My Solution to Polchinski's *String Theory* Books

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Appendix A. A Short Course on Path Integrals

A.1 (a). Following the harmonic oscillator example on p.339, choose a complete orthonormal basis of periodic eigenfunctions $\{\Phi_j\}$ such that $\Delta\Phi_j = (-\partial_u^2 + \omega^2)\Phi_j = \lambda_j \Phi_j$:

$$\Phi_0(u) = \left(\frac{1}{U}\right)^{1/2}, \quad \Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \begin{matrix}\sin\\ \cos\end{matrix} \frac{2j\pi u}{U}, \quad j = 1, 2, \cdots$$

Using the Pauli-Villars regulator ($\Omega \gg \omega$), the functional determinant becomes (up to a formal constant):

$$\det\Delta = \prod_{j=-\infty}^{+\infty} \lambda_j = \omega^2 \left(\prod_{j=1}^{\infty} \frac{4j^2\pi^2 + \omega^2 U^2}{U^2}\right)^2 \rightsquigarrow \omega^2 \left(\prod_{j=1}^{\infty} \frac{j^2\pi^2 + \omega^2 U^2/4}{j^2\pi^2 + \Omega^2 U^2/4}\right)^2 = \omega^2 \left(\frac{\Omega \sinh\frac{\omega U}{2}}{\omega \sinh\frac{\Omega U}{2}}\right)^2.$$

As $\Omega \to \infty$,

$$\mathrm{Tr}\exp(-\hat{H}U) = \int [\mathrm{d}q]_P \exp(-S_E) = (\det\Delta)^{-1/2} \sim \frac{1}{2\sinh\frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U - \ln\Omega\right).$$

The normalization is fixed by requiring $\mathrm{Tr}\exp(-\hat{H}U) \to \exp(-E_0 U)$ as $U \to \infty$, which implies that $\mathrm{Tr}\exp(-\hat{H}U) = (2\sinh\omega U/2)^{-1}$. Indeed,

$$\mathrm{Tr}\exp(-\hat{H}U) = \sum_{j=0}^{\infty} \exp\left(-(j+1/2)\omega U\right) = \frac{1}{2\sinh\frac{\omega U}{2}}.$$

The counterterm is composed of vacuum energy $\int \frac{1}{2}\Omega \mathrm{d}u$ and wavefunction renormalization $-\ln\Omega$.

(b). For anti-periodic configurations, a basis of eigenfunctions is:

$$\Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \begin{matrix}\sin\\ \cos\end{matrix} \frac{(2j+1)\pi u}{U}, \quad j = 1, 2, \cdots$$

Proceeding as before,

$$\det\Delta = \left(\prod_{j=1}^{\infty} \frac{(2j+1)^2\pi^2 + \omega^2 U^2}{U^2}\right)^2 \rightsquigarrow \left(\frac{\cosh\frac{\omega U}{2}}{\cosh\frac{\Omega U}{2}}\right)^2.$$

The trace $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}]$ has the form:

$$\int [\mathrm{d}q]_A \exp(-S_E) \sim \frac{1}{2\cosh\frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U\right), \quad \text{as } \Omega \to \infty.$$

The normalization is fixed by $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] \to \exp(-E_0 U) = (2\cosh\omega U/2)^{-1}$ as $U \to \infty$. Therefore,

$$\mathrm{Tr}\left[\exp(-\hat{H}U)\hat{R}\right] = \frac{1}{2\cosh\frac{\omega U}{2}} = \sum_{j=0}^{\infty} (-)^j \exp\left(-(j+1/2)\omega U\right).$$

This time, there is no counterterm for wavefunction renormalization.

Remark. The role of renormalization here is to hide the details of divergence cancellation, so that one may focus on the essentials (i.e. calculation of eigenvalues). This is similar to what happens in QFT: instead of $\det \Delta$, we compute Feynman diagrams (propagators) using eigenvalues; instead of known ground states, we use experimental data (relations between amplitudes of various processes) to fix the normalization. The price we pay is that, in general, renormalizability needs to be proved.

Thoughts. In (b), there is no "wavefunction renormalization". At the same time, the "ground state" $q(u) = 0$ doesn't contribute since it violates the boundary conditions. Are these two facts somehow related?