**33.** Let $F(x) = 1 - \exp(-\alpha x^\beta)$ for $x \geq 0, \alpha > 0, \beta > 0$, and $F(x) = 0$ for $x < 0$. Show that $F$ is a cdf, and find the corresponding density.

$\boxed{\text{Right Continuity - } F(x) \text{ must be continuous}}$

For $x \geq 0$, $F(x)$ is continuos and when $x<0, F(x)=0$

$\boxed{\text{Limit at negative infinity}}$

$\lim_{x \to -\infty} F(x) = \underline{0}$ because $F(x)=0$ when $x<0$

$\boxed{\text{Limit at positive infinity}}$

$\lim_{x \to +\infty} F(x) = 1 - \lim \exp(-ax^\beta)$, as $x \to \infty, -ax^\beta \to \infty$

$\to \exp(-ax^\beta) \to 0 \to \lim_{x \to +\infty} F(x) = 1 - 0 = \underline{1}$

$\boxed{\text{Non-Decreasing}}$

$F'(x) = \frac{d}{dx}\left(1 - \exp(-ax^\beta)\right) = a\beta x^{\beta-1} \exp(-ax^\beta)$

Since $a>0, \beta>0$, $\exp(-ax^\beta) > 0$ for $x \geq 0$, $F'(x) > 0$ for all $x \geq 0$ which indicates $F(x)$ is non-decreasing

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Corresponding density (PDF)

$f(x) = F'(x) = \frac{d}{dx} F(x) = \underline{a\beta x^{\beta-1} \exp(-ax^\beta) \text{ for } x \geq 0}$

$\underline{f(x) = 0 \quad \text{for } x<0}$

**37.** A line segment of length 1 is cut once at random. What is the probability that the longer piece is more than twice the length of the shorter piece?

line is cut at point $x$ where $0 < x < 1$

There will be two pieces $x$ (left) and $(1-x)$ (right)

Two Cases:

- $x$ is shorter (a)
- $x$ is longer (b)

(a) $1 - x > 2x \rightarrow 1 > 3x \rightarrow \frac{1}{3} > x$

(b) $x > 2(1-x) \rightarrow x > 2 - 2x \rightarrow x + 2x > 2 \rightarrow 3x > 2$

$\rightarrow x > \frac{2}{3}$

So $x < \frac{1}{3}$ or $x > \frac{2}{3}$

P(longer piece more that 2x shorter) =

$= \left(\frac{1}{3} - 0\right) + \left(1 - \frac{2}{3}\right) = \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$

**40.** Suppose that $X$ has the density function $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise.

**a.** Find $c$. **b.** Find the cdf. **c.** What is $P(.1 \le X < .5)$?

a

$f(x) = cx^2 \quad 0 \le x \le 1$

$f'(x) = \int_0^1 cx^2\,dx = c\int_0^1 x^2\,dx = c\frac{x^3}{3}\Big|_0^1 = c\frac{1}{3}$

$c \times \frac{1}{3} = 1 \rightarrow \boxed{c=3}$

b

$F(x) = \int_0^x f(t)\,dt = \int_0^x 3t^2\,dt = t^3\Big|_0^x = x^3$

$\boxed{\text{CDF} = x^3 \quad 0 \le x \le 1}$

c

$P(0.1 \le X \le 0.5) = F(0.5) - F(0.1)$

$= 0.5^3 - 0.1^3$

$= 0.125 - 0.001$

$= 0.124$

$\boxed{P(0.1 \le X \le 0.5) = 0.124}$

**43.** Find the probability density for the distance from an event to its nearest neighbor for a Poisson process in three-dimensional space.

Volume: $\frac{4}{3}\pi r^3$ Rate $= \lambda$

$$\left.\begin{array}{l} P(R>r) = \exp\left(-\lambda \frac{4}{3}\pi r^3\right) \\ P(R\le r) = 1-\exp\left(-\lambda \frac{4}{3} r^3\right) \end{array}\right\} \to \text{CDF}$$

PDF

$$f_R(r) = \frac{d}{dr} P(R\le r) = \frac{d}{dr}\left(1-\exp\left(-\lambda \frac{4}{3}\pi r^3\right)\right)$$

$$= \lambda\, 3\cdot\frac{4}{3}\pi r^2 \cdot \exp\left(-\lambda \frac{4}{3}\pi r^3\right)$$

$$= \boxed{\lambda 4\pi r^2 \exp\left(-\lambda \frac{4}{3}\pi r^3\right)}$$

**66.** Let $f(x) = \alpha x^{-\alpha-1}$ for $x \geq 1$ and $f(x) = 0$ otherwise, where $\alpha$ is a positive parameter. Show how to generate random variables from this density from a uniform random number generator.

$$F(x) = \int_1^x a t^{-a-1}\, dt = a\int_1^x t^{-a-1}\, dt = a\left[\frac{t^{-a-1+1}}{-a-1+1}\right]_1^x$$

$$= a\left[-\frac{t^{-a}}{a}\right]_1^x = -t^{-a}\Big|_1^x = -x^{-a} - (-1^{-a})$$

$$= -x^{-a} + 1^{-a} = \underline{1 - x^{-a}}$$

Invert CDF $\rightarrow F(x) = u \sim \text{Uniform}(0,1)$

$$1 - x^{-a} = u \rightarrow 1 - u = x^{-a} \rightarrow x = (1-u)^{-1/a}$$

So if $u$ is a uniform random variable between 0 and 1:

$x = (1-u)^{-1/a}$ is a random variable drawn from the distribution:

$f(x) = ax^{-a-1}$ for $x \geq 1$

- Other Problem. The Rockwell hardness of a metal is determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose the Rockwell hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3

a. If a specimen is acceptable only if its hardness is between 67 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

b. If the acceptable range of hardness is $(70 - c, 70 + c)$, for what value of $c$ would 95% of all specimens have acceptable hardness?

c. What is the probability that at most eight of ten independently selected specimens have a hardness of less than 73.84?

a $Z_1 = \frac{67-70}{3} = -1 \quad Z_2 = \frac{75-70}{3} = \frac{5}{3} \approx 1.67$

$P(Z_1 = -1) = 0.1587$ , $P(Z_2 = 1.67) = 0.9525$

$P(Z_1 = 67 \le X \le Z_2 = 75) = 0.9525 - 0.1587 = \boxed{0.7938}$

b $P(70 - c \le X \le 70 + c) = 0.95 \to c = ?$

Z score for middle 95% are $\pm 1.96$

$Z = \frac{X - M}{\sigma} \to \frac{X - 70}{\sigma} \to \frac{c}{\sigma} \to \frac{c}{3} = 1.96 \to \boxed{c = 5.88}$

$Z_1 = 70 - 5.88 = 64.12$ , $Z_2 = 70 + 5.88 = 75.88$

Range $= (64.12, 75.88)$

c $Z = \frac{73.84 - 70}{3} = 1.28 \quad P(Z = 1.28) = 0.8997$

$Y \sim Bin(10, 0.8997)$

$P(Y \le 8) = \sum_{x=1}^{8} \binom{n}{x} 0.8997^{x} \cdot (1 - 0.8997)^{n-x}$

$= \boxed{0.26506}$