MAST90104 - Lecture 6 Supplementary

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Consider a less than full-rank model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where assumptions (I) and (V) hold. How do I perform hypothesis tests?

Step 1: Frame your hypothesis as linear combination(s) of $\boldsymbol{\beta}$

In particular, frame it as

$$H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{0}, \quad \text{against} \ \ ; H_1 : \mathbf{L}\boldsymbol{\beta} \neq \mathbf{0}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{t}_1^T \\ \vdots \\ \mathbf{t}_m^T \end{pmatrix}.$$

Make sure that $m \leq r$, where $r = r(\mathbf{X})$.

Step 1: Frame your hypothesis as linear combination(s) of $\boldsymbol{\beta}$

Example: Consider the one-way ANCOVA with interaction:

$$y_{ij} = \mu + \tau_i + \beta x_{ij} + \xi_i x_{ij} + \epsilon_{ij},$$

where we have $I = 3$ levels for our factor τ. Suppose we want to test whether interaction is signficant, i.e., whether different populations should have the same slope. Under interaction model:

$$\text{Slope for level } i = \beta + \xi_i.$$

Hence, we want to test $\xi_1 = \xi_2 = \xi_3$, or equivalent $\xi_1 = \xi_2$ AND $\xi_1 = \xi_3$.

Step 1: Frame your hypothesis as linear combination(s) of $\boldsymbol{\beta}$

Now $\boldsymbol{\beta} = (\mu, \tau_1, \tau_2, \tau_3, \beta, \xi_1, \xi_2, \xi_3)^T$. Our hypothesis can be expressed as:

$$H_0 : \mathbf{L}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{t}_1^T \boldsymbol{\beta} \\ \mathbf{t}_2^T \boldsymbol{\beta} \end{pmatrix} = \mathbf{0}.$$

where $\mathbf{t}_1^T = (0, 0, 0, 0, 0, 1, -1, 0)$ and $\mathbf{t}_2^T = (0, 0, 0, 0, 0, 1, 0, -1)$.

Step 2: Write down your reparameterised full rank model

Let $r = r(\mathbf{X})$. Choose a p by r matrix $\mathbf{C}$ such that $\widetilde{\mathbf{X}} = \mathbf{XC}$ is full rank.

$\mathbf{C}$ can be chosen using `contr.treatment` or `contr.sum`.

Write down your full rank model: $\mathbf{y} = \widetilde{\mathbf{X}}\boldsymbol{\gamma} + \epsilon$.

Express $\boldsymbol{\gamma}$ as a linear combination of $\boldsymbol{\beta}$.

Step 2: Write down your reparameterised full rank model

Example: Back to one-way ANCOVA example.

$$\mathbf{X} = [\mathbf{Z}_1 | \mathbf{Z}_2]$$

where $\mathbf{Z}_1$ is the one-way ANOVA part of the design matrix and $\mathbf{Z}_2$ is the continuous prediction regression part.
Let's say we choose

$$\mathbf{C} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{Q} \end{pmatrix}$$

based on `contr.treatment`, where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is $\mathbf{X}$? And what is $\widetilde{\mathbf{X}}$

Example: By forcing the constraint $\mathbf{X}\beta = \widetilde{\mathbf{X}}\gamma$, we have

$$
\begin{aligned}
\gamma_1 &= \mu + \tau_1 \\
\gamma_2 &= \tau_2 - \tau_1 \\
\gamma_3 &= \tau_3 - \tau_1 \\
\gamma_4 &= \beta + \xi_1 \\
\gamma_5 &= \xi_2 - \xi_1 \\
\gamma_6 &= \xi_3 - \xi_1
\end{aligned}
$$

Step 3: Check that your hypothesis is testable

Our null hypothesis is $H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{0}$. Definition 6.7 requires us to check that

- RHS of our null hypothesis is $\mathbf{0}$.
- For each $\mathbf{t}_j^T\boldsymbol{\beta}$, there exists a r-dimensional vector $\widetilde{\mathbf{t}}_j$ such that $\mathbf{t}_j^T\boldsymbol{\beta} = \widetilde{\mathbf{t}}_j^T\gamma$. Proof left as an exercise.
- $\{\mathbf{t}_1, \ldots, \mathbf{t}_m\}$ is a linearly independent set.

Step 3: Check that your hypothesis is testable

Example: Our hypothesis is $H_0 : \mathbf{L}\boldsymbol{\beta} = \mathbf{0}$, where the first entry on LHS is $\xi_1 - \xi_2$ and the second entry is $\xi_1 - \xi_3$.

- RHS of our null hypothesis is indeed $\mathbf{0}$.
- $\xi_1 - \xi_2 = \gamma_4 - \gamma_5$ and $\xi_1 - \xi_3 = \gamma_4 - \gamma_6$. Hence

$$\mathbf{L}\boldsymbol{\beta} = \widetilde{\mathbf{L}}\boldsymbol{\gamma} = \begin{pmatrix} \widetilde{\mathbf{t}}_1^T \gamma \\ \widetilde{\mathbf{t}}_2^T \gamma \end{pmatrix} = \mathbf{0},$$

  where $\widetilde{\mathbf{t}}_1^T = (0, 0, 0, 0, -1, 0)$ and $\widetilde{\mathbf{t}}_1^T = (0, 0, 0, 0, 0, -1)$
- Clearly, $\{\mathbf{t}_1, \mathbf{t}_2\}$ is a linearly independent set.

Here, $\widetilde{\mathbf{L}}$ allows us to express $\mathbf{L}\boldsymbol{\beta}$ as linear combinations of $\boldsymbol{\gamma}$.

Step 4: Compute estimate for $\mathbf{L}\widehat{\boldsymbol{\beta}}$

Under assumptions (I) and (V), the UMVUE for γ is:

$$\widehat{\gamma} = (\widetilde{\mathbf{X}}^T\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}^T\mathbf{y}.$$

Since $\mathbf{L}\boldsymbol{\beta} = \widetilde{\mathbf{L}}\boldsymbol{\gamma}$, our UMVUE for $\mathbf{L}\boldsymbol{\beta}$ is

$$\mathbf{L}\widehat{\boldsymbol{\beta}} = \widetilde{\mathbf{L}}\widehat{\boldsymbol{\gamma}}.$$

Hence, a shortcut to computing $\mathbf{L}\widehat{\boldsymbol{\beta}}$ is to simply compute $\widetilde{\mathbf{L}}\widehat{\boldsymbol{\gamma}}$.

Step 5: Compute the F-statistic

Compute the residual sum of squares:

$$SS_{res} = \|\mathbf{y} - \widetilde{\mathbf{X}}\widehat{\gamma}\|^2.$$

Under assumption (I) and (V), SS_{res}/σ^2 has a central chi-square distribution with df $= n - r$.

Plug SS_{res}, $\widetilde{\mathbf{X}}$, $\mathbf{L}$, $\widetilde{\mathbf{L}}$, and $\widehat{\beta}$ into F-statistic formula:

$$f = \frac{(\mathbf{L}\widehat{\beta})^T[\widetilde{\mathbf{L}}(\widetilde{\mathbf{X}}^T\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{L}}^T]^{-1}\mathbf{L}\widehat{\beta}/m}{SS_{Res}/(n-r)},$$

where $r(\mathbf{L}) = m$.

Under H_0, $f \sim F_{m,n-r}$.