Lecture 8
Multivariate Distributions

Chao Song

College of Ecology

Lanzhou University

October 9, 2025

Conditional distributions

Let X and Y have a joint discrete distribution with PMF $f(x, y)$ on space S. Say the marginal PMF are $f_X(x)$ and $f_Y(y)$ respectively. Let event $A = \{X = x\}$ and event $B = \{Y = y\}$. Thus $A \cap B = \{X = x, Y = y\}$. Because $P(A \cap B) = P(X = x, Y = y) = f(x, y)$ and $P(B) = P(Y = y) = f_Y(y)$, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{f(x, y)}{f_Y(y)}$$

Definition: The conditional probability mass function of X, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x, y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$

Conditional distributions

Example: Let X and Y have the joint PMF

$$f(x, y) = \frac{x+y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

Find the conditional distribution $g(x|y)$.

We first calculate marginal PMF of y:

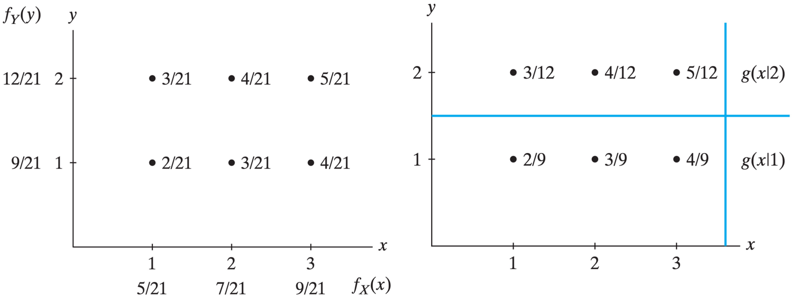
$$f_Y(y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{y+2}{7}, \quad y = 1, 2$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(x+y)/21}{(y+2)/7} = \frac{x+y}{3y+6}$$

Conditional distribution

Similar to conditional probability, we can visualize the joint, marginal, and conditional PMF.



(Graphic illustration of joint, marginal and conditional PMF.)

Conditional expectation

Because conditional PMF is a PMF, we thus can define conditional expectation the same way we define mathematical expectation:

$$E\left[u(Y)|X = x\right] = \sum_y u(y)g(y|x)$$

Conditional mean and conditional variance are defined by

$$\mu_{Y|X} = E(Y|X) = \sum_y yg(y|x)$$

$$\sigma^2_{Y|X} = E\left[(Y - \mu_{Y|X})^2|X\right] = \sum_y (y - \mu_{Y|X})^2 g(y|x)$$

Conditional expectation

Example: Let X and Y have a multinomial PMF with parameters n, p_X, and p_Y. That is,

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_x - p_Y)^{n-x-y}$$

What is the conditional mean of X given Y?

We know that the marginal distribution of Y is binomial,. i.e.,

$$f_Y(y) = \frac{n!}{y!(n-y)!} p_Y^y (1 - p_Y)^{n-y}$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(n-y)!}{x!(n-y-x)!} (\frac{p_X}{1-p_Y})^x (1 - \frac{p_X}{1-p_Y})^{n-y-x}$$

This is a binomial distribution with parameters $n - y$ and $\frac{p_X}{1-p_Y}$. Thus, the conditional mean is $(n-y)\frac{p_X}{1-p_Y}$.

Multivariate distribution of continuous random variables

The idea of joint distributions of discrete random variables can be extended to that of continuous random variables. The **joint probability density function** of two continuous random variables is an integrable function $f(x, y)$ such that

- $f(x, y) \geqslant 0$, where $f(x, y) = 0$ when (x, y) is not in the space of X and Y;
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$;
- $P(X, Y) \in A = \int \int_{A} f(x, y) dxdy$

Multivariate distribution of continuous random variables

The **marginal probability density** function of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy, \quad x \in S_X;$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx, \quad y \in S_Y;$$

X and Y are **independent** if and only if $f(x, y) = f_X(x)f_Y(y)$

Multivariate distribution of continuous random variables

The correlation coefficient of two continuous random variables X and Y is defined in the same way as the discrete random variables as

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The **Conditional probability density function** of X, given that $Y = y$, is

$$f(x|y) = \frac{f(x, y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Multivariate distribution of continuous random variables

Example: Let X and Y have the joint PDF

$$f(x, y) = 1, \quad x \leqslant y \leqslant x + 1, \quad 0 \leqslant x \leqslant 1.$$

Find the marginal PDF and the correlation coefficient of X and Y.

The marginal PDFs of X and Y are

$$f_X(x) = \int_x^{x+1} 1 dy = 1, \quad 0 \leqslant x \leqslant 1$$

$$f_Y(y) = \begin{cases} \displaystyle\int_0^y 1 dx = y, \quad 0 \leqslant y \leqslant 1, \\ \displaystyle\int_{y-1}^1 1 dx = 2 - y, \quad 1 \leqslant y \leqslant 2. \end{cases}$$

Multivariate distribution of continuous random variables

The mean and variance of X and Y are

$$\mu_X = \int_0^1 x \cdot 1 dx = \frac{1}{2}$$

$$\mu_Y = \int_0^1 y \cdot y dy + \int_1^2 y \cdot (2-y) dy = \frac{1}{3} + \frac{2}{3} = 1$$

$$E(X^2) = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3}$$

$$E(Y^2) = \int_0^1 y^2 \cdot y dy + \int_1^2 y^2 \cdot (2-y) dy = \frac{7}{6}$$

$$E(XY) = \int_0^1 \int_x^{x+1} xy \cdot 1 dy dx = \int_0^1 \frac{1}{2} x(2x+1) dx = \frac{7}{12}$$

Multivariate distribution of continuous random variables

$$\sigma_X^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$
$$\sigma_Y^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$$
$$\sigma_{XY} = \frac{7}{12} - \left(\frac{1}{2}\right)(1) = \frac{1}{12}$$

Therefore, the correlation coefficient is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1/12}{\sqrt{(1/12)(1/6)}} = \frac{\sqrt{2}}{2}$$

Multivariate normal distribution

A very commonly used multivariate distribution is the multivariate normal distribution. Random variables X and Y have a bivariate normal distribution if its joint PDF is

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x, y)}{2}\right],$$

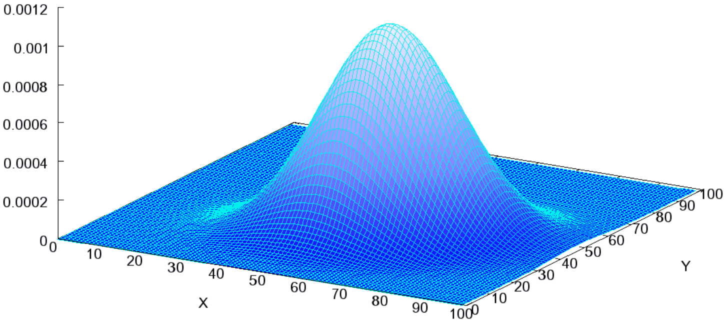
where

$$q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

Here, μ_X and μ_Y are the mean of X and Y, σ_X and σ_Y are the standard deviation of X and Y, and ρ is the correlation coefficient.

Multivariate normal distribution

A bivariate normal distribution has a typical PDF figure as follows.



Multivariate normal distribution

If random variables X and Y have a bivariate normal distribution, then the marginal distribution of X and Y are both normal.

$$
\begin{aligned}
q(x,y) &= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \\
&= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X} - \rho\frac{y-\mu_Y}{\sigma_Y}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \\
&= \frac{1}{\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2
\end{aligned}
$$

Multivariate normal distribution

Thus, the marginal distribution of Y is

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right]dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] \\
&\quad \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2\right]dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right](\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}) \\
&= \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]
\end{aligned}$$

Thus, the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$. Using the procedure, it is obvious that $X \sim N(\mu_X, \sigma_X^2)$.

Multivariate normal distribution

If If random variables X and Y have a bivariate normal distribution, then the conditional distribution of X given Y is normal.

The joint PDF is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right],$$

where

$$q(x,y) = \frac{1}{\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2$$

The marginal PDF of Y is

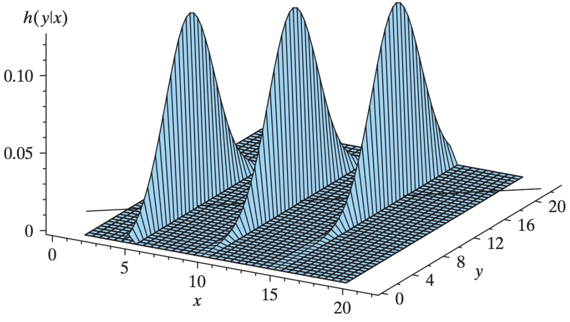
$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$$

Multivariate normal distribution

The conditional distribution of X given Y is thus

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{[x-\mu_X-\rho(\sigma_X/\sigma_Y)(y-\mu_Y)]^2}{2\sigma_X^2(1-\rho^2)}\right]$$

Thus, $g(x|y)$ is $N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), (1-\rho^2)\sigma_X^2\right)$.



(Illustration of conditional distribution of a bivariate normal distribution)

Multivariate normal distribution

We can extend the case of bivariate normal distribution to more than two variables. For more k variables, we write the PDF of a multivariate normal distribution in matrix notation:

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right)$$

where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are column vectors of the variables and its means, $\boldsymbol{\Sigma}$ is the $k \times k$ variance covariance matrix, i.e.,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}; \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}; \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2_{X_1} & \sigma_{X_1 X_2} & \cdots & \sigma_{X_1 X_k} \\ \sigma_{X_1 X_1} & \sigma^2_{X_2} & \cdots & \sigma_{X_2 X_k} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_k X_1} & \sigma^2_{X_k X_2} & \cdots & \sigma^2_{X_k} \end{bmatrix}$$