# homework2

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1. $\nabla^2 x_n = \nabla(\nabla x_n) = \nabla(x_n - x_{n-1}) = x_n - 2x_{n-1} + x_{n-2}$

$= 5^n - 2\cdot 5^{n-1} + 5^{n-2} = 16\cdot 5^{n-2}$

$\delta^2 x_n = \delta(\delta x_n) = \delta(x_{n+\frac{1}{2}} - x_{n-\frac{1}{2}}) = x_{n+1} - 2x_n + x_{n-1}$

$= 5^{n+1} - 2\cdot 5^n + 5^{n-1} = 16\cdot 5^{n-1}$

2. $f[x_0, x_1] = \frac{-5-(-1)}{-1-0} = 4$ $f[x_0, x_2] = \frac{-5-1}{-1-2} = 2$ $f[x_0, x_3] = \frac{-5-11}{-1-3} = 4$

$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_0, x_2]}{x_1 - x_2} = -1$

$f[x_0, x_1, x_3] = \frac{f[x_0, x_1] - f[x_0, x_3]}{x_1 - x_3} = 0$

$f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_0, x_1, x_3]}{x_2 - x_3} = 1$

$N_3(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

$= -5 + 4(x+1) - (x+1)x + (x+1)x(x-2)$

$= x^3 - 2x^2 + x - 1$

设新增的点为$(a, b)$

$f[x_0, x_4] = \frac{b+5}{a+1}$ $f[x_0, x_1, x_4] = \frac{4 - \frac{b+5}{a+1}}{-b} = \frac{b-4a+1}{(a+1)b}$

$f[x_0, x_1, x_2, x_4] = \frac{f[x_0, x_1, x_2] - f[x_0, x_1, x_4]}{x_2 - x_4}$

$= \frac{\frac{b-4a+1}{(a+1)b} + 1}{b} = \frac{ab+2b-4a+1}{(a+1)b^2}$

$$f[x_0,x_1,x_2,x_3,x_4] = \frac{f[x_0,x_1,x_2,x_3]-f[x_0,x_1,x_2,x_4]}{x_3-x_4}$$

$$= \frac{1-\frac{ab+2b-4a+1}{(a+1)b^2}}{11-b}$$

$$= \frac{ab^2+b^2-ab-2b+4a-1}{(a+1)b^2(11-b)}$$

$$N_4(x) = x^3-2x^2+x-1+\boxed{\frac{ab^2+b^2-ab-2b+4a-1}{(a+1)b^2(11-b)}}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

系数是不为零的常数就行

3. $$R_1(x) = \frac{f''(\xi)}{2}(x-x_k)(x-x_{k+1})$$

$$|R_1(x)| \le \frac{1}{2}\max_{30^\circ\le\xi\le 60^\circ}|f''(\xi)| \cdot \max_{x_k\le x\le x_{k+1}}|(x-x_k)(x-x_{k+1})|$$

$$= \frac{1}{2}\cdot \sin 60^\circ \cdot \frac{h^2}{4} = 3.30\times 10^{-7}$$

$$L_1(x) = \frac{x-x_{k+1}}{x_k-x_{k+1}}y_k + \frac{x-x_k}{x_{k+1}-x_k}y_{k+1}$$

$$|\Delta y_k| \le \frac{1}{2}\times 10^{-6}$$

$$|\Delta A| \le \max_{x_k\le x\le x_{k+1}}\left|\frac{x-x_{k+1}}{x_k-x_{k+1}}\Delta y_k + \frac{x-x_k}{x_{k+1}-x_k}\Delta y_{k+1}\right|$$

$$= 5\times 10^{-7}$$

总误差 $\le |R_1(x)|+|\Delta A| = 8.30\times 10^{-7}$

4. 设 $P(x) = (ax+b)(x-1)(x-2)(x-3) = ax^4-(6a-b)x^3+(11a-6b)x^2-(6a-11b)x-6b$

$$P'(x) = 4ax^3-3(6a-b)x^2+2(11a-6b)x-(6a-11b)$$

$$P'(1) = 2a+2b = 0 \quad \cdots ①$$

$$P''(x) = 12ax^2-6(6a-b)x+2(11a-6b)$$

$$P''(1) = -2a-6b = 4 \quad \cdots ②$$

联立①②得 $\begin{cases} a = 1 \\ b = -1 \end{cases}$

$$P(x) = (x-1)^2(x-2)(x-3)$$

$R_4(x) = P(x) - f(x)$ $x=1$ 为三重根，$x=2,3$ 为一重根

由于 $f \in C^{(5)}[1,3]$ 则 $\exists\, \xi \in [1,3]$ 使得

$$R_4(x) = \frac{f^{(5)}(\xi)}{5!}(x-1)^3(x-2)(x-3)$$

令 $h(x) = (x-1)^3(x-2)(x-3)$ $x \in [1,3]$

$$h'(x) = 3(x-1)^2(x-2)(x-3) + (x-1)^3(x-3) + (x-1)^3(x-2)$$

$$= (x-1)^2(3x^2-15x+18+x^2-4x+3+x^2-3x+2)$$

$$= (x-1)^2(5x^2-22x+23)$$

令 $h'(x)=0 \Rightarrow x_{1,2} = \frac{11 \pm \sqrt{6}}{5}$

$h(x)$ 在 $[1, \frac{11-\sqrt{6}}{5})$ 单增 $(\frac{11-\sqrt{6}}{5}, \frac{11+\sqrt{6}}{5})$ 单减 $(\frac{11+\sqrt{6}}{5}, 3)$ 单增

$h(1)=0$ $h(\frac{11-\sqrt{6}}{5}) = 0.134$ $h(\frac{11+\sqrt{6}}{5}) = -1.032$ $h(3)=0$.

$$|R_4(x)| \le \left|\frac{f^{(5)}(\xi)}{5!}\right| \cdot \max_{1\le x\le 3}|h(x)|$$

$$\le \frac{M}{120} \cdot 1.032$$

$$= 8.6 \times 10^{-3} M$$

5 设数字图像上的灰度值函数为 $f(x,y)$

① 最近邻插值

$$f(x,y) = \begin{cases} g(u,v) & u<x<u+\frac{1}{2},\ v<y<v+\frac{1}{2} \\ g(u+1,v) & u+\frac{1}{2}<x<u+1,\ v<y<v+\frac{1}{2} \\ g(u,v+1) & u<x<u+\frac{1}{2},\ v+\frac{1}{2}<y<v+1 \\ g(u+1,v+1) & u+\frac{1}{2}<x<u+1,\ v+\frac{1}{2}<y<v+1 \end{cases}$$

$$R(x,y)=f(x,y)-g(x,y)$$

$$|R(x,y)|\le \max\left|\frac{\partial g}{\partial x}\right|\cdot\max|\Delta x|+\max\left|\frac{\partial g}{\partial y}\right|\cdot\max|\Delta y|$$

$$=\frac{1}{2}(M_1+M_2)$$

② 双线性插值

先在 $x$ 方向上做线性插值得:

$$f(x,v)=(u+1-x)g(u,v)+(x-u)g(u+1,v)$$

$$f(x,v+1)=(u+1-x)g(u,v+1)+(x-u)g(u+1,v+1)$$

再在 $y$ 方向上做线性插值得.

$$f(x,y)=(v+1-y)f(x,v)+(y-v)f(x,v+1)$$

$$=(v+1-y)(u+1-x)g(u,v)+(v+1-y)(x-u)g(u+1,v)$$
$$+(y-v)(u+1-x)g(u,v+1)+(y-v)(x-u)g(u+1,v+1)$$

$$R_1(x)=f(x,v)-g(x,v)$$

$$|R_1(x)|\le\frac{1}{2}\max\left|\frac{\partial^2 g}{\partial x^2}\right|\cdot\max\left|(x-u)[x-(u+1)]\right|=\frac{1}{8}M_{11}$$

$$R_2(x)=f(x,v+1)-g(x,v+1)$$

$$|R_2(x)|\le\frac{1}{2}\max\left|\frac{\partial^2 g}{\partial x^2}\right|\cdot\max\left|(x-u)[x-(u+1)]\right|=\frac{1}{8}M_{11}$$

$$R(x,y)=f(x,y)-g(x,y)$$

$$|R(x,y)|\le\frac{1}{2}\max\left|\frac{\partial^2 g}{\partial y^2}\right|\cdot\max\left|(y-v)[y-(v+1)]\right|+\max\left|(v+1-y)R_1(x)+(y-v)R_2(x)\right|$$

$$=\frac{1}{8}M_{11}+\frac{1}{8}M_{22}$$