8/15/2018

$\boxed{\text{1st- pass proof (very rough)}}$

- Resembles Bertsekas 2000, Sec. 1.5, see argument starting with equation (1.17) in this section.
- We should make sure that we are using the CVaR-decomposition appropriately in our dynamic system context.

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$$V_k(x_k, y_k) = \min_{u_k} \Big\{ c(x_k) + \max_{Z} \sum_{\omega_k} Z(\omega_k)\, V_{k+1}\big(f_k(x_k, u_k, \omega_k),\, 1-(1-y_k)Z(\omega_k)\big)\, P_k(\omega_k) \Big\}$$

$$Z : D_k \to \left[0, \tfrac{1}{y_k}\right]$$

$$\sum_{\omega_k \in D_k} Z(\omega_k) P_k(\omega_k) = 1$$

$\mathcal{D}_k(y_k)$

$$V_k^*(x_k, y_k) := \min_{\underbrace{\{\mu_k, \dots, \mu_{N-1}\}}_{\pi^k}} \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \{\mu_k, \dots, \mu_{N-1}\}, x_k \right]$$

Let $\varepsilon > 0$. Define $\mu_k^\varepsilon(x_k)$ s.t.

$$\overbrace{c(x_k) + \max_{Z} \sum_{\omega_k} Z(\omega_k)\, V_{k+1}\big(f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k),\, 1-(1-y_k)Z(\omega_k)\big) P_k(\omega_k)}^{LHS_k^\varepsilon} \le V_k(x_k, y_k) + \varepsilon. \quad (1)$$

Define $V_k^\varepsilon(x_k, y_k) = \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \{\mu_k^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}, x_k \right]$.

Base case

$$V_{N-1}(x_{N-1}, y_{N-1}) = \min_{u_{N-1}} \Big\{ c(x_{N-1}) + \max_{Z} \sum_{\omega_{N-1}} Z(\omega_{N-1}) \underbrace{V_N\big(f_{N-1}(x_{N-1}, u_{N-1}, \omega_{N-1}),\, 1-(1-y_{N-1})Z(\omega_{N-1})\big)}_{= C_N(f_{N-1}(x_{N-1}, u_{N-1}, \omega_{N-1}))} P_{N-1}(\omega_{N-1}) \Big\}$$

$$Z : D_{N-1} \to \left[0, \tfrac{1}{y_{N-1}}\right]$$

$$\le \underbrace{c(x_{N-1}) + \max_{Z} \sum_{\omega_{N-1}} Z(\omega_{N-1})\, C_N\big(f_{N-1}(x_{N-1}, \mu_{N-1}^\varepsilon(x_{N-1}), \omega_{N-1})\big) P_{N-1}(\omega_{N-1})}_{= LHS_{N-1}^\varepsilon} \quad (2)$$

①

$$= \max_{\substack{Z \\ Z: D_{N-1} \to [0, \frac{1}{y_{N-1}}] \\ \sum_{\omega_{N-1} \in D_{N-1}} Z(\omega_{N-1}) P(\omega_{N-1}) = 1}} \sum_{\omega_{N-1}} \left(C(x_{N-1}) + C\Big(\underbrace{f_{N-1}(x_{N-1}, \mu^{\varepsilon}_{N-1}(x_{N-1}), \omega_{N-1})}_{x_N}\Big)\right) Z(\omega_{N-1}) P_{N-1}(\omega_{N-1})$$

$$= \max_{\substack{Z \\ Z: D_{N-1} \to [0, \frac{1}{y_{N-1}}] \\ \sum_{\omega_{N-1}} Z(\omega_{N-1}) P(\omega_{N-1}) = 1}} \mathbb{E}_Z \left[C(x_{N-1}) + C(x_N) \,\middle|\, \mu^{\varepsilon}_{N-1}, x_{N-1} \right]$$

dual rep for CVaR

$$= \mathrm{CVaR}_{y_{N-1}} \left[\sum_{i=N-1}^{N} C(x_i) \,\middle|\, \mu^{\varepsilon}_{N-1}, x_{N-1} \right]$$

$$= \underline{V^{\varepsilon}_{N-1}(x_{N-1}, y_{N-1})} \underset{(2)}{=} LHS^{\varepsilon}_k \underset{(1)}{\leq} V_{N-1}(x_{N-1}, y_{N-1}) + \varepsilon$$

$$\boxed{\therefore \; V_{N-1}(x_{N-1}, y_{N-1}) \leq V^{\varepsilon}_{N-1}(x_{N-1}, y_{N-1}) \leq V_{N-1}(x_{N-1}, y_{N-1}) + \varepsilon.}$$

analogous to 1.19 for $k := N-1$

$$V^*_{N-1}(x_{N-1}, y_{N-1}) := \min_{\mu_{N-1}} \mathrm{CVaR}_{y_{N-1}}\left[c(x_{N-1}) + c(x_N) \,\middle|\, \mu_{N-1}, x_{N-1} \right]$$

$$\leq \mathrm{CVaR}_{y_{N-1}}\left[c(x_{N-1}) + c(x_N) \,\middle|\, \mu^{\varepsilon}_{N-1}, x_{N-1} \right]$$

$$= V^{\varepsilon}_{N-1}(x_{N-1}, y_{N-1})$$

$$\overset{(2)}{\leq} \mathrm{LHS}^{\varepsilon}_{N-1}$$

$$\leq V_{N-1}(x_{N-1}, y_{N-1}) + \varepsilon$$

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{\substack{Z \\ Z : D_{N-1} \to [0, \frac{1}{y_{N-1}}] \\ \sum_{\omega_{N-1}} Z(\omega_{N-1}) P(\omega_{N-1}) = 1}} \sum_{\omega_{N-1}} Z(\omega_{N-1}) \underbrace{V_N(\overset{x_N}{f_{N-1}(x_{N-1}, u_{N-1}, \omega_{N-1})}, \cdot)}_{\overset{\shortparallel}{c(x_N)}} P_{N-1}(\omega_{N-1}) \right\} + \varepsilon$$

$$= \min_{u_{N-1}} \left\{ \underbrace{\max_{Z} \sum_{\omega_{N-1}} Z(\omega_{N-1}) P_{N-1}(\omega_{N-1}) \left(c(x_{N-1}) + c(x_N) \right)}_{\substack{\max_{Z} \mathbb{E}_Z[c(x_{N-1}) + c(x_N)] \\ Z : D_{N-1} \to [0, \frac{1}{y_{N-1}}]}} \right\} + \varepsilon$$

dual rep for CVaR

$$= \min_{u_{N-1}} \mathrm{CVaR}_{y_{N-1}}\left[c(x_{N-1}) + c(x_N) \,\middle|\, x_{N-1}, u_{N-1} \right] + \varepsilon$$

Analogous to 1.20
$K := N-1$

$$= V^*_{N-1}(x_{N-1}, y_{N-1}) + \varepsilon.$$

$$\therefore \boxed{V^*_{N-1}(x_{N-1}, y_{N-1}) \leq V^{\varepsilon}_{N-1}(x_{N-1}, y_{N-1}) \leq V^*_{N-1}(\cdot, \cdot) + \varepsilon}$$

$$V_k^\varepsilon(x_k, y_k) = \mathrm{CVaR}_{y_k}\left[\underbrace{c(x_k) + \sum_{i=k+1}^{N} c(x_i)}_{\text{lives in sigma algebra assoc. w/ times } k \text{ thru } N,\ \mathcal{F}} \,\middle|\, \{\mu_k^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}, x_k \right]$$

$\therefore$ Eqns hold for $k = N-1$.
ASSUME they ~~hold for index~~ $k+1$. (k, $k+1$, $\dots$, N)
Need to show that they hold for index k.

CVaR decomposition

$$= \max_{Z} \mathbb{E}\left[Z \cdot \underbrace{\mathrm{CVaR}_{1-(1-y_k)Z}\left[c(x_k) + \sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\mu_k^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}, x_{k+1} \right]}_{c(x_k) + \mathrm{CVaR}\left[\sum_{k+1}^{N} c(x_i) \,\middle|\, \dots, x_{k+1}\right]} \,\middle|\, \overbrace{\{\mu_k^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}}^{\pi_k^\varepsilon}, x_k \right]$$

$Z : D_k \to [0, \frac{1}{y_k}]$

$\sum_{\omega_k} Z(\omega_k) P_k(\omega_k) = 1$

$x_{k+1} = f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k)$

$$\mathbb{E}\left[Z \left(c(x_k) + \mathrm{CVaR}_{1-(1-y_k)Z}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\mu_k^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}, x_{k+1} \right] \right) \,\middle|\, \pi_k^\varepsilon, x_k \right]$$

$\parallel$ μ_k is not being used

$$c(x_k) + \mathrm{CVaR}_{1-(1-y_k)Z}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\mu_{k+1}^\varepsilon, \dots, \mu_{N-1}^\varepsilon\}, x_{k+1} \right]$$

$\parallel$ Def

$$V_{k+1}^\varepsilon(x_{k+1}, 1-(1-y_k)Z)$$

$$\mathbb{E}\left[Z \left(c(x_k) + V_{k+1}^\varepsilon\left(f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k), 1-(1-y_k)Z \right) \right) \,\middle|\, \pi_k^\varepsilon, x_k \right]$$

$$= c(x_k) + \max_{\substack{Z \\ Z : D_k \to [0, \frac{1}{y_k}]}} \mathbb{E}\left[Z \cdot \underbrace{V_{k+1}^\varepsilon\left(f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k), 1-(1-y_k)Z \right)}_{\le V_{k+1}(\cdot\,,\,\cdot) + (N-k-1)\varepsilon} \,\middle|\, \pi_k^\varepsilon, x_k \right]$$

induction hyp

$$\leq c(x_k) + \max_{\substack{Z \\ Z : D_k \to [0, \frac{1}{y_k}] \\ \mathbb{E}[Z]=1}} \mathbb{E}\left[Z \cdot V_{k+1}\left(f_k(x_k, \mu_k^{\varepsilon}(x_k), w_k), 1-(1-y_k)Z \right) \Big| \pi_k^{\varepsilon}, x_k \right] + (N-k-1)\varepsilon$$

$$\underset{(1)}{\leq} V_k(x_k, y_k) + \varepsilon$$

$$\leq V_k(x_k, y_k) + \varepsilon + (N-k-1)\varepsilon$$

$$= V_k(x_k, y_k) + (N-k)\varepsilon .$$

(analagous to part of 1.19)

$$\boxed{\therefore \quad V_k^{\varepsilon}(x_k, y_k) \leq V_k(x_k, y_k) + (N-k)\varepsilon .}$$

Key idea (analogous to * p.47 Bertsekas)

$$V_k^{\varepsilon}(x_k, y_k) = c(x_k) + \max_{\substack{Z \\ Z : D_k \to [0, \frac{1}{y_k}] \\ \mathbb{E}[Z]=1}} \mathbb{E}\left[Z \cdot V_{k+1}^{\varepsilon}\left(f_k(x_k, \mu_k^{\varepsilon}(x_k), w_k), 1-(1-y_k)Z \right) \Big| \pi_k^{\varepsilon}, x_k \right]$$

$$V_k^\varepsilon(x_k, y_k) = c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{w_k \in D_k} Z(w_k) \cdot \underbrace{V_{k+1}^\varepsilon\left(f_k(x_k, \mu_k^\varepsilon(x_k), w_k), 1-(1-y_k)Z(w_k)\right)}_{\geq V_{k+1}(\cdot,\cdot)} \cdot P_k(w_k)$$

by induction hypothesis

$$\geq c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{w_k \in D_k} Z(w_k) \cdot V_{k+1}\left(f_k(x_k, \mu_k^\varepsilon(x_k), w_k), 1-(1-y_k)Z(w_k)\right) \cdot P_k(w_k)$$

$$\geq \min_{u_k} \left\{ c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{w_k \in D_k} Z(w_k) \cdot V_{k+1}\left(f_k(x_k, u_k, w_k), 1-(1-y_k)Z(w_k)\right) \cdot P_k(w_k) \right\}$$

$$= V_k(x_k, y_k)$$

$\therefore$ $\boxed{V_k^\varepsilon(x_k, y_k) \geq V_k(x_k, y_k).}$

$\therefore$ $\boxed{\text{Induction step is complete for 1.19}}$

$\mathcal{D}_k(y_k)$: the set of all random variables $Z : D_k \to [0, \frac{1}{y_k}]$, such that $\mathbb{E}[Z] = 1$.

$$V_k^\varepsilon(x_k, y_k) = c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{\omega_k \in D_k} Z(\omega_k) \cdot \underbrace{\boxed{V_{k+1}^\varepsilon\left(f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k), 1-(1-y_k)Z(\omega_k)\right)}}_{\le V_{k+1}(\cdot,\cdot) + (N-k-1)\varepsilon \text{ by induction hypothesis}} \cdot P_k(\omega_k)$$

$$\le \underbrace{\boxed{c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{\omega_k \in D_k} Z(\omega_k) \cdot V_{k+1}\left(f_k(x_k, \mu_k^\varepsilon(x_k), \omega_k), 1-(1-y_k)Z(\omega_k)\right) \cdot P_k(\omega_k)}}_{\underset{(i)}{\le} V_k(x_k, y_k) + \varepsilon} + (N-k-1)\varepsilon$$

$$\le V_k(x_k, y_k) + \varepsilon + (N-k-1)\varepsilon$$

$$= V_k(x_k, y_k) + (N-k)\varepsilon$$

$$= \min_{u_k} \left\{ c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{\omega_k \in D_k} Z(\omega_k) \cdot V_{k+1}\left(f_k(x_k, u_k, \omega_k), 1-(1-y_k)Z(\omega_k)\right) \cdot P_k(\omega_k) \right\} + (N-k)\varepsilon$$

$$\le c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{\omega_k \in D_k} Z(\omega_k) \cdot \boxed{V_{k+1}\left(f_k(x_k, \mu_k(x_k), \omega_k), 1-(1-y_k)Z(\omega_k)\right)} \cdot P(\omega_k) + (N-k)\varepsilon \quad \forall \mu_k$$

$\parallel$ by induction hypothesis

$$V_{k+1}^*\left(f_k(x_k, \mu_k(x_k), \omega_k), 1-(1-y_k)Z(\omega_k)\right)$$

$$\parallel$$

$$\min_{\{\mu_{k+1}, \mu_{k+2}, \ldots, \mu_{N-1}\}} \mathrm{CVaR}_{1-(1-y_k)Z(\omega_k)}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\mu_{k+1}, \ldots, \mu_{N-1}\}, x_{k+1} = f_k(x_k, \mu_k(x_k), \omega_k)\right]$$

$$\le$$

$$\mathrm{CVaR}_{1-(1-y_k)Z(\omega_k)}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\underline{\mu_{k+1}}, \ldots, \mu_{N-1}\}, x_{k+1} = f_k(x_k, \mu_k(x_k), \omega_k)\right]$$

μ_k is already given making this explicit.

$$\parallel$$

$$\mathrm{CVaR}_{1-(1-y_k)Z(\omega_k)}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \{\underline{\mu_k}, \ldots, \mu_{N-1}\}, x_{k+1} = f_k(x_k, \mu_k(x_k), \omega_k)\right]$$

$$\forall \{\mu_{k+1}, \ldots, \mu_{N-1}\}.$$

$$\leq \quad c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \sum_{w_k \in D_k} Z(w_k) \cdot CVaR_{1-(1-y_k)Z(w_k)}\left[\sum_{i=k+1}^{N} c(x_i) \,\Big|\, \{\mu_k,\dots,\mu_{N-1}\}, x_{k+1} = f_k(x_k, \mu_k(x_k), w_k)\right] \cdot P_k(w_k) + (N-k)\varepsilon$$

$$\forall \{\mu_k,\dots,\mu_{N-1}\} = \pi^k$$

$$= \quad c(x_k) + \max_{Z \in \mathcal{D}_k(y_k)} \mathbb{E}\left[Z \cdot CVaR_{1-(1-y_k)Z}\left[\sum_{i=k+1}^{N} c(x_i) \,\Big|\, \pi^k, x_{k+1}\right] \,\Big|\, \pi^k, x_k\right] + (N-k)\varepsilon$$

decomposition result

$$= \quad c(x_k) + CVaR_{y_k}\left[\sum_{i=k+1}^{N} c(x_i) \,\Big|\, \pi^k, x_k\right] + (N-k)\varepsilon \qquad \forall \pi^k$$

$$= \quad CVaR_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \pi^k, x_k\right] + (N-k)\varepsilon \qquad \forall \pi^k$$

We have shown $\quad V_k^{\varepsilon}(x_k, y_k) \leq CVaR_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \pi^k, x_k\right] + (N-k)\varepsilon \qquad \forall \pi^k$

$$\Rightarrow \quad V_k^{\varepsilon}(x_k, y_k) \leq \min_{\pi^k} CVaR_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \pi^k, x_k\right] + (N-k)\varepsilon$$

$$= V_k^{*}(x_k, y_k) + (N-k)\varepsilon$$

$$\boxed{\therefore \quad V_k^{\varepsilon}(x_k, y_k) \leq V_k^{*}(x_k, y_k) + (N-k)\varepsilon.}$$

(induction step for part of (1.20) is proved)

$$V_k^{*}(x_k, y_k) = \min_{\pi^k} CVaR_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \pi^k, x_k\right] \leq CVaR_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\Big|\, \pi^k_{\varepsilon} = \{\mu_k^{\varepsilon},\dots,\mu_{N-1}^{\varepsilon}\}, x_k\right] = V_k^{\varepsilon}(x_k, y_k).$$

$$\boxed{\therefore \quad V_k^{*}(x_k, y_k) \leq V_k^{\varepsilon}(x_k, y_k).}$$

$\boxed{\therefore \text{ proved induction step for } 1.20.}$

Thus,

$$\left.\begin{aligned} V_k(x_k, y_k) &\le V_k^{\varepsilon}(x_k, y_k) \le V_k(x_k, y_k) + (N-k)\,\varepsilon \\ V_k^{*}(x_k, y_k) &\le V_k^{\varepsilon}(x_k, y_k) \le V_k^{*}(x_k, y_k) + (N-k)\,\varepsilon \end{aligned}\right\} \text{ holds for } \varepsilon > 0.$$

Take $\varepsilon \to 0$, to get $V_k^{*}(x_k, y_k) = V_k(x_k, y_k)$.