ST 501: Fundamentals Statistical Inference I

Conditional probability and independence

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Conditional probability

Definition
Let Ω be a sample space and P a probability measure on Ω. Let B be an event with $P(B) > 0$. Then for any $A \subset \Omega$, the conditional probability of A given B, denoted as $P(A \mid B)$ is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Example Let Ω be the sample space associated with rolling a fair, six-faced die. Let A be the event that the top face shows 1 and let B be the event that the top face shows an odd number. Then $A = \{1\}$, $B = \{1, 3, 5\}$ and

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{1\})}{P(\{1, 3, 5\})} = \frac{1/6}{1/2} = \frac{1}{2}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\{1\})}{P(\{1\})} = 1$$

We note the obvious fact that, in general, $P(A \mid B) \neq P(B \mid A)$.

Example Consider the following 2×2 contingency table listing disease status vs exposure status.

Disease/Exposure	Yes Exposure	No Exposure	Row Total
Yes Disease	a	b	$a+b$
No Disease	c	d	$c+d$
Column Total	$a+c$	$b+d$	

Let B be the event that a randomly selected person is expose to the contaminant and A be the event that a randomly selected person contracted the disease. Then

$$P(A) = \frac{a+b}{a+b+c+d}; \quad P(B) = \frac{a+c}{a+b+c+d};$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{a}{a+c} = \left(\frac{a}{a+b+c+d}\right) / \left(\frac{a+c}{a+b+c+d}\right)$$

Conditional probability is a probability measure

Proposition

Let Ω be a sample space and P a probability measure on Ω. Then for any event B with $P(B) > 0$, $P(\cdot|B)$ is a probability measure on Ω, i.e.,

- $P(A \mid B) \geq 0$ *for all events* $A \subset \Omega$
- $P(\Omega \mid B) = 1$
- *For any sequence of disjoint events* $A_1, A_2, \ldots,$

$$P((A_1 \cup A_2 \cup \ldots) \mid B) = \sum_i P(A_i \mid B) \quad \textit{(countable additivity)}$$

See also problem 1.64 in your textbook.

We now verify the countable additivity condition. Indeed,

$$
\begin{aligned}
P((A_1 \cup A_2 \cup \cdots \cup \cdots) \mid B) &= \frac{P(A_1 \cup A_2 \cup \cdots) \cap B)}{P(B)} \\
&= \frac{P((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots)}{P(B)} \\
&= \frac{1}{P(B)} \sum_i P(A_i \cap B) = \sum_i P(A_i \mid B)
\end{aligned}
$$

since, if the $\{A_i\}$ are disjoint, the $\{A_i \cap B\}$ are also disjoint.

EXAMPLE 2.26 A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05

Find the following probabilities

- $P(A \mid B)$
- $P(A \mid B \cup C)$
- $P(A \mid \text{read at least once})$
- $P(A \cup B \mid C)$

Multiplication rule

Proposition

Let Ω *be a sample space and* $B \subset \Omega$ *satisfy* $P(B) > 0$*. Then*

$$P(A \cap B) = P(A \mid B) \times P(B)$$

for all $A \in \Omega$*. Furthermore, if* $A_1, A_2, \dots, A_n$ *is a sequence of* n *events with* $P(A_2 \cap A_3 \cap \cdots \cap A_n) > 0$ *then*

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_n) \times \prod_{i=1}^{n-1} P(A_i \mid A_{i+1} \cap \cdots \cap A_n).$$

See also problem 1.45 in your textbook.

63. For customers purchasing a refrigerator at a certain appliance store, let A be the event that the refrigerator was manufactured in the U.S., B be the event that the refrigerator had an icemaker, and C be the event that the customer purchased an extended warranty. Relevant probabilities are

$$P(A) = .75 \quad P(B|A) = .9 \quad P(B|A') = .8$$

$$P(C|A \cap B) = .8 \quad P(C|A \cap B') = .6$$

$$P(C|A' \cap B) = .7 \quad P(C|A' \cap B') = .3$$

Find the following probabilities

- $P(A \cap B \cap C)$
- $P(B \cap C)$
- $P(C)$

Law of Total Probability and Bayes Law

Proposition

Let Ω be a sample space. Let $A_1, A_2, \ldots, A_n$ be a partition of Ω. Then for any event $B \subset \Omega$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i) \times P(A_i) \quad \textit{(total probability)},$$

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B \mid A_i) \times P(A_i)}{\sum_j P(B \mid A_j) \times P(A_j)} \quad \textit{(Bayes Law)}.$$

Example A certain item is produced on one of three machines, M_1, M_2, and M_3. The percentages of items produced on these machines are 50%, 30% and 20%, respectively. In addition, 4%, 2%, and 4% of the items produced by each machine are defective, respectively. What is the probability that a randomly chosen item is defective ? What is the probability that, given that an item is defective, that it was produced by machine M_1 ?

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A. Let B be the event that the item is defective, and A_i be the event that the item was produced by machine M_i. Then

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) \\ &= 0.5 \times 0.04 + 0.3 \times 0.02 + 0.2 \times 0.04 = 0.034 \end{aligned}$$

Furthermore

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B \mid A_1) \times P(A_1)}{P(B)} = \frac{0.04 \times 0.5}{0.034} = \frac{10}{17}$$

Example A disease is taking the world by storm. It is estimated that currently roughly 2% of the population has the disease. Assume that there is a test T that can diagnose the disease; the test returns a $+$ if the person is suspected of having the disease. Assume that T is correct 95% of the time if the patient has the disease and is wrong 10% of the time if the patient does not have the disease. A person is selected for testing; what is the probability that the patient has the disease, given that the test shows up positive ?

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Let A be the event that the test shows up positive and B be the event that the person has the disease. Then

$$P(A) = P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B}) = 0.117$$

We therefore have

$$P(B \mid A) = \frac{P(A \mid B) \times P(B)}{P(A)} = \frac{0.95 \times 0.02}{0.117} \approx 0.162$$

61. Components of a certain type are shipped to a supplier in batches of ten. Suppose that 50% of all such batches contain no defective components, 30% contain one defective component, and 20% contain two defective components. Two components from a batch are randomly selected and tested. What are the probabilities associated with 0, 1, and 2 defective components being in the batch under each of the following conditions?

a. Neither tested component is defective.

b. One of the two tested components is defective. [*Hint:* Draw a tree diagram with three first-generation branches for the three different types of batches.]

Let B_i denote the event that the two components came from a batch with i defect(s).

Let D_0 be the event that neither tested components is defective while D_1 be the event that one of the tested components is defective.

For part (a) we are interested in

$$P(B_i \mid D_0) = \frac{P(D_0 \mid B_i)P(B_i)}{\sum_{j=0}^{2} P(D_0 \mid B_j)P(B_j)}$$

We now have

$$P(D_0 \mid B_0) = 1, \quad P(D_0 \mid B_1) = \frac{\binom{9}{2}}{\binom{10}{2}}, \quad P(D_0 \mid B_2) = \frac{\binom{8}{2}}{\binom{10}{2}}.$$

For part (b) we are interested in

$$P(B_i \mid D_1) = \frac{P(D_1 \mid B_i)P(B_i)}{\sum_{j=0}^{2} P(D_1 \mid B_j)P(B_j)}$$

We also have

$$P(D_1 \mid B_0) = 0, \quad P(D_1 \mid B_1) = \frac{9}{45}, \quad P(D_1 \mid B_2) = \frac{16}{45}$$

Smears

In a population of women, a proportion p have abnormal cells on the cervix. The pap test entails taking a sample of cells from the surface of the cervix and examining the sample to detect any abnormality. The Pap test has the following property

- In a case where abnormal cells are present, the sample will fail to include any with probability q.
- In a sample including abnormal cells, examination fails to observe them with probability r.
- In a sample free of abnormal cells, normal cells are wrongly classified as abnormal with probability s.

Q. If a randomly selected woman has a Pap smear test

- What is the probability that the result is wrong ?
- If an abnormality is reported, what is the probability that there were no abnormal cells present ?

Q. If a randomly selected woman has a Pap smear test

- What is the probability that the result is wrong ?
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Let E be the event that the test is wrong, A be the event that abnormal cells are present and S be the event that the sample include abnormal cells when they are present. Then

$$P(A) = p; \quad P(\bar{S} \mid A) = q;$$
$$P(E \mid A \cap S) = P(E \mid S) = r; \quad P(E \mid A \cap \bar{S}) = 1 - s.$$

We therefore have

$$\begin{aligned} P(E) &= P(E \mid A)P(A) + P(E \mid \bar{A})P(\bar{A}) \\ &= P(E \mid A) \times p + P(E \mid \bar{A}) \times (1-p) \\ &= P(E \mid A) \times p + s \times (1-p) \end{aligned}$$

We next consider $P(E \mid A)$. Then

$$\begin{aligned} P(E \mid A) &= P(E \mid A \cap S)P(S \mid A) + P(E \mid A \cap \bar{S})P(\bar{S} \mid A) \\ &= r \times (1 - q) + (1 - s) \times q \end{aligned}$$

Combining the above expressions, we get

$$P(E) = p((1 - q)r + (1 - s)q) + (1 - p)s.$$

Let D be the event that an abnormality is reported. We want to find $P(\bar{A} \mid D)$. We first need to compute $P(D)$

$$
\begin{aligned}
P(D) &= P(D \mid A) \times P(A) + P(D \mid \bar{A}) \times P(\bar{A}) \\
&= P(D \cap S \mid A) \times p + P(D \cap \bar{S} \mid A) \times p + s \times (1-p) \\
&= (1-r) \times (1-q) \times p + s \times q \times p + s \times (1-p)
\end{aligned}
$$

We therefore have

$$
P(\bar{A} \mid D) = \frac{P(\bar{A} \cap D)}{P(D)} = \frac{(1-p)s}{pqs + p(1-q)(1-r) + (1-p)s}.
$$

Monty Hall Problem

Q. You are at a game show, where there are three doors. Two of the doors have goats behind them, and the remaining door has a beat up 1990 Toyota Camry. The host asked you to pick a door. After you picked one door, the host open another door, revealing a goat behind it. The host then offered you the opportunity to switch doors. Supposed you really loved Toyota Camry and wants to win one. Should you switch ?

Monty Hall Problem

Q. You are at a game show, where there are three doors. Two of the doors have goats behind them, and the remaining door has a beat up 1990 Toyota Camry. The host asked you to pick a door. After you picked one door, the host open another door, revealing a goat behind it. The host then offered you the opportunity to switch doors. Supposed you really loved Toyota Camry and wants to win one. Should you switch ?

A. Let us first assume that the host protocol is that

- If behind the door you picked is a goat, then the host will always show the remaining door that contains a goat.
- If behind the door you picked is a car, then the host will choose one of the remaining door (with equal probabilities) and show you that door.

Suppose, without loss of generality, you picked door 1. Let A be the event that door 1 contains the car. Then $P(A) = 1/3$.

If door 1 contains the car, and you switched, then the probability of you lose, i.e., $P(\text{wins} \mid A) = 0$.

In contrast, if door A does not contains the car and you switched, then $P(\text{wins} \mid \bar{A}) = 1$. Therefore

$$P(\text{wins}) = P(\text{wins} \mid A) \times P(A) + P(\text{wins} \mid \bar{A}) \times P(\bar{A}) = \frac{2}{3}$$

By switching, you increased your probability of winning to 2/3 (from the original 1/3)

Q2. What happen if the host protocol is that “whatever door you picked, the host will show one of the two remaining door with equal probabilities” ?

Suppose you flip three fair coins. At least two are alike, and the chances that the remaining coin is a head or a tail are equal so the chance that all three coins are the same is 1/2.

More specifically, suppose the two coins that are guaranteed to be alike are heads. The probability that the remaining coin is head is 1/2, so the probability that all three coins are heads is 1/2. Similarly, suppose the two coins that are guaranteed to be alike are tails. The probability that the remaining coin is tail is 1/2 and so the probability that all three coins are tails is 1/2.

Q. What is wrong with this reasoning ?

A. The statement that ”the chances that the remaining coin is a head or a tail are equal” is an unconditional probability.

However, the statement that ”at least two are alike, and the chances that the remaining coin is a head or a tail are equal” is talking about a conditional probability.

Explicit calculations yield

$$P(\text{remaining is H} \mid \geq 2 \text{ are H}) = \frac{P(\{HHH\})}{P(\{HHH, HHT, THH, HTH\})} = \frac{1}{4}$$

$$P(\text{remaining is T} \mid \geq 2 \text{ are H}) = \frac{P(\{HHT, THH, HTH\})}{P(\{HHH, HHT, THH, HTH\})} = \frac{3}{4}$$

$$P(\text{remaining is T} \mid \geq 2 \text{ are T}) = \frac{P(\{TTT\})}{P(\{TTT, TTH, THT, HTT\})} = \frac{1}{4}$$

$$P(\text{remaining is H} \mid \geq 2 \text{ are T}) = \frac{P(\{TTH, THT, HTT\})}{P(\{TTT, TTH, THT, HTT\})} = \frac{3}{4}$$

so the statement that ”at least two are alike, and the chances that the remaining coin is a head or a tail are equal” is wrong

Independence of two events

Definition

Let Ω be a sample space and P a probability measure on Ω. Then for any events $A \subset \Omega$ and $B \subset \Omega$, A and B are independent if and only if either of the following conditions are true.

- $P(A \mid B) = P(A)$ whenever $P(B) > 0$
- $P(B \mid A) = P(B)$ whenever $P(A) > 0$
- $P(A \cap B) = P(A) \times P(B)$

Two events A and B are said to be dependent if A and B are not independent.

Important Independence and disjointedness are two completely different concepts.

In particular, suppose A and B are disjoint. Then $P(A \cap B) = 0$. In order for A and B to also be independent, i.e.,

$$P(A \cap B) = 0 = P(A) \times P(B),$$

it is necessary that either $P(A) = 0$ or $P(B) = 0$ (or both).

In other words, if A and B are disjoint, then knowing that B occurs necessarily implies that A cannot occurs, i.e., $P(A \mid B) = 0$. In contrast, if A and B are independent, then knowing B occurs does not change the probability that A occurs, i.e., $P(A \mid B) = P(A)$.

Example Three brands of coffee, say X, Y, and Z are to be ranked according to taste by a judge. Consider the events

- A: Brand X is preferred to brand Y
- B: Brand X is ranked first (best)
- C: Brand X is ranked second.
- D: Brand X is ranked third (worst).

Assume the sample points are equally likely, then

$$
\begin{aligned}
P(A) &= P(\{(X,Y,Z),(X,Z,Y),(Z,X,Y)\}) = 1/2 \\
P(A \mid B) &= \frac{P(\{(X,Y,Z),(X,Z,Y)\})}{P(\{(X,Y,Z),(X,Z,Y)\})} = 1 \\
P(A \mid C) &= \frac{P(\{Z,X,Y\})}{P(\{(Y,X,Z),(Z,X,Y)\})} = 1/2 \\
P(A \mid D) &= \frac{P(\emptyset)}{P(\{(Y,Z,X),(Z,Y,X)\})} = 0
\end{aligned}
$$

Proposition

Let A and B be independent events. Then

- *A and $\bar{B}$ are independent events*
- *$\bar{A}$ and B are independent events*
- *$\bar{A}$ and $\bar{B}$ are independent events.*

Indeed, we have

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A \cap B) \\ &= P(A) - P(A) \times P(B) \\ &= P(A) \times (1 - P(B)) = P(A) \times P(\bar{B}) \end{aligned}$$

See also problem 1.65 in your textbook.

Similarly,

$$\begin{aligned}
P(\bar{A} \cap \bar{B}) &= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - P(A) - P(B) + P(A) \times P(B) \\
&= (1 - P(A)) \times (1 - P(B)) = P(\bar{A}) \times P(\bar{B})
\end{aligned}$$

Independence of $n \geq 2$ events

Definition

Let Ω be a sample space and P a probability measure on Ω. Then for any finite number of events $A_1, A_2, \ldots, A_n$, we say that the A_i are mutually independent if, for all non-empty set $S \subset \{1, 2, \ldots, n\}$, we have

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$$

Example A_1, A_2 and A_3 are independent if and only if

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2), P(A_1 \cap A_3) = P(A_1) \times P(A_3),$$
$$P(A_2 \cap A_3) = P(A_2) \times P(A_3);$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

Example We toss a fair coin twice. Define the events

- A: The first throw shows H
- B: The second throw shows H
- C: Both throw yields the same faces (both H or both T).

Then $C = \{HH, TT\}$ and $P(C) = 1/2$. Hence

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4$$
$$P(A \cap B \cap C) = 1/4 \neq P(A) \times P(B) \times P(C)$$

The events A, B and C are thus pairwise independent but not mutually independent.

Example There is an urn with m white balls and n black balls. We sample r balls, with replacement from the urn. Let A_i be the event that the ith draw returns a white ball. Then the $A_1, A_2, \ldots, A_r$ are mutually independent.

Example There is an urn with m white balls and n black balls. We sample r balls without replacement, from the urn. Let A_i be the event that the ith draw returns a white ball. Then the $A_1, A_2, \ldots, A_r$ are not mutually independent.

Probabilities calculations are greatly simplified when the events are mutually independent. A natural question is then

Q. Is there a simple and general way to determine if/when the events are independent ?

The answer is sadly no. More specifically, except in a few simple cases, mutual independence is a consequence of either an implicit or an explicit assumption, e.g., the above examples of sampling with and without replacement.

Example Suppose Jack and Jill have three cars, and on any given winter morning, the cars are, independently of each other, operational with with probabilities, 0.9, 0.95 and 0.99.

Let A_i be the event that the ith car runs.

The probability that all three cars are operational is

$$P(A_1 \cap A_2 \cap A_3) = 0.9 \times 0.95 \times 0.99 = 0.84645.$$

The probability that none of the car runs is operational is

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) = (1 - 0.9) \times (1 - 0.95) \times (1 - 0.99) = 0.0005$$

The probability that at least one of the car is operational is

$$P(A_1 \cup A_2 \cup A_3) = 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) = 0.99995$$

The probability that exactly one of the car is operational is

$$\begin{aligned} P(\text{exactly 1}) &= P(A_1 \cap \bar{A}_2 \cap \bar{A}_3) + P(\bar{A}_1 \cap A_2 \cap \bar{A}_3) + P(\bar{A}_1 \cap \bar{A}_2 \cap A_3) \\ &= 0.9 \cdot 0.05 \cdot 0.01 + 0.1 \cdot 0.95 \cdot 0.01 + 0.1 \cdot 0.05 \cdot 0.99 \\ &= 0.00635. \end{aligned}$$

The probability that exactly two of the car is operational is

$$\begin{aligned} P(\text{exactly 2}) &= P(A_1 \cap A_2 \cap \bar{A}_3) + P(A_1 \cap \bar{A}_2 \cap A_3) + P(\bar{A}_1 \cap A_2 \cap A_3) \\ &= 0.9 \cdot 0.95 \cdot 0.01 + 0.9 \cdot 0.05 \cdot 0.99 + 0.1 \cdot 0.95 \cdot 0.99 \\ &= 0.14715. \end{aligned}$$

Note how the mutual independence of the events allows for simple computations of the probabilities.

77. An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are defective independently of one another, each with the same probability.

a. If 15% of all seams need reworking, what is the probability that a rivet is defective?

b. How small should the probability of a defective rivet be to ensure that only 10% of all seams need reworking?

Let p be the probability that a rivet is defective.

The probability that none of the 25 rivets in a given seam is defective is $(1-p)^{25}$.

The probability that a seam needs reworking is then $1-(1-p)^{25}$.

For part (a) we want to find p such that

$$1-(1-p)^{25} = 0.15 \Longrightarrow p = 1-(0.85)^{1/25} \approx 0.006$$

For part (b) we want

$$1-(1-p)^{25} \leq 0.1 \Longrightarrow p \leq 1-0.9^{0.25} \approx 0.004.$$

First player advantage

Example Coin A gives heads with probability s and coin B gives heads with probability t. Alice and Bob take turns, alternating the throwing of these coins, with Alice throwing coin A and Bob throwing coin B. Alice wins if coin A lands up head first, while Bob wins if coin B lands up head first. Alice starts the game. What is the probability that Alice wins ?

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Define the events

- A_1: The first coin throw (coin A) is a head
- A_2: The first and second throws is TH (coin A is tail, coin B is head)
- A_3: The first and second throws is TT (coin A is tail, coin B is tail).

The events A_1, A_2 and A_3 is a partition of the sample space. Let A be the event that Alice wins. Then

$$\begin{aligned} P(A) &= P(A \mid A_1)P(A_1) + P(A \mid A_2)P(A_2) + P(A \mid A_3)P(A_3) \\ &= 1 \times P(A_1) + 0 \times P(A_2) + P(A) \times P(A_3) \\ &= s + (1-s) \times (1-t) \times P(A) \end{aligned}$$

as $P(A \mid A_3) = P(A)$ (the game restarts again if neither Alice nor Bob wins after their first turn). We therefore have

$$P(A) = \frac{s}{1-(1-s)(1-t)} = \frac{s}{s+t-st}.$$

Let $s = 0.35$ (Alice's coin is biased against heads), and $t = 0.5$ (Bob's coin is fair). In this case, $P(A) \approx 0.519 > 0.5$. Even with a strongly biased coin, Alice has an advantage by starting first.

Accidents vs Insurance

Q. In any given year, the probability that a random selected male driver has an accident that lead to a claim from his insurance company is p and is independent of the accidents/claims (for the same driver) in other years.

Similarly, the probability that a randomly selected female driver has an accident that lead to a claim is q and this is also independent of the accidents/claims in other years.

Assume $q < p$ and that there are roughly equal number of male and female drivers insured with the insurance company.

A driver is selected at random. What is

- the probability that the driver makes a claim this year
- the probability that the driver makes a claim in two consecutive years ?
- the probability that the drive makes another claim next year, given that he/she makes a claim this year ?

Let A_1 be the event that the selected driver makes a claim this year. Then

$$P(A_1) = P(A_1 \cap \text{male driver}) + P(A_1 \cap \text{female driver}) = (p+q)/2$$

Let A_2 be the event that the selected driver makes a claim next year. Then $P(A_2) = P(A_1)$. Furthermore,

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1 \cap A_2 \cap \text{male driver}) + P(A_1 \cap A_2 \cap \text{female driver}) \\ &= \frac{1}{2}p^2 + \frac{1}{2}q^2 \neq P(A_1) \times P(A_2) \end{aligned}$$

Finally, we have

$$P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{(p^2+q^2)/2}{(p+q)/2} = \frac{p^2+q^2}{p+q} > P(A_2)$$

In summary, the independence of the claims is thus a conditional probability statement (conditional on the gender) and not an unconditional probability statement.

In particular the probability that a randomly selected person makes a claim this year, given that they made a claim the previous year, is larger than the probability that another randomly selected person makes a claim this year.