作业十二

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8. $$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) = \frac{1}{n}\sum_{i=1}^{n}E(X_i) = 0$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i) = \frac{1}{n^2}\times n\times\frac{2^2}{12} = \frac{1}{3n}$$

9. $$Cov(X_i-\bar{X}, X_j-\bar{X}) = Cov(X_i, X_j) - Cov(X_i, \bar{X}) - Cov(X_j, \bar{X}) + D\bar{X}$$

$$= 0 - \frac{1}{n}DX_1 - \frac{1}{n}DX_1 + \frac{1}{n^2}nDX_1$$

$$= -\frac{1}{n}DX_1$$

$$D(X_i-\bar{X}) = DX_i + D\bar{X} - 2Cov(X_i, \bar{X}) = DX_1 + \frac{1}{n}DX_1 - 2\cdot\frac{1}{n}DX_1 = \frac{n-1}{n}DX_1$$

$$\therefore r = \frac{-\frac{1}{n}DX_1}{\frac{n-1}{n}DX_1} = -\frac{1}{n-1}$$

12. $$Cov(\bar{X}, S^2) = E[(\bar{X}-\mu)S^2]$$

$$(\bar{X}-\mu)S^2 = \frac{1}{n-1}\left[(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^3\right]$$

又 $$E[(\bar{X}-\mu)(X_i-\mu)^2] = \frac{1}{n}E(X_2-\mu)^3 + E\left[\sum_{j\neq i}(X_j-\mu)^2\right]$$

$$= \frac{V_3}{n}$$

$$E(\bar{X}-\mu)^3 = \frac{1}{n^3}E\left[\sum_{i=1}^{n}(X_i-\mu)^3\right] = \frac{V_3}{n^2}$$

$$\therefore E[(\bar{X}-\mu)S^2] = \frac{1}{n-1}\left(V_3 - \frac{V_3}{n}\right) = \frac{V_3}{n}$$

16 均匀分布 $U(0,5)$ 的均值为 $\frac{5}{2}$，方差为 $\frac{25}{12}$

样本容量为25

则样本均值 $\bar{X}$ 的渐近分布为 $N(\frac{5}{2}, \frac{1}{12})$

23. $P(X\le k)=\sum_{i=1}^{k} pq^{i-1}=1-q^k,\quad k=1,2,\cdots$

$P(X_{(n)}\le k)=P(X_1\le k, X_2\le k,\cdots,X_n\le k)$

$=(P(X_1\le k))^n=(1-q^k)^n \qquad k=1,2,\cdots$

$P(X_{(n)}\le k-1)=(1-q^{k-1})^n \qquad k=1,2,\cdots$

$X_{(n)}$的分布列为

$P(X_{(n)}=k)=P(X_{(n)}\le k)-P(X_{(n)}\le k-1)$

$=(1-q^k)^n-(1-q^{k-1})^n \qquad k=1,2,\cdots$

$P(X\ge k)=1-P(X\le k-1)=q^{k-1} \qquad k=1,2,\cdots$

$P(X_{(1)}\ge k)=(P(X_1\ge k))^n=q^{n(k-1)} \qquad k=1,2,\cdots$

$P(X_{(1)}\ge k+1)=q^{nk} \qquad k=1,2,\cdots$

$X_{(1)}$的分布列为

$P(X_{(1)}=k)=P(X_{(1)}\ge k)-P(X_{(1)}\ge k+1)$

$=q^{n(k-1)}(1-q^n) \qquad k=1,2,\cdots$

26. $F(x)=\int_0^x 6t(1-t)\,dt=3x^2-2x^3=x^2(3-2x),\quad 0\le x\le 1$

$1-F(x)=1-x^2(3-2x)=2x^3-3x^2+1$

$=(1-x)^2(2x+1),\quad 0\le x\le 1$

故样本中位数 $m_{0.5}=X_{(5)}$ 的密度函数为

$p_{m_{0.5}}(x)=\binom{9}{4}(F(x))^4\binom{5}{1}p(x)(1-F(x))^4$

$=3780x^9(1-x)^9(3-2x)^4(2x+1)^4$

32. $F(x)=x^3,\ 0<x\le 1$

$(X_{(2)}, X_{(4)})$ 的联合密度为

$$P(x,y)=\frac{5!}{1!\,1!\,1!\,1!\,1!}x^3(y^3-x^3)(1-y^3)\cdot 3x^2\cdot 3y^2,\quad 0<x<y<1$$

令 $\begin{cases} u=\frac{x}{y} \\ v=y \end{cases}$，其逆变换为 $\begin{cases} x=uv \\ y=v \end{cases}$

$$|J|=\left\|\begin{matrix} v & u \\ 0 & 1 \end{matrix}\right\|=v$$

由 $0<x<y<1$ 得 $0<u<1$，$0<v<1$

$\left(\frac{X_{(2)}}{X_{(4)}}, X_{(4)}\right)$ 的联合密度为

$$P(u,v)=P(uv,v)v=120v^3u^3(v^3-u^3v^3)(1-v^3)\cdot 3v^2u^2\cdot 3v^2v$$

$$=1080u^5(1-u^3)v^{11}(1-v^3)$$

$$U=\frac{X_{(2)}}{X_{(4)}}\sim P_1(u)=\int_0^1 P(u,v)\,dv$$

$$=1080u^5(1-u^3)\int_0^1 v^{11}(1-v^3)\,dv$$

$$=18u^5(1-u^3),\quad 0<u<1$$

$$V=X_{(4)}\sim P_2(v)=\int_0^1 P(u,v)\,du$$

$$=1080v^{11}(1-v^3)\int_0^1 u^5(1-u^3)\,du$$

$$=60v^{11}(1-v^3),\quad 0<v<1$$

$$P(u,v)=P_1(u)\cdot P_2(v)$$

$\frac{X_{(2)}}{X_{(4)}}$ 与 $X_{(4)}$ 独立

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2. $\bar{X} \sim N(\mu, \frac{16}{n})$

$$P(|\bar{X}-\mu|<1) = P\left(\left|\frac{\bar{X}-\mu}{\sqrt{\frac{16}{n}}}\right| < \frac{1}{\sqrt{\frac{16}{n}}}\right)$$

$$= 2\phi\left(\frac{\sqrt{n}}{4}\right) - 1$$

$$\geq 0.95$$

$\Rightarrow n \geq 61.47$

n 至少为 62

9. $X_1+X_2 \sim N(0, 2\sigma^2)$

$X_1-X_2 \sim N(0, 2\sigma^2)$

$$\left(\frac{X_1+X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1)$$

$$\left(\frac{X_1-X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1)$$

又 $\because$ $Cov(X_1+X_2, X_1-X_2) = Var(X_1) - Var(X_2) = 0$

$\therefore$ X_1+X_2 与 X_1-X_2 独立

$$Y = \left(\frac{X_1+X_2}{X_1-X_2}\right)^2 = \frac{\left(\frac{X_1+X_2}{\sqrt{2}\sigma}\right)^2}{\left(\frac{X_1-X_2}{\sqrt{2}\sigma}\right)^2} \sim F(1,1)$$

11 $\bar{X} \sim N(\mu_1, \frac{\sigma^2}{n})$ $\quad c(\bar{X}-\mu_1) \sim N(0, \frac{c^2\sigma^2}{n})$

$\bar{Y} \sim N(\mu_2, \frac{2\sigma^2}{m})$ $\quad d(\bar{y}-\mu_2) \sim N(0, \frac{d^2\sigma^2}{m})$

$\bar{X}$ 与 S_x^2 i.d 且 $\frac{(n-1)S_x^2}{\sigma^2} \sim \chi^2(n-1)$

$\bar{Y}$ 与 S_y^2 i.d. 且 $\frac{(m-1)S_y^2}{\sigma^2} \sim \chi^2(m-1)$

若 $\bar{x}, \bar{y}$ i.d

则 $\bar{x}, \bar{y}, S_x^2, S_y^2$ i.d.

故 $c(\bar{x}-\mu_1)+d(\bar{y}-\mu_2) \sim N(0, \frac{c^2\sigma^2}{n}+\frac{d^2\sigma^2}{m})$

$$\Rightarrow u = \frac{c(\bar{x}-\mu_1)+d(\bar{y}-\mu_2)}{\sigma\sqrt{\frac{c^2}{n}+\frac{d^2}{m}}} \sim N(0,1)$$

$$V = \frac{(n-1)S_x^2}{\sigma^2} + \frac{(m-1)S_y^2}{\sigma^2} \sim \chi^2(m+n-2)$$

且 u, v i.d.

则有 $t = \frac{u}{\sqrt{\frac{V}{m+n-2}}} \sim t(n+m-2)$

其中 $$t = \frac{u}{\sqrt{\frac{V}{m+n-2}}} = \frac{\frac{c(\bar{x}-\mu_1)+d(\bar{y}-\mu_2)}{\sigma\sqrt{\frac{c^2}{n}+\frac{d^2}{m}}}}{\frac{1}{\sigma}\sqrt{\frac{(n-1)S_x^2+(m-1)S_y^2}{n+m-2}}}$$

$$= \frac{c(\bar{x}-\mu_1)+d(\bar{y}-\mu_2)}{S_w\sqrt{\frac{c^2}{n}+\frac{d^2}{m}}}$$

$$\sim t(n+m-2)$$

12. $X_{n+1} \sim N(\mu, \sigma^2)$

$\overline{X_n} \sim N(\mu, \frac{\sigma^2}{n})$

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2(n-1)$$

且 $X_{n+1}, \overline{X_n}, S_n^2$ 相互独立

则 $X_{n+1}-\overline{X_n} \sim N(0, \sigma^2+\frac{\sigma^2}{n}) = N(0, \frac{n+1}{n}\sigma^2)$

故 $t=\dfrac{\dfrac{X_{n+1}-\overline{X}_n}{\sqrt{\frac{n+1}{n}}}}{S_n}=\dfrac{\dfrac{X_{n+1}-\overline{X}_n}{\sqrt{\frac{n+1}{n}\sigma^2}}}{\sqrt{\frac{S_n^2}{\sigma^2}}}\sim t(n-1)$

当 $c=\sqrt{\frac{n}{n+1}}$ 时，$t_c=c\dfrac{X_{n+1}-\overline{X}_n}{S_n}\sim t(n-1)$

19. 设 $X\sim F(x)$ 为连续总体.

设 $Y=F(X)$ 为另一总体，$y_1,y_2,\cdots,y_n$ 为其样本

则 $T=-2\sum_{i=1}^{n}\ln y_i$ 且 y_i 与 X_i 一一对应

记 Y 的分布函数为 $G(y)$

$G(y)=P(F(X)\le y)$

由于 $F(X)$ 连续 严格单增

故 $\exists F^{-1}$ s.t. $x=F^{-1}(y)$

$\therefore G(y)=P(X\le F^{-1}(y))=F(F^{-1}(y))=y$

要使 T 有意义，则 $y_i>0$

$g(y)=G'(y)=1\Rightarrow y_i\le 1$

$\therefore Y\sim U[0,1]$

记 $Z=-2\ln Y\Rightarrow Y=e^{-\frac{z}{2}}$ $(z\ge 0)$

$h(z)=g(y)\left|\frac{dY}{dz}\right|=g(e^{-\frac{z}{2}})\cdot\left|\frac{de^{-\frac{z}{2}}}{dz}\right|$

$=\frac{1}{2}e^{-\frac{z}{2}}$

即 $Z\sim E(\frac{1}{2})=\Gamma(1,\frac{1}{2})$ 且 Z_i iid

则 $T=\sum_{i=1}^{n}Z_i\sim\Gamma(n,\frac{1}{2})=\chi^2(2n)$

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2. $T \sim P(n\lambda)$

$$P(X_1 = x_1, \cdots, X_n = x_n \mid T = t) = \frac{P(X_1 = x_1, \cdots, X_{n-1} = x_{n-1}, X_n = t - \sum_{i=1}^{n-1} x_i)}{P(T=t)}$$

$$= \frac{\prod_{i=1}^{n-1} P(X_i = x_i) \cdot P(X_n = t - \sum_{i=1}^{n-1} x_i)}{\frac{(n\lambda)^t}{t!} e^{-n\lambda}}$$

$$= \frac{\prod_{i=1}^{n-1} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \cdot \frac{\lambda^{x_n}}{x_n!} e^{-\lambda}}{\frac{(n\lambda)^t}{t!} e^{-n\lambda}}$$

$$= \frac{t!}{n^t \prod_{i=1}^{n} x_1!}$$

与λ无关，$T = \sum_{i=1}^{n} x_i$ 是充分统计量.

4. $T = \sum_{i=1}^{n} x_i \sim N(n\mu, n)$

$$P_\mu(x_1, x_2, \cdots, x_n \mid T = t) = \frac{P_\mu(x_1, x_2, \cdots, x_n)}{P_\mu(t)}$$

$$= \frac{(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2}}{(2\pi n)^{-\frac{1}{2}} e^{-\frac{1}{2n}(t - n\mu)^2}}$$

$$= \frac{(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}(\sum_{i=1}^{n} x_i^2 - 2\mu t + n\mu^2)}}{(2\pi n)^{-\frac{1}{2}} e^{-\frac{1}{2n}(t^2 - 2n\mu t + n^2\mu^2)}}$$

$$= \sqrt{n}\,(2\pi)^{-\frac{n-1}{2}} e^{-\frac{1}{2}(\sum_{i=1}^{n} x_i^2 - \frac{t^2}{n})}$$

与μ无关，$T = \sum_{i=1}^{n} x_i$ 是充分统计量

8. 样本的联合密度函数为

$$p(x_1, x_2, \cdots, x_n; \theta) = \left(\frac{1}{2\theta}\right)^n e^{-\frac{\sum_{i=1}^{n}|x_i|}{\theta}}$$

取 $T = \sum_{i=1}^{n}|x_i|$ $\quad g(t;\theta) = \left(\frac{1}{2\theta}\right)^n e^{-\frac{t}{\theta}}$

$h(x_1, x_2, \cdots, x_n) = 1$

故 $T = \sum_{i=1}^{n}|x_i|$ 为 θ 的充分统计量.

12. $$\prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{\theta^n} \cdot \mathbb{1}\{\theta \le x_1, \cdots, x_n \le 2\theta\}$$

$$= \frac{1}{\theta^n} \cdot \mathbb{1}\{\theta \le x_{(1)} \le x_{(n)} \le 2\theta\}$$

故充分统计量为 $T = (x_{(1)}, x_{(n)})$

15. $$\prod_{i=1}^{n} f(x_i;\theta) = C^n(\theta) \prod_{i=1}^{n} h(x_i) \cdot e^{\sum_{i=1}^{n}\sum_{j=1}^{k} Q_j(\theta) T_j(x_i)}$$

$$= C^n(\theta)\, e^{\sum_{j=1}^{k}\left(Q_j^n(\theta) \cdot \sum_{i=1}^{n} T_j(x_i)\right)} \cdot \prod_{i=1}^{n} h(x_i)$$

故取 $T(x) = \left(\sum_{j=1}^{n} T_1(x_j), \cdots, \sum_{j=1}^{n} T_k(x_j)\right)$

$$g(T(x);\theta) = C^n(\theta) \cdot e^{\sum_{j=1}^{k}\left(Q_j^n(\theta) T_j(x)\right)}$$

$$H(x) = \prod_{i=1}^{n} h(x_i)$$

则 $\prod_{i=1} f(x_i;\theta) = H(x) \cdot g(T(x);\theta)$

故 $T(x)$ 为充分统计量.