62. If X and Y are independent standard normal random variables, find $P(X^2 + Y^2 \leq 1)$.

$X^2 + Y^2 \sim Chi(2) = Exp(\lambda = 1)$

$P(X^2 + Y^2 \leq 1) = ?$

$F(t) = 1 - e^{-t}, \ t > 0$

$P(X^2 + Y^2 \leq 1) = F(1) = 1 - e^{-1}$

$P(X^2 + Y^2 \leq 1) = 1 - \frac{1}{e} \approx \boxed{0.632}$

56. Let $S = \sum_{k=1}^{n} X_k$, where the X_k are as in Problem 55. Find the covariance and the correlation of S and T.

$E[S] = n\mu \;,\; E[T] = \mu \cdot \frac{n(n+1)}{2}$

$ST = \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} kX_i\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} kX_iX_k \rightarrow E[X_iX_k] = E[X_i]E[X_k] \;,\; E[X_i^2] = \sigma^2 + \mu^2$

$E[ST] = \sum_{i=1}^{n}\sum_{k=1}^{n} kE[X_iX_k] = \sum_{i=1}^{n} (k=i)k(\sigma^2+\mu^2) + (k \neq i)k\mu^2$

When $i=k$, $E[X_i^2] = \sigma^2+\mu^2 \rightarrow \sum_{i=1}^{n} i(\sigma^2+\mu^2) = (\sigma^2+\mu^2)\cdot\frac{n(n+1)}{2}$

When $i \neq k$, $E[X_iX_k] = \mu^2 \rightarrow \sum_{i=1}^{n}\sum_{k=1}^{n} k\mu^2 = \mu^2\cdot(n-1)\cdot\frac{n(n+1)}{2}$

$E[ST] = (\sigma^2+\mu^2)\cdot\frac{n(n+1)}{2} + \mu^2\cdot(n-1)\cdot\frac{n(n+1)}{2}$

$E[S]E[T] = n\mu\cdot\mu\cdot\frac{n(n+1)}{2} = \mu^2\cdot\frac{n^2(n+1)}{2}$

$Cov(S,T) = E[ST] - E[S]E[T]$

$$= \boxed{(\sigma^2+\mu^2)\cdot\frac{n(n+1)}{2} + \mu^2\cdot(n-1)\cdot\frac{n(n+1)}{2} - \mu^2\cdot\frac{n^2(n+1)}{2}}$$

$Corr(S,T) = \frac{Cov(S,T)}{\sqrt{Var(S)Var(T)}} \;,\; Var(S) = n\cdot\sigma^2$

$Var(T) = \sigma^2\sum_{k=1}^{n} k^2 = \sigma^2\cdot\frac{n(n+1)(2n+1)}{6}$

$$\boxed{Corr(S,T) = \frac{(\sigma^2+\mu^2)\cdot\frac{n(n+1)}{2} + \mu^2\cdot(n-1)\cdot\frac{n(n+1)}{2} - \mu^2\cdot\frac{n^2(n+1)}{2}}{\sqrt{n\sigma^2\cdot\sigma^2\cdot\frac{n(n+1)(2n+1)}{6}}}}$$

4. Suppose that the number of traffic accidents, N, in a given period of time is distributed as a Poisson random variable with $E(N) = 100$. Use the normal approximation to the Poisson to find Δ such that $P(100 - \Delta < N < 100 + \Delta) \approx .9$.

$$M = \lambda = 100 \qquad \sigma^2 = \lambda = 100 \qquad \sigma = \sqrt{100} = 10$$

$$P(100-\Delta < N < 100+\Delta) \Rightarrow P(100-\Delta < Z < 100+\Delta) \approx 0.9$$

$$P\left(-\frac{\Delta}{\sigma} < Z < \frac{\Delta}{\sigma}\right) = 0.9$$

$$P(-z < Z < z) = 0.9 \rightarrow z = \Phi^{-1}(0.95)$$

$$z = \frac{\Delta}{\sigma} \rightarrow \Delta = z \cdot \sigma$$

$$\sigma = 10$$

$$\boxed{\Delta \approx 16.45}$$

10. A six-sided die is rolled 100 times. Using the normal approximation, find the probability that the face showing a six turns up between 15 and 20 times. Find the probability that the sum of the face values of the 100 trials is less than 300.

a

$n = 100 \quad , \quad p = \frac{1}{6}$

$\mu = np = 16.67$

$\sigma = \sqrt{np(1-p} = 3.73$

$P(15 \leq X \leq 20) = P(14.5 \leq Z \leq 20.5)$

$= \boxed{P\left(\frac{14.5 - 16.67}{3.73} \leq Z \leq \frac{20.5 - 16.67}{3.73}\right)} \approx 56.8\%$

b

$\mu = n \cdot$ mean of 1 roll $= 100 \cdot 3.5 = 350$

$\sigma = \sqrt{n \cdot \text{var of roll}} = \sqrt{100 \cdot \frac{35}{12}} = 17.58$

$\boxed{P(S < 300) = P\left(Z < \frac{300 - \mu}{\sigma}\right) \text{ where } Z = \frac{S - \mu}{\sigma}}$

$\approx 0.17\%$

Additional problem 1: Suppose X is uniformly distributed on $[0, 1]$. Given $X = x$, Y is uniformly distributed on $[0, x]$. Compute $\mathbb{E}[X \mid Y = y]$.

$X \sim \text{Uniform}(0,1)$ so $f_X(x) = 1$ for $x \in [0,1]$

$f_{Y|X}(y|x) = \frac{1}{x} \quad 0 \le y \le x$

$f_Y(y) = \int_y^1 f_X(x) f_{Y|X}(y|x)\,dx = \int_y^1 \frac{1}{x}\,dx$

$f_Y(y) = \int_y^1 \frac{1}{x}\,dx = \ln(1) - \ln(y) = -\ln(y), \quad 0 < y < 1$

$E[X|Y=y] = \frac{\int_y^1 x f_{X|Y}(x|y)\,dx}{f_Y(y)}$

$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$

$f_{Y|X}(y|x) = \frac{1}{x}, \quad f_X(x) = x, \quad f_Y(y) = -\ln(y)$

$f_{X|Y}(x|y) = \frac{1/x}{-\ln y} = \frac{1}{-x\ln(y)} \quad x \in [y,1]$

$\int_y^1 x f_{X|Y}(x|y)\,dx = \int_y^1 x \cdot \frac{1}{-x\ln(y)}\,dx = \frac{1}{-\ln(y)} \int_y^1 1\,dx$

$= \frac{\frac{1-y}{-\ln(y)}}{-\ln y} = \frac{1-y}{-\ln(y)} \cdot \frac{1}{-\ln(y)} = \boxed{\frac{1-y}{(\ln(y))^2} \quad 0 < y < 1}$

- Additional problem 2: Delta method and normal approximation. Let $X_1, X_2, \dots$ be iid Bernoulli random variable with success probabability $0 < p < 1$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the normal approximation for for

$$\sqrt{\frac{\bar{X}}{1-\bar{X}}}.$$

$E[\bar{X}] = p$, $Var(\bar{X}) = \frac{p(1-p)}{n}$, $\bar{X} \sim N\left(p, \frac{p(1-p)}{n}\right)$

$Var(g(\bar{X})) \approx [g'(E[\bar{X}])]^2 \cdot Var(\bar{X})$

$g(\bar{X}) = \sqrt{\frac{\bar{X}}{1-\bar{X}}}$ $\quad g'(x) = \frac{d}{dx}\sqrt{\frac{x}{1-x}} = \frac{1}{2}\cdot\left(\frac{x}{1-x}\right)^{-1/2} \cdot \frac{(1-x)+x}{(1-x)^2}$

$$= \frac{1}{2}\sqrt{\frac{1-x}{x}} \cdot \frac{1}{(1-x)^2} = \frac{1}{2\sqrt{x(1-x)^3}}, \quad g'(p) = \frac{1}{2\sqrt{p(1-p)^3}}$$

$$Var(g(\bar{X})) \approx [g'(p)^2] \cdot Var(\bar{X})$$

$$= \left(\frac{1}{2\sqrt{p(1-p)^3}}\right)^2 \cdot \frac{p(1-p)}{n}$$

$$= \frac{1}{4p(1-p)^3} \cdot \frac{p(1-p)}{n} = \frac{1}{4n(1-p)^2}$$

$g(\bar{X}) \sim N\left(g(p), \frac{1}{4n(1-p)^2}\right.$ where $g(p) = \sqrt{\frac{p}{1-p}}$

$$\boxed{\sqrt{\frac{\bar{X}}{1-\bar{X}}} \sim N\left(\sqrt{\frac{p}{1-p}}, \frac{1}{2\sqrt{n}(1-p}\right)}$$