Equivalence of typeclass methods under laws

Why flatMap is "equivalent" to flatten and map

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Equivalent formulations of typeclasses I

Monads can be defined via `pure`, `map`, and `flatten`, *or* via `pure` and `flatMap`
It is often said that these methods are "equivalent":

- P. Wadler, "*Monads for functional programming*" (1995)

> Often, monads are defined not in terms of *unit* and $\star$, but rather in terms of *unit*, *join*, and *map* [10, 13]. The three monad laws are replaced by the first seven of the eight laws above. If one defines $\star$ by the eighth law, then the three monad laws follow. Hence the two definitions are equivalent.

Applicative functors may be defined via `ap` and `pure` *or* via `map2` and `pure`

- P. Chuisano and R. Bjarnason, "*Functional programming in Scala*"

EXERCISE 12.2

Hard: The name *applicative* comes from the fact that we can formulate the `Applicative` interface using an alternate set of primitives, `unit` and the function `apply`, rather than `unit` and `map2`. Show that this formulation is equivalent in expressiveness by defining `map2` and `map` in terms of `unit` and `apply`. Also establish that `apply` can be implemented in terms of `map2` and `unit`.

```
trait Applicative[F[_]] extends Functor[F] {

  def apply[A,B](fab: F[A => B])(fa: F[A]): F[B]
  def unit[A](a: => A): F[A]

  def map[A,B](fa: F[A])(f: A => B): F[B]
  def map2[A,B,C](fa: F[A], fb: F[B])(f: (A,B) => C): F[A]
}
```

Define in terms of `map2` and `unit`.

Define in terms of `apply` and `unit`.

This must be right, but questions remain...

- What does it mean that `x` is "equivalent in expressiveness" to `y`?
- How can it be that `map2[A, B, C]` is "equivalent" to `ap[A, B]`?

Equivalent formulations of typeclasses II. More examples

We know that `flatMap` is *equal* to the composition of `flatten` and `map`

Also, `flatten` can be expressed via `flatMap`

```
def flatten[A](ffa: F[F[A]]): F[A] = ffa.flatMap(identity)
def flatMap[A, B](p: F[A])(f: A => F[B]): F[B] = p.map(f).flatten
```

$$\text{flatten} = \text{flatMap}\,(\text{id}) \quad , \qquad p \triangleright \text{flatMap}\,(f) = p \triangleright f^{\uparrow F} \triangleright \text{flatten}$$

The `pure` method can be expressed via `wu` and vice versa:

```
def wu: F[Unit] = pure(())
def pure[A](a: A): F[A] = wu.map(_ => a)
```

$$\text{wu} = \text{pure}(1) \quad , \qquad \text{pure}\,(a) = \text{wu} \triangleright (\_ \rightarrow a)^{\uparrow F}$$

The `filter` method can be expressed via `deflate` and vice versa:

```
def deflate[A](foa: F[Option[A]]): F[A] =
  foa.filter(_.nonEmpty).map(_.get)
def filter[A](fa: F[A])(p: A => Boolean): F[A] =
  deflate(fa.map { a => if (p(a)) Some(a) else None } )
```

Notation: $x \triangleright f$ means `f(x)` or in Scala 2.13, `x.pipe(f)`

$f^{\uparrow F}$ means `_.map(f)` for the functor `F`

Confusing issue 1: “equivalence” of values?

- Yes, we can express `flatten` through `flatMap`, but so what?
- Is `5` “equivalent” to `10` in expressive power?

```
def five: Int = ten / 2
def ten: Int = five * 2
```

Confusing issue 2: "equivalence" of functions with different sets of type parameters?

- How can `pure[A]: A => F[A]` and `wu: F[Unit]` be equivalent?
  - An extra type parameter means there are many more implementations
- Example of a `pure` that is *not* equivalent to `wu`:

```
def badPure[A](x: A): List[A] = x match {
  case i: Int   => List(i + 123)
  case _        => List(x)
}
```

- The corresponding `wu = List(())`
- We cannot restore `badPure` from `wu`: the corresponding `pure` is `List(_)`

Equivalence under naturality laws I: example

The problem with `badPure` is that it does not work the same for all types

- The code of `badPure` is not **fully parametric**
- To enforce full parametricity, we require a naturality law:
  `pure(f(x)) == pure(x).map(f)`

$$x \triangleright f \triangleright \text{pure} = x \triangleright \text{pure} \triangleright f^{\uparrow F} \quad , \qquad f^{:A\to B} \fcomp \text{pure}^B = \text{pure}^A \fcomp f^{\uparrow F}$$

$$\begin{tikzcd}
\mathtt{A} \arrow[r, "\mathtt{pure[A]}"] \arrow[d, "\mathtt{f}"'] & \mathtt{F[A]} \arrow[d, "\mathtt{_.map(f)}\text{ for }\mathtt{F}"] \\
\mathtt{B} \arrow[r, "\mathtt{pure[B]}"] & \mathtt{F[B]}
\end{tikzcd}$$

Equivalence under naturality laws II: general formulation

The precise meaning of the "equivalent expressive power" of `pure[A]` and `wu`:

- the set of all functions `pure[A]: A => F[A]` satisfying the naturality law is in a one-to-one correspondence with the set of all values `wu: F[Unit]`

Proof:

1. Start with `pure` that satisfies the naturality law; define `wu = pure(())`; then define `pure2(x) = wu.map(_ => x)`. Show that `pure2 == pure`:

$$\text{for an arbitrary } x: \quad \text{pu}_2(x) = \underline{\text{wu}} \triangleright (\_ \to x)^{\uparrow F} = 1 \triangleright \underline{\text{pu} \triangleright (\_ \to x)^{\uparrow F}}$$

$$\text{use naturality law}: \quad = \underline{1 \triangleright (\_ \to x)} \triangleright \text{pu} = x \triangleright \text{pu} = \text{pu}\,(x) \quad .$$

2. Start with `wu: F[Unit]`; define `pure(x) = wu.map(_ => x)`; then define `wu2 = pure(())`. Show that `wu2 == wu`:

$$\text{wu}_2 = \text{pure}\,(1) = \text{wu} \triangleright (\underline{\_ \to 1})^{\uparrow F} = \text{wu} \triangleright \underline{\text{id}^{\uparrow F}} = \text{wu} \triangleright \text{id} = \text{wu}$$

The function `pure(x) = wu.map(_ => x)` satisfies the naturality law:

$$\text{pure}\,(x) \triangleright f^{\uparrow F} = \text{wu} \triangleright (\_ \to \underline{x)^{\uparrow F} \triangleright f^{\uparrow F}} = \text{wu} \triangleright (\_ \to x \triangleright f)^{\uparrow F}$$

$$= \text{wu} \triangleright (\_ \to f(x))^{\uparrow F} = \text{pure}\,(f(x))$$

Equivalence under naturality laws III: general pattern

To prove the equivalence of `p: P[A, B, C]` and `q: Q[A, B, C]` under assumption of some naturality laws:

- Implement functions `p2q` and `q2p`:

```
def p2q[A, B, C]: P[A, B, C] => Q[A, B, C] = ...
def q2p[A, B, C]: Q[A, B, C] => P[A, B, C] = ...
```

- Show that `q2p(p2q(p)) == p` and `p2q(q2p(q)) == q`
- Show that `p2q(p)` satisfies `q`'s laws, and `q2p(q)` satisfies `p`'s laws

The "set of `p: P[A, B, C]` satisfying a law" is a **refined type**

- The Scala compiler cannot verify laws automatically
- Testing cannot verify laws since type parameters cannot be arbitrary
- Laws must be verified via symbolic reasoning about code

Equivalence under naturality laws IV: further examples

- Equivalence of `flatten[A]` and `flatMap[A, B]` requires a naturality law for `flatMap[A, B]` with respect to `B`

```
p.flatMap(f andThen g) == p.map(f).flatMap(g)
```

$$\text{flatMap}\,(f\,{}^{\circ}_{9}\,g) = f^{\uparrow F}\,{}^{\circ}_{9}\,\text{flatMap}\,(g)$$

- Equivalence of `ap` and `zip` requires a naturality law for each of them

```
def ap[A, B]: F[A => B] => F[A] => F[B] = ...
ap(r)(p).map(f) == ap(r.map(x => x andThen f))(p)
def zip[A, B]: (F[A], F[B]) => F[(A, B)] = ...
zip(p.map(f), q) == zip(p, q).map { case (a, b) => (f(a), b) }
```

Conclusions

- Formulated the "equivalence of expressive power" rigorously
  - It is a one-to-one correspondence between *refined types*
- In most cases, the equivalence holds only after imposing naturality laws
- Naturality laws constrain code and may eliminate a type parameter
  - Naturality laws will hold automatically for fully parametric code