## Problem 1 (10pts)

A fair coin is tossed $n$ times. Let $X$ be the number of heads in this $n$ toss. Given $X = x$, we generate a Poisson random variable $Y$ with rate parameter $\lambda = x$. Find Var[$Y$]. Your answer should only depends on $n$.

**Hint** Law of total variation. $\text{Var}[Y] = \mathbb{E}[\text{Var}[Y \mid X]] + \text{Var}[\mathbb{E}[Y \mid X]]$.

$X \sim Bin(n=n, P=0.5) \quad \lambda = X \rightarrow Y|X = x \sim Poisson(x)$

$Var(Y|X) = X \rightarrow E[X] = E[X]$

$E[X] = np = n \cdot 0.5$

$E[Y|X] = X \rightarrow Var(X) = np \cdot (1-P) = n \cdot 0.5 \cdot 0.5 = n \cdot 0.25$

$Var(Y) = \frac{n}{2} + \frac{n}{4} = \frac{2n}{4} + \frac{n}{4} = \boxed{\frac{3n}{4}}$

## Problem 3 (15pts)

A factory that manufactures televisions operates in three shifts. Shift 1 produced 10% of the TVs, while Shift 2 and Shift 3 each produced 45% of the TVs. The rate of defective TVs produced by Shift 1, 2, and 3 are 2%, 4%, and 6%, respectively.

(a) (5pts) What proportion of all TVs produced are **defective** ?
(b) (5pts) You test a TV and found out that it is not defective. What is the probability it was made by Shift 1 ?
(c) (5pts) What proportion of TVs are made by Shift 1 **or** is not defective ? Here or is an **inclusive** or.

a $P(Defective) = P(D|S1)P(S1) + P(D|S2)P(S2) + P(D|S3)P(S3)$

$P(D|S1) = 0.02 \quad P(S1) = 0.1$

$P(D|S2) = 0.04 \quad P(S2) = 0.45$

$P(D|S3) = 0.06 \quad P(S3) = 0.45$

$P(Defective) = (0.02)(0.1) + (0.04)(0.45) + (0.06)(0.45)$

b $P(S1|D') = \frac{P(D'|S1)P(S1)}{P(D')} \rightarrow P(D'|S1) = 1 - 0.02 = 0.98$

$P(D') = 1 - P(D) = 1 - 0.47 = 0.53$

$= \boxed{\frac{0.98 \cdot 0.1}{0.53}}$

c $P(S1 \cup D') = P(S1) + P(D') - P(S1 \cap D')$

$P(S1) = 0.1 \quad P(D') = 0.53 \quad P(S1 \cap D') = 0.98 \times 0.1 = 0.098$

$P(S1 \cup D') = 0.1 + 0.53 - 0.098$

## Problem 2 (20pts)

Let $(X, Y)$ be a bivariate random variable with **joint** probability density function (pdf)

$$f(x,y) = \begin{cases} 6x & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) (5pts) Write down a simple and **explicit** expression for the **marginal** pdf of $X$.
(b) (5pts) Are $X$ and $Y$ **independent** ? Justify your answer.
(c) (5pts) Calculate $P(X + Y \le 0.6)$
(d) (5pts) Calculate $\mathbb{E}[\frac{X}{Y}]$.

**Note** Be careful of the limits of integrations, especially for part (c).

a $f_X(x) = \int_{y=x}^{1} f(x,y)\,dy = \int_{y=x}^{1} 6x\,dy = 6x \int_{y=x}^{1} 1\,dy$

$= 6x[y]_x^1 = 6x(1-x) \quad \boxed{f_X(x) = 6x(1-x) \quad 0 \le x \le 1}$

b $f(x,y) = f_X(x) f_Y(y)$ ?

$f_X(x) = 6x(1-x)$

$f_Y(y) = \int_{x=0}^{y} f(x,y)\,dx = \int_{x=0}^{y} 6x\,dx = 6\int_{x=0}^{y} x\,dx$

$= 6\left[\frac{x^2}{2}\right]_0^y = \boxed{3y^2}$

$f(x,y) = 6x \neq 6x(1-x) \cdot 3y^2$ $\boxed{\text{X and Y are not}}$

c $P(X+Y \le 0.6) = \int_0^1 \int_0^{\min(y, 0.6-y)} f(x,y)\,dx\,dy = \int_0^{0.6}\int_0^{0.6-y} 6x\,dx\,dy$

$= \int_0^{0.6} 3(0.36 - 1.2y + y^2)\,dy$

d $E[X|Y=y] = \frac{\int_0^y x f(x,y)\,dx}{\int_0^y f(x,y)\,dx}$

top: $\int_0^y x(6x)\,dx = 6\int_0^y x^2\,dx = 6\left[\frac{x^3}{3}\right]_0^y = 6 \cdot \frac{y^3}{3} = 2y^3$

bottom: $\int_0^y 6x\,dx = 6\int_0^y x\,dx = 6\left[\frac{x^2}{2}\right]_0^y = 6\frac{y^2}{2} = 3y^2$

So $\rightarrow \frac{2y^3}{3y^2} = \boxed{\frac{2y}{3}}$

## Problem 8 (10pts)

Let $X$ and $Y$ have **joint** probability density function

$$f(x,y) = 6(y-x) \quad \text{for } 0 \le x \le y \le 1$$

Compute $\mathbb{E}[X - Y]$.

$E[X] = \int_0^1 \int_0^y x \cdot f(x,y)\,dx\,dy$

$f(x,y) = 6(y-x)$

$E[X] = \int_0^1\int_0^y x \cdot 6(y-x)\,dx\,dy$

Distribute x

$E[X] = 6\int_0^1\int_0^1 (xy - x^2)\,dx\,dy$

$\rightarrow \int_0^y (xy - x^2)\,dx = \frac{y^3}{2} - \frac{y^3}{3} = \frac{y^3}{6}$

$E[X] = 6\int_0^1 \frac{y^3}{6}\,dy = \int_0^1 y^3\,dy$

$= \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{4} = E[X]$

$E[Y] = \int_0^1\int_0^y y \cdot f(x,y)\,dx\,dy$

$= \int_0^1\int_0^y y(6y - x)\,dx\,dy$

$= 6\int_0^1\int_0^y (y^2 - yx)\,dx\,dy = \int_0^y (y^2 - yx)\,dx = y^3 - \frac{y^3}{2}$

$= \frac{y^3}{2}$

$E[Y] = 6\int_0^1 \frac{y^3}{2}\,dy = 3\int_0^1 y^3\,dy = \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{4} \cdot 3 = \frac{3}{4}$

$E[X-Y] = E[X] - E[Y]$

$= \frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = \boxed{-\frac{1}{2}}$

## Problem 5 (20pts)

Let $X_1, X_2, \ldots$ be independent and identically distributed **Bernoulli** random variables with the same success probability $p$. Let $Z_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$.

(a) (10pts) What is the **limiting** distribution of $n^{1/2}(Z_n - p)$ as $n \to \infty$ ?
(b) (10pts) Suppose $p \neq 1/2$. Find the limiting distribution of $Z_n(1 - Z_n)$ as $n \to \infty$ ? (Delta method)

a $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ where $X_i \sim Bernoulli(p)$

$E[X_i] = p \quad Var(X_i) = p(1-p)$

By CLT: $\sqrt{n}(Z_n - p) \xrightarrow{d} N(0, Var(X_i))$

So $\Rightarrow N(0, p(1-p))$

b $g(x) = x(1-x)$ and $g(Z_n) = Z_n(1-Z_n)$ maps sample proportion to Bernoulli RV variance.

Taylor Expansion - Delta Method linearizes $g(Z_n)$

$g(Z_n) \approx g(p) + g'(p)(Z_n - p)$ where $g'(x) = 1-2x$

$Z_n$ is asymptotically normal:

$n^{1/2}(Z_n - p) \xrightarrow{d} N(0, p(1-p))$

So $n^{1/2}(Z_n - g(p)) \xrightarrow{d} N(0, [g'(p)]^2 p(1-p))$

$g'(p) = 1-2p$ Variance of Limiting Distribution

$\hookrightarrow (1-2p)^2 p(1-p)$

$\boxed{\text{So } N(0, (1-2p)^2 p(1-p))}$

## Problem 4 (10pts)

Dirty Harry's experience is that 10% of the packages he mails do not reach their destination. He has bought two books for 25 dollars apiece and wants to mail them to his sister. If he sends them in one package, the stamp is 6 dollars, while for separate packages the stamp is 4 dollars for each package. To minimize his expected total money lost (possible loss of books + postage), should he send one or two packages ?

Scenario 1 - 1 Package

Probability of Package Getting Lost: 0.1

Loss if package is lost: 25+25=50

Cost of package: 6 dollars

Expected Loss = P(Loss) x Loss Amount + Package Cost

= 0.1 x 50 + 6 = \$11

Scenario 2 - 2 Packages

Probability of Package Getting Lost: 0.1

Loss if package is lost: 25

Cost of package: 4+4 = 8 dollars

Possible Outcomes:

1. Neither Package Lost
   $P = (1-0.1)(1-0.1) = 0.9^2 = 0.81 \rightarrow$ Loss = 0
2. One gets lost
   $P = 2 \cdot (0.1)(1-0.1) = 2 \times 0.1 \times 0.9 = 0.18 \rightarrow$ Loss = 25
3. Both Lost
   $P = 0.1 \times 0.1 = 0.01$ Loss = 50

Expected Loss = 0.81 x 0 + 0.18 x 25 + 0.01 x 50 = 13\$

Single Package is better

## Problem 6 (10pts)

Let $X$ be a gamma random variable with shape parameters $\alpha = 4$ and $\beta = 1$. Let $Y = e^X$. Write down the expression for the **probability density function** of $Y$.

**Hint** The pdf of a Gamma random variable $X$ with shape parameter $\alpha = n$ and scale parameter $\beta = 1$ is

$$f(x) = \frac{x^{n-1}e^{-x}}{(n-1)!}, \quad \text{for } x \geq 0$$

$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$

$\ln(y) = x$ and $\frac{dx}{dy} = \frac{1}{y}$

$f_Y(y) = f_X(\ln(y)) \cdot \frac{1}{y}, \quad y > 0$

$f_X(x) = \frac{x^3 e^{-x}}{6}$ with $x = \ln(y)$

$f_Y(y) = \frac{(\ln(y))^3 e^{-\ln(y)}}{6} \cdot \frac{1}{y}, \quad y > 0$

$e^{-\ln(y)} = \frac{1}{y}$

$f_Y(y) = \frac{(\ln(y))^3}{6y^2}, \quad y > 0$

## Problem 10 (10pts)

There are two urns; call these urn $A$ and urn $B$. There are 5 balls in urn $A$ with 3 of them being red and 2 being black. There are 6 balls in urn $B$ with 2 of them being red and 4 being black. We flip a **fair** coin once and if the coin landed head we draw a ball **at random** from urn $A$, while if the coin landed tail we choose a ball **at random** from urn $B$. **Given** that we drawn a **red** ball, what is the probability that the coin landed head ?

Urn A: 5, 3R, 2B

Urn B: 6, 2R, 4B

if head → Urn B

if tail → Urn A

$P(H \mid R) = \frac{P(R \mid H)P(H)}{P(R)}$

$P(H) = 0.5$

$P(R \mid H) = \frac{2}{6} = \frac{1}{3}$

$P(R \mid T) = \frac{3}{5}$

$R(R) = P(R \mid H)P(H) + P(R \mid T)(P \mid T)$

$= \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{1}{6} + \frac{3}{10} = \frac{10}{60} + \frac{18}{60} = \frac{28}{60} = \frac{7}{15}$

$P(H \mid R) = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{7}{15}} = \frac{\frac{1}{6}}{\frac{7}{15}} = \frac{1}{6} \cdot \frac{15}{7} = \frac{15}{42} = \boxed{\frac{5}{14}}$

## Problem 9 (20pts)

Let $(X, Y)$ has a uniform density in the half circle, i.e.,

$$f(x,y) = \frac{2}{\pi}, \qquad y \geq 0, x^2 + y^2 \leq 1.$$

and $f(x,y) = 0$ otherwise.

(a) (10pts) Find $\mathbb{E}[Y]$.
(b) (10pts) Find $\mathbb{E}[X \mid Y]$.

(a) $E[Y] = \iint_{\text{region}} y \cdot f(x,y)\, dx\, dy$

$x = r\cos\theta, \; y = r\sin\theta$

$r \in [0,1], \; \theta \in [0,\pi]$

$f(r,\theta) = \frac{2}{\pi}$ with $dx\,dy = r\,dr\,d\theta$

$E[Y] = \int_0^{\pi}\int_0^1 r\sin\theta \frac{2}{\pi} r\,dr\,d\theta$

$= \frac{2}{\pi}\int_0^{\pi} \sin\theta\, d\theta \int_0^1 r^2\, dr$

$\int_0^1 r^2\, dr = \frac{r^3}{3}\Big|_0^1 = \frac{1}{3}$

$\int_0^{\pi} \sin\theta\, d\theta = [-\cos\theta]_0^{\pi} = -\cos(\pi) + \cos(0) = 2$

$E[Y] = \frac{2}{\pi} \cdot 2 \cdot \frac{1}{3} = \frac{4}{3\pi}$

(b) $E[X \mid Y] = 0$

By symmetry of uniform dist over half circle, for $Y = y$, the $X$ values are uniformly distributed between $-\sqrt{1-y^2}$ and $\sqrt{1-y^2}$

The symmetry about $X = 0$ implies $E[X] = 0$

## Problem 11 (15pts)

Sales delay is the elapsed time between the manufacture of a product and its sale. Suppose we can model sales delay as a **log normal** random variable $X$ with parameters $\mu = 2$ and $\sigma = 0.2$ (the unit for $X$ here is in months).

(a) (5pts) What is the probability that delay time exceeds 8 months ?
(b) (5pts) What is the probability that delay time exceeds 10 months **given** that it already exceeds 6 months ?
(c) (5pts) Suppose we selected 10 items at random. What is the **expected** number of items (among these 10 items) that have delay time exceeding 8 months ?

**Note:** All answers to this problem can (and should be) expressed in terms of the **cumulative distribution function** (cdf) for the **standard normal** random variable $Z \sim \mathcal{N}(0,1)$. That is, your answer should be expressed in terms of the cdf

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-z^2/2}\,\mathrm{d}z.$$

For example, a valid answer to part (a) is $\Phi(c)$ where $c$ is some number.

(a) $P(X>8) = P(\ln(X) > \ln(8))$, $Y = \ln(X)$, $Y \sim N(\mu = 2, \sigma^2 = 0.2^2)$

$P(\ln(X) > \ln(8)) = P\left(Z > \frac{\ln(8) - \mu}{\sigma}\right) \to P(X > 8) = 1 - \Phi\left(\frac{\ln(8) - \mu}{\sigma}\right)$

(b) $P(X > 10 \mid X > 6) = \frac{P(X > 10 \cap X > 6)}{P(X > 6)} = \frac{P(X > 10)}{P(X > 6)}$

$c_1$: $P(X > 10) = 1 - \Phi\left(\frac{\ln(10) - 2}{0.2}\right)$ $\quad c_2$: $P(X > 6) = 1 - \Phi\left(\frac{\ln(6) - 2}{0.2}\right)$

$P(X > 10 \mid X > 6) = \frac{1 - \Phi(c_1)}{1 - \Phi(c_2)}$

(c) $P(X > 8) = 1 - \Phi\left(\frac{\ln(8) - 2}{0.2}\right)$

Expected Number $= n \cdot P(X > 8)$

$n = 10$ $\quad = 10 \cdot \left(1 - \Phi\left(\frac{\ln(8) - 2}{0.2}\right)\right)$

## Problem 12 (10pts)

Write down, for each of the following statements (statements (a) through (e)) whether the statement is **true** or **false**.

(a) Given **only** the **marginal** distributions of two random variables $X$ and $Y$, we can find $\mathbb{E}[XY]$.

(b) Suppose $X$ and $Y$ are two random variables with $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$. Then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

(c) Suppose $X$ and $Y$ are independent random variables. Then $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$.

(d) Let $A$, $B$, and $C$ be events. If $P(A \cap B) = P(A) \times P(B)$ and $P(B \cap C) = P(B) \times P(C)$ then $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$.

(e) Let $A$ and $B$ be events. Then $P(A \mid B) \geq P(A)$ if and only if $P(\bar{A} \mid B) \leq P(\bar{A})$.

(a) False → need joint distribution

(b) True

(c) True

(d) False → Pairwise Independence ≠ Mutual Independence

(e) True

## Problem 16 (20pts)

Let $X$ denote the temperature at which a certain chemical reaction takes place. Suppose that $X$ has pdf

$$f(x) = \begin{cases} \frac{1}{9}(4 - x^2) & \text{if } -1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) (10pts) Write down a simple expression for the cdf (cumulative distribution function) of $X$.
(b) (5pts) What is the probability that the temperature is **positive** when the reaction occurs ?
(c) (5pts) Compute $\mathbb{E}[X]$, the **expected temperature**, at which a reaction will take place.

(a) CDF: $F(x) = P(X < x) = \int_{-\infty}^{x} f(t)\,dt$

$F(x) = \int_{-1}^{x} \frac{1}{9}(4 - t^2)\,dt = \frac{1}{9}\left(4t - \frac{t^3}{3}\right)\Big|_{-1}^{x} = \frac{1}{9}\left(4x - \frac{x^3}{3} + \frac{13}{3}\right)$

(b) $P(X > 0) = 1 - P(X \leq 0) = 1 - F(0)$

$F(0) = \frac{1}{9}\left(4(0) - \frac{0^3}{3} + \frac{13}{3}\right) = \frac{1}{9} \cdot \frac{13}{3} = \frac{13}{27}$

$1 - \frac{13}{27} = \boxed{\frac{14}{27}}$

(c) $E[X] = \int_{-1}^{2} x f(x)\,dx$

$= \int_{-1}^{2} x \frac{1}{9}(4 - x^2)\,dx = \frac{1}{9}\int_{-1}^{2}(4x - x^3)\,dx$

$= \frac{1}{9}\left(\int_{-1}^{2} 4x\,dx - \int_{-1}^{2} x^3\,dx\right) = \frac{1}{9}\left(6 - \frac{15}{4}\right) = \boxed{\frac{1}{4}}$

## Problem 15 (10pts)

In repeated throws of a **fair** die (so all six faces of the die are equally likely to appear), let $X$ be the throw in which the first six is obtained and $Y$ the throw in which the second six is obtained. For example, $X = 3$ and $Y = 7$ indicates that the first six appeared in the third throw and the second six appeared in the seventh throw.

(a) (5pts) Write down the **joint** probability mass function for $X$ and $Y$. That is, write down a simple expressions for the function $p(x,y) = P(X = x, Y = y)$
(b) (5pts) Compute $\mathbb{E}[Y - X]$.

**Hint** Think memory-less. Especially for part (b).

(a) $P(X = x) = \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6}$

$P(Y = y \mid X = x) = \left(\frac{5}{6}\right)^{y-x-1} \cdot \frac{1}{6}$

Joint PMF: $P(X = x \mid Y = y) = P(X = x) \cdot P(Y = y \mid X = x)$

$= \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{y-x-1} \cdot \frac{1}{6}$

$\left(\frac{5}{6}\right)^{y-2} \cdot \frac{1}{36} \quad 1 \leq x < y$

(b) $p = \frac{1}{6} \to E[\text{Geometric}(p)] = \frac{1}{p}$

$E[Y - X] = \frac{1}{1/6} = 6$

$\boxed{E[Y - X] = 6}$