3. INTRODUCTION TO COMPLEX NUMBERS

1. Complex numbers

2. Complex numbers and the complex plane

3. Operations on complex numbers

practice: $z = 1+3i$, $w = 3-2i$, $\bar{z} = 1-3i$, $\bar{w} = 3+2i$

$\bar{z} + \bar{w} = 4 - i$

$wz = (1+3i)(3-2i) = 3 + 6 + 9i - 2i = 9 + 7i$

$1/z = \frac{1}{1+3i} = \frac{1-3i}{(1+3i)(1-3i)} = \frac{1-3i}{1+9} = 0.1 - 0.3i$

4. Complex conjugation

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z} \cdot \bar{w}$

practice: $\bar{z} = a - bi = -z = -a - bi$, $a = 0$

5. Absolute value

$|a+bi| = \sqrt{a^2+b^2}$

$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 = |z|^2$

practice: $z = 1+3i$, $|z| = \sqrt{10}$

$w = 3-2i$, $|w| = \sqrt{9+4} = \sqrt{13}$

6. Some useful identities

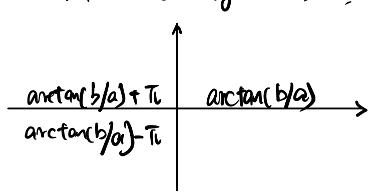
$\mathrm{Re}\, z = \frac{z+\bar{z}}{2}$, $\mathrm{Im}\, z = \frac{z-\bar{z}}{2i}$, $\bar{\bar{z}} = z$. $z\bar{z} = |z|^2$

$\mathrm{Re}(cz) = c\,\mathrm{Re}\, z$, $\mathrm{Im}(cz) = c\,\mathrm{Im}(z)$, c: real number

7. Polar form

$a = r\cos\theta$, $b = r\sin\theta$ $a + bi = r(\cos\theta + i\sin\theta)$

$r = |z|$, $\theta = \mathrm{Arg}(a+bi)$, $-\pi < \theta \le \pi$, → principle value of the argument



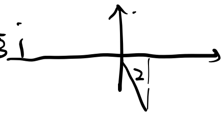
practice: $z = -1 - i$, $\theta = -\frac{3}{4}\pi$

8. Euler's formula

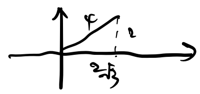
$e^{i\theta} = \cos(\theta) + i\cdot\sin(\theta)$

practice: $z = re^{i\theta}$. $z = -6 - 6\sqrt{3}i$

$2e^{-i\frac{\pi}{3}}$: $1-\sqrt{3}\,i$



$-4e^{i\frac{7\pi}{6}} = 4e^{i\frac{\pi}{6}}$



$= 2\sqrt{3} + 2i$

9. Exponential law

$e^{it_1} \cdot e^{it_2} = (\cos t_1 + i\sin t_1)(\cos t_2 + i\sin t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2 + i(\sin t_1 \cos t_2 + \sin t_2 \cos t_1)$

$= \cos(t_1+t_2) + i\sin(t_1+t_2) = e^{i(t_1+t_2)}$

10. Differential equation for exponential

$y(t) = f(t) + ig(t)$, $y'(t) = f'(t) + ig'(t)$ $\int y(t)\,dt = \int f(t)\,dt + i\int g(t)\,dt$

$\frac{de^{it}}{dt} = \frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t = i(i\sin t + \cos t) = ie^{it}$

$e^{i0} = \cos 0 + i\sin 0 = 1$

11. Operations in polar form

multiplication:	$(r_1e^{i\theta_1})(r_2e^{i\theta_2})$	=	$r_1r_2e^{i(\theta_1+\theta_2)}$	(multiply absolute values, add angles)
reciprocal:	$\frac{1}{re^{i\theta}}$	=	$\frac{1}{r}e^{-i\theta}$	
division:	$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}$	=	$\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$	(divide absolute values, subtract angles)
n^{th} power:	$(re^{i\theta})^n$	=	$r^ne^{in\theta}$	for any integer n
complex conjugation:	$\overline{re^{i\theta}}$	=	$re^{-i\theta}$.	

12. Worked examples

12.1. $(1+i)^6 = (\sqrt{2}e^{i\frac{\pi}{4}})^6 = 8e^{i\frac{6}{4}\pi} = -8$

12.2 $\frac{1+i\sqrt{3}}{\sqrt{3}+i} = \frac{2e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}} = e^{i\frac{\pi}{6}}$

Practice: $(1+i)^4 = (\sqrt{2}e^{i\frac{\pi}{4}})^4 = 4e^{i\pi} = -4$

13. Summary

RECITATION 3

1. Complex numbers
2. Operations with complex numbers

$(1+\sqrt{3}i)(a+bi) = (a-\sqrt{3}b) + (\sqrt{3}a+b)i$

$1+\sqrt{3}i = 2(\frac{1}{2}+\frac{\sqrt{3}}{2}i) = 2e^{i\frac{\pi}{3}}$

$a+bi = re^{i\theta} = r\cos\theta + ir\sin\theta$. $a = r\cos\theta$, $b = r\sin\theta$

$|(1+\sqrt{3}i)(a+bi)| = \sqrt{(a-\sqrt{3}b)^2+(\sqrt{3}a+b)^2} = \sqrt{a^2-2\sqrt{3}ab+3b^2+3a^2+2\sqrt{3}ab+b^2} = 2\sqrt{a^2+b^2} = 2r$

$\arg((1+\sqrt{3}i)(a+bi)) = \frac{\pi}{3}+\theta$

$1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2e^{i\frac{\pi}{3}}$

3 Modulus and argement.

$z = a+bi = re^{i\theta}$ $\qquad |z^n| = |z|^n$

$z^n = r^n e^{in\theta}$ $\qquad \therefore \arg(z^n) = n\arg(z)$

$|z|^n = r^n$

$|z^n| = r^n$

4. Operations with complex exponentials

$\sin(4t) = \dfrac{e^{i4t}-e^{-i4t}}{2i} = \dfrac{(e^{it})^4-(e^{-it})^4}{2i} = [(e^{it})^2-(e^{-t})^2]$

$(e^{it})^4 = e^{4it}$

$(\cos t + i\sin t)^4 = \cos 4t + i\sin 4t = 8\cos^4 t - 8\cos^2 t + 8i\sin t\cos^3 t - 4i\sin t\cos t + 1$

$\therefore \sin 4t = 8\sin t\cos^3 t - 4\sin t\cos t$

4. THE COMPLEX EXPONENTIAL FUNCTION

1. complex exponential function
2. The complex exponential function
3. (Optional) Proof of Uniqueness
4. Graphing the complex exponential function
5. Application of complex exponential to integration

$\int e^{x}\cos x\,dx = \int e^{x}\dfrac{e^{ix}-e^{-ix}}{2}dx = \frac{1}{2}\int [e^{x(i-1)} - e^{-x(i+1)}]\,dx$

$= \dfrac{1}{2(i-1)}\int e^{x(i-1)}\,dx(i-1) - \dfrac{1}{-2(i+1)}\int e^{-x(i+1)}\,d\,x(-i-1)$

$= \dfrac{1}{2(i-1)}e^{x(i-1)} + \dfrac{1}{2(i+1)}e^{-x(i+1)}$

6. Complex roots of polynomials

7. Finding roots

$z^5 = -32$, $(re^{i\theta})^5 = 32e^{i\pi}$, $r^5 e^{i5\theta} = 32e^{i\pi}$, $r^5 = 32$ and $5\theta = \pi + 2\pi k$.

$\therefore r = 2$, $\theta = \frac{\pi + 2\pi k}{5}$ for some integer k

$\therefore z = 2e^{i(\pi + 2\pi k)/5}$ for some integer k.

$k = 0, 1, 2, 3, 4$

8. Complex roots activity

$z^5 - z = 0$, $z(z^4 - 1) = 0$

9. Root of unit

$\sqrt[n]{1} = \sqrt[n]{e^{i2k\pi}} = e^{i\frac{2k\pi}{n}}$

10. Worked examples

$\sqrt[3]{1} = e^{i\frac{2}{3}k\pi}$, $k = 0, 1, 2.$, $e^0 = 1$, $e^{i\frac{2}{3}\pi} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $e^{i\frac{4}{3}\pi} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\sqrt[4]{i} = \sqrt[4]{e^{i(\frac{\pi}{2} + 2k\pi)}} = e^{i(\frac{\pi}{8} + \frac{k\pi}{2})}$, $k = 0, 1, 2, 3$.

$x^6 - 2x^3 + 2 = 0$ $\quad t = x^3$. $\quad t^2 - 2t + 2 = 0$, $t = 1 \pm i$

when $t = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, $x^3 = 2^{\frac{1}{2}} e^{i(\frac{\pi}{4} + 2k\pi)}$ $\quad x = 2^{\frac{1}{6}} e^{i(\frac{\pi}{12} + \frac{2}{3}k\pi)}$, $k = 0, 1, 2$

$t = 1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$, $x^3 = 2^{\frac{1}{2}} e^{i(-\frac{\pi}{4} + 2k\pi)}$ $\quad x = 2^{\frac{1}{6}} e^{i(-\frac{\pi}{12} + \frac{2}{3}k\pi)}$, $k = 0, 1, 2$

11. Review and worked examples

$z = -2 + 3i = a + ib = re^{i\theta}$

$r = |z| = \sqrt{a^2 + b^2} = \sqrt{13}$, $a + ib = r\cos\theta + ir\sin\theta$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a} = -\frac{3}{2}$

$\theta = \tan^{-1}(-\frac{3}{2})$

$3e^{i\frac{\pi}{6}} = 3\cos\frac{\pi}{6} + i \cdot 3\sin\frac{\pi}{6} = \frac{3}{2}\sqrt{3} + \frac{3}{2}i$

$\frac{1}{-2+3i} = \frac{1}{\sqrt{13}e^{i\theta}} = \frac{1}{\sqrt{13}} e^{-i\theta}$ $\quad \theta = \tan^{-1}(-\frac{3}{2})$

$(1)^{\frac{1}{3}} = (e^{i2k\pi})^{\frac{1}{3}} = e^{i\frac{2}{3}k\pi}$, $k = 0, 1, 2$

RECITATION 4

1. Complex exponential
2. Roots of unity

$z^n = 1 = e^{i2k\pi} \qquad z = e^{i\frac{2k\pi}{n}}$

$n=4, \quad z = e^{i\frac{k}{2}\pi}, \qquad k=0,1,2,3,$

$n=6, \quad z = e^{i\frac{k}{3}\pi}, \qquad k=0,1,2,3,4,5$

$\sqrt[3]{i} = e^{i(\frac{\pi}{2}+2k\pi)\frac{1}{3}} = e^{i(\frac{\pi}{6}+\frac{2}{3}k\pi)}, \quad k=0,1,2,$

3. Trajectories

$e^{-t} \cdot e^{2\pi i t}$

$= e^{-t}(\cos(2\pi t) + i\sin(2\pi t))$

4. Trajectory questions

$e^{(a+ib)t} = e^{at} \cdot e^{ibt}$

5. Integration

$\int e^{rt} dt = \frac{1}{r}\int e^{rt} d(rt) = \frac{1}{r}e^{rt} + C$

$\int e^{rt}\cos(3t)dt = \mathrm{Re}\int e^{rt} e^{i3t} dt = \mathrm{Re}\int e^{(r+3i)t} dt = \mathrm{Re}\,\frac{e^{(r+3i)t}}{r+3i} + C$

$r=2, \quad \therefore \quad = \mathrm{Re}\,\frac{e^{(2+3i)t}(2-3i)}{(2+3i)(2-3i)} + C = \mathrm{Re}\,\frac{e^{2t}[\cos(3t)+i\sin(3t)](2-3i)}{13} + C$

$= \frac{e^{2t}}{13}\left(2\cos(3t) + 3\sin(3t)\right) + C$

5. HOMOGENEOUS 2ND ORDER LINER ODES WITH CONSTANT COEFFICIENTS

1. Second order homogeneous Linear ODEs
2. Modeling
3. Superposition for homogeneous solutions

All linear combinations of homogeneous solutions are homogeneous solutions.

4. All solutions to a second order linear homogeneous ODE
5. Modelling: a spring-mass-dashpot-system

$m\ddot{x} = -kx - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

6. Characteristic polynomial

To solve ODE, find 2 solutions.

Basic method:

try $y = e^{rt}$ (t : ind var). Plug in : $r^2 \cancel{e^{rt}} + Ar\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$, $r^2 + Ar + B = 0 = P(r)$

$P(r) = 0$: characteristic equation

example: $\ddot{x} + x = 0$. $y = e^{rt}$. $r^2 + 1 = 0$

7. Two distinct real roots

example: $\ddot{y} + 5\dot{y} + 6y = 0$. Set $y = e^{rt}$, $P(r) = r^2 + 5r + 6 = (r+2)(r+3) = 0$

$\therefore e^{-2t}, e^{-3t}$, $\therefore$ general solution : $c_1 e^{-2t} + c_2 e^{-3t}$

Suppose : $y(0) = 0$, $\dot{y}(0) = 1$.

$\therefore c_1 + c_2 = 0$. $-2c_1 - 3c_2 = 1$ $\therefore c_1 = 1, c_2 = -1$. $\therefore y(t) = e^{-2t} - e^{-3t}$

8. Complex roots $y = e^{(a+bi)t} = e^{at}(\cos(bt) + i \cdot \sin(bt))$

Solution: $y = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$

9. Undamped system revisited

10. Optimal: explicit relation between the real and complex solutions.

$y = c_1 y_1 + c_2 y_2$

General: $y = c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t}$ which are real solutions?

change $i \to -i$ to see whether it stays the same

$c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t}$

$\downarrow$ want to the same

$\bar{c}_1 e^{(a-bi)t} + \bar{c}_2 e^{(a+bi)t}$

$\begin{cases} c_2 = \bar{c}_1 \\ \bar{c}_2 = c_1 \end{cases}$ $\therefore$ Real Solutions $(c+id)e^{(a+bi)t} + (c-id)e^{(a-bi)t}$

$e^{at}[c(e^{ibt} + e^{-ibt}) + id(e^{ibt} - e^{-ibt})]$,

$= e^{at}[c \cdot 2\cos bt + id \cdot 2i \cdot \sin bt]$

$= 2e^{at}[c \cdot \cos(bt) - d\sin(bt)]$

$\cos a = \dfrac{e^{ia} + e^{-ia}}{2}$

$\sin a = \dfrac{e^{ia} - e^{-ia}}{2i}$

11. Summary on Complex solutions

12. Review: worked examples

Example 1. $\ddot{x}+5\dot{x}+4x=0$ $\quad r^2+5r+4=0$. $\quad (r+1)(r+4)=0$, $\quad c_1e^{-x}+c_2e^{-4x}$

Example 2. $\ddot{x}+4\dot{x}+5x=0$ $\quad r^2+4r+5=0$. $\quad \frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$ $\quad z(t)=e^{(-2+i)t}$. $\quad \bar{z}(t)=e^{(-2-i)t}$

$z(t)=e^{-2t}e^{it}=e^{-2t}(\cos t+i\sin t)$ $\quad$ Re: $e^{-2t}\cos t$. $\quad$ Im: $e^{-2t}\sin t$

$\therefore x(t)=c_1e^{-2t}\cos t+c_2e^{-2t}\sin t$

Example 3. $\ddot{x}+\dot{x}+x=0$. $\quad r^2+r+1=0$. $\quad \frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

Basis: $z(t)=e^{\frac{-1+i\sqrt{3}}{2}t}$, $\quad \bar{z}(t)=e^{\frac{-1-i\sqrt{3}}{2}t}$

Basis of real solution: $x_1(t)=e^{-\frac{t}{2}}\cos(\frac{\sqrt{3}}{2}t)$, $\quad x_2(t)=e^{-\frac{t}{2}}\sin(\frac{\sqrt{3}}{2}t)$

General real solution: $x(t)=e^{-\frac{t}{2}}(c_1\cdot\cos(\frac{\sqrt{3}}{2}t)+c_2\cdot\sin(\frac{\sqrt{3}}{2}t))$

RECITATION 5

1. Exponential solutions, damping conditions
2. Spring-mass-dashpot system: distinct real roots

① $r^2+br+16=0$

② $b^2-4\times16>0$. $\quad b>8$.

③ $\frac{-b\pm\sqrt{b^2-64}}{2}$

④ $x(0)=1$, $\quad x'(0)=\frac{b}{2}$

$x=c_1e^{pt}+c_2e^{qt}$ $\qquad x(0)=c_1+c_2=1$

$x'=c_1pe^{pt}+c_2qe^{qt}$ $\qquad x'(0)=c_1\frac{-b-\sqrt{b^2-64}}{2}+c_2\frac{-b+\sqrt{b^2-64}}{2}=-\frac{b}{2}$

$c_1(-b-\sqrt{b^2-64})+(1-c_1)(-b+\sqrt{b^2-64})=-b$

$c_1(-b-\sqrt{b^2-64}+b-\sqrt{b^2-64})=-b+b-\sqrt{b^2-64}$

$c_1(-2\sqrt{b^2-64})=-\sqrt{b^2-64}$

$c_1=\frac{1}{2}=c_2$

⑤

3. Complex distinct roots

② $x=c_1e^{-\frac{b}{2}t}\cos(\frac{\sqrt{64-b^2}}{2})t+c_2e^{-\frac{b}{2}t}\sin(\frac{\sqrt{64-b^2}}{2})t$

$x(0)=1$, $\quad x'(0)=-\frac{b}{2}$

$x(0) = C_1 = 1$

$x'(0) = C_1\left(-\frac{b}{2}\right)e^{-\frac{b}{2}t}\cos\left(\frac{\sqrt{64-b^2}}{2}t\right) + C_2 e^{-\frac{b}{2}t}\cdot\frac{\sqrt{64-b^2}}{2}\cdot\cos\left(\frac{\sqrt{64-b^2}}{2}t\right) = -\frac{b}{2}$

$C_2 = 0$

③

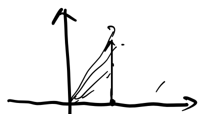
$\frac{\sqrt{64-b^2}}{2}\cdot t = \pi \qquad t = \frac{2\pi}{\sqrt{64-b^2}}$

PART A HOMEWORK 2

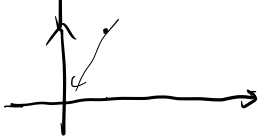
1. Lecture 3.

3-1 $\bar{z} = a - bi = -z = -a - bi \quad . \quad a = 0 \qquad z = bi$

3-3



3-4

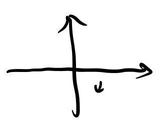


3-6. $5\left(\frac{1+it}{1-it}\right) = 5\left(\frac{(1+it)(1+it)}{(1-it)(1+it)}\right) = 5\,\frac{1-t^2+2it}{1+t^2} \qquad Re:\ 5\left(\frac{1-t^2}{1+t^2}\right)$

2. Lecture 4

4-1 $\frac{1}{7i+8t^2} = \frac{8t^2-7i}{(8t^2+7i)(8t^2-7i)} = \frac{8t^2-7i}{64t^4+49}$

4-2. te^{-3it}

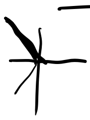


4-3 $\left|e^{-x-yi}\right| = e^{-x}$

4-4

4-5 $r^2+2r+2=0 \quad \frac{-2\pm\sqrt{4-4\times2}}{2} = -1\pm i$

$-1-i$



4-6. $(1+i\sqrt{3})^6 = 2^6\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^6 = 2^6\left(e^{i\frac{\pi}{3}}\right)^6 = 2^6 e^{i2\pi} = 2^6 = 8\times8 = 64$

4-7 $a_1z + a_2z^2 + \cdots a_nz^n = 0$

4-7

5.0/5.0 points (graded)

Suppose that $f(z)$ is a nonzero polynomial with real coefficients such that $f(-7-2i)=0$. Which of the following **must** be true? (Select **all** possible answers.)

- [ ] $f(7+2i)=0$
- [ ] $f(7-2i)=0$
- [x] $f(-7+2i)=0$
- [x] the degree of f is at least 2
- [ ] the degree of f is exactly 2
- [ ] the polynomial f has no real roots

✔

Solution:

Since $-7-2i$ is a root of f, so must be its complex conjugate $-7+2i$; thus $f(-7+2i)=0$ is true.

In particular, f has at least 2 roots, so the total number of roots counted with multiplicity is at least 2. By the fundamental theorem of algebra, that total number equals the degree of f, so the degree is at least 2.

The polynomial f might be

$$f(z)=(z-(-7-2i))(z-(-7+2i))=(z+7+2i)(z+7-2i)=(z+7)^2+2^2=z^2+14z+53.$$

In this case, using the first expression for $f(z)$ shows that $f(7+2i)\neq 0$. Similarly $f(7-2i)=0$ does not have to be true.

On the other hand, f might be

$$f(z)=z(z-(-7-2i))(z-(-7+2i))=z(z^2+14z+53)=z^3+14z^2+53z,$$

which is of degree 3. Also, this f has a real root, namely 0.

3. Lecture 5

提交 您已经尝试了4次，总共可以尝试4次 Show answer

5-1. $m\ddot{x}=-b\dot{x}$ $\quad m\ddot{x}+b\dot{x}=0$, $\quad m\dot{v}+bv=0$, $\quad \dot{v}=-\frac{b}{m}v$, $\quad v=ce^{-\frac{b}{m}t}$

5-2. $r^2-7r-8=0$. $\quad (r-8)(r+1)=0$. $\quad r=8$, $r=-1$

What happens in the spring-mass-dashpot model if the cart is attached to a dashpot, but there is no spring and no external force? Assume that the initial velocity of the cart is nonzero. Check all that apply.

- [x] The cart never changes direction.
- [ ] The cart oscillates, changing direction infinitely many times.
- [x] The cart never stops moving.
- [x] The cart's speed tends to 0.
- [x] The cart's acceleration tends to 0.
- [x] The ODE governing the motion is a homogeneous linear ODE.

✔

Solution:

The ODE is a homogeneous linear ODE; the cart never stops moving; the cart never changes direction; the cart's speed tends to 0; the cart's acceleration tends to 0.
For a general spring-mass-dashpot system, the ODE is

$$m\ddot{x}+b\dot{x}+kx=F_{\text{external}}(t),$$

but in the system here, there is no term kx for the spring, and no external force, so the ODE simplifies to

$$m\ddot{x}+b\dot{x}=0.$$

This is a homogeneous linear ODE, so (a) is true. It can be solved by the general method for homogeneous linear ODEs with constant coefficients, but it is easier to set $v := \dot{x}$, which satisfies the **first-order** homogeneous ODE

$$m\dot{v}+bv=0.$$

This leads to

$$\begin{aligned}\dot{v}&=-\frac{b}{m}v\\ v&=ce^{-(b/m)t}\end{aligned}$$

for some number c, which must be nonzero since the initial velocity is nonzero. At every t, the sign of v is the same as the sign of c, so the cart never stops moving, and never changes direction; this means it cannot oscillate. Also, $v\to 0$ as $t\to\infty$. Finally, the acceleration is $\dot{v}=-(b/m)\,ce^{(-b/m)t}$, which also tends to 0 as $t\to\infty$. (Alternative way to see (f): $\dot{v}=-\frac{b}{m}v$, so if $v\to 0$, then $\dot{v}\to 0$ too.)

PARTB HOMEWORK2

1. Complex numbers.

① $z^4=-4=4e^{i(\pi+2k\pi)}$ $\quad z=\sqrt{2}\,e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}$, $\quad k=0,1,2,3$

$z^2+2z+2=0$ $\quad r^2+2r+2=0$ $\quad \frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$.

② $\frac{3}{1-i} = \frac{3(1+i)}{1+1} = \frac{3}{2}(1+i) = \frac{3\sqrt{2}}{2} e^{i\frac{\pi}{4}}$

③ $e^{1-\pi i} = e \cdot e^{-i\pi} = -e$

④ $e^{3+\frac{\pi}{2}i} = e^3 \cdot e^{i\frac{\pi}{2}} = i e^3$

2. Complex exponential function

① $f(t) = \cos(2\pi t) = Re\, e^{zt} = Re\, e^{(a+bi)t} = Re\, e^{at}(\cos bt + i \sin bt) = e^{at}\cos bt$

$a = 0 \quad b = 2\pi$

② $f(t) = e^{-t} \qquad a = -1,\ b = 0$

③ $f(t) = e^{-t}\cos(2\pi t) \qquad a = -1,\ b = 2\pi$

④ $f(t) = 1 \quad a = 0,\ b = 0$

⑤ $z(t) = \frac{e^{4it}}{1+i} = \frac{1}{(1+i)(1-i)} e^{4it} \cdot (1-i) = \frac{1}{2} e^{i4t} \cdot \sqrt{2}\, e^{-\frac{\pi}{4}i}$

⑥ $z(t) = \frac{\sqrt{2}}{2} e^{i(4t-\frac{\pi}{4})}$

⑦ $z'(t) = \frac{\sqrt{2}}{2} \cdot 4i \cdot e^{i(4t-\frac{\pi}{4})} = 2\sqrt{2}i\, e^{i(4t-\frac{\pi}{4})}$

⑧ $z'(t) = 2\sqrt{2}\, e^{\frac{\pi}{2}i} e^{i(4t-\frac{\pi}{4})} = 2\sqrt{2}\, e^{i(4t+\frac{\pi}{4})} =$

$Im(z'(t)) = 2\sqrt{2} \sin(4t + \frac{\pi}{4})$

⑨ $Im(z(t)) = \frac{\sqrt{2}}{2} \sin(4t - \frac{\pi}{4}) = f(t)$

⑩ $f'(t) = 2\sqrt{2} \cos(4t - \frac{\pi}{4}) = 2\sqrt{2} \sin(4t + \frac{\pi}{4})$

3. Swing door

① $0.5\ddot{x} + 1.5\dot{x} + 0.625x = 0$

$\frac{1}{2}\ddot{x} + \frac{3}{2}\dot{x} + \frac{5}{8}x = 0$

$4\ddot{x} + 12\dot{x} + 5x = 0$

② $4r^2 + 12r + 5 = 0 \qquad (2r+5)(2r+1) = 0 \qquad r_1 = -\frac{5}{2},\ r_2 = -\frac{1}{2}$

$x = C_1 e^{-\frac{5}{2}t} + C_2 e^{-\frac{1}{2}t} \qquad \dot{x} = -\frac{5}{2}C_1 e^{-\frac{5}{2}t} - \frac{1}{2}C_2 e^{-\frac{1}{2}t}$

$\begin{cases} x(0) = C_1 + C_2 = x_0 \\ \dot{x}(0) = -\frac{5}{2}C_1 - \frac{1}{2}C_2 = v_0 \end{cases} \qquad \begin{cases} C_1 + C_2 = x_0 \\ 5C_1 + C_2 = -2v_0 \end{cases} \qquad 4C_1 = -2v_0 - x_0$

$C_1 = \frac{-2v_0 - x_0}{4}, \quad C_2 = x_0 - C_1 = \frac{2v_0 + 5x_0}{4}$

④ $\frac{-2v_0 - 0.25}{4} e^{-\frac{5}{2}t} + \frac{2v_0 + 1.25}{4} e^{-\frac{t}{2}} = 0$

$(-2v_0 - 0.25) + (2v_0 + 1.25) e^{2t} = 0$

$$e^{2t} = \frac{2v_0+0.25}{2v_0+1.25} > 1$$

$\begin{cases} 2v_0+0.25>0 \\ 2v_0+1.25>0. \\ 2v_0+0.25 > 2v_0+1.25 \end{cases}$ ✗

$\begin{cases} 2v_0+1.25<0 \\ 2v_0+0.25<0 \\ 2v_0+0.25 < 2v_0+1.25 \end{cases}$ ✓

$v_0 < -0.625$ ✓

⑤ $0.5\ddot{x} + 0.25\dot{x} + 2.04x = 0$

⑥ $x = e^{-0.25t}(C_1\cos 2t + C_2 \sin 2t)$

$C_1 = 0.$

$x = e^{-0.25t}\cdot C_2 \sin 2t$

$\dot{x} = -0.25\, e^{-0.25t}\cdot C_2 \sin 2t + e^{-0.25t}\cdot 2C_2\cdot\cos 2t$

$\dot{x}(0) = 2C_2 = 1 \Rightarrow C_2 = \frac{1}{2}$

$\therefore\ x(t) = e^{-0.25t}\, 0.5 \sin(2t)$

⑦ $t = 2.46 - 0.84 = 1.58$

$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{1.58\times 2} = \frac{\pi}{1.58}$

⑧