2

RIGID MOTIONS AND HOMOGENEOUS TRANSFORMATIONS

A large part of robot kinematics is concerned with the establishment of various coordinate systems to represent the positions and orientations of rigid objects, and with transformations among these coordinate systems. Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation. In this chapter we study the operations of rotation and translation, and introduce the notion of homogeneous transformations.[1] Homogeneous transformations combine the operations of rotation and translation into a single matrix multiplication, and are used in Chapter 3 to derive the so-called forward kinematic equations of rigid manipulators.

We begin by examining representations of points and vectors in a Euclidean space equipped with multiple coordinate frames. Following this, we introduce the concept of a rotation matrix to represent relative orientations among coordinate frames. Then we combine these two concepts to build homogeneous transformation matrices, which can be used to simultaneously represent the position and orientation of one coordinate frame relative to another. Furthermore, homogeneous transformation matrices can be used to perform coordinate transformations. Such transformations allow us to represent various quantities in different coordinate frames, a facility that we will often exploit in subsequent chapters.

[1]Since we make extensive use of elementary matrix theory, the reader may wish to review Appendix B before beginning this chapter.

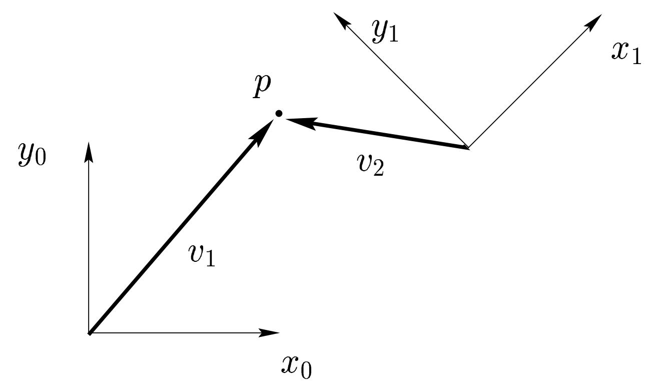

Fig. 2.1 Two coordinate frames, a point p, and two vectors v_1 and v_2.

2.1 REPRESENTING POSITIONS

Before developing representation schemes for points and vectors, it is instructive to distinguish between the two fundamental approaches to geometric reasoning: the *synthetic* approach and the *analytic* approach. In the former, one reasons directly about geometric entities (e.g., points or lines), while in the latter, one represents these entities using coordinates or equations, and reasoning is performed via algebraic manipulations.

Consider Figure 2.1. This figure shows two coordinate frames that differ in orientation by an angle of 45°. Using the synthetic approach, without ever assigning coordinates to points or vectors, one can say that x_0 is perpendicular to y_0, or that $v_1 \times v_2$ defines a vector that is perpendicular to the plane containing v_1 and v_2, in this case pointing out of the page.

In robotics, one typically uses analytic reasoning, since robot tasks are often defined using Cartesian coordinates. Of course, in order to assign coordinates it is necessary to specify a coordinate frame. Consider again Figure 2.1. We could specify the coordinates of the point p with respect to either frame $o_0x_0y_0$ or frame $o_1x_1y_1$. In the former case, we might assign to p the coordinate vector $(5, 6)^T$, and in the latter case $(-2.8, 4.2)^T$. So that the reference frame will always be clear, we will adopt a notation in which a superscript is used to denote the reference frame. Thus, we would write

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

Geometrically, a point corresponds to a specific location in space. We stress here that p is a geometric entity, a point in space, while both p^0 and p^1 are coordinate vectors that represent the location of this point in space with respect to coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, respectively.

Since the origin of a coordinate system is just a point in space, we can assign coordinates that represent the position of the origin of one coordinate system with respect to another. In Figure 2.1, for example, we have

$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

In cases where there is only a single coordinate frame, or in which the reference frame is obvious, we will often omit the superscript. This is a slight abuse of notation, and the reader is advised to bear in mind the difference between the geometric entity called p and any particular coordinate vector that is assigned to represent p. The former is independent of the choice of coordinate systems, while the latter obviously depends on the choice of coordinate frames.

While a point corresponds to a specific location in space, a *vector* specifies a direction and a magnitude. Vectors can be used, for example, to represent displacements or forces. Therefore, while the point p is not equivalent to the vector v_1, the displacement from the origin o_0 to the point p is given by the vector v_1. In this text, we will use the term *vector* to refer to what are sometimes called *free vectors*, i.e., vectors that are not constrained to be located at a particular point in space. Under this convention, it is clear that points and vectors are not equivalent, since points refer to specific locations in space, but a vector can be moved to any location in space. Under this convention, two vectors are equal if they have the same direction and the same magnitude.

When assigning coordinates to vectors, we use the same notational convention that we used when assigning coordinates to points. Thus, v_1 and v_2 are geometric entities that are invariant with respect to the choice of coordinate systems, but the representation by coordinates of these vectors depends directly on the choice of reference coordinate frame. In the example of Figure 2.1, we would obtain

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, \qquad v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, \qquad v_2^1 = \begin{bmatrix} -2.89 \\ 4.2 \end{bmatrix}$$

Coordinate Convention

In order to perform algebraic manipulations using coordinates, it is essential that all coordinate vectors be defined with respect to the same coordinate frame. In the case of free vectors, it is enough that they be defined with respect to "parallel" coordinate frames, i.e. frames whose respective coordinate axes are parallel, since only their magnitude and direction are specified and not their absolute locations in space.

Using this convention, an expression of the form $v_1^1 + v_2^2$, where v_1^1 and v_2^2 are as in Figure 2.1, is not defined since the frames $o_0x_0y_0$ and $o_1x_1y_1$ are not parallel. Thus, we see a clear need, not only for a representation system that allows points to be expressed with respect to various coordinate systems, but also for a mechanism that allows us to transform the coordinates of points that are expressed in one coordinate system into the appropriate coordinates with

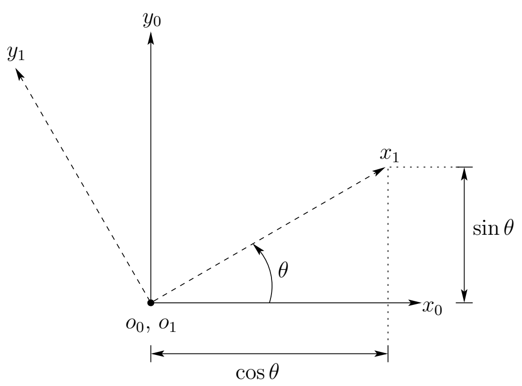

Fig. 2.2 Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

respect to some other coordinate frame. Such coordinate transformations and their derivations are the topic for much of the remainder of this chapter.

2.2 REPRESENTING ROTATIONS

In order to represent the relative position and orientation of one rigid body with respect to another, we will rigidly attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames. In Section 2.1 we saw how one can represent the position of the origin of one frame with respect to another frame. In this section, we address the problem of describing the orientation of one coordinate frame relative to another frame. We begin with the case of rotations in the plane, and then generalize our results to the case of orientations in a three dimensional space.

2.2.1 Rotation in the plane

Figure 2.2 shows two coordinate frames, with frame $o_1x_1y_1$ being obtained by rotating frame $o_0x_0y_0$ by an angle θ. Perhaps the most obvious way to represent the relative orientation of these two frames is to merely specify the angle of rotation, θ. There are two immediate disadvantages to such a representation. First, there is a discontinuity in the mapping from relative orientation to the value of θ in a neighborhood of $\theta = 0$. In particular, for $\theta = 2\pi - \epsilon$, small changes in orientation can produce large changes in the value of θ (i.e., a rotation by ϵ causes θ to "wrap around" to zero). Second, this choice of representation does not scale well to the three dimensional case.

A slightly less obvious way to specify the orientation is to specify the coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame

$o_0x_0y_0$[2]:

$$R_1^0 = \left[x_1^0 | y_1^0\right]$$

where x_1^0 and y_1^0 are the coordinates in frame $o_0x_0y_0$ of unit vectors x_1 and y_1, respectively. A matrix in this form is called a **rotation matrix**. Rotation matrices have a number of special properties that we will discuss below.

In the two dimensional case, it is straightforward to compute the entries of this matrix. As illustrated in Figure 2.2,

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

which gives

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

Note that we have continued to use the notational convention of allowing the superscript to denote the reference frame. Thus, R_1^0 is a matrix whose column vectors are the coordinates of the (unit vectors along the) axes of frame $o_1x_1y_1$ expressed relative to frame $o_0x_0y_0$.

Although we have derived the entries for R_1^0 in terms of the angle θ, it is not necessary that we do so. An alternative approach, and one that scales nicely to the three dimensional case, is to build the rotation matrix by projecting the axes of frame $o_1x_1y_1$ onto the coordinate axes of frame $o_0x_0y_0$. Recalling that the dot product of two unit vectors gives the projection of one onto the other, we obtain

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}$$

which can be combined to obtain the rotation matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

Thus the columns of R_1^0 specify the direction cosines of the coordinate axes of $o_1x_1y_1$ relative to the coordinate axes of $o_0x_0y_0$. For example, the first column $(x_1 \cdot x_0, x_1 \cdot y_0)^T$ of R_1^0 specifies the direction of x_1 relative to the frame $o_0x_0y_0$. Note that the right hand sides of these equations are defined in terms of geometric entities, and not in terms of their coordinates. Examining Figure 2.2 it can be seen that this method of defining the rotation matrix by projection gives the same result as was obtained in Equation (2.1).

If we desired instead to describe the orientation of frame $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ (i.e., if we desired to use the frame $o_1x_1y_1$ as the reference frame), we would construct a rotation matrix of the form

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

[2]We will use x_i, y_i to denote both coordinate axes and unit vectors along the coordinate axes depending on the context.

Table 2.2.1: Properties of the Matrix Group $SO(n)$

- $R \in SO(n)$
- $R^{-1} \in SO(n)$
- $R^{-1} = R^T$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

Since the inner product is commutative, (i.e. $x_i \cdot y_j = y_j \cdot x_i$), we see that

$$R_0^1 = (R_1^0)^T$$

In a geometric sense, the orientation of $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ is the inverse of the orientation of $o_1x_1y_1$ with respect to the frame $o_0x_0y_0$. Algebraically, using the fact that coordinate axes are always mutually orthogonal, it can readily be seen that

$$(R_1^0)^T = (R_1^0)^{-1}$$

The column vectors of R_1^0 are of unit length and mutually orthogonal (Problem 2-4). Such a matrix is said to be **orthogonal**. It can also be shown (Problem 2-5) that $\det R_1^0 = \pm 1$. If we restrict ourselves to right-handed coordinate systems, as defined in Appendix B, then $\det R_1^0 = +1$ (Problem 2-5). It is customary to refer to the set of all such $n \times n$ matrices by the symbol $SO(n)$, which denotes the **Special Orthogonal group of order** n. The properties of such matrices are summarized in Table 2.2.1.

To provide further geometric intuition for the notion of the inverse of a rotation matrix, note that in the two dimensional case, the inverse of the rotation matrix corresponding to a rotation by angle θ can also be easily computed simply by constructing the rotation matrix for a rotation by the angle $-\theta$:

$$\begin{aligned}
\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T
\end{aligned}$$

2.2.2 Rotations in three dimensions

The projection technique described above scales nicely to the three dimensional case. In three dimensions, each axis of the frame $o_1x_1y_1z_1$ is projected onto coordinate frame $o_0x_0y_0z_0$. The resulting rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

As was the case for rotation matrices in two dimensions, matrices in this form are orthogonal, with determinant equal to 1. In this case, 3×3 rotation matrices belong to the group $SO(3)$. The properties listed in Table 2.2.1 also apply to rotation matrices in $SO(3)$.

Example 2.1

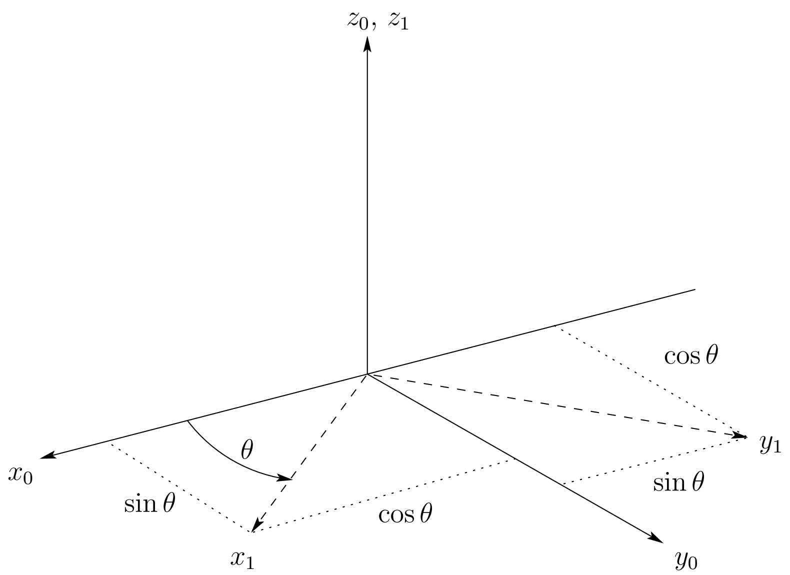

Fig. 2.3 Rotation about z_0 by an angle θ.

Suppose the frame $o_1x_1y_1z_1$ is rotated through an angle θ about the z_0-axis, and it is desired to find the resulting transformation matrix R_1^0. Note that by convention the positive sense for the angle θ is given by the right hand rule; that is, a positive rotation by angle θ about the z-axis would advance a right-hand *threaded screw along the positive z-axis*[3]*. From Figure 2.3 we see that*

$$x_1 \cdot x_0 = \cos\theta, \qquad y_1 \cdot x_0 = -\sin\theta,$$
$$x_1 \cdot y_0 = \sin\theta, \qquad y_1 \cdot y_0 = \cos\theta$$

and

$$z_0 \cdot z_1 = 1$$

while all other dot products are zero. Thus the rotation matrix R_1^0 has a particularly simple form in this case, namely

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

◇

The Basic Rotation Matrices

The rotation matrix given in Equation (2.2) is called a **basic rotation matrix** (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of R_1^0 to denote the matrix. It is easy to verify that the basic rotation matrix $R_{z,\theta}$ has the properties

$$R_{z,0} = I \tag{2.3}$$
$$R_{z,\theta}R_{z,\phi} = R_{z,\theta+\phi} \tag{2.4}$$

which together imply

$$\left(R_{z,\theta}\right)^{-1} = R_{z,-\theta} \tag{2.5}$$

Similarly the basic rotation matrices representing rotations about the x and y-axes are given as (Problem 2-8)

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2.6}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.7}$$

which also satisfy properties analogous to Equations (2.3)-(2.5).

Example 2.2

[3]See also Appendix B.

Consider the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ shown in Figure 2.4. Projecting the unit vectors x_1, y_1, z_1 onto x_0, y_0, z_0 gives the coordinates of x_1, y_1, z_1 in the $o_0x_0y_0z_0$ frame. We see that the coordinates of x_1 are $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$, the coordinates of y_1 are $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)^T$ and the coordinates of z_1 are $(0,1,0)^T$. The rotation matrix R_1^0 specifying the orientation of $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ has these as its column vectors, that is,

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \tag{2.8}$$

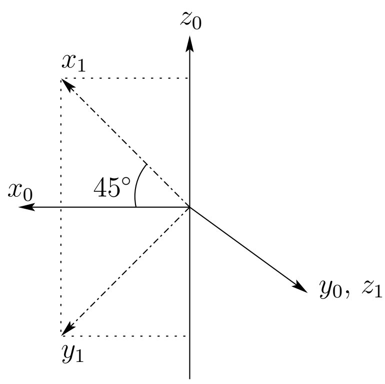

Fig. 2.4 Defining the relative orientation of two frames.

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2.3 ROTATIONAL TRANSFORMATIONS

Figure 2.5 shows a rigid object S to which a coordinate frame $o_1x_1y_1z_1$ is attached. Given the coordinates p^1 of the point p (i.e., given the coordinates of p with respect to the frame $o_1x_1y_1z_1$), we wish to determine the coordinates of p relative to a fixed reference frame $o_0x_0y_0z_0$. The coordinates $p^1 = (u, v, w)^T$ satisfy the equation

$$p = ux_1 + vy_1 + wz_1$$

In a similar way, we can obtain an expression for the coordinates p^0 by projecting the point p onto the coordinate axes of the frame $o_0x_0y_0z_0$, giving

$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

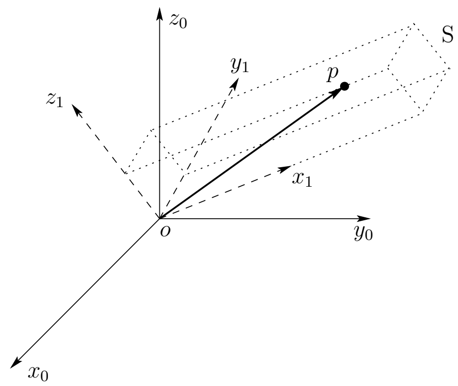

Fig. 2.5 Coordinate frame attached to a rigid body.

Combining these two equations we obtain

$$\begin{aligned} p^0 &= \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix} \\ &= \begin{bmatrix} ux_1 \cdot x_0 + vy_1 \cdot x_0 + wz_1 \cdot x_0 \\ ux_1 \cdot y_0 + vy_1 \cdot y_0 + wz_1 \cdot y_0 \\ ux_1 \cdot z_0 + vy_1 \cdot z_0 + wz_1 \cdot z_0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{aligned}$$

But the matrix in this final equation is merely the rotation matrix R_1^0, which leads to

$$p^0 = R_1^0 p^1 \tag{2.9}$$

Thus, the rotation matrix R_1^0 can be used not only to represent the orientation of coordinate frame $o_1x_1y_1z_1$ with respect to frame $o_0x_0y_0z_0$, but also to transform the coordinates of a point from one frame to another. If a given point is expressed relative to $o_1x_1y_1z_1$ by coordinates p^1, then $R_1^0 p^1$ represents the **same point** expressed relative to the frame $o_0x_0y_0z_0$.

We can also use rotation matrices to represent rigid motions that correspond to pure rotation. Consider Figure 2.6. One corner of the block in Figure 2.6(a) is located at the point p_a in space. Figure 2.6(b) shows the same block after it has been rotated about z_0 by the angle π. In Figure 2.6(b), the same corner of the block is now located at point p_b in space. It is possible to derive the coordinates for p_b given only the coordinates for p_a and the rotation matrix that corresponds to the rotation about z_0. To see how this can be accomplished, imagine that a coordinate frame is rigidly attached to the block in Figure 2.6(a), such that

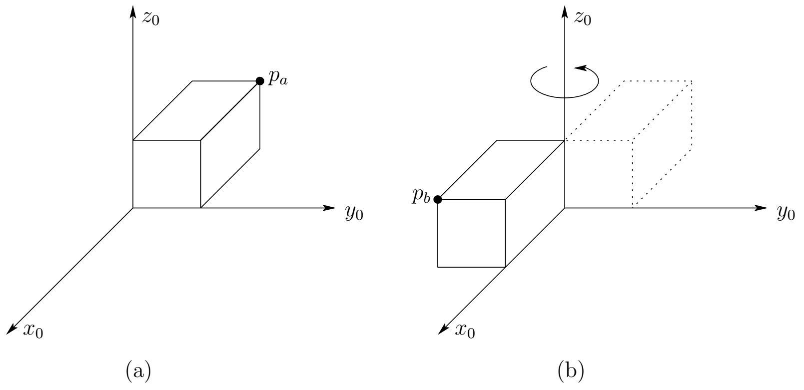

Fig. 2.6 The block in (b) is obtained by rotating the block in (a) by π about z_0.

it is coincident with the frame $o_0x_0y_0z_0$. After the rotation by π, the block's coordinate frame, which is rigidly attached to the block, is also rotated by π. If we denote this rotated frame by $o_1x_1y_1z_1$, we obtain

$$R_1^0 = R_{z,\pi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the local coordinate frame $o_1x_1y_1z_1$, the point p_b has the coordinate representation p_b^1. To obtain its coordinates with respect to frame $o_0x_0y_0z_0$, we merely apply the coordinate transformation Equation (2.9), giving

$$p_b^0 = R_{z,\pi}p_b^1$$

The key thing to notice is that the local coordinates, p_b^1, of the corner of the block do not change as the block rotates, since they are defined in terms of the block's own coordinate frame. Therefore, when the block's frame is aligned with the reference frame $o_0x_0y_0z_0$ (i.e., before the rotation is performed), the coordinates $p_b^1 = p_a^0$, since before the rotation is performed, the point p_a is coincident with the corner of the block. Therefore, we can substitute p_a^0 into the previous equation to obtain

$$p_b^0 = R_{z,\pi}p_a^0$$

This equation shows us how to use a rotation matrix to represent a rotational motion. In particular, if the point p_b is obtained by rotating the point p_a as defined by the rotation matrix R, then the coordinates of p_b with respect to the reference frame are given by

$$p_b^0 = Rp_a^0$$

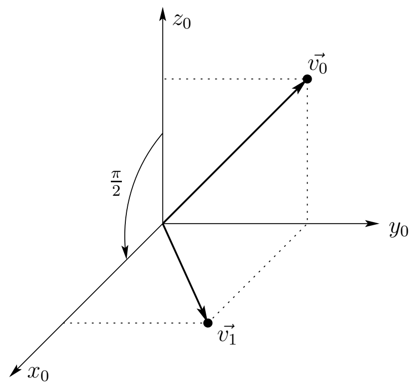

Fig. 2.7 Rotating a vector about axis y_0.

This same approach can be used to rotate vectors with respect to a coordinate frame, as the following example illustrates.

Example 2.3

The vector v with coordinates $v^0 = (0,1,1)^T$ is rotated about y_0 by $\frac{\pi}{2}$ as shown in Figure 2.7. The resulting vector v_1 has coordinates given by

$$v_1^0 = R_{y,\frac{\pi}{2}} v^0 \tag{2.10}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2.11}$$

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Thus, as we have now seen, a third interpretation of a rotation matrix R is as an operator acting on vectors in a fixed frame. In other words, instead of relating the coordinates of a fixed vector with respect to two different coordinate frames, Equation (2.10) can represent the coordinates in $o_0x_0y_0z_0$ of a vector v_1 that is obtained from a vector v by a given rotation.

Summary

We have seen that a rotation matrix, either $R \in SO(3)$ or $R \in SO(2)$, can be interpreted in three distinct ways:

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.

2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.

3. It is an operator taking a vector and rotating it to a new vector in the same coordinate system.

The particular interpretation of a given rotation matrix R that is being used must then be made clear by the context.

2.3.1 Similarity Transformations

A coordinate frame is defined by a set of **basis vectors**, for example, unit vectors along the three coordinate axes. This means that a rotation matrix, as a coordinate transformation, can also be viewed as defining a change of basis from one frame to another. The matrix representation of a general linear transformation is transformed from one frame to another using a so-called **similarity transformation**[4]. For example, if A is the matrix representation of a given linear transformation in $o_0x_0y_0z_0$ and B is the representation of the same linear transformation in $o_1x_1y_1z_1$ then A and B are related as

$$B = (R_1^0)^{-1} A R_1^0 \tag{2.12}$$

where R_1^0 is the coordinate transformation between frames $o_1x_1y_1z_1$ and $o_0x_0y_0z_0$. In particular, if A itself is a rotation, then so is B, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

Example 2.4

Henceforth, whenever convenient we use the shorthand notation $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ for trigonometric functions. Suppose frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the rotation

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

as shown in Figure 2.4. If $A = R_{z,\theta}$ relative to the frame $o_0x_0y_0z_0$, then, relative to frame $o_1x_1y_1z_1$ we have

$$B = (R_1^0)^{-1} A^0 R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

In other words, B is a rotation about the z_0-axis but expressed relative to the frame $o_1x_1y_1z_1$. This notion will be useful below and in later sections.
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[4] See Appendix B.

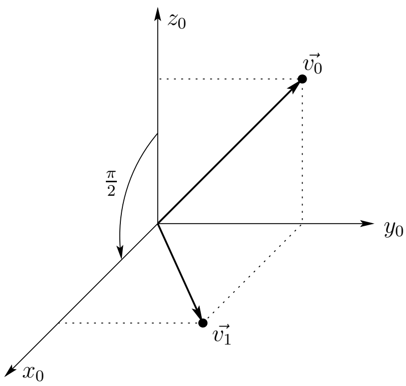

Fig. 2.8 Coordinate Frames for Example 2.4.

2.4 COMPOSITION OF ROTATIONS

In this section we discuss the composition of rotations. It is important for subsequent chapters that the reader understand the material in this section thoroughly before moving on.

2.4.1 Rotation with respect to the current frame

Recall that the matrix R_1^0 in Equation (2.9) represents a rotational transformation between the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Suppose we now add a third coordinate frame $o_2x_2y_2z_2$ related to the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ by rotational transformations. A given point p can then be represented by coordinates specified with respect to any of these three frames: p^0, p^1 and p^2. The relationship among these representations of p is

$$p^0 = R_1^0 p^1 \tag{2.13}$$
$$p^1 = R_2^1 p^2 \tag{2.14}$$
$$p^0 = R_2^0 p^2 \tag{2.15}$$

where each R_j^i is a rotation matrix. Substituting Equation (2.14) into Equation (2.13) results in

$$p^0 = R_1^0 R_2^1 p^2 \tag{2.16}$$

Note that R_1^0 and R_2^0 represent rotations relative to the frame $o_0x_0y_0z_0$ while R_2^1 represents a rotation relative to the frame $o_1x_1y_1z_1$. Comparing Equations (2.15) and (2.16) we can immediately infer

$$R_2^0 = R_1^0 R_2^1 \tag{2.17}$$

Equation (2.17) is the composition law for rotational transformations. It states that, in order to transform the coordinates of a point p from its representation

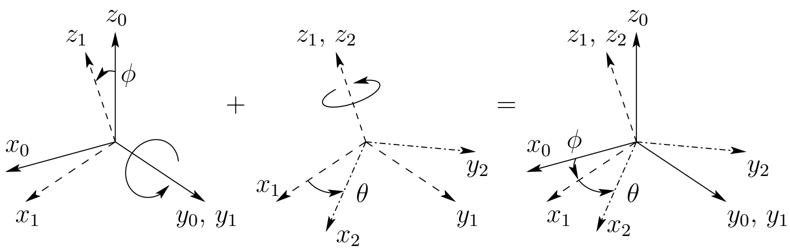

Fig. 2.9 Composition of rotations about current axes.

p^2 in the frame $o_2x_2y_2z_2$ to its representation p^0 in the frame $o_0x_0y_0z_0$, we may first transform to its coordinates p^1 in the frame $o_1x_1y_1z_1$ using R_2^1 and then transform p^1 to p^0 using R_1^0.

We may also interpret Equation (2.17) as follows. Suppose initially that all three of the coordinate frames coincide. We first rotate the frame $o_2x_2y_2z_2$ relative to $o_0x_0y_0z_0$ according to the transformation R_1^0. Then, with the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ coincident, we rotate $o_2x_2y_2z_2$ relative to $o_1x_1y_1z_1$ according to the transformation R_2^1. In each case we call the frame relative to which the rotation occurs the **current frame**.

Example 2.5

Suppose a rotation matrix R represents a rotation of angle ϕ about the current y-axis followed by a rotation of angle θ about the current z-axis. Refer to Figure 2.9. Then the matrix R is given by

$$
\begin{aligned}
R &= R_{y,\phi}R_{z,\theta} \qquad (2.18)\\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}
\end{aligned}
$$

◇

It is important to remember that the order in which a sequence of rotations are carried out, and consequently the order in which the rotation matrices are multiplied together, is crucial. The reason is that rotation, unlike position, is not a vector quantity and so rotational transformations do not commute in general.

Example 2.6

Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current

y-axis. Then the resulting rotation matrix is given by

$$
\begin{aligned}
R' &= R_{z,\theta}R_{y,\phi} \\
&= \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned} \tag{2.19}
$$

Comparing Equations (2.18) and (2.19) we see that $R \neq R'$.
◇

2.4.2 Rotation with respect to the fixed frame

Many times it is desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames. For example we may wish to perform a rotation about x_0 followed by a rotation about y_0 (and not y_1!). We will refer to $o_0x_0y_0z_0$ as the **fixed frame**. In this case the composition law given by Equation (2.17) is not valid. It turns out that the correct composition law in this case is simply to multiply the successive rotation matrices *in the reverse order* from that given by Equation (2.17). Note that the rotations themselves are not performed in reverse order. Rather they are performed about the fixed frame instead of about the current frame.

To see why this is so, suppose we have two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ related by the rotational transformation R_1^0. If $R \in SO(3)$ represents a rotation relative to $o_0x_0y_0z_0$ we know from Section 2.3.1 that the representation for R in the **current** frame $o_1x_1y_1z_1$ is given by $(R_1^0)^{-1}RR_1^0$. Therefore, applying the composition law for rotations about the current axis yields

$$
R_2^0 = R_1^0\left[(R_1^0)^{-1}RR_1^0\right] = RR_1^0 \tag{2.20}
$$

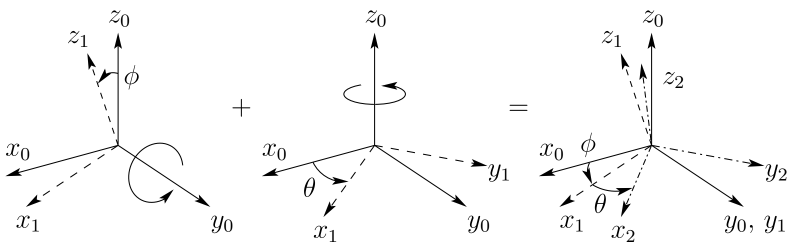

Fig. 2.10 Composition of rotations about fixed axes.

Example 2.7

Referring to Figure 2.10, suppose that a rotation matrix R represents a rotation of angle ϕ about y_0 followed by a rotation of angle θ about the fixed z_0.

The second rotation about the fixed axis is given by $R_{y,-\phi}R_{z,\theta}R_{y,\phi}$, which is the basic rotation about the z-axis expressed relative to the frame $o_1x_1y_1z_1$ using a similarity transformation. Therefore, the composition rule for rotational transformations gives us

$$
\begin{aligned}
p^0 &= R_{y,\phi}p^1 \\
&= R_{y,\phi}\left[R_{y,-\phi}R_{z,\theta}R_{y,\phi}\right]p^2 \\
&= R_{z,\theta}R_{y,\phi}p^2
\end{aligned}
\tag{2.21}
$$

It is not necessary to remember the above derivation, only to note by comparing Equation (2.21) with Equation (2.18) that we obtain the same basic rotation matrices, but in the reverse order.

◇

Summary
We can summarize the rule of composition of rotational transformations by the following recipe. Given a fixed frame $o_0x_0y_0z_0$ a current frame $o_1x_1y_1z_1$, together with rotation matrix R_1^0 relating them, if a third frame $o_2x_2y_2z_2$ is obtained by a rotation R performed relative to the **current frame** then **post-multiply** R_1^0 by $R = R_2^1$ to obtain

$$R_2^0 \quad = \quad R_1^0 R_2^1 \tag{2.22}$$

If the second rotation is to be performed relative to the **fixed frame** then it is both confusing and inappropriate to use the notation R_2^1 to represent this rotation. Therefore, if we represent the rotation by R, we **premultiply** R_1^0 by R to obtain

$$R_2^0 \quad = \quad R R_1^0 \tag{2.23}$$

In each case R_2^0 represents the transformation between the frames $o_0x_0y_0z_0$ and $o_2x_2y_2z_2$. The frame $o_2x_2y_2z_2$ that results in Equation (2.22) will be different from that resulting from Equation (2.23).

Using the above rule for composition of rotations, it is an easy matter to determine the result of multiple sequential rotational transformations.

Example 2.8
Suppose R is defined by the following sequence of basic rotations in the order specified:

1. *A rotation of θ about the current x-axis*
2. *A rotation of ϕ about the current z-axis*
3. *A rotation of α about the fixed z-axis*
4. *A rotation of β about the current y-axis*
5. *A rotation of δ about the fixed x-axis*

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or post-multiply as the case may be to obtain

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.24}$$

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2.5 PARAMETERIZATIONS OF ROTATIONS

The nine elements r_{ij} in a general rotational transformation R are not independent quantities. Indeed a rigid body possesses at most three rotational

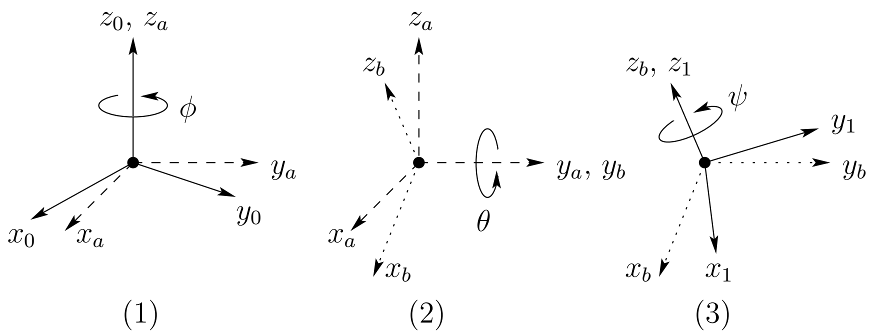

Fig. 2.11 Euler angle representation.

degrees-of-freedom and thus at most three quantities are required to specify its orientation. This can be easily seen by examining the constraints that govern the matrices in $SO(3)$:

$$\sum_i r_{ij}^2 = 1, \quad j \in \{1,2,3\} \tag{2.25}$$

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j \tag{2.26}$$

Equation (2.25) follows from the fact the the columns of a rotation matrix are unit vectors, and Equation (2.26) follows from the fact that columns of a rotation matrix are mutually orthogonal. Together, these constraints define six independent equations with nine unknowns, which implies that there are three free variables.

In this section we derive three ways in which an arbitrary rotation can be represented using only three independent quantities: the **Euler Angle** representation, the **roll-pitch-yaw** representation, and the **axis/angle** representation.

2.5.1 Euler Angles

A common method of specifying a rotation matrix in terms of three independent quantities is to use the so-called **Euler Angles**. Consider the fixed coordinate frame $o_0x_0y_0z_0$ and the rotated frame $o_1x_1y_1z_1$ shown in Figure 2.11. We can specify the orientation of the frame $o_1x_1y_1z_1$ relative to the frame $o_0x_0y_0z_0$ by three angles (ϕ, θ, ψ), known as Euler Angles, and obtained by three successive rotations as follows: First rotate about the z-axis by the angle ϕ. Next rotate about the current y-axis by the angle θ. Finally rotate about the current z-axis by the angle ψ. In Figure 2.11, frame $o_ax_ay_az_a$ represents the new coordinate frame after the rotation by ϕ, frame $o_bx_by_bz_b$ represents the new coordinate frame after the rotation by θ, and frame $o_1x_1y_1z_1$ represents the final frame, after the rotation by ψ. Frames $o_ax_ay_az_a$ and $o_bx_by_bz_b$ are shown in the figure only to help you visualize the rotations.

In terms of the basic rotation matrices the resulting rotational transformation R_1^0 can be generated as the product

$$\begin{aligned} R_{ZYZ} &= R_{z,\phi}R_{y,\theta}R_{z,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \end{aligned} \tag{2.27}$$

The matrix R_{ZYZ} in Equation (2.27) is called the ZYZ-**Euler Angle Transformation**.

The more important and more difficult problem is the following: Given a matrix $R \in SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

determine a set of Euler angles ϕ, θ, and ψ so that

$$R = R_{ZYZ} \tag{2.28}$$

This problem will be important later when we address the inverse kinematics problem for manipulators. In order to find a solution for this problem we break it down into two cases.

First, suppose that not both of r_{13}, r_{23} are zero. Then from Equation (2.28) we deduce that $s_\theta \neq 0$, and hence that not both of r_{31}, r_{32} are zero. If not both r_{13} and r_{23} are zero, then $r_{33} \neq \pm 1$, and we have $c_\theta = r_{33}$, $s_\theta = \pm\sqrt{1 - r_{33}^2}$ so

$$\theta = \text{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \tag{2.29}$$

or

$$\theta = \text{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right) \tag{2.30}$$

where the function atan2 is the **two-argument arctangent function** defined in Appendix A.

If we choose the value for θ given by Equation (2.29), then $s_\theta > 0$, and

$$\phi = \text{atan2}(r_{13}, r_{23}) \tag{2.31}$$

$$\psi = \text{atan2}(-r_{31}, r_{32}) \tag{2.32}$$

If we choose the value for θ given by Equation (2.30), then $s_\theta < 0$, and

$$\phi = \text{atan2}(-r_{13}, -r_{23}) \tag{2.33}$$

$$\psi = \text{atan2}(r_{31}, -r_{32}) \tag{2.34}$$

Thus there are two solutions depending on the sign chosen for θ.

If $r_{13} = r_{23} = 0$, then the fact that R is orthogonal implies that $r_{33} = \pm 1$, and that $r_{31} = r_{32} = 0$. Thus R has the form

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{2.35}$$

If $r_{33} = 1$, then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$. In this case Equation (2.27) becomes

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the sum $\phi + \psi$ can be determined as

$$\begin{aligned} \phi + \psi &= \operatorname{atan2}(r_{11}, r_{21}) \\ &= \operatorname{atan2}(r_{11}, -r_{12}) \end{aligned} \tag{2.36}$$

Since only the sum $\phi + \psi$ can be determined in this case there are infinitely many solutions. In this case, we may take $\phi = 0$ by convention. If $r_{33} = -1$, then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$. In this case Equation (2.27) becomes

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.37}$$

The solution is thus

$$\phi - \psi = \operatorname{atan2}(-r_{11}, -r_{12}) \tag{2.38}$$

As before there are infinitely many solutions.

2.5.2 Roll, Pitch, Yaw Angles

A rotation matrix R can also be described as a product of successive rotations about the principal coordinate axes x_0, y_0, and z_0 taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote ϕ, θ, ψ, and which are shown in Figure 2.12.

We specify the order of rotation as $x - y - z$, in other words, first a yaw about x_0 through an angle ψ, then pitch about the y_0 by an angle θ, and finally roll about the z_0 by an angle ϕ[5]. Since the successive rotations are relative to

[5]It should be noted that other conventions exist for naming the roll, pitch and yaw angles.

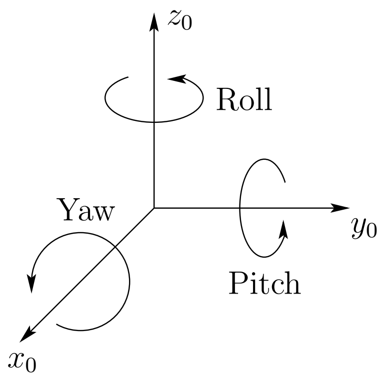

Fig. 2.12 Roll, pitch, and yaw angles.

the fixed frame, the resulting transformation matrix is given by

$$\begin{aligned} R_{XYZ} &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix} \end{aligned} \tag{2.39}$$

Of course, instead of yaw-pitch-roll relative to the fixed frames we could also interpret the above transformation as roll-pitch-yaw, in that order, each taken with respect to the current frame. The end result is the same matrix as in Equation (2.39).

The three angles, ϕ, θ, ψ, can be obtained for a given rotation matrix using a method that is similar to that used to derive the Euler angles above. We leave this as an exercise for the reader.

2.5.3 Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices. Let $k = (k_x, k_y, k_z)^T$, expressed in the frame $o_0x_0y_0z_0$, be a unit vector defining an axis. We wish to derive the rotation matrix $R_{k,\theta}$ representing a rotation of θ about this axis.

There are several ways in which the matrix $R_{k,\theta}$ can be derived. Perhaps the simplest way is to note that the axis define by the vector k is along the z-axis following the rotational transformation $R_1^0 = R_{z,\alpha}R_{y,\beta}$. Therefore, a rotation about the axis k can be computed using a similarity transformation as

$$R_{k,\theta} = R_1^0 R_{z,\theta} R_1^{0^{-1}} \tag{2.40}$$

$$= R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha} \tag{2.41}$$

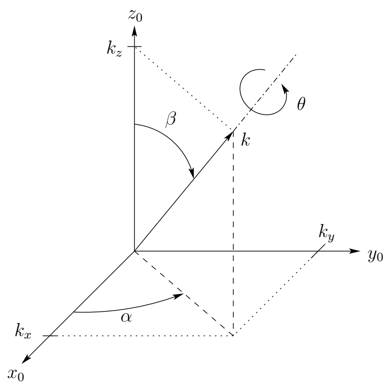

Fig. 2.13 Rotation about an arbitrary axis.

From Figure 2.13, we see that

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \tag{2.42}$$

$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \tag{2.43}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2} \tag{2.44}$$

$$\cos\beta = k_z \tag{2.45}$$

Note that the final two equations follow from the fact that k is a unit vector. Substituting Equations (2.42)-(2.45) into Equation (2.41) we obtain after some lengthy calculation (Problem 2-17)

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right] \tag{2.46}$$

where $v_\theta = \text{ vers } \theta = 1 - c_\theta$.

In fact, any rotation matrix $R \in S0(3)$ can be represented by a single rotation about a suitable axis in space by a suitable angle,

$$R = R_{k,\theta} \tag{2.47}$$

where k is a unit vector defining the axis of rotation, and θ is the angle of rotation about k. The matrix $R_{k,\theta}$ given in Equation (2.47) is called the **axis/angle representation** of R. Given an arbitrary rotation matrix R with components r_{ij}, the equivalent angle θ and equivalent axis k are given by the expressions

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)\end{aligned} \tag{2.48}$$

where Tr denotes the trace of R, and

$$k = \frac{1}{2\sin\theta}\left[\begin{array}{c} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12}\end{array}\right] \tag{2.49}$$

These equations can be obtained by direct manipulation of the entries of the matrix given in Equation (2.46). The axis/angle representation is not unique since a rotation of $-\theta$ about $-k$ is the same as a rotation of θ about k, that is,

$$R_{k,\theta} = R_{-k,-\theta} \tag{2.50}$$

If $\theta = 0$ then R is the identity matrix and the axis of rotation is undefined.

Example 2.9

Suppose R is generated by a rotation of 90° about z_0 followed by a rotation of 30° about y_0 followed by a rotation of 60° about x_0. Then

$$\begin{aligned}R &= R_{x,60}R_{y,30}R_{z,90} \\ &= \left[\begin{array}{ccc} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4}\end{array}\right]\end{aligned} \tag{2.51}$$

We see that $Tr(R) = 0$ and hence the equivalent angle is given by Equation (2.48) as

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \tag{2.52}$$

The equivalent axis is given from Equation (2.49) as

$$k = \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right)^T \tag{2.53}$$

◇

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis k and the equivalent angle θ. However, since the equivalent axis k is given as a unit vector only

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Fig. 2.14 Homogeneous transformations in two dimensions.

two of its components are independent. The third is constrained by the condition that k is of unit length. Therefore, only three independent quantities are required in this representation of a rotation R. We can represent the equivalent axis/angle by a single vector r as

$$r \quad = \quad (r_x, r_y, r_z)^T = (\theta k_x, \theta k_y, \theta k_z)^T \tag{2.54}$$

Note, since k is a unit vector, that the length of the vector r is the equivalent angle θ and the direction of r is the equivalent axis k.

Remark 2.1 *One should be careful not to interpret the representation in Equation (2.54) to mean that two axis/angle representations may be combined using standard rules of vector algebra as doing so would imply that rotations commute which, as we have seen, in not true in general.*

2.6 RIGID MOTIONS

We have seen how to represent both positions and orientations. We combine these two concepts in this section to define a **rigid motion** and, in the next section, we derive an efficient matrix representation for rigid motions using the notion of homogeneous transformation.

Definition 2.1 *A rigid motion is an ordered pair (d, R) where $d \in \mathbb{R}^3$ and $R \in SO(3)$. The group of all rigid motions is known as the* **Special Euclidean Group** *and is denoted by $SE(3)$. We see then that $SE(3) = \mathbb{R}^3 \times SO(3)$.*[a]

[a]The definition of rigid motion is sometimes broadened to include **reflections**, which correspond to $\det R = -1$. We will always assume in this text that $\det R = +1$, i.e. that $R \in SO(3)$.

A rigid motion is a pure translation together with a pure rotation. Referring to Figure 2.14 we see that if frame $o_1x_1y_1z_1$ is obtained from frame $o_0x_0y_0z_0$ by first applying a rotation specified by R_1^0 followed by a translation given (with respect to $o_0x_0y_0z_0$) by d_1^0, then the coordinates p^0 are given by

$$p^0 = R_1^0 p^1 + d_1^0 \tag{2.55}$$

Two points are worth noting in this figure. First, note that we cannot simply add the vectors p^0 and p^1 since they are defined relative to frames with different orientations, i.e. with respect to frames that are not parallel. However, we are able to add the vectors p^1 and $R_1^0 p^1$ precisely because multiplying p^1 by the orientation matrix R_1^0 expresses p^1 in a frame that is parallel to frame $o_0x_0y_0z_0$. Second, it is not important in which order the rotation and translation are performed.

If we have the two rigid motions

$$p^0 = R_1^0 p^1 + d_1^0 \tag{2.56}$$

and

$$p^1 = R_2^1 p^2 + d_2^1 \tag{2.57}$$

then their composition defines a third rigid motion, which we can describe by substituting the expression for p^1 from Equation (2.57) into Equation (2.56)

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \tag{2.58}$$

Since the relationship between p^0 and p^2 is also a rigid motion, we can equally describe it as

$$p^0 = R_2^0 p^2 + d_2^0 \tag{2.59}$$

Comparing Equations (2.58) and (2.59) we have the relationships

$$R_2^0 = R_1^0 R_2^1 \tag{2.60}$$
$$d_2^0 = d_1^0 + R_1^0 d_2^1 \tag{2.61}$$

Equation (2.60) shows that the orientation transformations can simply be multiplied together and Equation (2.61) shows that the vector from the origin o_0 to the origin o_2 has coordinates given by the sum of d_1^0 (the vector from o_0 to o_1 expressed with respect to $o_0x_0y_0z_0$) and $R_1^0 d_2^1$ (the vector from o_1 to o_2, expressed in the orientation of the coordinate system $o_0x_0y_0z_0$).

2.7 HOMOGENEOUS TRANSFORMATIONS

One can easily see that the calculation leading to Equation (2.58) would quickly become intractable if a long sequence of rigid motions were considered. In this section we show how rigid motions can be represented in matrix form so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

In fact, a comparison of Equations (2.60) and (2.61) with the matrix identity

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_1^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_1^2 + d_1^0 \\ 0 & 1 \end{bmatrix} \tag{2.62}$$

where 0 denotes the row vector $(0, 0, 0)$, shows that the rigid motions can be represented by the set of matrices of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; R \in SO(3),\ d \in \mathbb{R}^3 \tag{2.63}$$

Transformation matrices of the form given in Equation (2.63) are called **homogeneous transformations**. A homogeneous transformation is therefore nothing more than a matrix representation of a rigid motion and we will use $SE(3)$ interchangeably to represent both the set of rigid motions and the set of all 4×4 matrices H of the form given in Equation (2.63)

Using the fact that R is orthogonal it is an easy exercise to show that the inverse transformation H^{-1} is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \tag{2.64}$$

In order to represent the transformation given in Equation (2.55) by a matrix multiplication, we must augment the vectors p^0 and p^1 by the addition of a fourth component of 1 as follows,

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} \tag{2.65}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \tag{2.66}$$

The vectors P^0 and P^1 are known as **homogeneous representations** of the vectors p^0 and p^1, respectively. It can now be seen directly that the transformation given in Equation (2.55) is equivalent to the (homogeneous) matrix equation

$$P^0 = H_1^0 P^1 \tag{2.67}$$

A set of **basic homogeneous transformations** generating $SE(3)$ is given by

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.68}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.69}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad Rot_{x,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.70}$$

for translation and rotation about the x, y, z-axes, respectively.

The most general homogeneous transformation that we will consider may be written now as

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_x & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.71}$$

In the above equation $n = (n_x, n_y, n_z)^T$ is a vector representing the direction of x_1 in the $o_0x_0y_0z_0$ system, $s = (s_x, s_y, s_z)^T$ represents the direction of y_1, and $a = (a_x, a_y, a_z)^T$ represents the direction of z_1. The vector $d = (d_x, d_y, d_z)^T$ represents the vector from the origin o_0 to the origin o_1 expressed in the frame $o_0x_0y_0z_0$. The rationale behind the choice of letters n, s and a is explained in Chapter 3.

Composition Rule for Homogeneous Transformations

The same interpretation regarding composition and ordering of transformations holds for 4×4 homogeneous transformations as for 3×3 rotations. Given a homogeneous transformation H_1^0 relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$

Example 2.10

The homogeneous transformation matrix H that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, *followed by a rotation by angle θ about the current z-axis, is given by*

$$
\begin{aligned}
H &= Rot_{x,\alpha}Trans_{x,b}Trans_{z,d}Rot_{z,\theta} \\
&= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

◇

The homogeneous representation given in Equation (2.63) is a special case of homogeneous coordinates, which have been extensively used in the field of computer graphics. There, one is interested in scaling and/or perspective transformations in addition to translation and rotation. The most general homogeneous transformation takes the form

$$
H = \left[\begin{array}{c|c} R_{3\times3} & d_{3\times1} \\ \hline f_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} Rotation & Translation \\ \hline perspective & scale\ factor \end{array}\right] \tag{2.72}
$$

For our purposes we always take the last row vector of H to be $(0, 0, 0, 1)$, although the more general form given by (2.72) could be useful, for example, for interfacing a vision system into the overall robotic system or for graphic simulation.

2.8 CHAPTER SUMMARY

In this chapter, we have seen how matrices in $SE(n)$ can be used to represent the relative position and orientation of two coordinate frames for $n = 2, 3$. We have adopted a notional convention in which a superscript is used to indicate a reference frame. Thus, the notation p^0 represents the coordinates of the point p relative to frame 0.

The relative orientation of two coordinate frames can be specified by a rotation matrix, $R \in SO(n)$, with $n = 2, 3$. In two dimensions, the orientation of frame 1 with respect to frame 0 is given by

$$
R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

in which θ is the angle between the two coordinate frames. In the three dimensional case, the rotation matrix is given by

$$
R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}
$$

In each case, the columns of the rotation matrix are obtained by projecting an axis of the target frame (in this case, frame 1) onto the coordinate axes of the reference frame (in this case, frame 0).

The set of $n \times n$ rotation matrices is known as the special orthogonal group of order n, and is denoted by $SO(n)$. An important property of these matrices is that $R^{-1} = R^T$ for any $R \in SO(n)$.

Rotation matrices can be used to perform coordinate transformations between frames that differ only in orientation. We derived rules for the composition of rotational transformations as

$$R_2^0 = R_1^0 R$$

for the case where the second transformation, R, is performed relative to the current frame and

$$R_2^0 = R R_1^0$$

for the case where the second transformation, R, is performed relative to the fixed frame.

In the three dimensional case, a rotation matrix can be parameterized using three angles. A common convention is to use the Euler angles (ϕ, θ, ψ), which correspond to successive rotations about the z, y and z axes. The corresponding rotation matrix is given by

$$R(\phi, \theta, \psi) = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

Roll, pitch and yaw angles are similar, except that the successive rotations are performed with respect to the fixed, world frame instead of being performed with respect to the current frame.

Homogeneous transformations combine rotation and translation. In the three dimensional case, a homogeneous transformation has the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; R \in SO(3),\ d \in \mathbb{R}^3$$

The set of all such matrices comprises the set $SE(3)$, and these matrices can be used to perform coordinate transformations, analogous to rotational transformations using rotation matrices.

The interested reader can find deeper explanations of these concepts in a variety of sources, including [4] [18] [29] [62] [54] [75].

1. Using the fact that $v_1 \cdot v_2 = v_1^T v_2$, show that the dot product of two free vectors does not depend on the choice of frames in which their coordinates are defined.

2. Show that the length of a free vector is not changed by rotation, i.e., that $\|v\| = \|Rv\|$.

3. Show that the distance between points is not changed by rotation i.e., that $\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$.

4. If a matrix R satisfies $R^T R = I$, show that the column vectors of R are of unit length and mutually perpendicular.

5. If a matrix R satisfies $R^T R = I$, then
   a) show that $\det R = \pm 1$
   b) Show that $\det R = \pm 1$ if we restrict ourselves to right-handed coordinate systems.

6. Verify Equations (2.3)-(2.5).

7. A **group** is a set X together with an operation $*$ defined on that set such that

   - $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
   - $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
   - There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$.
   - For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$.

   Show that SO(n) with the operation of matrix multiplication is a group.

8. Derive Equations (2.6) and (2.7).

9. Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

10. Consider the following sequence of rotations:

    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the current z-axis.
    (c) Rotate by ψ about the world y-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

11. Consider the following sequence of rotations:
    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the world z-axis.
    (c) Rotate by ψ about the current x-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

12. Consider the following sequence of rotations:
    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the current z-axis.
    (c) Rotate by ψ about the current x-axis.
    (d) Rotate by α about the world z-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

13. Consider the following sequence of rotations:
    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the world z-axis.
    (c) Rotate by ψ about the current x-axis.
    (d) Rotate by α about the world z-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

14. Find the rotation matrix representing a roll of $\frac{\pi}{4}$ followed by a yaw of $\frac{\pi}{2}$ followed by a pitch of $\frac{\pi}{2}$.

15. If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

16. Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$ and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}; R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

    Find the matrix R_3^2.

17. Verify Equation (2.46).

18. If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

19. Let $k = \frac{1}{\sqrt{3}}(1,1,1)^T$, $\theta = 90°$. Find $R_{k,\theta}$.

20. Show by direct calculation that $R_{k,\theta}$ given by Equation (2.46) is equal to R given by Equation (2.51) if θ and k are given by Equations (2.52) and (2.53), respectively.

21. Compute the rotation matrix given by the product

$$R_{x,\theta} R_{y,\phi} R_{z,\pi} R_{y,-\phi} R_{x,-\theta}$$

22. Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

23. Find the rotation matrix corresponding to the set of Euler angles $\left\{\frac{\pi}{2}, 0, \frac{\pi}{4}\right\}$. What is the direction of the x_1 axis relative to the base frame?

24. Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

25. Unit magnitude complex numbers (i.e., $a + ib$ such that $a^2 + b^2 = 1$) can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

26. Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

27. Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned} -1 &= i^2 = j^2 = k^2, \\ i &= jk = -kj, \\ j &= ki = -ik, \\ k &= ij = -ji \end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = (n_x, n_y, n_z)^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, i.e., that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

28. Using $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$, and the results from Section 2.5.3, determine the rotation matrix R that corresponds to the rotation represented by the quaternion (q_0, q_1, q_2, q_3).

29. Determine the quaternion Q that represents the same rotation as given by the rotation matrix R.

30. The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $= (q_1, q_2, q_3)^T$. Show that the product of two quaternions, $Z = XY$ is given by

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

Hint: perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

31. Show that $Q_I = (1, 0, 0, 0)$ is the identity element for unit quaternion multiplication, i.e., that $QQ_I = Q_I Q = Q$ for any unit quaternion Q.

32. The conjugate Q^* of the quaternion Q is defined as

$$Q^* = (q_0, -q_1, -q_2, -q_3)$$

Show that Q^* is the inverse of Q, i.e., that $Q^* Q = QQ^* = (1, 0, 0, 0)$.

33. Let v be a vector whose coordinates are given by $(v_x, v_y, v_z)^T$. If the quaternion Q represents a rotation, show that the new, rotated coordinates of v are given by $Q(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component.

34. Let the point p be rigidly attached to the end effector coordinate frame with local coordinates (x, y, z). If Q specifies the orientation of the end effector frame with respect to the base frame, and T is the vector from the base frame to the origin of the end effector frame, show that the coordinates of p with respect to the base frame are given by

$$Q(0, x, y, z)Q^* + T \tag{2.73}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component.

35. Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin O_1 with respect to the original frame in each case?

36. Consider the diagram of Figure 2.15. Find the homogeneous transfor-

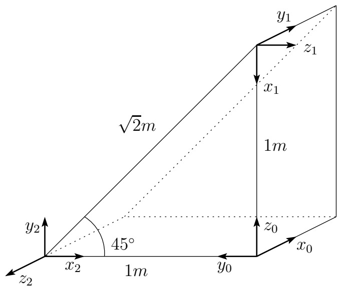

Fig. 2.15 Diagram for Problem 2-36.

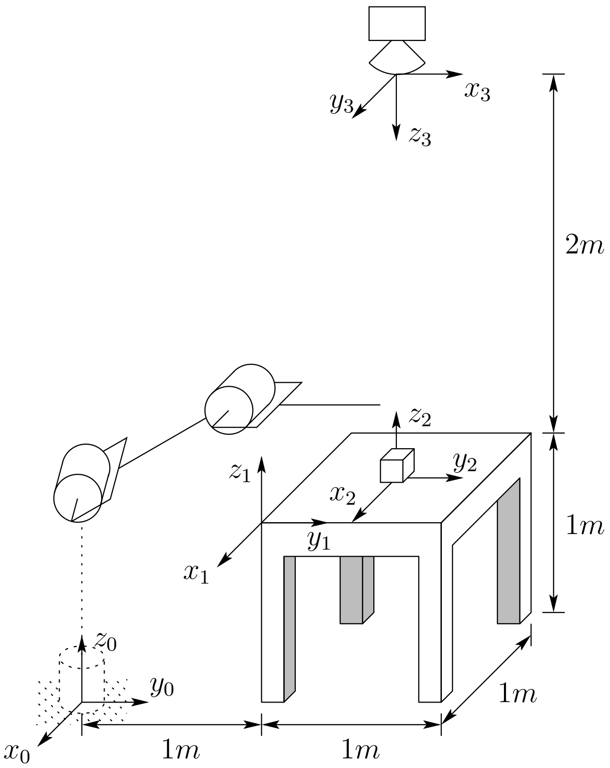

Fig. 2.16 Diagram for Problem 2-37.

mations H_1^0, H_2^0, H_2^1 representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0, H_2^1$.

37. Consider the diagram of Figure 2.16. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a

side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2m above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

38. In Problem 37, suppose that, after the camera is calibrated, it is rotated $90°$ about z_3. Recompute the above coordinate transformations.

39. If the block on the table is rotated $90°$ about z_2 and moved so that its center has coordinates $(0, .8, .1)^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.

40. Consult an astronomy book to learn the basic details of the Earth's rotation about the sun and about its own axis. Define for the Earth a local coordinate frame whose z-axis is the Earth's axis of rotation. Define $t = 0$ to be the exact moment of the summer solstice, and the global reference frame to be coincident with the Earth's frame at time $t = 0$. Give an expression $R(t)$ for the rotation matrix that represents the instantaneous orientation of the earth at time t. Determine as a function of time the homogeneous transformation that specifies the Earth's frame with respect to the global reference frame.

41. In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product

$$H = Rot_{x,\alpha}\text{Trans}_{x,b}\text{Trans}_{z,d}Rot_{z,\theta}$$

Determine which pairs of the four matrices on the right hand side commute. Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix, H.

3

FORWARD AND INVERSE KINEMATICS

In this chapter we consider the forward and inverse kinematics for serial link manipulators. The problem of kinematics is to describe the motion of the manipulator without consideration of the forces and torques causing the motion. The kinematic description is therefore a geometric one. We first consider the problem of *forward kinematics*, which is to determine the position and orientation of the end-effector given the values for the joint variables of the robot. The *inverse kinematics* problem is to determine the values of the joint variables given the end-effector position and orientation.

3.1 KINEMATIC CHAINS

As described in Chapter 1, a robot manipulator is composed of a set of links connected together by joints. The joints can either be very simple, such as a revolute joint or a prismatic joint, or they can be more complex, such as a ball and socket joint. (Recall that a revolute joint is like a hinge and allows a relative rotation about a single axis, and a prismatic joint permits a linear motion along a single axis, namely an extension or retraction.) The difference between the two situations is that, in the first instance, the joint has only a single degree-of-freedom of motion: the angle of rotation in the case of a revolute joint, and the amount of linear displacement in the case of a prismatic joint. In contrast, a ball and socket joint has two degrees-of-freedom. In this book it is assumed throughout that all joints have only a single degree-of-freedom. This assumption does not involve any real loss of generality, since joints such as a ball and socket joint (two degrees-of-freedom) or a spherical wrist (three degrees-of-freedom) can always be thought of as a succession of single degree-of-freedom joints with links of length zero in between.

With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number; the angle of rotation in the case of a revolute joint or the displacement in the case of a prismatic joint. The objective of forward kinematic analysis is to determine the *cumulative* effect of the entire set of joint variables, that is, to determine the position and orientation of the end effector given the values of these joint variables. The objective of inverse kinematic analysis is, in contrast, to determine the values for these joint variables given the position and orientation of the end effector frame.

A robot manipulator with n joints will have $n + 1$ links, since each joint connects two links. We number the joints from 1 to n, and we number the links from 0 to n, starting from the base. By this convention, joint i connects link $i - 1$ to link i. We will consider the location of joint i to be fixed with respect to link $i - 1$. *When joint i is actuated, link i moves.* Therefore, link 0 (the first link) is fixed, and does not move when the joints are actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted on a mobile platform or on an autonomous vehicle), but we will not consider this case in the present chapter, since it can be handled easily by slightly extending the techniques presented here.

With the i^{th} joint, we associate a *joint variable*, denoted by q_i. In the case of a revolute joint, q_i is the angle of rotation, and in the case of a prismatic joint, q_i is the joint displacement:

$$q_i = \left\{ \begin{array}{ll} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{array} \right. \tag{3.1}$$

To perform the kinematic analysis, we attach a coordinate frame rigidly to each link. In particular, we attach $o_ix_iy_iz_i$ to link i. This means that, whatever motion the robot executes, the coordinates of each point on link i are constant when expressed in the i^{th} coordinate frame. Furthermore, when joint i is actuated, link i and its attached frame, $o_ix_iy_iz_i$, experience a resulting motion. The frame $o_0x_0y_0z_0$, which is attached to the robot base, is referred to as the inertial frame. Figure 3.1 illustrates the idea of attaching frames rigidly to links in the case of an elbow manipulator.

Now suppose A_i is the homogeneous transformation matrix that expresses the position and orientation of $o_ix_iy_iz_i$ with respect to $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. The matrix A_i is not constant, but varies as the configuration of the robot is changed. However, the assumption that all joints are either revolute or prismatic means that A_i is a function of only a single joint variable, namely q_i. In other words,

$$A_i \quad = \quad A_i(q_i) \tag{3.2}$$

Now the homogeneous transformation matrix that expresses the position and orientation of $o_jx_jy_jz_j$ with respect to $o_ix_iy_iz_i$ is called, by convention, a **trans-**

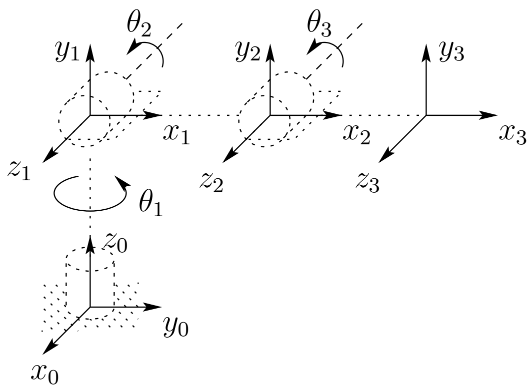

Fig. 3.1 Coordinate frames attached to elbow manipulator

formation matrix, and is denoted by T_j^i. From Chapter 2 we see that

$$T_j^i = \begin{cases} A_{i+1}A_{i+2}\dots A_{j-1}A_j & \text{if } i < j \\ I & \text{if } i = j \\ (T_i^j)^{-1} & \text{if } j > i \end{cases} \tag{3.3}$$

By the manner in which we have rigidly attached the various frames to the corresponding links, it follows that the position of any point on the end-effector, when expressed in frame n, is a constant independent of the configuration of the robot. Denote the position and orientation of the end-effector with respect to the inertial or base frame by a three-vector o_n^0 (which gives the coordinates of the origin of the end-effector frame with respect to the base frame) and the 3×3 rotation matrix R_n^0, and define the homogeneous transformation matrix

$$H = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix} \tag{3.4}$$

Then the position and orientation of the end-effector in the inertial frame are given by

$$H = T_n^0 = A_1(q_1)\cdots A_n(q_n) \tag{3.5}$$

Each homogeneous transformation A_i is of the form

$$A_i = \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

Hence

$$T_j^i = A_{i+1}\cdots A_j = \begin{bmatrix} R_j^i & o_j^i \\ 0 & 1 \end{bmatrix} \tag{3.7}$$

The matrix R^i_j expresses the orientation of $o_j x_j y_j z_j$ relative to $o_i x_i y_i z_i$ and is given by the rotational parts of the A-matrices as

$$R^i_j = R^i_{i+1} \cdots R^{j-1}_j \tag{3.8}$$

The coordinate vectors o^i_j are given recursively by the formula

$$o^i_j = o^i_{j-1} + R^i_{j-1} o^{j-1}_j \tag{3.9}$$

These expressions will be useful in Chapter 4 when we study Jacobian matrices.

In principle, that is all there is to forward kinematics; determine the functions $A_i(q_i)$, and multiply them together as needed. However, it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, such as the Denavit-Hartenberg representation of a joint, and this is the objective of the next section.

3.2 FORWARD KINEMATICS: THE DENAVIT-HARTENBERG CONVENTION

In this section we develop the **forward** or **configuration kinematic equations** for rigid robots. The forward kinematics problem is concerned with the relationship between the individual joints of the robot manipulator and the position and orientation of the tool or end-effector. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints.

We will develop a set of conventions that provide a systematic procedure for performing this analysis. It is, of course, possible to carry out forward kinematics analysis even without respecting these conventions, as we did for the two-link planar manipulator example in Chapter 1. However, the kinematic analysis of an n-link manipulator can be extremely complex and the conventions introduced below simplify the analysis considerably. Moreover, they give rise to a universal language with which robot engineers can communicate.

A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg, or DH convention. In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations

$$\begin{aligned} A_i &= Rot_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}Rot_{x,\alpha_i} \qquad (3.10) \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where the four quantities θ_i, a_i, d_i, α_i are parameters associated with link i and joint i. The four parameters a_i, α_i, d_i, and θ_i in (3.10) are generally given the names **link length**, **link twist**, **link offset**, and **joint angle**, respectively. These names derive from specific aspects of the geometric relationship between two coordinate frames, as will become apparent below. Since the matrix A_i is a function of a single variable, it turns out that three of the above four quantities are constant for a given link, while the fourth parameter, θ_i for a revolute joint and d_i for a prismatic joint, is the joint variable.

From Chapter 2 one can see that an arbitrary homogeneous transformation matrix can be characterized by six numbers, such as, for example, three numbers to specify the fourth column of the matrix and three Euler angles to specify the upper left 3×3 rotation matrix. In the DH representation, in contrast, there are only *four* parameters. How is this possible? The answer is that, while frame i is required to be rigidly attached to link i, we have considerable freedom in choosing the origin and the coordinate axes of the frame. For example, it is not necessary that the origin, o_i, of frame i be placed at the physical end of link i. In fact, it is not even necessary that frame i be placed within the physical link; frame i could lie in free space — so long as frame i is *rigidly attached* to link i. By a clever choice of the origin and the coordinate axes, it is possible to cut down the number of parameters needed from six to four (or even fewer in some cases). In Section 3.2.1 we will show why, and under what conditions, this can be done, and in Section 3.2.2 we will show exactly how to make the coordinate frame assignments.

3.2.1 Existence and uniqueness issues

Clearly it is not possible to represent any arbitrary homogeneous transformation using only four parameters. Therefore, we begin by determining just which homogeneous transformations can be expressed in the form (3.10). Suppose we

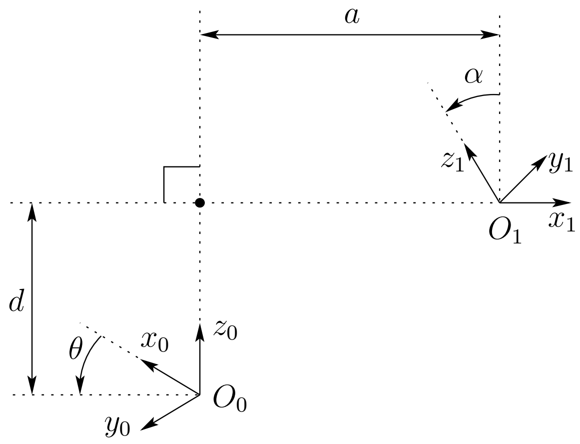

Fig. 3.2 Coordinate frames satisfying assumptions DH1 and DH2

are given two frames, denoted by frames 0 and 1, respectively. Then there exists a unique homogeneous transformation matrix A that takes the coordinates from frame 1 into those of frame 0. Now suppose the two frames have the following two additional features.

DH Coordinate Frame Assumptions

(DH1) The axis x_1 is perpendicular to the axis z_0.

(DH2) The axis x_1 intersects the axis z_0.

These two properties are illustrated in Figure 3.2. Under these conditions, we claim that there exist unique numbers a, d, θ, α such that

$$A = Rot_{z,\theta}\mathrm{Trans}_{z,d}\mathrm{Trans}_{x,a}Rot_{x,\alpha} \tag{3.11}$$

Of course, since θ and α are angles, we really mean that they are unique to within a multiple of 2π. To show that the matrix A can be written in this form, write A as

$$A = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix} \tag{3.12}$$

If (DH1) is satisfied, then x_1 is perpendicular to z_0 and we have $x_1 \cdot z_0 = 0$. Expressing this constraint with respect to $o_0x_0y_0z_0$, using the fact that the first column of R_1^0 is the representation of the unit vector x_1 with respect to frame 0, we obtain

$$\begin{aligned} 0 &= x_1^0 \cdot z_0^0 \\ &= [r_{11}, r_{21}, r_{31}] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = r_{31} \end{aligned}$$

Since $r_{31} = 0$, we now need only show that there exist *unique* angles θ and α such that

$$R_1^0 \quad = \quad R_{x,\theta} R_{x,\alpha} = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \tag{3.13}$$

The only information we have is that $r_{31} = 0$, but this is enough. First, since each row and column of R_1^0 must have unit length, $r_{31} = 0$ implies that

$$\begin{aligned} r_{11}^2 + r_{21}^2 &= 1, \\ r_{32}^2 + r_{33}^2 &= 1 \end{aligned}$$

Hence there exist unique θ and α such that

$$(r_{11}, r_{21}) = (c_\theta, s_\theta), \qquad (r_{33}, r_{32}) = (c_\alpha, s_\alpha)$$

Once θ and α are found, it is routine to show that the remaining elements of R_1^0 must have the form shown in (3.13), using the fact that R_1^0 is a rotation matrix.

Next, assumption (DH2) means that the displacement between o_0 and o_1 can be expressed as a linear combination of the vectors z_0 and x_1. This can be written as $o_1 = o_0 + dz_0 + ax_1$. Again, we can express this relationship in the coordinates of $o_0x_0y_0z_0$, and we obtain

$$\begin{aligned} o_1^0 &= o_0^0 + dz_0^0 + ax_1^0 \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} ac_\theta \\ as_\theta \\ d \end{bmatrix} \end{aligned}$$

Combining the above results, we obtain (3.10) as claimed. Thus, we see that four parameters are sufficient to specify any homogeneous transformation that satisfies the constraints (DH1) and (DH2).

Now that we have established that each homogeneous transformation matrix satisfying conditions (DH1) and (DH2) above can be represented in the form (3.10), we can in fact give a physical interpretation to each of the four quantities in (3.10). The parameter a is the distance between the axes z_0 and z_1, and is measured along the axis x_1. The angle α is the angle between the axes z_0 and z_1, measured in a plane normal to x_1. The positive sense for α is determined from z_0 to z_1 by the right-handed rule as shown in Figure 3.3. The parameter d is the perpendicular distance from the origin o_0 to the intersection of the x_1 axis with z_0 measured along the z_0 axis. Finally, θ is the angle between x_0 and x_1 measured in a plane normal to z_0. These physical interpretations will prove useful in developing a procedure for assigning coordinate frames that satisfy the constraints (DH1) and (DH2), and we now turn our attention to developing such a procedure.

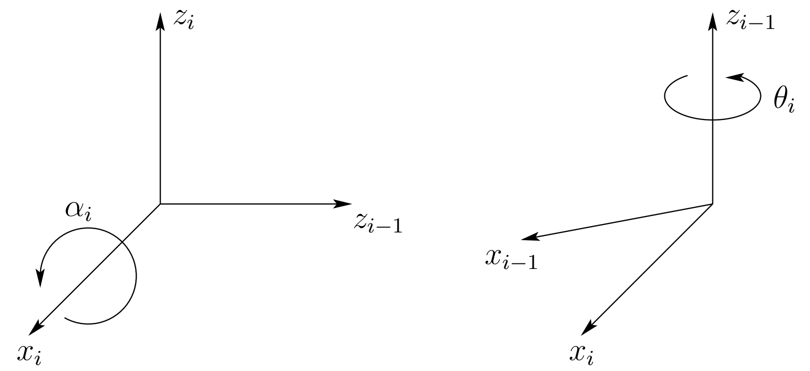

Fig. 3.3 Positive sense for α_i and θ_i

3.2.2 Assigning the coordinate frames

For a given robot manipulator, one can always choose the frames $0, \ldots, n$ in such a way that the above two conditions are satisfied. In certain circumstances, this will require placing the origin o_i of frame i in a location that may not be intuitively satisfying, but typically this will not be the case. In reading the material below, it is important to keep in mind that the choices of the various coordinate frames are not unique, even when constrained by the requirements above. Thus, it is possible that different engineers will derive differing, but equally correct, coordinate frame assignments for the links of the robot. It is very important to note, however, that the end result (i.e., the matrix T_n^0) will be the same, regardless of the assignment of intermediate link frames (assuming that the coordinate frames for link n coincide). We will begin by deriving the general procedure. We will then discuss various common special cases where it is possible to further simplify the homogeneous transformation matrix.

To start, note that the choice of z_i is arbitrary. In particular, from (3.13), we see that by choosing α_i and θ_i appropriately, we can obtain any arbitrary direction for z_i. Thus, for our first step, we assign the axes $z_0, \ldots, z_{n-1}$ in an intuitively pleasing fashion. Specifically, we assign z_i to be the axis of actuation for joint $i+1$. Thus, z_0 is the axis of actuation for joint 1, z_1 is the axis of actuation for joint 2, etc. There are two cases to consider: (i) if joint $i+1$ is revolute, z_i is the axis of revolution of joint $i+1$; (ii) if joint $i+1$ is prismatic, z_i is the axis of translation of joint $i+1$. At first it may seem a bit confusing to associate z_i with joint $i+1$, but recall that this satisfies the convention that we established above, namely that joint i is fixed with respect to frame i, and that when joint i is actuated, link i and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion.

Once we have established the z-axes for the links, we establish the base frame. The choice of a base frame is nearly arbitrary. We may choose the origin o_0 of the base frame to be any point on z_0. We then choose x_0, y_0 in any convenient manner so long as the resulting frame is right-handed. This sets up frame 0.

Once frame 0 has been established, we begin an iterative process in which we define frame i using frame $i - 1$, beginning with frame 1. Figure 3.4 will be useful for understanding the process that we now describe.

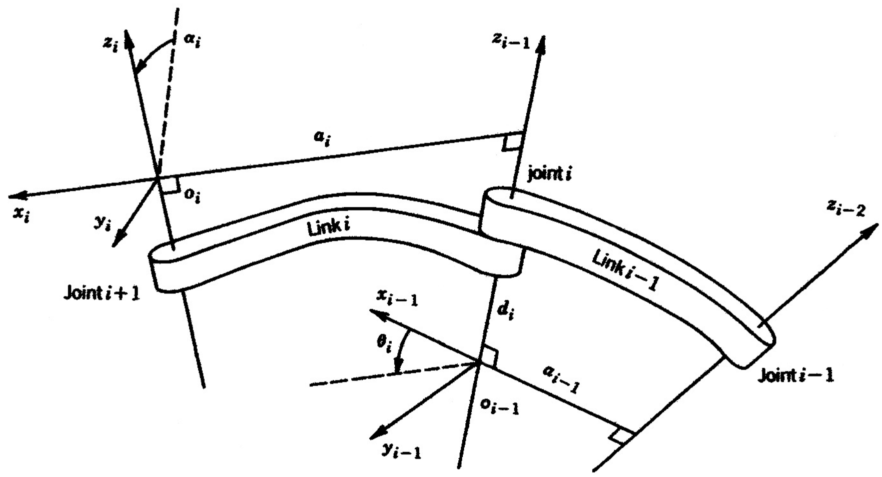

Fig. 3.4 Denavit-Hartenberg frame assignment

In order to set up frame i it is necessary to consider three cases: (i) the axes z_{i-1}, z_i are not coplanar, (ii) the axes z_{i-1}, z_i intersect (iii) the axes z_{i-1}, z_i are parallel. Note that in both cases (ii) and (iii) the axes z_{i-1} and z_i are coplanar. This situation is in fact quite common, as we will see in Section 3.2.3. We now consider each of these three cases.

(i) z_{i-1} and z_i are not coplanar: If z_{i-l} and z_i are not coplanar, then there exists a *unique* line segment perpendicular to both z_{i-1} and z_i such that it connects both lines and it has minimum length. The line containing this common normal to z_{i-1} and z_i defines x_i, and the point where this line intersects z_i is the origin o_i. By construction, both conditions (DH1) and (DH2) are satisfied and the vector from o_{i-1} to o_i is a linear combination of z_{i-1} and x_i. The specification of frame i is completed by choosing the axis y_i to form a right-handed frame. Since assumptions (DH1) and (DH2) are satisfied the homogeneous transformation matrix A_i is of the form (3.10).

(ii) z_{i-1} is parallel to z_i: If the axes z_{i-1} and z_i are parallel, then there are infinitely many common normals between them and condition (DH1) does not specify x_i completely. In this case we are free to choose the origin o_i anywhere along z_i. One often chooses o_i to simplify the resulting equations. The axis x_i is then chosen either to be directed from o_i toward z_{i-1}, along the common

normal, or as the opposite of this vector. A common method for choosing o_i is to choose the normal that passes through o_{i-1} as the x_i axis; o_i is then the point at which this normal intersects z_i. In this case, d_i would be equal to zero. Once x_i is fixed, y_i is determined, as usual by the right hand rule. Since the axes z_{i-1} and z_i are parallel, α_i will be zero in this case.

(iii) z_{i-1} intersects z_i: In this case x_i is chosen normal to the plane formed by z_i and z_{i-1}. The positive direction of x_i is arbitrary. The most natural choice for the origin o_i in this case is at the point of intersection of z_i and z_{i-1}. However, any convenient point along the axis z_i suffices. Note that in this case the parameter a_i equals 0.

This constructive procedure works for frames $0, \ldots, n-1$ in an n-link robot. To complete the construction, it is necessary to specify frame n. The final coordinate system $o_n x_n y_n z_n$ is commonly referred to as the **end-effector** or **tool frame** (see Figure 3.5). The origin o_n is most often placed symmetrically between the fingers of the gripper. The unit vectors along the x_n, y_n, and z_n axes are labeled as n, s, and a, respectively. The terminology arises from fact that the direction a is the **approach** direction, in the sense that the gripper typically approaches an object along the a direction. Similarly the s direction is the **sliding** direction, the direction along which the fingers of the gripper slide to open and close, and n is the direction **normal** to the plane formed by a and s.

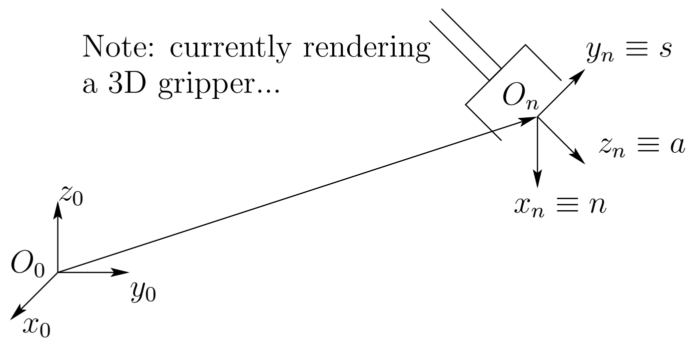

Fig. 3.5 Tool frame assignment

In most contemporary robots the final joint motion is a rotation of the end-effector by θ_n and the final two joint axes, z_{n-1} and z_n, coincide. In this case, the transformation between the final two coordinate frames is a translation along z_{n-1} by a distance d_n followed (or preceded) by a rotation of θ_n about z_{n-1}. This is an important observation that will simplify the computation of the inverse kinematics in the next section.

Finally, note the following important fact. In all cases, whether the joint in question is revolute or prismatic, the quantities a_i and α_i are always constant for all i and are characteristic of the manipulator. If joint i is prismatic, then

θ_i is also a constant, while d_i is the i^{th} joint variable. Similarly, if joint i is revolute, then d_i is constant and θ_i is the i^{th} joint variable.

3.2.3 Examples

In the DH convention the only variable angle is θ, so we simplify notation by writing c_i for $\cos\theta_i$, etc. We also denote $\theta_1+\theta_2$ by θ_{12}, and $\cos(\theta_1+\theta_2)$ by c_{12}, and so on. In the following examples it is important to remember that the DH convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

Example 3.1 Planar Elbow Manipulator

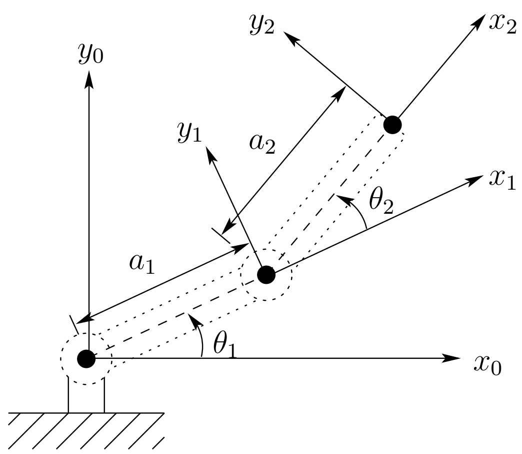

Fig. 3.6 Two-link planar manipulator. The z-axes all point out of the page, and are not shown in the figure

Consider the two-link planar arm of Figure 3.6. The joint axes z_0 and z_1 are normal to the page. We establish the base frame $o_0x_0y_0z_0$ as shown. The origin is chosen at the point of intersection of the z_0 axis with the page and the direction of the x_0 axis is completely arbitrary. Once the base frame is established, the $o_1x_1y_1z_1$ frame is fixed as shown by the DH convention, where the origin o_1 has been located at the intersection of z_1 and the page. The final frame $o_2x_2y_2z_2$ is fixed by choosing the origin o_2 at the end of link 2 as shown. The DH parameters are shown in Table 3.1. The A-matrices are determined

Table 3.1 Link parameters for 2-link planar manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

from (3.10) as

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The T-matrices are thus given by

$$T_1^0 = A_1$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the first two entries of the last column of T_2^0 are the x and y components of the origin o_2 in the base frame; that is,

$$x = a_1c_1 + a_2c_{12}$$
$$y = a_1s_1 + a_2s_{12}$$

are the coordinates of the end-effector in the base frame. The rotational part of T_2^0 gives the orientation of the frame $o_2x_2y_2z_2$ relative to the base frame.
$\diamond$

Example 3.2 Three-Link Cylindrical Robot *Consider now the three-link cylindrical robot represented symbolically by Figure 3.7. We establish o_0 as shown at joint 1. Note that the placement of the origin o_0 along z_0 as well as the direction of the x_0 axis are arbitrary. Our choice of o_0 is the most natural, but o_0 could just as well be placed at joint 2. The axis x_0 is chosen normal to the page. Next, since z_0 and z_1 coincide, the origin o_1 is chosen at joint 1 as shown. The x_1 axis is normal to the page when $\theta_1 = 0$ but, of course its*

Table 3.2 Link parameters for 3-link cylindrical manipulator

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

direction will change since θ_1 is variable. Since z_2 and z_1 intersect, the origin o_2 is placed at this intersection. The direction of x_2 is chosen parallel to x_1 so that θ_2 is zero. Finally, the third frame is chosen at the end of link 3 as shown. The DH parameters are shown in Table 3.2. The corresponding A and

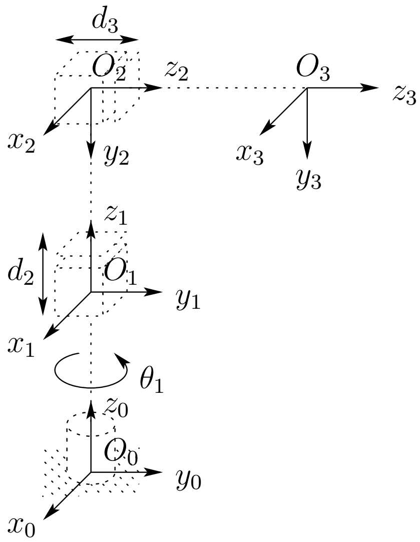

Fig. 3.7 Three-link cylindrical manipulator

T matrices are

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

◇

Example 3.3 Spherical Wrist

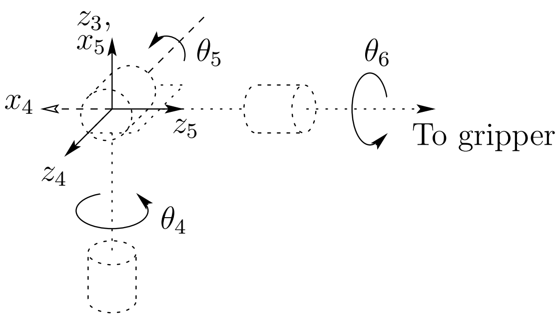

Fig. 3.8 The spherical wrist frame assignment

The spherical wrist configuration is shown in Figure 3.8, in which the joint axes z_3, z_4, z_5 intersect at o. The DH parameters are shown in Table 3.3. The Stanford manipulator is an example of a manipulator that possesses a wrist of this type.

We show now that the final three joint variables, θ_4, θ_5, θ_6 are the Euler angles ϕ, θ, ψ, respectively, with respect to the coordinate frame $o_3x_3y_3z_3$. To see this we need only compute the matrices A_4, A_5, and A_6 using Table 3.3 and

Table 3.3 DH parameters for spherical wrist

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

the expression (3.10). This gives

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying these together yields

$$\begin{aligned} T_6^3 &= A_4A_5A_6 \\ &= \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{3.15}$$

Comparing the rotational part R_6^3 of T_6^3 with the Euler angle transformation (2.27) shows that $\theta_4, \theta_5, \theta_6$ can indeed be identified as the Euler angles ϕ, θ and ψ with respect to the coordinate frame $o_3x_3y_3z_3$.

$\diamond$

Example 3.4 Cylindrical Manipulator with Spherical Wrist

Suppose that we now attach a spherical wrist to the cylindrical manipulator of Example 3.2 as shown in Figure 3.9. Note that the axis of rotation of joint 4

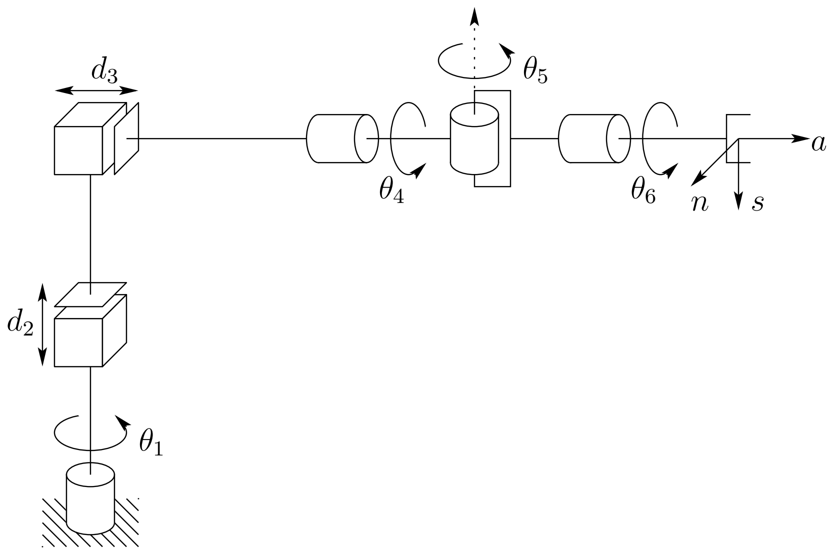

Fig. 3.9 Cylindrical robot with spherical wrist

is parallel to z_2 and thus coincides with the axis z_3 of Example 3.2. The implication of this is that we can immediately combine the two previous expression (3.14) and (3.15) to derive the forward kinematics as

$$T_6^0 = T_3^0 T_6^3 \tag{3.16}$$

with T_3^0 given by (3.14) and T_6^3 given by (3.15). Therefore the forward kinematics of this manipulator is described by

$$T_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.17}$$

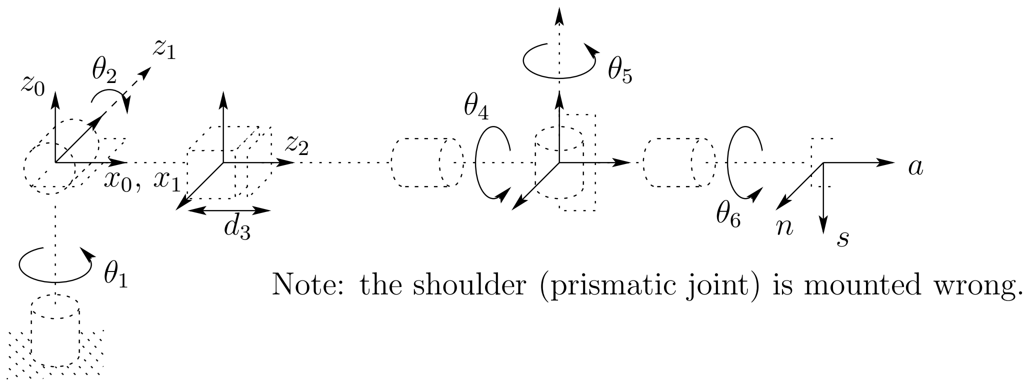

Fig. 3.10 DH coordinate frame assignment for the Stanford manipulator

in which

$$
\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2
\end{aligned}
$$

Notice how most of the complexity of the forward kinematics for this manipulator results from the orientation of the end-effector while the expression for the arm position from (3.14) is fairly simple. The spherical wrist assumption not only simplifies the derivation of the forward kinematics here, but will also greatly simplify the inverse kinematics problem in the next chapter.
◇

Example 3.5 Stanford Manipulator

Consider now the Stanford Manipulator shown in Figure 3.10. This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

Table 3.4 DH parameters for Stanford Manipulator

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\theta^\star$
2	d_2	0	$+90$	$\theta^\star$
3	$d^\star$	0	0	0
4	0	0	-90	$\theta^\star$
5	0	0	$+90$	$\theta^\star$
6	d_6	0	0	$\theta^\star$

* joint variable

We first establish the joint coordinate frames using the DH convention as shown. The DH parameters are shown in the Table 3.4.

It is straightforward to compute the matrices A_i as

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.18}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.19}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.20}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.21}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.22}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.23}$$

T_6^0 is then given as

$$
\begin{aligned}
T_6^0 &= A_1 \cdots A_6 & (3.24)\\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} & (3.25)
\end{aligned}
$$

where

$$
\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6)\\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6)\\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6\\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6)\\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6)\\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6\\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5\\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5\\
r_{33} &= -s_2c_4s_5 + c_2c_5\\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5)\\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2)\\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}
$$

◇

Example 3.6 SCARA Manipulator

As another example of the general procedure, consider the SCARA manipulator of Figure 3.11. This manipulator, which is an abstraction of the AdeptOne robot of Figure 1.14, consists of an RRP arm and a one degree-of-freedom wrist, whose motion is a roll about the vertical axis. The first step is to locate and label the joint axes as shown. Since all joint axes are parallel we have some freedom in the placement of the origins. The origins are placed as shown for convenience. We establish the x_0 axis in the plane of the page as shown. This is completely arbitrary and only affects the zero configuration of the manipulator, that is, the position of the manipulator when $\theta_1 = 0$.

The joint parameters are given in Table 3.5, and the A-matrices are as fol-

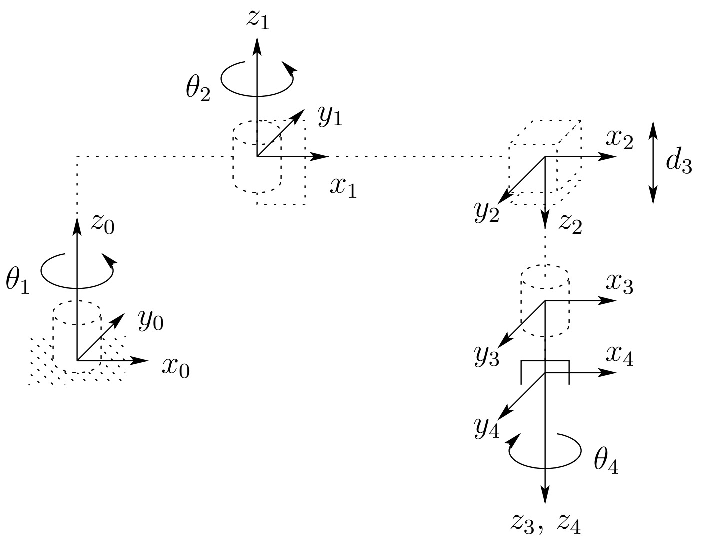

Fig. 3.11 DH coordinate frame assignment for the SCARA manipulator

Table 3.5 Joint parameters for SCARA

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$

* joint variable

lows.

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.26}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.27}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.28}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.29}$$

The forward kinematic equations are therefore given by

$$\begin{aligned} T_4^0 &= A_1 \cdots A_4 \\ &= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{3.30}$$

◇

3.3 INVERSE KINEMATICS

In the previous section we showed how to determine the end-effector position and orientation in terms of the joint variables. This section is concerned with the inverse problem of finding the joint variables in terms of the end-effector position and orientation. This is the problem of **inverse kinematics**, and it is, in general, more difficult than the forward kinematics problem.

In this chapter, we begin by formulating the general inverse kinematics problem. Following this, we describe the principle of kinematic decoupling and how it can be used to simplify the inverse kinematics of most modern manipulators. Using kinematic decoupling, we can consider the position and orientation problems independently. We describe a geometric approach for solving the positioning problem, while we exploit the Euler angle parameterization to solve the orientation problem.

3.3.1 The General Inverse Kinematics Problem

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \tag{3.31}$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \ldots, q_n) = H \tag{3.32}$$

where

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n) \tag{3.33}$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \ldots, q_n$ so that $T_n^0(q_1, \ldots, q_n) = H$.

Equation (3.32) results in twelve nonlinear equations in n unknown variables, which can be written as

$$T_{ij}(q_1, \ldots, q_n) = h_{ij}, \qquad i = 1, 2, 3, \quad j = 1, \ldots, 4 \tag{3.34}$$

where T_{ij}, h_{ij} refer to the twelve nontrivial entries of T_n^0 and H, respectively. (Since the bottom row of both T_n^0 and H are (0,0,0,1), four of the sixteen equations represented by (3.32) are trivial.)

Example 3.7

Recall the Stanford manipulator of Example 3.3.5. Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.35}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations:

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

If the values of the nonzero DH parameters are $d_2 = 0.154$ and $d_6 = 0.263$, one solution to this set of equations is given by:

$$\theta_1 = \pi/2, \quad \theta_2 = \pi/2, \quad d_3 = 0.5, \quad \theta_4 = \pi/2, \quad \theta_5 = 0, \quad \theta_6 = \pi/2.$$

Even though we have not yet seen how one might derive this solution, it is not difficult to verify that it satisfies the forward kinematics equations for the Stanford arm.

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The equations in the preceding example are, of course, much too difficult to solve directly in closed form. This is the case for most robot arms. Therefore, we need to develop efficient and systematic techniques that exploit the particular kinematic structure of the manipulator. Whereas the forward kinematics problem always has a unique solution that can be obtained simply by evaluating the forward equations, the inverse kinematics problem may or may not have a solution. Even if a solution exists, it may or may not be unique. Furthermore, because these forward kinematic equations are in general complicated nonlinear functions of the joint variables, the solutions may be difficult to obtain even when they exist.

In solving the inverse kinematics problem we are most interested in finding a closed form solution of the equations rather than a numerical solution. Finding a closed form solution means finding an explicit relationship:

$$q_k = f_k(h_{11}, \ldots, h_{34}), \qquad k = 1, \ldots, n \tag{3.36}$$

Closed form solutions are preferable for two reasons. First, in certain applications, such as tracking a welding seam whose location is provided by a vision system, the inverse kinematic equations must be solved at a rapid rate, say every 20 milliseconds, and having closed form expressions rather than an iterative search is a practical necessity. Second, the kinematic equations in general have multiple solutions. Having closed form solutions allows one to develop rules for choosing a particular solution among several.

The practical question of the existence of solutions to the inverse kinematics problem depends on engineering as well as mathematical considerations. For example, the motion of the revolute joints may be restricted to less than a full 360 degrees of rotation so that not all mathematical solutions of the kinematic equations will correspond to physically realizable configurations of the manipulator. We will assume that the given position and orientation is such that at least one solution of (3.32) exists. Once a solution to the mathematical equations is identified, it must be further checked to see whether or not it satisfies all constraints on the ranges of possible joint motions. For our purposes, we henceforth assume that the given homogeneous matrix H in (3.32) corresponds to a configuration within the manipulator's workspace with an attainable orientation. This guarantees that the mathematical solutions obtained correspond to achievable configurations.

3.3.2 Kinematic Decoupling

Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as **inverse position kinematics**, and **inverse orientation kinematics**. To put it another way, for a six-DOF manipulator with a spherical wrist, the inverse kinematics problem may be separated into two simpler problems, namely first finding the position of the intersection of the wrist axes, hereafter called the **wrist center**, and then finding the orientation of the wrist.

For concreteness let us suppose that there are exactly six degrees-of-freedom and that the last three joint axes intersect at a point o_c. We express (3.32) as two sets of equations representing the rotational and positional equations

$$R_6^0(q_1, \ldots, q_6) = R \tag{3.37}$$
$$o_6^0(q_1, \ldots, q_6) = o \tag{3.38}$$

where o and R are the desired position and orientation of the tool frame, expressed with respect to the world coordinate system. Thus, we are given o and R, and the inverse kinematics problem is to solve for $q_1, \ldots, q_6$.

The assumption of a spherical wrist means that the axes z_3, z_4, and z_5 intersect at o_c and hence the origins o_4 and o_5 assigned by the DH-convention will always be at the wrist center o_c. Often o_3 will also be at o_c, but this is not necessary for our subsequent development. The important point of this assumption for the inverse kinematics is that motion of the final three links about these axes will not change the position of o_c, and thus, the position of the wrist center is thus a function of only the first three joint variables.

The origin of the tool frame (whose desired coordinates are given by o) is simply obtained by a translation of distance d_6 along z_5 from o_c (see Table 3.3). In our case, z_5 and z_6 are the same axis, and the third column of R expresses the direction of z_6 with respect to the base frame. Therefore, we have

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.39}$$

Thus in order to have the end-effector of the robot at the point with coordinates given by o and with the orientation of the end-effector given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center o_c have coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.40}$$

and that the orientation of the frame $o_6x_6y_6z_6$ with respect to the base be given by R. If the components of the end-effector position o are denoted o_x, o_y, o_z and the components of the wrist center o_c^0 are denoted x_c, y_c, z_c then (3.40) gives the relationship

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix} \tag{3.41}$$

Using Equation (3.41) we may find the values of the first three joint variables. This determines the orientation transformation R_3^0 which depends only on these first three joint variables. We can now determine the orientation of the end-effector relative to the frame $o_3x_3y_3z_3$ from the expression

$$R = R_3^0 R_6^3 \tag{3.42}$$

as

$$R_6^3 \quad = \quad (R_3^0)^{-1}R = (R_3^0)^T R \tag{3.43}$$

As we shall see in Section 3.3.4, the final three joint angles can then be found as a set of Euler angles corresponding to R_6^3. Note that the right hand side of (3.43) is completely known since R is given and R_3^0 can be calculated once the first three joint variables are known. The idea of kinematic decoupling is illustrated in Figure 3.12.

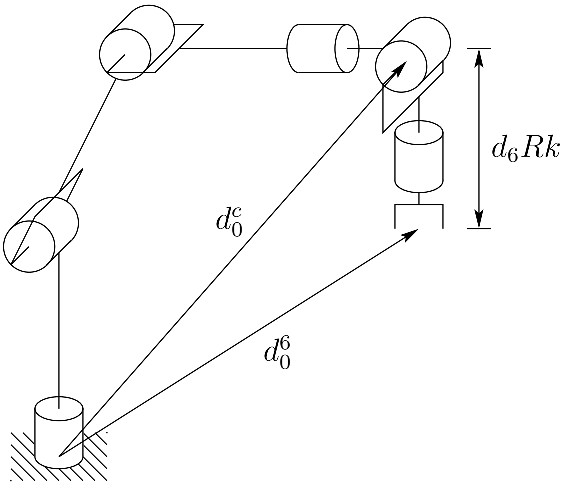

Fig. 3.12 Kinematic decoupling

3.3.3 Inverse Position: A Geometric Approach

For the common kinematic arrangements that we consider, we can use a geometric approach to find the variables, q_1, q_2, q_3 corresponding to o_c^0 given by (3.40). We restrict our treatment to the geometric approach for two reasons. First, as we have said, most present manipulator designs are kinematically simple, usually consisting of one of the five basic configurations of Chapter 1 with a spherical wrist. Indeed, it is partly due to the difficulty of the general inverse kinematics problem that manipulator designs have evolved to their present state. Second, there are few techniques that can handle the general inverse kinematics problem for arbitrary configurations. Since the reader is most likely to encounter robot configurations of the type considered here, the added difficulty involved in treating the general case seems unjustified. The interested reader can find more detailed treatment of the general case in [32] [34] [61] [72].

In general the complexity of the inverse kinematics problem increases with the number of nonzero link parameters. For most manipulators, many of the a_i, d_i are zero, the α_i are 0 or $\pm\pi/2$, etc. In these cases especially, a geometric approach is the simplest and most natural. The general idea of the geometric approach is to solve for joint variable q_i by projecting the manipulator onto the $x_{i-1} - y_{i-1}$ plane and solving a simple trigonometry problem. For example, to solve for θ_1, we project the arm onto the $x_0 - y_0$ plane and use trigonometry to find θ_1. We will illustrate this method with two important examples: the articulated and spherical arms.

3.3.3.1 Articulated Configuration Consider the elbow manipulator shown in Figure 3.13, with the components of o_c^0 denoted by x_c, y_c, z_c. We project o_c onto the $x_0 - y_0$ plane as shown in Figure 3.14.

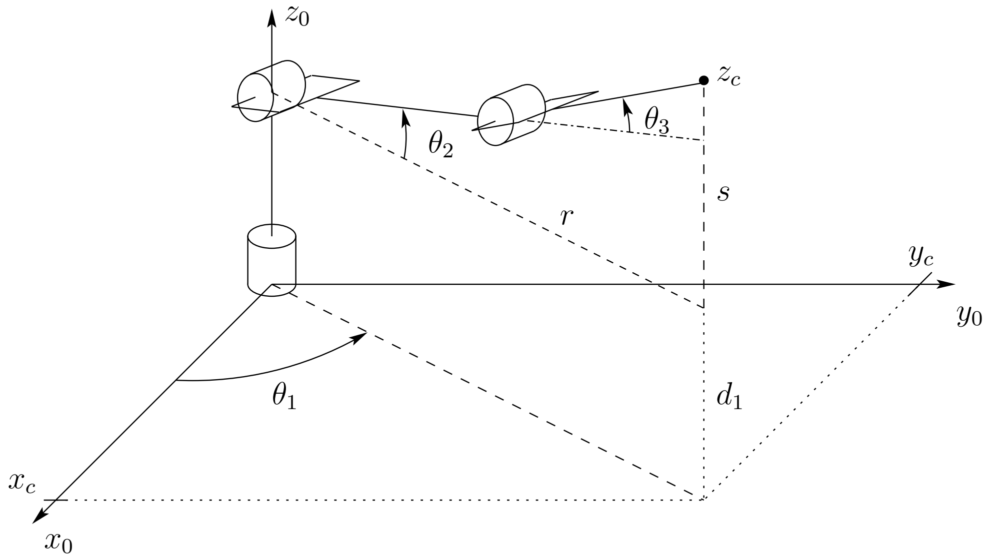

Fig. 3.13 Elbow manipulator

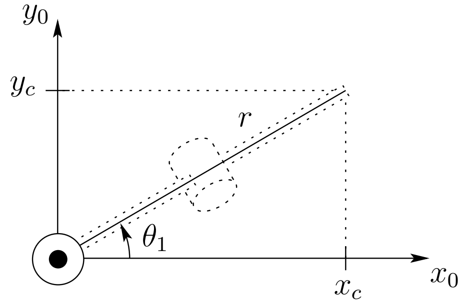

Fig. 3.14 Projection of the wrist center onto $x_0 - y_0$ plane

We see from this projection that

$$\theta_1 = \operatorname{atan2}(x_c, y_c) \tag{3.44}$$

in which $\operatorname{atan2}(x, y)$ denotes the two argument arctangent function defined in Chapter 2.

Note that a second valid solution for θ_1 is

$$\theta_1 = \pi + \operatorname{atan2}(x_c, y_c) \tag{3.45}$$

Of course this will, in turn, lead to different solutions for θ_2 and θ_3, as we will see below.

These solutions for θ_1, are valid unless $x_c = y_c = 0$. In this case (3.44) is undefined and the manipulator is in a singular configuration, shown in Figure 3.15. In this position the wrist center o_c intersects z_0; hence any value of θ_1 leaves o_c

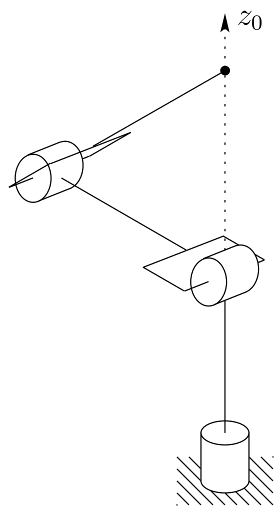

Fig. 3.15 Singular configuration

fixed. There are thus infinitely many solutions for θ_1 when o_c intersects z_0.

If there is an offset $d \neq 0$ as shown in Figure 3.16 then the wrist center cannot intersect z_0. In this case, depending on how the DH parameters have been assigned, we will have $d_2 = d$ or $d_3 = d$. In this case, there will, in general, be only two solutions for θ_1. These correspond to the so-called **left arm** and **right arm** configurations as shown in Figures 3.17 and 3.18. Figure 3.17 shows the left arm configuration. From this figure, we see geometrically that

$$\theta_1 = \phi - \alpha \tag{3.46}$$

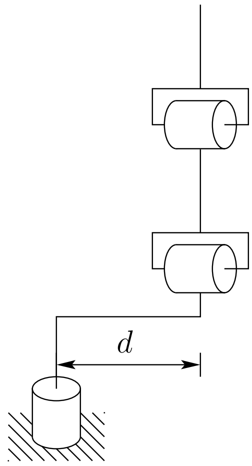

Fig. 3.16 Elbow manipulator with shoulder offset

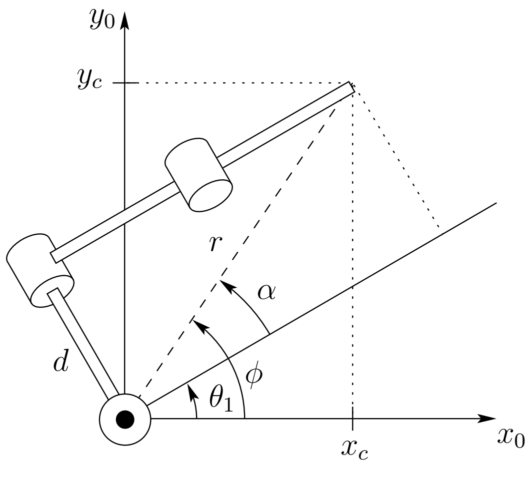

Fig. 3.17 Left arm configuration

where

$$\phi = \operatorname{atan2}(x_c, y_c) \tag{3.47}$$

$$\begin{aligned} \alpha &= \operatorname{atan2}\left(\sqrt{r^2 - d^2}, d\right) \\ &= \operatorname{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right) \end{aligned} \tag{3.48}$$

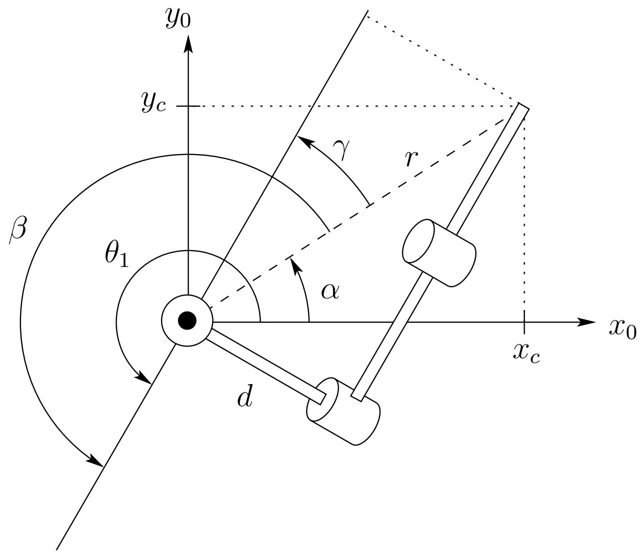

Fig. 3.18 Right arm configuration

The second solution, given by the right arm configuration shown in Figure 3.18 is given by

$$\theta_1 = \operatorname{atan2}(x_c, y_c) + \operatorname{atan2}\left(-\sqrt{r^2 - d^2}, -d\right) \tag{3.49}$$

To see this, note that

$$\begin{aligned}
\theta_1 &= \alpha + \beta & (3.50)\\
\alpha &= \operatorname{atan2}(x_c, y_c) & (3.51)\\
\beta &= \gamma + \pi & (3.52)\\
\gamma &= \operatorname{atan2}(\sqrt{r^2 - d^2}, d) & (3.53)
\end{aligned}$$

which together imply that

$$\beta = \operatorname{atan2}\left(-\sqrt{r^2 - d^2}, -d\right) \tag{3.54}$$

since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$.

To find the angles θ_2, θ_3 for the elbow manipulator, given θ_1, we consider the plane formed by the second and third links as shown in Figure 3.19. Since the motion of links two and three is planar, the solution is analogous to that of the two-link manipulator of Chapter 1. As in our previous derivation (cf. (1.7) and

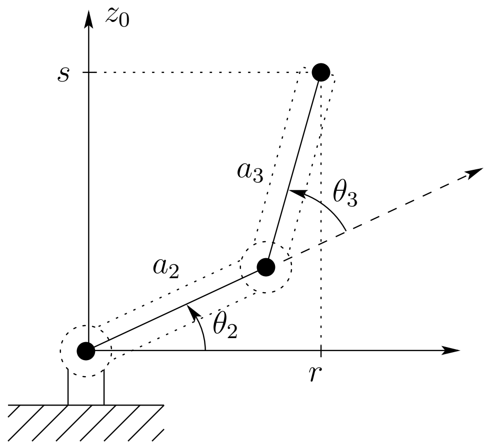

Fig. 3.19 Projecting onto the plane formed by links 2 and 3

(1.8)) we can apply the law of cosines to obtain

$$\begin{aligned} \cos\theta_3 &= \frac{r^2+s^2-a_2^2-a_3^2}{2a_2a_3} \qquad (3.55) \\ &= \frac{x_c^2+y_c^2-d^2+(z_c-d_1)^2-a_2^2-a_3^2}{2a_2a_3} := D \end{aligned}$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c - d_1$. Hence, θ_3 is given by

$$\theta_3 = \operatorname{atan2}\left(D, \pm\sqrt{1-D^2}\right) \qquad (3.56)$$

The two solutions for θ_3 correspond to the elbow-up position and elbow-down position, respectively.

Similarly θ_2 is given as

$$\begin{aligned} \theta_2 &= \operatorname{atan2}(r,s) - \operatorname{atan2}(a_2+a_3c_3, a_3s_3) \qquad (3.57) \\ &= \operatorname{atan2}\left(\sqrt{x_c^2+y_c^2-d^2}, z_c-d_1\right) - \operatorname{atan2}(a_2+a_3c_3, a_3s_3) \end{aligned}$$

An example of an elbow manipulator with offsets is the PUMA shown in Figure 3.20. There are four solutions to the inverse position kinematics as shown. These correspond to the situations left arm-elbow up, left arm–elbow down, right arm–elbow up and right arm–elbow down. We will see that there are two solutions for the wrist orientation thus giving a total of eight solutions of the inverse kinematics for the PUMA manipulator.

3.3.3.2 Spherical Configuration We next solve the inverse position kinematics for a three degree of freedom spherical manipulator shown in Figure 3.21. As

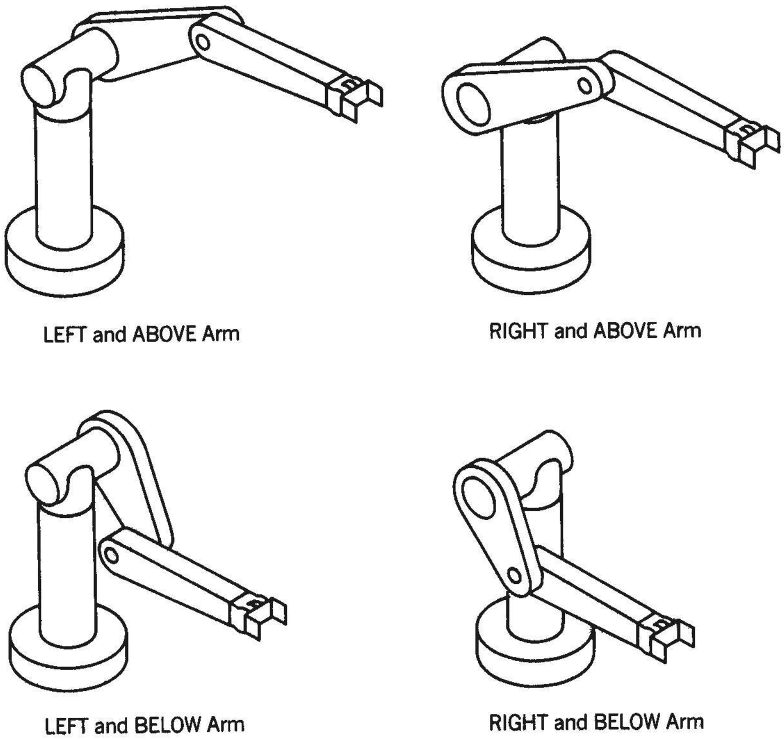

Fig. 3.20 Four solutions of the inverse position kinematics for the PUMA manipulator

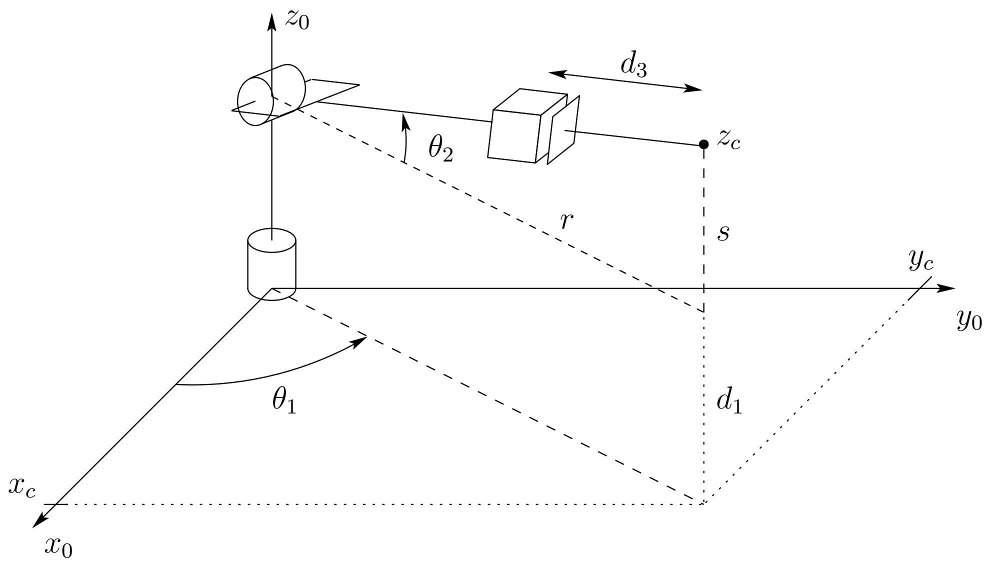

Fig. 3.21 Spherical manipulator

in the case of the elbow manipulator the first joint variable is the base rotation and a solution is given as

$$\theta_1 \quad = \quad \text{atan2}(x_c, y_c) \tag{3.58}$$

provided x_c and y_c are not both zero. If both x_c and y_c are zero, the configuration is singular as before and θ_1 may take on any value. As in the case of the elbow manipulator, a second solution for θ_1 is given by

$$\theta_1 = \pi + \text{atan2}(x_c, y_c). \tag{3.59}$$

The angle θ_2 is given from Figure 3.21 as

$$\theta_2 \quad = \quad \text{atan2}(r, s) + \frac{\pi}{2} \tag{3.60}$$

where $r^2 = x_c^2 + y_c^2$, $s = z_c - d_1$.

The linear distance d_3 is found as

$$d_3 = \sqrt{r^2 + s^2} \quad = \quad \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2} \tag{3.61}$$

The negative square root solution for d_3 is disregarded and thus in this case we obtain two solutions to the inverse position kinematics as long as the wrist center does not intersect z_0. If there is an offset then there will be left and right arm configurations as in the case of the elbow manipulator (Problem 3-25).

Table 3.6 Link parameters for the articulated manipulator of Figure 3.13

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*

* variable

3.3.4 Inverse Orientation

In the previous section we used a geometric approach to solve the inverse position problem. This gives the values of the first three joint variables corresponding to a given position of the wrist origin. The inverse orientation problem is now one of finding the values of the final three joint variables corresponding to a given orientation with respect to the frame $o_3x_3y_3z_3$. For a spherical wrist, this can be interpreted as the problem of finding a set of Euler angles corresponding to a given rotation matrix R. Recall that equation (3.15) shows that the rotation matrix obtained for the spherical wrist has the same form as the rotation matrix for the Euler transformation, given in (2.27). Therefore, we can use the method developed in Section 2.5.1 to solve for the three joint angles of the spherical wrist. In particular, we solve for the three Euler angles, ϕ, θ, ψ, using Equations (2.29) – (2.34), and then use the mapping

$$\begin{aligned} \theta_4 &= \phi \\ \theta_5 &= \theta \\ \theta_6 &= \psi \end{aligned}$$

Example 3.8 Articulated Manipulator with Spherical Wrist

The DH parameters for the frame assignment shown in Figure 3.13 are summarized in Table 3.6. Multiplying the corresponding A_i matrices gives the matrix R_3^0 for the articulated or elbow manipulator as

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix} \tag{3.62}$$

The matrix $R_6^3 = A_4A_5A_6$ is given as

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} \tag{3.63}$$

The equation to be solved for the final three variables is therefore

$$R_6^3 = (R_3^0)^T R \tag{3.64}$$

and the Euler angle solution can be applied to this equation. For example, the three equations given by the third column in the above matrix equation are given by

$$
\begin{aligned}
c_4 s_5 &= c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33} && (3.65)\\
s_4 s_5 &= -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33} && (3.66)\\
c_5 &= s_1 r_{13} - c_1 r_{23} && (3.67)
\end{aligned}
$$

Hence, if not both of the expressions (3.65), (3.66) are zero, we obtain θ_5 from (2.29) and (2.30) as

$$
\theta_5 = \mathrm{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right) \qquad (3.68)
$$

If the positive square root is chosen in (3.68), then θ_4 and θ_6 are given by (2.31) and (2.32), respectively, as

$$
\begin{aligned}
\theta_4 &= \mathrm{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33},\\
&\qquad -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) && (3.69)\\
\theta_6 &= \mathrm{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22}) && (3.70)
\end{aligned}
$$

The other solutions are obtained analogously. If $s_5 = 0$, then joint axes z_3 and z_5 are collinear. This is a singular configuration and only the sum $\theta_4 + \theta_6$ can be determined. One solution is to choose θ_4 arbitrarily and then determine θ_6 using (2.36) or (2.38).

◇

3.3.5 Examples

Example 3.9 Elbow Manipulator - Complete Solution

To summarize the geometric approach for solving the inverse kinematics equations, we write give here one solution to the inverse kinematics of the six degree-of-freedom elbow manipulator shown in Figure 3.13 which has no joint offsets and a spherical wrist.

Given

$$
o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}; \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad (3.71)
$$

then with

$$
\begin{aligned}
x_c &= o_x - d_6 r_{13} && (3.72)\\
y_c &= o_y - d_6 r_{23} && (3.73)\\
z_c &= o_z - d_6 r_{33} && (3.74)
\end{aligned}
$$

a set of DH joint variables is given by

$$\theta_1 = \text{atan2}(x_c, y_c) \quad (3.75)$$

$$\theta_2 = \text{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{atan2}(a_2 + a_3c_3, a_3s_3) \quad (3.76)$$

$$\theta_3 = \text{atan2}\left(D, \pm\sqrt{1 - D^2}\right),$$

$$\text{where } D = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} \quad (3.77)$$

$$\theta_4 = \text{atan2}(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \quad (3.78)$$

$$\theta_5 = \text{atan2}\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right) \quad (3.79)$$

$$\theta_6 = \text{atan2}(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}) \quad (3.80)$$

The other possible solutions are left as an exercise (Problem 3-24).
◇

Example 3.10 SCARA Manipulator

As another example, we consider the SCARA manipulator whose forward kinematics is defined by T_4^0 from (3.30). The inverse kinematics solution is then given as the set of solutions of the equation

$$T_4^1 = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.81)$$

We first note that, since the SCARA has only four degrees-of-freedom, not every possible H from $SE(3)$ allows a solution of (3.81). In fact we can easily see that there is no solution of (3.81) unless R is of the form

$$R = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (3.82)$$

and if this is the case, the sum $\theta_1 + \theta_2 - \theta_4$ is determined by

$$\theta_1 + \theta_2 - \theta_4 = \alpha = \text{atan2}(r_{11}, r_{12}) \quad (3.83)$$

Projecting the manipulator configuration onto the $x_0 - y_0$ plane immediately yields the situation of Figure 3.22. We see from this that

$$\theta_2 = \text{atan2}\left(c_2, \pm\sqrt{1 - c_2}\right) \quad (3.84)$$

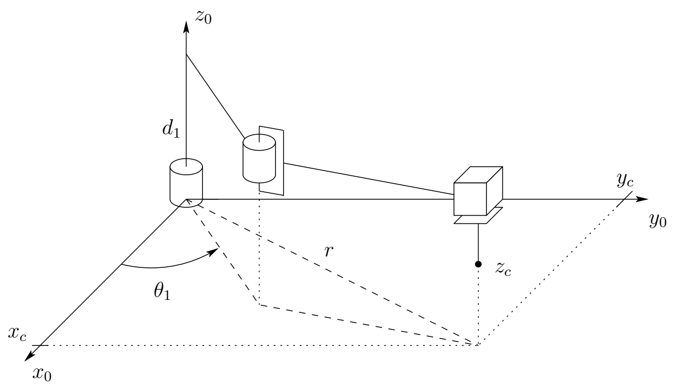

Fig. 3.22 SCARA manipulator

where

$$\begin{aligned} c_2 &= \frac{o_x^2 + o_y^2 - a_1^2 - a_2^2}{2a_1a_2} && (3.85) \\ \theta_1 &= \operatorname{atan2}(o_x, o_y) - \operatorname{atan2}(a_1 + a_2c_2, a_2s_2) && (3.86) \end{aligned}$$

We may then determine θ_4 *from (3.83) as*

$$\begin{aligned} \theta_4 &= \theta_1 + \theta_2 - \alpha && (3.87) \\ &= \theta_1 + \theta_2 - \operatorname{atan2}(r_{11}, r_{12}) \end{aligned}$$

Finally d_3 *is given as*

$$d_3 = o_z + d_4 \qquad (3.88)$$

◇

3.4 CHAPTER SUMMARY

In this chapter we studied the relationships between joint variables, q_i and the position and orientation of the end effector. We begain by introducing the Denavit-Hartenberg convention for assigning coordinate frames to the links of a serial manipulator. We may summarize the procedure based on the DH convention in the following algorithm for deriving the forward kinematics for any manipulator.

Step l: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-handed frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-handed frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = a$ along the direction z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = s$ in the direction of the gripper closure and set $x_n = n$ as $s \times a$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-handed frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i.

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1}. θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

This DH convention defines the forward kinematics equations for a manipulator, i.e., the mapping from joint variables to end effector position and orientation. To control a manipulator, it is necessary to solve the inverse problem, i.e., given a position and orientation for the end effector, solve for the corresponding set of joint variables. In this chapter, we have considered the special case of manipulators for which kinematic decoupling can be used (e.g., a manipulator with a sperical wrist). For this class of manipulators the determination of the inverse kinematics can be summarized by the following algorithm.

Step 1: Find q_1, q_2, q_3 such that the wrist center o_c has coordinates given by

$$o_c^0 \quad = \quad o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.89}$$

Step 2: Using the joint variables determined in Step 1, evaluate R_3^0.

Step 3: Find a set of Euler angles corresponding to the rotation matrix

$$R_6^3 = (R_3^0)^{-1}R = (R_3^0)^T R \tag{3.90}$$

In this chapter, we demonstrated a geometric approach for Step 1. In particular, to solve for joint variable q_i, we project the manipulator (including the wrist center) onto the $x_{i-1} - y_{i-1}$ plane and use trigonometry to find q_i.

3.5 NOTES AND REFERENCES

Kinematics and inverse kinematics have been the subject of research in robotics for many years. Some of the seminal work in these areas can be found in [9] [14] [20] [22] [43] [44] [60] [71] [35] [76] [2] [32] [34] [44] [45] [60] [61] [66] [72].

Problems

1. Verify the statement after Equation (3.14) that the rotation matrix R has the form (3.13) provided assumptions DH1 and DH2 are satisfied.

2. Consider the three-link planar manipulator shown in Figure 3.23. Derive

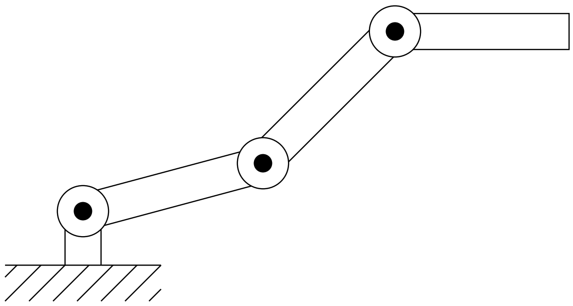

Fig. 3.23 Three-link planar arm of Problem 3-2

the forward kinematic equations using the DH-convention.

3. Consider the two-link cartesian manipulator of Figure 3.24. Derive the

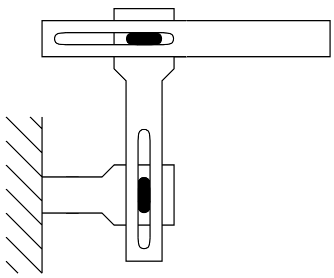

Fig. 3.24 Two-link cartesian robot of Problem 3-3

forward kinematic equations using the DH-convention.

4. Consider the two-link manipulator of Figure 3.25 which has joint 1 revolute and joint 2 prismatic. Derive the forward kinematic equations using the DH-convention.

5. Consider the three-link planar manipulator of Figure 3.26 Derive the forward kinematic equations using the DH-convention.

6. Consider the three-link articulated robot of Figure 3.27. Derive the for-

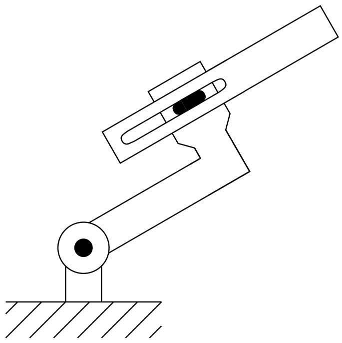

Fig. 3.25 Two-link planar arm of Problem 3-4

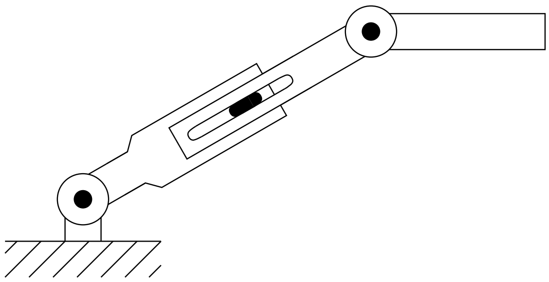

Fig. 3.26 Three-link planar arm with prismatic joint of Problem 3-5

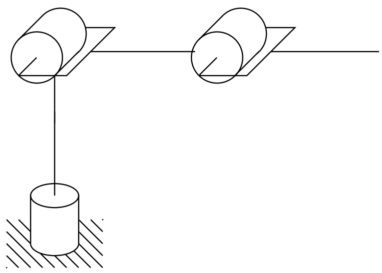

Fig. 3.27 Three-link articulated robot

ward kinematic equations using the DH-convention.

7. Consider the three-link cartesian manipulator of Figure 3.28. Derive the forward kinematic equations using the DH-convention.

8. Attach a spherical wrist to the three-link articulated manipulator of Problem 3-6. as shown in Figure 3.29. Derive the forward kinematic equations for this manipulator.

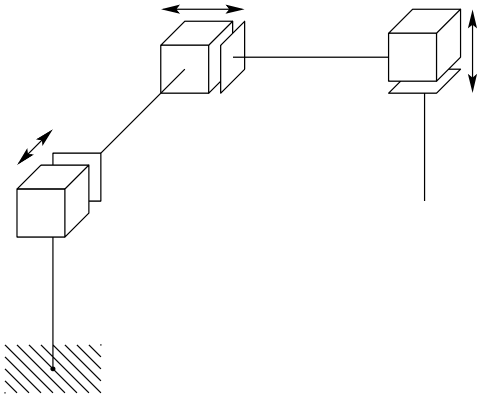

Fig. 3.28 Three-link cartesian robot

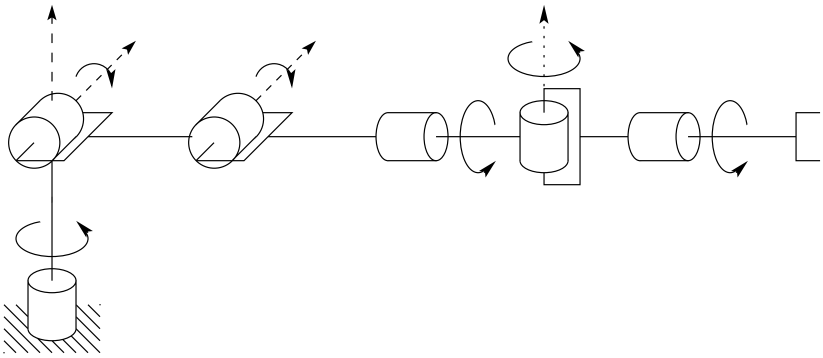

Fig. 3.29 Elbow manipulator with spherical wrist

9. Attach a spherical wrist to the three-link cartesian manipulator of Problem 3-7 as shown in Figure 3.30. Derive the forward kinematic equations for this manipulator.

10. Consider the PUMA 260 manipulator shown in Figure 3.31. Derive the complete set of forward kinematic equations, by establishing appropriate DH coordinate frames, constructing a table of link parameters, forming the A-matrices, etc.

11. Repeat Problem 3-9 for the five degree-of-freedom Rhino XR-3 robot shown in Figure 3.32. (Note: you should replace the Rhino wrist with the sperical wrist.)

12. Suppose that a Rhino XR-3 is bolted to a table upon which a coordinate frame $o_s x_s y_s z_s$ is established as shown in Figure 3.33. (The frame $o_s x_s y_x z_s$ is often referred to as the **station frame**.) Given the base frame

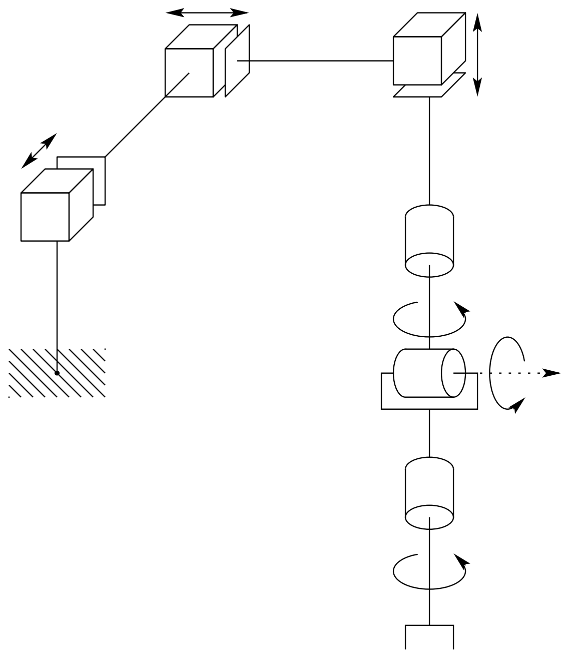

Fig. 3.30 Cartesian manipulator with spherical wrist

that you established in Problem 3-11, find the homogeneous transformation T_0^s relating the base frame to the station frame. Find the homogeneous transformation T_5^s relating the end-effector frame to the station frame. What is the position and orientation of the end-effector in the station frame when $\theta_1 = \theta_2 = \cdots = \theta_5 = 0$?

13. Consider the GMF S-400 robot shown in Figure 3.34 Draw the symbolic representation for this manipulator. Establish DH-coordinate frames and write the forward kinematic equations.

14. Given a desired position of the end-effector, how many solutions are there to the inverse kinematics of the three-link planar arm shown in Figure 3.35? If the orientation of the end-effector is also specified, how many solutions are there? Use the geometric approach to find them.

15. Repeat Problem 3-14 for the three-link planar arm with prismatic joint of Figure 3.36.

16. Solve the inverse position kinematics for the cylindrical manipulator of Figure 3.37.

17. Solve the inverse position kinematics for the cartesian manipulator of Figure 3.38.

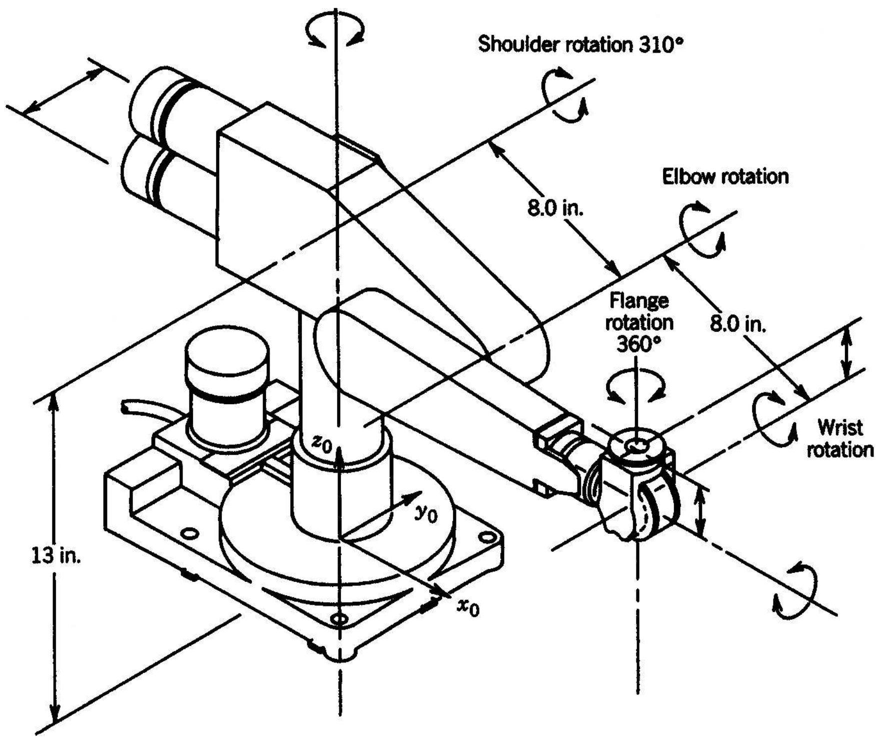

Fig. 3.31 PUMA 260 manipulator

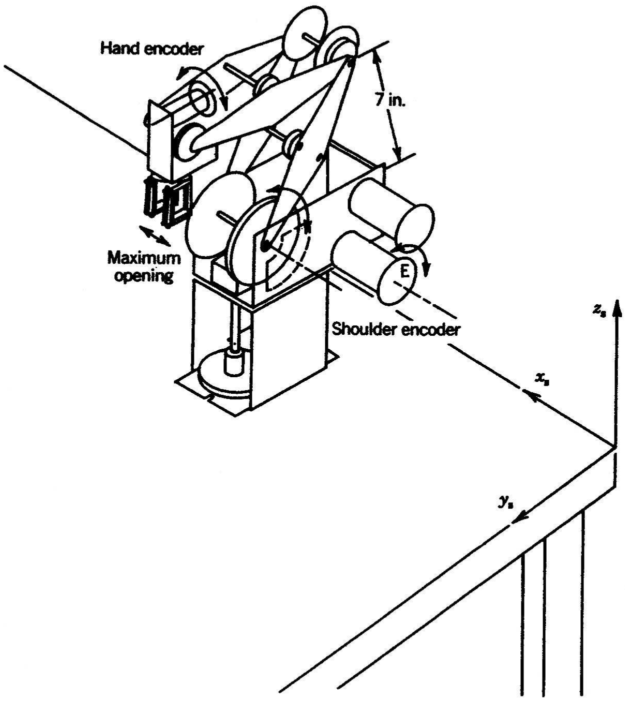

Fig. 3.32 Rhino XR-3 robot

18. Add a spherical wrist to the three-link cylindrical arm of Problem 3-16 and write the complete inverse kinematics solution.

19. Repeat Problem 3-16 for the cartesian manipulator of Problem 3-17.

20. Write a computer program to compute the inverse kinematic equations for the elbow manipulator using Equations (3.75)-(3.80). Include procedures for identifying singular configurations and choosing a particular solution when the configuration is singular. Test your routine for various special cases, including singular configurations.

21. The Stanford manipulator of Example 3.3.5 has a spherical wrist. Therefore, given a desired position O and orientation R of the end-effector,

    a) Compute the desired coordinates of the wrist center O_c^0.

    b) Solve the inverse position kinematics, that is, find values of the first three joint variables that will place the wrist center at O_c. Is the solution unique? How many solutions did you find?

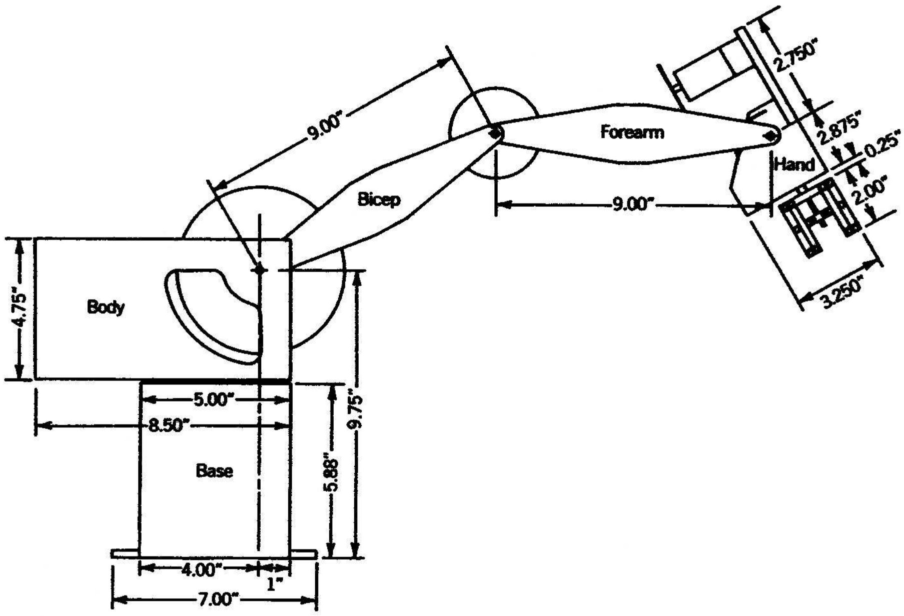

Fig. 3.33 Rhino robot attached to a table. From: *A Robot Engineering Textbook*, by Mohsen Shahinpoor. Copyright 1987, Harper & Row Publishers, Inc

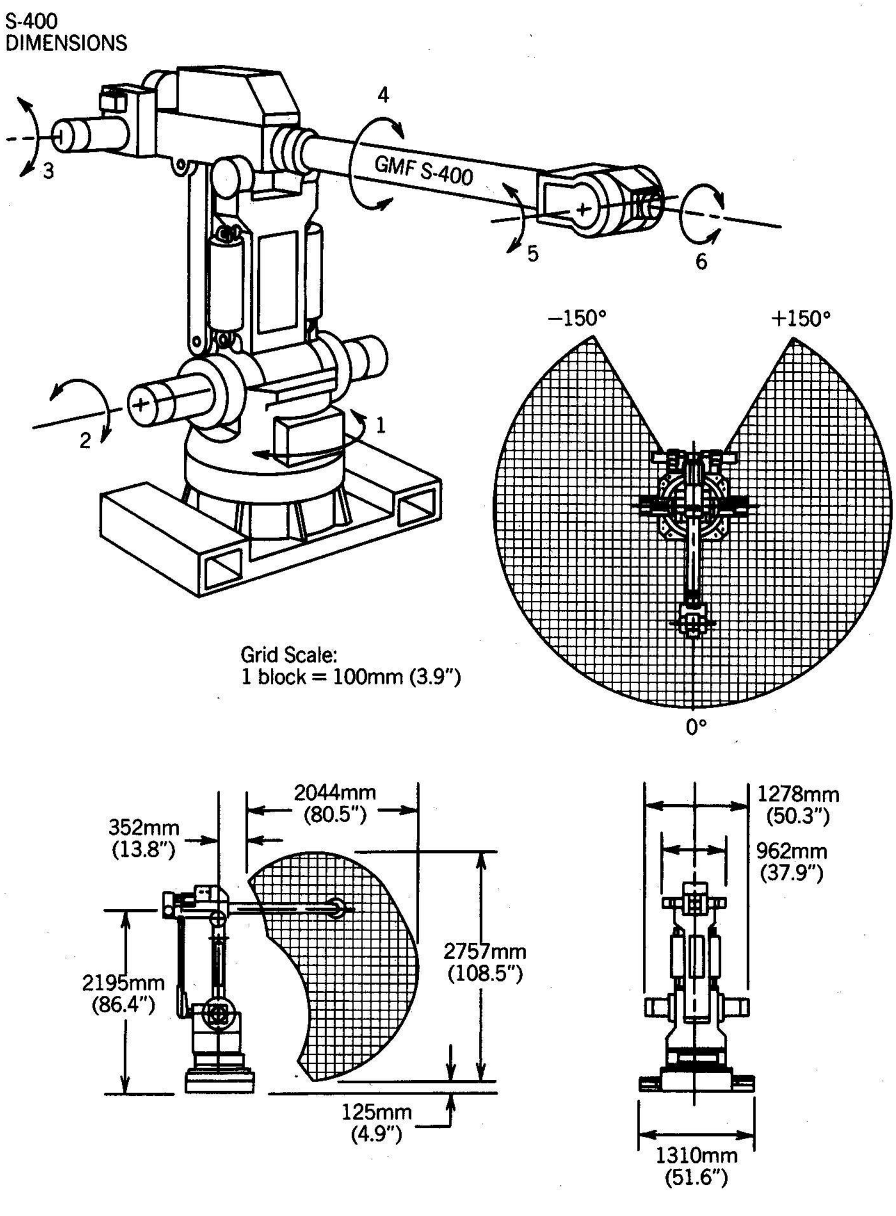

Fig. 3.34 GMF S-400 robot. (Courtesy GMF Robotics.)

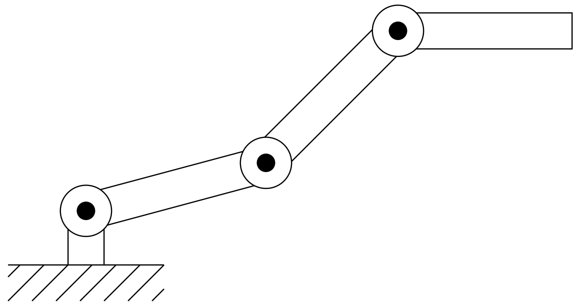

Fig. 3.35 Three-link planar robot with revolute joints.

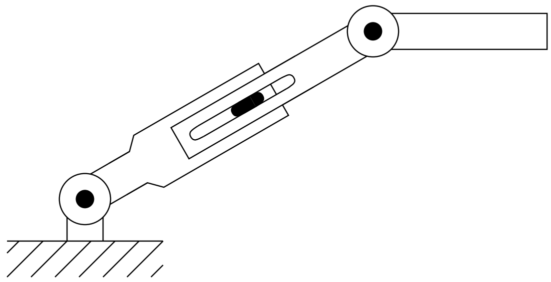

Fig. 3.36 Three-link planar robot with prismatic joint

c) Compute the rotation matrix R_3^0. Solve the inverse orientation problem for this manipulator by finding a set of Euler angles corresponding to R_6^3 given by (3.63).

22. Repeat Problem 3-21 for the PUMA 260 manipulator of Problem 3-9, which also has a spherical wrist. How many total solutions did you find?

23. Solve the inverse position kinematics for the Rhino robot.

24.). Find all other solutions to the inverse kinematics of the elbow manipulator of Example 3.9.

25. . Modify the solutions θ_1 and θ_2 for the spherical manipulator given by Equations (3.58) and (3.60) in the case of a shoulder offset.

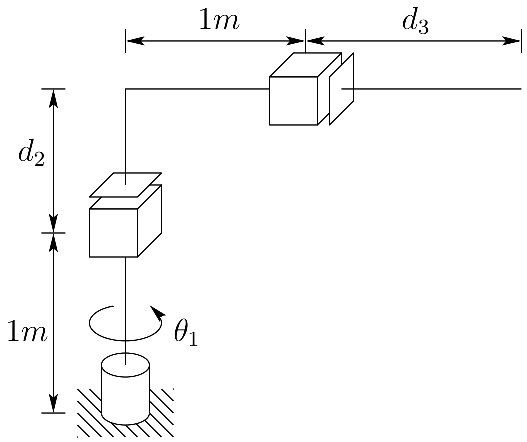

Fig. 3.37 Cylindrical configuration

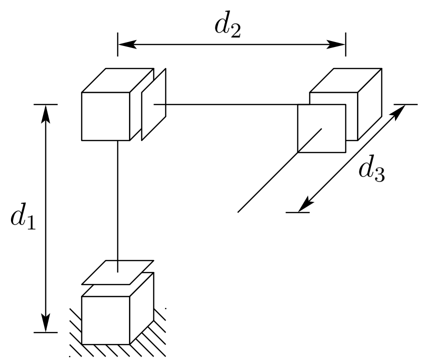

Fig. 3.38 Cartesian configuration