My Solution to Polchinski's *String Theory* Books

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Appendix A. A Short Course on Path Integrals

A.1 (a). Following the harmonic oscillator example on p.339, choose a complete orthonormal basis of periodic eigenfunctions $\{\Phi_j\}$ such that $\Delta\Phi_j = (-\partial_u^2 + \omega^2)\Phi_j = \lambda_j\Phi_j$:

$$\Phi_0(u) = \left(\frac{1}{U}\right)^{1/2}, \quad \Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \begin{matrix}\sin\\ \cos\end{matrix}\, \frac{2j\pi u}{U}, \quad j = 1, 2, \cdots$$

Using the Pauli-Villars regulator ($\Omega \gg \omega$), the functional determinant becomes (up to a formal constant):

$$\det\Delta = \prod_{j=-\infty}^{+\infty} \lambda_j = \omega^2 \left(\prod_{j=1}^{\infty} \frac{4j^2\pi^2 + \omega^2U^2}{U^2}\right)^2 \rightsquigarrow \omega^2 \left(\prod_{j=1}^{\infty} \frac{4j^2\pi^2 + \omega^2U^2}{4j^2\pi^2 + \Omega^2U^2}\right)^2 = \omega^2 \left(\frac{\Omega\sinh\frac{\omega U}{2}}{\omega\sinh\frac{\Omega U}{2}}\right)^2.$$

As $\Omega \to \infty$,

$$\mathrm{Tr}\exp(-\hat{H}U) = \int [\mathrm{d}q]_P \exp(-S_E) \sim (\det\Delta)^{-1/2} \sim \frac{1}{2\sinh\frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U - \ln\Omega\right).$$

The normalization is fixed by requiring $\mathrm{Tr}\exp(-\hat{H}U) \sim \exp(-E_0U) = \exp(-\omega U/2)$ as $U \to \infty$, which implies that $\mathrm{Tr}\exp(-\hat{H}U) = (2\sinh\omega U/2)^{-1}$. Indeed,

$$\mathrm{Tr}\exp(-\hat{H}U) = \sum_{j=0}^{\infty} \exp\left(-(j+1/2)\omega U\right) = \frac{1}{2\sinh\frac{\omega U}{2}}.$$

The counterterm consists of the vacuum energy $\int \frac{1}{2}\Omega\mathrm{d}u$ and a wavefunction renormalization $-\ln\Omega$.

(b). For anti-periodic configurations, a basis of eigenfunctions is:

$$\Phi_{\pm j}(u) = \left(\frac{2}{U}\right)^{1/2} \begin{matrix}\sin\\ \cos\end{matrix}\, \frac{(2j+1)\pi u}{U}, \quad j = 1, 2, \cdots$$

Proceeding as before,

$$\det\Delta = \left(\prod_{j=1}^{\infty} \frac{(2j+1)^2\pi^2 + \omega^2U^2}{U^2}\right)^2 \rightsquigarrow \left(\prod_{j=1}^{\infty} \frac{(2j+1)^2\pi^2 + \omega^2U^2}{(2j+1)^2\pi^2 + \Omega^2U^2}\right)^2 = \left(\frac{\cosh\frac{\omega U}{2}}{\cosh\frac{\Omega U}{2}}\right)^2.$$

As $\Omega \to \infty$

$$\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] = \int [\mathrm{d}q]_A \exp(-S_E) \sim (\det\Delta)^{-1/2} \sim \frac{1}{2\cosh\frac{\omega U}{2}} \exp\left(-S_{\mathrm{ct}} + \frac{1}{2}\Omega U\right).$$

The normalization is fixed by requiring $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] \sim \exp(-E_0U) = \exp(-\omega U/2)$ as $U \to \infty$, which implies that $\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}] = (2\cosh\omega U/2)^{-1}$. Indeed,

$$\mathrm{Tr}\left[\exp(-\hat{H}U)\hat{R}\right] = \sum_{j=0}^{\infty} (-)^j \exp\left(-(j+1/2)\omega U\right) = \frac{1}{2\cosh\frac{\omega U}{2}}.$$

This time, the counterterm consists only of the vacuum energy.

Remark. The role of renormalization here is to hide the details of divergence cancellation, so that one may focus on the essentials (i.e. computing the eigenvalues). This is similar to what happens in QFT, where instead of $\det\Delta$, we compute Feynman diagrams (propagators), and instead of known ground states, we use experimental data (relations between amplitudes of various processes) to fix the normalization. The price we pay is that, in general, renormalizability needs to be proved.

Thoughts. In (b), there is no wavefunction renormalization. At the same time, the "ground state" $q(u) = \text{const.}$ doesn't contribute since it violates the boundary conditions. Are these two facts somehow related?

A.2 Including possible counterterms, the Euclidean action:

$$S_E = \int_0^U \mathrm{d}u\,\left((\partial_u q^*)(\partial_u q) + \omega^2 q^* q\right) + S_{\text{ct}} \implies \int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \exp(-S_{\text{ct}})\,(\det\Delta)^{-1}.$$

A basis of eigenvectors satisfying the boundary condition $q(U) = q(0)e^{i\theta}$ is

$$\Phi_j(u) = \left(\frac{1}{U}\right)^{1/2} \exp\left(iu\frac{\theta + 2j\pi}{U}\right), \quad j = 0, \pm1, \pm2, \cdots$$

The functional determinant is regularized using the Pauli-Villars regulator:

$$\det\Delta = \prod_{j=-\infty}^{+\infty} \frac{(\theta+2j\pi)^2 + \omega^2U^2}{U^2} \rightsquigarrow \prod_{j=-\infty}^{+\infty} \frac{(\theta+2j\pi)^2 + \omega^2U^2}{(\theta+2j\pi)^2 + \Omega^2U^2} = \frac{\cosh\omega U - \cos\theta}{\cosh\Omega U - \cos\theta}.$$

As $\Omega \to \infty$,

$$\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \frac{1}{2(\cosh\omega U - \cos\theta)} \exp\left(-S_{\text{ct}} + \Omega U\right).$$

To fix the normalization, notice that $\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) = \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}]$, where the twist operator is defined by $\hat{R}_\theta|q\rangle = |e^{i\theta}q\rangle$. This is because

$$\begin{aligned}
\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) &= \int \mathrm{d}q\mathrm{d}q^* \int [\mathrm{d}q\mathrm{d}q^*]_{q,0}^{e^{i\theta}q,U} \exp(-S_E) \\
&= \int \mathrm{d}q\mathrm{d}q^* \langle e^{i\theta}q, U|q, 0\rangle_E = \int \mathrm{d}q\mathrm{d}q^* \langle q, U|\hat{R}_{-\theta}|q, 0\rangle_E \\
&= \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}].
\end{aligned}$$

Therefore, for two harmonic oscillators, $\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) \sim \exp(-E_0U) = \exp(-\omega U)$ as $U \to \infty$. Hence the counterterm is the vacuum energy $2\int \frac{1}{2}\Omega \mathrm{d}u$ corresponding to the two "virtual" harmonic oscillators with frequency Ω.

$$\int [\mathrm{d}q\mathrm{d}q^*]_\theta \exp(-S_E) = \mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}] = \frac{1}{2(\cosh\omega U - \cos\theta)}.$$

To compare with the operational method, define $a_\pm := (a_1 \pm ia_2)/\sqrt{2}$ so that $a_\pm^\dagger = (a_1^\dagger \mp ia_2^\dagger)/\sqrt{2}$.

$$\hat{H} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) = \omega(a_+^\dagger a_+ + a_-^\dagger a_- + 1).$$

Since $\hat{q} \sim (a_1 + a_1^\dagger) + i(a_2 + a_2^\dagger) = (a_+ + a_-^\dagger)$ and $\hat{R}_{-\theta}\hat{q}\hat{R}_\theta = e^{i\theta}\hat{q}$, the holomorphic and anti-holomorphic operators (now considered independent) transform as $\hat{R}_{-\theta}a_\pm\hat{R}_\theta = e^{\pm i\theta}a_\pm$. Label the states with $a_\pm$ quantum numbers such that $|m,n\rangle := \frac{a_+^{\dagger m}a_-^{\dagger n}}{\sqrt{m!n!}}|0,0\rangle$.

$$\begin{aligned}
\mathrm{Tr}[\exp(-\hat{H}U)\hat{R}_{-\theta}] &= \sum_{m,n=0}^{\infty} \langle m,n|\exp(-\hat{H}U)\hat{R}_{-\theta}|m,n\rangle \\
&= \sum_{m,n=0}^{\infty} \langle m,n|\exp(-\omega(m+n+1)U - i\theta(m-n))|m,n\rangle \\
&= \frac{1}{2(\cosh\omega U - \cos\theta)}.
\end{aligned}$$

If $\theta = 0(\pi)$, the answer reduces to

$$\frac{1}{2(\cosh \omega U \mp 1)} = \left(2 \begin{matrix}\sinh \\ \cosh\end{matrix} \frac{\omega U}{2}\right)^{-2},$$

which (due to the multiplicative nature of traces / partition functions) is indeed the expected result for two independent harmonic oscillators.

A.3 Fourier transform with respect to σ_1:

$$x_j(\sigma_2) := \int_0^{2\pi} e^{ij2\pi\sigma_1} X(\sigma_1, \sigma_2) \mathrm{d}\sigma_1 \implies X(\sigma_1, \sigma_2) = \sum_{j=-\infty}^{+\infty} e^{-ij2\pi\sigma_1} x_j(\sigma_2).$$

The partition function factorizes because

$$[\mathrm{d}X]_{P_1 P_2} = \prod_{j=-\infty}^{+\infty} [\mathrm{d}x_j]_{P_2}$$

and with periodic boundary conditions,

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left[X(-\partial_1^2 - \partial_2^2 + m^2)X\right] = \frac{1}{4\pi\alpha'} \sum_{j=-\infty}^{+\infty} \int \mathrm{d}\sigma_2 \left[x_j(-\partial_2^2 + m^2 + 4\pi^2 j^2) x_j\right].$$

Put together,

$$\int [\mathrm{d}X]_{P_1 P_2} \exp(-S) = \prod_{j=-\infty}^{+\infty} \int [\mathrm{d}x_j]_{P_2} \exp(-S_j), \quad S_j := \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma_2 \left[x_j(-\partial_2^2 + m^2 + 4\pi^2 j^2) x_j\right].$$

Each factor corresponds to a harmonic oscillator trace as in **A.1**.

A.4 Uncompactify the circle, so that the particle starts at $\tilde{\phi}_i = \phi_i$ and ends at $\tilde{\phi}_f = \phi_f + nL$, where $n \in \mathbb{Z}$. All paths need to be summed:

$$K(\phi_f, \phi_i) = \sum_n \mathcal{C}_n \tilde{K}(\phi_f + nL, \phi_i),$$

where $\tilde{K}$ is the propagator of an unconfined particle. Unitarity implies twisted periodicity, since $\phi_f + L$ and ϕ_f label the same point on the circle:

$$K(\phi_f + L, \phi_i) = e^{i\delta} K(\phi_f, \phi_i).$$

In terms of the uncompactification expansion, LHS is understood as the limit of $\sum \mathcal{C}_n \tilde{K}(\phi_f + nL + \Delta, \phi_i)$ as $\Delta \to L^-$. If the twisted periodicity were to hold for any ϕ_f, ϕ_i, the coefficients must satisfy $\mathcal{C}_n = e^{-i\delta}\mathcal{C}_{n-1}$. Therefore, $K(\phi_f, \phi_i) = \mathcal{C}_0 \sum_n e^{-in\delta} \tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i)$.

Now, the unconfined propagator is easily determined. The Hamiltonian is expressed using the conjugate coordinate $\pi := -i\partial/\partial\phi$:

$$\tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) = \int [\mathrm{d}\pi \mathrm{d}\phi]_{\tilde{\phi}_i, 0}^{\tilde{\phi}_f, T} \exp\left(i \int_0^T \mathrm{d}t \left(\pi\dot{\phi} - \frac{1}{2}\pi^2\right)\right) = \int [\mathrm{d}\phi]_{\tilde{\phi}_i, 0}^{\tilde{\phi}_f, T} \exp\left(i \int_0^T \frac{\dot{\phi}^2}{2} \mathrm{d}t\right).$$

Following the standard procedure, the classical contribution is found to be

$$S_{\mathrm{cl}} = \frac{1}{2} \int_0^T \dot{\phi}_{\mathrm{cl}}^2 \mathrm{d}t = \frac{(\tilde{\phi}_f - \tilde{\phi}_i)^2}{2T}.$$

The quantum variations satisfy $\phi(0) = \phi(T) = 0$, so $\phi_k(t) \sim \sin k\pi t/T$, with $k = 1, 2, \cdots$

$$\det \Delta = \prod_{k=1}^{\infty} \frac{4k^2\pi^2}{T^2} \rightsquigarrow \prod_{k=1}^{\infty} \frac{4k^2\pi^2}{4k^2\pi^2 + M^2T^2} = \frac{MT}{2\sinh \frac{MT}{2}}.$$

As $T \to \infty$,

$$\tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) \sim (\det \Delta)^{-1/2} e^{iS_{\rm cl}} \sim \left(\frac{1}{T}\right)^{1/2} \exp\left(iS_{\rm cl} - S_{\rm ct} + \frac{1}{2}MT - \ln M\right).$$

The finite normalization is fixed by requiring

$$\lim_{T\to 0} \tilde{K}(\tilde{\phi}_f, \phi_i) = \delta(\tilde{\phi}_f - \tilde{\phi}_i) \implies \tilde{K}(\tilde{\phi}_f, \tilde{\phi}_i) = \frac{e^{i(\tilde{\phi}_f - \tilde{\phi}_i)^2/2T}}{\sqrt{2\pi i T}}.$$

For the propagator on the circle, the normalization is fixed by requiring

$$\lim_{T\to 0} K(\phi_f, \phi_i) = \sum_n \delta(\phi_f - \phi_i + nL) \implies \mathcal{C}_0 = e^{-i(\phi_f - \phi_i)\delta/L}.$$

Finally,

$$K(\phi_f, \phi_i) = \frac{1}{\sqrt{2\pi i T}} \sum_n \exp\left(i\frac{(\phi_f - \phi_i + nL)^2}{2T} - i\frac{(\phi_f - \phi_i + nL)\delta}{L}\right).$$

A.5 For the periodic case,

$$\int \mathrm{d}\psi \langle \psi, U|\psi, 0\rangle = \int [\mathrm{d}\psi \mathrm{d}\chi]_P \exp\left(-\int_0^U \mathrm{d}u \left(\chi\dot{\psi} + m\chi\psi\right)\right).$$

A basis of eigenfunctions of $\Delta = -\partial_u - m$:

$$\psi_k \sim \exp\frac{+2ik\pi u}{U}, \quad \Delta\psi_k = \left(-m - \frac{2ik\pi}{U}\right)\psi_k, \quad k \in \mathbb{Z},$$
$$\chi_k \sim \exp\frac{-2ik\pi u}{U}, \quad \Delta^T\chi_k = \left(-m + \frac{2ik\pi}{U}\right)\chi_k, \quad k \in \mathbb{Z}.$$

Regularization:

$$\det \Delta = \prod_{k=-\infty}^{+\infty} \frac{-mU - 2ik\pi}{U} \rightsquigarrow \prod_{k=-\infty}^{+\infty} \frac{-mU - 2ik\pi}{-MU - 2ik\pi} = \frac{\sinh \frac{mU}{2}}{\sinh \frac{MU}{2}}.$$

As $M \to \infty$,

$$\int [\mathrm{d}\psi \mathrm{d}\chi]_P \exp(-S_E) = 2\sinh\frac{mU}{2} \exp\left(-S_{\rm ct} - \frac{1}{2}MU\right).$$

The finite normalization is fixed by requiring $\mathrm{Tr}[(-)^F \exp(-\hat{H}U)] \sim \exp(-E_0 U)$ as $U \to \infty$. The energy eigenstates are $\hat{H}|\uparrow\rangle = m|\uparrow\rangle$ and $\hat{H}|\downarrow\rangle = 0$.

$$\int [\mathrm{d}\psi \mathrm{d}\chi]_P \exp(-S_E) = 2e^{-mU/2} \sinh\frac{mU}{2} = 1 - e^{-mU},$$

which matches the sum over states.

For the anti-periodic case,

$$\int \mathrm{d}\psi \langle \psi, U|\psi, 0\rangle = \int [\mathrm{d}\psi \mathrm{d}\chi]_A \exp\left(-\int_0^U \mathrm{d}u \left(\chi\dot{\psi} + m\chi\psi\right)\right).$$

A basis of eigenfunctions of $\Delta = -\partial_u - m$:

$$\psi_k \sim \exp\frac{+i(2k+1)\pi u}{U}, \quad \Delta\psi_k = \left(-m - \frac{2ik\pi}{U}\right)\psi_k, \quad k \in \mathbb{Z},$$
$$\chi_k \sim \exp\frac{-i(2k+1)\pi u}{U}, \quad \Delta^T\chi_k = \left(-m + \frac{2ik\pi}{U}\right)\chi_k, \quad k \in \mathbb{Z}.$$

Regularization:

$$\det\Delta = \prod_{k=-\infty}^{+\infty} \frac{-mU - i(2k+1)\pi}{U} \rightsquigarrow \prod_{k=-\infty}^{+\infty} \frac{-mU - i(2k+1)\pi}{-MU - i(2k+1)\pi} = \frac{\cosh\frac{mU}{2}}{\cosh\frac{MU}{2}}.$$

As $M \to \infty$,

$$\int [\mathrm{d}\psi\mathrm{d}\chi]_A \exp(-S_E) = 2\cosh\frac{mU}{2}\exp\left(-S_{\mathrm{ct}} - \frac{1}{2}MU\right).$$

The finite normalization is fixed by requiring $\mathrm{Tr}\exp(-\hat{H}U) \sim \exp(-E_0 U)$ as $U \to \infty$. The energy eigenstates are $\hat{H}|\uparrow\rangle = m|\uparrow\rangle$ and $\hat{H}|\downarrow\rangle = 0$.

$$\int [\mathrm{d}\psi\mathrm{d}\chi]_A \exp(-S_E) = 2e^{-mU/2}\cosh\frac{mU}{2} = 1 + e^{-mU},$$

which matches the sum over states.

Chapter 1. A First Look at Strings

1.1 (a). Use diff invariance to fix $\tau = x^0$ so that $\dot{X}^0 = 1$ and $\dot{X}^i = v^i$. The action then becomes

$$S_{\mathrm{pp}} = -m\int \mathrm{d}t\sqrt{1-\mathbf{v}^2} \approx \int \mathrm{d}t\left(\frac{1}{2}m\mathbf{v}^2 - m\right).$$

Comparing with the standard interpretation that $L = T - V$, it's clear that the potential energy is the rest mass.

(b). Again, take $\tau = x^0$. In the non-relativistic limit, time flows uniformly throughout space, $\partial_\sigma t = 0$. The induced metric is easily determined:

$$\begin{aligned} h_{\tau\tau} &= (1,\mathbf{v}) \cdot_{\gamma_{ab}} (1,\mathbf{v}) = -1 + \mathbf{v}^2, \\ h_{\sigma\tau} &= (1,\mathbf{v}) \cdot_{\gamma_{ab}} (0,\partial_\sigma\mathbf{X}) = \mathbf{v}\cdot\partial_\sigma\mathbf{X}, \\ h_{\sigma\sigma} &= (0,\partial_\sigma\mathbf{X}) \cdot_{\gamma_{ab}} (0,\partial_\sigma\mathbf{X}) = (\partial_\sigma\mathbf{X})^2. \end{aligned}$$

The Nambu-Goto action becomes

$$\begin{aligned} S_{\mathrm{NG}} &= -\frac{1}{2\pi\alpha'}\int \mathrm{d}t\mathrm{d}\sigma\sqrt{(1-\mathbf{v}^2)(\partial_\sigma\mathbf{X})^2 + (\mathbf{v}\cdot\partial_\sigma\mathbf{X})^2} \\ &\approx -\frac{1}{2\pi\alpha'}\int \mathrm{d}t\mathrm{d}\sigma|\partial_\sigma\mathbf{X}|\left(1 + \frac{(\mathbf{v}\cdot\partial_\sigma\mathbf{X})^2 - \mathbf{v}^2(\partial_\sigma\mathbf{X})^2}{2(\partial_\sigma\mathbf{X})^2}\right). \end{aligned}$$

The interpretation of the above formula is as follows: $\partial_\sigma\mathbf{X})\mathrm{d}\sigma$ is the length element $\mathrm{d}\vec{\ell}$ along the string. Decompose the velocity into $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$ according to its direction with respect to the string. By definition, $\mathbf{v}_\perp\cdot\partial_\sigma\mathbf{X} = 0$ and $\mathbf{v}_\parallel\cdot\partial_\sigma\mathbf{X} = \pm|\mathbf{v}_\parallel||\partial_\sigma\mathbf{X}|$. This way, the parenthesis evaluates to $(1 - \mathbf{v}_\perp^2/2)$.

$$S_{\mathrm{NG}} \approx \int \mathrm{d}t\mathrm{d}\ell\frac{1}{2\pi\alpha'}\left(\frac{\mathbf{v}_\perp^2}{2} - 1\right).$$

Comparing with the standard interpretation that $L = T - V$ and recalling from (a) that $V = m$, it's clear that only transverse velocity contributes to the kinetic energy, and the mass density is read off to be $1/2\pi\alpha'$.

1.2 The ends of an open string satisfy the Neumann boundary conditions:

$$\partial^\sigma X^\mu|_{\sigma=0,\ell} = 0.$$

The equations of motion $T^{\sigma\sigma} = 0$ and $T^{\sigma\tau} = 0$ at the ends of the string imply:

$$\begin{aligned} \frac{1}{2}\gamma^{\sigma\sigma}\dot{X}^\mu\dot{X}_\mu &= \partial^\sigma X^\mu\partial^\sigma X_\mu = 0, \\ \frac{1}{2}\gamma^{\sigma\tau}\dot{X}^\mu\dot{X}_\mu &= \partial^\sigma X^\mu\partial^\tau X_\mu = 0. \end{aligned}$$

Since γ_{ab} is non-degenerate, $\gamma^{\sigma\sigma}$ and $\gamma^{\sigma\tau}$ cannot both vanish. Hence, at least one of the above equation implies $\dot{X}^\mu\dot{X}_\mu = 0$, i.e. the ends of the string move at light speed.