MAST90104 - Lecture 9

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Model fit used deviance as well as Akaike Information Criterion.

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Maximum likelihood, link functions and deviance again feature prominently.

Multinomial logit model

Suppose $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ are independent random vectors of counts with:

$$\begin{aligned}
\mathbf{Y}_i &\sim \text{multinomial}(m_i, \mathbf{p}_i) \quad \text{with } \mathbf{p}_i = (p_{i1}, \ldots, p_{iJ}) \\
\mathbb{P}(\mathbf{Y}_i = \mathbf{y}_i) &= \frac{m_i!}{y_{i1}! \cdots y_{iJ}!} p_{i1}^{y_{i1}} \cdots p_{iJ}^{y_{iJ}} \quad \text{for } \mathbf{y}_i \geq 0, \textstyle\sum_j y_{ij} = m_i
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A *multinomial logit model* supposes that

$$p_{ij} = \frac{e^{\eta_{ij}}}{\sum_{k=1}^{J} e^{\eta_{ik}}}.$$

where $\eta_{ij} = \mathbf{x}_i^T \boldsymbol{\beta}_j$ for predictor variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and parameter vectors $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_J$ (J distinct parameter vectors!).

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Hence, $(p_{i1}, \ldots, p_{iJ}) := \text{softmax}(\eta_{i1}, \ldots, \eta_{iJ})$.

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Hence, $(p_{i1}, \dots, p_{iJ}) := \text{softmax}(\eta_{i1}, \dots, \eta_{iJ})$.

Thus the linear predictor η_{ij} is the log of the odds of the probability p_{ij}. $e^{\eta_{ij}}$ can be interpreted as the rate at which an outcome of type j occurs.

Non-identifiability

Problem: $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_J$ is not identifiable. Example: say we have oracle knowledge of $(p_{i1}, \ldots, p_{iJ})$. If $J = 3$ and linear predictor $= \beta_j x$,

$$\begin{aligned}(p_{i1}, \ldots, p_{iJ}) &= \text{softmax}\{\beta_1 x_i, \ldots, \beta_J x_i\} \\ &= \text{softmax}\{(\beta_1 + c)x_i, \ldots, (\beta_J + c)x_i\}\end{aligned}$$

where c is some constant.

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where c is some constant.

Solution: This is fixed by setting $\boldsymbol{\beta}_1 = \mathbf{0}$.

This is equivalent to dividing top and bottom by $e^{\eta_{i1}}$.

Non-identifiability

If $J = 2$ then

$$p_{i1} = \frac{1}{1 + e^{\mathbf{x}_i^T \boldsymbol{\beta}_2}}, \quad p_{i2} = \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}_2}}{1 + e^{\mathbf{x}_i^T \boldsymbol{\beta}_2}}$$

which is just a binomial regression model with responses y_{i2}, parameters $\boldsymbol{\beta}_2$, and a logit link.

Maximum Likelihood reigns

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Nested models can be evaluated by comparing the difference of their deviances to a χ^2 distribution, whose degrees of freedom are the difference in the number of parameters.

Example 1996 American National Election Study: Data description and aim

10 variable subset of the 1996 American National Election Study, part of a series by Rosenstone, Kinder, and Miller from the University of Michigan Institute for Social Research.

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Levels of party affiliation and income

```
library(faraway)
data(nes96)
levels(nes96$PID)

## [1] "strDem"  "weakDem" "indDem"  "indind"  "indRep"  "weakRep" "strRep"

levels(nes96$income)

##  [1] "$3Kminus"   "$3K-$5K"    "$5K-$7K"    "$7K-$9K"    "$9K-$10K"
##  [6] "$10K-$11K"  "$11K-$12K"  "$12K-$13K"  "$13K-$14K"  "$14K-$15K"
## [11] "$15K-$17K"  "$17K-$20K"  "$20K-$22K"  "$22K-$25K"  "$25K-$30K"
## [16] "$30K-$35K"  "$35K-$40K"  "$40K-$45K"  "$45K-$50K"  "$50K-$60K"
## [21] "$60K-$75K"  "$75K-$90K"  "$90K-$105K" "$105Kplus"

sPID <- nes96$PID
# recode party affiliation as Republican, Democrat or Independent
levels(sPID) <- c("Democrat","Democrat","Independent","Independent",
"Independent","Republican","Republican")
# recode income as midpoint of group to make it numerical
inca <- c(1.5,4,6,8,9.5,10.5,11.5,12.5,13.5,14.5,16,18.5,21,23.5,
27.5,32.5,37.5,42.5,47.5,55,67.5,82.5,97.5,115)
nincome <- inca[unclass(nes96$income)]
```

Plotting voting preference against education

```
table(nes96$educ, sPID)
```

```
##         sPID
##          Democrat Independent Republican
##   MS            9           3          1
##   HSdrop       29          14          9
##   HS          108          63         77
##   Coll         74          40         73
##   CCdeg        34          24         32
##   BAdeg        81          55         91
##   MAdeg        45          40         42
```

```
prop.table(table(nes96$educ, sPID), 1)
```

```
##         sPID
##            Democrat Independent Republican
##   MS     0.69230769  0.23076923 0.07692308
##   HSdrop 0.55769231  0.26923077 0.17307692
##   HS     0.43548387  0.25403226 0.31048387
##   Coll   0.39572193  0.21390374 0.39037433
##   CCdeg  0.37777778  0.26666667 0.35555556
##   BAdeg  0.35682819  0.24229075 0.40088106
##   MAdeg  0.35433071  0.31496063 0.33070866
```

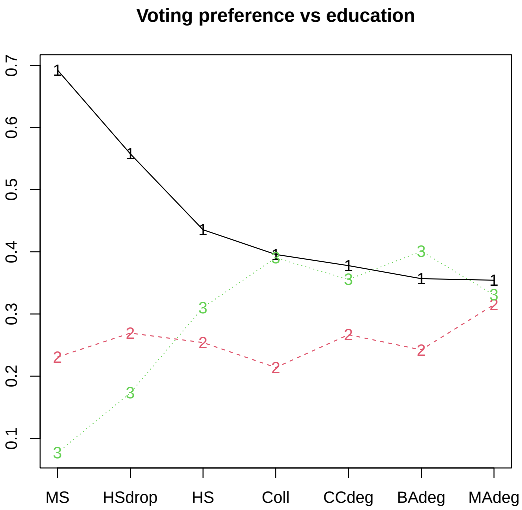

Figure: Party affiliation versus education. 1 (black) is Democrat, 2 (pink) is Independent and 3 (green) is Republican

Plotting voting preference against income and age

```
# plotting voting preference against income
matplot(prop.table(table(nincome, sPID), 1), type="o")
# plotting Party affiliation agains age;
# need to group age values
matplot(prop.table(table(cut(nes96$age, 6), sPID), 1),
type="o")
```

```
levels(cut(nes96$age, 6))

## [1] "(18.9,31]" "(31,43]" "(43,55]" "(55,67]" "(67,79]" "(79,91.1]"
```

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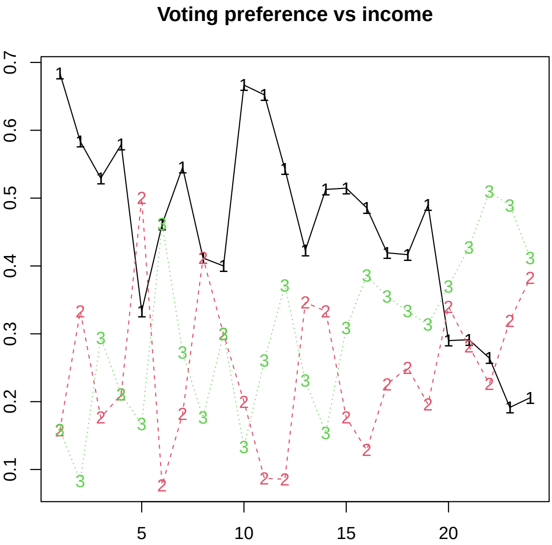

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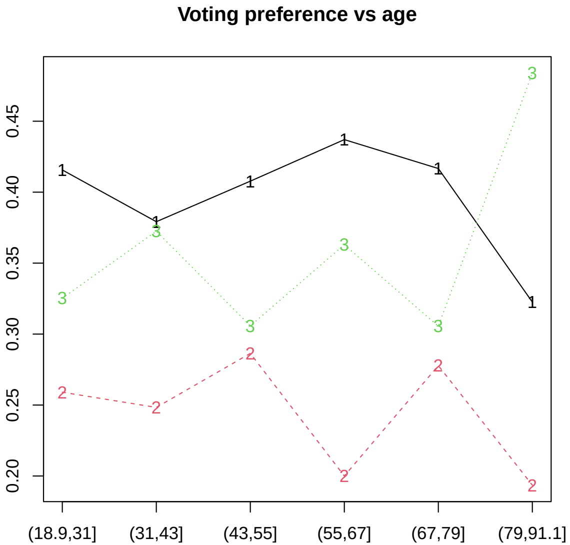


Fitting a model

```
library(nnet)
mmod <- multinom(sPID ~ age + educ + nincome, nes96)

## # weights:  30 (18 variable)
## initial  value 1037.090001
## iter  10 value 990.568608
## iter  20 value 984.319052
## final  value 984.166272
## converged
```

Results

```
summary(mmod)

## Call:
## multinom(formula = sPID ~ age + educ + nincome, data = nes96)
##
## Coefficients:
##             (Intercept)          age     educ.L     educ.Q    educ.C
## Independent   -1.197260 0.0001534525 0.06351451 -0.1217038 0.1119542
## Republican    -1.642656 0.0081943691 1.19413345 -1.2292869 0.1544575
##                  educ^4     educ^5      educ^6    nincome
## Independent -0.07657336  0.1360851  0.15427826 0.01623911
## Republican  -0.02827297 -0.1221176 -0.03741389 0.01724679
##
## Std. Errors:
##             (Intercept)         age    educ.L    educ.Q    educ.C
## Independent   0.3265951 0.005374592 0.4571884 0.4142859 0.3498491
## Republican    0.3312877 0.004902668 0.6502670 0.6041924 0.4866432
##                educ^4    educ^5    educ^6     nincome
## Independent 0.2883031 0.2494706 0.2171578 0.003108585
## Republican  0.3605620 0.2696036 0.2031859 0.002881745
##
## Residual Deviance: 1968.333
## AIC: 2004.333
```

Backward selection using AIC

```
# model selection using AIC
mmodi <- step(mmod)

## Start:  AIC=2004.33
## sPID ~ age + educ + nincome
##
## trying - age
## # weights:  27 (16 variable)
## initial  value 1037.090001
## iter  10 value 988.896864
## iter  20 value 985.822223
## final  value 985.812737
## converged
## trying - educ
## # weights:  12 (6 variable)
## initial  value 1037.090001
## iter  10 value 992.269502
## final  value 992.269484
## converged
```

Backward selection using AIC

```
## trying - nincome
## # weights:  27 (16 variable)
## initial  value 1037.090001
## iter  10 value 1009.025560
## iter  20 value 1006.961593
## final  value 1006.955275
## converged
##           Df      AIC
## - educ     6 1996.539
## - age     16 2003.625
## <none>    18 2004.333
## - nincome 16 2045.911
## # weights:  12 (6 variable)
## initial  value 1037.090001
## iter  10 value 992.269502
## final  value 992.269484
## converged
##
```

Backward selection using AIC

```
## Step:  AIC=1996.54
## sPID ~ age + nincome
##
## trying - age
## # weights:  9 (4 variable)
## initial  value 1037.090001
## final  value 992.712152
## converged
## trying - nincome
## # weights:  9 (4 variable)
## initial  value 1037.090001
## final  value 1020.425203
## converged
##            Df      AIC
## - age       4 1993.424
## <none>      6 1996.539
## - nincome   4 2048.850
## # weights:  9 (4 variable)
## initial  value 1037.090001
## final  value 992.712152
## converged
```

Backward selection using AIC

```
##
## Step:  AIC=1993.42
## sPID ~ nincome
##
## trying - nincome
## # weights:  6 (2 variable)
## initial  value 1037.090001
## final  value 1020.636052
## converged
##           Df      AIC
## <none>     4 1993.424
## - nincome  2 2045.272
```

Results

```
summary(mmodi)

## Call:
## multinom(formula = sPID ~ nincome, data = nes96)
##
## Coefficients:
##             (Intercept)    nincome
## Independent  -1.1749331 0.01608683
## Republican   -0.9503591 0.01766457
##
## Std. Errors:
##             (Intercept)     nincome
## Independent   0.1536103 0.002849738
## Republican    0.1416859 0.002652532
##
## Residual Deviance: 1985.424
## AIC: 1993.424
```

Nested models comparison using Deviance

```r
# model selection using likelihood ratios
mmode <- multinom(sPID ~ age + nincome, nes96)

## # weights:  12 (6 variable)
## initial  value 1037.090001
## iter  10 value 992.269502
## final  value 992.269484
## converged

deviance(mmode) - deviance(mmod)

## [1] 16.20642

mmod$edf

## [1] 18

mmode$edf

## [1] 6

pchisq(16.206, mmod$edf - mmode$edf, lower=FALSE)

## [1] 0.181982
```

Fitted probabilities

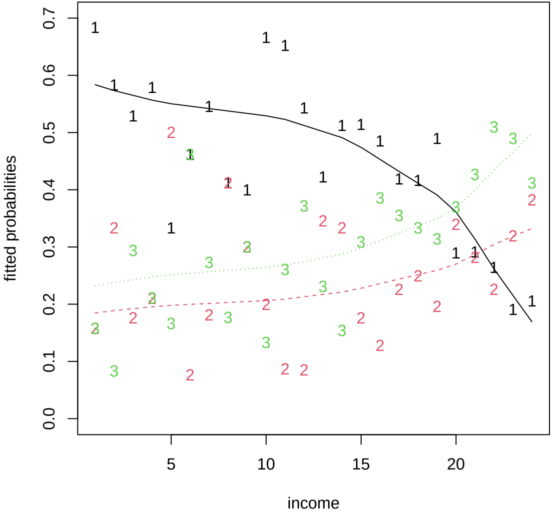
```
predict(mmodi, data.frame(nincome=inca), type="probs")
```

```
##     Democrat Independent Republican
## 1  0.5836466   0.1846557  0.2316977
## 2  0.5733047   0.1888271  0.2378682
## 3  0.5649837   0.1921708  0.2428455
## 4  0.5566253   0.1955183  0.2478565
## 5  0.5503347   0.1980300  0.2516353
## 6  0.5461317   0.1997045  0.2541638
## 7  0.5419219   0.2013787  0.2566993
## 8  0.5377060   0.2030524  0.2592415
## 9  0.5334846   0.2047254  0.2617901
## 10 0.5292582   0.2063972  0.2643446
## 11 0.5229106   0.2089023  0.2681871
## 12 0.5123151   0.2130684  0.2746165
## 13 0.5017076   0.2172194  0.2810730
## 14 0.4910976   0.2213511  0.2875513
## 15 0.4741402   0.2279116  0.2979482
## 16 0.4530281   0.2360027  0.3109692
## 17 0.4320800   0.2439428  0.3239772
## 18 0.4113683   0.2517021  0.3369297
## 19 0.3909623   0.2592525  0.3497852
## 20 0.3610676   0.2701312  0.3688012
## 21 0.3136199   0.2868931  0.3994870
## 22 0.2614599   0.3044513  0.4340888
## 23 0.2152314   0.3190178  0.4657508
## 24 0.1691487   0.3322310  0.4986204
```

Plot of fitted probabilities

```r
matplot(predict(mmodi, data.frame(nincome=inca),
type="probs"),
type="l", ylim=c(0, .7))
matplot(prop.table(table(nincome, sPID), 1), add=TRUE)
```

Figure: Voting preference versus income with fitted values



Ordinal data

Suppose that we have independent observations $Y_i \in \{1, \ldots, J\}$, representing ordered categories. Put

$$p_{ij} = \mathbb{P}(Y_i = j) \text{ and } \gamma_{ij} = \mathbb{P}(Y_i \leq j).$$

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Instead we suppose that for some link function g

$$g(\gamma_{ij}) = \theta_j - \mathbf{x}_i^T \boldsymbol{\beta}, \quad \text{for } j = 1, \ldots, J-1.$$

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and $\gamma_{iJ} = 1$.
Note:

- g is usually the logit, probit or complementary log log link;
- $\boldsymbol{\beta}$ does not depend on j;
- the $\mathbf{x}_i^T \boldsymbol{\beta}$ term must not include an intercept.

Interpretation of ordinal model

Suppose that response Y_i is a discretised version of some continuous r.v. $\tilde{Y}_i$, where the distribution of $\tilde{Y}_i$ is given by

$$\tilde{Y}_i \sim Z_i + \mathbf{x}_i^T \boldsymbol{\beta} \text{ for some } Z_i,$$

where $Z_i \sim F_Z$ are iid random variables.

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where $Z_i \sim F_Z$ are iid random variables. The θ_j are then defined by $\mathbb{P}(Y_i \leq j) = \mathbb{P}(\tilde{Y}_i \leq \theta_j)$, which gives

$$\gamma_{ij} = \mathbb{P}(Y_i \leq j)$$

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where $Z_i \sim F_Z$ are iid random variables. The θ_j are then defined by $\mathbb{P}(Y_i \leq j) = \mathbb{P}(\tilde{Y}_i \leq \theta_j)$, which gives

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Put $g = F_Z^{-1}$ to get our model. Like the multinomial logit model, we can fit an ordinal model using maximum likelihood.

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It is possible to introduce both scale and location parameters so that logistic distributions are possible with any mean or variance.

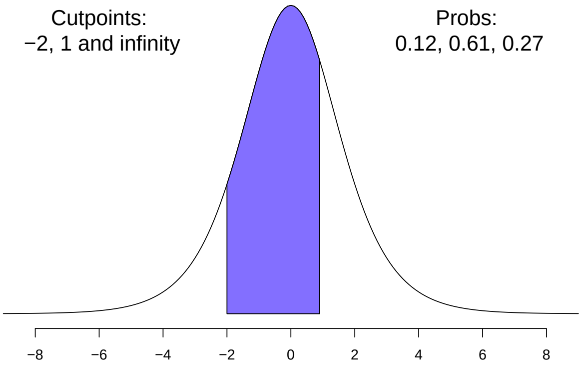

Figure: Standard Logistic

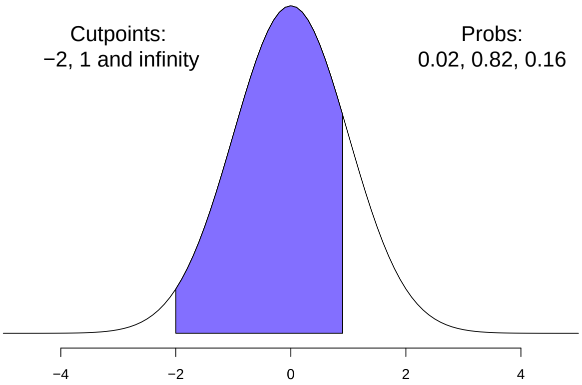

Figure: Normal mean 0, variance 1

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Certain kinds of latent variable models based on the logistic distribution are commonly used in education and called *Rasch* models.

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The R library MASS contains the commamd `polr` which does ordinal logistic regression by default.

Fitting the ordinal model

```
library(MASS)
omod <- polr(sPID ~ age + educ + nincome, nes96)
summary(omod)

##
## Re-fitting to get Hessian

## Call:
## polr(formula = sPID ~ age + educ + nincome, data = nes96)
##
## Coefficients:
##              Value Std. Error  t value
## age       0.005775   0.003887  1.48581
## educ.L    0.724087   0.384388  1.88374
## educ.Q   -0.781361   0.351173 -2.22500
## educ.C    0.040168   0.291762  0.13767
## educ^4   -0.019925   0.232429 -0.08573
## educ^5   -0.079413   0.191533 -0.41462
## educ^6   -0.061104   0.157747 -0.38735
## nincome   0.012739   0.002140  5.95187
##
## Intercepts:
##                         Value   Std. Error t value
## Democrat|Independent     0.6449  0.2435     2.6479
## Independent|Republican  1.7374  0.2493     6.9694
##
## Residual Deviance: 1984.211
## AIC: 2004.211
```

Compare to categorical

```
summary(mmod)

## Call:
## multinom(formula = sPID ~ age + educ + nincome, data = nes96)
##
## Coefficients:
##             (Intercept)          age     educ.L     educ.Q    educ.C
## Independent   -1.197260 0.0001534525 0.06351451 -0.1217038 0.1119542
## Republican    -1.642656 0.0081943691 1.19413345 -1.2292869 0.1544575
##                  educ^4     educ^5      educ^6    nincome
## Independent -0.07657336  0.1360851  0.15427826 0.01623911
## Republican  -0.02827297 -0.1221176 -0.03741389 0.01724679
##
## Std. Errors:
##             (Intercept)         age    educ.L    educ.Q    educ.C
## Independent   0.3265951 0.005374592 0.4571884 0.4142859 0.3498491
## Republican    0.3312877 0.004902668 0.6502670 0.6041924 0.4866432
##                educ^4    educ^5    educ^6     nincome
## Independent 0.2883031 0.2494706 0.2171578 0.003108585
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```

Model selection using AIC

```
omod2 <- step(omod)

## Start:  AIC=2004.21
## sPID ~ age + educ + nincome
##
##           Df    AIC
## - educ     6 2002.8
## <none>       2004.2
## - age      1 2004.4
## - nincome  1 2038.6
##
## Step:  AIC=2002.83
## sPID ~ age + nincome
##
##           Df    AIC
## - age      1 2001.4
## <none>       2002.8
## - nincome  1 2047.2
##
## Step:  AIC=2001.36
## sPID ~ nincome
##
##           Df    AIC
## <none>       2001.4
## - nincome  1 2045.3
```

So just as with the unordered model, voter preference is modelled on income.

Summary of selected model

```
summary(omod2)

##
## Re-fitting to get Hessian

## Call:
## polr(formula = sPID ~ nincome, data = nes96)
##
## Coefficients:
##            Value Std. Error t value
## nincome 0.01312   0.001971   6.657
##
## Intercepts:
##                           Value   Std. Error t value
## Democrat|Independent     0.2091  0.1123      1.8627
## Independent|Republican  1.2916  0.1201     10.7526
##
## Residual Deviance: 1995.363
## AIC: 2001.363
```

For a person with income \$50,000, the fitted log odds of being Democrat is $0.2091 - 0.01312 \times 50$, and the fitted log odds of being Democrat or Independent is $1.2916 - 0.01312 \times 50$.

Model confirmation using LR Test

```
deviance(omod2) - deviance(omod)

## [1] 11.15136

omod$edf

## [1] 10

omod2$edf

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pchisq(11.151, omod$edf - omod2$edf, lower=FALSE)

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So we cannot reject the null hypothesis that all of the parameters in the full model (model with all predictors), other than those in the selected model, are 0.

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Comparison of fitted probabilities

```
matplot(predict(omod2, data.frame(nincome=inca), type="probs"),
type="l", ylim=c(0, .7))
matplot(predict(mmodi, data.frame(nincome=inca),
type="probs"),
type="l", ylim=c(0, .7),add=TRUE)
matplot(prop.table(table(nincome, sPID), 1), add=TRUE)
```

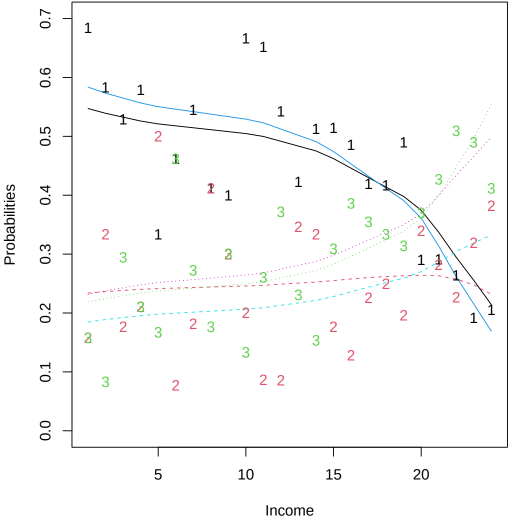

Figure: Ordinal fitted probs. with unordered & data

Contingency tables: two-way tables

Example: Semiconductor production

	die contaminated		
wafer quality	no	yes	
good	320	14	334
bad	80	36	116
	400	50	450

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cholesterol	heart disease yes	heart disease no	
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In each case we ask if the two factors are dependent or not?

Let y_{ij} be the number of observations with factor 1 at level i and factor 2 at level j:

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	factor 2			
factor 1	1	2	3	
1	y_{11}	y_{12}	y_{13}	$y_{1\cdot}$
2	y_{21}	y_{22}	y_{23}	$y_{2\cdot}$
	$y_{\cdot 1}$	$y_{\cdot 2}$	$y_{\cdot 3}$	$y_{\cdot\cdot}$

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2	y_{21}	y_{22}	y_{23}	$y_{2\cdot}$
	$y_{\cdot 1}$	$y_{\cdot 2}$	$y_{\cdot 3}$	$y_{\cdot\cdot}$

Let π_{ij} be the probability that an observation has factor 1 at level i and factor 2 at level j.

$$\begin{aligned}
\pi_{i\cdot} &= \textstyle\sum_j \pi_{ij} \text{ is prob. an obs. has factor 1 at level } i \\
\pi_{\cdot j} &= \textstyle\sum_i \pi_{ij} \text{ is prob. an obs. has factor 2 at level } j \\
\pi_{\cdot\cdot} &= \textstyle\sum_i \pi_{i\cdot} = \sum_j \pi_{\cdot j} = 1
\end{aligned}$$

We want to know if $\pi_{ij} = \pi_{i\cdot}\pi_{\cdot j}$ for all i and j?

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They often given the same or similar conclusions, so attention will be confined to the multinomial and product multinomial models.

Multinomial model - as number of balls in urns

Suppose we have a fixed number of balls $y_{..}$ and we throw each ball into one of the IJ urns. Each ball lands in urn (i, j) with probability π_{ij}. The number of balls in each urn can be modelled as:

$$
\begin{aligned}
(y_{ij})_{i=1,\ldots,I;j=1\ldots,J} \mid y_{..} &\sim \text{multinomial}\left(y_{..}, (\pi_{ij})_{i=1,\ldots,I;j=1\ldots,J}\right) \\
\mathbb{P}(Y_{ij} = y_{ij} \text{ for all } i \text{ and } j \mid y_{..}) &= \frac{y_{..}!}{\prod_{ij} y_{ij}!} \prod_{ij} \pi_{ij}^{y_{ij}}
\end{aligned}
$$

Product multinomial

Rearrange the IJ urns into an I by J grid. Fix the number of balls $y_{\cdot j}$ for column j. observations are independent. Then, for each i,

$$(y_{ij})_{i=1,\ldots,I} \mid y_{\cdot j} \sim \text{multinomial}(y_{\cdot j}, (\pi_{i|j})_{i=1,\ldots,I})$$

where

$$\pi_{i|j} = \mathbb{P}(\text{observe factor } 1 = i \text{ given factor } 2 = j) = \frac{\pi_{ij}}{\pi_{\cdot j}}.$$

Testing independence: multinomial model

H_0 $\pi_{ij} = \pi_{i\cdot}\pi_{\cdot j}$ (independent factors)

H_1 π_{ij} unrestricted.

As our test statistic we use the log likelihood ratio for the model under H_0 compared to the model under H_1.
This is just the deviance, since the model under H_1 is the saturated model.

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Testing independence: product multinomial model

H_0 $\pi_{i|j} = \pi_{i\cdot}$ (equivalently $\pi_{ij} = \pi_{i\cdot}\pi_{\cdot j}$)

H_1 $\pi_{i|j}$ unrestricted.

We have a multinomial logistic regression model, where the i-th response is

$$Y_{ij} \mid Y_{\cdot j} \sim \text{multinomial}(y_{\cdot j}, \pi_{i|j})$$

and under H_0 we have that $\pi_{i|j} = \pi_{i\cdot}$ does not depend on j. That is, for some β_i,

$$\pi_{i|j} = \frac{e^{\beta_i}}{\sum_k e^{\beta_k}} \quad (= \pi_{i\cdot})$$

Testing for H_0 is then equivalent to testing if the model with just an intercept is adequate.

Maximum Likelihood Estimators

The MLE $\hat{p}_k$ in a multinomial distribution with J categories and m balls for the probability, p_k of category k is $\hat{p}_k = y_k/m$ because the log likelihood is proportional to

$$y_1 \log(1 - \sum_{j=2}^{J} p_j) + \sum_{k=2}^{J} y_k \log(p_k).$$

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Thus, under the independence hypothesis, the MLE for the probability of the *ij* cell is the row total times the column total divided by the square of the table total.

The deviance can be computed using this or through the logistic fit with only an intercept term.

Wafers example

Data were collected as part of a quality improvement study at a semiconductor factory.
A sample of wafers was drawn and cross-classified according to whether a particle was found on the die that produced the wafer and whether the wafer was good or bad

Data frame and table

```
y <- c(320, 14, 80, 36)
particle <- gl(2, 1, 4, labels=c("no","yes"))
quality <- gl(2, 2, 4, labels=c("good","bad"))
(wafer <- data.frame(y, particle, quality))

##     y particle quality
## 1 320       no    good
## 2  14      yes    good
## 3  80       no     bad
## 4  36      yes     bad

(ov <- xtabs(y ~ quality + particle))

##        particle
## quality  no yes
##    good 320  14
##    bad   80  36
```

Marginal proportions

```r
# multinomial model
# marginal proportions for particle values
(pp <- prop.table( xtabs(y ~ particle)))

## particle
##        no       yes
## 0.8888889 0.1111111

# marginal proportions for quality values
(qp <- prop.table( xtabs(y ~ quality)))

## quality
##      good       bad
## 0.7422222 0.2577778
```

Independence fitted values

```r
# multinomial model under independence
# # fitted values
(fv <- outer(qp,pp)*450)

##         particle
## quality        no      yes
##     good 296.8889 37.11111
##     bad  103.1111 12.88889

# deviance (on 1 d.f.)
2*sum(ov*log(ov/fv))

## [1] 54.03045

pchisq(54.03, 1, lower.tail=FALSE)

## [1] 1.974517e-13
```

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```
# pearson's chisquared stat
sum((ov-fv)^2/fv)

## [1] 62.81231

summary(ov)

## Call: xtabs(formula = y ~ quality + particle)
## Number of cases in table: 450
## Number of factors: 2
## Test for independence of all factors:
##  Chisq = 62.81, df = 1, p-value = 2.274e-15
```

Via Logistic Fit

```r
# product multinomial model
(m <- matrix(y, nrow=2))

##      [,1] [,2]
## [1,]  320   80
## [2,]   14   36

modb <- glm(m ~ 1, family=binomial)
deviance(modb)

## [1] 54.03045
```