matches is much larger than would be expected by chance alone. This requires a chance model; a simple one stipulates that the nucleotide at each site of fragment 1 occurs randomly with probabilities p_{A1}, p_{G1}, p_{C1}, p_{T1}, and that the second fragment is similarly composed with probabilities $p_{A2}, \ldots, p_{T2}$. What is the chance that the fragments match at a particular site if in fact the identity of the nucleotide on fragment 1 is independent of that on fragment 2? The match probability can be calculated using the law of total probability:

$$\begin{aligned} P(\text{match}) &= P(\text{match}|\text{A on fragment 1})P(\text{A on fragment 1}) + \\ &\qquad \ldots + P(\text{match}|\text{T on fragment 1})P(\text{T on fragment 1}) \\ &= p_{A2}p_{A1} + p_{G2}p_{G1} + p_{C2}p_{C1} + p_{T2}p_{T1} \end{aligned}$$

The problem of determining the probability that they match at k out of a total of n sites is discussed later. ■

1.7 Concluding Remarks

This chapter provides a simple axiomatic development of the mathematical theory of probability. Some subtle issues that arise in a careful analysis of infinite sample spaces have been neglected. Such issues are typically addressed in graduate-level courses in measure theory and probability theory. Certain philosophical questions have also been avoided. One might ask what is meant by the statement "The probability that this coin will land heads up is $\frac{1}{2}$." Two commonly advocated views are the **frequentist approach** and the **Bayesian approach.** According to the frequentist approach, the statement means that if the experiment were repeated many times, the long-run average number of heads would tend to $\frac{1}{2}$. According to the Bayesian approach, the statement is a quantification of the speaker's uncertainty about the outcome of the experiment and thus is a personal or subjective notion; the probability that the coin will land heads up may be different for different speakers, depending on their experience and knowledge of the situation. There has been vigorous and occasionally acrimonious debate among proponents of various versions of these points of view.

In this and ensuing chapters, there are many examples of the use of probability as a model for various phenomena. In any such modeling endeavor, an idealized mathematical theory is hoped to provide an adequate match to characteristics of the phenomenon under study. The standard of adequacy is relative to the field of study and the modeler's goals.

1.8 Problems

1. A coin is tossed three times and the sequence of heads and tails is recorded.

a. List the sample space.

b. List the elements that make up the following events: (1) $A =$ at least two heads, (2) $B =$ the first two tosses are heads, (3) $C =$ the last toss is a tail.

c. List the elements of the following events: (1) A^c, (2) $A \cap B$, (3) $A \cup C$.

2. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

a. List the sample space.
b. List the elements that make up the following events: (1) $A =$ the sum of the two values is at least 5, (2) $B =$ the value of the first die is higher than the value of the second, (3) $C =$ the first value is 4.
c. List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.

3. An urn contains three red balls, two green balls, and one white ball. Three balls are drawn without replacement from the urn, and the colors are noted in sequence. List the sample space. Define events A, B, and C as you wish and find their unions and intersections.

4. Draw Venn diagrams to illustrate De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

5. Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

6. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions in this chapter.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

7. Prove Bonferroni's inequality:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

8. Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

9. The weather forecaster says that the probability of rain on Saturday is 25% and that the probability of rain on Sunday is 25%. Is the probability of rain during the weekend 50%? Why or why not?

10. Make up another example of Simpson's paradox by changing the numbers in Example B of Section 1.4.

11. The first three digits of a university telephone exchange are 452. If all the sequences of the remaining four digits are equally likely, what is the probability that a randomly selected university phone number contains seven distinct digits?

12. In a game of poker, five players are each dealt 5 cards from a 52-card deck. How many ways are there to deal the cards?

13. In a game of poker, what is the probability that a five-card hand will contain (a) a straight (five cards in unbroken numerical sequence), (b) four of a kind, and (c) a full house (three cards of one value and two cards of another value)?

14. The four players in a bridge game are each dealt 13 cards. How many ways are there to do this?

15. How many different meals can be made from four kinds of meat, six vegetables, and three starches if a meal consists of one selection from each group?

16. How many different letter arrangements can be obtained from the letters of the word *statistically*, using all the letters?

17. In acceptance sampling, a purchaser samples 4 items from a lot of 100 and rejects the lot if 1 or more are defective. Graph the probability that the lot is accepted as a function of the percentage of defective items in the lot.

18. A lot of n items contains k defectives, and m are selected randomly and inspected. How should the value of m be chosen so that the probability that at least one defective item turns up is .90? Apply your answer to (a) $n = 1000$, $k = 10$, and (b) $n = 10{,}000$, $k = 100$.

19. A committee consists of five Chicanos, two Asians, three African Americans, and two Caucasians.

a. A subcommittee of four is chosen at random. What is the probability that all the ethnic groups are represented on the subcommittee?
b. Answer the question for part (a) if a subcommittee of five is chosen.

20. A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

21. A fair coin is tossed five times. What is the probability of getting a sequence of three heads?

22. A standard deck of 52 cards is shuffled thoroughly, and n cards are turned up. What is the probability that a face card turns up? For what value of n is this probability about .5?

23. How many ways are there to place n indistinguishable balls in n urns so that exactly one urn is empty?

24. If n balls are distributed randomly into k urns, what is the probability that the last urn contains j balls?

25. A woman getting dressed up for a night out is asked by her significant other to wear a red dress, high-heeled sneakers, and a wig. In how many orders can she put on these objects?

26. The game of Mastermind starts in the following way: One player selects four pegs, each peg having six possible colors, and places them in a line. The second player then tries to guess the sequence of colors. What is the probability of guessing correctly?

27. If a five-letter word is formed at random (meaning that all sequences of five letters are equally likely), what is the probability that no letter occurs more than once?

28. How many ways are there to encode the 26-letter English alphabet into 8-bit binary words (sequences of eight 0s and 1s)?

29. A poker player is dealt three spades and two hearts. He discards the two hearts and draws two more cards. What is the probability that he draws two more spades?

30. A group of 60 second graders is to be randomly assigned to two classes of 30 each. (The random assignment is ordered by the school district to ensure against any bias.) Five of the second graders, Marcelle, Sarah, Michelle, Katy, and Camerin, are close friends. What is the probability that they will all be in the same class? What is the probability that exactly four of them will be? What is the probability that Marcelle will be in one class and her friends in the other?

31. Six male and six female dancers perform the Virginia reel. This dance requires that they form a line consisting of six male/female pairs. How many such arrangements are there?

32. A wine taster claims that she can distinguish four vintages of a particular Cabernet. What is the probability that she can do this by merely guessing? (She is confronted with four unlabeled glasses.)

33. An elevator containing five people can stop at any of seven floors. What is the probability that no two people get off at the same floor? Assume that the occupants act independently and that all floors are equally likely for each occupant.

34. Prove the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m-n}{n-k} = \binom{m}{n}$$

(*Hint*: How can each of the summands be interpreted?)

35. Prove the following two identities both algebraically and by interpreting their meaning combinatorially.

a. $\binom{n}{r} = \binom{n}{n-r}$

b. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

36. What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$?

37. What is the coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$?

38. A child has six blocks, three of which are red and three of which are green. How many patterns can she make by placing them all in a line? If she is given three white blocks, how many total patterns can she make by placing all nine blocks in a line?

39. A monkey at a typewriter types each of the 26 letters of the alphabet exactly once, the order being random.

a. What is the probability that the word *Hamlet* appears somewhere in the string of letters?

b. How many independent monkey typists would you need in order that the probability that the word appears is at least .90?

40. In how many ways can two octopi shake hands? (There are a number of ways to interpret this question—choose one.)

41. A drawer of socks contains seven black socks, eight blue socks, and nine green socks. Two socks are chosen in the dark.

a. What is the probability that they match?
b. What is the probability that a black pair is chosen?

42. How many ways can 11 boys on a soccer team be grouped into 4 forwards, 3 midfielders, 3 defenders, and 1 goalie?

43. A software development company has three jobs to do. Two of the jobs require three programmers, and the other requires four. If the company employs ten programmers, how many different ways are there to assign them to the jobs?

44. In how many ways can 12 people be divided into three groups of 4 for an evening of bridge? In how many ways can this be done if the 12 consist of six pairs of partners?

45. Show that if the conditional probabilities exist, then

$$\begin{aligned} &P(A_1 \cap A_2 \cap \cdots \cap A_n) \\ &\quad = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \end{aligned}$$

46. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, a ball is drawn from urn B.

a. What is the probability that a red ball is drawn?
b. If a red ball is drawn, what is the probability that the coin landed heads up?

47. Urn A has four red, three blue, and two green balls. Urn B has two red, three blue, and four green balls. A ball is drawn from urn A and put into urn B, and then a ball is drawn from urn B.

a. What is the probability that a red ball is drawn from urn B?
b. If a red ball is drawn from urn B, what is the probability that a red ball was drawn from urn A?

48. An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

a. What is the probability that the second ball drawn is white?
b. If the second ball drawn is white, what is the probability that the first ball drawn was red?

49. A fair coin is tossed three times.

a. What is the probability of two or more heads given that there was at least one head?
b. What is the probability given that there was at least one tail?

50. Two dice are rolled, and the sum of the face values is six. What is the probability that at least one of the dice came up a three?

51. Answer Problem 50 again given that the sum is less than six.

52. Suppose that 5 cards are dealt from a 52-card deck and the first one is a king. What is the probability of at least one more king?

53. A fire insurance company has high-risk, medium-risk, and low-risk clients, who have, respectively, probabilities .02, .01, and .0025 of filing claims within a given year. The proportions of the numbers of clients in the three categories are .10, .20, and .70, respectively. What proportion of the claims filed each year come from high-risk clients?

54. This problem introduces a simple meteorological model, more complicated versions of which have been proposed in the meteorological literature. Consider a sequence of days and let R_i denote the event that it rains on day i. Suppose that $P(R_i \mid R_{i-1}) = \alpha$ and $P(R_i^c \mid R_{i-1}^c) = \beta$. Suppose further that only today's weather is relevant to predicting tomorrow's; that is, $P(R_i \mid R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i \mid R_{i-1})$.

a. If the probability of rain today is p, what is the probability of rain tomorrow?
b. What is the probability of rain the day after tomorrow?
c. What is the probability of rain n days from now? What happens as n approaches infinity?

55. This problem continues Example D of Section 1.5 and concerns occupational mobility.

a. Find $P(M_1 \mid M_2)$ and $P(L_1 \mid L_2)$.
b. Find the proportions that will be in the three occupational levels in the third generation. To do this, assume that a son's occupational status depends on his father's status, but that given his father's status, it does not depend on his grandfather's.

56. A couple has two children. What is the probability that both are girls given that the oldest is a girl? What is the probability that both are girls given that one of them is a girl?

57. There are three cabinets, A, B, and C, each of which has two drawers. Each drawer contains one coin; A has two gold coins, B has two silver coins, and C has one gold and one silver coin. A cabinet is chosen at random, one drawer is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

58. A teacher tells three boys, Drew, Chris, and Jason, that two of them will have to stay after school to help her clean erasers and that one of them will be able to leave. She further says that she has made the decision as to who will leave and who will stay at random by rolling a special three-sided Dungeons and Dragons die. Drew wants to leave to play soccer and has a clever idea about how to increase his chances of doing so. He figures that one of Jason and Chris will certainly stay and asks the teacher to tell him the name of one of the two who will stay. Drew's idea is that if, for example, Jason is named, then he and Chris are left and they each have a probability .5 of leaving; similarly, if Chris is named, Drew's probability of leaving is still .5. Thus, by merely asking the teacher a question, Drew will increase his probability of leaving from $\frac{1}{3}$ to $\frac{1}{2}$. What do you think of this scheme?

59. A box has three coins. One has two heads, one has two tails, and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped, and comes up heads.

a. What is the probability that the coin chosen is the two-headed coin?
b. What is the probability that if it is thrown another time it will come up heads?
c. Answer part (a) again, supposing that the coin is thrown a second time and comes up heads again.

60. A factory runs three shifts. In a given day, 1% of the items produced by the first shift are defective, 2% of the second shift's items are defective, and 5% of the third shift's items are defective. If the shifts all have the same productivity, what percentage of the items produced in a day are defective? If an item is defective, what is the probability that it was produced by the third shift?

61. Suppose that chips for an integrated circuit are tested and that the probability that they are detected if they are defective is .95, and the probability that they are declared sound if in fact they are sound is .97. If .5% of the chips are faulty, what is the probability that a chip that is declared faulty is sound?

62. Show that if $P(A \mid E) \geq P(B \mid E)$ and $P(A \mid E^c) \geq P(B \mid E^c)$, then $P(A) \geq P(B)$.

63. Suppose that the probability of living to be older than 70 is .6 and the probability of living to be older than 80 is .2. If a person reaches her 70th birthday, what is the probability that she will celebrate her 80th?

64. If B is an event, with $P(B) > 0$, show that the set function $Q(A) = P(A \mid B)$ satisfies the axioms for a probability measure. Thus, for example,

$$P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) - P(A \cap C \mid B)$$

65. Show that if A and B are independent, then A and B^c as well as A^c and B^c are independent.

66. Show that $\emptyset$ is independent of A for any A.

67. Show that if A and B are independent, then

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

68. If A is independent of B and B is independent of C, then A is independent of C. Prove this statement or give a counterexample if it is false.

69. If A and B are disjoint, can they be independent?

70. If $A \subset B$, can A and B be independent?

71. Show that if A, B, and C are mutually independent, then $A \cap B$ and C are independent and $A \cup B$ and C are independent.

72. Suppose that n components are connected in series. For each unit, there is a backup unit, and the system fails if and only if both a unit and its backup fail. Assuming that all the units are independent and fail with probability p, what is the probability that the system works? For $n = 10$ and $p = .05$, compare these results with those of Example F in Section 1.6.

73. A system has n independent units, each of which fails with probability p. The system fails only if k or more of the units fail. What is the probability that the system fails?

74. What is the probability that the following system works if each unit fails independently with probability p (see Figure 1.5)?

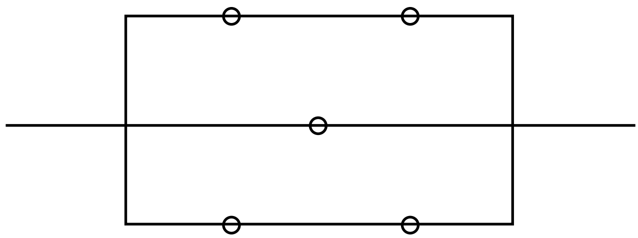

FIGURE **1.5**

75. This problem deals with an elementary aspect of a simple branching process. A population starts with one member; at time $t = 1$, it either divides with probability p or dies with probability $1 - p$. If it divides, then both of its children behave independently with the same two alternatives at time $t = 2$. What is the probability that there are no members in the third generation? For what value of p is this probability equal to .5?

76. Here is a simple model of a queue. The queue runs in discrete time ($t = 0, 1, 2, \ldots$), and at each unit of time the first person in the queue is served with probability p and, independently, a new person arrives with probability q. At time $t = 0$, there is one person in the queue. Find the probabilities that there are 0, 1, 2, 3 people in line at time $t = 2$.

77. A player throws darts at a target. On each trial, independently of the other trials, he hits the bull's-eye with probability .05. How many times should he throw so that his probability of hitting the bull's-eye at least once is .5?

78. This problem introduces some aspects of a simple genetic model. Assume that genes in an organism occur in pairs and that each member of the pair can be either of the types a or A. The possible genotypes of an organism are then AA, Aa, and aa (Aa and aA are equivalent). When two organisms mate, each independently contributes one of its two genes; either one of the pair is transmitted with probability .5.

a. Suppose that the genotypes of the parents are AA and Aa. Find the possible genotypes of their offspring and the corresponding probabilities.

b. Suppose that the probabilities of the genotypes AA, Aa, and aa are p, $2q$, and r, respectively, in the first generation. Find the probabilities in the second and third generations, and show that these are the same. This result is called the Hardy-Weinberg Law.

c. Compute the probabilities for the second and third generations as in part (b) but under the additional assumption that the probabilities that an individual of type AA, Aa, or aa survives to mate are u, v, and w, respectively.

79. Many human diseases are genetically transmitted (for example, hemophilia or Tay-Sachs disease). Here is a simple model for such a disease. The genotype aa is diseased and dies before it mates. The genotype Aa is a carrier but is not diseased. The genotype AA is not a carrier and is not diseased.

a. If two carriers mate, what are the probabilities that their offspring are of each of the three genotypes?

b. If the male offspring of two carriers is not diseased, what is the probability that he is a carrier?

c. Suppose that the nondiseased offspring of part (b) mates with a member of the population for whom no family history is available and who is thus assumed to have probability p of being a carrier (p is a very small number). What are the probabilities that their first offspring has the genotypes AA, Aa, and aa?

d. Suppose that the first offspring of part (c) is not diseased. What is the probability that the father is a carrier in light of this evidence?

80. If a parent has genotype Aa, he transmits either A or a to an offspring (each with a $\frac{1}{2}$ chance). The gene he transmits to one offspring is independent of the one he transmits to another. Consider a parent with three children and the following events: $A = \{$children 1 and 2 have the same gene$\}$, $B = \{$children 1 and 3 have the same gene$\}$, $C = \{$children 2 and 3 have the same gene$\}$. Show that these events are pairwise independent but not mutually independent.

2.4 Concluding Remarks

This chapter introduced the concept of a random variable, one of the fundamental ideas of probability theory. A fully rigorous discussion of random variables requires a background in measure theory. The development here is sufficient for the needs of this course.

Discrete and continuous random variables have been defined, and it should be mentioned that more general random variables can also be defined and are useful on occasion. In particular, it makes sense to consider random variables that have both a discrete and a continuous component. For example, the lifetime of a transistor might be 0 with some probability $p > 0$ if it does not function at all; if it does function, the lifetime could be modeled as a continuous random variable.

2.5 Problems

1. Suppose that X is a discrete random variable with $P(X = 0) = .25$, $P(X = 1) = .125$, $P(X = 2) = .125$, and $P(X = 3) = .5$. Graph the frequency function and the cumulative distribution function of X.

2. An experiment consists of throwing a fair coin four times. Find the frequency function and the cumulative distribution function of the following random variables: (a) the number of heads before the first tail, (b) the number of heads following the first tail, (c) the number of heads minus the number of tails, and (d) the number of tails times the number of heads.

3. The following table shows the cumulative distribution function of a discrete random variable. Find the frequency function.

k	$F(k)$
0	0
1	.1
2	.3
3	.7
4	.8
5	1.0

4. If X is an integer-valued random variable, show that the frequency function is related to the cdf by $p(k) = F(k) - F(k-1)$.

5. Show that $P(u < X \leq v) = F(v) - F(u)$ for any u and v in the cases that (a) X is a discrete random variable and (b) X is a continuous random variable.

6. Let A and B be events, and let I_A and I_B be the associated indicator random variables. Show that

$$I_{A\cap B} = I_A I_B = \min(I_A, I_B)$$

and

$$I_{A\cup B} = \max(I_A, I_B)$$

7. Find the cdf of a Bernoulli random variable.

8. Show that the binomial probabilities sum to 1.

9. For what values of p is a two-out-of-three majority decoder better than transmission of the message once?

10. Appending three extra bits to a 4-bit word in a particular way (a Hamming code) allows detection and correction of up to one error in any of the bits. If each bit has probability .05 of being changed during communication, and the bits are changed independently of each other, what is the probability that the word is correctly received (that is, 0 or 1 bit is in error)? How does this probability compare to the probability that the word will be transmitted correctly with no check bits, in which case all four bits would have to be transmitted correctly for the word to be correct?

11. Consider the binomial distribution with n trials and probability p of success on each trial. For what value of k is $P(X = k)$ maximized? This value is called the **mode** of the distribution. (*Hint:* Consider the ratio of successive terms.)

12. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses?

13. A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 12 items or more correct.

a. What is the probability that the student passes?
b. Answer the question in part (a) again, assuming that the student can eliminate two of the choices on each question.

14. Two boys play basketball in the following way. They take turns shooting and stop when a basket is made. Player A goes first and has probability p_1 of making a basket on any throw. Player B, who shoots second, has probability p_2 of making a basket. The outcomes of the successive trials are assumed to be independent.

a. Find the frequency function for the total number of attempts.
b. What is the probability that player A wins?

15. Two teams, A and B, play a series of games. If team A has probability .4 of winning each game, is it to its advantage to play the best three out of five games or the best four out of seven? Assume the outcomes of successive games are independent.

16. Show that if n approaches ∞ and r/n approaches p and m is fixed, the hypergeometric frequency function tends to the binomial frequency function. Give a heuristic argument for why this is true.

17. Suppose that in a sequence of independent Bernoulli trials, each with probability of success p, the number of failures up to the first success is counted. What is the frequency function for this random variable?

18. Continuing with Problem 17, find the frequency function for the number of failures up to the rth success.

19. Find an expression for the cumulative distribution function of a geometric random variable.

20. If X is a geometric random variable with $p = .5$, for what value of k is $P(X \leq k) \approx .99$?

21. If X is a geometric random variable, show that

$$P(X > n + k - 1 | X > n - 1) = P(X > k)$$

In light of the construction of a geometric distribution from a sequence of independent Bernoulli trials, how can this be interpreted so that it is "obvious"?

22. Three identical fair coins are thrown simultaneously until all three show the same face. What is the probability that they are thrown more than three times?

23. In a sequence of independent trials with probability p of success, what is the probability that there are r successes before the kth failure?

24. (Banach Match Problem) A pipe smoker carries one box of matches in his left pocket and one box in his right. Initially, each box contains n matches. If he needs a match, the smoker is equally likely to choose either pocket. What is the frequency function for the number of matches in the other box when he first discovers that one box is empty?

25. The probability of being dealt a royal straight flush (ace, king, queen, jack, and ten of the same suit) in poker is about 1.3×10^{-8}. Suppose that an avid poker player sees 100 hands a week, 52 weeks a year, for 20 years.

a. What is the probability that she is never dealt a royal straight flush dealt?
b. What is the probability that she is dealt exactly two royal straight flushes?

26. The university administration assures a mathematician that he has only 1 chance in 10,000 of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years, and always rides the elevator up to his office when he first arrives, what is the probability that he will never be trapped? That he will be trapped once? Twice? Assume that the outcomes on all the days are mutually independent (a dubious assumption in practice).

27. Suppose that a rare disease has an incidence of 1 in 1000. Assuming that members of the population are affected independently, find the probability of k cases in a population of 100,000 for $k = 0, 1, 2$.

28. Let $p_0, p_1, \ldots, p_n$ denote the probability mass function of the binomial distribution with parameters n and p. Let $q = 1 - p$. Show that the binomial probabilities can be computed recursively by $p_0 = q^n$ and

$$p_k = \frac{(n - k + 1)p}{kq} p_{k-1}, \qquad k = 1, 2, \ldots, n$$

Use this relation to find $P(X \leq 4)$ for $n = 9000$ and $p = .0005$.

41. Find the upper and lower quartiles of the exponential distribution.

42. Find the probability density for the distance from an event to its nearest neighbor for a Poisson process in the plane.

43. Find the probability density for the distance from an event to its nearest neighbor for a Poisson process in three-dimensional space.

44. Let T be an exponential random variable with parameter λ. Let X be a discrete random variable defined as $X = k$ if $k \leq T < k + 1, k = 0, 1, \ldots.$ Find the frequency function of X.

45. Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = .1$.

a. Find the probability that the lifetime is less than 10.
b. Find the probability that the lifetime is between 5 and 15.
c. Find t such that the probability that the lifetime is greater than t is .01.

46. T is an exponential random variable, and $P(T < 1) = .05$. What is λ?

47. If $\alpha > 1$, show that the gamma density has a maximum at $(\alpha - 1)/\lambda$.

48. Show that the gamma density integrates to 1.

49. The gamma function is a generalized factorial function.

a. Show that $\Gamma(1) = 1$.
b. Show that $\Gamma(x + 1) = x\Gamma(x)$. (*Hint:* Use integration by parts.)
c. Conclude that $\Gamma(n) = (n - 1)!$, for $n = 1, 2, 3, \ldots.$
d. Use the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to show that, if n is an odd integer,

$$\Gamma\left(\frac{n}{2}\right) = \frac{\sqrt{\pi}(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}$$

50. Show by a change of variables that

$$\begin{aligned}\Gamma(x) &= 2\int_0^\infty t^{2x-1}e^{-t^2}\,dt \\ &= \int_{-\infty}^\infty e^{xt}e^{-e^t}\,dt\end{aligned}$$

51. Show that the normal density integrates to 1. (*Hint:* First make a change of variables to reduce the integral to that for the standard normal. The problem is then to show that $\int_{-\infty}^\infty \exp(-x^2/2)\,dx = \sqrt{2\pi}$. Square both sides and reexpress the problem as that of showing

$$\left(\int_{-\infty}^\infty \exp(-x^2/2)\,dx\right)\left(\int_{-\infty}^\infty \exp(-y^2/2)\,dy\right) = 2\pi$$

Finally, write the product of integrals as a double integral and change to polar coordinates.)

52. Suppose that in a certain population, individuals' heights are approximately normally distributed with parameters $\mu = 70$ and $\sigma = 3$ in.

a. What proportion of the population is over 6 ft. tall?
b. What is the distribution of heights if they are expressed in centimeters? In meters?

53. Let X be a normal random variable with $\mu = 5$ and $\sigma = 10$. Find (a) $P(X > 10)$, (b) $P(-20 < X < 15)$, and (c) the value of x such that $P(X > x) = .05$.

54. If $X \sim N(\mu, \sigma^2)$, show that $P(|X - \mu| \leq .675\sigma) = .5$.

55. $X \sim N(\mu, \sigma^2)$, find the value of c in terms of σ such that $P(\mu - c \leq X \leq \mu + c) = .95$.

56. If $X \sim N(0, \sigma^2)$, find the density of $Y = |X|$.

57. $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, where $a < 0$, show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

58. If U is uniform on [0, 1], find the density function of $\sqrt{U}$.

59. If U is uniform on $[-1, 1]$, find the density function of U^2.

60. Find the density function of $Y = e^Z$, where $Z \sim N(\mu, \sigma^2)$. This is called the **lognormal density,** since $\log Y$ is normally distributed.

61. Find the density of cX when X follows a gamma distribution. Show that only λ is affected by such a transformation, which justifies calling λ a scale parameter.

62. Show that if X has a density function f_X and $Y = aX + b$, then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$$

63. Suppose that Θ follows a uniform distribution on the interval $[-\pi/2, \pi/2]$. Find the cdf and density of $\tan \Theta$.

64. A particle of mass m has a random velocity, V, which is normally distributed with parameters $\mu = 0$ and σ. Find the density function of the kinetic energy, $E = \frac{1}{2}mV^2$.

65. How could random variables with the following density function be generated from a uniform random number generator?

$$f(x) = \frac{1 + \alpha x}{2}, \qquad -1 \leq x \leq 1, \qquad -1 \leq \alpha \leq 1$$

66. Let $f(x) = \alpha x^{-\alpha-1}$ for $x \geq 1$ and $f(x) = 0$ otherwise, where α is a positive parameter. Show how to generate random variables from this density from a uniform random number generator.

67. The **Weibull** cumulative distribution function is

$$F(x) = 1 - e^{-(x/\alpha)^\beta}, \qquad x \geq 0, \qquad \alpha > 0, \qquad \beta > 0$$

a. Find the density function.

b. Show that if W follows a Weibull distribution, then $X = (W/\alpha)^{\beta}$ follows an exponential distribution.

c. How could Weibull random variables be generated from a uniform random number generator?

68. If the radius of a circle is an exponential random variable, find the density function of the area.

69. If the radius of a sphere is an exponential random variable, find the density function of the volume.

70. Let U be a uniform random variable. Find the density function of $V = U^{-\alpha}$, $\alpha > 0$. Compare the rates of decrease of the tails of the densities as a function of α. Does the comparison make sense intuitively?

71. This problem shows one way to generate discrete random variables from a uniform random number generator. Suppose that F is the cdf of an integer-valued random variable; let U be uniform on [0, 1]. Define a random variable $Y = k$ if $F(k-1) < U \leq F(k)$. Show that Y has cdf F. Apply this result to show how to generate geometric random variables from uniform random variables.

72. One of the most commonly used (but not one of the best) methods of generating pseudorandom numbers is the linear congruential method, which works as follows. Let x_0 be an initial number (the "seed"). The sequence is generated recursively as

$$x_n = (ax_{n-1} + c) \bmod m$$

a. Choose values of a, c, and m, and try this out. Do the sequences "look" random?

b. Making good choices of a, c, and m involves both art and theory. The following are some values that have been proposed: (1) $a = 69069$, $c = 0$, $m = 2^{31}$; (2) $a = 65539$, $c = 0$, $m = 2^{31}$. The latter is an infamous generator called RANDU. Try out these schemes, and examine the results.

For example, if $n = 100$ and $\alpha = .95$, this probability is .96. In words, this means that the probability is .96 that the range of 100 independent random variables covers 95% or more of the probability mass, or, with probability .96, 95% of all further observations from the same distribution will fall between the minimum and maximum. This statement does not depend on the actual form of the distribution. ■

3.8 Problems

1. The joint frequency function of two discrete random variables, X and Y, is given in the following table:

	x			
y	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

a. Find the marginal frequency functions of X and Y.
b. Find the conditional frequency function of X given $Y = 1$ and of Y given $X = 1$.

2. An urn contains p black balls, q white balls, and r red balls; and n balls are chosen without replacement.

a. Find the joint distribution of the numbers of black, white, and red balls in the sample.
b. Find the joint distribution of the numbers of black and white balls in the sample.
c. Find the marginal distribution of the number of white balls in the sample.

3. Three players play 10 independent rounds of a game, and each player has probability $\frac{1}{3}$ of winning each round. Find the joint distribution of the numbers of games won by each of the three players.

4. A sieve is made of a square mesh of wires. Each wire has diameter d, and the holes in the mesh are squares whose side length is w. A spherical particle of radius r is dropped on the mesh. What is the probability that it passes through? What is the probability that it fails to pass through if it is dropped n times? (Calculations such as these are relevant to the theory of sieving for analyzing the size distribution of particulate matter.)

5. (Buffon's Needle Problem) A needle of length L is dropped randomly on a plane ruled with parallel lines that are a distance D apart, where $D \geq L$. Show that the probability that the needle comes to rest crossing a line is $2L/(\pi D)$. Explain how this gives a mechanical means of estimating the value of π.

6. A point is chosen randomly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the marginal densities of the x and y coordinates of the point.

7. Find the joint and marginal densities corresponding to the cdf

$$F(x, y) = (1-e^{-\alpha x})(1-e^{-\beta y}), \qquad x \geq 0, \qquad y \geq 0, \qquad \alpha > 0, \qquad \beta > 0$$

8. Let X and Y have the joint density

$$f(x, y) = \frac{6}{7}(x + y)^2, \qquad 0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

a. By integrating over the appropriate regions, find (i) $P(X > Y)$, (ii) $P(X + Y \leq 1)$, (iii) $P(X \leq \frac{1}{2})$.
b. Find the marginal densities of X and Y.
c. Find the two conditional densities.

9. Suppose that (X, Y) is uniformly distributed over the region defined by $0 \leq y \leq 1 - x^2$ and $-1 \leq x \leq 1$.

a. Find the marginal densities of X and Y.
b. Find the two conditional densities.

10. A point is uniformly distributed in a unit sphere in three dimensions.

a. Find the marginal densities of the x, y, and z coordinates.
b. Find the joint density of the x and y coordinates.
c. Find the density of the xy coordinates conditional on $Z = 0$.

11. Let U_1, U_2, and U_3 be independent random variables uniform on [0, 1]. Find the probability that the roots of the quadratic $U_1x^2 + U_2x + U_3$ are real.

12. Let

$$f(x, y) = c(x^2 - y^2)e^{-x}, \qquad 0 \leq x < \infty, \qquad -x \leq y < x$$

a. Find c.
b. Find the marginal densities.
c. Find the conditional densities.

13. A fair coin is thrown once; if it lands heads up, it is thrown a second time. Find the frequency function of the total number of heads.

14. Suppose that

$$f(x, y) = xe^{-x(y+1)}, \qquad 0 \leq x < \infty, \qquad 0 \leq y < \infty$$

a. Find the marginal densities of X and Y. Are X and Y independent?
b. Find the conditional densities of X and Y.

15. Suppose that X and Y have the joint density function

$$f(x, y) = c\sqrt{1 - x^2 - y^2}, \qquad x^2 + y^2 \leq 1$$

a. Find c.

b. Sketch the joint density.
c. Find $P(X^2 + Y^2) \leq \frac{1}{2}$.
d. Find the marginal densities of X and Y. Are X and Y independent random variables?
e. Find the conditional densities.

16. What is the probability density of the time between the arrival of the two packets of Example E in Section 3.4?

17. Let (X, Y) be a random point chosen uniformly on the region $R = \{(x, y) : |x| + |y| \leq 1\}$.

a. Sketch R.
b. Find the marginal densities of X and Y using your sketch. Be careful of the range of integration.
c. Find the conditional density of Y given X.

18. Let X and Y have the joint density function

$$f(x, y) = k(x - y), \qquad 0 \leq y \leq x \leq 1$$

and 0 elsewhere.

a. Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.
b. Find k.
c. Find the marginal densities of X and Y.
d. Find the conditional densities of Y given X and X given Y.

19. Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β, respectively. Find (a) $P(T_1 > T_2)$ and (b) $P(T_1 > 2T_2)$.

20. If X_1 is uniform on [0, 1], and, conditional on X_1, X_2, is uniform on $[0, X_1]$, find the joint and marginal distributions of X_1 and X_2.

21. An instrument is used to measure very small concentrations, X, of a certain chemical in soil samples. Suppose that the values of X in those soils in which the chemical is present is modeled as a random variable with density function $f(x)$. The assay of a soil reports a concentration only if the chemical is first determined to be present. At very low concentrations, however, the chemical may fail to be detected even if it is present. This phenomenon is modeled by assuming that if the concentration is x, the chemical is detected with probability $R(x)$. Let Y denote the concentration of a chemical in a soil in which it has been determined to be present. Show that the density function of Y is

$$g(y) = \frac{R(y)f(y)}{\int_0^\infty R(x)f(x)\,dx}$$

22. Consider a Poisson process on the real line, and denote by $N(t_1, t_2)$ the number of events in the interval (t_1, t_2). If $t_0 < t_1 < t_2$, find the conditional distribution of $N(t_0, t_1)$ given that $N(t_0, t_2) = n$. (*Hint:* Use the fact that the numbers of events in disjoint subsets are independent.)

23. Suppose that, conditional on N, X has a binomial distribution with N trials and probability p of success, and that N is a binomial random variable with m trials and probability r of success. Find the unconditional distribution of X.

24. Let P have a uniform distribution on [0, 1], and, conditional on $P = p$, let X have a Bernoulli distribution with parameter p. Find the conditional distribution of P given X.

25. Let X have the density function f, and let $Y = X$ with probability $\frac{1}{2}$ and $Y = -X$ with probability $\frac{1}{2}$. Show that the density of Y is symmetric about zero—that is, $f_Y(y) = f_Y(-y)$.

26. Spherical particles whose radii have the density function $f_R(r)$ are dropped on a mesh as in Problem 4. Find an expression for the density function of the particles that pass through.

27. Prove that X and Y are independent if and only if $f_{X|Y}(x|y) = f_X(x)$ for all x and y.

28. Show that $C(u, v) = uv$ is a copula. Why is it called "the independence copula"?

29. Use the Farlie-Morgenstern copula to construct a bivariate density whose marginal densities are exponential. Find an expression for the joint density.

30. For $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$, show that $C(u, v) = \min(u^{1-\alpha}v, uv^{1-\beta})$ is a copula (the Marshall-Olkin copula). What is the joint density?

31. Suppose that (X, Y) is uniform on the disk of radius 1 as in Example E of Section 3.3. Without doing any calculations, argue that X and Y are not independent.

32. Continuing Example E of Section 3.5.2, suppose you had to guess a value of θ. One plausible guess would be the value of θ that maximizes the posterior density. Find that value. Does the result make intuitive sense?

33. Suppose that, as in Example E of Section 3.5.2, your prior opinion that the coin will land with heads up is represented by a uniform density on [0, 1]. You now spin the coin repeatedly and record the number of times, N, until a heads comes up. So if heads comes up on the first spin, $N = 1$, etc.

a. Find the posterior density of Θ given N.
b. Do this with a newly minted penny and graph the posterior density.

34. This problem continues Example E of Section 3.5.2. In that example, the prior opinion for the value of Θ was represented by the uniform density. Suppose that the prior density had been a beta density with parameters $a = b = 3$, reflecting a stronger prior belief that the chance of a 1 was near $\frac{1}{2}$. Graph this prior density. Following the reasoning of the example, find the posterior density, plot it, and compare it to the posterior density shown in the example.

35. Find a newly minted penny. Place it on its edge and spin it 20 times. Following Example E of Section 3.5.2, calculate and graph the posterior distribution. Spin another 20 times, and calculate and graph the posterior based on all 40 spins. What happens as you increase the number of spins?

36. Let $f(x) = (1 + \alpha x)/2$, for $-1 \le x \le 1$ and $-1 \le \alpha \le 1$.

a. Describe an algorithm to generate random variables from this density using the rejection method.
b. Write a computer program to do so, and test it out.

37. Let $f(x) = 6x^2(1 - x)^2$, for $-1 \le x \le 1$.

a. Describe an algorithm to generate random variables from this density using the rejection method. In what proportion of the trials will the acceptance step be taken?
b. Write a computer program to do so, and test it out.

38. Show that the number of iterations necessary to generate a random variable using the rejection method is a geometric random variable, and evaluate the parameter of the geometric frequency function. Show that in order to keep the number of iterations small, $M(x)$ should be chosen to be close to $f(x)$.

39. Show that the following method of generating discrete random variables works (D. R. Fredkin). Suppose, for concreteness, that X takes on values $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$. Let U be a uniform random variable. If $U < p_0$, return $X = 0$. If not, replace U by $U - p_0$, and if the new U is less than p_1, return $X = 1$. If not, decrement U by p_1, compare U to p_2, etc.

40. Suppose that X and Y are discrete random variables with a joint probability mass function $p_{XY}(x, y)$. Show that the following procedure generates a random variable $X \sim p_{X|Y}(x|y)$.

a. Generate $X \sim p_X(x)$.
b. Accept X with probability $p(y|X)$.
c. If X is accepted, terminate and return X. Otherwise go to Step a.

Now suppose that X is uniformly distributed on the integers $1, 2, \ldots, 100$ and that given $X = x$, Y is uniform on the integers $1, 2, \ldots, x$. You observe $Y = 44$. What does this tell you about X? Simulate the distribution of X, given $Y = 44$, 1000 times and make a histogram of the value obtained. How would you estimate $E(X|Y = 44)$?

41. How could you extend the procedure of the previous problem in the case that X and Y are continuous random variables?

42. **a.** Let T be an exponential random variable with parameter λ; let W be a random variable independent of T, which is ± 1 with probability $\frac{1}{2}$ each; and let $X = WT$. Show that the density of X is

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

which is called the **double exponential density.**

b. Show that for some constant c,

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \le c e^{-|x|}$$

Use this result and that of part (a) to show how to use the rejection method to generate random variables from a standard normal density.

43. Let U_1 and U_2 be independent and uniform on [0, 1]. Find and sketch the density function of $S = U_1 + U_2$.

44. Let N_1 and N_2 be independent random variables following Poisson distributions with parameters λ_1 and λ_2. Show that the distribution of $N = N_1 + N_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

45. For a Poisson distribution, suppose that events are independently labeled A and B with probabilities $p_A + p_B = 1$. If the parameter of the Poisson distribution is λ, show that the number of events labeled A follows a Poisson distribution with parameter $p_A\lambda$.

46. Let X and Y be jointly continuous random variables. Find an expression for the density of $Z = X - Y$.

47. Let X and Y be independent standard normal random variables. Find the density of $Z = X + Y$, and show that Z is normally distributed as well. (*Hint:* Use the technique of completing the square to help in evaluating the integral.)

48. Let T_1 and T_2 be independent exponentials with parameters λ_1 and λ_2. Find the density function of $T_1 + T_2$.

49. Find the density function of $X + Y$, where X and Y have a joint density as given in Example D in Section 3.3.

50. Suppose that X and Y are independent discrete random variables and each assumes the values 0, 1, and 2 with probability $\frac{1}{3}$ each. Find the frequency function of $X + Y$.

51. Let X and Y have the joint density function $f(x, y)$, and let $Z = XY$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|}\, dy$$

52. Find the density of the quotient of two independent uniform random variables.

53. Consider forming a random rectangle in two ways. Let U_1, U_2, and U_3 be independent random variables uniform on [0, 1]. One rectangle has sides U_1 and U_2, and the other is a square with sides U_3. Find the probability that the area of the square is greater than the area of the other rectangle.

54. Let X, Y, and Z be independent $N(0, \sigma^2)$. Let Θ, Φ, and R be the corresponding random variables that are the spherical coordinates of (X, Y, Z):

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned}$$

$$0 \le \phi \le \pi, \qquad 0 \le \theta \le 2\pi$$

Find the joint and marginal densities of Θ, Φ, and R. (*Hint:* $dx\, dy\, dz = r^2 \sin\phi\, dr\, d\theta\, d\phi$.)

55. A point is generated on a unit disk in the following way: The radius, R, is uniform on $[0, 1]$, and the angle Θ is uniform on $[0, 2\pi]$ and is independent of R.

a. Find the joint density of $X = R\cos\Theta$ and $Y = R\sin\Theta$.
b. Find the marginal densities of X and Y.
c. Is the density uniform over the disk? If not, modify the method to produce a uniform density.

56. If X and Y are independent exponential random variables, find the joint density of the polar coordinates R and Θ of the point (X, Y). Are R and Θ independent?

57. Suppose that Y_1 and Y_2 follow a bivariate normal distribution with parameters $\mu_{Y_1} = \mu_{Y_2} = 0$, $\sigma_{Y_1}^2 = 1$, $\sigma_{Y_2}^2 = 2$, and $\rho = 1/\sqrt{2}$. Find a linear transformation $x_1 = a_{11}y_1 + a_{12}y_2$, $x_2 = a_{21}y_1 + a_{22}y_2$ such that x_1 and x_2 are independent standard normal random variables. (*Hint:* See Example C of Section 3.6.2.)

58. Show that if the joint distribution of X_1 and X_2 is bivariate normal, then the joint distribution of $Y_1 = a_1X_1 + b_1$ and $Y_2 = a_2X_2 + b_2$ is bivariate normal.

59. Let X_1 and X_2 be independent standard normal random variables. Show that the joint distribution of

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 + b_1 \\ Y_2 &= a_{21}X_1 + a_{22}X_2 + b_2 \end{aligned}$$

is bivariate normal.

60. Using the results of the previous problem, describe a method for generating pseudorandom variables that have a bivariate normal distribution from independent pseudorandom uniform variables.

61. Let X and Y be jointly continuous random variables. Find an expression for the joint density of $U = a + bX$ and $V = c + dY$.

62. If X and Y are independent standard normal random variables, find $P(X^2 + Y^2 \leq 1)$.

63. Let X and Y be jointly continuous random variables.

a. Develop an expression for the joint density of $X + Y$ and $X - Y$.
b. Develop an expression for the joint density of XY and Y/X.
c. Specialize the expressions from parts (a) and (b) to the case where X and Y are independent.

64. Find the joint density of $X + Y$ and X/Y, where X and Y are independent exponential random variables with parameter λ. Show that $X + Y$ and X/Y are independent.

65. Suppose that a system's components are connected in series and have lifetimes that are independent exponential random variables with parameters λ_i. Show that the lifetime of the system is exponential with parameter $\sum \lambda_i$.

66. Each component of the following system (Figure 3.19) has an independent exponentially distributed lifetime with parameter λ. Find the cdf and the density of the system's lifetime.

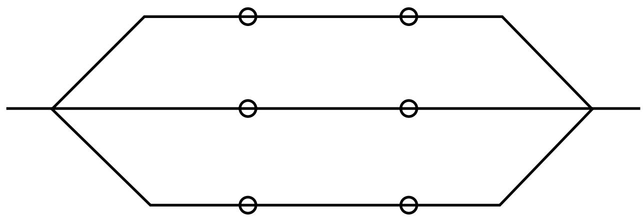

FIGURE **3.19**

67. A card contains n chips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential with parameter λ, find the density function of the card's lifetime.

68. Suppose that a queue has n servers and that the length of time to complete a job is an exponential random variable. If a job is at the top of the queue and will be handled by the next available server, what is the distribution of the waiting time until service? What is the distribution of the waiting time until service of the next job in the queue?

69. Find the density of the minimum of n independent Weibull random variables, each of which has the density

$$f(t) = \beta\alpha^{-\beta}t^{\beta-1}e^{-(t/\alpha)^{\beta}}, \qquad t \geq 0$$

70. If five numbers are chosen at random in the interval [0, 1], what is the probability that they all lie in the middle half of the interval?

71. Let $X_1, \dots, X_n$ be independent random variables, each with the density function f. Find an expression for the probability that the interval $(-\infty, X_{(n)}]$ encompasses at least $100\nu\%$ of the probability mass of f.

72. Let $X_1, X_2, \dots, X_n$ be independent continuous random variables each with cumulative distribution function F. Show that the joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F(x, y) = F^n(y) - [F(y) - F(x)]^n, \qquad x \leq y$$

73. If $X_1, \dots, X_n$ are independent random variables, each with the density function f, show that the joint density of $X_{(1)}, \dots, X_{(n)}$ is

$$n!f(x_1)f(x_2)\cdots f(x_n), \qquad x_1 < x_2 < \cdots < x_n$$

74. Let U_1, U_2, and U_3 be independent uniform random variables.

a. Find the joint density of $U_{(1)}$, $U_{(2)}$, and $U_{(3)}$.

b. The locations of three gas stations are independently and randomly placed along a mile of highway. What is the probability that no two gas stations are less than $\frac{1}{3}$ mile apart?

75. Use the differential method to find the joint density of $X_{(i)}$ and $X_{(j)}$, where $i < j$.

76. Prove Theorem A of Section 3.7 by finding the cdf of $X_{(k)}$ and differentiating. (*Hint:* $X_{(k)} \leq x$ if and only if k or more of the X_i are less than or equal to x. The number of X_i less than or equal to x is a binomial random variable.)

77. Find the density of $U_{(k)} - U_{(k-1)}$ if the U_i, $i = 1, \ldots, n$ are independent uniform random variables. This is the density of the spacing between adjacent points chosen uniformly in the interval [0, 1].

78. Show that

$$\int_0^1 \int_0^y (y-x)^n \, dx \, dy = \frac{1}{(n+1)(n+2)}$$

79. If T_1 and T_2 are independent exponential random variables, find the density function of $R = T_{(2)} - T_{(1)}$.

80. Let $U_1, \ldots, U_n$ be independent uniform random variables, and let V be uniform and independent of the U_i.

a. Find $P(V \leq U_{(n)})$.
b. Find $P(U_{(1)} < V < U_{(n)})$.

81. Do both parts of Problem 80 again, assuming that the U_i and V have the density function f and the cdf F, with F^{-1} uniquely defined. *Hint:* $F(U_i)$ has a uniform distribution.

Evaluating these derivatives at (μ_X, μ_Y) and using the preceding result, we find, if $\mu_X \neq 0$,

$$\begin{aligned} E(Z) &\approx \frac{\mu_Y}{\mu_X} + \sigma_X^2 \frac{\mu_Y}{\mu_X^3} - \frac{\sigma_{XY}}{\mu_X^2} \\ &= \frac{\mu_Y}{\mu_X} + \frac{1}{\mu_X^2}\left(\sigma_X^2 \frac{\mu_Y}{\mu_X} - \rho\sigma_X\sigma_Y\right) \end{aligned}$$

From this equation, we see that the difference between $E(Z)$ and μ_Y/μ_X depends on several factors. If σ_X and σ_Y are small—that is, if the two concentrations are measured quite accurately—the difference is small. If μ_X is small, the difference is relatively large. Finally, correlation between X and Y affects the difference.

We now consider the variance. Again using the preceding result and evaluating the partial derivatives at (μ_X, μ_Y), we find

$$\begin{aligned} \mathrm{Var}(Z) &\approx \sigma_X^2 \frac{\mu_Y^2}{\mu_X^4} + \frac{\sigma_Y^2}{\mu_X^2} - 2\sigma_{XY}\frac{\mu_Y}{\mu_X^3} \\ &= \frac{1}{\mu_X^2}\left(\sigma_X^2 \frac{\mu_Y^2}{\mu_X^2} + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y\frac{\mu_Y}{\mu_X}\right) \end{aligned}$$

From this equation, we see that the ratio is quite variable when μ_X is small, paralleling the results in Example A, and that correlation between X and Y, if of the same sign as μ_Y/μ_X, decreases $\mathrm{Var}(Z)$. ■

4.7 Problems

1. Show that if a random variable is bounded—that is, $|X| < M < \infty$—then $E(X)$ exists.

2. If X is a discrete uniform random variable—that is, $P(X = k) = 1/n$ for $k = 1, 2, \ldots, n$—find $E(X)$ and $\mathrm{Var}(X)$.

3. Find $E(X)$ and $\mathrm{Var}(X)$ for Problem 3 in Chapter 2.

4. Let X have the cdf $F(x) = 1 - x^{-\alpha}$, $x \geq 1$.

a. Find $E(X)$ for those values of α for which $E(X)$ exists.
b. Find $\mathrm{Var}(X)$ for those values of α for which it exists.

5. Let X have the density

$$f(x) = \frac{1 + \alpha x}{2}, \qquad -1 \leq x \leq 1, \qquad -1 \leq \alpha \leq 1$$

Find $E(X)$ and $\mathrm{Var}(X)$.

6. Let X be a continuous random variable with probability density function $f(x) = 2x, 0 \leq x \leq 1$.

a. Find $E(X)$.
b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.
c. Use Theorem A in Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).
d. Find Var(X) according to the definition of variance given in Section 4.2. Also find Var(X) by using Theorem B of Section 4.2.

7. Let X be a discrete random variable that takes on values 0, 1, 2 with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively.

a. Find $E(X)$.
b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.
c. Use Theorem A of Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).
d. Find Var(X) according to the definition of variance given in Section 4.2. Also find Var(X) by using Theorem B in Section 4.2.

8. Show that if X is a discrete random variable, taking values on the positive integers, then $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$. Apply this result to find the expected value of a geometric random variable.

9. This is a simplified inventory problem. Suppose that it costs c dollars to stock an item and that the item sells for s dollars. Suppose that the number of items that will be asked for by customers is a random variable with the frequency function $p(k)$. Find a rule for the number of items that should be stocked in order to maximize the expected income. (*Hint:* Consider the difference of successive terms.)

10. A list of n items is arranged in random order; to find a requested item, they are searched sequentially until the desired item is found. What is the expected number of items that must be searched through, assuming that each item is equally likely to be the one requested? (Questions of this nature arise in the design of computer algorithms.)

11. Referring to Problem 10, suppose that the items are not equally likely to be requested but have known probabilities $p_1, p_2, \ldots, p_n$. Suggest an alternative searching procedure that will decrease the average number of items that must be searched through, and show that in fact it does so.

12. If X is a continuous random variable with a density that is symmetric about some point, ξ, show that $E(X) = \xi$, provided that $E(X)$ exists.

13. If X is a nonnegative continuous random variable, show that

$$E(X) = \int_0^{\infty} [1 - F(x)]\, dx$$

Apply this result to find the mean of the exponential distribution.

14. Let X be a continuous random variable with the density function

$$f(x) = 2x, \qquad 0 \le x \le 1$$

a. Find $E(X)$.
b. Find $E(X^2)$ and $\text{Var}(X)$.

15. Suppose that two lotteries each have n possible numbers and the same payoff. In terms of expected gain, is it better to buy two tickets from one of the lotteries or one from each?

16. Suppose that $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Show that $E(Z) = 0$ and $\text{Var}(Z) = 1$.

17. Find (a) the expectation and (b) the variance of the kth-order statistic of a sample of n independent random variables uniform on [0, 1]. The density function is given in Example C in Section 3.7.

18. If $U_1, \ldots, U_n$ are independent uniform random variables, find $E(U_{(n)} - U_{(1)})$.

19. Find $E(U_{(k)} - U_{(k-1)})$, where the $U_{(i)}$ are as in Problem 18.

20. A stick of unit length is broken into two pieces. Find the expected ratio of the length of the longer piece to the length of the shorter piece.

21. A random square has a side length that is a uniform [0, 1] random variable. Find the expected area of the square.

22. A random rectangle has sides the lengths of which are independent uniform random variables. Find the expected area of the rectangle, and compare this result to that of Problem 21.

23. Repeat Problems 21 and 22 assuming that the distribution of the lengths is exponential.

24. Prove Theorem A of Section 4.1.2 for the discrete case.

25. If X_1 and X_2 are independent random variables following a gamma distribution with parameters α and λ, find $E(R^2)$, where $R^2 = X_1^2 + X_2^2$.

26. Referring to Example B in Section 4.1.2, what is the expected number of coupons needed to collect r different types, where $r < n$?

27. If n men throw their hats into a pile and each man takes a hat at random, what is the expected number of matches? (*Hint:* Express the number as a sum.)

show answer.

28. Suppose that n enemy aircraft are shot at simultaneously by m gunners, that each gunner selects an aircraft to shoot at independently of the other gunners, and that each gunner hits the selected aircraft with probability p. Find the expected number of aircraft hit by the gunners.

29. Prove Corollary A of Section 4.1.1.

30. Find $E[1/(X + 1)]$, where X is a Poisson random variable.

31. Let X be uniformly distributed on the interval $[1, 2]$. Find $E(1/X)$. Is $E(1/X) = 1/E(X)$?

32. Let X have a gamma distribution with parameters α and λ. For those values of α and λ for which it is defined, find $E(1/X)$.

33. Prove Chebyshev's inequality in the discrete case.

34. Let X be uniform on [0, 1], and let $Y = \sqrt{X}$. Find $E(Y)$ by (a) finding the density of Y and then finding the expectation and (b) using Theorem A of Section 4.1.1.

35. Find the mean of a negative binomial random variable. (*Hint:* Express the random variable as a sum.)

36. Consider the following scheme for group testing. The original lot of samples is divided into two groups, and each of the subgroups is tested as a whole. If either subgroup tests positive, it is divided in two, and the procedure is repeated. If any of the groups thus obtained tests positive, test every member of that group. Find the expected number of tests performed, and compare it to the number performed with no grouping and with the scheme described in Example C in Section 4.1.2.

37. For what values of p is the group testing of Example C in Section 4.1.2 inferior to testing every individual?

38. This problem continues Example A of Section 4.1.2.

a. What is the probability that a fragment is the leftmost member of a contig?
b. What is the expected number of fragments that are leftmost members of contigs?
c. What is the expected number of contigs?

39. Suppose that a segment of DNA of length 1,000,000 is to be shotgun sequenced with fragments of length 1000.

a. How many fragment would be needed so that the chance of an individual site being covered is greater than 0.99?
b. With this choice, how many sites would you expect to be missed?

40. A child types the letters Q, W, E, R, T, Y, randomly producing 1000 letters in all. What is the expected number of times that the sequence QQQQ appears, counting overlaps?

41. Continuing with the previous problem, how many times would we expect the word "TRY" to appear? Would we be surprised if it occurred 100 times? (*Hint:* Consider Markov's inequality.)

42. Let X be an exponential random variable with standard deviation σ. Find $P(|X - E(X)| > k\sigma)$ for $k = 2, 3, 4$, and compare the results to the bounds from Chebyshev's inequality.

43. Show that $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)$.

44. If X and Y are independent random variables with equal variances, find $\text{Cov}(X + Y,\ X - Y)$.

45. Find the covariance and the correlation of N_i and N_j, where $N_1,\ N_2,\ \ldots,\ N_r$ are multinomial random variables. (*Hint:* Express them as sums.)

46. If $U = a + bX$ and $V = c + dY$, show that $|\rho_{UV}| = |\rho_{XY}|$.

47. If X and Y are independent random variables and $Z = Y - X$, find expressions for the covariance and the correlation of X and Z in terms of the variances of X and Y.

48. Let U and V be independent random variables with means μ and variances σ^2. Let $Z = \alpha U + V\sqrt{1 - \alpha^2}$. Find $E(Z)$ and ρ_{UZ}.

49. Two independent measurements, X and Y, are taken of a quantity μ. $E(X) = E(Y) = \mu$, but σ_X and σ_Y are unequal. The two measurements are combined by means of a weighted average to give

$$Z = \alpha X + (1 - \alpha)Y$$

where α is a scalar and $0 \le \alpha \le 1$.

a. Show that $E(Z) = \mu$.
b. Find α in terms of σ_X and σ_Y to minimize $\text{Var}(Z)$.
c. Under what circumstances is it better to use the average $(X + Y)/2$ than either X or Y alone?

50. Suppose that X_i, where $i = 1,\ \ldots,\ n$, are independent random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let $\overline{X} = n^{-1}\ \sum_{i=1}^{n}\ X_i$. Show that $E(\overline{X}) = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$.

51. Continuing Example E in Section 4.3, suppose there are n securities, each with the same expected return, that all the returns have the same standard deviations, and that the returns are uncorrelated. What is the optimal portfolio vector? Plot the risk of the optimal portfolio versus n. How does this risk compare to that incurred by putting all your money in one security?

52. Consider two securities, the first having $\mu_1 = 1$ and $\sigma_1 = 0.1$, and the second having $\mu_2 = 0.8$ and $\sigma_2 = 0.12$. Suppose that they are negatively correlated, with $\rho = -0.8$.

a. If you could only invest in one security, which one would you choose, and why?
b. Suppose you invest 50% of your money in each of the two. What is your expected return and what is your risk?
c. If you invest 80% of your money in security 1 and 20% in security 2, what is your expected return and your risk?
d. Denote the expected return and its standard deviation as functions of π by $\mu(\pi)$ and $\sigma(\pi)$. The pair $(\mu(\pi), \sigma(\pi))$ trace out a curve in the plane as π varies from 0 to 1. Plot this curve.
e. Repeat **b–d** if the correlation is $\rho = 0.1$.

53. Show that $\text{Cov}(X,\ Y) \le \sqrt{\text{Var}(X)\text{Var}(Y)}$.

54. Let X, Y, and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2, and σ_Z^2, respectively. Let

$$U = Z + X$$
$$V = Z + Y$$

Find $\mathrm{Cov}(U,\ V)$ and ρ_{UV}.

55. Let $T = \sum_{k=1}^{n} kX_k$, where the X_k are independent random variables with means μ and variances σ^2. Find $E(T)$ and $\mathrm{Var}(T)$.

56. Let $S = \sum_{k=1}^{n} X_k$, where the X_k are as in Problem 55. Find the covariance and the correlation of S and T.

57. If X and Y are independent random variables, find $\mathrm{Var}(XY)$ in terms of the means and variances of X and Y.

58. A function is measured at two points with some error (for example, the position of an object is measured at two times). Let

$$X_1 = f(x) + \varepsilon_1$$
$$X_2 = f(x + h) + \varepsilon_2$$

where ε_1 and ε_2 are independent random variables with mean zero and variance σ^2. Since the derivative of f is

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

it is estimated by

$$Z = \frac{X_2 - X_1}{h}$$

a. Find $E(Z)$ and $\mathrm{Var}(Z)$. What is the effect of choosing a value of h that is very small, as is suggested by the definition of the derivative?

b. Find an approximation to the mean squared error of Z as an estimate of $f'(x)$ using a Taylor series expansion. Can you find the value of h that minimizes the mean squared error?

c. Suppose that f is measured at three points with some error. How could you construct an estimate of the second derivative of f, and what are the mean and the variance of your estimate?

59. Let $(X,\ Y)$ be a random point uniformly distributed on a unit disk. Show that $\mathrm{Cov}(X,\ Y) = 0$, but that X and Y are not independent.

60. Let Y have a density that is symmetric about zero, and let $X = SY$, where S is an independent random variable taking on the values $+1$ and -1 with probability $\frac{1}{2}$ each. Show that $\mathrm{Cov}(X,\ Y) = 0$, but that X and Y are not independent.

61. In Section 3.7, the joint density of the minimum and maximum of n independent uniform random variables was found. In the case $n = 2$, this amounts to X and Y, the minimum and maximum, respectively, of two independent random variables uniform on [0, 1], having the joint density

$$f(x, y) = 2, \qquad 0 \le x \le y$$

a. Find the covariance and the correlation of X and Y. Does the sign of the correlation make sense intuitively?
b. Find $E(X|Y = y)$ and $E(Y|X = x)$. Do these results make sense intuitively?
c. Find the probability density functions of the random variables $E(X|Y)$ and $E(Y|X)$.
d. What is the linear predictor of Y in terms of X (denoted by $\hat{Y} = a + bX$) that has minimal mean squared error? What is the mean square prediction error?
e. What is the predictor of Y in terms of X [$\hat{Y} = h(X)$] that has minimal mean squared error? What is the mean square prediction error?

62. Let X and Y have the joint distribution given in Problem 1 of Chapter 3.

a. Find the covariance and correlation of X and Y.
b. Find $E(Y|X = x)$ for $x = 1, 2, 3, 4$. Find the probability mass function of the random variable $E(Y|X)$.

63. Let X and Y have the joint distribution given in Problem 8 of Chapter 3.

a. Find the covariance and correlation of X and Y.
b. Find $E(Y|X = x)$ for $0 \le x \le 1$.

64. Let X and Y be jointly distributed random variables with correlation ρ_{XY}; define the *standardized* random variables $\tilde{X}$ and $\tilde{Y}$ as $\tilde{X} = (X - E(X))/\sqrt{\text{Var}(X)}$ and $\tilde{Y} = (Y - E(Y))/\sqrt{\text{Var}(Y)}$. Show that $\text{Cov}(\tilde{X}, \tilde{Y}) = \rho_{XY}$.

65. How has the assumption that N and the X_i are independent been used in Example D of Section 4.4.1?

66. A building contains two elevators, one fast and one slow. The average waiting time for the slow elevator is 3 min. and the average waiting time of the fast elevator is 1 min. If a passenger chooses the fast elevator with probability $\frac{2}{3}$ and the slow elevator with probability $\frac{1}{3}$, what is the expected waiting time? (Use the law of total expectation, Theorem A of Section 4.4.1, defining appropriate random variables X and Y.)

67. A random rectangle is formed in the following way: The base, X, is chosen to be a uniform [0, 1] random variable and after having generated the base, the height is chosen to be uniform on [0, X]. Use the law of total expectation, Theorem A of Section 4.4.1, to find the expected circumference and area of the rectangle.

68. Show that $E[\text{Var}(Y|X)] \le \text{Var}(Y)$.

69. Suppose that a bivariate normal distribution has $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Sketch the contours of the density and the lines $E(Y|X = x)$ and $E(X|Y = y)$ for $\rho = 0, .5$, and $.9$.

70. If X and Y are independent, show that $E(X|Y = y) = E(X)$.

71. Let X be a binomial random variable representing the number of successes in n independent Bernoulli trials. Let Y be the number of successes in the first m trials, where $m < n$. Find the conditional frequency function of Y given $X = x$ and the conditional mean.

72. An item is present in a list of n items with probability p; if it is present, its position in the list is uniformly distributed. A computer program searches through the list sequentially. Find the expected number of items searched through before the program terminates.

73. A fair coin is tossed n times, and the number of heads, N, is counted. The coin is then tossed N more times. Find the expected total number of heads generated by this process.

74. The number of offspring of an organism is a discrete random variable with mean μ and variance σ^2. Each of its offspring reproduces in the same manner. Find the expected number of offspring in the third generation and its variance.

75. Let T be an exponential random variable, and conditional on T, let U be uniform on $[0, T]$. Find the unconditional mean and variance of U.

76. Let the point (X, Y) be uniformly distributed over the half disk $x^2 + y^2 \leq 1$, where $y \geq 0$. If you observe X, what is the best prediction for Y? If you observe Y, what is the best prediction for X? For both questions, "best" means having the minimum mean squared error.

77. Let X and Y have the joint density

$$f(x, y) = e^{-y}, \qquad 0 \leq x \leq y$$

a. Find $\mathrm{Cov}(X, Y)$ and the correlation of X and Y.
b. Find $E(X|Y = y)$ and $E(Y|X = x)$.
c. Find the density functions of the random variables $E(X|Y)$ and $E(Y|X)$.

78. Show that if a density is symmetric about zero, its skewness is zero.

79. Let X be a discrete random variable that takes on values 0, 1, 2 with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

80. Let X be a continuous random variable with density function $f(x) = 2x$, $0 \leq x \leq 1$. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

81. Find the moment-generating function of a Bernoulli random variable, and use it to find the mean, variance, and third moment.

82. Use the result of Problem 81 to find the mgf of a binomial random variable and its mean and variance.

83. Show that if X_i follows a binomial distribution with n_i trials and probability of success $p_i = p$, where $i = 1, \dots, n$ and the X_i are independent, then $\sum_{i=1}^{n} X_i$ follows a binomial distribution.

84. Referring to Problem 83, show that if the p_i are unequal, the sum does not follow a binomial distribution.

85. Find the mgf of a geometric random variable, and use it to find the mean and the variance.

86. Use the result of Problem 85 to find the mgf of a negative binomial random variable and its mean and variance.

87. Under what conditions is the sum of independent negative binomial random variables also negative binomial?

88. Let X and Y be independent random variables, and let α and β be scalars. Find an expression for the mgf of $Z = \alpha X + \beta Y$ in terms of the mgf's of X and Y.

89. Let $X_1, X_2, \dots, X_n$ be independent normal random variables with means μ_i and variances σ_i^2. Show that $Y = \sum_{i=1}^{n} \alpha_i X_i$, where the α_i are scalars, is normally distributed, and find its mean and variance. (*Hint:* Use moment-generating functions.)

90. Assuming that $X \sim N(0, \sigma^2)$, use the mgf to show that the odd moments are zero and the even moments are

$$\mu_{2n} = \frac{(2n)!\sigma^{2n}}{2^n(n!)}$$

91. Use the mgf to show that if X follows an exponential distribution, cX $(c > 0)$ does also.

92. Suppose that Θ is a random variable that follows a gamma distribution with parameters λ and α, where α is an integer, and suppose that, conditional on Θ, X follows a Poisson distribution with parameter Θ. Find the unconditional distribution of $\alpha + X$. (*Hint:* Find the mgf by using iterated conditional expectations.)

93. Find the distribution of a geometric sum of exponential random variables by using moment-generating functions.

94. If X is a nonnegative integer-valued random variable, the **probability-generating function** of X is defined to be

$$G(s) = \sum_{k=0}^{\infty} s^k p_k$$

where $p_k = P(X = k)$.

a. Show that

$$p_k = \frac{1}{k!} \frac{d^k}{ds^k} G(s) \bigg|_{s=0}$$

b. Show that

$$\left.\frac{dG}{ds}\right|_{s=1} = E(X)$$

$$\left.\frac{d^2G}{ds^2}\right|_{s=1} = E[X(X-1)]$$

c. Express the probability-generating function in terms of moment-generating function.

d. Find the probability-generating function of the Poisson distribution.

95. Show that if X and Y are independent, their joint moment-generating function factors.

96. Show how to find $E(XY)$ from the joint moment-generating function of X and Y.

97. Use moment-generating functions to show that if X and Y are independent, then

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

98. Find the mean and variance of the compound Poisson distribution (Example H in Section 4.5).

99. Find expressions for the approximate mean and variance of $Y = g(X)$ for (a) $g(x) = \sqrt{x}$, (b) $g(x) = \log x$, and (c) $g(x) = \sin^{-1} x$.

100. If X is uniform on [10, 20], find the approximate and exact mean and variance of $Y = 1/X$, and compare them.

101. Find the approximate mean and variance of $Y = \sqrt{X}$, where X is a random variable following a Poisson distribution.

102. Two sides, x_0 and y_0, of a right triangle are independently measured as X and Y, where $E(X) = x_0$ and $E(Y) = y_0$ and $\mathrm{Var}(X) = \mathrm{Var}(Y) = \sigma^2$. The angle between the two sides is then determined as

$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Find the approximate mean and variance of Θ.

103. The volume of a bubble is estimated by measuring its diameter and using the relationship

$$V = \frac{\pi}{6}D^3$$

Suppose that the true diameter is 2 mm and that the standard deviation of the measurement of the diameter is .01 mm. What is the approximate standard deviation of the estimated volume?

104. The position of an aircraft relative to an observer on the ground is estimated by measuring its distance r from the observer and the angle θ that the line of sight from the observer to the aircraft makes with the horizontal. Suppose that the measurements, denoted by R and Θ, are subject to random errors and are independent of each other. The altitude of the aircraft is then estimated to be $Y = R \sin \Theta$.

a. Find an approximate expression for the variance of Y.

b. For given r, at what value of θ is the estimated altitude most variable?

size of the particle is $Y_1 = X_1 y_0$; after the second impact, the size is $Y_2 = X_2 X_1 y_0$; and after the nth impact, the size is

$$Y_n = X_n X_{n-1} \cdots X_2 X_1 y_0$$

Then

$$\log Y_n = \log y_0 + \sum_{i=1}^{n} \log X_i$$

and the central limit theorem applies to $\log Y_n$. ■

A similar construction is relevant to the theory of finance. Suppose that an initial investment of value v_0 is made and that returns occur in discrete time, for example, daily. If the return on the first day is R_1, then the value becomes $V_1 = R_1 v_0$. After day two the value is $V_2 = R_2 R_1 v_0$, and after day n the value is

$$V_n = R_n R_{n-1} \cdots R_1 v_0$$

The log value is thus

$$\log V_n = \log v_0 + \sum_{i=1}^{n} \log R_i$$

If the returns are independent random variables with the same distribution, then the distribution of $\log V_n$ is approximately normally distributed.

5.4 Problems

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Show that if $n^{-2} \sum_{i=1}^{n} \sigma_i^2 \to 0$, then $\overline{X} \to \mu$ in probability.

2. Let X_i be as in Problem 1 but with $E(X_i) = \mu_i$ and $n^{-1} \sum_{i=1}^{n} \mu_i \to \mu$. Show that $\overline{X} \to \mu$ in probability.

3. Suppose that the number of insurance claims, N, filed in a year is Poisson distributed with $E(N) = 10{,}000$. Use the normal approximation to the Poisson to approximate $P(N > 10{,}200)$.

4. Suppose that the number of traffic accidents, N, in a given period of time is distributed as a Poisson random variable with $E(N) = 100$. Use the normal approximation to the Poisson to find Δ such that $P(100 - \Delta < N < 100 + \Delta) \approx .9$.

5. Using moment-generating functions, show that as $n \to \infty$, $p \to 0$, and $np \to \lambda$, the binomial distribution with parameters n and p tends to the Poisson distribution.

6. Using moment-generating functions, show that as $\alpha \to \infty$ the gamma distribution with parameters α and λ, properly standardized, tends to the standard normal distribution.

7. Show that if $X_n \to c$ in probability and if g is a continuous function, then $g(X_n) \to g(c)$ in probability.

8. Compare the Poisson cdf and the normal approximation for (a) $\lambda = 10$, (b) $\lambda = 20$, and (c) $\lambda = 40$.

9. Compare the binomial cdf and the normal approximation for (a) $n = 20$ and $p = .2$, and (b) $n = 40$ and $p = .5$.

10. A six-sided die is rolled 100 times. Using the normal approximation, find the probability that the face showing a six turns up between 15 and 20 times. Find the probability that the sum of the face values of the 100 trials is less than 300.

11. A skeptic gives the following argument to show that there must be a flaw in the central limit theorem: "We know that the sum of independent Poisson random variables follows a Poisson distribution with a parameter that is the sum of the parameters of the summands. In particular, if n independent Poisson random variables, each with parameter n^{-1}, are summed, the sum has a Poisson distribution with parameter 1. The central limit theorem says that as n approaches infinity, the distribution of the sum tends to a normal distribution, but the Poisson with parameter 1 is not the normal." What do you think of this argument?

12. The central limit theorem can be used to analyze round-off error. Suppose that the round-off error is represented as a uniform random variable on $[-\frac{1}{2}, \frac{1}{2}]$. If 100 numbers are added, approximate the probability that the round-off error exceeds (a) 1, (b) 2, and (c) 5.

13. A drunkard executes a "random walk" in the following way: Each minute he takes a step north or south, with probability $\frac{1}{2}$ each, and his successive step directions are independent. His step length is 50 cm. Use the central limit theorem to approximate the probability distribution of his location after 1 h. Where is he most likely to be?

14. Answer Problem 13 under the assumption that the drunkard has some idea of where he wants to go so that he steps north with probability $\frac{2}{3}$ and south with probability $\frac{1}{3}$.

15. Suppose that you bet \$5 on each of a sequence of 50 independent fair games. Use the central limit theorem to approximate the probability that you will lose more than \$75.

16. Suppose that $X_1, \dots, X_{20}$ are independent random variables with density functions

$$f(x) = 2x, \qquad 0 \leq x \leq 1$$

Let $S = X_1 + \cdots + X_{20}$. Use the central limit theorem to approximate $P(S \leq 10)$.

17. Suppose that a measurement has mean μ and variance $\sigma^2 = 25$. Let $\overline{X}$ be the average of n such independent measurements. How large should n be so that $P(|\overline{X} - \mu| < 1) = .95$?

18. Suppose that a company ships packages that are variable in weight, with an average weight of 15 lb and a standard deviation of 10. Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variables, find the probability that 100 packages will have a total weight exceeding 1700 lb.

19. **a.** Use the Monte Carlo method with $n = 100$ and $n = 1000$ to estimate $\int_0^1 \cos(2\pi x)\, dx$. Compare the estimates to the exact answer.
b. Use Monte Carlo to evaluate $\int_0^1 \cos(2\pi x^2)\, dx$. Can you find the exact answer?

20. What is the variance of the estimate of an integral by the Monte Carlo method (Example A of Section 5.2)? [*Hint:* Find $E(\hat{I}^2(f))$.] Compare the standard deviations of the estimates of part (a) of previous problem to the actual errors you made.

21. This problem introduces a variation on the Monte Carlo integration technique of Example A of Section 5.2. Suppose that we wish to evaluate

$$I(f) = \int_a^b f(x)\, dx$$

Let g be a density function on $[a, b]$. Generate $X_1, \cdots, X_n$ from g and estimate I by

$$\hat{I}(f) = \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)}$$

a. Show that $E(\hat{I}(f)) = I(f)$.
b. Find an expression for $\mathrm{Var}(\hat{I}(f))$. Give an example for which it is finite and an example for which it is infinite. Note that if it is finite, the law of large numbers implies that $\hat{I}(f) \to I(f)$ as $n \to \infty$.
c. Show that if $a = 0$, $b = 1$, and g is uniform, this is the same Monte Carlo estimate as that of Example A of Section 5.2. Can this estimate be improved by choosing g to be other than uniform? (*Hint*: Compare variances.)

22. Use the central limit theorem to find Δ such that $P(|\hat{I}(f) - I(f)| \leq \Delta) = .05$, where $\hat{I}(f)$ is the Monte Carlo estimate of $\int_0^1 \cos(2\pi x)\, dx$ based on 1000 points.

23. An irregularly shaped object of unknown area A is located in the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Consider a random point distributed uniformly over the square; let $Z = 1$ if the point lies inside the object and $Z = 0$ otherwise. Show that $E(Z) = A$. How could A be estimated from a sequence of n independent points uniformly distributed on the square?

24. How could the central limit theorem be used to gauge the probable size of the error of the estimate of the previous problem? Denoting the estimate by $\hat{A}$, if $A = .2$, how large should n be so that $P(|\hat{A} - A| < .01) \approx .99$?

25. Let X be a continuous random variable with density function $f(x) = \frac{3}{2}x^2$, $-1 \leq x \leq 1$. Sketch this density function. Use the central limit theorem to sketch the approximate density function of $S = X_1 + \cdots + X_{50}$, where the X_i are independent random variables with density f. Similarly, sketch the approximate density functions of $S/50$ and $S/\sqrt{50}$. For each sketch, label at least three points on the horizontal axis.

26. Suppose that a basketball player can score on a particular shot with probability .3. Use the central limit theorem to find the approximate distribution of S, the number of successes out of 25 independent shots. Find the approximate probabilities that S is less than or equal to 5, 7, 9, and 11 and compare these to the exact probabilities.

27. Prove that if $a_n \to a$, then $(1 + a_n/n)^n \to e^a$.

28. Let f_n be a sequence of frequency functions with $f_n(x) = \frac{1}{2}$ if $x = \pm(\frac{1}{2})^n$ and $f_n(x) = 0$ otherwise. Show that $\lim f_n(x) = 0$ for all x, which means that the frequency functions do not converge to a frequency function, but that there exists a cdf F such that $\lim F_n(x) = F(x)$.

29. In addition to limit theorems that deal with sums, there are limit theorems that deal with extreme values such as maxima or minima. Here is an example. Let $U_1, \ldots, U_n$ be independent uniform random variables on [0, 1], and let $U_{(n)}$ be the maximum. Find the cdf of $U_{(n)}$ and a standardized $U_{(n)}$, and show that the cdf of the standardized variable tends to a limiting value.

30. Generate a sequence $U_1, U_2, \ldots, U_{1000}$ of independent uniform random variables on a computer. Let $S_n = \sum_{i=1}^{n} U_i$ for $n = 1, 2, \ldots, 1000$. Plot each of the following versus n:

a. S_n
b. S_n/n
c. $S_n - n/2$
d. $(S_n - n/2)/n$
e. $(S_n - n/2)/\sqrt{n}$

Explain the shapes of the resulting graphs using the concepts of this chapter.

COROLLARY **B**

Let $\overline{X}$ and S^2 be as given at the beginning of this section. Then

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

Proof

We simply express the given ratio in a different form:

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{\left(\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right)}{\sqrt{S^2/\sigma^2}}$$

The latter is the ratio of an $N(0, 1)$ random variable to the square root of an independent random variable with a χ^2_{n-1} distribution divided by its degrees of freedom. Thus, from the definition in Section 6.2, the ratio follows a t distribution with $n - 1$ degrees of freedom. ■

6.4 Problems

1. Prove Proposition A of Section 6.2.
2. Prove Proposition B of Section 6.2.
3. Let $\overline{X}$ be the average of a sample of 16 independent normal random variables with mean 0 and variance 1. Determine c such that
$$P(|\overline{X}| < c) = .5$$
4. If T follows a t_7 distribution, find t_0 such that (a) $P(|T| < t_0) = .9$ and (b) $P(T > t_0) = .05$.
5. Show that if $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$.
6. Show that if $T \sim t_n$, then $T^2 \sim F_{1,n}$.
7. Show that the Cauchy distribution and the t distribution with 1 degree of freedom are the same.
8. Show that if X and Y are independent exponential random variables with $\lambda = 1$, then X/Y follows an F distribution. Also, identify the degrees of freedom.
9. Find the mean and variance of S^2, where S^2 is as in Section 6.3.
10. Show how to use the chi-square distribution to calculate $P(a < S^2/\sigma^2 < b)$.
11. Let $X_1, \dots, X_n$ be a sample from an $N(\mu_X, \sigma^2)$ distribution and $Y_1, \dots, Y_m$ be an independent sample from an $N(\mu_Y, \sigma^2)$ distribution. Show how to use the F distribution to find $P(S^2_X/S^2_Y > c)$.