作业五

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23. (a)

X \ Y	0	1	2	3	X
1	$\frac{2}{27}$	0	0	$\frac{1}{27}$	$\frac{1}{9}$
2	$\frac{6}{27}$	$\frac{6}{27}$	$\frac{6}{27}$	0	$\frac{2}{3}$
3	0	$\frac{6}{27}$	0	0	$\frac{2}{9}$
Y	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$	

(b) $P(X=1, Y=1) = 0 \neq P_X(X=1)P_Y(Y=1) = \frac{1}{9} \times \frac{4}{9} = \frac{4}{81}$

X, Y 不独立

(c) $P(Y=0 \mid X=1) = \frac{2}{3}$ $\quad P(Y=1 \mid X=1) = 0$

$P(Y=2 \mid X=1) = 0$ $\quad P(Y=3 \mid X=1) = \frac{1}{3}$

$P(X=1 \mid Y=0) = \frac{1}{4}$ $\quad P(X=2 \mid Y=0) = \frac{3}{4}$

$P(X=3 \mid Y=0) = 0$

(d) $P(X=3 \mid Y=2) = 0$ $\quad P(Y=2 \mid X=3) = 0$

24. $\sum_k P[X=x_i \mid (Y=y_k) \cap (Z=z_j)]\, P(Y=y_k \mid Z=z_j)$

$$= \sum_k \frac{P(X=x_i, Y=y_k, Z=z_j)}{P(Y=y_k, Z=z_j)} \frac{P(Y=y_k, Z=z_j)}{P(Z=z_j)}$$

$$= \frac{\sum_k P(X=x_i, Y=y_k, Z=z_j)}{P(Z=z_j)} = \frac{P(X=x_i, Z=z_j)}{P(Z=z_j)}$$

$$= P(X=x_i \mid Z=z_j)$$

26. 由于 $\{\xi_i\}$ iid. 且方差有限

故 $\forall \underset{i\neq j}{i,j} \leq n+m$ 有 $Cov(\xi_i, \xi_j)=0$

则 $Cov(X,Y) = Cov(\sum_{k=1}^{n}\xi_k, \sum_{k=1}^{n}\xi_{m+k})$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(\xi_i, \xi_{m+j})$$

$$= \sum_{k=m+1}^{n} D(\xi_k)$$

$$D(X) = D(\sum_{k=1}^{n}\xi_k) = \sum_{k=1}^{n} D(\xi_k)$$

$$D(Y) = D(\sum_{k=1}^{n}\xi_{m+k}) = \sum_{k=1}^{n} D(\xi_{m+k})$$

记 $D(\xi_i) = M \quad \forall i \leq n+m$

则 $r_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{DX\,DY}} = \dfrac{(n-m)M}{nM} = \dfrac{n-m}{n}$

27. 将抛 n 次骰子视作 n 次独立同分布事件，每个记为 B.

$EX_B = \frac{1}{6} \qquad EX = \frac{n}{6} \qquad$ 同理 $EY = \frac{n}{6} \quad DY = \frac{5n}{36}$

$DX_B = \frac{5}{36} \qquad DX = \frac{5n}{36}$

$$cov(X_B, Y_B) = E(X_B Y_B) - EX_B \cdot EY_B = -\frac{1}{36}$$

故 $Cov(X,Y) = Cov(\sum_{i=1}^{n} X_{Bi}, \sum_{j=1}^{n} Y_{Bj})$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_{Bi}, Y_{Bj})$$

$$= \sum_{i=1}^{n} Cov(X_{Bi}, Y_{Bi})$$

$$= -\frac{n}{36}$$

$$r_{X,Y} = \frac{-\frac{n}{36}}{\frac{5n}{36}} = -\frac{1}{5}$$

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Y	1	2	3
$E(X\|Y)$	$\frac{13}{7}$	$\frac{28}{15}$	$\frac{11}{5}$
P	$\frac{7}{27}$	$\frac{15}{27}$	$\frac{5}{27}$

$$E(X|Y=y_i) = \sum_{j=1}^{3} x_j P(X=x_j|Y=y_i)$$

$$EX = E(E(X|Y)) = \frac{13}{27} + \frac{28}{27} + \frac{11}{27} = \frac{52}{27}$$

补充题

1. "$\Rightarrow$" X, Y 独立

则 $E(XY) = EX \cdot EY$

$$r_{X,Y} = \frac{Cov(X,Y)}{\sqrt{DX \cdot DY}} = \frac{E(XY) - EX \cdot EY}{\sqrt{DX \cdot DY}} = 0$$

X, Y 不相关

"$\Leftarrow$"

$$EX = x_1 p + x_2(1-p)$$

$$EY = y_1 q + y_2(1-q)$$

$$E(XY) = x_1 y_1 a + x_1 y_2 (p-a) + x_2 y_1 (q-a) + x_2 y_2 (1-p-q+a)$$

$r_{X,Y} = 0 \Rightarrow E(XY) - EX \cdot EY = 0$

$$x_1 y_1 a + x_1 y_2 (p-a) + x_2 y_1 (q-a) + x_2 y_2 (1-p-q+a)$$
$$= x_1 y_1 pq + x_1 y_2 p(1-q) + x_2 y_1 (1-p) q + x_2 y_2 (1-p)(1-q)$$

则 $a = pq$ $\quad p(1-q) = p-a$ $\quad (1-p)q = q-a$ $\quad (1-p)(1-q) = 1-p-q+a$

即 $P(XY) = P(X)P(Y)$ $\quad X, Y$ 独立

2. $Ef(x) = \int_0^{\infty} f(x)p(x)\,dx = \int_0^{c} f(x)p(x)\,dx + \int_c^{\infty} f(x)p(x)\,dx$

$$\geqslant \int_c^{\infty} f(x)p(x)\,dx$$

$$\geqslant \int_c^{\infty} f(c)p(x)\,dx$$

$$= f(c)\int_c^{\infty} p(x)\,dx$$

$$= f(c)P(x>c)$$

$$p(x>c) \leqslant \frac{Ef(x)}{f(c)}$$

3. $Cov(X-(\hat{a}Y+\hat{b}),Y) = E[(X-(\hat{a}Y+\hat{b})Y] - E[X-(\hat{a}Y+b)]EY$

其中 $\hat{a} = \dfrac{Cov(X,Y)}{DY}$, $EX = \hat{a}EY+\hat{b}$

$Cov(X-(\hat{a}Y+\hat{b}),Y) = E[(X-(\hat{a}Y+\hat{b})-EX+E(\hat{a}Y+\hat{b}))(Y-EY)]$

$$= E[(X-EX)-\hat{a}(Y-EY)(Y-EY)]$$

$$= E[(X-EX)(Y-EY)-\hat{a}(Y-EY)^2]$$

$$= E[(X-EX)(Y-EY)-Cov(X,Y)]$$

$$= E[(X-EX)(Y-EY)]-Cov(X,Y)$$

$$= 0$$