12. Let

$$f(x, y) = c(x^2 - y^2)e^{-x}, \qquad 0 \le x < \infty, \qquad -x \le y < x$$

a. Find c.
b. Find the marginal densities.
c. Find the conditional densities.

a $\int_0^\infty \int_{-x}^{x} f(x,y)\,dy\,dx = 1 \to \int_0^\infty \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy\,dx = 1$

$= e^{-x}\left(\int_{-x}^{x} x^2\,dy - \int_{-x}^{x} y^2\,dx\right) = e^{-x}\left(2x^3 - \frac{2x^3}{3}\right) = e^{-x}\frac{4x^3}{3}\cdot c$

$\to c\int_0^\infty e^{-x}\frac{4x^3}{3}\,dx = 1 \to \frac{4c}{3}\int_0^\infty x^3 e^{-x}\,dx = 1 \to \frac{4c}{3}\Gamma 4 = 1$

$\to \frac{4c}{3}3! = 1 \to 8c = 1 \to \boxed{c = \frac{1}{8}}$

b $f_X(x) = \int_{-x}^{x} f(x,y)\,dy = \int_{-x}^{x} \frac{1}{8}(x^2-y^2)e^{-x}\,dy = \frac{e^{-x}}{8}\left[x^2y - \frac{y^3}{3}\right]_{-x}^{x}$

$= \frac{e^{-x}}{8}\left[x^3 - \frac{x^3}{3} + x^3 - \frac{x^3}{3}\right] = \frac{e^{-x}}{8}\left(\frac{6x^3 - 2x^3}{3}\right) = \frac{4x^3e^{-x}}{8\times 3} =$

$= \boxed{\frac{x^3}{6}e^{-x}}$

$f_X(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty \frac{1}{8}(x^2-y^2)e^{-x}\,dx = \frac{1}{8}\int_0^\infty x^2e^{-x}\,dx - \frac{y^2}{8}\int_0^\infty e^{-x}\,dx$

$= \frac{2}{8}\left[(xe^{-x})_\infty^0 - \int_0^\infty -e^{-x}\,dx\right] - \frac{y^2}{8} = \frac{1}{4}(e^{-x})_\infty^0 - \frac{y^2}{8} = \boxed{\frac{2-y^2}{8}}$

c $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{8}(x^2-y^2)e^{-x}}{\frac{1}{6}x^3e^{-x}} = \frac{6}{8}\left(\frac{x^2-y^2}{x^3}\right) = \boxed{\frac{3}{4}\left[\frac{1}{x} - \frac{y^2}{x^3}\right]}$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(x^2-y^2)e^{-x}}{\frac{2-y^2}{8}} = \boxed{\frac{(x^2-y^2)e^{-x}}{2-y^2}}$

16. What is the probability density of the time between the arrival of the two packets of Example E in Section 3.4?

$f(t_1, t_2) = \frac{1}{T^2}$ for $0 \le t_1, t_2 \le T$

$D = |T_1 - T_2|$

Find $D \le d \rightarrow F_D(d)$ where $|T_1 - T_2| \le d$

Area given by $T^2 - (T-d)^2 = 2Td - d^2$

CDF of $F_D(d) = \frac{2Td - d^2}{T^2}$

$$f_D(d) = \frac{d}{dd}\left(\frac{2Td - d^2}{T^2}\right) = \frac{2T - 2d}{T^2} = \frac{2(T-d)}{T^2}$$

$$\boxed{f_D(d) = \frac{2(T-d)}{T^2}, \quad 0 \le d \le T}$$

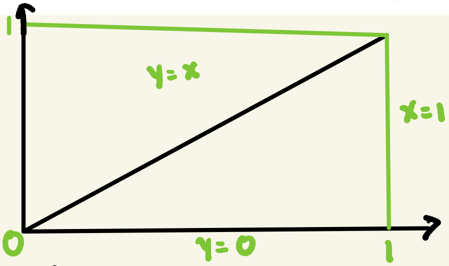
18. Let X and Y have the joint density function

$$f(x, y) = k(x - y), \qquad 0 \le y \le x \le 1$$

and 0 elsewhere.

a. Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.
b. Find k.
c. Find the marginal densities of X and Y.
d. Find the conditional densities of Y given X and X given Y.

a



b $\int_0^1\int_0^x k(x-y)\,dy\,dx = 1 \rightarrow \int_0^1\int_0^x k(x-y)\,dy\,dx =$

$= k\int_0^1 \left(\int_0^x (x-y)\,dy\right)dx$

$\int_0^x (x-y)\,dy = \left[xy - \frac{y^2}{2}\right]_0^x = x^2 - \frac{x^2}{2} = \frac{x^2}{2} \rightarrow \int_0^1\int_0^x k(x-y)\,dy\,dx = \int_0^1 \frac{x^2}{2}$

$k\int_0^1 \frac{x^2}{2}\,dx = k\cdot\frac{1}{2}\int_0^1 x^2\,dx = k\cdot\frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^1 = k\cdot\frac{1}{2}\cdot\frac{1}{3} = \frac{k}{6}$

$\frac{k}{6} = 1 \rightarrow \boxed{k=6}$

c $f_X(x) = \int_0^x f(x,y)\,dy = \int_0^x 6(x-y)\,dy = 6\cdot\frac{x^2}{2} = \boxed{3x^2 \text{ for } 0 \le x \le 1}$

$f_Y(y) = \int_y^1 f(x,y)\,dx = \int_y^1 6(x-y)\,dx = 6\int_0^{1-y} u\,du = 6\cdot\frac{(1-y)^2}{2} = \boxed{3(1-y)^2 \quad 0 \le y \le 1}$

d $f(y|x) = \frac{f(x,y)}{f(x)} = \frac{6(x-y)}{3x^2} = \frac{2(x-y)}{x^2}$

$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{6(x-y)}{3\cdot(1-2y+y^2)} = \frac{2(x-y)}{1-2y+y^2}$

$$f(y|x) = \begin{cases} \frac{2(x-y)}{x^2} & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x|y) = \begin{cases} \frac{2(x-y)}{1-2y+y^2} & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

23. Suppose that, conditional on N, X has a binomial distribution with N trials and probability p of success, and that N is a binomial random variable with m trials and probability r of success. Find the unconditional distribution of X.

$X \mid N = n \sim \text{Binomial}(n, p)$

PMF: $P(X=x \mid N=n) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \dots, n$

$N \sim \text{Binomial}(m, r)$

PMF: $P(N=n) = \binom{m}{n} r^n (r-1)^{m-n}$, $n = 0, 1, \dots, n$

$$P(X=x) = \sum_{n=0}^{m} P(X=x \mid N=n) \cdot P(N=n)$$

$$\Rightarrow P(X=x) = \sum_{n=0}^{m} \binom{n}{x} p^x (1-p)^{n-x} \cdot \binom{m}{n} r^n (1-r)^{m-n}$$

↳ Represents probability $X=x$ for a compound binomial distribution with random number of trials N. So..

$$\boxed{X \sim \text{Binomial}(m, pr)}$$

48. Let U and V be independent random variables with means μ and variances σ^2. Let $Z = \alpha U + V\sqrt{1-\alpha^2}$. Find $E(Z)$ and ρ_{UZ}.

$E(Z)$

$$E(Z) = E\left(aU + V\sqrt{1-a^2}\right)$$
$$= aE(U) + \sqrt{1-a^2}\,E(V)$$
$$= a\mu + \sqrt{1-a^2}\,\mu = \boxed{\mu\left(a + \sqrt{1-a^2}\right)}$$

ρ_{UZ}

$$\rho_{UZ} = \frac{\text{Cov}(U,Z)}{\sqrt{\text{Var}(U)\text{Var}(Z)}}$$

$$\text{Cov}(U,Z) = \text{Cov}\left(U, aU + V\sqrt{1-a^2}\right)$$
$$= a\,\text{Cov}(U,U) + \sqrt{1-a^2}\,\text{Cov}(U,V) = a\sigma^2$$
$$\rightarrow \text{Cov}(U,U) = \text{Var}(U) = \sigma^2 \text{ and } \text{Cov}(U,V) = 0$$

$$\text{Var}(Z) = \text{Var}\left(aU + V\sqrt{1-a^2}\right)$$
$$= a^2\text{Var}(U) + (1-a^2)\text{Var}(V)$$
$$= a^2\sigma^2 + (1-a^2)\sigma^2$$
$$= \sigma^2(a^2 + 1 - a^2)$$
$$= \sigma^2$$

$$\rho_{UZ} = \frac{\text{Cov}(U,Z)}{\sqrt{\text{Var}(U)\text{Var}(Z)}} = \frac{a\sigma^2}{\sqrt{\sigma^2 \cdot \sigma^2}} = \frac{a\sigma^2}{\sigma^2} = \boxed{a}$$

63. Let X and Y have the joint distribution given in Problem 8 of Chapter 3.

a. Find the covariance and correlation of X and Y.

b. Find $E(Y|X = x)$ for $0 \le x \le 1$.

$f(x,y) = \frac{6}{7}(x+y)^2 \quad 0 \le x \le 1 \quad 0 \le y \le 1$

a) $f_x(x) = \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right)$, $f_y(y) = \frac{6}{7}\left(y^2 + y + \frac{1}{3}\right)$

$E[X] = \int_0^1 x f_x(x)\, dx = \int_0^1 x \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right) dx = \frac{9}{14} = E[X] = E[Y]$

$E[XY] = \int_0^1 \int_0^1 xy \cdot \frac{6}{7}(x+y)^2\, dx\, dy = \int_0^1 \int_0^1 xy\left(x^2 + 2xy + y^2\right) dx\, dy$

$= \frac{6}{7}\left(\int_0^1 \int_0^1 x^3 y\, dx\, dy + 2\int_0^1 \int_0^1 x^2 y^2\, dx\, dy + \int_0^1 \int_0^1 x y^3\, dx\, dy\right)$

$= \frac{6}{7}\left(\frac{1}{8} + \frac{2}{9} + \frac{1}{8}\right) = \frac{17}{42} = E[XY]$

$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{17}{42} - \left(\frac{9}{14}\right)^2 = \boxed{-\frac{5}{588} = Cov(X,Y)}$

$Corr(X,Y) = \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\, Var(Y)}} \rightarrow Var(X) = Var(Y)$

$E[X^2] = \int_0^1 x^2 \cdot \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right) dx = \frac{6}{7}\int_0^1 \left(x^4 + x^3 + \frac{x^2}{3}\right) dx = \frac{6}{7}\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{9}\right) = \frac{101}{210}$

$Var(X) = E[X^2] - E[X]^2 = \frac{101}{210} - \left(\frac{81}{196}\right) = \frac{3515}{39396} \rightarrow Corr(X,Y) = \frac{-5/588}{3515/39396} = \boxed{-\frac{67}{703}}$

b) $f_{Y|X}(Y|X) = \frac{f(x,y)}{f_x(x)}$

$f_x(x) = \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right)$, $f_{Y|X}(Y|X) = \frac{\frac{6}{7}(x+y)^2}{\frac{6}{7}\left(x^2 + x + \frac{1}{3}\right)} = \frac{(x+y)^2}{x^2 + x + \frac{1}{3}}$

$E[Y|X=x] = \int_0^1 y f_{Y|X}(Y|X)\, dy$

$= \int_0^1 y \cdot \frac{(x+y)^2}{x^2 + x + \frac{1}{3}}\, dy = \frac{1}{x^2 + x + \frac{1}{3}} \int_0^1 y(x+y)^2\, dy = \frac{1}{x^2 + x + \frac{1}{3}} \int_0^1 \left(yx^2 + 2xy^2 + y^3\right) dy$

$= \frac{1}{x^2 + x + \frac{1}{3}}\left(\frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4}\right)$

$$\boxed{E[Y|X=x] = \frac{\frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4}}{x^2 + x + \frac{1}{3}} \quad \text{for } 0 \le x \le 1}$$