My Solution to Polchinski's *String Theory* Books

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Appendix A. A Short Course on Path Integrals

A.1 (a). Following the harmonic oscillator example on p.339, choose a complete orthonormal basis of periodic eigenfunctions $\{\Phi_j\}$ such that $\Delta\Phi_j = (-\partial_u^2 + \omega^2)\Phi_j = \lambda_j \Phi_j$:

$$\Phi_0(u) = \left(\frac{1}{U}\right)^{1/2}, \quad \Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \frac{\sin}{\cos} \frac{2j\pi u}{U}, \quad j = 1, 2, \cdots$$

Using the Pauli-Villars regulator ($\Omega \gg \omega$), the functional determinant becomes (up to a formal constant):

$$\det \Delta = \prod_{j=-\infty}^{+\infty} \lambda_j = \omega^2 \left(\prod_{j=1}^{\infty} \frac{4j^2\pi^2 + \omega^2 U^2}{U^2} \right)^2 \rightsquigarrow \omega^2 \left(\prod_{j=1}^{\infty} \frac{4j^2\pi^2 + \omega^2 U^2}{4j^2\pi^2 + \Omega^2 U^2} \right)^2 = \omega^2 \left(\frac{\Omega \sinh \frac{\omega U}{2}}{\omega \sinh \frac{\Omega U}{2}} \right)^2.$$

As $\Omega \to \infty$,

$$\mathrm{Tr} \exp(-\hat{H}U) = \int [\mathrm{d}q]_P \exp(-S_E) \sim (\det \Delta)^{-1/2} \sim \frac{1}{2 \sinh \frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U - \ln \Omega \right).$$

The normalization is fixed by requiring $\mathrm{Tr} \exp(-\hat{H}U) \sim \exp(-E_0 U) = \exp(-\omega U/2)$ as $U \to \infty$, which implies that $\mathrm{Tr} \exp(-\hat{H}U) = (2 \sinh \omega U/2)^{-1}$. Indeed,

$$\mathrm{Tr} \exp(-\hat{H}U) = \sum_{j=0}^{\infty} \exp\left(-(j + 1/2)\omega U\right) = \frac{1}{2 \sinh \frac{\omega U}{2}}.$$

The counterterm consists of the vacuum energy $\int \frac{1}{2}\Omega \mathrm{d}u$ and a wavefunction renormalization $-\ln \Omega$.

(b). For anti-periodic configurations, a basis of eigenfunctions is:

$$\Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \frac{\sin}{\cos} \frac{(2j+1)\pi u}{U}, \quad j = 1, 2, \cdots$$

Proceeding as before,

$$\det \Delta = \left(\prod_{j=1}^{\infty} \frac{(2j+1)^2\pi^2 + \omega^2 U^2}{U^2} \right)^2 \rightsquigarrow \left(\prod_{j=1}^{\infty} \frac{(2j+1)^2\pi^2 + \omega^2 U^2}{(2j+1)^2\pi^2 + \Omega^2 U^2} \right)^2 = \left(\frac{\cosh \frac{\omega U}{2}}{\cosh \frac{\Omega U}{2}} \right)^2.$$

As $\Omega \to \infty$

$$\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] = \int [\mathrm{d}q]_A \exp(-S_E) \sim (\det \Delta)^{-1/2} \sim \frac{1}{2 \cosh \frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U \right).$$

The normalization is fixed by requiring $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] \sim \exp(-E_0 U) = \exp(-\omega U/2)$ as $U \to \infty$, which implies that $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] = (2 \cosh \omega U/2)^{-1}$. Indeed,

$$\mathrm{Tr}\left[\exp(-\hat{H}U)\hat{R}\right] = \sum_{j=0}^{\infty} (-)^j \exp\left(-(j + 1/2)\omega U\right) = \frac{1}{2 \cosh \frac{\omega U}{2}}.$$

This time, the counterterm consists only of the vacuum energy.

Remark. The role of renormalization here is to hide the details of divergence cancellation, so that one may focus on the essentials (i.e. computing the eigenvalues). This is similar to what happens in QFT, where instead of $\det\Delta$, we compute Feynman diagrams (propagators), and instead of known ground states, we use experimental data (relations between amplitudes of various processes) to fix the normalization. The price we pay is that, in general, renormalizability needs to be proved.

Thoughts. In (b), there is no wavefunction renormalization. At the same time, the "ground state" $q(u) = \text{const.}$ doesn't contribute since it violates the boundary conditions. Are these two facts somehow related?

A.2 Including possible counterterms, the Euclidean action:

$$S_E = \int_0^U \mathrm{d}u\, \left((\partial_u q^*)(\partial_u q) + \omega^2 q^* q\right) + S_{\text{ct}} \implies \int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \exp(-S_{\text{ct}})\, (\det\Delta)^{-1}.$$

A basis of eigenvectors satisfying the boundary condition $q(U) = q(0)e^{i\theta}$ is

$$\Phi_j(u) = \left(\frac{1}{U}\right)^{1/2} \exp\left(iu\frac{\theta + 2j\pi}{U}\right), \quad j = 0, \pm1, \pm2, \cdots$$

The functional determinant is regularized using the Pauli-Villars regulator:

$$\det\Delta = \prod_{j=-\infty}^{+\infty} \frac{(\theta+2j\pi)^2 + \omega^2 U^2}{U^2} \rightsquigarrow \prod_{j=-\infty}^{+\infty} \frac{(\theta+2j\pi)^2 + \omega^2 U^2}{(\theta+2j\pi)^2 + \Omega^2 U^2} = \frac{\cosh\omega U - \cos\theta}{\cosh\Omega U - \cos\theta}.$$

As $\Omega \to \infty$,

$$\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \frac{1}{2(\cosh\omega U - \cos\theta)} \exp\left(-S_{\text{ct}} + \Omega U\right).$$

To fix the normalization, notice that $\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) = \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}]$, where the twist operator is defined by $\hat{R}_\theta|q\rangle = |e^{i\theta}q\rangle$. This is because

$$\begin{aligned}
\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) &= \int \mathrm{d}q\mathrm{d}q^* \int [\mathrm{d}q\mathrm{d}q^*]_{q,0}^{e^{i\theta}q,U} \exp(-S_E) \\
&= \int \mathrm{d}q\mathrm{d}q^* \langle e^{i\theta}q, U|q, 0\rangle_E = \int \mathrm{d}q\mathrm{d}q^* \langle q, U|\hat{R}_{-\theta}|q, 0\rangle_E \\
&= \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}].
\end{aligned}$$

Therefore, for two harmonic oscillators, $\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \exp(-E_0 U) = \exp(-\omega U)$ as $U \to \infty$. Hence the counterterm is the vacuum energy $2\int \frac{1}{2}\Omega \mathrm{d}u$ corresponding to the two "virtual" harmonic oscillators with frequency Ω.

$$\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) = \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}] = \frac{1}{2(\cosh\omega U - \cos\theta)}.$$

To compare with the operational method, define $a_\pm := (a_1 \pm ia_2)/\sqrt{2}$ so that $a_\pm^\dagger = (a_1^\dagger \mp ia_2^\dagger)/\sqrt{2}$.

$$\hat{H} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) = \omega(a_+^\dagger a_+ + a_-^\dagger a_- + 1).$$

Since $\hat{q} \sim (a_1 + a_1^\dagger) + i(a_2 + a_2^\dagger) = (a_+ + a_-^\dagger)$ and $\hat{R}_{-\theta}\hat{q}\hat{R}_\theta = e^{i\theta}\hat{q}$, the holomorphic and anti-holomorphic operators (now considered independent) transform as $\hat{R}_{-\theta}a_\pm\hat{R}_\theta = e^{\pm i\theta}a_\pm$. Label the states with $a_\pm$ quantum numbers such that $|m,n\rangle := \frac{a_+^{\dagger m} a_-^{\dagger n}}{\sqrt{m!n!}}|0,0\rangle$.

$$\begin{aligned}
\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}] &= \sum_{m,n=0}^{\infty} \langle m,n|\exp(-\hat{H}U)\hat{R}_{-\theta}|m,n\rangle \\
&= \sum_{m,n=0}^{\infty} \langle m,n|\exp(-\omega(m+n+1)U - i\theta(m-n))|m,n\rangle \\
&= \frac{1}{2(\cosh\omega U - \cos\theta)}.
\end{aligned}$$

If $\theta = 0(\pi)$, the answer reduces to

$$\frac{1}{2(\cosh \omega U \mp 1)} = \left(2 \begin{matrix}\sinh \\ \cosh\end{matrix} \frac{\omega U}{2}\right)^{-2},$$

which (due to the multiplicative nature of traces / partition functions) is indeed the expected result for two independent harmonic oscillators.

A.3 Fourier transform with respect to σ_1:

$$x_j(\sigma_2) := \int_0^{2\pi} e^{ij2\pi\sigma_1} X(\sigma_1, \sigma_2) \mathrm{d}\sigma_1 \implies X(\sigma_1, \sigma_2) = \sum_{j=-\infty}^{+\infty} e^{-ij2\pi\sigma_1} x_j(\sigma_2).$$

The partition function factorizes because

$$[\mathrm{d}X]_{P_1 P_2} = \prod_{j=-\infty}^{+\infty} [\mathrm{d}x_j]_{P_2}$$

and with periodic boundary conditions,

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left[X(-\partial_1^2 - \partial_2^2 + m^2)X\right] = \frac{1}{4\pi\alpha'} \sum_{j=-\infty}^{+\infty} \int \mathrm{d}\sigma_2 \left[x_j(-\partial_2^2 + m^2 + 4\pi^2 j^2)x_j\right].$$

Put together,

$$\int [\mathrm{d}X]_{P_1 P_2} \exp(-S) = \prod_{j=-\infty}^{+\infty} \int [\mathrm{d}x_j]_{P_2} \exp(-S_j), \quad S_j := \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma_2 \left[x_j(-\partial_2^2 + m^2 + 4\pi^2 j^2)x_j\right].$$

Each factor corresponds to a harmonic oscillator trace as in **A.1**.

A.4 Uncompactify the circle, so that the particle starts at $\tilde{\phi}_i = \phi_i$ and ends at $\tilde{\phi}_f = \phi_f + nL$, where $n \in \mathbb{Z}$. All paths need to be summed:

$$K(\phi_f, \phi_i) = \sum_n \mathcal{C}_n \tilde{K}(\phi_f + nL, \phi_i),$$

where $\tilde{K}$ is the propagator of an unconfined particle. Unitarity implies twisted periodicity, since $\phi_f + L$ and ϕ_f label the same point on the circle:

$$K(\phi_f + L, \phi_i) = e^{i\delta} K(\phi_f, \phi_i).$$

In terms of the uncompactification expansion, LHS is understood as the limit of $\sum \mathcal{C}_n \tilde{K}(\phi_f + nL + \Delta, \phi_i)$ as $\Delta \to L^-$. If the twisted periodicity were to hold for any ϕ_f, ϕ_i, the coefficients must satisfy $\mathcal{C}_n = e^{-i\delta}\mathcal{C}_{n-1}$. Therefore, $K(\phi_f, \phi_i) = \mathcal{C}_0 \sum_n e^{-in\delta} \tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i)$.

Now, the unconfined propagator is easily determined. The Hamiltonian is expressed using the conjugate coordinate $\pi := -i\partial/\partial\phi$:

$$\tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) = \int [\mathrm{d}\pi \mathrm{d}\phi]_{\tilde{\phi}_i, 0}^{\tilde{\phi}_f, T} \exp\left(i \int_0^T \mathrm{d}t \left(\pi\dot{\phi} - \frac{1}{2}\pi^2\right)\right) = \int [\mathrm{d}\phi]_{\tilde{\phi}_i, 0}^{\tilde{\phi}_f, T} \exp\left(i \int_0^T \frac{\dot{\phi}^2}{2} \mathrm{d}t\right).$$

Following the standard procedure, the classical contribution is found to be

$$S_{\mathrm{cl}} = \frac{1}{2} \int_0^T \dot{\phi}_{\mathrm{cl}}^2 \mathrm{d}t = \frac{(\tilde{\phi}_f - \tilde{\phi}_i)^2}{2T}.$$

The quantum variations satisfy $\phi(0) = \phi(T) = 0$, so $\phi_k(t) \sim \sin k\pi t/T$, with $k = 1, 2, \cdots$

$$\det \Delta = \prod_{k=1}^{\infty} \frac{4k^2\pi^2}{T^2} \rightsquigarrow \prod_{k=1}^{\infty} \frac{4k^2\pi^2}{4k^2\pi^2 + M^2T^2} = \frac{MT}{2\sinh \frac{MT}{2}}.$$

As $T \to \infty$,

$$\tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) \sim (\det \Delta)^{-1/2} e^{iS_{\rm cl}} \sim \left(\frac{1}{T}\right)^{1/2} \exp\left(iS_{\rm cl} - S_{\rm ct} + \frac{1}{2}MT - \ln M\right).$$

The finite normalization is fixed by requiring

$$\lim_{T\to 0} \tilde{K}(\tilde{\phi}_f, \phi_i) = \delta(\tilde{\phi}_f - \tilde{\phi}_i) \implies \tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) = \frac{e^{i(\tilde{\phi}_f - \tilde{\phi}_i)^2/2T}}{\sqrt{2\pi i T}}.$$

For the propagator on the circle, the normalization is fixed by requiring

$$\lim_{T\to 0} K(\phi_f, \phi_i) = \sum_n \delta(\phi_f - \phi_i + nL) \implies \mathcal{C}_0 = e^{-i(\phi_f - \phi_i)\delta/L}.$$

Finally,

$$K(\phi_f, \phi_i) = \frac{1}{\sqrt{2\pi i T}} \sum_n \exp\left(i\frac{(\phi_f - \phi_i + nL)^2}{2T} - i\frac{(\phi_f - \phi_i + nL)\delta}{L}\right).$$

A.5 For the periodic case,

$$\int \mathrm{d}\psi \langle \psi, U | \psi, 0\rangle = \int [\mathrm{d}\psi \mathrm{d}\chi]_P \exp\left(-\int_0^U \mathrm{d}u \left(\chi\dot{\psi} + m\chi\psi\right)\right).$$

A basis of eigenfunctions of $\Delta = -\partial_u - m$:

$$\psi_k \sim \exp\frac{+2ik\pi u}{U}, \quad \Delta\psi_k = \left(-m - \frac{2ik\pi}{U}\right)\psi_k, \quad k \in \mathbb{Z},$$
$$\chi_k \sim \exp\frac{-2ik\pi u}{U}, \quad \Delta^T\chi_k = \left(-m + \frac{2ik\pi}{U}\right)\chi_k, \quad k \in \mathbb{Z}.$$

Regularization:

$$\det\Delta = \prod_{k=-\infty}^{+\infty} \frac{-mU - 2ik\pi}{U} \rightsquigarrow \prod_{k=-\infty}^{+\infty} \frac{-mU - 2ik\pi}{-MU - 2ik\pi} = \frac{\sinh\frac{mU}{2}}{\sinh\frac{MU}{2}}.$$

As $M \to \infty$,

$$\int [\mathrm{d}\psi\mathrm{d}\chi]_P \exp(-S_E) = 2\sinh\frac{mU}{2} \exp\left(-S_{\rm ct} - \frac{1}{2}MU\right).$$

The finite normalization is fixed by requiring $\mathrm{Tr}[(-)^F \exp(-\hat{H}U)] \sim \exp(-E_0 U)$ as $U \to \infty$. The energy eigenstates are $\hat{H}|\uparrow\rangle = m|\uparrow\rangle$ and $\hat{H}|\downarrow\rangle = 0$.

$$\int [\mathrm{d}\psi\mathrm{d}\chi]_P \exp(-S_E) = 2e^{-mU/2}\sinh\frac{mU}{2} = 1 - e^{-mU},$$

which matches the sum over states.

For the anti-periodic case,

$$\int \mathrm{d}\psi \langle \psi, U | \psi, 0\rangle = \int [\mathrm{d}\psi \mathrm{d}\chi]_A \exp\left(-\int_0^U \mathrm{d}u \left(\chi\dot{\psi} + m\chi\psi\right)\right).$$

A basis of eigenfunctions of $\Delta = -\partial_u - m$:

$$\psi_k \sim \exp\frac{+i(2k+1)\pi u}{U}, \quad \Delta\psi_k = \left(-m - \frac{2ik\pi}{U}\right)\psi_k, \quad k \in \mathbb{Z},$$
$$\chi_k \sim \exp\frac{-i(2k+1)\pi u}{U}, \quad \Delta^T\chi_k = \left(-m + \frac{2ik\pi}{U}\right)\chi_k, \quad k \in \mathbb{Z}.$$

Regularization:

$$\det \Delta = \prod_{k=-\infty}^{+\infty} \frac{-mU - i(2k+1)\pi}{U} \rightsquigarrow \prod_{k=-\infty}^{+\infty} \frac{-mU - i(2k+1)\pi}{-MU - i(2k+1)\pi} = \frac{\cosh \frac{mU}{2}}{\cosh \frac{MU}{2}}.$$

As $M \to \infty$,

$$\int [\mathrm{d}\psi \mathrm{d}\chi]_A \exp(-S_E) = 2\cosh \frac{mU}{2} \exp\left(-S_{\mathrm{ct}} - \frac{1}{2}MU\right).$$

The finite normalization is fixed by requiring $\mathrm{Tr}\exp(-\hat{H}U) \sim \exp(-E_0 U)$ as $U \to \infty$. The energy eigenstates are $\hat{H}|\uparrow\rangle = m|\uparrow\rangle$ and $\hat{H}|\downarrow\rangle = 0$.

$$\int [\mathrm{d}\psi \mathrm{d}\chi]_A \exp(-S_E) = 2e^{-mU/2} \cosh \frac{mU}{2} = 1 + e^{-mU},$$

which matches the sum over states.

Chapter 1. A First Look at Strings

1.1 (a). Use diffeomorphism invariance to fix $\tau = X^0 \equiv t$ so that $\dot{X}^0 = 1$ and $\dot{X}^i = v^i$. The action becomes

$$S_{\mathrm{pp}} = -m \int \mathrm{d}t \sqrt{1 - \mathbf{v}^2} \approx \int \mathrm{d}t \left(\frac{1}{2}m\mathbf{v}^2 - m\right).$$

Interpret the Lagrangian as $L = T - V$; the potential energy is the rest mass.

(b). Again, take $\tau = X^0$. Recall the definition of the induced metric $h_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

$$\begin{aligned} h_{\tau\tau} &= (1, \mathbf{v}) \cdot (1, \mathbf{v}) = -1 + \mathbf{v}^2, \\ h_{\tau\sigma} &= (1, \mathbf{v}) \cdot (0, \partial_\sigma \mathbf{X}) = \mathbf{v} \cdot \partial_\sigma \mathbf{X}, \\ h_{\sigma\sigma} &= (0, \partial_\sigma \mathbf{X}) \cdot (0, \partial_\sigma \mathbf{X}) = (\partial_\sigma \mathbf{X})^2. \end{aligned}$$

The Nambu-Goto action becomes

$$\begin{aligned} S_{\mathrm{NG}} &= -\frac{1}{2\pi\alpha'} \int \mathrm{d}t\mathrm{d}\sigma \sqrt{(1 - \mathbf{v}^2)(\partial_\sigma \mathbf{X})^2 + (\mathbf{v} \cdot \partial_\sigma \mathbf{X})^2} \\ &\approx -\frac{1}{2\pi\alpha'} \int \mathrm{d}t\mathrm{d}\sigma |\partial_\sigma \mathbf{X}| \left(1 + \frac{(\mathbf{v} \cdot \partial_\sigma \mathbf{X})^2 - \mathbf{v}^2(\partial_\sigma \mathbf{X})^2}{2(\partial_\sigma \mathbf{X})^2}\right). \end{aligned}$$

To understand the above formula, consider the embedded length $|\mathrm{d}\mathbf{X}| = |\partial_\sigma \mathbf{X}|\mathrm{d}\sigma$ of the string in Minkowski spacetime at a fixed time t. Decompose the velocity $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$ according to its direction with respect to the string. By definition, $\mathbf{v}_\perp \cdot \partial_\sigma \mathbf{X} = 0$ and $\mathbf{v}_\parallel \cdot \partial_\sigma \mathbf{X} = \pm|\mathbf{v}_\parallel||\partial_\sigma \mathbf{X}|$. This way, the parenthesis evaluates to $(1 - \mathbf{v}_\perp^2/2)$.

$$S_{\mathrm{NG}} \approx \int \mathrm{d}t |\mathrm{d}\mathbf{X}| \frac{1}{2\pi\alpha'} \left(\frac{\mathbf{v}_\perp^2}{2} - 1\right).$$

Comparing with $L = T - V$ and recalling from (a) that the potential energy is the rest mass, the mass density is found to be $T = 1/2\pi\alpha'$. Then, it's clear that only transverse velocity contributes to the kinetic energy.

1.2 The ends of an open string satisfy the Neumann boundary conditions:

$$\partial^\sigma X^\mu|_{\sigma=0,\ell} = 0.$$

The equations of motion $T^{\sigma\sigma} = 0$ and $T^{\sigma\tau} = 0$ at the ends of the string imply:

$$\begin{aligned} \frac{1}{2}\gamma^{\sigma\sigma} \dot{X}^\mu \dot{X}_\mu &= \partial^\sigma X^\mu \partial^\sigma X_\mu = 0, \\ \frac{1}{2}\gamma^{\sigma\tau} \dot{X}^\mu \dot{X}_\mu &= \partial^\sigma X^\mu \partial^\tau X_\mu = 0. \end{aligned}$$

Since γ_{ab} is non-degenerate, $\gamma^{\sigma\sigma}$ and $\gamma^{\sigma\tau}$ cannot both vanish. Hence, at least one of the above equation implies $\dot{X}^\mu \dot{X}_\mu = 0$, i.e., the ends of the string move at the speed of light.

1.3 Under a Weyl transformation $\gamma_{ab} \to \gamma'_{ab} = e^{2\omega}\gamma_{ab}$, the induced metric $\mathrm{d}s$ on the boundary transforms as $\mathrm{d}s \to \mathrm{d}s' = e^{\omega}\mathrm{d}s$, since the boundary is one-dimensional. The unit vectors are normalized such that $\gamma_{ab}t^a t^b = \gamma^{ab}n_a n_b = \mp 1$, where the upper sign corresponds to time-like boundaries and the lower sign corresponds to space-like boundaries. Under a Weyl transformation, due to the unit-length constraint, $t'^a = e^{-\omega}t^a$ and $n'_a = e^{\omega}n_a$.

On the other hand, it's easily checked that $\Gamma'^b_{ac} = \Gamma^b_{ac} + \delta^b_a\partial_c\omega + \delta^b_c\partial_a\omega - \gamma_{ac}\partial^b\omega$. Putting everything together and using $t^a n_a = 0$,

$$k' = \pm t'^a n'_b(\partial_a t'^b + \Gamma'^b_{ac}t'^c) = e^{-\omega}(k + n_b\partial^b\omega) = e^{-\omega}(k + n_b\partial^b\omega).$$

$$\begin{aligned}
\chi' &= \frac{1}{4\pi}\int_M \mathrm{d}\tau\mathrm{d}\sigma\sqrt{|\gamma|}(R - 2\nabla^2\omega) + \frac{1}{2\pi}\int_{\partial M}\mathrm{d}s(k + n_b\partial^b\omega) \\
&= \chi - \frac{1}{2\pi}\int_M \mathrm{d}\tau\mathrm{d}\sigma\sqrt{|\gamma|}\nabla^2\omega + \frac{1}{2\pi}\int_M \mathrm{d}\tau\mathrm{d}\sigma\sqrt{|\gamma|}\nabla_b\partial^b\omega \\
&= \chi,
\end{aligned}$$

where, in the last step, we used the fact that $\partial^b\omega = \nabla^b\omega$ for the scalar field ω.

1.4 At level $m^2 = 1/\alpha'$ ($N = 2$), the raising operators form irreducible representations of $SO(D-2)$:

vector	$\boldsymbol{(D-2)}$	α^i_{-2}
scalar	$\boldsymbol{1}$	$\sum_i(\alpha^i_{-1})^2$
traceless symmetric tensor	$\boldsymbol{D(D-3)/2}$	$\alpha^i_{-1}\alpha^j_{-1} - \frac{1}{D-2}\sum_i(\alpha^i_{-1})^2$

Counting dimensions, these fit into the traceless symmetric tensor representation of $SO(D-1)$ nicely.

$$\underbrace{\boldsymbol{(D-2)} \oplus \boldsymbol{1} \oplus \frac{\boldsymbol{D(D-3)}}{\boldsymbol{2}}}_{SO(D-2)} = \underbrace{\frac{\boldsymbol{(D+1)(D-2)}}{\boldsymbol{2}}}_{SO(D-1)}.$$

At level $m^2 = 2/\alpha'$ ($N = 3$), the raising operators are $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}$, $\alpha^i_{-2}\alpha^j_{-1}$ and α^i_{-3}. Dimension-counting suggests these form a traceless symmetric rank-3 tensor and an antisymmetric rank-2 tensor.

$$\frac{D(D-1)(D-2)}{6} + (D-2)^2 + (D-2) = \underbrace{\frac{\boldsymbol{(D-1)(D-2)(D+3)}}{\boldsymbol{6}} \oplus \frac{\boldsymbol{(D-1)(D-2)}}{\boldsymbol{2}}}_{SO(D-1)}.$$

At higher levels, it's easier to use the Young tableaux. For instance, at level $N = 4$,

4	(1)	(1)
31	$(1) \otimes (1)$	$(2) \oplus () \oplus (11)$
22	$(1) \otimes_S (1)$	$(2) \oplus ()$
211	$(1) \otimes [(1) \otimes_S (1)]$	$(3) \oplus (1) \oplus (21) \oplus (1)$
1111	$(1) \otimes_S (1) \otimes_S (1) \otimes_S (1)$	$(4) \oplus (2) \oplus ()$

where the first column indicates which raising operators to use (a 3 means an α^i_{-3}), the second column illustrates the symmetry relations between them, and the final column lists the irreducible representations. The irreducible representations are labeled by (ab), the Young diagram with a boxes in the first row and b boxes in the second row. Using the $SO(D-1) \supset SO(D-2)$ branching rules (e.g., $(2) = (2) \oplus (1) \oplus ()$ and $(21) = (21) \oplus (11) \oplus (2) \oplus (1)$), we see that these states form the $SO(D-1)$ representation $(4) \oplus (21) \oplus (2) \oplus ()$.

1.5 Using the exponential regulator,

$$\sum_{n=1}^{+\infty}(n-\theta)e^{-\epsilon(n-\theta)} = -\partial_\epsilon\sum_{n=1}^{\infty}e^{-\epsilon(n-\theta)} = \partial_\epsilon\frac{e^{\epsilon\theta}}{1-e^{\epsilon}} = \frac{1}{\epsilon^2} - \frac{1}{12}(1 - 6\theta + 6\theta^2) + \mathcal{O}(\epsilon).$$

Remark. Knowing that any divergence would be removed by renormalization, we may also use zeta-function regularization. However, a naive guess fails to give the correct answer:

$$\sum_{n=1}^{+\infty}(n-\theta)=\sum_{n=1}^{+\infty}n-\theta\sum_{n=1}^{+\infty}1=\lim_{s\to-1}\left(\sum_{n=1}^{+\infty}\frac{1}{n^s}-\theta\sum_{n=1}^{+\infty}\frac{1}{n^{s+1}}\right)=\zeta(-1)-\theta\zeta(0)=-\frac{1}{12}+\frac{\theta}{2}.$$

To get the correct answer, perform the Taylor expansion after applying the regulator:

$$\begin{aligned}\sum_{n=1}^{+\infty}(n-\theta)&=\lim_{s\to-1}\sum_{n=1}^{+\infty}\frac{1}{(n-\theta)^s}=\lim_{s\to-1}\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}+\frac{s\theta}{n^{1+s}}+\frac{s(1+s)\theta^2}{2n^{2+s}}+\underbrace{\frac{s(1+s)(2+s)\theta^3}{6n^{2+s}}+\cdots}_{\lim_{s\to-1}=0\cdot\text{finite}=0}\right)\\&=\zeta(-1)-\theta\zeta(0)-\frac{\theta^2}{2}\lim_{s\to-1}(1+s)\zeta(2+s)=-\frac{1}{12}+\frac{\theta}{2}-\frac{\theta^2}{2}.\end{aligned}$$

1.6 From the boundary condition $\partial^\sigma X^+|_{\sigma=0,\ell}=0$, we still have $\gamma_{\tau\sigma}|_{\sigma=0,\ell}=0$, which implies $\gamma_{\tau\sigma}\equiv 0$. Therefore, everything proceeds as p.19, except for a different boundary condition for X^{25}.

For X^I with $I=2,\cdots,24$, the mode expansion is the same as the Neumann open string, with homogeneous boundary conditions: (set $c=1$)

$$X^I(\tau,\sigma)=\underbrace{x^I+\frac{p^I}{p^+}\tau+i\sqrt{\frac{1}{\pi T}}\sum_{n\neq0}\frac{\alpha_n^I}{n}\cos\frac{n\pi\sigma}{\ell}\exp\frac{-in\pi\tau}{\ell}}_{\text{general sol.}}.$$

For X^{25} satisfying the Dirichlet boundary condition, a special solution is $\tilde{X}^{25}(\tau,\sigma)=y\sigma/\ell$.

$$X^{25}(\tau,\sigma)=\underbrace{\frac{y}{\ell}\sigma}_{\text{special sol.}}+\underbrace{\sqrt{\frac{1}{\pi T}}\sum_{n\neq0}\frac{\beta_n}{n}\sin\frac{n\pi\sigma}{\ell}\exp\frac{-in\pi\tau}{\ell}}_{\text{general sol.}}.$$

The mass spectrum satisfies $H=(p^Ip^I+m^2)/2p^+$ where $I=2,\cdots,24$, since the average momentum $p^{25}=0$.

$$\begin{aligned}H&=\frac{T}{2}\int_0^\ell \mathrm{d}\sigma\left(\partial_\tau X^i\partial_\tau X^i+\partial_\sigma X^i\partial_\sigma X^i\right)\\&=\frac{T}{2}\int_0^\ell \mathrm{d}\sigma\left(\frac{p^Ip^I}{p^{+2}}+\frac{\pi^2}{\ell^2}\frac{1}{\pi T}(2\alpha_{-n}^I\alpha_n^I+23n+2\beta_{-n}\beta_n+n)+\frac{y^2}{\ell^2}\right).\end{aligned}$$

Define the level operator

$$N:=\sum_{n=1}^{+\infty}(\alpha_{-n}^I\alpha_n^I+\beta_{-n}\beta_n),$$

$$\implies m^2=\frac{1}{\alpha'}\left(N+12\sum_{n=1}^{+\infty}n\right)+y^2T^2=\frac{1}{\alpha'}(N-1)+y^2T^2.$$

1.7 Same as before, the mode expansion for X^I is still

$$X^I(\tau,\sigma)=\underbrace{x^I+\frac{p^I}{p^+}\tau+i\sqrt{\frac{1}{\pi T}}\sum_{n\neq0}\frac{\alpha_n^I}{n}\cos\frac{n\pi\sigma}{\ell}\exp\frac{-in\pi\tau}{\ell}}_{\text{general sol.}},$$

while for X^{25},

$$X^{25}(\tau,\sigma)=\underbrace{\sqrt{\frac{1}{\pi T}}\sum_{n=1}^{+\infty}\frac{\beta_{n-1/2}}{n-1/2}\sin\frac{(n-1/2)\pi\sigma}{\ell}\exp\frac{-i(n-1/2)\pi\tau}{\ell}+\text{c.c.}}_{\text{general sol.}}$$

The mass spectrum satisfies $H = (p^I p^I + m^2)/2p^+$ where $I = 2, \cdots, 24$, since the average momentum $p^{25} = 0$.

$$\begin{aligned}
H &= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma \left(\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i\right) \\
&= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma \left(\frac{p^I p^I}{p^{+2}} + \frac{\pi^2}{\ell^2}\frac{1}{\pi T}(2\alpha^I_{-n}\alpha^I_n + 23n + 2\beta_{-(n-1/2)}\beta_{n-1/2} + n - 1/2)\right).
\end{aligned}$$

Define the level operator

$$N := \sum_{n=1}^{+\infty}(\alpha^I_{-n}\alpha^I_n + \beta_{-(n-1/2)}\beta_{n-1/2}),$$

$$\implies m^2 = \frac{1}{\alpha'}\left(N + \frac{23}{2}\sum_{n=1}^{+\infty} n + \frac{1}{2}\sum_{n=1}^{+\infty}(n - 1/2)\right) = \frac{1}{\alpha'}\left(N - \frac{15}{16}\right).$$

In the last step, it's crucial to evaluate the regulated infinite sum for each dimension separately, because the regulator and hence the regulated sum is not linear.

1.8 By choosing the reference curve $\sigma = 0$ on the world-sheet to be perpendicular to constant-τ slices, it's possible to fix $\gamma_{\tau\sigma}|_{\sigma=0} = 0$, which (together with the equation of motion $\partial_\sigma^2 \gamma_{\tau\sigma} = 0$ and periodicity $\gamma_{\tau\sigma}|_{\sigma=0} = \gamma_{\tau\sigma}|_{\sigma=\ell}$) implies $\gamma_{\tau\sigma} \equiv 0$.

Mode expansion:

$$X^I(\tau,\sigma) = \underbrace{x^I + \frac{p^I}{p^+}\tau + i\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^I_n e^{-2in\pi\sigma/\ell} + \tilde{\alpha}^I_n e^{+2in\pi\sigma/\ell}}{2}\exp\frac{-2in\pi\tau}{\ell}}_{\text{general sol.}},$$

$$X^{25}(\tau,\sigma) = \underbrace{\frac{2\pi R}{\ell}\left(\sigma - \frac{\ell}{2}\right)}_{\text{special sol.}} + \underbrace{x^{25} + \frac{p^{25}}{p^+}\tau + i\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha^{25}_n e^{-2in\pi\sigma/\ell} + \tilde{\alpha}^{25}_n e^{+2in\pi\sigma/\ell}}{2}\exp\frac{-2in\pi\tau}{\ell}}_{\text{general sol.}}.$$

The mass spectrum satisfies $H = (p^i p^i + m^2)/2p^+$ where $i = 2, \cdots, 25$.

$$\begin{aligned}
H &= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma \left(\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i\right) \\
&= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma \left(\frac{p^i p^i}{p^{+2}} + \frac{4\pi^2}{\ell^2}\frac{2}{4\pi T}(2\alpha^i_{-n}\alpha^i_n + 2\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + 48n) + \frac{(2\pi R)^2}{\ell^2}\right).
\end{aligned}$$

Define the level operators:

$$N := \sum_{n=1}^{+\infty}\alpha^i_{-n}\alpha^i_n, \quad \tilde{N} := \sum_{n=1}^{+\infty}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n,$$

$$\implies m^2 = \frac{2}{\alpha'}\left(N + \tilde{N} + 24\sum_{n=1}^{+\infty} n\right) = \frac{2}{\alpha'}(N + \tilde{N} - 2).$$

The world-sheet momentum that generates σ-translations is

$$P = -\int_0^\ell \mathrm{d}\sigma \Pi_i \partial_\sigma X^i = -\frac{2\pi}{\ell}(N - \tilde{N}) - \int_0^\ell \Pi^{25}\frac{2\pi R}{\ell}\mathrm{d}\sigma = -\frac{2\pi}{\ell}(N - \tilde{N}) - p^{25}\frac{2\pi R}{\ell}.$$

Invariance under σ-translations implies

$$N - \tilde{N} + p^{25}R = 0.$$

Therefore, p^{25} is quantized as expected for a finite quantum system. Moreover, we see that for the winding string in periodic compactification, N and $\tilde{N}$ no longer have to be equal.

1.9 Same as before, the mode expansion is:

$$X^I(\tau,\sigma) = \underbrace{x^I + \frac{p^I}{p^+}\tau + i\sqrt{\frac{1}{\pi T}}\sum_{n\neq 0}\frac{1}{n}\frac{\alpha_n^I e^{-2in\pi\sigma/\ell} + \tilde{\alpha}_n^I e^{+2in\pi\sigma/\ell}}{2}\exp\frac{-2in\pi\tau}{\ell},}_{\text{general sol.}}$$

$$X^{25}(\tau,\sigma) = \underbrace{i\sqrt{\frac{1}{\pi T}}\sum_{n=1}^{+\infty}\frac{1}{n-1/2}\frac{\alpha_{n-1/2}^{25} e^{-2i(n-1/2)\pi\sigma/\ell} + \tilde{\alpha}_{n-1/2}^{25} e^{+2i(n-1/2)\pi\sigma/\ell}}{2}\exp\frac{-2i(n-1/2)\pi\tau}{\ell}}_{\text{general sol.}}.$$

The mass spectrum satisfies $H = (p^I p^I + m^2)/2p^+$ where $I = 2, \cdots, 24$, since the average momentum $p^{25} = 0$.

$$\begin{aligned} H &= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma\left(\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i\right) \\ &= \frac{T}{2}\int_0^\ell \mathrm{d}\sigma\left(\frac{p^I p^I}{p^{+2}} + \frac{4\pi^2}{\ell^2}\frac{2}{4\pi T}(2\alpha_{-n}^I\alpha_n^I + 23n + 2\alpha_{-(n-1/2)}^{25}\alpha_{n-1/2}^{25} + n - 1/2 + \text{terms with tilde})\right). \end{aligned}$$

Define the level operators:

$$N := \sum_{n=1}^{+\infty}(\alpha_{-n}^I\alpha_n^I + \alpha_{-(n-1/2)}^{25}\alpha_{n-1/2}^{25}), \quad \tilde{N} := \sum_{n=1}^{+\infty}(\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i + \tilde{\alpha}_{-(n-1/2)}^{25}\tilde{\alpha}_{n-1/2}^{25}),$$

$$\Longrightarrow\ m^2 = \frac{2}{\alpha'}\left(N + \tilde{N} + 23\sum_{n=1}^{+\infty} n + \sum_{n=1}^{+\infty}(n-1/2)\right) = \frac{2}{\alpha'}\left(N + \tilde{N} - \frac{15}{8}\right) = \frac{4}{\alpha'}\left(N - \frac{15}{16}\right).$$

In the last step, we used the constraint $P = -2\pi(N - \tilde{N})/\ell = 0$.