$\pi(\theta | x_1, x_2, x_3) = \frac{192 \theta^{-7}}{\int_8^{+\infty} 192 \theta^{-7} d\theta}$$

$$= 1572864 \, \theta^{-7} \quad (\theta > 8)$$

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2. σ^2 已知 $\Rightarrow$ μ 的置信区间为 $[\bar{x} - u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{x} + u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]$

长度为 $L = 2 u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$1-\alpha=95\% \Rightarrow \alpha=0.05 \Rightarrow u_{1-\frac{\alpha}{2}}=u_{0.975}=1.96$

$L\le k \Rightarrow n\ge \left(3.92\frac{\sigma}{k}\right)^2$

故样本容量最少为 $\left\lceil \left(3.92\frac{\sigma}{k}\right)^2 \right\rceil$

3. (1) $\left[\bar{y}-u_{0.975}\cdot\frac{1}{2},\ \bar{y}+u_{0.975}\cdot\frac{1}{2}\right]$

$\bar{y}=\frac{1}{4}\sum_{i=1}^{4}\ln x_i=\frac{1}{4}\ln\prod_{i=1}^{4}x_i=0.$

故置信区间为 $[-0.98, 0.98]$.

(2) $EX=Ee^{Y}=M_Y(1)=e^{\mu u+\frac{1}{2}u^2}\Big|_{u=1}=e^{\mu+\frac{1}{2}}$

$P(-0.98\le\mu\le 0.98)=95\%$

$\Rightarrow P(0.62\le e^{\mu+\frac{1}{2}}\le 4.39)=95\%$

故EX的置信区间为 $[0.62, 4.39]$.

12 $X\sim E(\lambda) \qquad f(x)=\lambda e^{-\lambda x}\ (x>0)$

$T=\sum_{i=1}^{n}x_i$ 为 λ 的充分统计量

T的值为 531

$T\sim\Gamma(n,\lambda)$

令 $Z=2\lambda T$ 则 $Z\sim\Gamma(n,\frac{1}{2})=\chi^2(2n)$

$P\left(\chi^2_{\frac{\alpha}{2}}(18)\le Z\le\chi^2_{1-\frac{\alpha}{2}}(18)\right)=1-\alpha=0.9$

$\Rightarrow P\left(\frac{2T}{\chi^2_{1-\frac{\alpha}{2}}(18)}\le\frac{1}{\lambda}\le\frac{2T}{\chi^2_{\frac{\alpha}{2}}(18)}\right)=0.9$

$\chi^2_{1-\frac{0.1}{2}}(18)=28.87 \qquad \chi^2_{\frac{0.1}{2}}(18)=9.39$

则置信区间为 $[36.79, 113.10]$.

$\chi^2_{1-0.1}(18) = 25.99$ 则置信下限为 40.86

$\chi^2_{0.1}(18) = 10.86$ 则置信上限为 97.79

16.

令 $u = \dfrac{(X_{(n)} + X_{(1)}) - 2\theta}{X_{(n)} - X_{(1)}}$

$V = X_{(n)} - X_{(1)}$

则 $\begin{cases} X_{(1)} = \dfrac{uv - v + 2\theta}{2} \\ X_{(n)} = \dfrac{uv + v + 2\theta}{2} \end{cases}$

$$J = \left\| \frac{\partial(X_{(1)}, X_{(n)})}{\partial(u,v)} \right\| = \begin{vmatrix} \frac{v}{2} & \frac{u-1}{2} \\ \frac{v}{2} & \frac{u+1}{2} \end{vmatrix} = \frac{v}{2}$$

$(X_{(1)}, X_{(n)})$ 的联合密度函数为

$$f(X_{(1)}, X_{(n)}; \theta-\tfrac{1}{2}, \theta+\tfrac{1}{2}) = n(n-1)(X_{(n)} - X_{(1)})^{n-2}, \quad \theta-\tfrac{1}{2} < X_{(1)} \le X_{(n)} < \theta+\tfrac{1}{2}$$

所以, (u,v) 的联合密度函数为

$$g(u,v; \theta-\tfrac{1}{2}, \theta+\tfrac{1}{2}) = \frac{n(n-1)}{2} v^{n-1}$$

$$\theta - \frac{1}{2} < \frac{uv - v + 2\theta}{2} \le \frac{uv + v + 2\theta}{2} < \theta + \frac{1}{2}$$

$$-1 < uv - v \le uv + v < 1$$

若 $u \ge 0$ 则 $0 < v < \dfrac{1}{1+u}$

若 $u < 0$ 则 $0 < v < \dfrac{1}{1-u}$

$$h(u) = I_{\{u\ge 0\}} \int_0^{\frac{1}{1+u}} \frac{n(n-1)}{2} v^{n-1} dv + I_{\{u<0\}} \int_0^{\frac{1}{1-u}} \frac{n(n-1)}{2} v^{n-1} dv$$

$$= \frac{n-1}{2(1+|u|)^n}$$

$\forall c>0 \quad P(|u|\le c) = \int_0^c \frac{n-1}{(1+u)^n}\,du = 1-(1+c)^{-(n-1)}$

取 $c_0 = 2^{-\frac{1}{n-1}}-1$

则 $P(-c_0<u<c_0)=1-\alpha$

即 $P\left(-c_0 \le \frac{X_{(n)}+X_{(1)}-2\theta}{X_{(n)}-X_{(1)}} \le c_0\right) = 1-\alpha$

θ 的置信水平为 $1-\alpha$ 的置信区间为

$$\left[\frac{X_{(n)}+X_{(1)}}{2}-\frac{(2^{-\frac{1}{n-1}}-1)(X_{(n)}-X_{(1)})}{2},\ \frac{X_{(n)}+X_{(1)}}{2}+\frac{(2^{-\frac{1}{n-1}}-1)(X_{(n)}-X_{(1)})}{2}\right]$$

19. (1) $p(x;\theta)=e^{-(x-\theta)} \quad (x>\theta)$

$\Rightarrow F(x;\theta) = \int_0^x e^{-(x-\theta)}dx$

$= 1-e^{-(x-\theta)} \quad (x>\theta)$

$f_{X_{(1)}}(x) = n\,(1-F(x;\theta))^{n-1}\cdot p(x;\theta)$

$= n\,e^{-(n-1)(x-\theta)}\cdot e^{-(x-\theta)}$

$= ne^{-n(x-\theta)} \quad (x>\theta)$

$X_{(1)}-\theta$ 的密度函数为

$g(x) = f(X_{(1)}-\theta=x) = f_{X_{(1)}}(x+\theta) = ne^{-nx} \quad (x>0)$

与 θ 无关

(2) $P(G_{\frac{\alpha}{2}} \le X_{(1)}-\theta \le G_{1-\frac{\alpha}{2}}) = 1-\alpha$

设 $G_{\frac{\alpha}{2}} = a$

$\int_0^a ne^{-nx}dx = \frac{\alpha}{2} \Rightarrow a = G_{\frac{\alpha}{2}} = -\frac{1}{n}\ln(1-\frac{\alpha}{2})$

同理 $c_{1-\frac{\alpha}{2}} = -\frac{1}{n}\ln\frac{\alpha}{2}$

故置信区间为 $[X_{(1)} + \frac{1}{n}\ln\frac{\alpha}{2},\ X_{(1)} + \frac{1}{n}\ln(1-\frac{\alpha}{2})]$