homework5

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1.

$$\begin{cases} y' = f(x,y) = -2y+2x^2+2x \\ y(0)=1 \end{cases}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f(x_{n+1},y_{n+1})\right]$$

$$= y_n + \frac{h}{2}\left[-2y_n+2x_n^2+2x_n-2y_{n+1}+2x_{n+1}^2+2x_{n+1}\right]$$

$$y_{n+1} = \frac{1-h}{h+1}y_n + \frac{h}{h+1}x_n^2 + \frac{h}{h+1}x_n + \frac{h}{h+1}x_{n+1}^2 + \frac{h}{h+1}x_{n+1}$$

$$= \frac{9}{11}y_n + \frac{1}{11}(x_n^2+x_n+x_{n+1}^2+x_{n+1})$$

x_n	0	0.1	0.2	0.3	0.4	0.5
数值解 y_n	1	0.828182	0.709421	0.637708	0.608125	0.616648
解析解 $y(x_n)$	1	0.828731	0.710320	0.638812	0.609329	0.617879
误差 $\lvert e\rvert$	0	5.49×10^{-4}	8.98×10^{-4}	1.103×10^{-3}	1.204×10^{-3}	1.231×10^{-3}

2.

$$y_{n+2} = by_n - (b-1)y_{n+1} + \frac{b+3}{4}hy'_{n+2} + \frac{3b+1}{4}hy'_n$$

$$= by(x_n) - (b-1)\left[y(x_n)+hy'(x_n)+\frac{h^2}{2}y^{(2)}(x_n)+\frac{h^3}{6}y^{(3)}(x_n)+\frac{h^4}{24}y^{(4)}(x_n)+\cdots\right]$$

$$+\frac{b+3}{4}h\left[y'(x_n)+2hy^{(2)}(x_n)+2h^2y^{(3)}(x_n)+\frac{4h^3}{3}y^{(4)}(x_n)+\cdots\right]+\frac{3b+1}{4}hy'(x_n)$$

$$= y(x_n)+2hy'(x_n)+2h^2y^{(2)}(x_n)+\frac{b+5}{3}h^3y^{(3)}(x_n)+\frac{7b+25}{24}h^4y^{(4)}(x_n)+\cdots$$

$$y(x_{n+2}) = y(x_n) + 2hy'(x_n) + 2h^2y^{(2)}(x_n) + \frac{4}{3}h^3y^{(3)}(x_n) + \frac{2}{3}h^4y^{(4)}(x_n) + \cdots$$

$$y_{n+2} - y(x_{n+2}) = \frac{b+1}{3}h^3y^{(3)}(x_n) + \frac{7b+9}{24}h^4y^{(4)}(x_n) + \cdots$$

当 $b \neq -1$ 时 $\quad y_{n+2} - y(x_{n+2}) \approx \frac{b+1}{3}h^3y^{(3)}(x_n) = O(h^3)$ 二阶精度

当 $b = -1$ 时 $\quad y_{n+2} - y(x_{n+2}) \approx \frac{h^4}{12}y^{(4)}(x_n) = O(h^4)$ 三阶精度

3. $\quad y_{n+3} = \frac{1}{8}(9y_{n+2} - y_n) + \frac{3}{8}h(y'_{n+3} + 2y'_{n+2} - y'_{n+1})$

$$= -\frac{1}{8}y(x_n) + \frac{9}{8}\left[y(x_n) + 2hy'(x_n) + 2h^2y^{(2)}(x_n) + \frac{4}{3}h^3y^{(3)}(x_n) + \frac{2}{3}h^4y^{(4)}(x_n) + \frac{4}{15}h^5y^{(5)}(x_n) + \cdots\right]$$

$$+ \frac{3}{8}h\left[y'(x_n) + 3hy^{(2)}(x_n) + \frac{9}{2}h^2y^{(3)}(x_n) + \frac{9}{2}h^3y^{(4)}(x_n) + \frac{27}{8}h^5y^{(5)}(x_n) + \cdots\right]$$

$$+ \frac{3}{4}h\left[y'(x_n) + 2hy^{(2)}(x_n) + 2h^2y^{(3)}(x_n) + \frac{4}{3}h^3y^{(4)}(x_n) + \frac{2}{3}h^4y^{(5)}(x_n) + \cdots\right]$$

$$- \frac{3}{8}h\left[y'(x_n) + hy^{(2)}(x_n) + \frac{h^2}{2}y^{(3)}(x_n) + \frac{h^3}{6}y^{(4)}(x_n) + \frac{h^4}{24}y^{(5)}(x_n) + \cdots\right]$$

$$= y(x_n) + 3hy'(x_n) + \frac{9}{2}h^2y^{(2)}(x_n) + \frac{9}{2}h^3y^{(3)}(x_n) + \frac{27}{8}h^4y^{(4)}(x_n) + \frac{41}{20}h^5y^{(5)}(x_n) + \cdots$$

$$y(x_{n+3}) = y(x_n) + 3hy'(x_n) + \frac{9}{2}h^2y^{(2)}(x_n) + \frac{9}{2}h^3y^{(3)}(x_n) + \frac{27}{8}h^4y^{(4)}(x_n) + \frac{81}{40}h^5y^{(5)}(x_n) + \cdots$$

$$y_{n+3} - y(x_{n+3}) = \frac{h^5}{40}y^{(5)}(x_n) = O(h^5)$$

局部截断误差主项为 $\frac{1}{40}h^5y^{(5)}(x_n)$，有四阶精度

4. $x^2 \in [0,1]$ $\quad y' = x^2+y^2 \le y^2+1$

$$\Rightarrow \frac{y'}{y^2+1} \le 1$$

$$\int_0^1 \frac{y'}{y^2+1}\,dx = \int_{y(0)}^{y(1)} \frac{1}{y^2+1}\,dy = \arctan y(1) - \arctan y(0) \le \int_0^1 dx = 1$$

$$\arctan y(0) \ge 0.3258 \quad \Rightarrow \quad y(0) \ge 0.3379$$

又$\because$ $y' = x^2+y^2 > 0$

$\therefore$ $x \in [0,1]$ 时，y单调递增. $0.3379 \le 0.3379 \le 4$

$$\begin{cases} \bar{y}_{n+1} = y_n + h\cdot f(x_n, y_n) \\ y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})] \end{cases}$$

方法累积误差：

$$\begin{cases} \bar{\Delta}_{n+1} \le (1+hM)\Delta_n + \frac{L}{2}h^2 \\ \Delta_{n+1} \le \Delta_n + \frac{h}{2}M\Delta_n + \frac{h}{2}M\bar{\Delta}_{n+1} + \frac{T}{12}h^3 \end{cases}$$

$$\Rightarrow \Delta_{n+1} \le \left(\frac{h^2M^2}{2} + hM + 1\right)\Delta_n + \left(\frac{LM}{4} + \frac{T}{12}\right)h^3$$

$$\Delta_{n+1} + \frac{\left(\frac{ML}{4}+\frac{T}{12}\right)h^3}{hM+\frac{h^2M^2}{2}} \le \left(\frac{h^2M^2}{2}+hM+1\right)\left(\Delta_n + \frac{\left(\frac{ML}{4}+\frac{T}{12}\right)h^3}{hM+\frac{h^2M^2}{2}}\right)$$

$$\le \left(\frac{h^2M^2}{2}+hM+1\right)^{n+1}\left(\Delta_0 + \frac{\left(\frac{ML}{4}+\frac{T}{12}\right)h^3}{hM+\frac{h^2M^2}{2}}\right)$$

$$\Rightarrow \Delta_n \le \left[\left(\frac{h^2M^2}{2}+hM+1\right)^n - 1\right]\frac{\left(\frac{ML}{4}+\frac{T}{12}\right)h^3}{hM+\frac{h^2M^2}{2}}$$

舍入累积误差:

$$\begin{cases} \bar{\delta}_{n+1} \leq (1+hM)\delta_n + \frac{1}{2}\cdot 10^{-m} \\ \delta_{n+1} \leq \delta_n + \frac{h}{2}M\delta_n + \frac{h}{2}M\bar{\delta}_{n+1} + \frac{1}{2}\cdot 10^{-m} \end{cases}$$

$$\Rightarrow \quad \delta_{n+1} \leq \left(\frac{h^2M^2}{2} + hM + 1\right)\delta_n + \left(\frac{hM}{2}+1\right)\cdot\frac{1}{2}\cdot 10^{-m}$$

$$\delta_{n+1} + \frac{1+\frac{hM}{2}}{hM+\frac{h^2M^2}{2}}\cdot\frac{1}{2}\cdot 10^{-m} \leq \left(\frac{h^2M^2}{2} + hM + 1\right)\left(\delta_n + \frac{1+\frac{hM}{2}}{hM+\frac{h^2M^2}{2}}\cdot\frac{1}{2}\cdot 10^{-m}\right)$$

$$\leq \left(\frac{h^2M^2}{2} + hM + 1\right)^{n+1}\left(\delta_0 + \frac{1+\frac{hM}{2}}{hM+\frac{h^2M^2}{2}}\cdot\frac{1}{2}\cdot 10^{-m}\right)$$

$$\Rightarrow \delta_n \leq \left[\left(\frac{h^2M^2}{2} + hM + 1\right)^n - 1\right]\cdot\left(\frac{1+\frac{hM}{2}}{hM+\frac{h^2M^2}{2}}\cdot\frac{1}{2}\cdot 10^{-m}\right)$$

其中 $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq M \quad \left|y^{(2)}(x)\right| \leq L \quad \left|y^{(3)}(x)\right| \leq T$

$\frac{\partial f}{\partial y}(x,y) = 2y \leq 8$ 取 $M=8$

$y^{(2)}(x) = 2x + 2yy' = 2x + 2y(x^2+y^2) \leq 138$ 取 $L=138$

$y^{(3)}(x) = 2 + 2(x^2+y^2)^2 + 2y(2x+2y(x^2+y^2)) \leq 1684$ 取 $T=1684$

要使结果精确到8位,总误差 $\Delta \leq \frac{1}{2}\times 10^{-8}$

$$\begin{cases} \delta_n \leq \left[(1+8h+32h^2)^n - 1\right]\frac{\frac{1249}{3}}{8+32h}h^2 \leq \frac{1}{4}\times 10^{-8} \\ \delta_n \leq \left[(1+8h+32h^2)^n - 1\right]\frac{1+4h}{8h+32h^2}\cdot\frac{1}{2}\cdot 10^{-m} \leq \frac{1}{4}\times 10^{-8} \end{cases}$$

$$\Rightarrow \begin{cases} h \le h_0 \\ m \ge m_0 \end{cases}$$

代入 $\begin{cases} \bar{y}_{n+1} = y_n + h \cdot f(x_n, y_n) \\ y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})] \end{cases}$

初值 $y(1) = 4$

迭代 $k = -\frac{1}{h_0}$ 次 每次精确到小数点后 $m_0 - 1$ 位

求得 $y_k = y(0)$