homework 3

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1. 设 $P_1^*(x) = a_0 + a_1 x$

$$\begin{cases} f'(x_1) = a_1 \\ f(1) - P_1^*(1) = f(4) - P_1^*(4) \\ f(1) - P_1^*(1) = -\left[f(x_1) - P_1^*(x_1)\right] \end{cases}$$

$$\ln 1 - a_0 - a_1 = \ln 4 - a_0 - 4a_1 \Rightarrow a_1 = \frac{\ln 4}{3}$$

$$f'(x_1) = \frac{1}{x_1} = a_1 \Rightarrow x_1 = \frac{1}{a_1} = \frac{3}{\ln 4}$$

$$\ln 1 - a_0 - \frac{\ln 4}{3} = -\left[\ln\frac{3}{\ln 4} - a_0 - 1\right]$$

$$\Rightarrow a_0 = \frac{-\frac{\ln 4}{3} - \ln\frac{\ln 4}{3} - 1}{2}$$

$$P_1^*(x) = \frac{-\frac{\ln 4}{3} - \ln\frac{\ln 4}{3} - 1}{2} + \frac{\ln 4}{3}x$$

$$\max_{1\le x\le 4}\left|f(x) - P_1^*(x)\right| = \left|\ln 1 - \frac{-\frac{\ln 4}{3} - \ln\frac{\ln 4}{3} - 1}{2} - \frac{\ln 4}{3}\right| = \frac{\frac{\ln 4}{3} - \ln\frac{\ln 4}{3} - 1}{2}$$

2. 令 $x = \frac{1}{2} + \frac{1}{2}t \quad t \in [-1, 1]$

则 $F(t) = f\left(\frac{1+t}{2}\right) = 3\left(\frac{t+1}{2}\right)^4 + 8\left(\frac{t+1}{2}\right)^3 + 3$.

$$= \frac{3}{16}t^4 + \frac{7}{4}t^3 + \frac{33}{8}t^2 + \frac{15}{4}t + \frac{67}{16}$$

$$F(t) - P_3^*(t) = k\,T_4(t) = k(8t^4 - 8t^2 + 1)$$

对比系数得，$k = \frac{3}{128}$

$$P_3^*(t) = \frac{7}{4}t^3 + \frac{69}{16}t^2 + \frac{15}{4}t + \frac{533}{128}$$

将 $t = 2x-1$ 代入得

$$P_3^*(x) = 14x^3 - \frac{15}{4}x^2 + \frac{3}{4}x + \frac{381}{128}$$

3 $\Phi = \text{span}\{P_0, P_1, P_2, P_3\}$

设 $S^*(x) = \sum_{j=0}^{3} a_j P_j(x)$

$$a_0 = \frac{1}{2}(f, P_0) = \frac{1}{2}\int_{-1}^{1} \cos\frac{\pi}{2}x \, dx = \frac{2}{\pi}$$

$$a_1 = \frac{3}{2}(f, P_1) = \frac{3}{2}\int_{-1}^{1} \cos\frac{\pi}{2}x \cdot x \, dx = 0$$

$$a_2 = \frac{5}{2}(f, P_2) = \frac{5}{2}\int_{-1}^{1} \cos\frac{\pi}{2}x \cdot \frac{3x^2-1}{2} \, dx = \frac{10}{\pi} - \frac{120}{\pi^3}$$

$$a_3 = \frac{7}{2}(f, P_3) = \frac{7}{2}\int_{-1}^{1} \cos\frac{\pi}{2}x \cdot \frac{5x^3-3x}{2} \, dx = 0$$

$$S^*(x) = \frac{2}{\pi} + \left(\frac{10}{\pi} - \frac{120}{\pi^3}\right)\left(\frac{3x^2-1}{2}\right)$$

$$= \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 - \frac{3}{\pi} + \frac{120}{\pi^3}$$

4. 设 $S^*(x) = -x + a_0$

$$I = \int_1^3 [S^*(x) - f(x)]^2 dx$$

$$= \int_1^3 \left[\frac{1}{x} + x - a_0\right]^2 dx$$

$$= \int_1^3 \left(\frac{1}{x^2} - \frac{2a_0}{x} + 2 + a_0^2 - 2a_0x + x^2\right) dx$$

$$= -\frac{1}{x} - 2a_0\ln x + (2 + a_0^2)x - a_0x^2 + \frac{1}{3}x^3 \Big|_1^3$$

$$= \frac{40}{3} - 2a_0\ln 3 - 8a_0 + 2a_0^2$$

$$\frac{dI}{da_0} = -2\ln 3 - 8 + 4a_0$$

当且仅当 $a_0 = \frac{1}{2}\ln 3 + 2$ 时平方逼近误差最小.

$$S^*(x) = -x + \frac{1}{2}\ln 3 + 2$$

5. (1) 设 $f(x)$ 为偶函数，$P_n^*(x)$ 为其最佳一致逼近

根据最佳一致逼近的定义

$$\delta(f, P_n^*) = \max_{-a\le x\le a} |f(x) - P_n^*(x)| = E_n.$$

$$\begin{cases} -E_n \le f(x) - P_n^*(x) \le E_n \\ -E_n \le f(-x) - P_n^*(-x) \le E_n \end{cases}$$

$$\Rightarrow \quad -E_n \le \frac{1}{2}[f(x) + f(-x)] - \frac{1}{2}[P_n^*(x) + P_n^*(-x)] \le E_n$$

$$-E_n \le f(x) - \frac{1}{2}[P_n^*(x) + P_n^*(-x)] \le E_n.$$

根据最佳一致逼近的唯一性

$$\frac{1}{2}[P_n^*(x) + P_n^*(-x)] = P_n^*(x)$$

$$\Rightarrow \quad P_n^*(-x) = P_n^*(x)$$

$P_n^*(x)$ 为偶函数

(2) $f(x) = e^{|2x|}$ 为偶函数 则其最佳一致逼近为偶函数.

① $P_1^*(x) = a_0 + a_1 x.$

$P_1^*(x) = P_1^*(-x) \Rightarrow a_1 = 0.$

$P_1^*(x) = a_0.$ 与 $e^{|2x|}$ 有3个正负相间的偏差点

$$\left(f(x) - P_1^*(x)\right)' = \begin{cases} 2e^{2x} & x>0 \\ -2e^{-2x} & x<0 \end{cases}$$

三个偏差点为 $-1, 0, 1$

$$f(1) - P_1^*(1) = -[f(0) - P_1^*(0)]$$

$$\Rightarrow \quad a_0 = \frac{e^2+1}{2}$$

$$P_1^*(x) = \frac{e^2+1}{2}$$

②
$$P_2^*(x) = a_0 + a_1 x + a_2 x^2$$
$$P_2^*(x) = P_2^*(-x) \Rightarrow a_1 = 0.$$
$$P_2^*(x) = a_0 + a_2 x^2.$$

根据对称性 $P_2^*(x)$ 与 $e^{|2x|}$ 至少有5个正负相间的偏差点且0为偏差点

$$\left(f(x) - P_2^*(x)\right)' = \begin{cases} 2e^{2x} - 2a_2 x & x>0 \\ -2e^{-2x} - 2a_2 x & x<0 \end{cases}$$

令 $g(x) = 2e^{2x} - 2a_2 x \quad x \in [0,1].$

$g(x)$ 存在零点，设 $g(x_1) = 0$

则 $-1, -x_1, 0, x_1, 1$ 为偏差点

$$\begin{cases} 2e^{2x_1} - 2a_2 x_1 = 0 & (x_1 > 0). \\ 1 - a_0 = e^2 - a_0 - a_2 \\ 1 - a_0 = -e^{2x_1} + a_0 + a_2 x_1^2 \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = e^2 - 1 \\ a_0 = \frac{1}{2}\left[e^{2x_1} - (e^2-1)x_1^2 + 1\right] \end{cases}$$

$$P_2^*(x) = \frac{1}{2}\left[e^{2x_1} - (e^2-1)x_1^2 + 1\right] + (e^2-1)x^2.$$

其中，x_1 满足超越方程：$2e^{2x_1} - 2(e^2-1)x_1 = 0$

③
$$P_3^*(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_3^*(x) = P_3^*(-x)$$

$$\Rightarrow a_1 = 0 \quad a_3 = 0$$

$$P_3^*(x) = P_2^*(x) = \frac{1}{2}\left[e^{2x_1} - (e^2-1)x_1^2 + 1\right] + (e^2-1)x^2$$

其中，x_1 满足超越方程：$2e^{2x_1} - 2(e^2-1)x_1 = 0$

6. $f(x)=e^{-x} \quad x\in[0,1]$

令 $x=\frac{1}{2}+\frac{1}{2}t \quad t\in[-1,1]$

$F(t)=f(\frac{1+t}{2})=e^{-\frac{1+t}{2}}$

$$|R_3(t)|=|F(t)-L_3(t)|\le \max_{-1\le\xi\le1}\left|\frac{F^{(4)}(\xi)}{4!}\right|\cdot\max_{-1\le t\le1}|w_4(t)|$$

$$=\max_{-1\le\xi\le1}\left|\frac{\frac{1}{2^4}e^{-\frac{1+\xi}{2}}}{4!}\right|\cdot\max_{-1\le t\le1}|w_4(t)|$$

$$=\frac{1}{384}\cdot\max_{-1\le t\le1}|w_4(t)|$$

取插值节点 $t_k=\cos\frac{2k+1}{8}\pi \quad k=0,1,2,3$

则 $w_4(t)=\frac{1}{2^3}T_4(t)$ 为最小偏差多项式

$$\max_{-1\le t\le1}|w_4(t)|=\max_{-1\le t\le1}\left|\frac{1}{8}(8t^4-8t^2+1)\right|=\frac{1}{8}$$

$$|R_3(t)|\le\frac{1}{384}\cdot\frac{1}{8}=\frac{1}{3072}\le\frac{1}{2}\times10^{-3}$$

$$x_k=\frac{1+t}{2}=\frac{1}{2}+\frac{1}{2}\cos\frac{2k+1}{8}\pi \quad k=0,1,2,3$$

$$L_3(x)=\sum_{k=0}^{3}\frac{w_4(x)}{(x-x_k)w_4'(x_k)}e^{-x_k}$$