作业十三

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1. $E(\hat{\mu}_1) = \frac{1}{2}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{6}E(X_3)$

$= \frac{1}{2}\mu + \frac{1}{3}\mu + \frac{1}{6}\mu = \mu$

$E(\hat{\mu}_2) = \frac{1}{3}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{3}E(X_3)$

$= \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu = \mu$

$E(\hat{\mu}_3) = \frac{1}{6}E(X_1) + \frac{1}{6}E(X_2) + \frac{2}{3}E(X_3)$

$= \frac{1}{6}\mu + \frac{1}{6}\mu + \frac{2}{3}\mu = \mu$

$\therefore$ 上述统计量均为该总体均值 μ 的无偏估计

设总体的方差为 σ^2

$Var(\hat{\mu}_1) = \frac{1}{4}Var(X_1) + \frac{1}{9}Var(X_2) + \frac{1}{36}Var(X_3)$

$= \frac{1}{4}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{36}\sigma^2 = \frac{7}{18}\sigma^2$

$Var(\hat{\mu}_2) = \frac{1}{9}Var(X_1) + \frac{1}{9}Var(X_2) + \frac{1}{9}Var(X_3)$

$= \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2 = \frac{1}{3}\sigma^2$

$Var(\hat{\mu}_3) = \frac{1}{36}Var(X_1) + \frac{1}{36}Var(X_2) + \frac{4}{9}Var(X_3)$

$= \frac{1}{36}\sigma^2 + \frac{1}{36}\sigma^2 + \frac{4}{9}\sigma^2 = \frac{1}{2}\sigma^2$

$$Var(\hat{\mu}_2) < Var(\hat{\mu}_1) < Var(\hat{\mu}_3)$$

$\hat{\mu}_3$ 的有效性最差.

2. $\because$ $X_1, X_2, \cdots, X_n$ i.i.d. $\sim Exp(\lambda)$

$\therefore$ $y = \sum_{i=1}^{n} X_i \sim Ga(n, \lambda)$

$$p(y; n, \lambda) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}, \quad y>0$$

$$E\left(\frac{1}{y}\right) = \int_0^\infty \frac{\lambda^n}{\Gamma(n)} y^{n-2} e^{-\lambda y} dy$$

$$= \frac{\lambda}{n-1} \int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\lambda y} dy$$

$$= \frac{\lambda}{n-1}$$

$$E\left(\frac{1}{\bar{x}}\right) = \frac{n\lambda}{n-1}$$

$\frac{1}{\bar{x}}$ 不是 λ 的无偏估计

5. $X \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$E(X) = \theta$

$Var(X) = \frac{1}{12}$

$E(\bar{X}) = \theta$

$\bar{X}$ 为 θ 的无偏估计

$Var(\bar{X}) = \frac{1}{12n}$

由于 $Y = X - (\theta - \frac{1}{2}) \sim U(0,1)$

令 $y_i = x_i - (\theta - \frac{1}{2}) \quad i = 1, 2, \cdots, n$

$$\frac{1}{2}(x_{(1)} + x_{(n)}) = \frac{1}{2}(y_{(1)} + y_{(n)}) + \theta - \frac{1}{2}$$

$$y_{(i)} \sim Be(i, n-i+1)$$

$$E(y_{(1)}) = \frac{1}{n+1} \qquad E(y_{(n)}) = \frac{n}{n+1}$$

$$E\left(\frac{1}{2}(x_{(1)} + x_{(n)})\right) = \frac{1}{2}E(y_{(1)} + y_{(n)}) + \theta - \frac{1}{2} = \theta$$

$\frac{1}{2}(x_{(1)} + x_{(n)})$ 为 θ 的无偏估计

$$y_{(n)} - y_{(1)} \sim Be(n-1, 2)$$

$$Var(y_{(1)}) = Var(y_{(n)}) = \frac{n}{(n+1)^2(n+2)}$$

$$Var(y_{(n)} - y_{(1)}) = \frac{2(n-1)}{(n+1)^2(n+2)}$$

$$Cov(y_{(1)}, y_{(n)}) = \frac{1}{2}[Var(y_{(n)}) + Var(y_{(1)}) - Var(y_{(n)} - y_{(1)})]$$

$$= \frac{1}{(n+1)^2(n+2)}$$

$$Var\left(\frac{1}{2}(x_{(1)} + x_{(n)})\right) = \frac{1}{4}[Var(y_{(1)}) + Var(y_{(n)}) + 2Cov(y_{(1)}, y_{(n)})]$$

$$= \frac{1}{4}\left[\frac{2n}{(n+1)^2(n+2)} + \frac{2}{(n+1)^2(n+2)}\right]$$

$$= \frac{1}{2(n+1)(n+2)}$$

$n > 2$ 时 $\quad \frac{1}{12n} > \frac{1}{2(n+1)(n+2)}$

$\frac{1}{2}(x_{(1)} + x_{(n)})$ 比 $\bar{x}$ 有效

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$$f_1(x) = 3\cdot\left(\frac{\theta-x}{\theta}\right)^2\cdot\frac{1}{\theta}$$

$$= \frac{3}{\theta^3}(\theta-x)^2,\quad 0<x<\theta$$

$$f_3(x) = 3\cdot\left(\frac{x}{\theta}\right)^2\cdot\frac{1}{\theta}$$

$$= \frac{3}{\theta^3}x^2,\quad 0<x<\theta$$

$$E(X_{(1)}) = \frac{3}{\theta^3}\int_0^\theta x(\theta-x)^2 d\theta$$

$$= \frac{\theta}{4}$$

$$E(X_{(3)}) = \frac{3}{\theta^3}\int_0^\theta x^3 dx$$

$$= \frac{3}{4}\theta$$

$$E(4X_{(1)}) = \theta$$

$$E(\tfrac{4}{3}X_{(3)}) = \theta$$

$4(X_{(1)})$ 与 $\frac{4}{3}X_{(3)}$ 均为 θ 的无偏估计

$$E(X_{(1)}^2) = \frac{1}{10}\theta^2$$

$$E(X_{(3)}^2) = \frac{3}{5}\theta^2$$

$$Var(4X_{(1)}) = \frac{3}{5}\theta^2$$

$$Var(\tfrac{4}{3}X_{(3)}) = \frac{\theta^2}{15}$$

$$Var\left(\frac{4}{3}X_{(3)}\right) < Var(4X_{(1)})$$

$\frac{4}{3}X_{(3)}$ 更有效

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2.
$$E(X) = \frac{\theta}{2} \Rightarrow \theta = 2E(X)$$

$$\overline{x} = \frac{0.5 + 1.3 + \cdots + 1.6}{10} = 1.34$$

θ 的矩估计为 $2\overline{x} = 2.68$.

3. (1)
$$EX = \sum_{k=0}^{N-1} kP(X=k)$$

$$= \sum_{k=1}^{N-1} k \cdot \frac{1}{N}$$

$$= \frac{N-1}{2}$$

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

令 $EX = \overline{x} \Rightarrow \hat{N}_{矩} = 2\overline{x} + 1 = \frac{2}{n}\sum_{i=1}^{n} x_i - 1$

又 $\because N \in N^*$

$\therefore \hat{N}_{矩} = \lceil 2\overline{x} \rceil + 1$

(2)
$$EX = \sum_{k=2}^{+\infty} kP(X=k)$$

$$= \sum_{k=2}^{+\infty} k(k-1)\theta^2(1-\theta)^{k-2}$$

$$= \theta^2 \sum_{k=2}^{+\infty} (p^k)'' \Big|_{p=1-\theta}$$

$$= \theta^2 \left(\sum_{k=2}^{+\infty} p^k\right)'' \Big|_{p=1-\theta}$$

$$= \theta^2 \left(\frac{p^2}{1-p}\right)'' \Big|_{p=1-\theta}$$

$$= \theta^2 \cdot \frac{2}{(1-p)^3} \Big|_{p=1-\theta}$$

$$= \frac{2}{\theta}$$

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

令 $EX = \overline{X} \Rightarrow \hat{\theta}_{矩} = \frac{2n}{\sum_{i=1}^{n} X_i} = \frac{2}{\overline{X}}$

又 $\because \theta \in (0,1)$

$\therefore \hat{\theta}_{矩} = \min\left\{\frac{2}{\overline{X}}, 1\right\}$

4. (1) $EX = \int_0^\theta x\, p(x;\theta)\, d\theta$

$$= \int_0^\theta x \cdot \frac{2}{\theta^2} (\theta - x)\, dx$$

$$= \frac{2}{\theta^2} \cdot \frac{1}{6} \theta^3$$

$$= \frac{\theta}{2}$$

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

令 $EX = \overline{X} \Rightarrow \theta = \frac{3}{n} \sum_{i=1}^{n} X_i = 3\overline{X} \quad (\overline{X} > 0)$

(2) $EX = \int_0^1 x\,p(x;\theta)\,dx$

$= \int_0^1 x(\theta+1)x^{\theta}\,dx$

$= \dfrac{\theta+1}{\theta+2}$

令 $EX = \overline{X} \Rightarrow \theta = \dfrac{1}{1-\overline{X}} - 2 \quad (\frac{1}{2} < \overline{X} < 1)$

(3) $EX = \int_0^1 x \cdot \sqrt{\theta}\ x^{\sqrt{\theta}-1}\,dx$

$= \dfrac{\sqrt{\theta}}{\sqrt{\theta}+1}$

令 $EX = \overline{X} \Rightarrow \hat{\theta}_{矩} = \left(\dfrac{1}{1-\overline{X}} - 1\right)^2 = \left(\dfrac{\overline{X}}{1-\overline{X}}\right)^2 \quad (0 < \overline{X} < 1)$

(4) $EX = \int_{\mu}^{+\infty} \frac{1}{\theta} x e^{-\frac{x-\mu}{\theta}}\,dx$

$= -x e^{-\frac{x-\mu}{\theta}}\Big|_{\mu}^{+\infty} + \left(-\theta e^{-\frac{x-\mu}{\theta}}\Big|_{\mu}^{+\infty}\right)$

$= \mu + \theta$

$EX^2 = \int_{\mu}^{+\infty} \frac{1}{\theta} x^2 e^{-\frac{x-\mu}{\theta}}\,dx$

$= -x^2 e^{-\frac{x-\mu}{\theta}}\Big|_{\mu}^{+\infty} + 2\theta EX$

$= \mu^2 + 2\theta\mu + 2\theta^2$

令 $\begin{cases} EX = \overline{X} \\ EX^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_{矩} = \overline{X} - S \\ \hat{\theta}_{矩} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2} = S \end{cases}$

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1. (1) $L(x;\theta) = \prod_{i=1}^{n} p(x_i;\theta)$

$$= (\sqrt{\theta})^n \prod_{i=1}^{n} x_i^{\sqrt{\theta}-1}$$

$$= \theta^{\frac{n}{2}} \left(\prod_{i=1}^{n} x_i\right)^{\sqrt{\theta}-1}$$

$$\ln L(\theta) = \frac{n}{2}\ln\theta + (\sqrt{\theta}-1)\ln\prod_{i=1}^{n} x_i$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n}{2\theta} + \frac{1}{2\sqrt{\theta}}\ln\prod_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{n^2}{\ln^2\prod_{i=1}^{n} x_i}$$ 且为极大值点

(2) $L(x;\theta) = \prod_{i=1}^{n} p(x_i;\theta)$

$$= (\theta c^{\theta})^n \left(\prod_{i=1}^{n} x_i\right)^{-(\theta+1)}$$

$$= \theta^n \cdot c^{n\theta} \cdot \left(\prod_{i=1}^{n} x_i\right)^{-(\theta+1)}$$

$$\ln L(\theta) = n\ln\theta + n\theta\ln c - (\theta+1)\ln\prod_{i=1}^{n} x_i$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n}{\theta} + n\ln c - \ln\prod_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{n}{\ln\prod_{i=1}^{n} x_i - n\ln c}$$ 且为极大值点

2.(1)

$$L(\theta) = c^n \theta^{nc} (x_1 x_2 \cdots x_n)^{-(c+1)} I_{\{x_{(1)} > \theta\}}$$

θ的最大似然估计为$x_{(1)}$

(2)

$$L(\theta) = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)}{\theta}}, \quad x_{(1)} > \mu$$

$$\ln L(\theta, \mu) = -n\ln\theta - \frac{\sum_{i=1}^{n}(x_i-\mu)}{\theta}$$

μ的最大似然估计为$\hat{\mu} = x_{(1)}$

令 $$\frac{\partial \ln L(\theta, \hat{\mu})}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})}{\theta^2} = 0.$$

$$\Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})}{n} = \bar{x} - x_{(1)}$$

(3)

$$L(\theta) = \left(\frac{1}{k\theta}\right)^n I_{\{\theta < x_{(1)} \le x_{(n)} < (k+1)\theta\}}$$

$$\frac{x_{(1)}}{k+1} < \theta < x_{(1)}$$

θ的最大似然估计为$\hat{\theta} = \frac{x_{(n)}}{k+1}$

6.

$$\hat{\mu} = \frac{1}{20}\sum_{i=1}^{20} \ln x_i = 3.0890$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\ln x_i - 3.0890)^2 = 0.5081$$

$$\hat{E}(X) = e^{3.0890 + \frac{0.5081}{2}} = 28.3053$$

7.(1)

$X \sim U(\theta, 2\theta)$

$$E(X) = \frac{3\theta}{2}$$

$$Var(X) = \frac{\theta^2}{12}$$

$$E(\bar{x}) = \frac{3\theta}{2}$$

$$Var(\bar{x}) = \frac{\theta^2}{12n}$$

$E(\hat{\theta}) = \frac{2}{3}E(\bar{X}) = \theta$

$\hat{\theta} = \frac{2}{3}\bar{X}$ 是 θ 的无偏估计

$Var(\hat{\theta}) = \frac{4}{9} \times \frac{\theta^2}{12n} = \frac{\theta^2}{27n} \to 0$

$\hat{\theta}$ 是 θ 的相合估计

(2) $L(\theta) = \left(\frac{1}{\theta}\right)^n I_{\{\theta < X_{(1)} \leq X_{(n)} < 2\theta\}}$

$\frac{X_{(n)}}{2} < \theta < X_{(1)}$

$\hat{\theta} = \frac{X_{(n)}}{2}$ 为 θ 的最大似然估计

$f(x) = n\left(\frac{x-\theta}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} = \frac{n}{\theta^n}(x-\theta)^{n-1}, \quad \theta < x < 2\theta$

$E(X_{(n)}) = \int_{\theta}^{2\theta} x \frac{n}{\theta^n}(x-\theta)^{n-1} dx$

$= \frac{n}{\theta^n}\int_0^{\theta}(t+\theta)t^{n-1} dt$

$= \frac{2n+1}{n+1}\theta$

$E(X_{(n)}^2) = \int_{\theta}^{2\theta} x^2 \frac{n}{\theta^n}(x-\theta)^{n-1} dx$

$= \frac{4n^2+8n+2}{(n+2)(n+1)}\theta^2$

$E(\hat{\theta}) = \frac{1}{2}E(X_{(n)}) = \frac{2n+1}{2(n+1)}\theta$

$\hat{\theta}$ 不是 θ 的无偏估计

$$Var(\hat{\theta}) = \frac{1}{4} Var(X_{(n)})$$

$$= \frac{n\theta^2}{4(n+1)^2(n+2)} \to 0 \quad (n \to \infty)$$

$\hat{\theta}$ 是 θ 的相合估计

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1. $MSE(\hat{g}) - MSE(\tilde{g}) = E(\hat{g} - g(\theta))^2 - E(\tilde{g} - g(\theta))^2$

$$= [D\hat{g} + (E\hat{g} - g(\theta))^2] - [D\tilde{g} + (E\tilde{g} - g(\theta))^2]$$

由于 $E\tilde{g} = E(E(\hat{g}|T)) = E\hat{g}$

故 $MSE(\hat{g}) - MSE(\tilde{g}) = D\hat{g} - D\tilde{g}$

$$= D\hat{g} - D(E(\hat{g}|T))$$

$$= D(E(\hat{g}|T)) + E(D(\hat{g}|T)) - D(E(\hat{g}|T))$$

$$= E(D(\hat{g}|T)) \geq 0$$

$\therefore MSE(\hat{g}) \geq MSE(\tilde{g})$

7 $\ln p(x;\theta) = \ln 2 + \ln\theta - 3\ln x - \frac{\theta}{x^2}$

$$\frac{\partial \ln p(x;\theta)}{\partial\theta} = \frac{1}{\theta} - \frac{1}{x^2}$$

$$\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$$

$$I(\theta) = -E\left(\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2}$$

8. $$\ln P(x;\theta) = \ln\theta + \theta\ln c - (\theta+1)\ln x$$

$$\frac{\partial \ln P(x;\theta)}{\partial\theta} = \frac{1}{\theta} + \ln c - \ln x$$

$$\frac{\partial^2 \ln P(x;\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$$

$$I(\theta) = -E\left(\frac{\partial^2 \ln P(x;\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2}$$