MAST90104: A first course in Statistical Learning

Week 2 Tutorial/Lab Solutions

Workshop questions

1. (a) Find the eigenvalues, and an associated eigenvector for each eigenvalue, of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Solution: The characteristic equation is

$$det(A - \lambda I) = (2 - \lambda)^2 - 4 = -4\lambda + \lambda^2 = \lambda(\lambda - 4) = 0,$$

so the eigenvalues are 0 and 4.
For $\lambda = 4$, we solve

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One such solution is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 0$, we solve

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One such solution is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(b) Find an orthogonal matrix P such that $P^T AP$ is diagonal.
Solution:

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(c) Write down $P^T AP$ for the P given in part (b).
Solution:

$$P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix}.$$

(a) Write down the trace of A.
Solution: $tr(A) = 1$.

(b) Are the columns of A linearly independent? Justify your answer.
Solution: The columns of A are not linearly independent: the second column is the sum of the first and third.

(c) Find the rank of A.
Solution: The first and third columns of A are not multiples of each other, so $r(A) = 2$.

3. Show that if X is of full rank, then

$$I - X(X^T X)^{-1} X^T$$

is an idempotent matrix.

Solution: In general, if A is idempotent then so is $I - A$, since

$$(I - A)(I - A) = I - A - A + A^2 = I - A - A + A = I - A.$$

To see that $X^T X$ is invertible, we know that $r(X^T X) = r(X) = k$ since X is of full rank. To see that $A = X(X^T X)^{-1} X^T$ is idempotent we just multiply it by itself:

$$[X(X^T X)^{-1} X^T][X(X^T X)^{-1} X^T] = X(X^T X)^{-1}(X^T X)(X^T X)^{-1} X^T = X(X^T X)^{-1} X^T.$$

4. Consider the matrix

$$X = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

(a) Show that X is idempotent.
Solution: $X^2 = X$

(b) What is the rank of X ?
Solution: Easy to see that $r(X) = 1$

5. Is

$$X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

orthogonal? If not, what value of c makes the matrix cX orthogonal?

Solution: The matrix is not orthogonal, as its columns do not form an orthonormal set (e.g. the first column has norm > 1). However they do form an *orthogonal* set, so we can just normalise each vector to produce an orthogonal matrix. This gives $c = \frac{1}{2}$.

6. For the following matrices, find the eigenvalues and eigenvectors

(a) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$
Solution: To find the eigenvalues, we solve the equation

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = 0.$$

This gives $\lambda = 1$ and $\lambda = -1$
To find the eigenvector associated with eigenvalue 1, we solve the system of equation

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This implies $x_2 = 0$ and x_1 can be any value. One solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Similarly, the eigenvector associated with eigenvalue -1 is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **Solution:** The eigenvalues are $\lambda = 1$ and $\lambda = 3$

The eigenvector associated with $\lambda = 1$ is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The eigenvector associated with $\lambda = 3$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

7. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 7/3 & -3/\sqrt{6} & 1/\sqrt{18} \\ -3/\sqrt{6} & 5/2 & -3/\sqrt{12} \\ 1/\sqrt{18} & -3/\sqrt{12} & 13/6 \end{bmatrix}.$$

   (a) Write down the characteristic equation for $\mathbf{A}$.

   (b) Show that $\{1, 2, 4\}$ is the solution set to the characteristic equation in (a).

   (c) Show that the set of orthonormal eigenvectors of $\mathbf{A}$ is

$$\left\{(1/\sqrt{3}, 1/\sqrt{2}, 1/\sqrt{6})^T, (1/\sqrt{3}, 0, -2/\sqrt{6})^T, (1/\sqrt{3}, -1/\sqrt{2}, 1/\sqrt{6})^T\right\}.$$

Solution:

(a)The characteristic equation is defined as $det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Now,

$$\begin{aligned} det(\mathbf{A} - \lambda\mathbf{I}) = (7/3 - \lambda)\det\left\{\begin{bmatrix} 5/2-\lambda & -3/\sqrt{12} \\ -3/\sqrt{12} & 13/6-\lambda \end{bmatrix}\right\} + 3/\sqrt{6}\det\left\{\begin{bmatrix} -3/\sqrt{6} & -3/\sqrt{12} \\ 1/\sqrt{18} & 13/6-\lambda \end{bmatrix}\right\} \\ + 1/\sqrt{18}\det\left\{\begin{bmatrix} -3/\sqrt{6} & 5/2-\lambda \\ 1/\sqrt{18} & -3/\sqrt{12} \end{bmatrix}\right\} \\ = (7/3-\lambda)\{(5/2-\lambda)(13/6-\lambda) - 9/12\} + 3/\sqrt{6}\{-3/\sqrt{6}\times(13/6-\lambda) + 3/\sqrt{216}\} \\ + 1/\sqrt{18}\{9/\sqrt{72} - 1/\sqrt{18}\times(5/2-\lambda)\} \\ = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \end{aligned}$$

(b) Let $C(\lambda) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8$. Then, check that $C(1) = 0$, $C(2) = 0$, and $C(4) = 0$.

(c) Easy to check that each vector in set has norm 1. To show that there are indeed eigenvectors for $\mathbf{A}$, we can check that:

$$\mathbf{A}\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix}$$

and

$$\mathbf{A}\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -2/\sqrt{6} \end{pmatrix} = 2\times\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -2/\sqrt{6} \end{pmatrix}$$

and

$$\mathbf{A}\begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix} = 4\times\begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{pmatrix}$$

Practical questions

1. Use R to find the number of integers that are divisible by 17 between 1 and 500

   Solution:

```
> x = 1:500
> sum(x%%17==0)


[1] 29
```

2. Suppose that `queue <- c("Steve", "Russell", "Alison", "Liam")` and that `queue` represents a supermarket queue with Steve first in line. Using R expressions update the supermarket queue as successively:

   (a) Barry arrives;

(b) Steve is served;
(c) Pam talks her way to the front with one item;
(d) Barry gets impatient and leaves;
(e) Alison gets impatient and leaves.

For the last case you should not assume that you know where in the queue Alison is standing.

Finally, using the function `which(x)`, find the position of Russell in the queue.

Note that when assigning a text string to a variable, it needs to be in quotes.

Solution:

```
> (queue <- c("S", "R", "A", "L"))

[1] "S" "R" "A" "L"

> # a
> (queue <- c(queue, "B"))

[1] "S" "R" "A" "L" "B"

> # b
> (queue <- queue[-1])

[1] "R" "A" "L" "B"

> # c
> (queue <- c("P", queue))

[1] "P" "R" "A" "L" "B"

> # d
> (queue <- queue[1:(length(queue)-1)])

[1] "P" "R" "A" "L"

> # e
> (queue <- queue[queue != "A"])

[1] "P" "R" "L"

> which(queue == "R")

[1] 2
```

3. The table below is taken from a clinic's database, that records the patients' name, age, and their waiting time. Create an R *data frame* with these information. Find the patients with the longest waiting time

Name	Age	Waiting time
Ron	23	5
Steve	24	7
Barry	20	2
Louise	30	3
Ann	25	5
Kristen	24	4
Emma	21	6

Solution:

Note that there are several ways to do this

```
> name = c('Ron','Steve','Barry','Louise','Ann','Kristen','Emma')
> age = c(23,24,20,30,25,24,21)
> waittime = c(5,7,2,3,5,4,6)
> data1 = data.frame(name,age,waittime)
> names(data1) = c('Name','Age','Waiting time')
```

To find the patient with the longest waiting time

```
> data1[which.max(data1$`Waiting time`),]
Name Age Waiting time
2 Steve 24            7
```

or

```
> data1[which.max(data1[,'Waiting time']),]
```

Note: Data frames in R can be interpreted as special "matrices" but the columns can be of different data types (string, numeric, ...). Try and see what happen when you run `(cbind(name, age, waittime))`

4. Let

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 7 & 6 & 8 \\ 3 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

(a) Give R expression that return A and B

Solution:

```
> A= matrix(c(2,7,3,4,6,1,0,8,4),nrow = 3,ncol = 3)
> B = matrix(c(1,-1,4,0,-1,0,2,0,1),nrow = 3,ncol = 3)
> A
[,1] [,2] [,3]
[1,]    2    4    0
[2,]    7    6    8
[3,]    3    1    4
> B
[,1] [,2] [,3]
[1,]    1    0    2
[2,]   -1   -1    0
[3,]    4    0    1
```

(b) Use R to compute AB, $B^T A$

Solution:

```
> A%*%B
[,1] [,2] [,3]
[1,]   -2   -4    4
[2,]   33   -6   22
[3,]   18   -1   10
> t(B)%*%A
[,1] [,2] [,3]
[1,]    7    2    8
[2,]   -7   -6   -8
[3,]    7    9    4
```

(c) Use R to find $det(A)$ and $r(B)$

Solution:

```
> det(A)
[1] 16
> rankMatrix(B)[1]
[1] 3
```

5. Use R to create 3 vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 5 \\ -4 \\ 6 \end{bmatrix}$$

(a) Create matrix $\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}$ (*Hint: use function cbind*)

(b) Create a vector of length 3, call it $\mathbf{z}$

(c) Add $\mathbf{z}$ to $\mathbf{A}$ as the last row (*Hint: use function rbind*)

Solution:

```
> x1 = c(2,4,1)
> x2= c(3,0,2)
> x3 = c(5,-4,6)
> A = cbind(x1,x2,x3)
> A
x1 x2 x3
[1,]  2  3  5
[2,]  4  0 -4
[3,]  1  2  6
> z = c(8,4,2)
> A = rbind(A,z)
```

6. Write a program to read in a square matrix and return its trace. *Hint: We first need to check whether the input is a square matrix.*

Suggested solution:

```
findtrace <- function(A){
   if(nrow(A) == ncol(A)){
      return(sum(diag(A)))
   }else{
      cat('Input has to be a square matrix')
   }
}
```

```
> A = matrix(rnorm(9),ncol = 3,nrow = 3)
> A
[,1]      [,2]       [,3]
[1,]  0.1572473 0.5020795 -0.4481195
[2,] -0.5456538 1.5682662 -1.3856881
[3,] -1.1990344 1.0687277 -0.3181188
> findtrace(A)
[1] 1.407395
```