6.1 Sinusoidal Functions

6.2 Solutions to the spring-mass-dashpot system

$m\ddot{x} + b\dot{x} + kx = 0$, $\quad mr^2 + br + k = 0$, $\quad r_1 = \dfrac{-b + i\sqrt{4mk - b^2}}{2m}$ $\quad r_2 = \dfrac{-b - i\sqrt{4mk - b^2}}{2m}$

6.3 Real solutions to spring-mass-dashpot system

damped frequency

natural frequency

6.4 From rectangular to polar form

6.5 Worked Example

$-\cos(5t) - \sqrt{3}\sin(5t) = -2\left(\frac{1}{2}\cos(5t) + \frac{\sqrt{3}}{2}\sin(5t)\right) = -2\left[\cos\frac{\pi}{3}\cos(5t) + \sin\frac{\pi}{3}\sin(5t)\right]$

$= -2\cos(5t - \frac{\pi}{3}) = 2\cos(5t + \frac{2}{3}\pi)$

6.6 The equivalence of the three forms.

if $c = Ae^{i\phi} = a + bi$, $\quad c \in \mathbb{C}$, $A > 0$, $a, b, \phi \in \mathbb{R}$ hold, then

$\mathrm{Re}(ce^{i\omega t}) = A\cos(\omega t - \phi) = a\cos(\omega t) + b\sin(\omega t)$ $\quad \omega > 0 \in \mathbb{R}$

6.7 Graphing sinusoidal functions

6.8 Trigonometric identity mathlet

$\sqrt{5} \cdot \left[\frac{1}{\sqrt{5}}\cos(\omega t) - \frac{2}{\sqrt{5}}\sin(\omega t)\right]$

6.9 Worked example

$\dfrac{e^{i\omega t}}{2+3i}$ $\qquad 2 + 3i = \sqrt{2^2 + 3^2}\, e^{i\phi}$, $\quad \tan\phi = \frac{3}{2}$ $\qquad \dfrac{e^{i\omega t}}{2+3i} = \dfrac{e^{i\omega t}}{\sqrt{13}e^{i\phi}} = \dfrac{1}{\sqrt{13}} e^{i(\omega t - \phi)}$

$e^{i\theta} = \cos\theta + i\sin\theta$, $\quad \mathrm{Re}\left\{\frac{1}{\sqrt{13}}\left[\cos(\omega t - \phi) + i\sin(\omega t - \phi)\right]\right\} = \frac{1}{\sqrt{13}}\cos(\omega t - \phi)$

$\mathrm{Re}\left\{\dfrac{[\cos(\omega t) + i\sin(\omega t)](2-3i)}{(2+3i)(2-3i)}\right\} = \mathrm{Re}\left\{\frac{1}{13}\left(2\cos(\omega t) + 3\sin(\omega t)\right) + i\{\cdots\}\right\} = \frac{1}{13}\left[2\cos(\omega t) + 3\sin(\omega t)\right]$

6.10. Application to spring-mass system

$ma = -kx$, $m\ddot{x} + kx = 0$ $m=1$, $k = \pi^2$,

6.11. Amplitude phase form of solutions to spring-mass-dashpot system

$y(t) = e^{-2t}(\cos t + 2\sin t) = \sqrt{5}\, e^{-2t}\left(\frac{1}{\sqrt{5}}\cos t + \frac{2}{\sqrt{5}}\sin t\right) = \sqrt{2}\, e^{-2t}\cos(t - \phi)$, ramp $=2$

6.12 Damped sinusoid revisited

6.13. Beats

Recitation 6

1. Sinusoidal functions

2. The different forms of a sinusoid function

① $\cos(2t) + \sin(2t) = \sqrt{2}\left[\cos\left(2t - \frac{\pi}{4}\right)\right]$

② $\cos(\pi t) - \sqrt{3}\sin(\pi t) = 2\left(\frac{1}{2}\cos(\pi t) - \frac{\sqrt{3}}{2}\sin(\pi t)\right) = 2\cos\left(\pi t + \frac{\pi}{3}\right)$

③ $5\cos\left(3t + \frac{3}{4}\pi\right) = 5\left(-\frac{\sqrt{2}}{2}\cos 3t - \frac{\sqrt{2}}{2}\sin 3t\right)$

3. Graphing a damped sinusoid

① $m\ddot{x} = -kx - b\dot{x}$, $mr^2 + br + k = 0$, $r^2 + \frac{1}{2}r + \frac{17}{16} = 0$ $\quad \frac{1}{4} - \frac{17}{4} = -\frac{16}{4} = -4$

$r = \frac{-\frac{1}{2} \pm 2i}{2} = -\frac{1}{4} \pm i$ $\quad x = e^{-\frac{1}{4}t}(c_1\cos(t) + c_2\sin(t))$, $\dot{x} = -\frac{1}{4}e^{-\frac{1}{4}t}(c_1\cos(t) + c_2\sin(t))$

$+ e^{-\frac{1}{4}t}(-c_1\sin(t) + c_2\cos t)$

$\begin{cases} x(0) = 1, \\ \dot{x}(0) = 0.75 \end{cases}$, $\begin{cases} x(0) = c_1 = 1 \\ \dot{x}(0) = -\frac{1}{4}c_1 + c_2 = 0.75 \end{cases}$ $\therefore \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$

$\therefore x(t) = e^{-\frac{1}{4}t}(\cos(t) + \sin(t)) = \sqrt{2}\, e^{-\frac{1}{4}t}\cos\left(t - \frac{\pi}{4}\right)$

② $\varepsilon(t) = \sqrt{2}\, e^{-\frac{1}{4}t}$ $\quad \varepsilon(0) = \sqrt{2}$. $\quad \frac{\varepsilon(t)}{\varepsilon_0} = \frac{\frac{\sqrt{2}}{2}}{\sqrt{2}} = \frac{1}{2}$, $\varepsilon(t) = \frac{\sqrt{2}}{2} = \sqrt{2}\, e^{-\frac{1}{4}t}$, $e^{-\frac{1}{4}t} = \frac{1}{2}$

$-\frac{1}{4}t = -\ln 2$ $\quad t = 4\ln 2$

③ $w = 2\pi\nu = \frac{2\pi}{T} = \frac{2\pi}{P} = 1$, $P = 2\pi$, $\nu = \frac{1}{2\pi}$. $t_0 = \frac{\pi}{4}$

④ $t - \frac{\pi}{4} = \frac{\pi}{2} + k\pi$, $t = \frac{3}{4}\pi + k\pi$, $k = 0, 1, 2, 3,$

$\therefore t = \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi, \frac{15}{4}\pi$

7. Damped harmonic Oscillators

7.1 Objectives

7.2 Simple harmonic Oscillator

Let us review what we have learned about the spring-mass-dashpot system with no external force. We will first review the cases where oscillations are present.

If there is **no damping**, the DE that models the position of the mass is

$$m\ddot{x} + kx = 0 \qquad (m, k > 0) \tag{4.64}$$

$$\text{standard linear form:} \quad \ddot{x} + \frac{k}{m}x = 0 \tag{4.65}$$

$$\text{or} \quad \ddot{x} + \omega_n^2 x = 0 \qquad \text{where } \omega_n := \sqrt{\frac{k}{m}} \tag{4.66}$$

Summary of results:

Characteristic polynomial: $P(r) = mr^2 + k$ (defined up to a constant multiple)

Roots: $\pm i\omega_n$, where $\omega_n = \sqrt{\frac{k}{m}}$.

Basis of solution space: $e^{i\omega_n t}, e^{-i\omega_n t}$.

Real-valued basis: $\cos\omega_n t, \sin\omega_n t$.

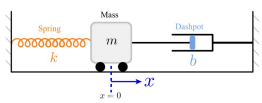
General real solution: $a\cos\omega_n t + b\sin\omega_n t$, where a, b are real constants

$= A\cos(\omega_n t - \phi)$, where $A > 0$ and ϕ are real constants.

In other words, the real-valued solutions are all the sinusoidal functions of angular frequency ω_n.

This system, or any other system governed by the same DE, is also called a **simple harmonic oscillator** . The angular frequency ω_n is also called the **natural frequency** (or **resonant frequency**) of the oscillator.

7.3 Damped Harmonic Oscillator



When damping is present, the DE that models the position of the mass is

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad m, b, k > 0 \tag{4.67}$$

Summary of results:

Characteristic polynomial: $mr^2 + br + k$

Roots: $\dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ (by the quadratic formula)

$= -\dfrac{b}{2m} \pm \sqrt{\left(\dfrac{b}{2m}\right)^2 - \omega_n^2}$

There are three cases, depending on the sign of $b^2 - 4mk$. The behavior of the solutions in these 3 cases are qualitatively different in these 3 cases.

7.4 Case1: underdamped

Case 1: $b^2 < 4mk$ **underdamped**

There are two complex roots, and we will give names to the real and imaginary parts. Since the real part is always negative, we call it $-p$, with $p = \frac{b}{2m}$. The imaginary part is either positive or negative of the **damped frequency** ω_d, given by

$$\omega_d := \frac{\sqrt{4mk - b^2}}{2m} \tag{4.68}$$

$$= \sqrt{\omega_n^2 - p^2}, \qquad \text{where } \omega_n = \sqrt{\frac{k}{m}} \text{ is the natural frequency.} \tag{4.69}$$

Note that both p and ω_d are positive.

Summary of results:

Roots: $-p \pm i\omega_d$

Basis of solution space: $e^{(-p+i\omega_d)t}, \ e^{(-p-i\omega_d)t}$

Real-valued basis: $e^{-pt}\cos(\omega_d t), \ e^{-pt}\sin(\omega_d t)$.

General real solution: $e^{-pt}(a\cos(\omega_d t) + b\sin(\omega_d t))$, where a, b are real constants.

$= Ae^{-pt}\cos(\omega_d t - \phi)$ for some A and ϕ.

This is a sinusoid multiplied by a decaying exponential. Each nonzero solution tends to 0, but changes sign infinitely many times along the way. The system is called **underdamped** , because there was not enough damping to eliminate the oscillation completely.

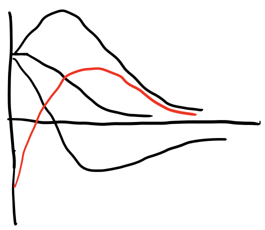
The damping not only causes the solution to decay exponentially, but also **changes the frequency of the sinusoid**. The new angular frequency, ω_d, is called **damped (angular) frequency** (or sometimes *pseudo (angular) frequency* as in the video).

The damped frequency ω_d is less than the natural (undamped) frequency ω_n, as evident from the formula $\omega_d = \sqrt{\omega_n^2 - p^2}$. When $b = 0$, ω_d is the same as ω_n.

The damped solutions are not actually periodic: they don't repeat exactly, because of the decay. Therefore, $2\pi/\omega_d$ is called the **pseudo-period** .

7.5. Concept check

7.6. Case 2: overdamped



6. Case 2: overdamped

Bookmark this page

Case 2: $b^2 > 4mk$ **overdamped** .

In this case, the roots $\frac{-b \pm \sqrt{b^2-4mk}}{2m}$ are real and distinct. Both roots are negative, since $\sqrt{b^2-4mk} < b$. Call them $-s_1$ and $-s_2$.

General real solution: $ae^{-s_1 t} + be^{-s_2 t}$, where a, b are real constants.

As in all the other damped cases, all solutions tend to 0 as $t \to +\infty$. The term corresponding to the **less negative** root eventually controls the rate of return to equilibrium. The system is called **overdamped** ; there is so much damping that it is slowing the return to equilibrium.

7.7 Case 3: Critically damped

Case 3: $b^2 = 4mk$ **critically damped** .

The critically damped case happens when $b^2 = 4mk$, at the border between the underdamped case when $b^2 < 4mk$, and the overdamped case, when $b^2 > 4mk$.

There is a repeated (negative) real root, which we denote by $-p$ with $p = \frac{b}{2m}$ as before. The repeated root gives only one exponential solution $e^{-bt/2m}$. What is another solution?

For a second order (homogeneous linear constant coefficient) ODE with repeated (real) characteristic roots r, a basis of the solutions is

$$e^{rt}, \quad te^{rt}, \tag{4.70}$$

giving the general (real) solution $c_1 e^{rt} + c_2 te^{rt}$.

7.8 Summary

The DE that models the position of the mass in a spring-mass-dashpot system is

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad m, b, k > 0, \tag{4.71}$$

The following table summarizes the various damping conditions for a spring-mass-dashpot system.

Case	Roots	Situation
$b = 0$	two complex roots $\pm i\omega_n$	undamped (simple harmonic oscillator)
$b^2 < 4mk$	two complex roots $-p \pm i\omega_d$	underdamped (damped oscillator)
$b^2 = 4mk$	repeated real root $-p, -p$	critically damped
$b^2 > 4mk$	distinct real roots $-s_1, -s_2$	overdamped

In the table above, $p = \frac{b}{2m}$, and ω_d and ω_n are related as follows:

$$\omega_d = \frac{\sqrt{4mk - b^2}}{2m} \tag{4.72}$$

$$= \sqrt{\omega_n^2 - p^2}, \quad \text{where } \omega_n = \sqrt{\frac{k}{m}}. \tag{4.73}$$

7.9. Damped vibrations mathlet

7.10. Damping conditions examples and practice : example I

$\ddot{x} + \dot{x} + 3x = 0$ $\quad r^2 + r + 3 = 0$ $\quad b^2 - 4mk = 1 - 4 \times 3 = -11$

7.11. example II

$\frac{-4 \pm 2}{2} = -3, -1$

$\ddot{x} + 4\dot{x} + 3x = 0$ $\quad b^2 - 4mk = 16 - 4 \times 3 = 4$ $\quad x(t) = c_1 e^{-t} + c_2 e^{-3t}$, $\dot{x}(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

$x(0) = c_1 + c_2 = 1$ $\quad \dot{x}(0) = -c_1 - 3c_2 = 0$ $\quad c_2 = -\frac{1}{2}$, $c_1 = \frac{3}{2}$

7.12. Example III.

$\ddot{x}+4\dot{x}+4x=0$. $\quad x = c_1 e^{-2t} + c_2 t e^{-2t}$ $\quad \dot{x}(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$x(0) = c_1 = 1$. $\quad \dot{x}(0) = -2c_1 + c_2 = 0$ $\quad c_1 = 1$, $c_2 = 2$ ✓

7.13. Comparing damping qualitatively

Recitation 7

1. Damped harmonic oscillations
2. Damping vibrations.
3. Real life application: the rocking motion of a boat

① $I\ddot{\theta} - k\theta = 0$, $Ir^2 - k = 0$. $\quad r^2 = \frac{k}{I}$ $\quad r = \pm\sqrt{\frac{k}{I}}$. $\quad \theta = c_1 e^{-\sqrt{\frac{k}{I}}t} + c_2 e^{\sqrt{\frac{k}{I}}t}$

$\dot{\theta} = c_1\left(-\sqrt{\frac{k}{I}}\right)e^{-\sqrt{\frac{k}{I}}t} + c_2\sqrt{\frac{k}{I}}\,e^{\sqrt{\frac{k}{I}}t}$

$\theta(0) = c_1 + c_2 = 0$

$\dot{\theta}(0) = -c_1\sqrt{\frac{k}{I}} + \sqrt{\frac{k}{I}}\,c_2 = c$. $\quad 2c_2 = c\sqrt{\frac{I}{k}}$. $\quad$, $c_2 = \frac{c}{2}\sqrt{\frac{I}{k}}$, $= -c_1$

② $I\ddot{\theta} + k\theta = 0$. $\quad Ir^2 + k = 0$ $\quad r^2 = -\frac{k}{I}$. $\quad r = \pm i\sqrt{\frac{k}{I}}$. $\quad \cdots \cos\left(\sqrt{\frac{k}{I}}\,t\right)$ $\quad \omega = \frac{2\pi}{T}$.

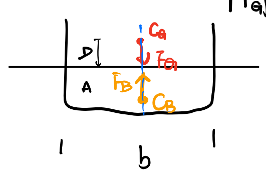
$\therefore \sqrt{\frac{k}{I}} = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

4. Gravity and Buoyancy

$I\ddot{\theta} + k\theta = 0$

Gravity (Newton) F_G

Buoyancy (Archimedes)



$|\vec{F}_G| = |\vec{F}_B|$

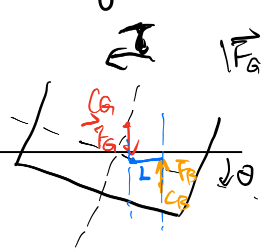
W: weight of boat

5. Details of modeling a rocking boat

$\tau = -|\vec{F}_G|\cdot L$

$L \approx \left(\frac{b^3}{12A} - D\right)\theta$, $\theta \ll 1$

$\tau(\theta) = -W\left(\frac{b^3}{12A} - D\right)\theta$



$|\vec{F}_G| = |\vec{F}_B| = W$

Stable equilibrium:

$\frac{b^3}{12A} - D > 0$ $\quad$ short wide boats.

Unstable:

$\frac{b^3}{12A} - D < 0$ $\quad$ tall, skinny boats

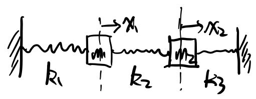
TOPIC: boat.

to be revised

8. Higher order Linear ODEs

8.1 Higher order homogeneous linear ODEs

8.2 Motivation: the coupled oscillator



Spring 1: x_1; spring 2: x_2-x_1; spring 3: $-x_2$; no damping

Force on mass 1: $F_1=-k_1x_1$; $F_2=k_2(x_2-x_1)$

$\therefore$ $m_1\ddot{x}_1 = F_1+F_2 = -k_1x_1+k_2(x_2-x_1) = -(k_1+k_2)x_1+k_2x_2$ ①

Force on mass 2: $-F_2$; $F_3=-k_3x_2$

$\therefore$ $m_2\ddot{x}_2 = -F_2+F_3 = -k_2(x_2-x_1)-k_3x_2 = k_2x_1-(k_3+k_2)x_2$ ②

$\therefore$ From eq ①: $x_2 = \frac{m_1\ddot{x}_1}{k_2}+\frac{k_1+k_2}{k_2}x_1$ ③

Substitute eq ③ into eq ②, we get

$$\frac{m_1m_2}{k_2}x_1^{(4)} + \left(\frac{m_2(k_1+k_2)}{k_2}+\frac{m_1(k_2+k_3)}{k_2}\right)\ddot{x}_1 + \left(-k_2+\frac{(k_1+k_2)(k_2+k_3)}{k_2}\right)x_1=0$$

$$\therefore\ m_1m_2x_1^{(4)} + (m_2(k_1+k_2)+m_1(k_2+k_3))\ddot{x}_1 + (k_1k_2+k_1k_3+k_2k_3)x_1=0$$

8.3. Span

The span is the linear algebra term for the phrase "all linear combinations."

Definition 3.1 Suppose that $f_1,\dots,f_n$ are functions. The **span** of $f_1,\dots,f_n$ is the set of **all linear combinations** of $f_1,\dots,f_n$:

$$\text{Span}(f_1,\dots,f_n) := \{\text{all functions } c_1f_1+\cdots+c_nf_n,\ \text{ where } c_1,\dots,c_n \text{ are numbers}\}.$$

Warning: Sometimes we allow the numbers $c_1,\dots,c_n$ to be **complex** and sometimes we allow them to be only **real**.

8.4 Vector spaces

The span of a set of functions is always a *vector space.*

What is a vector space?

Example 4.1 The 2-dimensional (x,y)-plane is a vector space. The vectors are the ordered pairs (x,y) that add component-wise: $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$. There is a zero element $(0,0)$, and you can scale vectors by real numbers: $a(x,y)=(ax,ay)$.

Definition 4.2 Suppose that S is a set of elements called **vectors** equipped with two laws: a law for adding vectors in S, called **vector addition**, and a law for multiplying vectors in S by real numbers, called **scalar multiplication.** Then S is a **real vector space** if **all** of the following are true:

1. There is a **zero vector** in S: a vector $\mathbf{0}$ in S such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for every vector $\mathbf{v}$ in S.
2. (*Closed under scalar multiplication.*) Multiplying any one vector in S by a real number gives another vector in S: if $\mathbf{v}$ is in S and a is a real number, then $a\mathbf{v}$ is in S too.
3. (*Closed under vector addition.*) Adding any two vectors in S gives another vector in S: if $\mathbf{v}$ and $\mathbf{w}$ are in S, then the vector $\mathbf{v}+\mathbf{w}$ is in S.

Remark: This definition can also be used to define a **complex vector space**; it's the same thing as a real vector space, but scalar multiplication is defined for complex numbers instead of just real numbers.

Example 4.3 The set of all real-valued functions forms a real vector space. The constant 0-function is the zero vector. Vector addition is addition of functions point-wise $(f+g)(x)=f(x)+g(x)$. Scalar multiplication is multiplying a function by a real number at each point, e.g. $-2f(x)=(-2)(f(x))$.

Non-example:
The set of all nonnegative functions is not a real vector space because x^2 is in this set, but $(-1)\cdot x^2=-x^2$ is not.

Non-example:
The set containing only the x- and y-axes is not a vector space because if we add two nonzero elements we don't always stay in the space, e.g. $(1,0)+(0,1)=(1,1)$ is not on either axes.

Last, but not least, recall:

Theorem 4.4 **The set of all homogeneous solutions to a linear ODE is a vector space.**

8.5. Linear independence

Definition 5.1 Vectors $v_1, \ldots, v_n$ are **linearly dependent** (think redundant) if at least one of them is a linear combination of the others. Otherwise, call them **linearly independent**.

Example 5.2 In a vector space consisting of functions, each function is a vector. The functions e^{2t}, e^{-3t} are linearly **independent**.

Note: For a set of 2 vectors, the definition of linear dependence and linear independence reduces to what we have learned before: v_1, v_2 are linearly dependent if one is a scalar multiple of the other. Otherwise, they are linearly independent.

Example 5.3 The functions $f_1 = e^{2t}$, $f_2 = e^{-3t}$, $f_3 = e^{2t} + e^{-3t}$ are linearly **dependent**. The third function is a linear combination of the first two:

$$f_3 = f_1 + f_2. \tag{4.95}$$

In fact, each function is a linear combination of the other two:

$$f_1 = e^{2t} = \left(e^{2t} + e^{-3t}\right) - e^{-3t} = f_3 - f_2, \tag{4.96}$$

and similarly $f_2 = f_3 - f_1$.

Equivalent definition: Vectors $v_1, \ldots, v_n$ are **linearly dependent** if there exist numbers $c_1, \ldots, c_n$ **not all zero** such that $c_1 v_1 + \cdots + c_n v_n = 0$.

In example 5.3, we wrote

$$f_3 = f_1 + f_2, \quad f_1 = f_3 - f_2, \quad f_2 = f_3 - f_1. \tag{4.97}$$

A more symmetric way to describe this is

$$f_1 + f_2 - f_3 = 0. \tag{4.98}$$

Example 5.4 The three functions e^{2t}, e^{-3t}, $5e^{2t} + 7e^{-3t}$ are linearly **dependent**, because

$$(-5)\, e^{2t} + (-7)\, e^{-3t} + (1)\left(5e^{2t} + 7e^{-3t}\right) = 0.$$

The meaning of linear dependence is that at least one of the functions on the list is redundant (and can be expressed as a linear combination of the others.)

8.6. Basis.

When discussing second order ODEs, we called a set of 2 linearly independent homogeneous solutions a *basis* for all homogeneous solutions. Below is the general definition of a basis.

Definition 6.1 A **basis** of a vector space S is a list of vectors $v_1, v_2, \ldots, v_n$ such that

1. $\mathrm{Span}\,(v_1, v_2, \ldots, v_n) = S$, and
2. The vectors $v_1, v_2, \ldots, v_n$ are linearly **independent.**

Example 6.2 The vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ form a basis for $\mathbb{R}^3$ (a real vector space). This is because any vector (x, y, z) can be written as (real) linear combinations of these 3 basis vectors, and these 3 basis vectors are linearly independent: $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$ **only** if c_1, c_2, c_3 **all** equal zero.

The vector spaces that are relevant to differential equations are vector spaces of functions. A basis of a vector space of functions is given as a list of functions $f_1, f_2, \ldots, f_n$. Think of the functions in the basis as "basic building blocks". Condition 1 says that every function in S can be built from $f_1, f_2, \ldots$. Condition 2 says that there is no redundancy in the list (no building block could have been built from the others).

Example 6.3 The functions e^{2t}, e^{-3t} form a basis for the space of solutions to the homogeneous equation $\ddot{y} + \dot{y} - 6y = 0$.

Key point: The vector space of solutions to a homogeneous ODE consists of infinitely many functions. To describe it compactly, we give a **basis** of the vector space. In this case, the basis has only 2 functions.

The plural of basis is **bases,** pronounced BAY-sees.

Fact: A vector space has many different bases.

Example 6.4 The vectors $(1,0)$, $(0,1)$ is a basis for $\mathbb{R}^2$ (a real vector space), but so is any other pair of linearly independent vectors, such as $(1,0)$, $(1,1)$.

Example 6.5 The space of solutions to $\ddot{y} + y = 0$ is spanned both by the basis e^{it}, e^{-it}, and the basis $\cos(t)$, $\sin(t)$. (Here, we allow complex linear combinations of the basis functions.)

8.7. Dimension

It turns out that although a vector space can have different bases, each basis has **the same number of vectors** in it.

Definition 7.1 The **dimension** of a vector space is the number of vectors in **any** basis.

Example 7.2 The vector space $\mathbb{R}^3$ is a 3-dimensional real vector space, since there are 3 vectors in the basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Example 7.3 The space of solutions to the homogeneous equation $\ddot{y} + \dot{y} - 6y = 0$ is 2-dimensional, since the basis e^{2t}, e^{-3t} has 2 elements. (Every other basis for the same vector space also has 2 elements.)

Example 7.4 The space of solutions to $\dot{y} = 3y$ is 1-dimensional.

In the examples involving differential equations above, the dimension equals the order of the homogeneous linear ODE. It turns out that this holds in general:

Dimension theorem. The dimension of the space of solutions to an n^{th} order homogeneous ODE with constant coefficients is n.

In other words, the number of parameters needed in the general solution to an n^{th} order homogeneous ODE with constant coefficients is n.

(The theorem is also true for n^{th} order homogeneous linear ODEs with variable coefficients provided that they are continuous and the leading coefficient is never zero.)

This theorem is a consequence of the existence and uniqueness theorem as we will see shortly.

8.8. Solving a homogeneous linear ODE with constant coefficients

Equipped with the linear algebra notions, we now generalize the solution method we used for second order homogeneous constant coefficient ODEs, such as $\ddot{y} - \dot{y} + 6y = 0$, to solve homogeneous constant coefficient ODEs of any order.

Given

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = 0, \tag{4.99}$$

where $a_n, \ldots, a_0$ are **real** constants, do the following:

1. Write down the **characteristic equation**

$$a_n r^n + \cdots + a_1 r + a_0 = 0,$$

in which the coefficient of r^j is the coefficient of $y^{(j)}$ from the ODE. The left hand side is called the **characteristic polynomial** $P(r)$. For example, $y^{(4)} - 2y^{(2)} + y = 0$ has characteristic polynomial $r^4 - 2r^2 + 1$.

2. Factor $P(r)$ as

$$P(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n)$$

where the n roots $r_1, \ldots, r_n$ are (possibly complex) numbers. There are guaranteed to be n (possibly complex) roots counted with multiplicity by the fundamental theorem of algebra. The roots may be all distinct or some roots may be repeated.

3. If $r_1, \ldots, r_n$ are **distinct**, then the functions $e^{r_1 t}, \ldots, e^{r_n t}$ form a basis for the vector space of solutions to the constant coefficient ODE. In other words, the general solution is

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}.$$

Note that complex roots always appear in pairs of conjugates, and if some of the roots are complex, the coefficients $c_1, \ldots, c_n$ will have to be complex as well.

4. If $r_1, \ldots, r_n$ are **not distinct**, then $e^{r_1 t}, \ldots, e^{r_n t}$ cannot be a basis since some of these functions are redundant (definitely not linearly independent)! If a particular root r is repeated m times, then

$$\text{replace} \quad \overbrace{e^{rt}, \ e^{rt}, \ e^{rt}, \ \ldots, \ e^{rt}}^{m \text{ copies}} \tag{4.100}$$

$$\text{by} \quad e^{rt}, \ te^{rt}, \ t^2 e^{rt}, \ \ldots, \ t^{m-1} e^{rt}. \tag{4.101}$$

We'll explain why this works in the next lecture.

Important: In all cases,

$$\text{Number of roots of } P(r) \text{ counted with multiplicity} = \text{Order of ODE} = \text{Number of functions in basis,}$$

as asserted in the dimension theorem.

Example 1: $y^{(6)} + 6y^{(5)} + 9y^{(4)} = 0$

The characteristic polynomial: $r^6 + 6r^5 + 9r^4 = r^4(r+3)^2$. the roots are: $0, 0, 0, 0, -3, -3$

$\therefore$ the basis : $e^{0t}, te^{0t}, t^2e^{0t}, t^3e^{0t}, e^{-3t}, te^{-3t}$,

which simplifies to : $1, t, t^2, t^3, e^{-3t}, te^{-3t}$.

$\therefore$ general solution: $c_1 + c_2t + c_3t^2 + c_4t^3 + c_5e^{-3t} + c_6te^{-3t}$

8.9. Review : complex roots

8.10. Higher order ODEs with complex roots

Example 10.1 Find a basis of solutions to

$$y^{(3)} + 3\ddot{y} + 9\dot{y} - 13y = 0$$

consisting of **real-valued** functions.

Solution:
The characteristic polynomial is $P(r) := r^3 + 3r^2 + 9r - 13$. Checking the divisors of -13 (as instructed by the rational root test), we find that 1 is a root, so $r - 1$ is a factor. Long division (or solving for unknown coefficients) produces the other factor:

$$P(r) = (r-1)(r^2 + 4r + 13).$$

The quadratic formula, or completing the square (rewriting the second factor as $(r+2)^2 + 9$), shows that the roots of $P(r)$ are $1, -2+3i, -2-3i$. Thus $e^t, e^{(-2+3i)t}, e^{(-2-3i)t}$ form a basis of solutions. But the last two are not real-valued!

So instead replace $y = e^{(-2+3i)t}$ and $\overline{y} = e^{(-2-3i)t}$ by $\mathrm{Re}(y)$ and $\mathrm{Im}(y)$, found by expanding

$$\begin{aligned} e^{(-2+3i)t} &= e^{-2t}e^{i(3t)} \\ &= e^{-2t}(\cos(3t) + i\sin(3t)) \\ &= e^{-2t}\cos(3t) + i\,e^{-2t}\sin(3t). \end{aligned}$$

Thus

$$e^t, \quad e^{-2t}\cos(3t), \quad e^{-2t}\sin(3t)$$

is another basis, this time consisting of real-valued functions.

Example 10.2 Suppose the roots —with multiplicity— of the characteristic polynomial of a certain homogeneous constant coefficient linear equation are

$$3,\ 4,\ 4,\ 4,\ 5 \pm 2i,\ 5 \pm 2i.$$

Give the general real solution to the equation. What is the order of the equation?

Solution:
A basis for the real valued solutions is given by

$$\{e^{3t}, e^{4t}, te^{4t}, t^2e^{4t}, e^{5t}\cos(2t), e^{5t}\sin(2t), te^{5t}\cos(2t), te^{5t}\sin(2t)\}.$$

(For each repeated root we added a power of t to the basic solution.) Using superposition, the general solution is a linear combination of these basis solutions

$$\begin{aligned} x(t) = \ & c_1e^{3t} + c_2e^{4t} + c_3te^{4t} + c_4t^2e^{4t} \\ & + c_5e^{5t}\cos(2t) + c_6e^{5t}\sin(2t) + c_7te^{5t}\cos(2t) + c_8te^{5t}\sin(2t). \end{aligned}$$

There are 8 roots, so the order of the differential equation is 8.

We can obtain real solutions to linear ODEs with nonconstant real coefficients as well:

Complex basis vs. real-valued basis. Let $y(t)$ be a complex-valued function of a real-valued variable t. If y and $\overline{y}$ are part of a basis of solutions to a homogeneous linear system of ODEs with **real coefficients**, then

$$\text{replacing} \quad y, \overline{y} \quad \text{by} \quad \mathrm{Re}(y), \mathrm{Im}(y)$$

gives a new basis of the homogeneous solutions.

To know that the new list, which includes $\mathrm{Re}(y)$ and $\mathrm{Im}(y)$, is a basis, we need to check two things:

1. **The span of the new list is the set of all solutions.**
   This is true, because any solution is a linear combination of the old basis, and can be converted to a linear combination of the new list by substituting

$$y = \mathrm{Re}(y) + i\,\mathrm{Im}(y), \qquad \overline{y} = \mathrm{Re}(y) - i\,\mathrm{Im}(y).$$

2. **The new list is linearly independent.**
   If not, say

$$c_1\mathrm{Re}(y) + c_2\mathrm{Im}(y) + \cdots = 0,$$

then substituting

$$\mathrm{Re}(y) = \frac{y + \overline{y}}{2}, \qquad \mathrm{Im}(y) = \frac{y - \overline{y}}{2i}$$

would show that the old basis was linearly dependent, which is impossible for a basis.

8.11. Existence and Uniqueness Theorem

We now know how to find the general solution to a DE of the form

$$a_ny^{(n)} + \cdots + a_1\dot{y} + a_0y = 0, \qquad (4.102)$$

but what do we need in order to pin down one specific solution? The answer is provided by the existence and uniqueness theorem below, which we have only stated for first order linear ODEs before:

Existence and uniqueness theorem for a linear ODE.

Let $p_{n-1}(t), \ldots, p_0(t), q(t)$ be continuous functions on an open interval I. Let $a \in I$, and let $b_0, \ldots, b_{n-1}$ be given numbers. Then there exists a unique solution on I to the n^{th} order linear ODE

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,\dot{y} + p_0(t)\,y = q(t)$$

satisfying the n initial conditions

$$y(a) = b_0, \quad \dot{y}(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

As before, **existence** means that there is **at least** one solution; **uniqueness** means that there is **only** one solution.

The theorem says that, given an n^{th} order linear ODE, a single solution can be found by giving n numbers, namely the values of $y(a), \dot{y}(a), \ldots, y^{(n-1)}(a)$, the initial conditions.

Remark 11.1 The existence and uniqueness theorem in the homogeneous case explains the dimension theorem we stated previously. Each solution $y(t)$ determines a set of initial conditions, given by

$$y(t) \to \left(y(a), \dot{y}(a), \ldots, y^{(n-1)}(a)\right).$$

On the other hand, the existence and uniqueness theorem says that each set of initial conditions specifies exactly one solution. The sequence of numbers $\left(y(a), \dot{y}(a), \ldots, y^{(n-1)}(a)\right)$ constitutes a vector in $\mathbb{R}^n$, and the 1-to-1 correspondence between solutions and sets of initial conditions given by existence and uniqueness theorem proves the dimension theorem.

Recitation 8.

2. Span

3. Third order linear ODE

$x^{(3)} - \dot{x} = 0$, $r^3 - r = 0$ $r(r^2-1) = 0$. $r = 0, 1, -1$

$x(t) = c_1 + c_2 e^t + c_3 e^{-t}$ $x'(t) = c_2 e^t - c_3 e^{-t}$ $x''(t) = c_2 e^t + c_3 e^{-t}$

$x(0) = c_1 + c_2 + c_3 = 0$, $\dot{x}(0) = c_2 - c_3 = 1$ $x''(0) = c_2 + c_3 = 1$ $c_1 = -1$, $c_2 = 1$, $c_3 = 0$

$x(t) = -1 + e^t$

4. Find the simplest DE

$(r+2)^3 ((r+1)^2 + 9) = 0$ $e^{(-1+3i)t}$

$r^5 + 8r^4 + 34r^3 + 92r^2 + 136r + 80$

Part A Homework 3

6-1. $\sqrt{2} e^{i(\frac{\pi}{4} + 2k\pi)} = \sqrt{2} \cdot e^{i(\frac{\pi}{4} + k\pi)}$, let $k = -1$, $\sqrt{2}\, e^{-\frac{3}{4}\pi i} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = -1 - i$

6-2 $e^{at - i\frac{\pi}{2}}$, say, $\lambda = a + bi$, $\therefore$ $e^{at - i\frac{\pi}{2}} = e^{(a+bi)t - i\frac{\pi}{2}} = e^{at + i(bt - \frac{\pi}{2})}$

$= e^{at}\left[\cos(bt - \frac{\pi}{2}) + i \sin(bt - \frac{\pi}{2})\right]$,

$\mathrm{Im} = e^{at} \cdot \sin(bt - \frac{\pi}{2}) = -e^{at} \cos(bt) = -e^{-3t}\cos(2t)$ $\therefore$ $a = -3$, $b = 2$

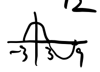
6-3 $\underbrace{e^{-4t}}\ \underline{e^{i5t}}$

6-4 $ce^{i\omega t} = (a+bi)(\cos\omega t + i \sin\omega t)$,

$\mathrm{Re} = a\cos\omega t - b\sin\omega t = 3\cos\omega t - \sin\omega t$, $\therefore a = 3$, $b = 1$

6-5. $A = 4$. $\omega = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$, $0 = A\cos(-\phi)$ $\phi = -\frac{\pi}{2}$

$4\cos(\frac{\pi}{6}t - \phi)$



7-1 $\ddot{x} + 4x = 0$ $r^2 + 4 = 0$, $r = \pm 2i$, $> e^{i\frac{\pi}{2} + k\pi}$

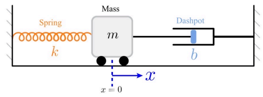
7-2. $\ddot{x} + \pi^2 x = 0$, $r^2 + \pi^2 = 0$, $r = \pm \pi i$. $x(t) = c_1\cos(\pi t) + c_2 \sin(\pi t)$

$x(0) = c_1 = 1$, $\dot{x}(0) = c_2 \cdot \pi = \sqrt{3}\pi$. $c_1 = 1$, $c_2 = \sqrt{3}$.

$x(t) = \cos(\pi t) + \sqrt{3}\sin(\pi t) = 2(\cdots)$

7-3 $m\ddot{x} + b\dot{x} + kx = 0$, $4\ddot{x} + b\dot{x} + 200 = 0$. $\frac{-b \pm \sqrt{b^2 - 3200}}{8}$

$\omega = \frac{\sqrt{b^2 - 3200}}{8} = \frac{2\pi}{T} = \frac{2\pi}{2\pi/5} = 5$, $3200 - b^2 = 1600$



When damping is present, the DE that models the position of the mass is

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad m, b, k > 0 \tag{4.67}$$

Summary of results:

Characteristic polynomial: $mr^2 + br + k$

Roots: $\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ (by the quadratic formula)

$= -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_n^2}$

There are three cases, depending on the sign of $b^2 - 4mk$. The behavior of the solutions in these 3 cases are qualitatively different in these 3 cases.

Case 1: $b^2 < 4mk$ **underdamped**

There are two complex roots, and we will give names to the real and imaginary parts. Since the real part is always negative, we call it $-p$, with $p = \frac{b}{2m}$. The imaginary part is either positive or negative of the **damped frequency** ω_d, given by

$$\omega_d := \frac{\sqrt{4mk - b^2}}{2m} \tag{4.68}$$

$$= \sqrt{\omega_n^2 - p^2}, \qquad \text{where } \omega_n = \sqrt{\frac{k}{m}} \text{ is the natural frequency.} \tag{4.69}$$

Note that both p and ω_d are positive.

Summary of results:

Roots: $-p \pm i\omega_d$

Basis of solution space: $e^{(-p+i\omega_d)t}$, $e^{(-p-i\omega_d)t}$

Real-valued basis: $e^{-pt}\cos(\omega_d t)$, $e^{-pt}\sin(\omega_d t)$.

General real solution: $e^{-pt}(a\cos(\omega_d t) + b\sin(\omega_d t))$, where a, b are real constants.

$= Ae^{-pt}\cos(\omega_d t - \phi)$ for some A and ϕ.

This is a sinusoid multiplied by a decaying exponential. Each nonzero solution tends to 0, but changes sign infinitely many times along the way. The system is called **underdamped** , because there was not enough damping to eliminate the oscillation completely.

The damping not only causes the solution to decay exponentially, but also **changes the frequency of the sinusoid**. The new angular frequency, ω_d, is called **damped (angular) frequency** (or sometimes *pseudo (angular) frequency* as in the video).

The damped frequency ω_d is less than the natural (undamped) frequency ω_n, as evident from the formula $\omega_d = \sqrt{\omega_n^2 - p^2}$. When $b = 0$, ω_d is the same as ω_n.

The damped solutions are not actually periodic: they don't repeat exactly, because of the decay. Therefore, $2\pi/\omega_d$ is called the **pseudo-period** .

8-1 e^{-4t}, te^{-4t}, $e^{(4+3i)t}$, $e^{(4-3i)t}$

$(r+4)^2((r-4)^2+9) = 0 = r^4 - 23r^2 + 72r + 400$

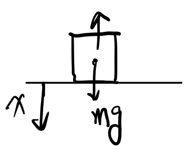
8-3 $C_1(t^2+2) + C_2(3t+5) = C_1t^2 + 3C_2t + 2C_1 + 5C_2 = 3t^2 - 6t + a$

$C_1 = 3$, $C_2 = -2$, $a = 2C_1 + 5C_2 = 6 - 10 = -4$

Part B Homework 3

1. Cylindrical buoy

①



$x(0) = 0$ $\dot{x}(0) = 0$

$m\ddot{x} = G - F = \rho_b \pi r^2 h g - \rho_w \pi r^2 x g = \rho_b \pi r^2 h \cdot \ddot{x}$

$\ddot{x} = g - \frac{\rho_w g x}{\rho_b h} = g - \frac{2gx}{h}$

② $h = 2m$

$\ddot{x} = g(1-x)$ $\ddot{x} + gx - g = 0$ $r^2 + g = 0$ $r = \pm i\sqrt{g}$. $\cos(\sqrt{g}t) + \sin(\sqrt{g}t)$

$T = \frac{2\pi}{w} = \frac{2\pi}{\sqrt{g}}$

③ $T = 2.5$

$\ddot{x} = g - \frac{\rho_w g x}{\rho_b h}$ $m = \underset{\text{kg/m}^3}{\underline{\rho_b}} \pi R^2 \underset{m}{\underline{h}} = 50$ kg

$= g - \frac{\rho_w \cdot g x \cdot \pi R^2}{50}$

$\sqrt{\frac{\rho_w \cdot g \cdot \pi R^2}{50}} = \frac{2\pi}{T} = \frac{2\pi}{5/2} = \frac{4}{5}\pi$, $\sqrt{20g \cdot \pi R^2} = \frac{4}{5}\pi$, $20g \cdot \pi R^2 = \frac{16}{25}\pi^2$

$5\,\cancel{20}g \cdot R^2 = \frac{\cancel{16}^{\,4}}{25}\pi$ $R = \sqrt{\frac{4\pi}{125g}} \approx 10.1$ cm

2. Damping

① $\frac{1}{2}\ddot{x}+b\dot{x}+\frac{1}{2}x=0$, $\ddot{x}+2b\dot{x}+x=0$ $\frac{-2b\pm\sqrt{4b^2-4}}{2}=-b\pm\sqrt{b^2-1}$.

$b<1$ $c_1 e^{-bt}\cos(\sqrt{1-b^2}t)+c_2e^{-bt}\sin(\sqrt{1-b^2}t)$ $\dot{x}(t)=-c_1 b e^{-bt}\cos(\ )+c_2e^{-bt}\cdot(\sqrt{1-b^2})\cos(\ )$

$x(0)=c_1=1$ $\dot{x}(0)=-c_1 b+c_2\sqrt{1-b^2}=-1$, $c_2=\frac{b-1}{\sqrt{1-b^2}}=-\frac{(1-b)}{\sqrt{1-b^2}}=-\sqrt{\frac{1-b}{1+b}}$

$b=1$ $(r+1)^2=0$ $r=-1$, $c_1e^{-t}+c_2te^{-t}$

$x(0)=c_1=1$, $\dot{x}(0)=-c_1+c_2=-1$, $c_1=1$, $c_2=0$. $x=e^{-t}$

$b>1$, $-b\pm\sqrt{b^2-1}$, $c_1e^{(-b+\sqrt{b^2-1})t}+c_2e^{(-b-\sqrt{b^2-1})t}$

$x(0)=c_1+c_2=1$, $\dot{x}(0)=c_1(-b+\sqrt{b^2-1})+c_2(-b-\sqrt{b^2-1})=-1$

$c_1(-b+\sqrt{b^2-1})+c_2(-b+\sqrt{b^2-1})=-b+\sqrt{b^2-1}$,

$c_2\cdot 2\sqrt{b^2-1}=-b+\sqrt{b^2-1}+1$, $c_2=\frac{-b+\sqrt{b^2-1}+1}{2\sqrt{b^2-1}}=\frac{-(b-1)+\sqrt{b^2-1}}{2\sqrt{b^2-1}}$

$c_2=\frac{1}{2}\left(-\sqrt{\frac{b-1}{b+1}}+1\right)$ $c_1=1-c_2=\frac{1}{2}\left(\sqrt{\frac{b-1}{b+1}}+1\right)$

②

③ $e^{-b}\cos(\sqrt{1-b^2})-e^{-b}\sqrt{\frac{1-b}{1+b}}\sin(\sqrt{1-b^2})$ $b\to1^-$, $e^{-b}=e^{-1}$

$\frac{1}{2}\left(1+\sqrt{\frac{b-1}{b+1}}\right)e^{(-b+\sqrt{b^2-1})}+\frac{1}{2}\left(1-\sqrt{\frac{b-1}{b+1}}\right)e^{-b-\sqrt{b^2-1}}$ $b\to1^+$, $e^{-b}=e^{-1}$

3. Application of sinusoids

4. Coupled oscillator

$y^{(4)}+4\ddot{y}+3y=0$ $r^4+4r^2+3=0$ $(r^2+1)(r^2+3)=0$, $-i$, $+i$, $-\sqrt{3}i$ $\sqrt{3}i$