Equations

1 Reference

Book: Compressibility, Turbulence and High Speed Flow (2nd Edition) By Gatski, Thomas B.; Bonnet, Jean-Paul

5.3.1.1 Turbulent Stress and Kinetic Energy Transport Equations

2 Favre-averaged Reynolds-Stress Equation for compressible flow

The transport equation for the turbulent stress in the compressible case are obtained in a manner analogous to the incompressible formulation. The dependent variables are decomposed into Favre-mean and fluctuating components and the equations are then Reynolds-averaged (e.g. Gatski, 1996). Turbulent stress transport models are used in flow predictions.

$$\frac{D\tau_{ij}}{Dt} = \frac{\partial(\tau_{ij})}{\partial t} + \frac{\partial}{\partial x_k}(\widetilde{u}_k \tau_{ij}) = P_{ij} + T_{ij} + \Pi^d_{ij} + \Pi^t_{ij} - \epsilon_{ij} + M_{ij} + D_{ij}$$

where,

$$P_{ij} = -(\tau_{ik}\frac{\partial \widetilde{u_j}}{\partial x_k} + \frac{\partial \widetilde{u_i}}{\partial x_k}\tau_{kj})$$

$$T_{ij} = -\frac{\partial}{\partial x_k}(\overline{\rho u''_i u''_j u''_k})$$

$$\Pi^d_{ij} = \overline{p'(\frac{\partial u''_i}{\partial x_j} + \frac{\partial u''_j}{\partial x_i})} = \overline{p'(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})}$$

$$\Pi^t_{ij} = -\frac{\partial}{\partial x_k}(\delta_{ik}\overline{p' u''_j} + \overline{p' u''_i}\delta_{jk})$$

$$\epsilon_{ij} = \overline{\sigma'_{ik}\frac{\partial u''_j}{\partial x_k}} + \overline{\sigma'_{jk}\frac{\partial u''_i}{\partial x_k}} = \overline{\sigma'_{ik}\frac{\partial u'_j}{\partial x_k}} + \overline{\sigma'_{jk}\frac{\partial u'_i}{\partial x_k}}$$

$$M_{ij} = \overline{u''_i}(\frac{\partial \overline{\sigma}_{jk}}{\partial x_k} - \frac{\partial \overline{p}}{\partial x_j}) + \overline{u''_j}(\frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \frac{\partial \overline{p}}{\partial x_i}) = \frac{\overline{\rho' u'_i}}{\overline{\rho}}(\frac{\partial \overline{p}}{\partial x_j} - \frac{\partial \overline{\sigma}_{jk}}{\partial x_k}) + \frac{\overline{\rho' u'_j}}{\overline{\rho}}(\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial \overline{\sigma}_{ik}}{\partial x_k})$$

$$D_{ij} = \frac{\partial}{\partial x_k}(\overline{\sigma''_{ik} u''_j + \sigma''_{jk} u''_i})$$

As the equation shows, the transport of the turbulent stress is controlled by a balance among its Production term P_{ij}, Turbulent Transport term T_{ij}, Pressure-Dilatation term Π^d_{ij}, Pressure Transport term Π^t_{ij}, Viscous Dissipation term ϵ_{ij}, Density fluctuation term M_{ij}, Viscous Diffusion D_{ij}.

The dimension for each part is $\frac{kg}{m \cdot s^3} = \frac{pa}{s}$

2.1 Favre-averaged Reynolds-stress tensor τ_{ij}

$$\tau_{ij} = \overline{\rho u''_i u''_j}$$

$$\tau_{11} = \tau_{xx} = \overline{\rho u'' u''} = \overline{\rho u^2} - \frac{(\overline{\rho u})^2}{\overline{\rho}} = \overline{\rho u^2} - \widetilde{u}\overline{\rho u}$$

$$\tau_{22} = \tau_{yy} = \overline{\rho v'' v''} = \overline{\rho v^2} - \frac{(\overline{\rho v})^2}{\overline{\rho}} = \overline{\rho v^2} - \widetilde{v}\overline{\rho v}$$

$\tau_{33} = \tau_{zz} = \overline{\rho w''w''} = \overline{\rho w^2} - \frac{(\overline{\rho w})^2}{\overline{\rho}} = \overline{\rho w^2} - \widetilde{w}\overline{\rho w}$

$\tau_{12} = \tau_{xy} = \overline{\rho u''v''} = \overline{\rho uv} - \overline{\rho u}\frac{\overline{\rho v}}{\overline{\rho}} = \overline{\rho uv} - \overline{\rho u}\widetilde{v}$

$\tau_{13} = \tau_{xz} = \overline{\rho u''w''} = \overline{\rho uw} - \overline{\rho u}\frac{\overline{\rho w}}{\overline{\rho}} = \overline{\rho uw} - \overline{\rho u}\widetilde{w}$

$\tau_{23} = \tau_{yz} = \overline{\rho v''w''} = \overline{\rho vw} - \overline{\rho v}\frac{\overline{\rho w}}{\overline{\rho}} = \overline{\rho vw} - \overline{\rho v}\widetilde{w}$

2.2 Viscous-stress tensor σ_{ij}

$\sigma_{ij} = 2\mu(S_{ij} - \frac{\delta_{ij}}{3}S_{kk})$

$\sigma_{11} = \sigma_{xx} = 2\mu S_{xx} - \frac{2}{3}\mu S_{xx} - \frac{2}{3}\mu S_{yy} - \frac{2}{3}\mu S_{zz} = \frac{4}{3}\mu S_{xx} - \frac{2}{3}\mu S_{yy} - \frac{2}{3}\mu S_{zz} = \frac{4}{3}\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\frac{\partial w}{\partial z}$

$\sigma_{22} = \sigma_{yy} = 2\mu S_{yy} - \frac{2}{3}\mu S_{xx} - \frac{2}{3}\mu S_{yy} - \frac{2}{3}\mu S_{zz} = -\frac{2}{3}\mu S_{xx} + \frac{4}{3}\mu S_{yy} - \frac{2}{3}\mu S_{zz} = -\frac{2}{3}\mu\frac{\partial u}{\partial x} + \frac{4}{3}\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\frac{\partial w}{\partial z}$

$\sigma_{33} = \sigma_{zz} = 2\mu S_{zz} - \frac{2}{3}\mu S_{xx} - \frac{2}{3}\mu S_{yy} - \frac{2}{3}\mu S_{zz} = -\frac{2}{3}\mu S_{xx} - \frac{2}{3}\mu S_{yy} + \frac{4}{3}\mu S_{zz} = -\frac{2}{3}\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\frac{\partial v}{\partial y} + \frac{4}{3}\mu\frac{\partial w}{\partial z}$

$\sigma_{12} = \sigma_{xy} = 2\mu(S_{xy}) = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$

$\sigma_{13} = \sigma_{xz} = 2\mu(S_{xz}) = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$

$\sigma_{23} = \sigma_{yz} = 2\mu(S_{yz}) = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$

2.3 Strain-rate S_{ij}

$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$

$S_{11} = S_{xx} = \frac{\partial u}{\partial x}$

$S_{22} = S_{yy} = \frac{\partial v}{\partial y}$

$S_{33} = S_{zz} = \frac{\partial w}{\partial z}$

$S_{12} = S_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$

$S_{13} = S_{xz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})$

$S_{23} = S_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$

2.4 Production P_{ij}

The Production rate term is a function of the Reynolds stress and the mean velocity gradient.

$P_{ij} = -(\tau_{ik}\frac{\partial \widetilde{u_j}}{\partial x_k} + \frac{\partial \widetilde{u_i}}{\partial x_k}\tau_{kj})$

$P_{11} = P_{xx} = -2(\tau_{xx}\frac{\partial \widetilde{u}}{\partial x} + \tau_{xy}\frac{\partial \widetilde{u}}{\partial y} + \tau_{xz}\frac{\partial \widetilde{u}}{\partial z}) = -2(\overline{\rho u''u''}\frac{\partial \widetilde{u}}{\partial x} + \overline{\rho u''v''}\frac{\partial \widetilde{u}}{\partial y} + \overline{\rho u''w''}\frac{\partial \widetilde{u}}{\partial z})$

$P_{22} = P_{yy} = -2(\tau_{xy}\frac{\partial \widetilde{v}}{\partial x} + \tau_{yy}\frac{\partial \widetilde{v}}{\partial y} + \tau_{yz}\frac{\partial \widetilde{v}}{\partial z}) = -2(\overline{\rho u''v''}\frac{\partial \widetilde{v}}{\partial x} + \overline{\rho v''v''}\frac{\partial \widetilde{v}}{\partial y} + \overline{\rho v''w''}\frac{\partial \widetilde{v}}{\partial z})$

$P_{33} = P_{zz} = -2(\tau_{xz}\frac{\partial \widetilde{w}}{\partial x} + \tau_{yz}\frac{\partial \widetilde{w}}{\partial y} + \tau_{zz}\frac{\partial \widetilde{w}}{\partial z}) = -2(\overline{\rho u''w''}\frac{\partial \widetilde{w}}{\partial x} + \overline{\rho v''w''}\frac{\partial \widetilde{w}}{\partial y} + \overline{\rho w''w''}\frac{\partial \widetilde{w}}{\partial z})$

$P_{12} = P_{xy} = -(\tau_{xy}\frac{\partial \widetilde{u}}{\partial x} + \tau_{yy}\frac{\partial \widetilde{u}}{\partial y} + \tau_{yz}\frac{\partial \widetilde{u}}{\partial z}) - (\tau_{xx}\frac{\partial \widetilde{v}}{\partial x} + \tau_{xy}\frac{\partial \widetilde{v}}{\partial y} + \tau_{xz}\frac{\partial \widetilde{v}}{\partial z}) = -(\overline{\rho u''v''}\frac{\partial \widetilde{u}}{\partial x} + \overline{\rho v''v''}\frac{\partial \widetilde{u}}{\partial y} + \overline{\rho v''w''}\frac{\partial \widetilde{u}}{\partial z}) -$

$(\overline{\rho u''u''}\frac{\partial \widetilde{v}}{\partial x}+\overline{\rho u''v''}\frac{\partial \widetilde{v}}{\partial y}+\overline{\rho u''w''}\frac{\partial \widetilde{v}}{\partial z})$

$P_{13} = P_{xz} = -(\tau_{xz}\frac{\partial \widetilde{u}}{\partial x}+\tau_{yz}\frac{\partial \widetilde{u}}{\partial y}+\tau_{zz}\frac{\partial \widetilde{u}}{\partial z}) - (\tau_{xx}\frac{\partial \widetilde{w}}{\partial x}+\tau_{xy}\frac{\partial \widetilde{w}}{\partial y}+\tau_{xz}\frac{\partial \widetilde{w}}{\partial z}) = -(\overline{\rho u''w''}\frac{\partial \widetilde{u}}{\partial x}+\overline{\rho v''w''}\frac{\partial \widetilde{u}}{\partial y}+\overline{\rho w''w''}\frac{\partial \widetilde{u}}{\partial z}) - (\overline{\rho u''u''}\frac{\partial \widetilde{w}}{\partial x}+\overline{\rho u''v''}\frac{\partial \widetilde{w}}{\partial y}+\overline{\rho u''w''}\frac{\partial \widetilde{w}}{\partial z})$

$P_{23} = P_{yz} = -(\tau_{xz}\frac{\partial \widetilde{v}}{\partial x}+\tau_{yz}\frac{\partial \widetilde{v}}{\partial y}+\tau_{zz}\frac{\partial \widetilde{v}}{\partial z}) - (\tau_{xy}\frac{\partial \widetilde{w}}{\partial x}+\tau_{yy}\frac{\partial \widetilde{w}}{\partial y}+\tau_{yz}\frac{\partial \widetilde{w}}{\partial z}) = -(\overline{\rho u''w''}\frac{\partial \widetilde{v}}{\partial x}+\overline{\rho v''w''}\frac{\partial \widetilde{v}}{\partial y}+\overline{\rho w''w''}\frac{\partial \widetilde{v}}{\partial z}) - (\overline{\rho u''v''}\frac{\partial \widetilde{w}}{\partial x}+\overline{\rho v''v''}\frac{\partial \widetilde{w}}{\partial y}+\overline{\rho v''w''}\frac{\partial \widetilde{w}}{\partial z})$

2.5 Turbulent Transport term T_{ij}

T_{ij} is the divergence of the triple correlation tensor, acting as a spatial redistribution term.

$T_{ij} = -\frac{\partial}{\partial x_k}(\overline{\rho u_i''u_j''u_k''})$

$T_{11} = T_{xx} = -\frac{\partial}{\partial x}(\overline{\rho u''u''u''}) - \frac{\partial}{\partial y}(\overline{\rho u''u''v''}) - \frac{\partial}{\partial z}(\overline{\rho u''u''w''})$

$T_{22} = T_{yy} = -\frac{\partial}{\partial x}(\overline{\rho v''v''u''}) - \frac{\partial}{\partial y}(\overline{\rho v''v''v''}) - \frac{\partial}{\partial z}(\overline{\rho v''v''w''})$

$T_{33} = T_{ww} = -\frac{\partial}{\partial x}(\overline{\rho w''w''u''}) - \frac{\partial}{\partial y}(\overline{\rho w''w''v''}) - \frac{\partial}{\partial z}(\overline{\rho w''w''w''})$

$T_{12} = T_{xy} = -\frac{\partial}{\partial x}(\overline{\rho u''v''u''}) - \frac{\partial}{\partial y}(\overline{\rho u''v''v''}) - \frac{\partial}{\partial z}(\overline{\rho u''v''w''})$

$T_{13} = T_{xz} = -\frac{\partial}{\partial x}(\overline{\rho u''w''u''}) - \frac{\partial}{\partial y}(\overline{\rho u''w''v''}) - \frac{\partial}{\partial z}(\overline{\rho u''w''w''})$

$T_{23} = T_{yz} = -\frac{\partial}{\partial x}(\overline{\rho v''w''u''}) - \frac{\partial}{\partial y}(\overline{\rho v''w''v''}) - \frac{\partial}{\partial z}(\overline{\rho v''w''w''})$

2.6 Pressure-Dilatation term Π_{ij}^d

Π_{ij}^d is the pressure-strain rate correlation tensor, which is traceless and represents inter-components transfer between Reynolds stress terms.

$\Pi_{ij}^d = \overline{p'(\frac{\partial u_i''}{\partial x_j}+\frac{\partial u_j''}{\partial x_i})} = \overline{p'(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i})}$

$\Pi_{11}^d = \Pi_{xx}^d = \overline{p'(\frac{\partial u'}{\partial x}+\frac{\partial u'}{\partial x})} = \overline{2p'(\frac{\partial u'}{\partial x})}$

$\Pi_{22}^d = \Pi_{yy}^d = \overline{p'(\frac{\partial v'}{\partial y}+\frac{\partial v'}{\partial y})} = \overline{2p'(\frac{\partial v'}{\partial y})}$

$\Pi_{33}^d = \Pi_{zz}^d = \overline{p'(\frac{\partial w'}{\partial z}+\frac{\partial w'}{\partial z})} = \overline{2p'(\frac{\partial w'}{\partial z})}$

$\Pi_{12}^d = \Pi_{xy}^d = \overline{p'(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x})}$

$\Pi_{13}^d = \Pi_{xz}^d = \overline{p'(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x})}$

$\Pi_{23}^d = \Pi_{yz}^d = \overline{p'(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y})}$

2.7 Pressure Transport term Π_{ij}^t

Π_{ij}^t is the divergence of the pressure-velocity correlation, and represents transport driven by pressure fluctuations.

$\Pi_{ij}^t = -\frac{\partial}{\partial x_k}(\delta_{ik}\overline{p'u_j''}+\overline{p'u_i''}\delta_{jk})$

$\Pi_{11}^t = \Pi_{xx}^t = -\frac{\partial}{\partial x}(2\overline{p'u''})$

$$\Pi^t_{22} = \Pi^t_{yy} = -\frac{\partial}{\partial y}(2\overline{p'v''})$$

$$\Pi^t_{33} = \Pi^t_{zz} = -\frac{\partial}{\partial z}(2\overline{p'w''})$$

$$\Pi^t_{12} = \Pi^t_{xy} = -\frac{\partial}{\partial x}(\overline{p'v''}) - \frac{\partial}{\partial y}(\overline{p'u''})$$

$$\Pi^t_{13} = \Pi^t_{xz} = -\frac{\partial}{\partial x}(\overline{p'w''}) - \frac{\partial}{\partial z}(\overline{p'u''})$$

$$\Pi^t_{23} = \Pi^t_{yz} = -\frac{\partial}{\partial y}(\overline{p'w''}) - \frac{\partial}{\partial z}(\overline{p'v''})$$

2.8 Viscous Dissipation term ϵ_{ij}

ϵ_{ij} is the viscous dissipation rate tensor. It acts as a destruction term of turbulent energy (and stresses).

$$\epsilon_{ij} = \overline{\sigma'_{ik}\frac{\partial u''_j}{\partial x_k}} + \overline{\sigma'_{jk}\frac{\partial u''_i}{\partial x_k}} = \overline{\sigma'_{ik}\frac{\partial u'_j}{\partial x_k}} + \overline{\sigma'_{jk}\frac{\partial u'_i}{\partial x_k}}$$

$$\epsilon_{11} = \epsilon_{xx} = 2[\overline{\sigma'_{xx}\frac{\partial u'}{\partial x}} + \overline{\sigma'_{xy}\frac{\partial u'}{\partial y}} + \overline{\sigma'_{xz}\frac{\partial u'}{\partial z}}]$$

$$\epsilon_{22} = \epsilon_{yy} = 2[\overline{\sigma'_{xy}\frac{\partial v'}{\partial x}} + \overline{\sigma'_{yy}\frac{\partial v'}{\partial y}} + \overline{\sigma'_{yz}\frac{\partial v'}{\partial z}}]$$

$$\epsilon_{33} = \epsilon_{zz} = 2[\overline{\sigma'_{xz}\frac{\partial w'}{\partial x}} + \overline{\sigma'_{yz}\frac{\partial w'}{\partial y}} + \overline{\sigma'_{zz}\frac{\partial w'}{\partial z}}]$$

$$\epsilon_{12} = \epsilon_{xy} = \overline{\sigma'_{xy}\frac{\partial u'}{\partial x}} + \overline{\sigma'_{yy}\frac{\partial u'}{\partial y}} + \overline{\sigma'_{yz}\frac{\partial u'}{\partial z}} + \overline{\sigma'_{xx}\frac{\partial v'}{\partial x}} + \overline{\sigma'_{xy}\frac{\partial v'}{\partial y}} + \overline{\sigma'_{xz}\frac{\partial v'}{\partial z}}$$

$$\epsilon_{13} = \epsilon_{xz} = \overline{\sigma'_{xz}\frac{\partial u'}{\partial x}} + \overline{\sigma'_{yz}\frac{\partial u'}{\partial y}} + \overline{\sigma'_{zz}\frac{\partial u'}{\partial z}} + \overline{\sigma'_{xx}\frac{\partial w'}{\partial x}} + \overline{\sigma'_{xy}\frac{\partial w'}{\partial y}} + \overline{\sigma'_{xz}\frac{\partial w'}{\partial z}}$$

$$\epsilon_{23} = \epsilon_{yz} = \overline{\sigma'_{xz}\frac{\partial v'}{\partial x}} + \overline{\sigma'_{yz}\frac{\partial v'}{\partial y}} + \overline{\sigma'_{zz}\frac{\partial v'}{\partial z}} + \overline{\sigma'_{xy}\frac{\partial w'}{\partial x}} + \overline{\sigma'_{yy}\frac{\partial w'}{\partial y}} + \overline{\sigma'_{yz}\frac{\partial w'}{\partial z}}$$

2.9 Density fluctuation term M_{ij}

Density fluctuations term accounts for the sum of the mean flow viscous stress diffusion term $(\overline{u''_i}\frac{\partial\bar{\sigma}_{jk}}{\partial x_k} + \overline{u''_j}\frac{\partial\bar{\sigma}_{ik}}{\partial x_k})$ and the pressure work done term $(-\overline{u''_i}\frac{\partial\bar{p}}{\partial x_j} - \overline{u''_j}\frac{\partial\bar{p}}{\partial x_i})$.

$$M_{ij} = \overline{u''_i}(\frac{\partial\bar{\sigma}_{jk}}{\partial x_k} - \frac{\partial\bar{p}}{\partial x_j}) + \overline{u''_j}(\frac{\partial\bar{\sigma}_{ik}}{\partial x_k} - \frac{\partial\bar{p}}{\partial x_i}) = \frac{\overline{\rho' u'_i}}{\bar{\rho}}(\frac{\partial\bar{p}}{\partial x_j} - \frac{\partial\bar{\sigma}_{jk}}{\partial x_k}) + \frac{\overline{\rho' u'_j}}{\bar{\rho}}(\frac{\partial\bar{p}}{\partial x_i} - \frac{\partial\bar{\sigma}_{ik}}{\partial x_k})$$

$$M_{11} = M_{xx} = 2\frac{\overline{\rho' u'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial x} - \frac{\partial\bar{\sigma}_{xx}}{\partial x} - \frac{\partial\bar{\sigma}_{xy}}{\partial y} - \frac{\partial\bar{\sigma}_{xz}}{\partial z}]$$

$$M_{22} = M_{yy} = 2\frac{\overline{\rho' v'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial y} - \frac{\partial\bar{\sigma}_{xy}}{\partial x} - \frac{\partial\bar{\sigma}_{yy}}{\partial y} - \frac{\partial\bar{\sigma}_{yz}}{\partial z}]$$

$$M_{33} = M_{zz} = 2\frac{\overline{\rho' w'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial z} - \frac{\partial\bar{\sigma}_{xz}}{\partial x} - \frac{\partial\bar{\sigma}_{yz}}{\partial y} - \frac{\partial\bar{\sigma}_{zz}}{\partial z}]$$

$$M_{12} = M_{xy} = \frac{\overline{\rho' u'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial y} - \frac{\bar{\sigma}_{xy}}{\partial x} - \frac{\bar{\sigma}_{yy}}{\partial y} - \frac{\bar{\sigma}_{yz}}{\partial z}] + \frac{\overline{\rho' v'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial x} - \frac{\bar{\sigma}_{xx}}{\partial x} - \frac{\bar{\sigma}_{xy}}{\partial y} - \frac{\bar{\sigma}_{xz}}{\partial z}]$$

$$M_{13} = M_{xz} = \frac{\overline{\rho' u'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial z} - \frac{\bar{\sigma}_{xz}}{\partial x} - \frac{\bar{\sigma}_{yz}}{\partial y} - \frac{\bar{\sigma}_{zz}}{\partial z}] + \frac{\overline{\rho' w'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial x} - \frac{\bar{\sigma}_{xx}}{\partial x} - \frac{\bar{\sigma}_{xy}}{\partial y} - \frac{\bar{\sigma}_{xz}}{\partial z}]$$

$$M_{23} = M_{yz} = \frac{\overline{\rho' v'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial z} - \frac{\bar{\sigma}_{xz}}{\partial x} - \frac{\bar{\sigma}_{yz}}{\partial y} - \frac{\bar{\sigma}_{zz}}{\partial z}] + \frac{\overline{\rho' w'}}{\bar{\rho}}[3\frac{\partial\bar{p}}{\partial y} - \frac{\bar{\sigma}_{xy}}{\partial x} - \frac{\bar{\sigma}_{yy}}{\partial y} - \frac{\bar{\sigma}_{yz}}{\partial z}]$$

2.10 Viscous Diffusion D_{ij}

D_{ij} is the viscous diffusion tensor. It is a molecular diffusion term acting to even out the turbulent stresses by spatial redistribution.

$D_{ij} = \frac{\partial}{\partial x_k}(\overline{(\sigma''_{ik}u''_j + \sigma''_{jk}u''_i)}) = \frac{\partial}{\partial x_k}\overline{(\sigma'_{ik}u'_j + \sigma'_{jk}u'_i)}$

$D_{11} = D_{xx} = \frac{\partial}{\partial x}(\overline{2\sigma''_{xx}u''}) + \frac{\partial}{\partial y}(\overline{2\sigma''_{xy}u''}) + \frac{\partial}{\partial z}(\overline{2\sigma''_{xz}u''})$

$D_{22} = D_{yy} = \frac{\partial}{\partial x}(\overline{2\sigma''_{xy}v''}) + \frac{\partial}{\partial y}(\overline{2\sigma''_{yy}v''}) + \frac{\partial}{\partial z}(\overline{2\sigma''_{yz}v''})$

$D_{33} = D_{zz} = \frac{\partial}{\partial x}(\overline{2\sigma''_{xz}w''}) + \frac{\partial}{\partial y}(\overline{2\sigma''_{yz}w''}) + \frac{\partial}{\partial z}(\overline{2\sigma''_{zz}w''})$

$D_{12} = D_{xy} = \frac{\partial}{\partial x}\overline{(\sigma''_{xx}v'' + \sigma''_{xy}u'')} + \frac{\partial}{\partial y}\overline{(\sigma''_{xy}v'' + \sigma''_{yy}u'')} + \frac{\partial}{\partial z}\overline{(\sigma''_{xz}v'' + \sigma''_{yz}u'')}$

$D_{13} = D_{xz} = \frac{\partial}{\partial x}\overline{(\sigma''_{xx}w'' + \sigma''_{xz}u'')} + \frac{\partial}{\partial y}\overline{(\sigma''_{xy}w'' + \sigma''_{yz}u'')} + \frac{\partial}{\partial z}\overline{(\sigma''_{xz}w'' + \sigma''_{zz}u'')}$

$D_{23} = D_{yz} = \frac{\partial}{\partial x}\overline{(\sigma''_{xy}w'' + \sigma''_{xz}v'')} + \frac{\partial}{\partial y}\overline{(\sigma''_{yy}w'' + \sigma''_{yz}v'')} + \frac{\partial}{\partial z}\overline{(\sigma''_{yz}w'' + \sigma''_{zz}v'')}$

3 Favre-averaged Turbulence Kinetic Energy equation for compressible flow

$\overline{\rho}\frac{DK}{Dt} = P + T + \Pi_d + \Pi_t - \epsilon + M + D$

where,

$P = \frac{P_{ii}}{2} = -\tau_{ik}\frac{\partial \widetilde{u_i}}{\partial x_k} = -\tau_{ik}\widetilde{S_{ki}}$ is the Production term.

$T = \frac{T_{ii}}{2} = -\frac{\partial}{\partial x_k}(\frac{1}{2}\overline{\rho}u''_i\widetilde{u''_i u''_k})$ is the Turbulent Transport term.

$\Pi^d = \frac{\Pi^d_{ii}}{2} = \overline{p'\frac{\partial u'_i}{\partial x_i}}$ is the Pressure-Dilatation term.

$\Pi^t = \frac{\Pi^t_{ii}}{2} = -\frac{\partial}{\partial x_k}(\overline{p'u'_i}\delta_{ik})$ is the Pressure Transport term.

$\epsilon = \frac{\epsilon_{ii}}{2} = \overline{\sigma'_{ik}\frac{\partial u'_i}{\partial x_k}}$ is the Viscous Dissipation term.

$M = \frac{M_{ii}}{2} = \overline{u''_i}(\frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \frac{\partial \overline{p}}{\partial x_i})$ is the term associated with Density fluctuations.

$D = \frac{D_{ii}}{2} = \frac{\partial}{\partial x_k}(\overline{\sigma'_{ik}u'_i})$ is the Viscous Diffusion term.

3.1 Production term P

$P = \frac{P_{ii}}{2} = -\tau_{ik}\frac{\partial \widetilde{u_i}}{\partial x_k} = -\tau_{ik}\widetilde{S_{ki}}$

$P = \frac{P_{xx}}{2} + \frac{P_{yy}}{2} + \frac{P_{zz}}{2}$

$= -\tau_{xx}\frac{\partial \widetilde{u}}{\partial x} - \tau_{xy}\frac{\partial \widetilde{u}}{\partial y} - \tau_{xz}\frac{\partial \widetilde{u}}{\partial z} - \tau_{xy}\frac{\partial \widetilde{v}}{\partial x} - \tau_{yy}\frac{\partial \widetilde{v}}{\partial y} - \tau_{yz}\frac{\partial \widetilde{v}}{\partial z} - \tau_{xz}\frac{\partial \widetilde{w}}{\partial x} - \tau_{yz}\frac{\partial \widetilde{w}}{\partial y} - \tau_{zz}\frac{\partial \widetilde{w}}{\partial z} = -\overline{\rho u''u''}\frac{\partial \widetilde{u}}{\partial x} - \overline{\rho u''v''}\frac{\partial \widetilde{u}}{\partial y} - \overline{\rho u''w''}\frac{\partial \widetilde{u}}{\partial z} - \overline{\rho u''v''}\frac{\partial \widetilde{v}}{\partial x} - \overline{\rho v''v''}\frac{\partial \widetilde{v}}{\partial y} - \overline{\rho v''w''}\frac{\partial \widetilde{v}}{\partial z} - \overline{\rho u''w''}\frac{\partial \widetilde{w}}{\partial x} - \overline{\rho v''w''}\frac{\partial \widetilde{w}}{\partial y} - \overline{\rho w''w''}\frac{\partial \widetilde{w}}{\partial z}$

$\frac{\partial \widetilde{u}}{\partial x} = \frac{1}{\overline{\rho}}\frac{\partial \overline{\rho u}}{\partial x} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\frac{\partial \overline{\rho}}{\partial x} = \frac{1}{\overline{\rho}}\overline{(\frac{\partial \rho u}{\partial x})} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\overline{(\frac{\partial \rho}{\partial x})}$

$\frac{\partial \widetilde{u}}{\partial y} = \frac{1}{\overline{\rho}}\frac{\partial \overline{\rho u}}{\partial y} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\frac{\partial \overline{\rho}}{\partial y} = \frac{1}{\overline{\rho}}\overline{(\frac{\partial \rho u}{\partial y})} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\overline{(\frac{\partial \rho}{\partial y})}$

$\frac{\partial \widetilde{u}}{\partial z} = \frac{1}{\overline{\rho}}\frac{\partial \overline{\rho u}}{\partial z} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\frac{\partial \overline{\rho}}{\partial z} = \frac{1}{\overline{\rho}}\overline{(\frac{\partial \rho u}{\partial z})} - \frac{\overline{\rho u}}{(\overline{\rho})^2}\overline{(\frac{\partial \rho}{\partial z})}$

$\frac{\partial \widetilde{v}}{\partial x} = \frac{1}{\overline{\rho}}\frac{\partial \overline{\rho v}}{\partial x} - \frac{\overline{\rho v}}{(\overline{\rho})^2}\frac{\partial \overline{\rho}}{\partial x} = \frac{1}{\overline{\rho}}\overline{(\frac{\partial \rho v}{\partial x})} - \frac{\overline{\rho v}}{(\overline{\rho})^2}\overline{(\frac{\partial \rho}{\partial x})}$

$$\frac{\partial \widetilde{v}}{\partial y} = \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho v}}{\partial y} - \frac{\overline{\rho v}}{(\overline{\rho})^2} \frac{\partial \overline{\rho}}{\partial y} = \frac{1}{\overline{\rho}} \overline{(\frac{\partial \rho v}{\partial y})} - \frac{\overline{\rho v}}{(\overline{\rho})^2} \overline{(\frac{\partial \rho}{\partial y})}$$

$$\frac{\partial \widetilde{v}}{\partial z} = \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho v}}{\partial z} - \frac{\overline{\rho v}}{(\overline{\rho})^2} \frac{\partial \overline{\rho}}{\partial z} = \frac{1}{\overline{\rho}} \overline{(\frac{\partial \rho v}{\partial z})} - \frac{\overline{\rho v}}{(\overline{\rho})^2} \overline{(\frac{\partial \rho}{\partial z})}$$

$$\frac{\partial \widetilde{w}}{\partial x} = \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho w}}{\partial x} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \frac{\partial \overline{\rho}}{\partial x} = \frac{1}{\overline{\rho}} \overline{(\frac{\partial \rho w}{\partial x})} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \overline{(\frac{\partial \rho}{\partial x})}$$

$$\frac{\partial \widetilde{w}}{\partial y} = \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho w}}{\partial y} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \frac{\partial \overline{\rho}}{\partial y} = \frac{1}{\overline{\rho}} \overline{(\frac{\partial \rho w}{\partial y})} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \overline{(\frac{\partial \rho}{\partial y})}$$

$$\frac{\partial \widetilde{w}}{\partial z} = \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho w}}{\partial z} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \frac{\partial \overline{\rho}}{\partial z} = \frac{1}{\overline{\rho}} \overline{(\frac{\partial \rho w}{\partial z})} - \frac{\overline{\rho w}}{(\overline{\rho})^2} \overline{(\frac{\partial \rho}{\partial z})}$$

3.2 Turbulent Transport term T

$$T = \frac{T_{ii}}{2} = -\frac{\partial}{\partial x_k}(\frac{1}{2}\overline{\rho} u_i'' \widetilde{u_i'' u_k''})$$

$$T = \frac{T_{xx}}{2} + \frac{T_{yy}}{2} + \frac{T_{zz}}{2}$$

$$= -\frac{\partial}{\partial x}(\frac{1}{2}\overline{\rho u'' u'' u''} + \frac{1}{2}\overline{\rho v'' v'' u''} + \frac{1}{2}\overline{\rho w'' w'' u''}) - \frac{\partial}{\partial y}(\frac{1}{2}\overline{\rho u'' u'' v''} + \frac{1}{2}\overline{\rho v'' v'' v''} + \frac{1}{2}\overline{\rho w'' w'' v''}) - \frac{\partial}{\partial z}(\frac{1}{2}\overline{\rho u'' u'' w''} + \frac{1}{2}\overline{\rho v'' v'' w''} + \frac{1}{2}\overline{\rho w'' w'' w''})$$

$$\overline{\rho u'' u'' u''} = \overline{\rho u^3} - 3\overline{\rho u^2}\frac{\overline{\rho u}}{\overline{\rho}} + 2\frac{(\overline{\rho u})^3}{(\overline{\rho})^2}$$

$$\overline{\rho v'' v'' u''} = \overline{\rho u v^2} - \overline{\rho v^2}\frac{\overline{\rho u}}{\overline{\rho}} - 2\overline{\rho u v}\frac{\overline{\rho v}}{\overline{\rho}} + 2\overline{\rho u}(\frac{\overline{\rho v}}{\overline{\rho}})^2$$

$$\overline{\rho w'' w'' u''} = \overline{\rho u w^2} - \overline{\rho w^2}\frac{\overline{\rho u}}{\overline{\rho}} - 2\overline{\rho u w}\frac{\overline{\rho w}}{\overline{\rho}} + 2\overline{\rho u}(\frac{\overline{\rho w}}{\overline{\rho}})^2$$

$$\overline{\rho u'' u'' w''} = \overline{\rho w u^2} - \overline{\rho u^2}\frac{\overline{\rho w}}{\overline{\rho}} - 2\overline{\rho u w}\frac{\overline{\rho u}}{\overline{\rho}} + 2\overline{\rho w}(\frac{\overline{\rho u}}{\overline{\rho}})^2$$

$$\overline{\rho v'' v'' w''} = \overline{\rho w v^2} - \overline{\rho v^2}\frac{\overline{\rho w}}{\overline{\rho}} - 2\overline{\rho v w}\frac{\overline{\rho v}}{\overline{\rho}} + 2\overline{\rho w}(\frac{\overline{\rho v}}{\overline{\rho}})^2$$

$$\overline{\rho w'' w'' w''} = \overline{\rho w^3} - 3\overline{\rho w^2}\frac{\overline{\rho w}}{\overline{\rho}} + 2\frac{(\overline{\rho w})^3}{(\overline{\rho})^2}$$

3.3 Pressure-Dilatation term Π_d

$$\Pi^d = \frac{\Pi^d_{ii}}{2} = \overline{p' \frac{\partial u_i'}{\partial x_i}}$$

$$\Pi^d = \frac{\Pi^d_{xx}}{2} + \frac{\Pi^d_{yy}}{2} + \frac{\Pi^d_{zz}}{2}$$

$$= \overline{p' \frac{\partial u'}{\partial x}} + \overline{p' \frac{\partial v'}{\partial y}} + \overline{p' \frac{\partial w'}{\partial z}}$$

$$\overline{p' \frac{\partial u'}{\partial x}} = \overline{p \frac{\partial u}{\partial x}} - \overline{p}\frac{\partial \overline{u}}{\partial x} = \overline{p \frac{\partial u}{\partial x}} - \overline{p}\overline{(\frac{\partial u}{\partial x})}$$

$$\overline{p' \frac{\partial v'}{\partial y}} = \overline{p \frac{\partial v}{\partial y}} - \overline{p}\frac{\partial \overline{v}}{\partial y} = \overline{p \frac{\partial v}{\partial y}} - \overline{p}\overline{(\frac{\partial v}{\partial y})}$$

$$\overline{p' \frac{\partial w'}{\partial z}} = \overline{p \frac{\partial w}{\partial z}} - \overline{p}\frac{\partial \overline{w}}{\partial z} = \overline{p \frac{\partial w}{\partial z}} - \overline{p}\overline{(\frac{\partial w}{\partial z})}$$

3.4 Pressure Transport term Π_t

$$\Pi^t = \frac{\Pi^t_{ii}}{2} = -\frac{\partial}{\partial x_k}(\overline{p' u_i'}\delta_{ik})$$

$$\Pi^t = \frac{\Pi^t_{xx}}{2} + \frac{\Pi^t_{yy}}{2} + \frac{\Pi^t_{zz}}{2}$$

$= -\frac{\partial}{\partial x}(\overline{p'u'}) - \frac{\partial}{\partial y}(\overline{p'v'}) - \frac{\partial}{\partial z}(\overline{p'w'})$

$\overline{p'u'} = \overline{pu} - \overline{p}(\overline{u})$

$\frac{\partial}{\partial x}(\overline{p'u'}) = \frac{\partial \overline{\rho u}}{\partial x} - \overline{p}\frac{\partial \overline{u}}{\partial x} - \overline{u}\frac{\partial \overline{p}}{\partial x} = (\overline{\frac{\partial \rho u}{\partial x}}) - \overline{p}(\overline{\frac{\partial u}{\partial x}}) - \overline{u}(\overline{\frac{\partial p}{\partial x}})$

$\overline{p'v'} = \overline{pv} - \overline{p}(\overline{v})$

$\frac{\partial}{\partial y}(\overline{p'v'}) = \frac{\partial \overline{\rho v}}{\partial y} - \overline{p}\frac{\partial \overline{v}}{\partial y} - \overline{v}\frac{\partial \overline{p}}{\partial y} = (\overline{\frac{\partial \rho v}{\partial y}}) - \overline{p}(\overline{\frac{\partial v}{\partial y}}) - \overline{v}(\overline{\frac{\partial p}{\partial y}})$

$\overline{p'w'} = \overline{pw} - \overline{p}(\overline{w})$

$\frac{\partial}{\partial z}(\overline{p'w'}) = \frac{\partial \overline{\rho w}}{\partial z} - \overline{p}\frac{\partial \overline{w}}{\partial z} - \overline{w}\frac{\partial \overline{p}}{\partial z} = (\overline{\frac{\partial \rho w}{\partial z}}) - \overline{p}(\overline{\frac{\partial w}{\partial z}}) - \overline{w}(\overline{\frac{\partial p}{\partial z}})$

3.5 Viscous Dissipation ϵ

$\epsilon = \frac{\epsilon_{ii}}{2} = \overline{\sigma'_{ik}\frac{\partial u'_i}{\partial x_k}}$

$\epsilon = \frac{\epsilon_{xx}}{2} + \frac{\epsilon_{yy}}{2} + \frac{\epsilon_{zz}}{2}$

$= \overline{\sigma'_{xx}\frac{\partial u'}{\partial x}} + \overline{\sigma'_{xy}\frac{\partial u'}{\partial y}} + \overline{\sigma'_{xz}\frac{\partial u'}{\partial z}} + \overline{\sigma'_{xy}\frac{\partial v'}{\partial x}} + \overline{\sigma'_{yy}\frac{\partial v'}{\partial y}} + \overline{\sigma'_{yz}\frac{\partial v'}{\partial z}} + \overline{\sigma'_{xz}\frac{\partial w'}{\partial x}} + \overline{\sigma'_{yz}\frac{\partial w'}{\partial y}} + \overline{\sigma'_{zz}\frac{\partial w'}{\partial z}}$

$\overline{\sigma'_{xx}\frac{\partial u'}{\partial x}} = \overline{\sigma_{xx}\frac{\partial u}{\partial x}} - \overline{\sigma_{xx}}(\overline{\frac{\partial u}{\partial x}})$

$\overline{\sigma'_{xy}\frac{\partial u'}{\partial y}} = \overline{\sigma_{xy}\frac{\partial u}{\partial y}} - \overline{\sigma_{xy}}(\overline{\frac{\partial u}{\partial y}})$

$\overline{\sigma'_{xz}\frac{\partial u'}{\partial z}} = \overline{\sigma_{xz}\frac{\partial u}{\partial z}} - \overline{\sigma_{xz}}(\overline{\frac{\partial u}{\partial z}})$

$\overline{\sigma'_{xy}\frac{\partial v'}{\partial x}} = \overline{\sigma_{xy}\frac{\partial v}{\partial x}} - \overline{\sigma_{xy}}(\overline{\frac{\partial v}{\partial x}})$

$\overline{\sigma'_{yy}\frac{\partial v'}{\partial y}} = \overline{\sigma_{yy}\frac{\partial v}{\partial y}} - \overline{\sigma_{yy}}(\overline{\frac{\partial v}{\partial y}})$

$\overline{\sigma'_{yz}\frac{\partial v'}{\partial z}} = \overline{\sigma_{yz}\frac{\partial v}{\partial z}} - \overline{\sigma_{yz}}(\overline{\frac{\partial v}{\partial z}})$

$\overline{\sigma'_{xz}\frac{\partial w'}{\partial x}} = \overline{\sigma_{xz}\frac{\partial w}{\partial x}} - \overline{\sigma_{xz}}(\overline{\frac{\partial w}{\partial x}})$

$\overline{\sigma'_{yz}\frac{\partial w'}{\partial y}} = \overline{\sigma_{yz}\frac{\partial w}{\partial y}} - \overline{\sigma_{yz}}(\overline{\frac{\partial w}{\partial y}})$

$\overline{\sigma'_{zz}\frac{\partial w'}{\partial z}} = \overline{\sigma_{zz}\frac{\partial w}{\partial z}} - \overline{\sigma_{zz}}(\overline{\frac{\partial w}{\partial z}})$

3.6 Density fluctuation term M

$M = \frac{M_{ii}}{2} = \overline{u''_i}(\frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \frac{\partial \overline{p}}{\partial x_i})$

$M = \frac{M_{xx}}{2} + \frac{M_{yy}}{2} + \frac{M_{zz}}{2}$

$= \overline{u''}(\frac{\partial \overline{\sigma}_{xx}}{\partial x} - \frac{\partial \overline{p}}{\partial x}) + \overline{u''}(\frac{\partial \overline{\sigma}_{xy}}{\partial y} - \frac{\partial \overline{p}}{\partial x}) + \overline{u''}(\frac{\partial \overline{\sigma}_{xz}}{\partial z} - \frac{\partial \overline{p}}{\partial x}) + \overline{v''}(\frac{\partial \overline{\sigma}_{xy}}{\partial x} - \frac{\partial \overline{p}}{\partial y}) + \overline{v''}(\frac{\partial \overline{\sigma}_{yy}}{\partial y} - \frac{\partial \overline{p}}{\partial y}) + \overline{v''}(\frac{\partial \overline{\sigma}_{yz}}{\partial z} - \frac{\partial \overline{p}}{\partial y}) + \overline{w''}(\frac{\partial \overline{\sigma}_{xz}}{\partial x} - \frac{\partial \overline{p}}{\partial z}) + \overline{w''}(\frac{\partial \overline{\sigma}_{yz}}{\partial y} - \frac{\partial \overline{p}}{\partial z}) + \overline{w''}(\frac{\partial \overline{\sigma}_{zz}}{\partial z} - \frac{\partial \overline{p}}{\partial z})$

3.7 Viscous Diffusion term D

$D = \frac{D_{ii}}{2} = \frac{\partial}{\partial x_k}(\overline{\sigma'_{ik}u'_i})$

$D = \frac{D_{xx}}{2} + \frac{D_{yy}}{2} + \frac{D_{zz}}{2}$

$= \frac{\partial}{\partial x}(\overline{\sigma'_{xx}u'} + \overline{\sigma'_{xy}v'} + \overline{\sigma'_{xz}w'}) + \frac{\partial}{\partial y}(\overline{\sigma'_{xy}u'} + \overline{\sigma'_{yy}v'} + \overline{\sigma'_{yz}w'}) + \frac{\partial}{\partial z}(\overline{\sigma'_{xz}u'} + \overline{\sigma'_{yz}v'} + \overline{\sigma'_{zz}w'})$

$\overline{\sigma'_{xx}u'} = \overline{\sigma_{xx}u} - \overline{\sigma_{xx}}(\overline{u})$

$\frac{\partial}{\partial x}(\overline{\sigma'_{xx}u'}) = \overline{\frac{\partial \sigma_{xx}u}{\partial x}} - \overline{\sigma_{xx}}(\overline{\frac{\partial u}{\partial x}}) - \overline{u}(\overline{\frac{\partial \sigma_{xx}}{\partial x}})$

$\overline{\sigma'_{xy}v'} = \overline{\sigma_{xy}v} - \overline{\sigma_{xy}}(\overline{v})$

$\frac{\partial}{\partial x}(\overline{\sigma'_{xy}v'}) = \overline{\frac{\partial \sigma_{xy}v}{\partial x}} - \overline{\sigma_{xy}}(\overline{\frac{\partial v}{\partial x}}) - \overline{v}(\overline{\frac{\partial \sigma_{xy}}{\partial x}})$

$\overline{\sigma'_{xz}w'} = \overline{\sigma_{xz}w} - \overline{\sigma_{xz}}(\overline{w})$

$\frac{\partial}{\partial x}(\overline{\sigma'_{xz}w'}) = \overline{\frac{\partial \sigma_{xz}w}{\partial x}} - \overline{\sigma_{xz}}(\overline{\frac{\partial w}{\partial x}}) - \overline{w}(\overline{\frac{\partial \sigma_{xz}}{\partial x}})$

$\overline{\sigma'_{xy}u'} = \overline{\sigma_{xy}u} - \overline{\sigma_{xy}}(\overline{u})$

$\frac{\partial}{\partial y}(\overline{\sigma'_{xy}u'}) = \overline{\frac{\partial \sigma_{xy}u}{\partial y}} - \overline{\sigma_{xy}}(\overline{\frac{\partial u}{\partial y}}) - \overline{u}(\overline{\frac{\partial \sigma_{xy}}{\partial y}})$

$\overline{\sigma'_{yy}v'} = \overline{\sigma_{yy}v} - \overline{\sigma_{yy}}(\overline{v})$

$\frac{\partial}{\partial y}(\overline{\sigma'_{yy}v'}) = \overline{\frac{\partial \sigma_{yy}v}{\partial y}} - \overline{\sigma_{yy}}(\overline{\frac{\partial v}{\partial y}}) - \overline{v}(\overline{\frac{\partial \sigma_{yy}}{\partial y}})$

$\overline{\sigma'_{yz}w'} = \overline{\sigma_{yz}w} - \overline{\sigma_{yz}}(\overline{w})$

$\frac{\partial}{\partial y}(\overline{\sigma'_{xz}u'}) = \overline{\frac{\partial \sigma_{xz}u}{\partial y}} - \overline{\sigma_{xz}}(\overline{\frac{\partial u}{\partial y}}) - \overline{u}(\overline{\frac{\partial \sigma_{xz}}{\partial z}})$

$\overline{\sigma'_{xz}u'} = \overline{\sigma_{xz}u} - \overline{\sigma_{xz}}(\overline{u})$

$\frac{\partial}{\partial z}(\overline{\sigma'_{xz}u'}) = \overline{\frac{\partial \sigma_{xz}u}{\partial z}} - \overline{\sigma_{xz}}(\overline{\frac{\partial u}{\partial z}}) - \overline{u}(\overline{\frac{\partial \sigma_{xz}}{\partial z}})$

$\overline{\sigma'_{yz}v'} = \overline{\sigma_{yz}v} - \overline{\sigma_{yz}}(\overline{v})$

$\frac{\partial}{\partial z}(\overline{\sigma'_{yz}v'}) = \overline{\frac{\partial \sigma_{yz}v}{\partial z}} - \overline{\sigma_{yz}}(\overline{\frac{\partial v}{\partial z}}) - \overline{v}(\overline{\frac{\partial \sigma_{yz}}{\partial z}})$

$\overline{\sigma'_{zz}w'} = \overline{\sigma_{zz}w} - \overline{\sigma_{zz}}(\overline{w})$

$\frac{\partial}{\partial z}(\overline{\sigma'_{zz}w'}) = \overline{\frac{\partial \sigma_{zz}w}{\partial z}} - \overline{\sigma_{zz}}(\overline{\frac{\partial w}{\partial z}}) - \overline{w}(\overline{\frac{\partial \sigma_{zz}}{\partial z}})$

4 Reference

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