9. Operators and Exponential Response

9.1 Objective

9.2 Operator Notation

The operator D

- A function takes an input number and returns another number.
- An **operator** takes an input **function** and returns another function.

For example, the **differential operator** $\frac{d}{dt}$ takes an input function $y(t)$ and returns $\frac{dy}{dt}$. This operator is also called D. For instance $De^{4t} = 4e^{4t}$. The operator D is **linear** , which means that

$$D(f+g) = Df + Dg, \qquad D(af) = a\,Df$$

for any functions f and g, and any number a. Because of this, D behaves well with respect to linear combinations, namely

$$D(c_1 f_1 + \cdots + c_n f_n) = c_1\,Df_1 + \cdots + c_n\,Df_n$$

for any numbers $c_1, \ldots, c_n$ and functions $f_1, \ldots, f_n$.

Definition 2.2 In general, a **linear operator** L is any operator that satisfies

$$L(f+g) = Lf + Lg, \qquad L(af) = a\,Lf$$

for any functions f and g, and any number a.

9.3 Multiplying and adding operators

$2\ddot{y} + 3\dot{y} + 5y = 0$, $P(r) = 2r^2 + 3r + 5$, can be written as $(2D^2 + 3D + 5)y = 0$, $P(D)y = 0$

Generally, $a_n y^{(n)} + \cdots + a_0 y = 0 \Rightarrow P(D)y = 0$, P is the characteristic polynomial

9.4 Time invariance

In this course we'll focus on polynomial differential operators with constant coefficients, that is operators of the form

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots a_1 D + a_0,$$

where all of the coefficients a_k are numbers (as opposed to functions of t). All operators of this form are linear. In addition to being linear operators, they are also **time-invariant** operators, which means:

If $x(t)$ solves $P(D)x = f(t)$, then $y(t) = x(t - t_0)$ solves $P(D)y = f(t - t_0)$.

In words, this says that "delaying the input signal $f(t)$ by t_0 seconds delays the output signal $x(t)$ by t_0 seconds." If we know that $x(t)$ is a solution to $P(D)x = f(t)$, we can solve $P(D)y = f(t - t_0)$ by replacing t by $t - t_0$ in $x(t)$. This is a useful property because it gives us the solutions to many differential equations for free.

A system that can be modeled using a linear time invariant operator is called an **LTI (linear time invariant)** system.

$\dot{x} + x = \cos t$. , $\dot{x} + x = 0$. , $\frac{dx}{dt} = -x$, $\frac{dx}{x} = -dt$. $\ln|x| = -t + c$,

$x = e^{-t+c} = ce^{-t}$

$x = u(t)e^{-t}$, $\dot{x} + x = \dot{u}(t)e^{-t} - e^{-t}u(t) + u(t)e^{-t} = \cos t$ ∴ $\dot{u}(t) = e^{t}\cos t$

∴ $u(t) = \int e^{t}\cos t\,dt = \int \cos t\,de^{t} = \frac{1}{2}e^{t}\sin(t) + \frac{1}{2}e^{t}\cos t + C$

9.5 Superposition for an inhomogeneous linear ODE

$Ly = f(x)$

Solution: $y_p + y_h$. ie, $y_p + C_1y_1 + C_2y_2$

To prove: all the $y_p + C_1y_1 + C_2y_2$ … are solution.

Proof: $L(y_p + C_1y_2 + C_2y_2) = \underset{f(x)}{L(y_p)} + \underbrace{L(C_1y_1 + C_2y_2)}_{0} = f(x)$

∴ stisfy $L(y) = f(x)$

To prove: no other solutions

$\underset{\text{Solution}}{u(x)}$ $\quad L(u) = f(x)$

$$L(y_p) = f(x)$$

$$L(u - y_p) = 0$$

$$\therefore \; u - y_p = c_1 \tilde{y}_1 + c_2 \tilde{y}_2$$

$$\therefore \; u = y_p + c_1 y_1 + c_2 y_2$$

9.6 Why exponential inputs?

$y'' + Ay' + By = f(x)$, Find a part soln y_p, (gen soln: $y = y_p + c_1 y_1 + c_2 y_2$

Important $f(x)$. e^{ax} ($a<0$). $\begin{cases} \cos\omega x \\ \sin\omega x \end{cases}$, $\begin{cases} e^{at}\cos\omega t \\ e^{at}\sin\omega t \end{cases}$. all spec cases of $e^{(a+i\omega)x}$ $\; = e^{\alpha x}$

9.7 Shortcut

$y'' + Ay' + By = f(x)$, $\underbrace{D^2 + AD + B}_{P(D)} = f(x)$, $P(D) \cdot e^{\alpha x}$

Proof: $(D^2 + AD + B) e^{\alpha x} = D^2 e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x} = \alpha^2 e^{\alpha x} + A\alpha e^{\alpha x} + Be^{\alpha x}$

obviously true for D, D^2. all true for $P(D)$ $\quad \therefore P(D) = \alpha^2 + A\alpha + B$

9.8 Exponential response formula

Question: For any polynomial P and number r, what is a particular solution to

$$P(D)\,y = e^{rt}?$$

Answer (superposition): We have

$$P(D)\,e^{rt} = P(r)\,e^{rt}.$$

This is off by a factor of $P(r)$. So multiply through by $1/P(r)$ and we can use linearity to get

$$P(D)\left(\frac{1}{P(r)}e^{rt}\right) = e^{rt}.$$

Conclusion: This is called the **Exponential Response Formula (ERF)**.

In words, for any polynomial P and any number r such that $P(r) \neq 0$,

$$\frac{1}{P(r)}e^{rt} \quad \text{is a particular solution to} \quad P(D)\,y = e^{rt}.$$

Remember, this is just one particular solution. To get the general solution, we need to add the solution to the associated homogeneous equation.

$$\ddot{y} + 7\dot{y} + 12y = -5e^{2t}$$

characteristic polynomial: $P(r) = r^2 + 7r + 12 = (r+3)(r+4)$, Roots: -3, -4.

general solution: $y_h = c_1 e^{-3t} + c_2 e^{-4t}$

ERF says: $\frac{1}{P(2)} e^{2t}$ is a particular solution to $P(D)y = e^{2t}$

ie, $\frac{1}{30} e^{2t}$ is particular solution to $\ddot{y} + 7\dot{y} + 12y = -e^{2t}$

$\therefore$ $\frac{1}{6} e^{2t}$ is particular solution to $\ddot{y} + 7\dot{y} + 12y = -5e^{2t}$

general solution: $y = y_p + y_h = \frac{1}{6} e^{2t} + c_1 e^{-3t} + c_2 e^{-4t}$

9.9 ERF and complex roots

example: $\ddot{x} + x = e^{i2t}$, $(D^2+1)x = e^{i2t}$, By ERF, the particular solution:

$$x_p = \frac{1}{r^2+1} e^{rt} = \frac{1}{-4+1} e^{i2t} = -\frac{1}{3} e^{i2t}$$

Problem 1: $\frac{1}{(-1)^2+1} e^{-t} = \frac{1}{2} e^{-t}$

Problem 2: $(D^2+1) y = e^{-t} - 3e^{2it}$, $\begin{cases} (D^2+1)y = e^{-t}. & \frac{1}{(-1)^2+1} e^{-t} = \frac{1}{2} e^{-t} \\ (D^2+1)y = e^{2it}. & \frac{1}{(2i)^2+1} e^{2it} = -\frac{1}{3} e^{2it} \end{cases}$

$\therefore$ $\frac{1}{2} e^{-t} + e^{2it}$

9.10 The generalized exponential response formula

The existence and uniqueness theorem says that

$$P(D)\, y = e^{rt}$$

should have a solution even if $P(r) = 0$ (when ERF does not apply). Let's start with the one case:

ERF′ Suppose that P is a polynomial and $P(r_0) = 0$, but $P'(r_0) \neq 0$ for some number r_0. Then

$$\boxed{x_p = \frac{1}{P'(r_0)} t e^{r_0 t} \quad \text{is a particular solution to} \quad P(D)\, x = e^{r_0 t}.}$$

Need to revisit.

Example 10.1 $\ddot{x} - 4x = e^{-2t}$.

The characteristic polynomial is $P(r) = r^2 - 4$ $\therefore$ $P(-2) = 0$. But $P'(-2) = 2(-2) = -4 \neq 0$,

$\therefore$ there is a case where we can apply ERF′, which gives us particular solution.

$$x_p = \frac{1}{P'(r_0)} t e^{r_0 t} = \frac{1}{-4} t e^{-2t}$$

Generalized exponential response formula. If P is a polynomial and r_0 is a number such that

$$P(r_0) = P'(r_0) = \cdots = P^{(m-1)}(r_0) = 0 \qquad P^{(m)}(r_0) \neq 0,$$

then

$$P(D)\left(t^m e^{r_0 t}\right) = P^{(m)}(r_0)\, e^{r_0 t}$$

Need to revisit

and

$$\boxed{y_p = \frac{1}{P^{(m)}(r_0)} t^m e^{r_0 t} \quad \text{is a particular solution to} \quad P(D)\, y = e^{r_0 t}.}$$

In other words, multiply the input signal by t^m, and then multiply by the number $1/P^{(m)}(r_0)$, where $P^{(m)}$ is the m^{th} derivative of P.

Example 10.2 $\ddot{x}+x=e^{it}$, $x(0)=1$, $\dot{x}(0)=0$

$P(r)=r^2+1$, roots: $\pm i$. Find smallest integer s for which $P^{(s)}(i)\neq 0$.

$P'(r)=2r$, $P'(i)=2i\neq 0$. $\therefore s=1$

$\therefore$ according to generalized ERF, $x_p(t)=\frac{1}{P'(r_0)}te^{r_0 t}=\frac{1}{2i}te^{it}$ is a particular solution

Since $\frac{1}{i}=e^{-i\frac{\pi}{2}}$ $\therefore x_p(t)=\frac{1}{2}te^{it-i\pi/2}$

Since the roots of $\pm i$, $\therefore$ the pair e^{it}, e^{-it} form the basis

$\therefore$ general solution: $x(t)=c_1e^{it}+c_2e^{-it}+\frac{1}{2}te^{it-i\pi/2}$

$\because x(0)=1=c_1+c_2$

$\because \dot{x}(t)=ic_1e^{it}-i(1-c_1)e^{-it}+\frac{1}{2}e^{it-i\pi/2}+\frac{1}{2}te^{it}$

$\therefore \dot{x}(0)=ic_1-i(1-c_1)+\frac{1}{2}e^{-i\pi/2}=ic_1-i(1-c_1)-\frac{i}{2}=0$

$\therefore c_1=\frac{3}{4}$, $c_2=\frac{1}{4}$.

$\therefore$ the general solution: $x(t)=\frac{3}{4}e^{it}+\frac{1}{4}e^{-it}+\frac{1}{2}te^{it-i\pi/2}$

9.11. Worked example

①. $\dot{x}+kx=1$ ② $\dot{x}+kx=e^{-5t}$ ③. $\dot{x}+kx=4+7e^{-5t}$

Find homogeneous Soln:

$P(r)=r+k=0$. $r=-k$. $\therefore x_h=ce^{-kt}$

Find Particular Soln:

①. $\dot{x}+kx=1$ $x_p=\frac{1}{k}$.

General Soln: $x=\frac{1}{k}+ce^{-kt}$

② $\dot{x}+kx=e^{-5t}$

ERF. $x_p=\frac{1}{-5+k}e^{-5t}$; General Soln: $x=\frac{e^{-5t}}{k-5}+ce^{-kt}$ $(k\neq 5)$

If $k=5$, $\dot{x}+5x=e^{-5t}$,

Variation of parameters, $x_h=ce^{-5t}$, take $c=1$, $x=u(t)e^{-5t}$

$\frac{d}{dt}(ue^{-5t})+5ue^{-5t}=e^{-5t}$, $\frac{du}{dt}e^{-5t}=e^{-5t}$, $\frac{du}{dt}=1$. $u=t+c$

$\therefore x=(t+c)e^{-5t}$ $(k=5)$

③ $\dot{x}+kx=4+7e^{-5t}=4\cdot 1+7\cdot e^{-5t}$

Particular soln, $x_p = 4 \cdot \frac{1}{k} + 7 \cdot \frac{e^{-5t}}{k-5}$

9.12. Review: Basis of homogeneous solutions with linear operators.

Example: $y''' - 10y'' + 31y' - 30y = 0$.

$P(r) = r^3 - 10r^2 + 31r - 30 = (r-2)(r-3)(r-5)$

e^{2t} is a solution since $P(D)e^{2t} = P(2)e^{2t} = 0e^{2t} = 0$.

$e^{3t} \qquad = 0$

$e^{5t} \qquad = 0$

Prove $\rightarrow$ linear independent

Is it possible to form: $e^{5t} = c_1 e^{2t} + c_2 e^{3t}$ for some numbers c_1, c_2?

$(D-2)(D-3)e^{5t} = (D-2)(D-3)(c_1 e^{2t} + c_2 e^{3t}) = c_1 (D-2)(D-3)e^{2t} + c_2 (D-2)(D-3)e^{3t}$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0$

The left hand side gives us: $(D-2)(D-3)e^{5t} = (5-2)(5-3)e^{5t} \neq 0$

$\therefore e^{5t}$ is not a linear combination of e^{2t} and e^{3t}

$\therefore e^{5t}, e^{2t}, e^{3t}$ form a base for a 3-dimensional space

9.13. Review: repeated roots with linear operators

Example 13.1 $\quad D^3 y = 0$.

$P(r) = r^3 = 0$. $\quad$ roots: $0, 0, 0$.

$D^2 y = c_1$, $\quad Dy = c_1 t + c_2$, $\quad y = c_1 \frac{t^2}{2} + c_2 t + c_3 = C_1 t^2 + c_2 t + c_3$

$\because t^2, t, 1$ are linearly independent, they form a basis for the space of solutions

Example 13.2 $\quad (D-5)^3 y = 0$

$P(r) = (r-5)^3 = 0$. $\quad$ roots: $5, 5, 5$

$(D-5)ue^{5t} = \dot{u}e^{5t} + 5ue^{5t} - 5ue^{5t} = \dot{u}e^{5t}$

$(D-5)^2 ue^{5t} = \ddot{u}e^{5t}$

$(D-5)^3 ue^{5t} = u^{(3)}e^{5t}$

In order for ue^{5t} to be a solution to $(D-5)^3 y = 0$, the function $u^{(3)}$ must be 0, ie, $u = a + bt + ct^2$. for some number a, b, c. $\therefore$ the solutions are

$ue^{5t} = \underbrace{ae^{5t}} + \underbrace{bte^{5t}} + \underbrace{ct^2e^{5t}}$

linearly independent

9.14. Worked examples.

Example 14.1 $2\ddot{x} + \dot{x} + x = 1 + 2e^t$

Characteristic Polynomial: $P(r) = 2r^2 + r + 1$, roots: $(-1 \pm \sqrt{7}i)/4$

$\therefore x_h = e^{-t/4}\left[c_1 \cos(\frac{\sqrt{7}}{4}t) + c_2 \sin(\frac{\sqrt{7}}{4}t)\right] = A e^{-t/4} \cos(\frac{\sqrt{7}}{4}t - \phi)$

$P(D)x = 1 + 2 \cdot e^t$. $\begin{cases} P(D)x = 1 = e^{0t} \rightarrow x_1 \\ P(D)x = e^t \rightarrow x_2 \end{cases}$ $\therefore x_p = x_1 + 2x_2$

$x_1 = \frac{1}{P(0)} e^{0t} = 1$

$x_2 = \frac{1}{P(1)} e^t = \frac{1}{2+1+1} e^t = \frac{e^t}{4}$ $\therefore x_p = 1 + \frac{e^t}{2}$

$\therefore x_g = x_h + x_p$

Example 14.2 $\ddot{x} + 8\dot{x} + 15x = e^{-5t}$

$P(r) = r^2 + 8r + 15 = 0$. $r = -5$, $r = -3$. $\because P(-5) = 0$, $\therefore$ we use general ERF

$P'(r) = 2r + 8$. $\because P'(-5) = -2$ $\therefore x_p = \frac{te^{-5t}}{P'(-5)} = -\frac{te^{-5t}}{2}$

Problem: $y^{(5)} - y = e^{at}$. $P(r) = r^5 - 1 = 0$. $r = 1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$

$P'(r) = 5r^4$.

9.15. Comparison of ERF and Variation of parameters.

ERF works for	Variation of Parameters works for
nth order DEs	1st order DEs (so far...)
Constant coefficient DEs (LTIs)	Any linear DE
Exponential input	Any system input

10. Complex Replacement, Gain and Phase Lag, Stability

10.1 Complex replacement

10.2 Boston Harbor Example

$\dot{x} = k(y - x)$

10.3 Complex replacement method

$y' + ky = k \cdot q_e(t)$, $\cos \omega t$. ω: angular frequency ~ complex oscillation in 2π

$q_e(t) = \cos \omega t$: find the response

$e^{i\omega t} = \cos\omega t + i\sin\omega t$, $\therefore \tilde{y}' + k\tilde{y} = ke^{i\omega t}$. $\tilde{y} = y_1 + iy_2 \rightarrow$ complex soln.

Find $\hat{y}$, then y_1 will solve orig real ODE

Complex replacement is a method for finding **a particular solution** to an inhomogeneous linear ODE

$$P(D)\,x = \cos\omega t,$$

where P is a real polynomial, and ω is a real number. (Note that in the video y was the unknown function.)

1. Write the right hand side of the equation $\cos\omega t$ as $\mathrm{Re}\,(e^{i\omega t})$:

$$P(D)\,x = \mathrm{Re}\,(e^{i\omega t}).$$

2. Replace the right hand side of the differential equation with the complex exponential $e^{i\omega t}$. We need a new variable for the solution, which will be a complex function. Give the name z for the unknown complex function. The **complexified** differential equation is this:

$$P(D)\,z = \underbrace{e^{i\omega t}}_{\text{complex replacement}}.$$

(Note that in the video $\tilde{y}$ was used instead of z.)

3. Use ERF (or generalized ERF if $P(i\omega) = 0$) to find a particular solution z_p to the complexified ODE.

$$z_p = \frac{e^{i\omega t}}{P(i\omega)}.$$

4. Compute $x_p = \mathrm{Re}\,(z_p)$. Then x_p is a particular solution to the **original** ODE.

10.4 Why does complex replacement work?

Complex replacement is helpful also with other real input signals, with any real-valued function that can be written as the real part of a reasonably simple complex input signal. Here are some simple examples that would be helpful to have memorized:

Real input signal	Complex replacement
$\cos\omega t$	$e^{i\omega t}$
$A\cos(\omega t - \phi)$	$Ae^{i(\omega t - \phi)}$
$e^{at}\cos\omega t$	$e^{(a+i\omega)t}$

Using complex arithmetic, there is a more complicated formula which can be derived as well (it was derived in Lecture 6: Sinusoids)

Real input signal	Complex replacement
$a\cos\omega t + b\sin\omega t$	$(a - bi)\,e^{i\omega t}$

Each function in the first column is the real part of the corresponding function in the second column. The nice thing about these examples is that the complex replacement is a constant times a complex exponential, so ERF (or generalized ERF) applies.

10.5. Worked examples using complex replacement

Example 5.1 $\ddot{x} + 4x = \cos(2t)$

$\ddot{z} + 4z = e^{i2t}$, $P(r) = r^2 + 4$, roots: $\pm 2i$, $P'(r) = 2r$,

$\therefore$ particular soln: $z_p = \dfrac{te^{i2t}}{P'(2i)} = \dfrac{te^{i2t}}{4i}$

$x_p = \mathrm{Re}(z_p) = \mathrm{Re}\left(\dfrac{t(\cos 2t + i\sin 2t)}{4i}\right) = \dfrac{t}{4}\sin(2t)$

Example 5.2 $\ddot{x} + \dot{x} + 2x = \cos(2t)$

$\ddot{z} + \dot{z} + 2z = e^{i2t}$, $P(r) = r^2 + r + 2$, $z_p = \dfrac{1}{P(2i)}e^{i2t} = \dfrac{1}{-2+2i}e^{i2t}$

$x_p = \mathrm{Re}(z_p) = \mathrm{Re}\left(\dfrac{1}{-2+2i}e^{i2t}\right)$

$$-2+2i = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\, e^{i3\pi/4}$$

$$z_p = \frac{e^{2it}}{2\sqrt{2}e^{i3\pi/4}} = \frac{1}{2\sqrt{2}} e^{i(2t-3\pi/4)} \quad \therefore x_p = \frac{1}{2\sqrt{2}}\cos\left(2t-\frac{3\pi}{4}\right)$$

10.6 Damped sinusoidal inputs

$y''-y'+2y = 10\, e^{-x}\sin x$, Find particular solu

$$(D^2-D+2)\tilde{y} = 10e^{-x+ix}, \quad \tilde{y}_p = \frac{10\, e^{(-1+i)x}}{(-1+i)^2-(-1+i)+2} = \frac{10\, e^{(-1+i)x}}{3-3i} = \frac{5}{3}(1+i)e^{-x}(\cos x + i\sin x)$$

$$\therefore y_p = \mathrm{Im}(\tilde{y}_p) = \frac{5}{3}e^{-x}(\cos x + \sin x)$$

10.7 Complex gain

$\dot{x}+kx = k\cos(\omega t)$, $\dot{z}+kz = ke^{i\omega t}$, one particular response:

$$z_p = \frac{Q(i\omega)}{P(i\omega)}e^{i\omega t} = \frac{k}{i\omega+k}e^{i\omega t}$$

☆☆

$$\text{Complex gain:} \quad G(\omega) = \frac{\text{complexified system response}}{\text{complexified system input}}$$

in this case $G(\omega) = \dfrac{k}{i\omega+k}$

and $z_p = G(\omega)e^{i\omega t} = \dfrac{k}{i\omega+k}e^{i\omega t}$

$$x_p = \mathrm{Re}(z_p) = \mathrm{Re}(G(\omega)e^{i\omega t}) = \mathrm{Re}\left(\frac{k}{i\omega+k}e^{i\omega t}\right)$$

$$G(\omega) = |G(\omega)|\, e^{-i\phi}$$

in this case, $|G(\omega)| = \dfrac{k}{\sqrt{\omega^2+k^2}}$

$$\therefore z_p = \frac{k}{\sqrt{\omega^2+k^2}}e^{i(\omega t-\phi)} \quad \text{and} \quad x_p = \frac{k}{\sqrt{\omega^2+k^2}}\cos(\omega t-\phi)$$

gain: $g(\omega) = \dfrac{k}{\sqrt{\omega^2+k^2}}$, phase lag $= -\left(\arg\dfrac{k}{k+i\omega}\right) = -\arg k + \arg(k+i\omega) = \arg(k+i\omega)$

General rule: gain: $g(\omega) = \dfrac{k}{\sqrt{\omega^2+k^2}}$, phase lag: $\phi = -\arg G(\omega)$

$P(D)x = Q(D)y$

In general: complex gain $= G = \dfrac{Q(i\omega)}{P(i\omega)}$ phase lag $= -\arg G = \arg P(i\omega) - \arg Q(i\omega)$

10.8. Driving through the spring

Apply complex gain to a mass-spring-dashpot system :

$\ddot{x} + b\dot{x} + kx = ky$, y: input signal → position of far end of spring.

x: displacement of the mass

Set $m=1$, $y = \cos(\omega t)$

① Complex gain check :

$\ddot{x} + b\dot{x} + 2x = 2\cos(t)$, gain $= \dfrac{2}{(i)^2 + bi + 2} = \dfrac{2}{1+bi}$

② Phase lag $\quad -\operatorname{Arg} G = -\operatorname{Arg}\left(\frac{2}{P(i)}\right) = -\left[\operatorname{Arg}(2) - \operatorname{Arg}(P(i))\right] = \operatorname{Arg}(P(i))$

10.9. The meaning of LTI

Example 9.1 If $P(D)x = \cos\omega x$ has $x_p = A\cos(\omega t - \phi)$ as particular soln.

shifting time by $a = \alpha/\omega$ shows that $P(D)x = \cos(\omega t - \alpha)$ has $x_p = A\cos(\omega t - \alpha - \phi)$ as a particular soln.

10.10. Worked example

(a) $\dot{x} + kx = \cos(\omega t)$

General soln : $x = x_h + x_p$.

complex replace : $\cos(\omega t) = \operatorname{Re}(e^{i\omega t})$, $\dot{z} + kz = e^{i\omega t}$, $x_p = \operatorname{Re}(z_p)$,

ERF : $z_p = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{i\omega + k} = \dfrac{k - i\omega}{k^2 + \omega^2}\left[\cos\omega t + i\sin\omega t\right]$

$\therefore x_p = \operatorname{Re}(z_p) = \dfrac{1}{k^2+\omega^2}\left[k\cos(\omega t) + \omega\sin(\omega t)\right]$

(b). $\dot{x} + kx = F\sin(\omega t)$

Complex replace : $F\sin(\omega t) = F \cdot \operatorname{Im}(e^{i\omega t})$ . . $\dot{z} + kz = F \cdot e^{i\omega t}$

$\therefore z_p = F\dfrac{(k - i\omega)}{k^2+\omega^2}(\cos\omega t + i\sin\omega t)$

$\therefore x_p = \operatorname{Im}(z_p) = \dfrac{F}{k^2+\omega^2}\left[-\omega\cos(\omega t) + k\sin(\omega t)\right]$

(c). $\dot{x} + kx = \cos(\omega t) + 3\sin(\omega t)$

Superposition principle : $\dot{x}_a + kx_a = \cos(\omega t)$, $\dot{x}_b + kx_b = 3\cdot\sin(\omega t)$

$$\frac{d(x_a+x_b)}{dt} + k(x_a+x_b) = \cos(\omega t) + 3\cdot\sin(\omega t)$$

$$\therefore x_p = \frac{1}{k^2+\omega^2}\left[k\cos\omega t + \omega\sin\omega t - 3\omega\cos\omega t + 3k\sin\omega t\right]$$

$$= \frac{1}{k^2+\omega^2}\left[(k-3\omega)\cos(\omega t) + (\omega+3k)\sin(\omega t)\right]$$

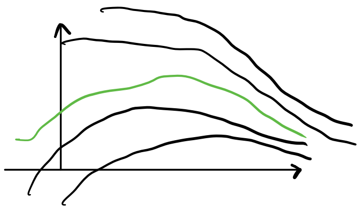
(d). $\dot{x} + kx = \cos(\omega t - \varphi)$

$\dot{x} + kx = \cos\left[\omega\left(t - \frac{\varphi}{\omega}\right)\right]$

$\therefore x_p = \frac{1}{k^2+\omega^2}\left[k\cos(\omega t - \varphi) + \omega\sin(\omega t - \varphi)\right]$

10.11. Stability

$$y' + ky = q(t) \rightarrow \text{Soln: } y = \underbrace{e^{-kt}\int q(t)e^{kt}\,dt}_{\text{steady-state soln (long term soln)}} + \underset{\substack{\downarrow \\ 0 \text{ as } t\to\infty\ (k>0)}}{Ce^{-kt}}$$



Example: $\ddot{x} + 7\dot{x} + 12x = 12f(t)$, $f(t) = \cos(2t)$

$P(r) = r^2 + 7r + 12$. roots: $-3, -4$, $\therefore x_h = c_1e^{-3t} + c_2e^{-4t}$

$$x = x_p + x_h = \underbrace{Re\left(\frac{12}{8+14i}e^{2it}\right)}_{\text{steady state soln}} + \underbrace{c_1e^{-3t} + c_2e^{-4t}}_{\text{transient}}$$

10.12. Tests for stability in a second-order system

Stability means that the long-term system behavior is independent of initial conditions.

Stability test in terms of roots. A constant coefficient linear ODE of *any order* is stable if and only if every root of the characteristic polynomial has negative real part.

We will explain this in the case of a second order system. For any second order system

$$m\ddot{x} + b\dot{x} + kx = 0$$

there are 3 cases for the roots of the characteristic polynomial.

1. The roots are complex conjugates: $a \pm bi$.
2. The roots are repeated and real: s, s.
3. The roots are distinct real numbers: r_1, r_2.

If the roots are complex conjugates, then the general solution to the homogeneous equation takes the form $e^{at}\left(A\cos\left(bt - \phi\right)\right)$. The homogeneous solution is transient only if the real part of the root is negative ($a < 0$). If the real part is zero $a = 0$, the homogeneous solution oscillates forever with constant amplitude; if the real part is positive ($a > 0$), the homogeneous solution oscillates and the amplitude grows exponentially as time goes on.

If the roots are real and repeated, then the homogeneous solution takes the form $(A + Bt)\,e^{st}$, and these solutions are transient only if $s < 0$.

If the roots are real and distinct, then the homogeneous solution takes the form $c_1e^{r_1t} + c_2e^{r_2t}$, and these solutions are transient only if $r_1, r_2 < 0$.

These conditions are collected in the table below. All three cases are covered by the statement that the real part of the roots of the characteristic polynomial must be negative.

Roots	General solution x_h	Condition for stability
complex $a \pm bi$	$e^{at}\left(c_1\cos(bt) + c_2\sin(bt)\right)$	$a < 0$
repeated real s, s	$e^{st}\left(c_1 + c_2t\right)$	$s < 0$
distinct real r_1, r_2	$c_1e^{r_1t} + c_2e^{r_2t}$	$r_1, r_2 < 0$

Problem: $x^{(3)} + \ddot{x} + \dot{x} - 3x = \sin 7t$

$r^3 + r^2 + r - 3 = (r-1)(r^2+2r+3)$

10.13. Coefficients and tests for stability

Stability test in terms of coefficients, 2^{nd} order case. Assume that a_0, a_1, a_2 are real numbers with $a_0 > 0$. The ODE

$$(a_0 D^2 + a_1 D + a_2)\, x = F(t)$$

is stable if and only if $a_1 > 0$ and $a_2 > 0$.

Recall the results for stability in terms of the roots of the characteristic polynomial.

Roots	General solution x_h	Condition for stability	Characteristic polynomial
complex $a \pm bi$	$e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$	$a < 0$	$r^2 - 2ar + (a^2 + b^2)$
repeated real s, s	$e^{st}(c_1 + c_2 t)$	$s < 0$	$r^2 - 2sr + s^2$
distinct real r_1, r_2	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1, r_2 < 0$	$r^2 - (r_1 + r_2)\, r + r_1 r_2$

By dividing by a_0, we can assume that $a_0 = 1$. Break into cases according to the table above.

- When the roots are complex, $a \pm bi$, we have $a < 0$ if and only if the coefficients $-2a$ and $a^2 + b^2$ are both positive.
- When the roots are s, s, we have $s < 0$ if and only if the coefficients $-2s$ and s^2 are both positive.
- When the roots are distinct and real $r_1 \neq r_2$, both roots are less than zero if and only if the coefficients $-(r_1 + r_2)$ and $r_1 r_2$ are both positive. (Knowing that $-(r_1 + r_2)$ is positive means that at least one of r_1, r_2 is negative; if moreover the product $r_1 r_2$ is positive, then the other root must be negative too.)

Remark 13.1 There is a generalization of the coefficient test to higher-order ODEs, called the **Routh–Hurwitz conditions for stability**, but the conditions are much more complicated.

10.14. Upshot of stability

10.15. Worked examples

① $2\ddot{x} + \dot{x} + x = e^{-t}\cos t$

$2\ddot{z} + \dot{z} + z = e^{(-1+i)t}$, ERF: $z_p = \dfrac{e^{(-1+i)t}}{P(-1+i)}$

$P(-1+i) = 2(-1+i)^2 + (-1+i) + 1 = -3i$

$z_p = \dfrac{i e^{(-1+i)t}}{3}$ ∴ $x_p = Re(z_p) = -\frac{1}{3}e^{-t}\sin(t)$

② $\ddot{x} + 8x = \cos(\omega t)$ $(\omega^2 \neq 8)$

$\ddot{x}_h + 8x_h = 0$, $P(r) = r^2 + 8$, $r = \pm\sqrt{8}\,i$, $x_h = c_1\cos(\sqrt{8}\,t) + c_2\sin(\sqrt{8}\,t)$

Complex replace: $\ddot{z} + 8z = e^{i\omega t}$ $z_p = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{-\omega^2 + 8} = \dfrac{1}{8-\omega^2}(\cos(\omega t) + i\sin(\omega t))$

∴ $x_p = \dfrac{1}{8-\omega^2}\cos(\omega t)$

∴ $x = x_h + x_p = c_1\cos(\sqrt{8}\,t) + c_2\sin(\sqrt{8}\,t) + \dfrac{1}{8-\omega^2}\cos(\omega t)$

③ $\ddot{x} + 2\dot{x} + 4x = \cos(3t)$

$\ddot{x}_h + 2\dot{x}_h + 4x_h = 0$. $P(r) = r^2 + 2r + 4 = 0$ $r = -1 \pm \sqrt{3}\,i$ ∴ $x_h = e^{-t}(c_3\cos(\sqrt{3}t) + c_4\sin(\sqrt{3}t))$

Complex replacement: $\ddot{z} + 2\dot{z} + 4z = e^{i3t}$, $z_p = \dfrac{e^{i3t}}{P(i3)} = \dfrac{e^{i3t}}{-5+6i} = \dfrac{-5-6i}{61}e^{i3t}$

∴ $x_p = Re(z_p) = \dfrac{1}{61}[-5\cos(3t) + 6\sin(3t)]$

∴ $x = x_h + x_p =$

11. Responce and frequency response

11.2 Warm up and review

$(D^2+9)x = \cos(\omega t)$. $\quad x_p = Re(z_p)$. $\quad z_p = \frac{e^{i\omega t}}{(i\omega)^2+9} = \frac{e^{i\omega t}}{9-\omega^2}$

$x_p = \frac{\cos(\omega t)}{9-\omega^2}$

$G = \frac{1}{9-\omega^2}$

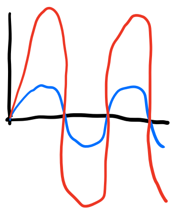
11.3 Near Resonance

$y'' + \omega_0^2 y = \cos\omega_1 t$ $\quad \underset{\text{Input}}{\omega_1} \neq \underset{\text{Natural}}{\omega_0}$

$(D^2+\omega_0^2)y = \cos(\omega_1 t)$, $\quad (D^2+\omega_0^2)\tilde{y} = e^{i\omega_1 t}$, $\quad \tilde{y}_p = \frac{e^{i\omega_1 t}}{(i\omega_1)^2+\omega_0^2} = \frac{e^{i\omega_1 t}}{\omega_0^2-\omega_1^2}$

$\therefore y_p = \frac{\cos(\omega_1 t)}{\omega_0^2-\omega_1^2}$

$\omega_1 \simeq \omega_0$, ampl big

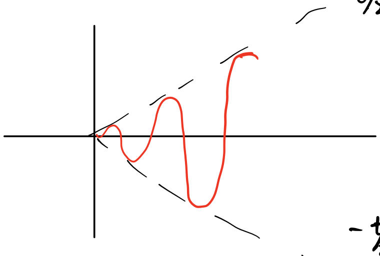


4. Pure resonance

If $\omega_1 = \omega_0$, $(D^2+\omega_0^2)y = \cos\omega_0 t$,

$(D^2+\omega_0^2)\tilde{y} = e^{i\omega_0 t}$ $\quad \therefore \tilde{y}_p = \frac{te^{i\omega_0 t}}{P'(i\omega_0)} = \frac{te^{i\omega_0 t}}{2i\omega_0}$

$\therefore y_p = Re(\tilde{y}_p) = \frac{t\sin(\omega_0 t)}{2\omega_0}$



$\ddot{z} + 4z = 2e^{i2t}$ $\quad P(r) = r^2+4$. $\quad \frac{\cancel{2}te^{i2t}}{i\cancel{4}\,2} = \frac{t}{2}\sin(2t)$

11.5 Resonance with damping

$\ddot{x} + 2p\dot{x} + \omega_n^2 x = 0$ $\quad 4p^2 - 4\omega_n^2 < 0$

$\frac{-2p \pm \sqrt{4p^2 - 4\omega_n^2}}{2} = -p \pm i\sqrt{\omega_n^2 - p^2}$ $\quad \omega_n^2 - p^2 = \omega_d^2$

w_0: natural undamped freq

w_1: natural damped freq

w_r: resonant freq

$w_0^2 - p^2 = w_1^2$

$\ddot{x} + 2p\dot{x} + w_0 x = \cos wt$

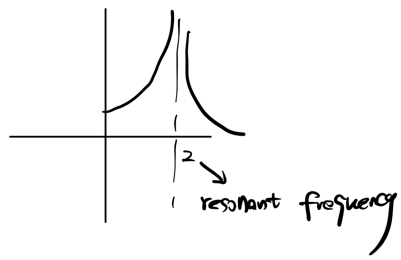
Prob: which w gives max amplitude for response $\rightarrow$ $w_r = \sqrt{w_0^2 - 2p^2}$

11.6 Frequency response and amplitude response

(a). $\ddot{x} + 4x = F_0 \cos(wt)$

$$x_p = \frac{F_0}{4-w^2}\cos(wt)$$

$$\text{gain} = \frac{\text{Output Amp}}{\text{Input Amp}} = \frac{F_0}{4-w^2} \cdot \frac{1}{F_0} = \frac{1}{4-w^2}$$



(b) $\ddot{x} + \dot{x} + 4x = F_0 \cos(wt)$

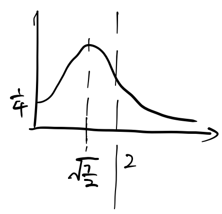
$$x_p = Re\left\{\frac{F_0 e^{iwt}}{P(iw)}\right\}$$

, $P(s) = s^2 + s + 4$, $P(iw) = (4-w^2) + iw$

$$x_p = Re\left\{\frac{F_0 e^{iwt}}{4-w^2+iw}\right\}.$$

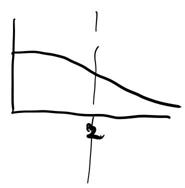
$$\text{output Amp} = \frac{F_0}{\sqrt{(4-w^2)^2 + w^2}}$$

$$g = \frac{1}{\sqrt{(4-w^2)^2 + w^2}}$$



(c). $\ddot{x} + 6\dot{x} + 4x = F_0 \cos(wt)$

$$g = \frac{1}{|P(iw)|} = \frac{1}{|4-w^2+6iw|} = \frac{1}{\sqrt{(4-w^2)^2 + 36w^2}}$$



Problem:

1 $\ddot{x} + 0.5\dot{x} + 3x = 0.5\cos(wt)$

$$G = \frac{1}{(iw)^2 + 0.5iw + 3} = \frac{1}{-w^2 + 0.5iw + 3}$$ X

$$g = \frac{1}{\sqrt{(3-w^2)^2 + 0.25w^2}}$$

$$\ddot{x} + 0.5\dot{x} + 3x = 0.5\cos(\omega t)$$

$$P(D)x = Q(D)y \quad , \quad P(D) = D^2 + 0.5D + 3 \; ; \quad Q(D) = 0.5D$$

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{0.5\omega i}{3 - \omega^2 + 0.5\omega i}$$

② $$\ddot{x} + 1.5\dot{x} + x = \cos(\omega t)$$

$$P(D) = D^2 + 1.5D + 1. \qquad Q(D) = D$$

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{1}{1 - \omega^2 + 1.5i\omega}$$

$$(D^4 + 2D^2 + 5)x = (D^2 + D)\cos(t)$$

$$G(\omega) = \frac{(i)^2 + i}{(i)^4 + 2(i)^2 + 5} = \frac{i - 1}{1 - 2 + 5} = \frac{i-1}{4}$$