12 Nonlinear DEs: Graphical Methods

12.2 Introduction

First order ODE, standard form: $y' = f(x,y)$

First order linear equation: $y' + P(x)y = Q(x)$

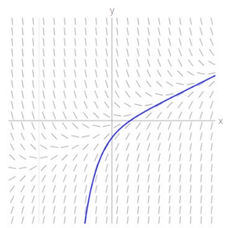
First order nonlinear equation: $y' = x - y^2$

12.3 Geometric view of DEs

Analytic : $y' = f(x,y)$ Geometric : Direction field

$y_1(x) \to$ sln

Definition 3.1 For a differential equation $y' = f(x,y)$, a **slope field** is a diagram which includes at each point (x,y) a short line element (or line segment) whose slope is the **value** $f(x,y)$.



A solution curve (blue) is tangent to each line segment that it touches.

The graph of a solution $y_1(x)$ to the DE in the xy-plane is called a **solution curve** or an **integral curve**.

An integral curve must be tangent to the slope field at every point: $y_1'(x) = f(x, y_1(x))$.

12.4 Slope field

12.5 First exercises

$y' = y - y^2$

(1) $y(0) = \frac{1}{2}$, $y' = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(2) $y(0) = -1$, $y(0) = -1 - 1 = -2$

(3) $y(0) = 2$, $y'(0) = 2 - 4 = -2$

12.6 Isoclines

Draw slope field.

Definition 6.1 For a number C, the C-**isocline** is the set of points in the (x,y)-plane such that the solution curve through that point has slope C. (Isocline means "same incline", or "same slope".)

Question 6.2 What is the equation for the C-isocline?

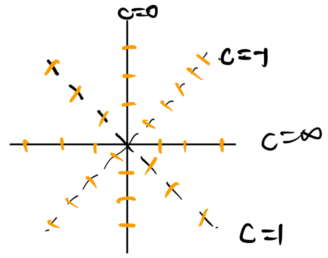
Answer: The ODE says that the slope of the solution curve through a point (x,y) is $f(x,y)$, so the **equation of the C-isocline** is

$$f(x,y) = C. \quad (6.1)$$

Computer: 1. Pick (x,y) [equally spaced];
2. $f(x,y) \leftarrow$ finds
3. Draws

Human: 1. Pick slope C.
2. $f(x,y) = C$ plot eqn

$y' = -\frac{x}{y}$, $-\frac{x}{y} = C$, $y = -\frac{1}{C}x$



$y: 2.$ $f(x,y) = -\frac{x}{y} = 2$

$y' = -\frac{x}{y}$, $(x,y) = (0,2)$, $y\,dy = -x\,dx$ $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, $\frac{1}{2} \times 4 = C = 2$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + 2$, $x^2 + y^2 = 4$ $y = \sqrt{4 - x^2}$

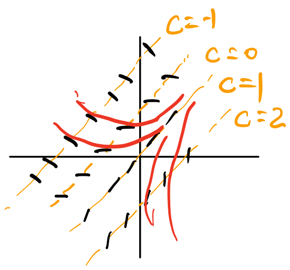
12.7 Zero Isoclines

$y' = y^2 - x$. $f(x,y) = 0 = y^2 - x$.

12.8 Isocline mathlet

12.9. Another worked example

$y' = 1 + x - y$ $f(x,y) = 1 + x - y = C$. , Isocline: $y = x + (1 - C)$



12.10. Existence and uniqueness

① Two integral curves cannot cross can't have 2 slopes

② Two integral curves cannot be tangent (touch) ×

Existence + uniqueness has one and only one s/n through (x_0, y_0)

Hyp: $f(x,y)$ continuous near (x_0, y_0) and $f_y(x,y)$ continuous near (x_0, y_0)

Theorem 10.1 Existence and uniqueness theorem for a first order (linear or nonlinear) ODE Consider a first order ODE

$$y' = f(x, y).$$

For any point (x_0, y_0), if $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0), then there is a unique solution to the first order DE through the point (x_0, y_0).

As a consequence of uniqueness, we have the following two geometric features:

1. Solutions curves cannot cross.

2. Solutions curves cannot become tangent to one another; that is, they cannot touch.

12.11 When Existence and Uniqueness fails

ex1. $xy' = y-1$, $x\frac{dy}{dx} = y-1$, $\frac{dy}{y-1} = \frac{1}{x}dx$ $(y\neq 1, x\neq 0)$, $\ln|y-1| = \ln|x| + C$

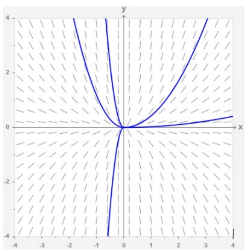
$|y-1| = C|x|$. $C>0$. $y-1 = \pm Cx$ $C>0$, $y = 1+Cx$ $C\neq 0$.

Bring back the solution $y=1$, we allow $c=0$ as well.

ex2. $y' = \frac{2y}{x}$ $\frac{dy}{dx} = \frac{2y}{x}$ $(x\neq 0)$, $\frac{dy}{y} = \frac{2dx}{x}$ (assuming $y\neq 0$) $\int \frac{dy}{y} = \int \frac{2dx}{x}$

$\ln|y| = 2\ln|x| + C$ $y = \pm e^{2\ln|x|+C}$, $y = \pm x^2 e^C$. $y = cx^2$ $(c\neq 0)$, $c = \pm e^C$

To bring back the solution $y=0$, we allow $c=0$ as well.



Several half parabolas as solution curves to $y' = \frac{2y}{x}$

Weird behavior happens along $x = 0$ where $y' = \frac{2y}{x}$ is not even defined:

- Through any point $(0, b)$ on the y-axis, there is **no** solution curve. The existence theorem does not apply.
- Geometrically, the parabolas become tangent at the origin. This would be ruled out by uniqueness if the uniqueness theorem applied. The full parabolas are not solutions; the solution curves are half parabolas defined for either all $x < 0$ or all $x > 0$.

Both existence and uniqueness apply to every point outside the y-axis. The rest of the plane (outside the y-axis) is covered with good solution curves, one through each point, none touching or crossing the others.

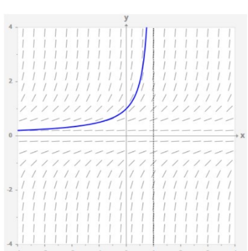
There is no connection between any half parabola on the left and any half parabola on the right.

12.12 Interval of validity

ex) $y' = y^2$, $y(0)=1$. $\frac{dy}{dx} = y^2$, $y^{-2}dy = dx$ $(y\neq 0)$ $\int y^{-2}dy = \int dx$, $\frac{y^{-1}}{-1} = x + C$

$y = \frac{-1}{x+C}$. $y(0) = 1 = \frac{-1}{0+C}$, $C = -1$. $y = \frac{-1}{x-1} = \frac{1}{1-x}$, where $-\infty < x < 1$

Therefore, here is the solution curve:



The solution approaches ∞ as $x \to 1^-$. We say that the solution **blows up** at $x = 1$. The domain of definition of the solution is $-\infty < x < 1$. It is also called the **interval of validity** of the solution. The **interval of validity** of a solution is the largest interval on which it can be defined.

The graph of $y = \frac{1}{1-x}$ consists of both branches of the hyperbola. But **be careful** , the full hyperbola is not a solution; only the branch through the point $(x, y) = (0, 1)$ is.

For nonlinear DEs, the interval of validity of a solution cannot be read off from the equation. It can be much smaller than the domain on which the equation is defined.

ex2. $y' = xy^4$, $y(0) = 1$.

$\frac{dy}{dx} = xy^4$, $\frac{dy}{y^4} = x\,dx$, $\frac{y^{-3}}{-3} = \frac{1}{2}x^2 + C$

$-\frac{1}{3} = C$, $\frac{1}{-3y^3} = \frac{1}{2}x^2 - \frac{1}{3}$

$\frac{1}{y^3} = -\frac{3}{2}x^2 + 1$, $y^3 = \frac{1}{1-\frac{3}{2}x^2} > 0$

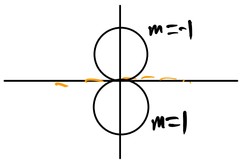
$1 - \frac{3}{2}x^2 > 0$ $\frac{3}{2}x^2 < 1$ $x^2 < \frac{2}{3}$

12.13 Worked example

$y' = \frac{-y}{x^2+y^2}$ write the slope field.

Isocline: $y' = m = \text{const}$, $-\frac{y}{x^2+y^2} = m$. $-\frac{1}{m}y = x^2+y^2$, $x^2+y^2+\frac{1}{m}y = 0$,

$x^2 + (y+\frac{1}{2m})^2 = \frac{1}{4m^2}$ → circle: $(0, -\frac{1}{2m})$. Rad: $\frac{1}{2|m|}$



12.14 Bernoulli equation

$\dot{y} + y^2 t - y = 0$ $y = \frac{1}{u}$ $y' = -\frac{1}{u^2}\dot{u}$, $y^2 = \frac{1}{u^2}$, $-\frac{1}{u^2}\dot{u} + \frac{1}{u^2}t - \frac{1}{u} = 0$

$\dot{u} - t + u = 0$ $\dot{u} = -u + t$

13. Autonomous Equations

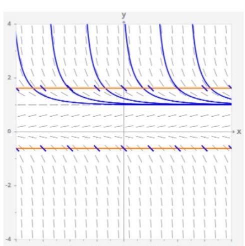
13.2 definition

$\frac{dy}{dt} = f(y)$ no t → autonomous

13.3 First properties

- Each isocline (in the (t, y)-plane) consists of one or more horizontal lines.
- Solution curves (in the (t, y)-plane) are horizontal translations of one another. That is, if $y(t)$ is a solution, then so is $y(t-a)$ for any a.

For example, here is the slope field for $\dot{y} = y(1-y)$ with the (-1)-isocline and a few solution curves:



The (-1)-isocline (orange) consists of 2 horizontal lines;
Solution curves (blue) are horizontal translations of one another.

Why are all translations of solutions also solutions? (A non-geometric argument)

Let $y(t)$ be a solution to $\dot{y} = f(y)$. We verify that $y(t-a)$ is also a solution.

Let $u = t - a$, then

$$\frac{d}{dt}y(t-a) = \frac{d}{dt}y(u(t)) = \frac{dy}{du}\frac{du}{dt} \quad \text{(chain rule)} \tag{6.25}$$

$$= f(y(u))(1) = f(y(t-a)). \tag{6.26}$$

Therefore $y(t-a)$ is also a solution.

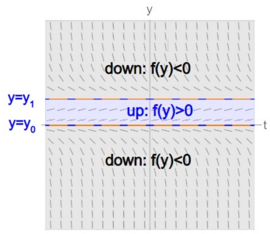
Hide

Ex 1. $\dot{y} = y(1-y) = -1$ $y - y^2 = -1$ $y^2 - y - 1 = 0$ $y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

13.4 Critical points

Definition 4.1 The values of y at which $f(y) = 0$ are called the **critical points** or **equilibria** of the autonomous equation $\dot{y} = f(y)$.

If y_0 is a critical point of an autonomous equation, then $y = y_0$ is a constant (or horizontal, or equilibrium) solution, because the derivative of a constant function is 0. The 0-isocline of an autonomous equation consists of all the constant solutions.



The constant solutions $y = y_0$ and $y = y_1$ (where y_0 and y_1 are critical points) divide the (t, y)-plane plane into "up" and "down" regions.

Recall that for any first order DE, the 0-isocline divides the (t, y)-plane into "up" regions, where $f > 0$ and solutions are increasing, and "down" regions, where $f < 0$ and solutions are decreasing. For autonomous equations, the qualitative behaviour of all solutions is encoded by the critical points and the signs of $f(y)$ in the intervals between the critical points.

To find the qualitative behaviour of solutions to $\dot{y} = f(y)$, we follow two steps:

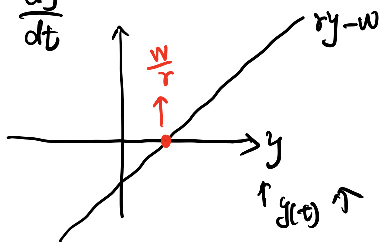
1. Find the critical points. That is, solve $f(y) = 0$.

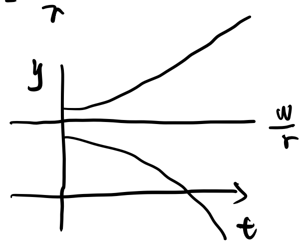
2. Determine the intervals of y in which $f(y) > 0$ and in which $f(y) < 0$. These are intervals in which solutions are increasing and decreasing respectively.

13.5. $\frac{dy}{dt} = ry - w$. $(r, w > 0)$

① critical point. $ry - w = 0$. $y = \frac{w}{r}$

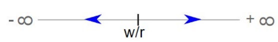
② $\frac{dy}{dt}$





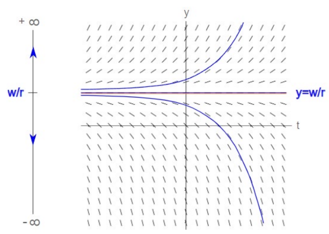
Definition 5.1 The **phase line** of a first order autonomous DE $\dot{y} = f(y)$ is a plot of the y-axis with all critical points and with an arrow in each interval between the critical points indicating whether solutions increase or decrease

In our example, the phase line is:



Phase line of $\dot{y} = ry - w$ $(r, w > 0)$

If the phase line is drawn vertically, the qualitative behaviour of solution curves can be read directly from it, as you see in the diagram below.



Vertical phase line (left); Solution curves on the ty-plane (right)

13.6. Logistic equation

Population behavior : $y(t)$. $\frac{dy}{dt} = ky$, k: growth rate

Simplest case: $k(y) = a - by$ $(a, b > 0$ constants$)$

$\dot{y} = k(y) \cdot y = (a - by)y = ay - by^2$ $(a, b > 0)$

13.7 Qualitative behavior

$\dot{y} = 3y - y^2$

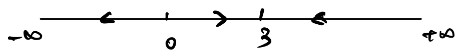
① Find critical points: $f(y) = 3y - y^2 = 0 \Rightarrow y=0,\ y=3$

∴ constant solutions are $y(t)=0$, $y(t)=3$

② determine the intervals of y in which $f(y)>0$ and in which $f(y)<0$.

$f(-1)<0$, $f(1)>0$, $f(4)<0$

∴ phase line



ex 1: $y(t) = \dfrac{3}{1+ce^{-3t}}$ $\qquad y(0) = \dfrac{3}{1+c} = 6$, $c = -0.5$

$y(t) = \dfrac{3}{1-\frac{1}{2}e^{-3t}}$

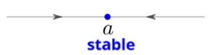
ex 2: $k = a - by$ $\qquad \dot{y} = (a-by)y$ $\qquad f(y) = ay - by^2 = 0$ $\qquad a = by$ $\qquad a = 6b$

$a=3$, $b=\frac{1}{2}$

13.8 Stability of critical points

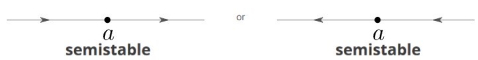
A critical point $x=a$ is called

- **stable** if solutions starting near it move towards it,



- **unstable** if solutions starting near it move away from it,

- **semistable** if the behavior depends on **which side** of the critical point the solution starts.



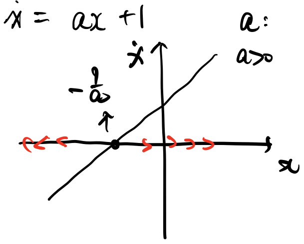
Remark 8.2 A solution corresponding to an unstable critical point is an example of a separatrix because it separates solutions having very different fates. In the example above, $y=0$ is a separatrix. A solution starting just **below** 0 tends to $-\infty$, while a solution starting just **above** 0 tends to 3: very different fates!

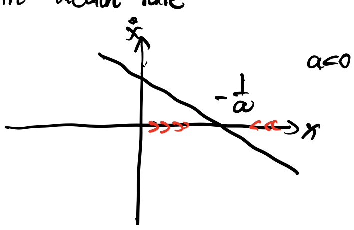
Summary: steps for understanding solutions to $\dot{y} = f(y)$ qualitatively.

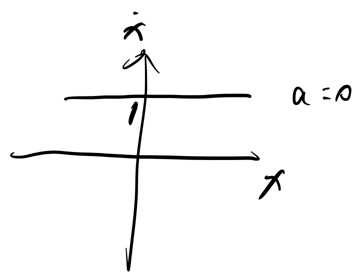
1. Find the critical points by solving $f(y) = 0$. These divide the y-axis into open intervals.
2. Determine the intervals of y in which $f(y) > 0$ and $f(y) < 0$, by either graphing $f(y)$ or evaluating $f(y)$ at one point in each interval.
3. Draw the phase line, which consists of a line marked with $-\infty$, the critical points, $+\infty$, and arrows between these.
4. Solutions starting at a critical point are constant.
5. Other solutions tend to the limit that the arrow points to as t increases. As t decreases, solutions tend to the limit that the arrow originates from.

12.9 Worked example

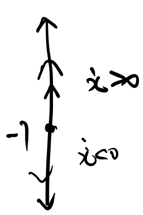
$\dot{x} = ax + 1$ $\qquad a$: birth-death rate

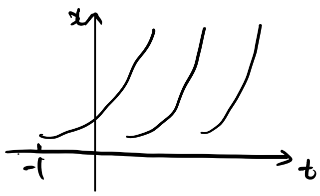


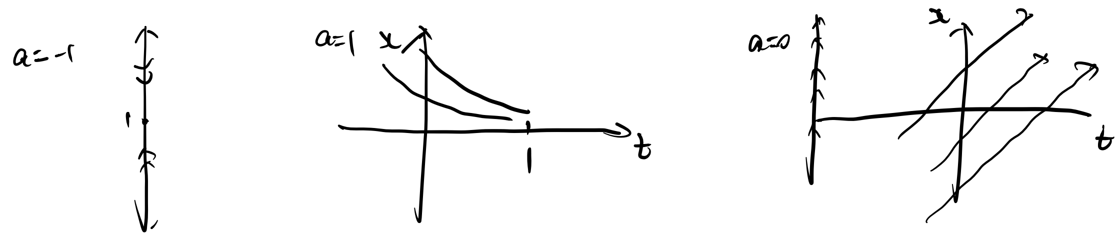




$a=1$



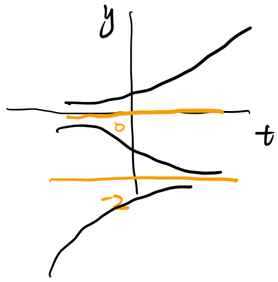


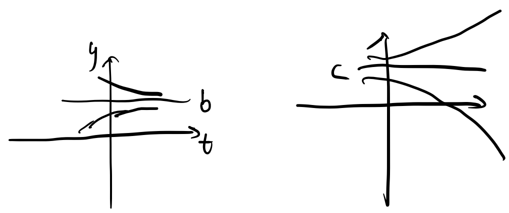


13.10 Phase line concept problem

① Phase line from DE

$\dot{y} = y^2+2y$, $f(y)=y^2+2y=0$, $y=0$, $y=-2$, $f(-3)=6$, $f(-1)=-1$, $f(1)=3$





13.11 Concept challenge

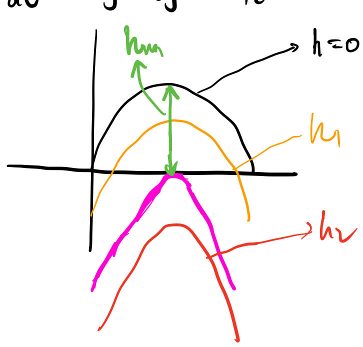
$\dot{y} = f(y) \neq 0$ $\quad \ddot{y} = f'(y)\cdot\dot{y}$ $\quad \ddot{y}=0$ at inflection point, so $f'(y)=0$ or $\dot{y}=0$

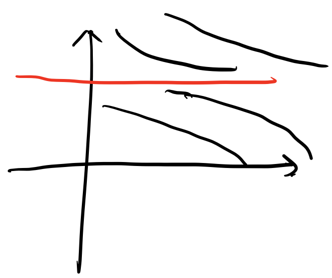
but $\dot{y} \neq 0$. $\therefore$ $f'(y)=0$

12.12. Harvesting models

harvest at constant time rate : h

$\frac{dy}{dt} = ay - by^2 - h$





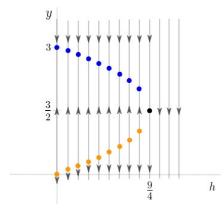
13.13. Bifurcation diagrams

Let us continue with our previous example of frogs being harvested from a pond. Their population is modeled by

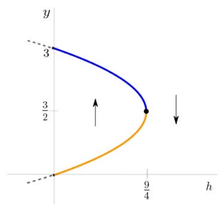
$$\dot{y} = 3y - y^2 - h \qquad (h \geq 0). \tag{6.61}$$

You have just seen that the phase lines corresponding to different values of h, can be very different, and they tell us how the frog population evolves for different harvesting rate. Our goal now is to summarize all this information in a single diagram.

If we draw phase lines vertically at several h-values in the (h, y)-plane, we get a diagram on the left below. If we draw phase lines at all h-values, we get a diagram on the right below.



Vertical phase lines of $\dot{y} = 3y - y^2 - h$ at discrete values of h



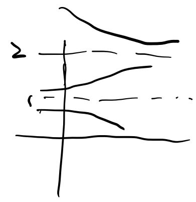
Bifurcation diagram for $\dot{y} = 3y - y^2 - h$

The diagram on the right is called a **bifurcation diagram**.

Definition 13.1 A **bifurcation diagram** of a family of autonomous equations depending on a parameter h is a plot of the values of the critical points as functions of h, along with one arrow in each **region** in the (h, y)-plane defined by the curve, indicating whether solutions are increasing or decreasing in that region.

As usual, the arrows tell us the stability of the critical points. In the diagram, we have also explicitly indicated the stability of the critical points by colors: **stable**, **unstable**, and **semistable**.

Ex 1. $h=2$. $\dot{y} = 3y - y^2 - 2$ $\quad 3y - y^2 - 2 = 0 \quad y = 1, 2$

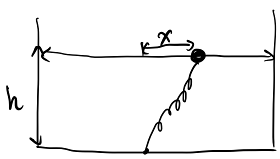


Ex 2 h $\quad \dot{y} = 3y - y^2 - h \quad 3y - y^2 - h = 0 \quad y^2 - 3y + h = 0$

$$y = \frac{3 \pm \sqrt{9-4h}}{2}.$$

13.14. Phase line and bifurcation diagram mathlet

13.15. Bead on wire



what is the equilibrium position of the bead?

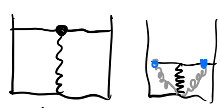
L: relaxed of spring

Two forces:

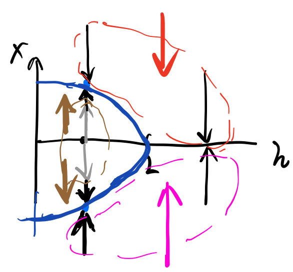
① Friction force: $f = -b\dot{x}$, $b > 0$

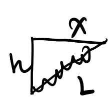
② Spring force: F_s

Horizontal force F_x



h>L h<L





$x = \pm\sqrt{L^2 - h^2}$

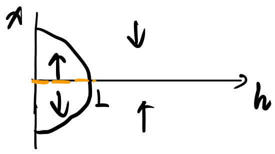
$$F_s = -k\Delta x = -k(\sqrt{h^2+x^2} - L) \cdot \frac{x}{\sqrt{h^2+x^2}} = -kx\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right) \quad (k > 0)$$

Newton's second Law

$$m\ddot{x} = -b\dot{x} - kx\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right), \quad \dot{x} = \frac{k}{b}x\left(\frac{L}{\sqrt{x^2+h^2}} - 1\right)$$

Equilibria: $(\dot{x}=0)$

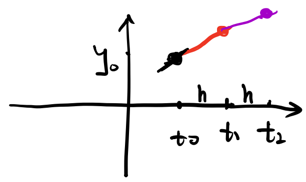
$x_0^* = 0$, $x_\pm^* = \pm\sqrt{L^2-h^2}$

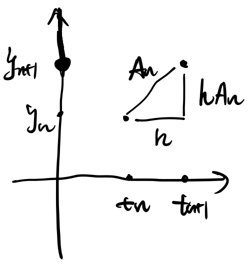


14. Numerical methods

14.2 Euler's method

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$





h: step size

$y_{n+1} - y_n = hA_n$, $y_{n+1} = y_n + hA_n$

Euler's equations

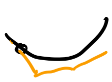
$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + hA_n \\ A_n = f(t_n, y_n) \end{cases}$$

EX: $\begin{cases} y' = x^2 - y^2, \\ y(0) = 1 \end{cases}$, $h = 0.1$

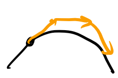
n	t_n	y_n	A_n	hA_n
0	0	1	-1	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82	-0.6324	-0.06324
3	0.3			

14.3 Overestimate, underestimate and concavity

$y_1(x)$



convex Too low $y'' > 0$



concave Too high $y'' < 0$

$y' = x^2 - y^2$

$y'' = 2x - 2yy'$

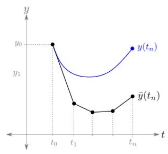
at $(0,1)$ $y(0) = 1$, $y'(0) = -1$ $y''(0) = 2\cdot 0 - 2\cdot 1\cdot(-1) = 2$

Recall in Euler's method, we start with the initial value problem

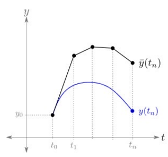
$$y' = f(t, y), \qquad y(t_0) = y_0.$$

Let $y(t)$ be the actual solution and let $\bar{y}(t)$ be the approximate solution given by Euler's method with stepsize $h.$ As before, $t_n = t_0 + nh$ is the time after n steps.

- If $y(t)$ is **convex** (or **concave up**), that is, if $\ddot{y} > 0,$ in the interval $[t_0, t_n]$, and in a region along the Euler polygon $\bar{y}(t)$, then $\bar{y}(t_n) < y(t_n)$.

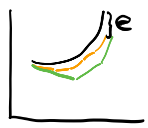


- If $y(t)$ is **concave** (or **concave down**), that is, if $\ddot{y} < 0,$ in the interval $[t_0, t_n]$, and in a region along the Euler polygon $\bar{y}(t)$, then $\bar{y}(t_n) > y(t_n)$.



14.4 Error and step size

Better method? → smaller step size



Error e depends on step size

$e \sim c \cdot h$ ↖ Euler first order method

halve the step size, → halve the error

$y'' = 0.5y' \qquad y''(-2) = 0.5\,(0.5 \times (-1) + 1) = 0.25 > 0$

$(-2, -1)$

$y'' = 2yy' - 2y' = (2y - 2)y' = 2(y-1)y' = 2 \cdot (-2) \cdot 2 < 0$

14.6. Worked example

14.7 Reliability

Question 7.1 How can we decide whether answers obtained numerically can be trusted?

Here are some heuristic tests. ("Heuristic" means that loosely speaking, these tests work in practice, but they are not proved to work always.)

- **Self-consistency:** Solution curves should not cross! If numerically computed solution curves cross, a smaller step size is needed. (E.g., try the mathlet "Euler's Method" with $\dot{y} = y^2 - x$, step size 1, and starting points $(0, 0)$ and $(0, 1/2)$.)
- **Convergence as $h \to 0$:** The estimate for $y(t)$ at a fixed later time t should converge to the true value as $h \to 0$. If shrinking h causes the estimate to change a lot, then h is probably not small enough yet. (E.g., try the mathlet "Euler's Method" with $y' = y^2 - x$ with starting point $(0, 0)$ and various step sizes.)
- **Structural stability:** Small changes in the DE's parameters should not change the outcome completely. If small changes in the parameters drastically change the outcome, the answer should not be trusted.
- **Stability:** Small changes in the DE's initial conditions do not change the outcome much. One reason for instability could be a separatrix, a curve such that nearby starting points on different sides lead to qualitatively different outcomes; this is not a fault of the numerical method, but rather an instability in the answer.

14.8 Change of variable

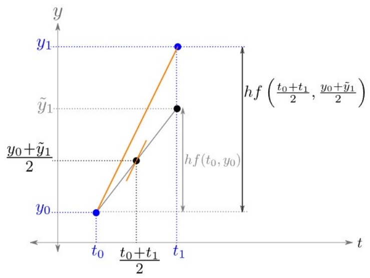
14.9. Runge-Kutta methods

Better methods:

The Runge–Kutta methods evaluate at more points on the interval $[t_0, t_0 + h]$ to get a better estimate of what happens to the slope over the course of that interval.

Below is an example of a **second-order Runge–Kutta method (RK2)** . It is also called the **midpoint method** or the **modified Euler method** .

Here is how one step of this method goes:



1. Starting from (t_0, y_0) , look ahead to see where one step of Euler's method would land, but do not go there! Call this temporary point $(t_1, \tilde{y}_1)$.

2. Find the **midpoint** between the starting point and the temporary point: $\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right)$.

3. Use the slope at this midpoint to find y_1

$$y_1 = y_0 + hf\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right).$$

Repeat the steps above using (t_1, y_1) as the starting point.

Here is a summary of the equations:

$$t_1 = t_0 + h \tag{6.77}$$

$$\tilde{y}_1 = y_0 + hf(t_0, y_0) \tag{6.78}$$

$$y_1 = y_0 + hf\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right) \tag{6.79}$$

$$(t_0, y_0) \mapsto (t_1, y_1). \tag{6.80}$$

Even better methods:

The **fourth-order Runge–Kutta method (RK4)** is similar, but more elaborate, averaging several slopes. It is the most commonly used method for solving DEs numerically. Some people simply call it **the** Runge–Kutta method. The mathlets we have been playing with use RK4 with a small step size to compute the "actual" solution to a DE.