12. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses?

9 heads in 10 toss $\rightarrow Bin(n=10, p=0.5)$

18 heads in 20 toss $\rightarrow Bin(n=20, p=0.5)$

$$P(X=x) = \binom{n}{x} \cdot p^x \cdot p^{n-x}$$

$$P(X=9) = \binom{10}{9} \cdot 0.5^9 \cdot 0.5^1 = 0.0098 \quad \frac{9 \text{ head}}{10 \text{ toss}}$$

$$P(X=18) = \binom{20}{18} \cdot 0.5^{18} \cdot 0.5^2 = 0.0062 \quad \frac{18 \text{ head}}{20 \text{ toss}}$$

Probability is higher for $P(X=9)$
9 heads in 10 tosses is more likely

13. A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 12 items or more correct.

a. What is the probability that the student passes?

b. Answer the question in part (a) again, assuming that the student can eliminate two of the choices on each question.

a

$$P(\text{Correct}) = p = \frac{1}{3}$$

$$\text{Bin}\left(n=20, p=\frac{1}{3}\right)$$

$$\boxed{P(X \geq 12) = \sum_{x=12}^{20} \binom{20}{x} \cdot \left(\frac{1}{3}\right)^{x} \cdot \left(\frac{2}{3}\right)^{20-x}}$$

b

$$P(\text{Correct}) = p = \frac{1}{2}$$

$$\text{Bin}\left(n=20, p=\frac{1}{2}\right)$$

$$\boxed{P(X \geq 12) = \sum_{x=12}^{26} \binom{20}{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{20-x}}$$

14. Two boys play basketball in the following way. They take turns shooting and stop when a basket is made. Player A goes first and has probability p_1 of making a basket on any throw. Player B, who shoots second, has probability p_2 of making a basket. The outcomes of the successive trials are assumed to be independent.

a. Find the frequency function for the total number of attempts.

b. What is the probability that player A wins?

(a) First basket Player A makes is on $2n-1$ attempt

So $P(\text{A makes basket}) = (1-p_1)(1-p_2)^{n-1} p_1$

First basket Player B makes is on $2n$ attempt

So $P(\text{B makes basket}) = (1-p_1)(1-p_2)^{n-1} p_2$

Frequency Function

odd $n: 2k-1$ ie. Player A makes basket

$$P(N=2k-1) = (1-p_1)(1-p_2)^{k-1} p_1$$

even $n: 2k$ ie Player B makes basket

$$P(N=2k) = (1-p_1)(1-p_2)^{k-1} p_2$$

(b) $P(\text{Player A wins}) = p_1 + (1-p_1)(1-p_2)p_1 + (1-p_1)^2(1-p_2)^2 p_1 + \dots$

$$= p_1 \sum_{k=0}^{\infty} [(1-p_1)(1-p_2)]^k$$

$$= \frac{p_1}{1-(1-p_1)(1-p_2)}$$

$$= \boxed{\frac{p_1}{p_1+p_2-p_1p_2}}$$

7. Let X be a discrete random variable that takes on values 0, 1, 2 with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively.

a. Find $E(X)$.

b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.

c. Use Theorem A of Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).

d. Find $\mathrm{Var}(X)$ according to the definition of variance given in Section 4.2. Also find $\mathrm{Var}(X)$ by using Theorem B in Section 4.2.

a $E[X] = \sum x \cdot p(x)$

$$= 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{8} = \frac{3}{8} + \frac{2}{8} = \boxed{\frac{5}{8}}$$

b $Y = x^2$

$$\boxed{P(Y = y = x^2) = \begin{cases} 1/2 & \to x = 0 \\ 3/8 & \to x = 1 \\ 1/8 & \to x = 4 \end{cases}} \quad E[Y] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 4 \cdot \frac{1}{8} = \frac{3}{8} + \frac{4}{8} = \boxed{\frac{7}{8}}$$

c Theorum A: $E[Y] = \sum_x g(x) \cdot p(x)$

$g(x) = x^2$ so $E[Y = x^2] = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{1}{8} = \boxed{\frac{7}{8}}$

d Definition: $\mathrm{Var}(x) = E([x - E(x)]^2)$

$$E([x - E(x)]^2) = \sum_x (x - E(x))^2 \cdot P(X = x)$$

$x = 0 \to (0 - \frac{5}{8})^2 = \frac{25}{64}$, $x = 1 \to (1 - \frac{5}{8})^2 = \frac{9}{64}$, $x = 2 \to (2 - \frac{5}{8})^2 = \frac{121}{64}$

$$E([x - E(x)]^2) = \frac{25}{64}\left(\frac{1}{2}\right) + \frac{9}{64}\left(\frac{3}{8}\right) + \frac{121}{64}\left(\frac{1}{8}\right) = \boxed{\frac{31}{64}}$$ – Var(x)

Theorum B: $\mathrm{Var}(x) = E[x^2] - (E[x])^2$

$E[x^2] = \frac{7}{8}$ from b, $E[x] = \frac{5}{8}$ from a so $\frac{7}{8} - \left(\frac{5}{8}\right)^2 = \frac{7}{8} - \frac{25}{64}$

$$= \frac{56}{64} - \frac{25}{64} = \boxed{\frac{31}{64}}$$ – Var(x)