

# **Lecture 7**

## **Multivariate Distributions**

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September 29, 2025

## Multivariate distribution

In many practical cases, it is possible, and often desirable, to take more than one measurement of a random observation. Moreover, we sometimes want to use these measurements to predict a third one. For example, we measure the GPA and extracurriculum activities of a student, and we give each of them a comprehensive evaluation score.

**Definition:** Let  $X$  and  $Y$  be two discrete random variables. Let  $S$  denote the two-dimensional space of  $X$  and  $Y$ . The probability that  $X = x$  and  $Y = y$  is denoted by  $f(x, y) = P(X = x, Y = y)$ . The function  $f(x, y)$  is called the joint probability mass function.

## Joint probability mass function

**Example:** Roll a pair of fair dice. For each of the 36 sampling points with probability  $1/36$ , let  $X$  denote the smaller and  $Y$  the larger outcome on the dice. For example, if the outcome is  $(3, 2)$ , then the observed values are  $X = 2, Y = 3$ . What is the joint PMF of  $X$  and  $Y$ ?

The event  $X = 2, Y = 3$  can happen in one of two ways  $(2, 3)$  or  $(3, 2)$ . So its probability is  $2/36$ . However, for event such as  $X = 2, Y = 2$ , it can only happen in one way. Thus, in general, the joint probability mass function is

$$f(x, y) = \begin{cases} \frac{1}{36} & x = y \\ \frac{1}{18} & x \neq y \end{cases}$$

## Multinomial distribution

Suppose we have three mutually exclusive and exhaustive ways for an experiment to end: perfect, seconds, and defective. We repeat the experiment  $n$  independent times and the probability  $p_X$ ,  $p_Y$ ,  $1 - p_X - p_Y$  of the three type of results. Let  $X$  and  $Y$  be the number of perfect and seconds. What is the joint probability mass function of  $X$  and  $Y$ ?

The probability of having  $x$  perfects,  $y$  seconds, and  $n - x - y$  defective is

$$p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

And it can be achieved in

$$\mathbf{C}_n^x \mathbf{C}_{n-x}^y = \frac{n!}{x!(n-x)!} \frac{(n-x)!}{y!(n-x-y)!} = \frac{n!}{x!y!(n-x-y)!}$$

Thus, the joint PMF is

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

## Marginal probability mass function

Let  $X$  and  $Y$  have the joint probability mass function  $f(x, y)$  with space  $S$ . The probability mass function of  $X$  alone is called the marginal probability mass function of  $X$  and is defined by

$$f_X(x) = \sum_y f(x, y) \quad x \in S_X$$

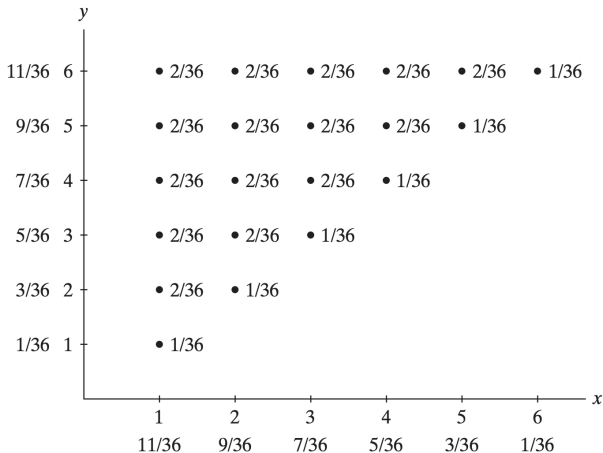
The random variables  $X$  and  $Y$  are independent if and only if, for every  $x \in S_X$  and  $y \in S_Y$ ,

$$f(x, y) = f_X(x)f_Y(y)$$

Otherwise,  $X$  and  $Y$  are said to be dependent.

## Marginal probability mass function

**Example:** In the dice rolling example mentioned above, what is the marginal probability mass function of  $X$  and  $Y$ ? Are  $X$  and  $Y$  independent?



## Marginal probability mass function

If  $X$  and  $Y$  has a multinomial distribution, are they independent?

It is easy to see by logic that  $X$  and  $Y$  both have a binomial distribution.

$$f_X(x) = \mathbf{C}_n^x p_X^x (1 - p_X)^{n-x}$$

$$f_Y(y) = \mathbf{C}_n^y p_Y^y (1 - p_Y)^{n-y}$$

Therefore,

$$f_X(x)f_Y(y) = \mathbf{C}_n^x \mathbf{C}_n^y p_X^x (1 - p_X)^{n-x} p_Y^y (1 - p_Y)^{n-y} \neq f(xy)$$

Thus,  $X$  and  $Y$  are not independent.

## Mathematical expectation

Let  $X_1$  and  $X_2$  be random variables of the discrete type with the joint PMF  $f(x_1, x_2)$  on the space  $S$ . If  $u(X_1, X_2)$  is a function of these two random variables, then

$$E[u(X_1, X_2)] = \sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2)$$

if it exists, is called the mathematical expectation of  $u(X_1, X_2)$ .

If  $u(X_1, X_2) = X_i$ , then  $E[u(X_1, X_2)] = E(X_i) = \mu_i$ ; if  $u(X_1, X_2) = (X_i - \mu_i)^2$ , then  $E[u(X_1, X_2)] = E[(X_i - \mu_i)^2] = \text{Var}(X_i)$



## Mathematical expectation

**Example:** There are eight chips in a bowl: three marked  $(0, 0)$ , two marked  $(1, 0)$ , two marked  $(0, 1)$ , and one marked  $(1, 1)$ . A player selects a chip at random and is given the sum of the two coordinates in dollars as a prize. What is the expected prize money a player can get?

Let  $X_1$  and  $X_2$  denote the two coordinates. Their joint PMF is

$$f(x, y) = \frac{3 - x_1 - x_2}{8}, \quad x_1 = 0, 1 \text{ and } x_2 = 0, 1$$

Thus,

$$\begin{aligned} E(X_1 + X_2) &= \sum_{x_2=0}^1 \sum_{x_1=0}^1 (x_1 + x_2) \frac{3 - x_1 - x_2}{8} \\ &= (0)\left(\frac{3}{8}\right) + (1)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{1}{8}\right) = \frac{3}{4} \end{aligned}$$

## Correlation coefficient

Let  $u(X, Y) = (X - \mu_X)(Y - \mu_Y)$ , then

$$E[u(X, Y)] = E[(X - \mu_X)(Y - \mu_Y)] = \text{Cov}(X, Y) = \sigma_{XY}$$

is called the covariance of  $X$  and  $Y$ .

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

is called the correlation coefficient of  $X$  and  $Y$ .

A commonly used formula to calculate covariance:

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\ &= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y \\ &= E(XY) - \mu_X \mu_Y\end{aligned}$$

## Correlation coefficient

**Example:** Let  $X$  and  $Y$  have the joint PMF

$$f(x, y) = \frac{x + 2y}{18}, \quad x = 1, 2 \text{ and } y = 1, 2$$

What is the correlation coefficient of  $X$  and  $Y$ ?

The marginal PMF are respectively

$$f_X(x) = \sum_{y=1}^2 \frac{x + 2y}{18} = \frac{x + 3}{9}$$

$$f_Y(y) = \sum_{x=1}^2 \frac{x + 2y}{18} = \frac{3 + 4y}{18}$$

The mean and variance of  $X$  are

$$\mu_X = \sum_{x=1}^2 x \frac{x + 3}{9} = (1) \frac{4}{9} + (2) \frac{5}{9} = \frac{14}{9}$$

## Correlation coefficient

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \sum_{x=1}^2 x^2 \frac{x+3}{9} - \left(\frac{14}{9}\right)^2 = \frac{20}{81}$$

Similarly, we get the mean and variance of  $Y$

$$\mu_Y = \frac{29}{18} \quad \sigma_Y^2 = \frac{77}{324}$$

The covariance of  $X$  and  $Y$

$$\begin{aligned} \text{Cov}(X, Y) &= \sum_{x=1}^2 \sum_{y=1}^2 xy \frac{x+2y}{18} - \frac{14}{9} \frac{29}{18} \\ &= (1)(1) \frac{3}{18} + (2)(1) \frac{4}{18} + (1)(2) \frac{5}{18} + (2)(2) \frac{6}{18} - \frac{14}{9} \frac{29}{18} \\ &= -\frac{1}{162} \\ \rho &= \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = -0.025 \end{aligned}$$

## Correlation coefficient

**Proposition:** If  $X$  and  $Y$  are independent,  $\text{Cov}(X, Y) = 0$ .

$$\begin{aligned} E(XY) &= \sum_{S_X} \sum_{S_Y} xyf(x, y) \\ &= \sum_{S_X} \sum_{S_Y} xyf_X(x)f_Y(y) \\ &= \sum_{S_X} xf_X(x) \sum_{S_Y} yf_Y(y) \\ &= \mu_X \mu_Y \end{aligned}$$

Thus, we have

$$\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y = 0$$

## Correlation coefficient

If  $\text{Cov}(X, Y) = 0$ , are  $X$  and  $Y$  necessarily independent?

**Example:** Let  $X$  and  $Y$  have the joint PMF

$$f(x, y) = \frac{1}{3}, \quad (x, y) = (0, 1), (1, 0), (2, 1).$$

It is easy to get the marginal PMF of  $X$  and  $Y$ :

$$f_X(x) = \frac{1}{3}, \quad x = 0, 1, 2; \quad f_Y(y) = \begin{cases} \frac{1}{3}, & y = 0 \\ \frac{2}{3}, & y = 1 \end{cases}$$

Thus,  $\mu_X = 1$  and  $\mu_Y = 2/3$ . Then

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - \mu_X \mu_Y \\ &= (0)(1)\frac{1}{3} + (1)(0)\frac{1}{3} + (2)(1)\frac{1}{3} - (1)\frac{2}{3} \\ &= 0 \end{aligned}$$

It is obvious that  $f(x, y) \neq f_X(x)f_Y(y)$ . Thus,  $X$  and  $Y$  are dependent.