作业八

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P154 10. $X \sim E(\frac{1}{5})$

$P(X \geq 10) = 1 - F(X<10)$

$= 1-(1-e^{-\frac{1}{3}X})\big|_{X=10}$

$= e^{-2}$

$Y \sim B(5, e^{-2})$

$EY = 5e^{-2}$

$P(Y \geq 1) = 1 - P(Y=0)$

$= 1-(1-e^{-2})^5$

$= 0.517$

11. $P(X>4) = \frac{1}{2} - P(2<X<4)$

$= 0.2$

$P(X<0) = P(X>4)$

$= 0.2$

12. $E(X) = \int_{-\infty}^{+\infty} x f(x) dx$

$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x e^{-x^2-2x-1} dx$

$\overset{u=x-1}{=\!=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (u+1) e^{-u^2} du$

$= \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-u^2} du$

$\overset{t=u^2}{=\!=} \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{1}{2} t^{-\frac{1}{2}} e^{-t} dt$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= 1$$

$$D(X) = EX^2 - (EX)^2$$

$$EX^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2+2x+1}\,dx$$

$$\underset{\overline{\overline{\phantom{u=x-1}}}}{u=x-1}\ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (u^2+2u+1)\, e^{-u^2}\,du$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} (u^2+1)\, e^{-u^2}\,du$$

$$\underset{\overline{\overline{\phantom{t=u^2}}}}{t=u^2}\ \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{1}{2}(t+1)\, t^{-\frac{1}{2}} e^{-t}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \left( \int_{0}^{+\infty} t^{\frac{1}{2}} e^{-t}\,dt + \int_{0}^{+\infty} t^{-\frac{1}{2}} e^{-t}\,dt \right)$$

$$= \frac{1}{\sqrt{\pi}} \left( \frac{1}{2}\sqrt{\pi} + \sqrt{\pi} \right)$$

$$= \frac{3}{2}$$

$$D(X) = \frac{3}{2} - 1 = \frac{1}{2}$$

14. 记 $Y = \frac{\bar{X}-\mu}{\sigma} \sim N(0,1)$

$$E|X-\mu| = \sigma E|Y|$$

$$= \sigma \left( \int_{-\infty}^{0} -y f(y)\,dy + \int_{0}^{+\infty} y f(y)\,dy \right)$$

$$= 2\sigma \int_{0}^{+\infty} \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$$

$$= \sigma \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy^2$$

$$= -2\sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \Big|_0^{+\infty}$$

$$= \sigma\sqrt{\frac{2}{\pi}}$$

16.


$$a = 2R\sin\frac{\theta}{2} \qquad \theta \sim U(0,\pi)$$

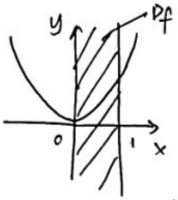
$$Ea = E\left(2R\sin\frac{\theta}{2}\right)$$

$$= 2R\int_0^{\pi} \sin\frac{\theta}{2} \cdot \frac{1}{\pi}\, d\theta$$

$$= \frac{2R}{\pi}\left(-2\cos\frac{\theta}{2}\right)\Big|_0^{\pi}$$

$$= \frac{4R}{\pi}$$

22.




$$f(x,y) = \begin{cases} 2xe^{-y} & (x,y)\in D_f \\ 0 & \text{其他} \end{cases}$$

$$\Delta = 4(X^2 - Y)$$

$$\Delta \geq 0 \Rightarrow Y \leq X^2.$$

$$P(Y \leq X^2) = \iint\limits_{y\leq x^2} f(x,y)\, dx dy$$

$$= \iint\limits_{\{y\leq x^2\}\cap D_f} 2xe^{-y}\, dx dy$$

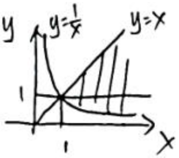
$$= \int_0^1 2x\, dx \int_0^{x^2} e^{-y}\, dy$$

$= \int_0^1 2(1-e^{-x^2})x\,dx$

$= 2\int_0^1 x\,dx - \int_0^1 e^{-x^2}dx^2$.

$= e^{-1}$

23.




$x \geq 1$时 $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$

$= \int_{\frac{1}{x}}^{x} \frac{1}{2x^2y}dy = \frac{1}{2x^2}\ln y\Big|_{\frac{1}{x}}^{x}$

$= \frac{\ln x}{x^2}$.

$$f_X(x) = \begin{cases} \frac{\ln x}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

$f_Y(y) = \int_{\frac{1}{y}}^{+\infty} \frac{1}{2x^2y}dx + \int_{y}^{+\infty} \frac{1}{2x^2y}dx$

$$= \begin{cases} \frac{1}{2} & 0 < y < 1 \\ \frac{1}{2y^2} & y \geq 1 \\ 0 & 其他 \end{cases}$$

$f(1,1) = \frac{1}{2}$

$f_X(1) = 0$

$f_Y(1) = \frac{1}{2}$

$f_X(1) \cdot f_Y(1) = 0 \neq f(1,1)$

$X, Y$ 不相互独立.

25 (a) $E X = \int_{-\infty}^{0} \frac{x}{2} e^{x} dx + \int_{0}^{+\infty} \frac{x}{2} e^{-x} dx$

$= \int_{-\infty}^{0} \frac{x}{2} e^{x} dx - \int_{-\infty}^{0} \frac{x}{2} e^{x} dx$

$= 0.$

$E(X^2) = \int_{-\infty}^{0} \frac{x^2}{2} e^{x} dx + \int_{0}^{+\infty} \frac{x^2}{2} e^{-x} dx$

$= \int_{-\infty}^{0} x^2 e^{x} dx$

$= 2.$

$D X = E(X^2) - (EX)^2$

$= 2.$

(b) $Cov(X, |X|) = E(X \cdot |X|) - EX \cdot E|X|$

$= \int_{-\infty}^{0} (-x^2) \frac{e^{x}}{2} dx + \int_{0}^{+\infty} x^2 \frac{e^{-x}}{2} dx$

$= \int_{-\infty}^{0} (-x^2) \frac{e^{x}}{2} dx = \int_{-\infty}^{0} x^2 \frac{e^{x}}{2} dx$

$= 0.$

$r_{X,|X|} = 0$ 不相关.

(c) $f_X(x) = \frac{1}{2} e^{-|x|}$

$f_{|X|}(x) = \begin{cases} \frac{1}{2} e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$

记 $Y = |X|$

则 $f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} e^{-x} & x = y \geq 0 \\ 0 & 其他 \end{cases}$

在$(0,+\infty)\times(0,+\infty)$区域内

$$f_{X,Y}(x,y)\neq f_X(x)\cdot f_{|X|}(x).$$

$X$与$|X|$不相互独立.

29. 设$\xi$，$\eta$分别表示在区间$[0,a]$上任取两点的坐标，则$\xi$及$\eta$都在$[0,a]$上服从均匀分布．其密度分别为

$$P_\xi(x)=\begin{cases}\frac{1}{a}, & 0\le x\le a\\ 0, & else\end{cases}$$

$$P_\eta(x)=\begin{cases}\frac{1}{a}, & 0\le y\le a\\ 0, & else\end{cases}$$

因为$\xi$与$\eta$是独立的，所以$(\xi,\eta)$的联合密度为

$$p(x,y)=\begin{cases}\frac{1}{a^2}, & 0\le x\le a,\ 0\le y\le a\\ 0, & else\end{cases}$$

因此

$$E|\xi-\eta|=\frac{1}{a^2}\int_0^a\int_0^a|x-y|\,dx\,dy$$

$$=\frac{1}{a^2}\left[\int_0^a dx\int_0^x(x-y)\,dy+\int_0^a dx\int_x^a(y-x)\,dy\right]$$

$$=\frac{1}{a^2}\left[\int_0^a\left(x^2-ax+\frac{1}{2}a^2\right)dx\right]$$

$$=\frac{a}{3}$$

$$E|\xi-\eta|^2=\frac{1}{a^2}\int_0^a\int_0^a(x-y)^2\,dx\,dy$$

$$=\frac{1}{3a^2}\int_0^a\left[x^3-(x-a)^3\right]dx$$

$$= \frac{1}{3a^2}\left[\frac{1}{4}x^4 - \frac{1}{4}(x-a)^4\right]\Big|_0^a$$

$$= \frac{a^2}{6}$$

$$D(|\xi-\eta|) = E|\xi-\eta|^2 - (E|\xi-\eta|)^2$$

$$= \frac{a^2}{6} - \frac{a^2}{9}$$

$$= \frac{a^2}{18}$$

32. $E(X) = 0$

$E(X^2) = 1$

$D(X) = 1$

$E(Y) = E(X^n)$

$$= \int_{-\infty}^{+\infty} x^n \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx$$

当n为奇数时，$E(Y) = 0$

当n为偶数时，令 $u = \frac{x^2}{2}$

$$E(Y) = \frac{2}{\sqrt{2\pi}} \cdot 2^{\frac{n-1}{2}} \int_0^{+\infty} u^{\frac{n-1}{2}} e^{-u} du$$

$$= \frac{1}{\sqrt{\pi}} \cdot 2^{\frac{n}{2}} \int_0^{+\infty} u^{\frac{n-1}{2}} e^{-u} du$$

$$= \frac{1}{\sqrt{\pi}} \cdot 2^{\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \cdot 2^{\frac{n}{2}} \cdot \frac{(n-1)!!}{2^{\frac{n}{2}}} \sqrt{\pi} = (n-1)!!$$

$$E(\bar{Y})=\begin{cases}0 & n\text{为奇}\\ (n-1)!! & n\text{为偶}\end{cases}$$

$$E(\bar{Y}^2)=E(X^{2n})=\int_{-\infty}^{+\infty}x^{2n}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$=(2n-1)!!$$

$$D(\bar{Y})=E(\bar{Y}^2)-(E\bar{Y})^2$$

$$=\begin{cases}(2n-1)!! & n\text{为奇}\\ (2n-1)!!-[(n-1)!!]^2 & n\text{为偶}\end{cases}$$

$$E(X\bar{Y})=E(XX^n)=E(X^{n+1})$$

$$=\begin{cases}n!! & n\text{为奇}\\ 0 & n\text{为偶}\end{cases}$$

$$\operatorname{cov}(X,\bar{Y})=E(X\bar{Y})-E(X)E(\bar{Y})$$

$$=\begin{cases}n!! & n\text{为奇}\\ 0 & n\text{为偶}\end{cases}$$

$$r_{X,Y}=\frac{\operatorname{cov}(X,\bar{Y})}{\sqrt{D(X)}\sqrt{D(\bar{Y})}}=\begin{cases}\dfrac{n!!}{\sqrt{(2n-1)!!}} & n\text{为奇}\\ 0 & n\text{为偶}\end{cases}$$

33. 令 $Y=\frac{X-\mu}{\sigma}$ 记 $m=\frac{a-\mu}{\sigma}$ $n=\frac{b-\mu}{\sigma}$

$$E(Y \mid m<Y<n)=\frac{E(Y\,1_{m<Y<n})}{P(m<Y<n)}$$

$$=\frac{\int_m^n y f(y)\,dy}{\Phi(n)-\Phi(m)}$$

$$=\frac{\int_m^n \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy}{\Phi(n)-\Phi(m)}$$

$$=\frac{-\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\Big|_{y^2=m^2}^{n^2}}{\Phi(n)-\Phi(m)}$$

$$=\frac{1}{\sqrt{2\pi}}\ \frac{e^{-\frac{m^2}{2}}-e^{-\frac{n^2}{2}}}{\Phi(n)-\Phi(m)}$$

$$E(X \mid a<X<b)=\sigma E(Y \mid m<Y<n)+\mu$$

$$=\mu+\frac{\sigma}{\sqrt{2\pi}}\cdot\frac{e^{-\frac{(a-\mu)^2}{2\sigma^2}}-e^{-\frac{(b-\mu)^2}{2\sigma^2}}}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$