Helmholtz Decomposition

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The Fourier coefficients of $u^{solenoidal}$ are given by

$$\hat{\mathbf{u}}^{solenoidal} = \hat{\mathbf{u}} \cdot \left(\mathbf{I} - \frac{\boldsymbol{\kappa}\boldsymbol{\kappa}}{\kappa^2} \right) \tag{1}$$

$$\hat{\mathbf{u}} \cdot \left(\mathbf{I} - \frac{\boldsymbol{\kappa}\boldsymbol{\kappa}}{\kappa^2} \right) = (\hat{u}, \hat{v}, \hat{w}) \begin{pmatrix} 1 - \frac{k_1 k_1}{k^2} & -\frac{k_1 k_2}{k^2} & -\frac{k_1 k_3}{k^2} \\ -\frac{k_1 k_2}{k^2} & 1 - \frac{k_2 k_2}{k^2} & -\frac{k_2 k_3}{k^2} \\ -\frac{k_1 k_3}{k^2} & -\frac{k_2 k_3}{k^2} & 1 - \frac{k_3 k_3}{k^2} \end{pmatrix} \tag{2}$$

$$\begin{aligned} \hat{u}_{solenoidal} &= \hat{u} \left(1 - \frac{k_1 k_1}{k^2} \right) + \hat{v} \left(-\frac{k_1 k_2}{k^2} \right) + \hat{w} \left(-\frac{k_1 k_3}{k^2} \right) \\ \hat{v}_{solenoidal} &= \hat{u} \left(-\frac{k_1 k_2}{k^2} \right) + \hat{v} \left(1 - \frac{k_2 k_2}{k^2} \right) + \hat{w} \left(-\frac{k_2 k_3}{k^2} \right) \\ \hat{w}_{solenoidal} &= \hat{u} \left(-\frac{k_1 k_3}{k^2} \right) + \hat{v} \left(-\frac{k_2 k_3}{k^2} \right) + \hat{w} \left(1 - \frac{k_3 k_3}{k^2} \right) \end{aligned} \tag{3}$$