Lecture 3
Continuous Random Variables

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Continuous random variables

Recall: The cumulative distribution function (CDF) of a random variable X is the function $F(x) : \mathbb{R} \to [0, 1]$ given by $F(x) = P(X \leqslant x)$

Definition 1: A random variable X is said to be of continuous type if its cumulative distribution function $F(x)$ is continuous on its support.

Definition 2: A random variable X is continuous if it takes infinite number of values within a interval or within the joint of several intervals on the real number.

Examples:

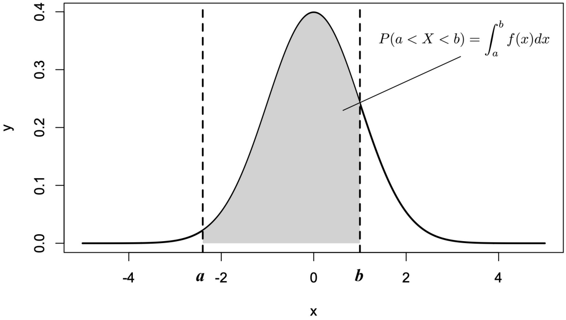
- length of time between machine failure;
- height of student in Lanzhou University;
- total biomass of sampling quadrants in grasslands;
- amount of precipitation received in a day.

Probability density function

Definition: Let $F(x)$ be the cumulative distribution function of a continuous random variable X. Probability density function, abbreviated as PDF, is a function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(s)ds$$

Probability density function **is not** a probability. The area under the probability density function curve is probability.



Deriving probability density function

For discrete random variables, we can directly derive the probability mass function. For continuous random variables, however, we derive probability density function by taking the derivative of the cumulative distribution function:

$$f(x) = \frac{dF(x)}{dx}$$

Example: Suppose we select a point at random in the interior of a circle of radius 1. Let X be the distance of the selected point from the origin. What is the probability density function of X?

Answer: Because a point is selected at random, the event $X \leqslant x$ is equivant to the point lying in a circle of radius x. Hence, the CDF of X is

$$F(x) = P(X \leqslant x) = \pi x^2/\pi = x^2,\ 0 \leqslant x \leqslant 1$$

The PDF of X is

$$f(x) = \frac{dF(x)}{dx} = 2x,\ 0 \leqslant x \leqslant 1$$

Mathematical expectation

Let $f(x)$ be the probability density function of continuous random variable X, the mathematical expectation of $u(x)$ is calculated as

$$E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

In particular, the mean and variance of a continuous distribution is calculated as

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx, \quad Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Example: X has a probability density function $f(x) = \lambda e^{-\lambda x},\ x \leqslant 0$. What is the expected value of X?

$$E(X) = \int_{0}^{\infty} x\lambda e^{-\lambda x}$$

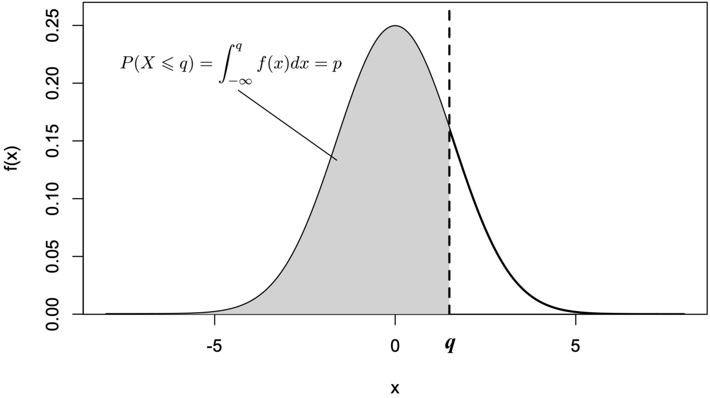
Integrating by parts with $u = \lambda x$ and $dv = e^{-\lambda x}dx$ so that $du = \lambda dx$ and $v = -\frac{1}{\lambda}e^{-\lambda x}$

$$E(X) = (-xe^{-\lambda x} - \frac{1}{\lambda}e^{-\lambda x})\Big|_0^{\infty} = \frac{1}{\lambda}$$

Quantile

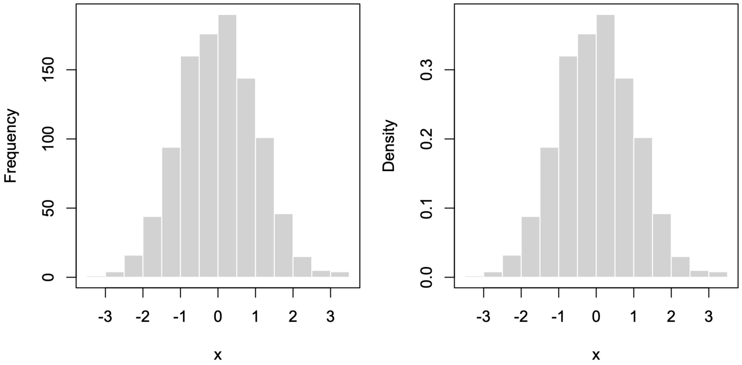
Definition: The 100*p*th quantile of a continuous distribution is a number q such that

$$\int_{-\infty}^{q} f(x)dx = p$$



Visualizing PDF empirically: histogram

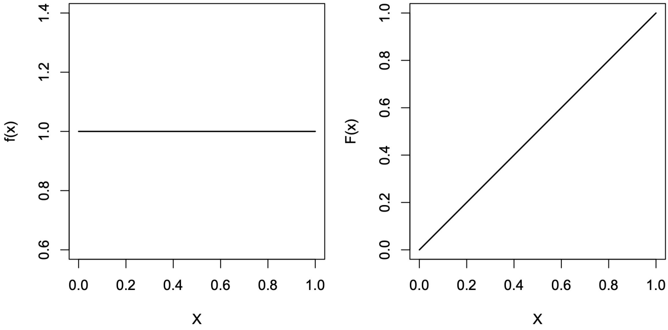
Histogram is an approximate representation of the distribution of numeric data. To construct a histogram, we bin the range of values into intervals and count the number of values in each interval. The y-axis of a histogram can be either **frequency** or **density** (i.e., relative frequency).



Uniform distribution

Definition: Let a and b be real numbers and $a < b$. A continuous random variable X has a uniform distribution, $U(a, b)$ if its probability density function is

$$f(x) = \frac{1}{b-a}, \ x \in (a, b)$$



PDF and CDF of a uniform distribution in (0, 1)

Uniform distribution

Example: A train leaves at 10:00AM from Lanzhou to Xining. A passenger arrives at the station randomly between 9:00AM and 10:00AM, what is the probability that he waits for less than 30 minutes?

Answer: Let X be the time he arrives at the station. X is uniformly distributed between 0 and 60 minutes. Its probability density function is

$$f(x) = \frac{1}{60},\ x \in (0, 60)$$

Let Y be his waiting time. Then

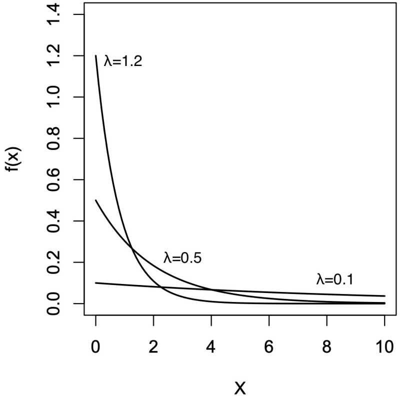
$$P(Y \leqslant 30) = P(30 < X \leqslant 60) = \int_{30}^{60} f(x)dx = \frac{1}{2}.$$

Exponential distribution

Definition: For a positive number λ, X has an exponential distribution if its probability density function is

$$f(x) = \lambda e^{-\lambda x}, \ x \geqslant 0.$$

The parameter λ is often referred to as the rate of the exponential distribution.



Exponential distribution

Exponential distribution models the time between two occurrence if the number of occurrences follows a Poisson distribution.

Proof: Let N_t follows a Poisson distribution with parameter λt. Let X be the waiting time between two events. We know that waiting time $X \leqslant t$ is equivalent to no event within time t, i.e., $N_t = 0$

$$\begin{aligned} F(t) &= P(X \leqslant t) \\ &= 1 - P(X > t) \\ &= 1 - P(N_t = 0) \\ &= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

With the cumulative distribution function $F(t)$, we can derive the probability density function

$$f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}$$

Exponential distribution

Theorem: Exponential distribution is **memoryless**, that is, if X has an exponential distribution, then

$$P(X > s + t | X > s) = P(X > t)$$

Proof: If X has an exponential distribution $f(x) = \lambda e^{-\lambda x}$

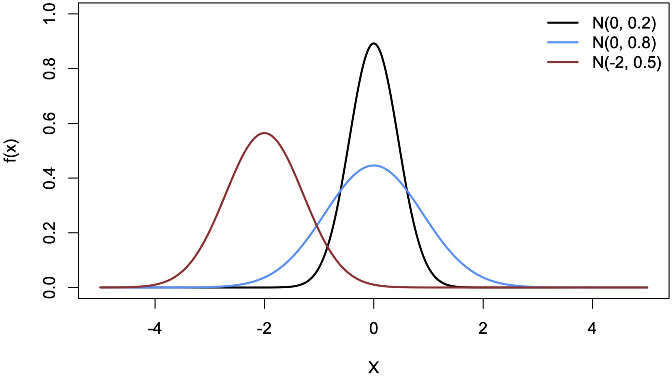
$$P(X > t) = \int_t^\infty \lambda e^{-\lambda x} dx = e^{-\lambda t}$$

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P(X > s + t, X > s)}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \end{aligned}$$

Normal distribution

Definition: A continuous random variable has a normal distribution with mean μ and variance σ^2, $N(\mu, \sigma^2)$, if it has a probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$$



Standard normal distribution

A normal distribution with **mean 0 and standard deviation 1** is a standard normal distribution. Its probability density function is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Standard normal distribution is useful in practice:

- If X is $N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is $N(0, 1)$.
- Calculating probability for a normally distributed variable can be converted to calculating the probability of a standard normal variable:

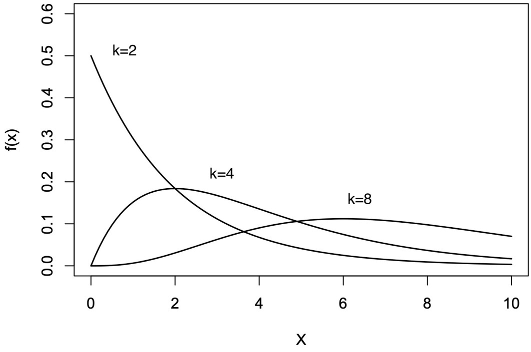
$$P(a < X < b) = P(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma})$$

Derived distributions: χ^2-distribution

Definition: If $Z_1, Z_2, \ldots, Z_k$ are independent, standard normal variables, i.e. $N(0,1)$, the sum of their squares

$$Q = \sum_{i=1}^{k} Z_i^2,$$

is distributed according to a χ^2-distribution with k degrees of freedom.

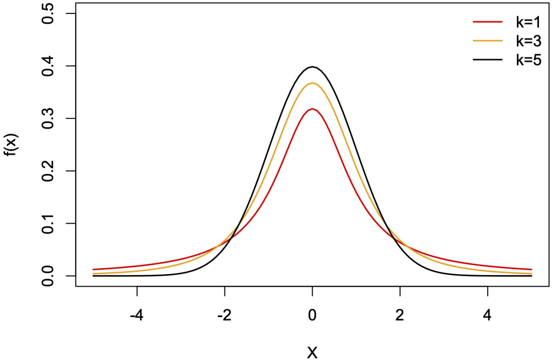


Derived distributions: $t-$distribution

Definition: If $X \sim N(0,1)$, $v \sim \chi^2(k)$ and the X and v are independent, than the random variable T defined below follows a $t-$distribution with k degrees of freedom.

$$T = \frac{X}{\sqrt{v/k}}$$

As the degrees of freedom increases, a t-distribution converge to a normal distribution.



Derived distributions: $F-$distribution

Definition: Let $S_1 \sim \chi^2(d_1)$, $S_2 \sim \chi^2(d_2)$, and the two random variables are independent, then the variable defined below follows a $F-$distribution with degrees of freedom d_1 and d_2

$$F = \frac{S_1/d_1}{S_2/d_2}$$