

Lecture 8

Multivariate Distributions

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Conditional distributions

Let X and Y have a joint discrete distribution with PMF $f(x, y)$ on space S .

Say the marginal PMF are $f_X(x)$ and $f_Y(y)$ respectively. Let event

$A = \{X = x\}$ and event $B = \{Y = y\}$. Thus $A \cap B = \{X = x, Y = y\}$.

Because $P(A \cap B) = P(X = x, Y = y) = f(x, y)$ and

$P(B) = P(Y = y) = f_Y(y)$, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{f(x, y)}{f_Y(y)}$$

Definition: The conditional probability mass function of X , given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x, y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$

Conditional distributions

Example: Let X and Y have the joint PMF

$$f(x, y) = \frac{x+y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

Find the conditional distribution $g(x|y)$.

We first calculate marginal PMF of y :

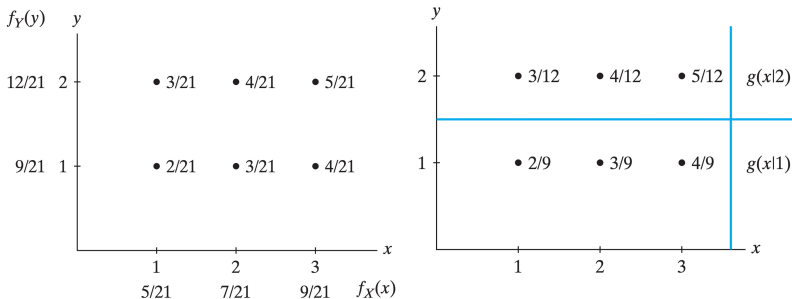
$$f_Y(y) = \sum_{x=1}^3 \frac{x+y}{21} = \frac{y+2}{7}, \quad y = 1, 2$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(x+y)/21}{(y+2)/7} = \frac{x+y}{3y+6}$$

Conditional distribution

Similar to conditional probability, we can visualize the joint, marginal, and conditional PMF.



(Graphic illustration of joint, marginal and conditional PMF.)

Conditional expectation

Because conditional PMF is a PMF, we thus can define conditional expectation the same way we define mathematical expectation:

$$E[u(Y)|X = x] = \sum_y u(y)g(y|x)$$

Conditional mean and conditional variance are defined by

$$\mu_{Y|X} = E(Y|X) = \sum_y yg(y|x)$$

$$\sigma_{Y|X}^2 = E[(Y - \mu_{Y|X})^2|X] = \sum_y (y - \mu_{Y|X})^2 g(y|x)$$

Conditional expectation

Example: Let X and Y have a multinomial PMF with parameters n , p_X , and p_Y . That is,

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

What is the conditional mean of X given Y ?

We know that the marginal distribution of Y is binomial, i.e.,

$$f_Y(y) = \frac{n!}{y!(n-y)!} p_Y^y (1 - p_Y)^{n-y}$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{(n-y)!}{x!(n-y-x)!} \left(\frac{p_X}{1-p_Y}\right)^x \left(1 - \frac{p_X}{1-p_Y}\right)^{n-y-x}$$

This is a binomial distribution with parameters $n - y$ and $\frac{p_X}{1-p_Y}$. Thus, the conditional mean is $(n - y) \frac{p_X}{1-p_Y}$.

Multivariate distribution of continuous random variables

The idea of joint distributions of discrete random variables can be extended to that of continuous random variables. The **joint probability density function** of two continuous random variables is an integrable function $f(x, y)$ such that

- $f(x, y) \geq 0$, where $f(x, y) = 0$ when (x, y) is not in the space of X and Y ;
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$;
- $P(X, Y) \in A = \int \int_A f(x, y) dx dy$

Multivariate distribution of continuous random variables

The **marginal probability density** function of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad x \in S_X;$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx, \quad y \in S_Y;$$

X and Y are **independent** if and only if $f(x, y) = f_X(x)f_Y(y)$

Multivariate distribution of continuous random variables

The correlation coefficient of two continuous random variables X and Y is defined in the same way as the discrete random variables as

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The **Conditional probability density function** of X , given that $Y = y$, is

$$f(x|y) = \frac{f(x, y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Multivariate distribution of continuous random variables

Example: Let X and Y have the joint PDF

$$f(x, y) = 1, \quad x \leq y \leq x + 1, \quad 0 \leq x \leq 1.$$

Find the marginal PDF and the correlation coefficient of X and Y .

The marginal PDFs of X and Y are

$$f_X(x) = \int_x^{x+1} 1 \, dy = 1, \quad 0 \leq x \leq 1$$
$$f_Y(y) = \begin{cases} \int_0^y 1 \, dx = y, & 0 \leq y \leq 1, \\ \int_{y-1}^1 1 \, dx = 2 - y, & 1 \leq y \leq 2. \end{cases}$$

Multivariate distribution of continuous random variables

The mean and variance of X and Y are

$$\mu_X = \int_0^1 x \cdot 1 dx = \frac{1}{2}$$

$$\mu_Y = \int_0^1 y \cdot y dy + \int_1^2 y \cdot (2 - y) dy = \frac{1}{3} + \frac{2}{3} = 1$$

$$E(X^2) = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3}$$

$$E(Y^2) = \int_0^1 y^2 \cdot y dy + \int_1^2 y^2 \cdot (2 - y) dy = \frac{7}{6}$$

$$E(XY) = \int_0^1 \int_x^{x+1} xy \cdot 1 dy dx = \int_0^1 \frac{1}{2} x(2x + 1) dx = \frac{7}{12}$$

Multivariate distribution of continuous random variables

$$\sigma_X^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$\sigma_Y^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$$

$$\sigma_{XY} = \frac{7}{12} - \left(\frac{1}{2}\right)(1) = \frac{1}{12}$$

Therefore, the correlation coefficient is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1/12}{\sqrt{(1/12)(1/6)}} = \frac{\sqrt{2}}{2}$$

Multivariate normal distribution

A very commonly used multivariate distribution is the multivariate normal distribution. Random variables X and Y have a bivariate normal distribution if its joint PDF is

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left[-\frac{q(x, y)}{2} \right],$$

where

$$q(x, y) = \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_X}{\sigma_X} \right)^2 - 2\rho \left(\frac{x-\mu_X}{\sigma_X} \right) \left(\frac{y-\mu_Y}{\sigma_Y} \right) + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 \right]$$

Here, μ_X and μ_Y are the mean of X and Y , σ_X and σ_Y are the standard deviation of X and Y , and ρ is the correlation coefficient.

Multivariate normal distribution

If random variables X and Y have a bivariate normal distribution, then the marginal distribution of X and Y are both normal.

$$\begin{aligned}q(x, y) &= \frac{1}{1 - \rho^2} \left[\left(\frac{x - \mu_X}{\sigma_X} \right)^2 - 2\rho \left(\frac{x - \mu_X}{\sigma_X} \right) \left(\frac{y - \mu_Y}{\sigma_Y} \right) + \left(\frac{y - \mu_Y}{\sigma_Y} \right)^2 \right] \\&= \frac{1}{1 - \rho^2} \left[\left(\frac{x - \mu_X}{\sigma_X} - \rho \frac{y - \mu_Y}{\sigma_Y} \right)^2 + (1 - \rho^2) \left(\frac{y - \mu_Y}{\sigma_Y} \right)^2 \right] \\&= \frac{1}{\sigma_X^2 (1 - \rho^2)} \left(x - \mu_X - \rho \frac{\sigma_X}{\sigma_Y} (y - \mu_Y) \right)^2 + \left(\frac{y - \mu_Y}{\sigma_Y} \right)^2\end{aligned}$$

Multivariate normal distribution

Thus, the marginal distribution of Y is

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x, y)}{2}\right] dx \\&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] \\&\quad \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2\right] dx \\&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] (\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}) \\&= \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]\end{aligned}$$

Thus, the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$. Using the procedure, it is obvious that $X \sim N(\mu_X, \sigma_X^2)$.

Multivariate normal distribution

If random variables X and Y have a bivariate normal distribution, then the conditional distribution of X given Y is normal.

The joint PDF is

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x, y)}{2}\right],$$

where

$$q(x, y) = \frac{1}{\sigma_X^2(1-\rho^2)} \left(x - \mu_X - \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right)^2 + \left(\frac{y - \mu_Y}{\sigma_Y}\right)^2$$

The marginal PDF of Y is

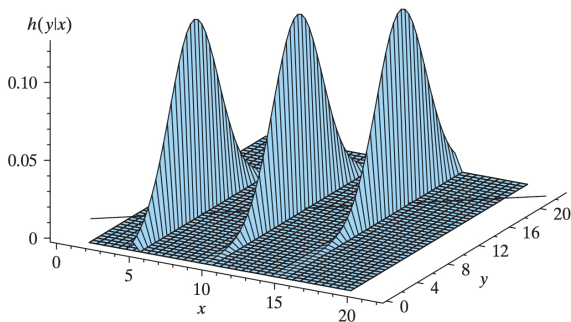
$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y - \mu_Y)^2}{2\sigma_Y^2}\right]$$

Multivariate normal distribution

The conditional distribution of X given Y is thus

$$g(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{\sigma_X \sqrt{2\pi} \sqrt{1 - \rho^2}} \exp \left[- \frac{[x - \mu_X - \rho(\sigma_X/\sigma_Y)(y - \mu_Y)]^2}{2\sigma_X^2(1 - \rho^2)} \right]$$

Thus, $g(x|y)$ is $N(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y), (1 - \rho^2)\sigma_X^2)$.



(Illustration of conditional distribution of a bivariate normal distribution)