作业九

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20.(a) X, Y 相互独立

$$f(x,y)=f_X(x)f_Y(y)=\begin{cases}\lambda\mu e^{-(\lambda x+\mu y)}, & x>0,\ y>0\\ 0, & \text{else}\end{cases}$$

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=f_X(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases}$$

(b) $$P(X\le Y)=\int_0^{+\infty}\int_x^{+\infty}\lambda\mu e^{-(\lambda x+\mu y)}\,dy\,dx=\int_0^{+\infty}\lambda e^{-(\lambda+\mu)x}\,dx$$

$$=\frac{\lambda}{\lambda+\mu}$$

$$P(X>Y)=1-P(X\le Y)=1-\frac{\lambda}{\lambda+\mu}=\frac{\mu}{\lambda+\mu}$$

$$F_Z(z)=\begin{cases}0 & z<0\\ \frac{\mu}{\lambda+\mu} & 0\le z<1\\ 1 & z\ge 1\end{cases}$$

28. $$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$$

故 $f(x,y)=15x^2y \quad (0<x<y,\ 0<y<1)$

$$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)\,dy=\int_x^1 15x^2y\,dy$$

$$=\frac{15}{2}x^2(1-x^2)\qquad(0<x<1)$$

故 $P(X>\frac12)=1-P(X\le\frac12)=1-\int_0^x f_X(t)\,dt\Big|_{x=\frac12}$

$=1-(\frac52x^3-\frac32x^5)\Big|_{x=\frac12}=\frac{47}{64}$

$$E(X)=\int_{-\infty}^{+\infty}x f_X(x)\,dx=\int_0^1\frac{15}{2}x^3(1-x^2)\,dx=\frac58$$

$$E(XY)=\iint_{D_f}xy\,f(x,y)\,dx\,dy=\int_0^1dx\int_x^1 15x^3y^2\,dy=\int_0^1(5x^3-5x^6)\,dx=\frac{15}{28}$$

30. $f(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in [-\pi, \pi] \\ 0 & \text{其他} \end{cases}$

$E(X) = \int_{-\pi}^{\pi} \sin\theta \cdot \frac{1}{2\pi} d\theta = 0$

$E(Y) = 0$

$EX^2 = \int_{-\pi}^{\pi} \sin^2\theta \cdot \frac{1}{2\pi} d\theta = \frac{1}{2}$

$EY^2 = \frac{1}{2}$

$E(XY) = \int_{-\pi}^{\pi} \sin\theta \cos\theta \cdot \frac{1}{2\pi} d\theta$

$= 0$

故 $Cov(X, Y) = 0 \Rightarrow r_{X,Y} = 0$

由于 $X^2 + Y^2 = 1$，故 X, Y 不相互独立

34. $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)dx}$

$= \frac{\lambda^2 e^{-\lambda x}}{\int_y^{+\infty} \lambda^2 e^{-\lambda x} dx} = \frac{\lambda^2 e^{-\lambda x}}{\lambda e^{-\lambda y}} \quad (\lambda > 0)$

$E(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx = \int_y^{+\infty} \lambda x e^{-\lambda(x-y)} dx$

$= \lambda e^{-\lambda(x-y)} = \lambda e^{\lambda y} \int_y^{+\infty} x e^{-\lambda x} dx$

$= \lambda e^{\lambda y} \left(\frac{y e^{-\lambda y}}{\lambda} + \frac{e^{-\lambda y}}{\lambda^2} \right)$

$= y + \frac{1}{\lambda} \quad (y > 0)$

$$D(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2$$

$$= \int_{-\infty}^{+\infty} x^2 f_{X|Y}(x|y)\,dx - (y+\frac{1}{\lambda})^2$$

$$= \int_{y}^{+\infty} \lambda x^2 e^{-\lambda(x-y)}\,dx - (y+\frac{1}{\lambda})^2$$

$$= \lambda e^{\lambda y} \int_{y}^{+\infty} x^2 e^{-\lambda x}\,dx - (y+\frac{1}{\lambda})^2$$

$$= \lambda e^{\lambda y} \left(\frac{y^2 e^{-\lambda y}}{\lambda} + \frac{2y e^{-\lambda y}}{\lambda^2} + \frac{2e^{-\lambda y}}{\lambda^3}\right) - (y+\frac{1}{\lambda})^2$$

$$= y^2 + \frac{2y}{\lambda} + \frac{2}{\lambda^2} - y^2 - \frac{2y}{\lambda} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

35.

$$f_Y(x) = \frac{1}{a}$$

$$f_X(x) = \frac{1}{a-y}$$

$$E(X|y) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{y}^{a} \frac{1}{a-y}\,dx$$

$$= \frac{a+y}{2}$$

$$E(X|Y) \sim U[\frac{a}{2}, a]$$

47. (a)

$$f(x+y) = f(x+y=z) = \int_0^z f(X=x, Y=z-x)\,dx$$

$$= \int_0^z \lambda^2 e^{-\lambda z}\,dx = \lambda^2 z e^{-\lambda z}$$

$$f(x+y=z) = \begin{cases} \lambda^2 z e^{-\lambda z} & z>0 \\ 0 & z\leq 0 \end{cases}$$

(b) $E(X \mid X+Y=z) = \int_0^\infty x \, f_{X|X+Y=z}(x)\, dx$

$$f_{X|X+Y=z} = \frac{d}{dx} F_{X|X+Y=z}(x)$$

$$F_{X|X+Y=z}(x) = \frac{P(X \le x,\ X+Y=z)}{P(X+Y=z)}$$

$$f_{X|X+Y=z}(x) = \frac{f_X(x,z)}{f_Z(z)}$$

$$\begin{aligned} f_{XZ}(x,z) &= f_{XY}(x,\ z-x) \\ &= \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} \\ &= \lambda^2 e^{-\lambda z} \qquad (x<z) \end{aligned}$$

$$f_{X|X+Y=z}(x) = \frac{1}{z} \quad (x<z)$$

$$E(X \mid X+Y=z) = \int_0^z \frac{x}{z}\,dx + \int_z^\infty 0\,dx = \frac{z}{2}$$

49. $f_X(x) = e^{-x} \quad (x>0)$

$$f_Y(x) = e^{-y} \quad (y>0)$$

$$f_{X,Y}(x-y) = f_X(x) \cdot f_Y(y) = e^{-(x+y)}$$

令 $\begin{cases} U = X + Y \\ V = \dfrac{X}{Y} \end{cases} \Rightarrow \begin{cases} X = \dfrac{UV}{V+1} \\ Y = \dfrac{U}{V+1} \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{\bar{u}^2}{(\bar{v}+1)^2}$$

$$f_{\bar{u},\bar{v}}(u,v) = \begin{cases} \frac{ue^{-u}}{(v+1)^2} & uv>0 \\ 0 \end{cases}$$

$$f_{\bar{v}}(u) = \begin{cases} \int_0^{+\infty} \frac{ue^{-u}}{(v+1)^2} dv = ue^{-u} & u>0 \\ 0 \end{cases}$$

$$f_{\bar{v}}(v) = \begin{cases} \int_0^{+\infty} \frac{ue^{-u}}{(v+1)^2} du = \frac{1}{(v+1)^2} & v>0 \\ 0 \end{cases}$$

$$f_{\bar{u},\bar{v}}(u,v) = f_{\bar{v}}(u)\cdot f_{\bar{v}}(v)$$

$\bar{u},\bar{v}$ 独立

50 $x \le y$ $\begin{cases} \bar{u}=\bar{X} \\ \bar{u}=\bar{Y} \end{cases} \Rightarrow \frac{\partial(\bar{X},\bar{Y})}{\partial(\bar{u},\bar{v})} = 1$

$x>y$ $\begin{cases} \bar{u}=\bar{Y} \\ \bar{v}=\bar{X} \end{cases} \Rightarrow \frac{\partial(\bar{X},\bar{Y})}{\partial(\bar{u},\bar{v})} = -1$

则 $f_{\bar{v},\bar{v}}(u,v) = f_{\bar{X},\bar{Y}}(u,v)\ 1 + f_{\bar{X},\bar{Y}}(v,u)\cdot|-1|$

$$= \frac{12}{7}u(u+v) + \frac{12}{7}v(u+v)$$

$$= \frac{12}{7}(u+v)^2 \quad 0<u\le v<1$$

$$f_{u,v}(u,v) = \begin{cases} \frac{12}{7}(u+v)^2 & 0<u\le v<1 \\ 0 \end{cases}$$

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2. $$\varphi_X(\theta) = \sum_k e^{i\theta x_k} \cdot p_k$$

$$= 0.8 + 0.1e^{-\frac{i\theta}{2}} + 0.1e^{\frac{i\theta}{2}}$$

$$= 0.8 + 0.2\cos\frac{\theta}{2}$$

7. (a) $$f_X(x) = \lambda e^{-\lambda x}$$

$$\varphi_X(\theta) = Ee^{i\theta X} = \int_0^a \lambda e^{x(i\theta x-\lambda)}\,d\theta = \frac{1}{1-i\theta}$$

(b) $$\varphi_{X-Y}(\theta) = Ee^{i\theta(XY)} = Ee^{i\theta X}\,Ee^{i\theta Y}$$

$$= \varphi_X(\theta)\cdot\varphi_Y(\theta)$$

$$= \frac{\lambda}{\lambda-i\theta}\cdot\frac{\lambda}{\lambda+i\theta}$$

$$= \frac{1}{1+\theta^2}$$

(c) $$f_{X-Y}(z) = \int_0^{+\infty} \lambda e^{-\lambda(y+z)}\cdot\lambda e^{-\lambda y}\,dy$$

$$= \frac{1}{z}e^{-z}$$

$$\Rightarrow \varphi_{X-Y}(\theta) = \int \frac{1}{z}e^{(i\theta-1)z}\,dz = \frac{1}{1+\theta^2}$$

$$\varphi_{X-Y}(\theta) = \frac{1}{1+\theta^2}$$

$$\Rightarrow f_{X-Y}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-i\theta z}\,d\theta = \frac{1}{z}e^{-z}$$

11. $\varphi_Z(\theta) = E\, e^{i\theta Z}$

$$= (1-p)\, E e^{i\theta(a\bar{X} + (1-a)\bar{Y})} + p\, E e^{i\theta(b\bar{X} + (1-b)\bar{Y})}$$

$$= (1-p)\, \varphi_{\bar{X}}(a\theta)\, \varphi_{\bar{Y}}((1-a)\theta) + p\, \varphi_{\bar{X}}(b\theta)\, \varphi_{\bar{Y}}((1-b)\theta)$$