Lecture 4
Transformation of Random Variables

Chao Song

College of Ecology
Lanzhou University

September 25, 2023

Transformation of discrete random variables

A problem often encountered in statistics is the following. We have a random variable X and we know its distribution. We are interested in a random variable Y which is some **transformation** of X, say $Y = g(X)$. We want to determine the distribution of Y.

Let X be the number of trials until we get the first success. Let p be the probability of success. The probability mass function of X is thus $P(X = x) = p(1-p)^{x-1}$. Let $Y = X - 1$, i.e., Y is the number of failures before first success. What is the PMF of Y?

$$\begin{aligned} P(Y = y) &= P(X - 1 = y) = P(X = y + 1) \\ &= p(1-p)^{y-1+1} = p(1-p)^{y} \end{aligned}$$

In general, for discrete random variable, we can directly use the probability mass function of the original random variable to derive the probability mass function of the transformed random variable.

Transformation of continuous random variables

Recall the theorem about standard normal distribution. If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{x-\mu}{\sigma}$ is $N(0, 1)$. Why is this the case?

Proof: The cumulative distribution function of Z is

$$
\begin{aligned}
P(Z \leqslant z) &= P(\frac{X-\mu}{\sigma} \leqslant z) = P(X \leqslant z\sigma + \mu) \\
&= \int_{-\infty}^{z\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx
\end{aligned}
$$

We now use the change of variable integration given by $w = (x-\mu)/\sigma$ (i.e., $x = w\sigma + \mu$) to obtain

$$P(Z \leqslant z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

Transformation of continuous random variables

Theorem: Let X be a continuous random variable with PDF $f_X(x)$ and support S_X. Let $Y = g(x)$, where $g(x)$ is a one-to-one differentiable function, on the support of X. Denote the inverse of g by $x = g^{-1}(y)$ and let $dx/dy = d[g^{-1}(y)]$. Then the PDF of Y is given by

$$f_Y(y) = f_X(g^{-1}(y))|\frac{dx}{dy}|$$

Proof: Since $g(x)$ is one-to-one and continuous, it is either monotonically increasing or decreasing. When it is strictly monotonically increasing, the CDF for Y is

$$F_Y(y) = P(Y \leqslant y) = P(g(x) \leqslant y) = P(x \leqslant g^{-1}(y)) = F_X(g^{-1}(y))$$

Hence the PDF of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(g^{-1}(y))\frac{dx}{dy} = f_X(g^{-1}(y))|\frac{dx}{dy}|$$

Similarly, when $g(x)$ is monotonically decreasing,

$$F_Y(y) = P(Y \leqslant y) = P(g(x) \leqslant y) = P(x \geqslant g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

Hence the PDF of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -f_X(g^{-1}(y))\frac{dx}{dy} = f_X(g^{-1}(y))|\frac{dx}{dy}|$$

log-normal distribution

Let $X \sim N(\mu, \sigma^2)$, then $Y = e^X$ has a log-normal distribution.

Proof: When $Y = e^X$, we have $X = \ln(Y)$. Using the general conclusions about transformation of continuous random variable, the PDF of Y is

$$
\begin{aligned}
f_Y(y) &= f_X(\ln(y))\left|\frac{dx}{dy}\right| \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}\left|\frac{d\ln(y)}{dy}\right| \\
&= \frac{1}{\sigma y\sqrt{2\pi}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}
\end{aligned}
$$

log-normal distributions

Let $X \sim N(\mu, \sigma^2)$, then $Y = e^X$ has a log-normal distribution. What is the mean and variance of Y?

$$
\begin{aligned}
E(Y) &= \int_0^\infty y f_Y(y) dy \\
&= \int_0^\infty y \frac{1}{\sigma y \sqrt{2\pi}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}} dy
\end{aligned}
$$

For convenience of integration, use change of variable $t = (\ln(y) - \mu)/\sigma$ so that $y = e^{\sigma t + \mu}$ and $dy = \sigma e^{\sigma t + \mu} dt$, we have

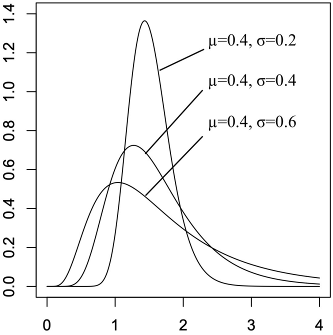
$$
\begin{aligned}
E(Y) &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \sigma e^{\sigma t + \mu} dt \\
&= e^{\mu + \frac{1}{2}\sigma^2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{(t-\sigma)^2}{2}} dt \\
&= e^{\mu + \frac{1}{2}\sigma^2}
\end{aligned}
$$

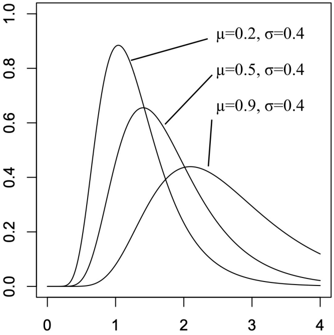
Similarly, we could calculate the variance of Y to be $Var(Y) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$.

log-normal distribution

If $X \sim N(\mu, \sigma^2)$, then $Y = e^X$ has a log-normal distribution with mean $e^{\mu+\frac{1}{2}\sigma^2}$ and variance $(e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$.

Note that if $X \sim N(\mu, \sigma^2)$, then the mean of $Y = e^X$ **is not** e^μ because e^X is a non-linear transformation.





χ^2-distribution

Let X follows a standard normal distribution. Find the PDF of $Y = X^2$

$$
\begin{aligned}
F_Y(y) &= P(Y \leqslant y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
&= \int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} dy - \int_{-\infty}^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} dy
\end{aligned}
$$

Thus, the PDF of Y is

$$
f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}
$$

This is the PDF of a χ^2-distribution with 1 degree of freedom.

Universality of the uniform

Let X be a continuous random variable and $F_X(x)$ be its cumulative distribution function. What is the PDF of $Y = F_X(x)$?

Using the method of distribution function, we have

$$\begin{aligned} F_Y(y) &= P(Y \leqslant y) = P(F_X(x) \leqslant y) = P(x \leqslant F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) = y \end{aligned}$$

Thus, the PDF of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = 1$$

Theorem: For a continuous random variable X, its cumulative distribution function $F_X(x)$ follows a uniform distribution between 0 and 1, $U(0, 1)$

Corollary: The fact that cumulative distribution function is $U(0, 1)$ provides a universal way to simulate continuous random variable. Specifically, one can draw random numbers from $U(0, 1)$ and then compute any random variable by the inverse of its cumulative distribution function.

Order statistics

Definition: Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the random variables sorted from the smallest to the largest. We call $X_{(j)}$ the jth order statistics of the random sample. We use $f_{(j)}$ and $F_{(j)}$ to denote its PDF and CDF respectively.

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample from a distribution. What is the probability density function of the maximum $X_{(n)}$?

$$
\begin{aligned}
F_{(n)}(x) &= P(X_{(n)} \leqslant x) = P(X_1 \leqslant x, \ldots X_n \leqslant x) \\
&= \prod_{i=1}^{n} P(X_i \leqslant x) = F_X(x)^n \\
f_n(x) &= \frac{d}{dx} F_{(n)}(x) = nF_X(x)^{n-1} f_X(x)
\end{aligned}
$$

Order statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample from a distribution. What is the probability density function of the minimum $X_{(1)}$?

$$\begin{aligned}
F_{(1)}(x) &= P(X_{(1)} \leqslant x) = 1 - P(X_{(1)} > x) \\
&= 1 - P(X_1 > x, \ldots, X_n > x) \\
&= 1 - \prod_{i=1}^{n} P(X_i > x) \\
&= 1 - \prod_{i=1}^{n} (1 - P(X_i \leqslant x)) \\
&= 1 - (1 - F_X(x))^n \\
f_{(1)}(x) &= \frac{d}{dx} F_{(1)}(x) = n(1 - F_X(x))^{n-1} f_X(x)
\end{aligned}$$

Moment generating function

Definition: Let X be a random variable, continuous or discrete. We define the moment generating function of X to be

$$m_X(t) = E(e^{tX})$$

Moment generating function, as its name suggests, can be used to find moments of a random variable.

$$\frac{d}{dt} m_X(t) = E(Xe^{tX}),$$

which when we evaluate at $t = 0$ becomes $E(X)$. More generally, the nth derivative of $m_X(t)$ evaluated at zero is the expected value of X^n, i.e., $m^{(n)}(0) = E(X^n)$

The moment generating function determines the distribution of X. From this point of view, knowing a random variables moment generating function gives as much information as its PDF or CDF.

Moment generating function

If $X \sim N(0, 1)$, what is the moment generating function of X?

$$
\begin{aligned}
m_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{Xt} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx \\
&= e^{\frac{t^2}{2}}
\end{aligned}
$$

Using similar technique with a change of variable in the integration, we can get the moment generating function for $X \sim N(\mu, \sigma^2)$ to be

$$
m_X(t) = e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}
$$

Moment generating function

Theorem: Let X and Y be random variables with moment generating functions $m_X(t)$ and $m_Y(t)$. if X and Y are independent, the moment generating function of $aX + bY$ is

$$m_{aX+bY}(t) = m_X(at)m_Y(bt)$$

Proof: According to the definition of moment generating function:

$$m_{aX+bY}(t) = E(e^{(aX+bY)t}) = E(e^{aXt+bYt}) = E(e^{Xat}e^{Ybt})$$

Because X and Y are independent, $E(e^{Xt}e^{Yt}) = E(e^{Xat})E(e^{Ybt})$. Thus

$$m_{aX+bY}(t) = m_X(at)m_Y(bt)$$

Methods of moment generating function

Because moment generating functions uniquely identifies a distribution. We can use the moment generating function to find the distribution of a transformed random variable.

Example: Recall that the moment generating function of $X \sim N(\mu, \sigma^2)$ is $m_X(t) = e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, what is the distribution of $X_1 + X_2$?

$$\begin{aligned} m_{X_1+X_2}(t) &= m_{X_1}(t) m_{X_2}(t) \\ &= e^{\mu_1 t} e^{\frac{1}{2}\sigma_1^2 t^2} e^{\mu_2 t} e^{\frac{1}{2}\sigma_2^2 t^2} \\ &= e^{(\mu_1+\mu_2)t} e^{\frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2} \end{aligned}$$

This is the moment generating function of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

Methods of moment generating function

Using the method of moment generating function, we get the following theorem about transformation of normally distributed random variables.

Theorem: if $X_1, \ldots, X_n$ are mutually independent normal variables with mean μ_i and variance σ_i^2, then the linear function

$$Y = \sum_{i=1}^{n} c_i Xi$$

has the normal distribution

$$N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

Theorem: if $X_1, X_2, \ldots, X_n$ are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, then the sample mean

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Method of moment generating function

If X has a Poisson distribution with parameter λ, its moment generating function is

$$\begin{aligned} E(e^{tX}) &= \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \\ &= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

If X_1 and X_2 are independent Poisson distributed random variables with parameters λ_1 and λ_2, what is the distribution of $X_1 + X_2$?

$$\begin{aligned} m_{X_1+X_2}(t) &= m_{X_1}(t) m_{X_2}(t) \\ &= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

Theorem: If X_1 and X_2 are independent Poisson distributed random variables with parameters λ_1 and λ_2, then $X_1 + X_2$ follows a Poisson distribution with parameter $\lambda_1 + \lambda_2$.