# homework 4

刘若涵 2020011126

1. $T_0^{(0)} = \frac{h}{2}[f(1)+f(1.6)] = -0.433333$

$T_0^{(1)} = \frac{1}{2}T_0^{(0)} + \frac{h}{2}f(1.3) = -0.385498$

$T_0^{(2)} = \frac{1}{2}T_0^{(1)} + \frac{h}{4}(f(1.15)+f(1.45)) = -0.371799$

$T_0^{(3)} = \frac{1}{2}T_0^{(2)} + \frac{h}{8}(f(1.075)+f(1.225)+f(1.375)+f(1.525)) = -0.368202$

根据公式 $T_m^{(k)} = \frac{4^m}{4^m-1}T_{m-1}^{(k+1)} - \frac{1}{4^m-1}T_{m-1}^{(k)}$

| $k$ | $T_0^{(k)}$ | $T_1^{(k)}$ | $T_2^{(k)}$ | $T_3^{(k)}$ |
|---|---|---|---|---|
| 0 | -0.433333 | | | |
| 1 | -0.385498 | -0.369553 | | |
| 2 | -0.371799 | -0.367233 | -0.367078 | |
| 3 | -0.368202 | -0.367003 | -0.366988 | -0.366987 |

$$\int_1^{1.6} \frac{x}{x^2-4}dx \approx -0.366987$$

2. 理论计算： $\int_0^1 \frac{2}{\sqrt[3]{x}}dx = 3x^{\frac{2}{3}}\Big|_0^1 = 3$

$$\int_0^1 \frac{\cos 2x}{\sqrt[3]{x}}dx = \int_0^1 \frac{1-\sin^2 x}{\sqrt[3]{x}}dx = \frac{3}{2} - 2\int_0^1 \frac{\sin^2 x}{\sqrt[3]{x}}dx$$

$$\int_0^1 \frac{2\sin^2 x}{\sqrt[3]{x}}dx = \frac{h}{6}\left[f(0) + 2\left(f(\tfrac{1}{3})+f(\tfrac{2}{3})\right) + 4\left(f(\tfrac{1}{6}) + 4f(\tfrac{1}{2}) + 4f(\tfrac{5}{6})\right) + f(1)\right]$$

$$= \frac{1}{6}\times\frac{1}{3}\times\left[0 + \frac{4\sin^2\frac{1}{3}}{\sqrt[3]{\frac{1}{3}}} + \frac{4\sin^2\frac{2}{3}}{\sqrt[3]{\frac{2}{3}}} + \frac{8\sin^2\frac{1}{6}}{\sqrt[3]{\frac{1}{6}}} + \frac{8\sin^2\frac{1}{2}}{\sqrt[3]{\frac{1}{2}}} + \frac{8\sin^2\frac{5}{6}}{\sqrt[3]{\frac{5}{6}}} + 2\sin^2 1\right]$$

$= 0.619942$

$\therefore \int \frac{\cos 2x}{\sqrt[3]{x}} dx = 0.880058$

3. (1) 设 $P = \int_0^1 f(x) dx$

令 $f(x) = 1$ 则 $P = \frac{1}{2} + C_1 = 1 \Rightarrow C_1 = \frac{1}{2}$

$$\left.\begin{array}{l}\text{令 } f(x) = x \text{ 则 } P = \frac{1}{2}x_0 + \frac{1}{2}x_1 = \frac{1}{2} \\ \text{令 } f(x) = x^2 \text{ 则 } P = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2 = \frac{1}{3}\end{array}\right\} \Rightarrow \begin{cases} x_0 = \frac{3-\sqrt{3}}{6} \\ x_1 = \frac{3+\sqrt{3}}{6} \end{cases}$$

当 $f(x) = x^3$ 时，$P = \frac{1}{4}$

而 $\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^3 + \frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^3 = \frac{1}{4} = P$

当 $f(x) = x^4$ 时，$P = \frac{1}{5}$

而 $\frac{1}{2}\left(\frac{3-\sqrt{3}}{6}\right)^4 + \frac{1}{2}\left(\frac{3+\sqrt{3}}{6}\right)^4 = \frac{7}{36} \neq \frac{1}{5}$

$\therefore$ 该求积公式具有三阶代数精度

(2) 设 $P = \int_0^h f(x) dx$

令 $f(x) = 1$ 则 $P = h$

令 $f(x) = x$ 则 $P = \frac{h^2}{2}$

令 $f(x) = x^2$ 则 $P = \frac{h^3}{2} - 2ah^3 = \frac{h^3}{3} \Rightarrow a = \frac{1}{12}$

当 $f(x) = x^3$ 时 $P = \frac{h^4}{4} = \frac{h^4}{2} - \frac{h^4}{4}$

当 $f(x) = x^4$ 时 $P = \frac{h^5}{5} \neq \frac{h^5}{2} - \frac{h^5}{3}$

$\therefore$ 该求积公式具有三阶代数精度

4. 将区间四等分，$I_1=\int_0^{\frac{\pi}{4}}\varphi^2\sin\varphi\,d\varphi$ $I_2=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\varphi^2\sin\varphi\,d\varphi$

$I_3=\int_{\frac{\pi}{2}}^{\frac{3}{4}\pi}\varphi^2\sin\varphi\,d\varphi$ $I_4=\int_{\frac{3}{4}\pi}^{\pi}\varphi^2\sin\varphi\,d\varphi$

两点高斯公式：$\int_{-1}^{1}f(x)\,dx=f(\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}})$

① 区间变换 $t=\frac{8}{\pi}\varphi-1 \Rightarrow \varphi=\frac{\pi}{8}(t+1)$

则 $I_1=\frac{\pi}{8}\int_{-1}^{1}(\frac{\pi}{8}(t+1)^2)\sin(\frac{\pi}{8}(t+1))\,dt$

令 $f_1(t)=(\frac{\pi}{8}(t+1)^2)\sin(\frac{\pi}{8}(t+1))$

则 $\frac{\pi}{8}(f(\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}}))=0.089263$

② 区间变换 $t=\frac{8}{\pi}\varphi-3 \Rightarrow \varphi=\frac{\pi}{8}(t+3)$

则 $I_2=\frac{\pi}{8}\int_{-1}^{1}(\frac{\pi}{8}(t+3)^2)\sin(\frac{\pi}{8}(t+3))\,dt$

令 $f_2(t)=(\frac{\pi}{8}(t+3)^2)\sin(\frac{\pi}{8}(t+3))$

则 $\frac{\pi}{8}(f(\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}}))=1.053751$

③ 区间变换 $t=\frac{8}{\pi}\varphi-5 \Rightarrow \varphi=\frac{\pi}{8}(t+5)$

则 $I_3=\frac{\pi}{8}\int_{-1}^{1}(\frac{\pi}{8}(t+5)^2)\sin(\frac{\pi}{8}(t+5))\,dt$

令 $f_3(t)=(\frac{\pi}{8}(t+5)^2)\sin(\frac{\pi}{8}(t+5))$

则 $\frac{\pi}{8}(f(\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}}))=2.702065$

④ 区间变换 $t=\frac{8}{\pi}\varphi-7 \Rightarrow \varphi=\frac{\pi}{8}(t+7)$

则 $I_4=\frac{\pi}{8}\int_{-1}^{1}(\frac{\pi}{8}(t+7)^2)\sin(\frac{\pi}{8}(t+7))\,dt$

令 $f_4(t)=(\frac{\pi}{8}(t+7)^2)\sin(\frac{\pi}{8}(t+7))$

则 $\frac{\pi}{8}(f(\frac{1}{\sqrt{3}})+f(-\frac{1}{\sqrt{3}}))=2.024768$

$\therefore \ I = 5.869847$

5. $L = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+(y')^2}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\frac{1}{\cos^4 x}}\, dx$

复化梯形公式中

$$f(x) = \sqrt{1+\frac{1}{\cos^4 x}} = \frac{\sqrt{1+\cos^4 x}}{\cos^2 x}$$

$$I = \sum_{k=0}^{n-1} \frac{h}{2}\left(f(x_k) + f(x_{k+1})\right)$$

$$= \frac{h}{2}\left[f(-\tfrac{\pi}{4}) + 2\sum_{k=1}^{n-1} f(x_k) + f(\tfrac{\pi}{4})\right]$$

其中，$h = \frac{\pi}{2n}$ $\quad x_k = -\frac{\pi}{4} + k\cdot\frac{\pi}{2n}$ $\quad k = 1, 2, \cdots, n-1$

方法误差：

$$R_n(f) = -\frac{(b-a)h^2}{12}\max\left(f''(\eta)\right)$$

$$= -\frac{\pi^3}{96n^2}\max\left(f''(\eta)\right)$$

$$\max\left(\frac{d^2}{dx^2}\left(\frac{\sqrt{1+\cos^4 x}}{\cos^2 x}\right)\right) = \frac{176}{5\sqrt{5}}$$

代入得 $\quad |R_n(f)| = \frac{5.0844}{n^2}$

保留m位小数时舍入误差.

$$|\Delta f_k| \leq \frac{1}{2} \times 10^{-m}$$

$$I_k = \frac{\pi}{4n}\left(f(x_k) + f(x_{k+1})\right) \leq \frac{\pi}{2n} \times \frac{1}{2} \times 10^{-m}$$

$$\delta = \frac{\pi}{4} \times 10^{-m}$$