1. Introduction to Differential Equations and Modeling

ODE $\begin{cases} \text{linear} \begin{cases} \text{homogeneous} \\ \text{inhomogeneous} \end{cases} \\ \text{nonlinear} \end{cases}$ How to determine?

Recitation 1

(1) $\dfrac{X_{i+1} - X_i}{\cancel{X_i}} = k$ $\quad$ 25 $\to$ 10 $\qquad \left.\dfrac{dx}{dt}\right|_{x=10} = 15$

10 $\quad \underline{k = 1.5}$.

$x_{i+1} - 100 = 100 \cdot k = 150$ $\qquad \left.\dfrac{dx}{dt}\right|_{x=100} = 150$

$x_{i+1} = 250$

$x_{i+1} - 1000 = 1000 \cdot k = 1500$ $\qquad \left.\dfrac{dx}{dt}\right|_{x=1000} = 1500$

$x_{i+1} = 2500$.

$\Rightarrow \dfrac{dx}{dt} = -a + kx$

(3)

$\dfrac{dx}{dt} = kx = x$

$X(t) = 2X(0)$.

$x(t) = e^{kt} + C$.

$\dfrac{dx}{dt} = ke^{kt}$

$t = 0$. $\quad x(0) = 1$

$x(t) = 2x(0) = 2 = e^{kt}$, $kt = \ln 2$., $t = \dfrac{\ln 2}{k}$

$x(k^{-1}) = e^{1}$

(4) $\frac{dx}{dt} = -a + kx \hookrightarrow \frac{dx}{kx-a} = dt$

$x(t) = \frac{a}{k} + ce^{kt}$, $x(0) = \frac{a}{k} + c = x_0$ $\quad C = x_0 - \frac{a}{k}$

$x(t) = \frac{a}{k} + (x_0 - \frac{a}{k})e^{kt}$

$100 + 2e^{0.2t}$

$100 - 2e^{0.2t}$

$x(t) = 0 = \frac{a}{k} + (x_0 - \frac{a}{k})e^{kt}$, $e^{kt} = \frac{a/k}{a/k - x_0}$

$kt = \ln(\frac{a}{a-kx_0})$, $t_e = \frac{1}{k}\ln(\frac{a}{a-kx_0})$

2. Solving First-order ODEs

Seperation of variables : $\frac{dy}{dt} = g(t) \cdot f(y)$

eg. $\dot{y} - 2ty = 0$

eg. $\frac{dy}{dx} = y^2$, $y(0) = 1$

$f(x) = 1$, $f(y) = y^2$.

$\frac{dy}{y^2} = dx$

$\int \frac{1}{y^2} dy = \int dx \quad (y \neq 0)$

$-\frac{1}{y} = x + C. \rightarrow y = \frac{-1}{x+C}$

$y(0) = 1 , \rightarrow -1 = 0 + C \rightarrow C = -1$

$-\frac{1}{y} = x - 1$, $y = \frac{1}{1-x}$

$\begin{cases} y = \frac{-1}{x+C} \\ y \neq 0 \end{cases}$, $y = 0$

Standard Linear Form

Standard linear form

Every **first-order linear** ODE can be written in **standard linear form** as follows:

$$\dot{y} + p(t)\, y = q(t),$$

where $p(t)$ and $q(t)$ can be any functions of t.

When the right hand side $q(t)$ is zero, we call the equation **homogeneous.** An equation that is not homogeneous is **inhomogeneous** .

$$\begin{aligned} \text{Homogeneous:} \quad \dot{y} + p(t)\, y &= 0 \\ \text{Inhomogeneous:} \quad \dot{y} + p(t)\, y &= q(t) \end{aligned}$$

Solving homogeneous first-order linear ODEs

The general solution of $\dot{y} + p(t)y = 0$ is : $y = ce^{-P(t)}$.

$\dot{y} + ky = x = 0$ $\quad \frac{dy}{dt} = -ky$ $\quad \frac{dy}{y} = -kdt$ $\quad \ln|y| = -kt$ $\quad |y| = e^{-kt + C_1}$, $y = C_2 e^{-kt}$

$t = 0$, $y = C_2 = 100$, $y = 100e^{-kt}$

Solving inhomogeneous equations: variation of parameters.

$\dot{y} + P(t)y = q(t)$

example: $t\dot{y} + 2y = t^5$ on the interval $(0, \infty)$

① $t\dot{y} + 2y = 0$, $\dot{y} + \frac{2}{t}y = 0$, $\frac{dy}{dt} + \frac{2}{t}y = 0$. $\frac{dy}{dt} = -\frac{2}{t}y$, $\frac{dy}{y} = -\frac{2}{t}dt$,

$\ln|y| = -2\ln t + C$ $\quad (t > 0)$, $y = ce^{-2\ln t}$, $y = ct^{-2}$

when $t = 0$, $y = 0$. ✓✓

② substitute $y = u(t)t^{-2}$ into the inhomogeneous equation: the left side is

$t\dot{y} + 2y = t(\dot{u}t^{-2} + u(-2t^{-3})) + 2ut^{-2} = t^{-1}\dot{u}$

$\therefore$ $t^{-1}\dot{u} = t^5$.

③ $\dot{u} = t^6$ $\quad u = \frac{1}{7}t^7 + c$

④. $y = ut^{-2} = (\frac{1}{7}t^7 + c)t^{-2} = \frac{1}{7}t^5 + ct^{-2}$

8. Solving inhomogeneous equations by integrating factor

$\boxed{y' + p(t)y = q(t) \quad \int p(t)\,dt = P(t).}$ $\quad e^{P(t)}\dot{y} + e^{P(t)} \cdot p(t)y = qe^{P(t)}$,

$\frac{d(y \cdot e^{P(t)})}{dt}$

$\therefore \frac{d(e^P \cdot y)}{dt} = qe^P$, $e^P \cdot y = \int q e^P dt$, $y = e^{-P} \cdot \int q e^P dt$

$\therefore y = e^{-P(t)} \cdot R(t) + C \cdot e^{-P(t)}$

example. $xy' - y = x^3$, $y' - \frac{1}{x} y = x^2$, $e^{-\ln x} y' - \frac{1}{x} e^{-\ln x} y = x^2 \cdot e^{-\ln x}$,

$\frac{d(ye^{-\ln x})}{dx} = x^2 e^{-\ln x}$, $ye^{-\ln x} = \int x^2 e^{-\ln x} dx$, $\frac{y}{x} = \int x^2 \frac{1}{x} dx = \int x dx = \frac{1}{2} x^2 + C$

$y = \frac{1}{2} x^3 + Cx$

9. Linear Combination

10. Superposition

11. Consequence of superposition for 1st order linear ODEs

$$\left.\begin{aligned} &\dot{x} + 2x = 0 && x = ce^{-2t} \\ &\dot{x} + 2x = 1 && x_p = \tfrac{1}{2} \\ &\dot{x} + 2x = t && x_p = \tfrac{t}{2} - \tfrac{1}{4} \\ &\dot{x} + 2x = e^{-2t} && x_p = te^{-2t} \end{aligned}\right\}$$

$\dot{x} + 2x = 5 + 6t - 7e^{-2t}$

$x = ce^{-2t} + \frac{1}{2} \times 5 + (\frac{t}{2} - \frac{1}{4}) \times 6 - 7te^{-2t}$

$x = (c - 7t) e^{-2t} + 1 + 3t$

12. How superposition fails for nonlinear ODEs

$\dot{x}_1 + \dot{x}_2 + P(t) \cdot (x_1 + x_2)^2 = q(t)$

$2x_1x_2 P(t)$

$\dot{x}_1 + P(t) x_1^2 = q(t)$

$\dot{x}_2 + P(t) x_2^2 = q(t)$

13. Newton's Law of Cooling and CSI

$T(t) = T_e + (T_0 - T_e) e^{-kt}$

14. Cell division revisited

$y_h(t) = y(0) e^{at}$ $\qquad \frac{dy}{dt} - ay = 0.$

$= C e^{at}$ $\qquad \frac{dy}{dt} - ay = -b \quad \Rightarrow y_p = \frac{b}{a}$

$\therefore \quad y = C_1 e^{at} + \frac{b}{a}$

15. Existence and uniqueness of solutions

16. Summary

Recitation 2

1. linear equations
2. Characteristics of an equation
3. Homogeneous equation

$y' - (\tan x) y = 1. \quad y' - (\tan x) y = 0 \quad \frac{dy}{dx} = (\tan x) y, \quad \frac{dy}{y} = (\tan x) dx$

$\ln|y| = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d(\cos x) = -\ln(\cos x) + C$

$|y| = \frac{1}{\cos x} + C, \quad y_h(0) = 1 + C = 1, \quad C = 0, \quad y_h = \frac{1}{\cos x}$

4. Inhomogeneous equation

$y = u(x) \cdot \frac{1}{\cos x} \qquad y' = u' \frac{1}{\cos x} + u \left(-\frac{-\sin x}{\cos^2 x}\right) = u' \frac{1}{\cos x} + u \frac{\sin x}{\cos^2 x}$

$\text{left} = \frac{u'}{\cos x} + \frac{u \tan x}{\cos x} - \tan x \cdot u \frac{1}{\cos x} = \frac{u'}{\cos x} = 1. \quad \therefore u' = \cos x$

$u = \sin x + C$

$\therefore \quad y = (\sin x + C) \cdot \frac{1}{\cos x} = \tan x + \frac{C}{\cos x}$

$y(0) = c = y_0 \quad \therefore \quad y = \tan x + \frac{y_0}{\cos x}$

5. A different equation

$y' - xy = 1$, $\quad p(x) = -x, \quad P(x) = \int p(x)dx = -\frac{1}{2}x^2$

I.F.: $e^{P(x)} = e^{-\frac{1}{2}x^2}$. $\quad e^{-\frac{1}{2}x^2}y' - xe^{-\frac{1}{2}x^2}y = e^{-\frac{1}{2}x^2}, \quad \frac{d(y \cdot e^{-\frac{1}{2}x^2})}{dx} = e^{-\frac{1}{2}x^2}$

$y \cdot e^{-\frac{1}{2}x^2} = \int e^{-\frac{1}{2}x^2}dx = \frac{\sqrt{2\pi}}{2}\operatorname{erf}\left(\frac{\sqrt{2}x}{2}\right) + C$

$y = \frac{\sqrt{2\pi}}{2}e^{\frac{1}{2}x^2}\operatorname{erf}\left(\frac{\sqrt{2}x}{2}\right) + Ce^{\frac{1}{2}x^2}$.

Part A Homework 1

1. Lecture 1

1-3 $\quad m\ddot{x} = -mg - b\dot{x} \quad x(t)$

$kg \cdot m/s^2 \quad m/s \quad b: kg/s$

2. Lecture 2

2-1. $\quad \dot{y}(0) = -\pi t$, $\quad \dot{y} + \frac{\pi t}{2}y^2 = -\frac{t}{2}$. $\quad \cancel{\dot{y} + \frac{\pi t}{2}y^2 = 0}, \quad \frac{\dot{y}}{y^2} = -\frac{\pi}{2}t$

$\frac{dy}{y^2} = -\frac{\pi}{2}t\,dt$. $\quad \frac{1}{y} = \frac{\pi}{4}t^2 + c \quad y = \frac{1}{\frac{\pi}{4}t^2 + c} \quad c=1$

$y = \frac{1}{\frac{\pi}{4}+1}$

$\frac{dy}{dt} = -\frac{\pi}{2}t(1+y^2)$. $\quad \frac{dy}{1+y^2} = -\frac{\pi}{2}t\,dt$, $\quad \tan y = -\frac{\pi}{4}t^2 + c$.

$y = \arctan\left(-\frac{\pi}{4}t^2 + c\right)$. $\quad y(0) = \arctan(c) = 1$. $\quad c = \frac{\pi}{4} + n\pi$

$y(1) = \arctan\left(-\frac{\pi}{4} + \frac{\pi}{4} + n\pi\right) = 0$

2-2 (a) $\quad y = e^{10t}$, $\quad \dot{y} = 10e^{10t}$, $\ddot{y} = 100e^{10t}$,

$3\ddot{y} + 7\dot{y} + 7y = (300 + 70 + 7)e^{10t} = q(t)$

2-3. (b) $\quad t^2 - 4t + 4 = t^2 + 1 - 4t + 3$

2-4 $\quad y \to t^9 \qquad y_p = t^9$

$\dot{y} + 5y = 0 \to y_h = ce^{-5t} \qquad \frac{dy}{y} = -5dt \qquad \ln y = -5t + c.$

$y = ce^{-5t}$

$y = y_h + y_p = ce^{-5t} + t^9$

$y(0) = 2. = c \qquad y = 2e^{-5t} + t^9$

$y(-1) = 2e^{5} - 1$

Part B. Homework 1.

1. Find the differential equations

$y_h = a\sin(2t) \qquad 2\cancel{a}\cos(2t) + P(t)\cdot\cancel{a}\sin(2t) = 0 \quad , \quad P(t) = \frac{-2}{\tan(2t)}$

2. Modelling a mass attached to two springs

$m\ddot{x} = -(k_2 + k_1)x$

3. Heating a house

$T(t) = T_e + (T_0 - T_e)e^{-kt}$

Initial $\quad$ ambient

$T(0) = 40 + (T_0 - 40)e^{-k\cdot 0} = 70 \quad , \quad T_0 = 70$

$T(24) = 40 + (T_0 - 40)e^{-k\cdot 24} = 55 \qquad 30e^{-24k} = 15 \quad , \quad e^{-24k} = \frac{1}{2}$

$-24k = \ln\frac{1}{2} = -\ln 2 \quad , \quad k = \frac{1}{24}\ln 2$

$T(t) = 40 + 30e^{-\frac{1}{24}\ln 2 \cdot t}$

$\dot{T} = -\frac{1}{24}\ln 2 \cdot e^{-\frac{1}{24}\ln 2 \cdot t}$

Newton's law of cooling:

$\dot{T} = k(40 - T) \quad , \quad \dot{T} + kT = 40k. \qquad T_p = 40 \quad , \quad T_h = ce^{-kt}$

$T = 40 + ce^{-kt}$

$T(0) = 40 + c = 70 \quad , \quad T(24) = 40 + ce^{-24k} = 55. \quad , \quad k = \frac{\ln 2}{24}$

$\dot{T} = \frac{\ln 2}{24}(40 - T)$

$\dot{T} = k(40 - T) + q$

$\dot{T} + kT = 40k + q.$

$T = u(t)e^{-kt}$ $\dot{T} = \dot{u}e^{-kt} + (-k)ue^{-kt}$

$\dot{T} + kT = \dot{u}e^{-kt} = 40k + q$, $\dot{u} = (40k+q)e^{kt}$

$u(t) = (\frac{40k+q}{k})e^{kt} + C$

$T = (40 + \frac{q}{k}) + ce^{-kt}$

$k(40 - 70) + q_{40} = 0.$ $\frac{-30}{15}$

$k(40 - 55) + q_{45} = 0$

4. Generous grandmother

$I \cdot X - q = \dot{x}$

$q(t) = 120t$ $120 \times 12t = 1440t$

$\dot{x} = I \cdot x - 1440t$

$\dot{x} - Ix = -1440t$ $x_h = ce^{It}$

$x(t) = u(t) \cdot e^{It}$ $\dot{x} = \dot{u}e^{It} + uIe^{It}$

$\text{left} = \dot{u}e^{It} = -1440t$ $\dot{u} = -1440t \cdot e^{-It}$,

$u = \int -1440te^{-It}dt = 1440 \int \frac{1}{I} t \, de^{-It} = \frac{1440}{I}(te^{-It} - \int e^{-It}dt)$

$= \frac{1440}{I}te^{-It} + \frac{1440}{I^2} \int e^{-It} d(-It)$

$$= \frac{1440}{I} t e^{-It} + \frac{1440}{I^2} e^{-It}$$

$$= \underline{\frac{1440}{I}\left(t + \frac{1}{I} \right) e^{-It} + C}$$

Verify:

$$-I\left(t + \frac{1}{I}\right) \cdot e^{-It} + e^{-It}$$

$$\underline{-\cancel{I}t e^{-It}} \cdot \frac{1440}{\cancel{I}} = -1440 t e^{-It}$$

$$x(t) = \frac{1440}{I}\left(t + \frac{1}{I}\right) + Ce^{It}$$

$t = 18.$

$$x(0) = \frac{1440}{I^2} + C$$

$$x(18) = \frac{1440}{I}\left(18 + \frac{1}{I}\right) + Ce^{18I} = 0. \checkmark$$