8/17/2018

start of 2nd-pass proof (and explanation (not yet finished)

"Shapiro" refers to Lectures on Stochastic Programming Modeling and Theory

"Chow 2015" refers to Risk-Sensitive and Robust Decision-Making: A CVaR Optimization Approach

"Bertsekas 2000" refers to Dynamic Programming and Optimal Control, Vol I

Please see Sec. 1.5 "Some Mathematical Issues." Our proof will resemble the argument starting with equation (1.17) in this section.

System model

$$x_{k+1} = f_k(x_k, u_k, w_k) \qquad k = 0, 1, \dots, N-1 \qquad (1)$$

$x_k \in S$, $u_k \in C$, $w_k \in D_k$ countable

$P_k(w_k)$ is the probability that the random disturbance takes on the value w_k at time k

$w_0, \dots, w_{N-1}$ are independent but may not be identically distributed

$\Pi := \{(\mu_0, \dots, \mu_{N-1}), \mu_i : S \to C\}$ set of admissible control policies

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Goal: To compute

$$J^*(x_0, \alpha) := \min_{\pi \in \Pi} \mathrm{CVaR}_\alpha\left[\sum_{k=0}^{N} c(x_k) \,\middle|\, x_0, \pi \right] \quad \forall x_0 \in S, \ \forall \alpha \in L \subset (0,1) \qquad (2)$$

$c : S \to \mathbb{R}$ bounded

random state at time k under policy π, subject to the dynamics (1), starting from non-random initial condition, $x_0 \in S$

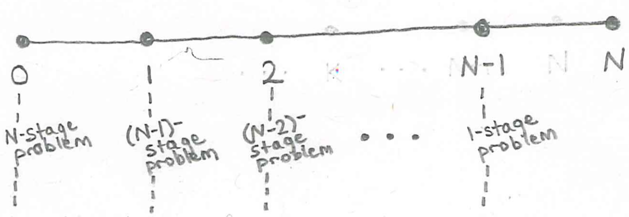
set of confidence levels

Next, we will develop machinery to compute (2).

Definitions

$$J_k^*(x_k, y_k) := \min_{\pi_k \in \Pi_k} \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right]$$

is the optimal cost for the $(N-k)$-stage problem that starts at state x_k, confidence level y_k, time k, and ends at time N.



$\Pi_k := \{(\mu_k, \dots, \mu_{N-1}), \mu_i : S \times L \to C\}$

set of admissible control policies for the $(N-k)$-stage problem.

Dual Representation of CVaR

(see Shapiro, eqs. (6.37) (6.40) (6.70); Shapiro calls CVaR, AV@R for Average Value-at-Risk; note that CVaR is a coherent risk measure; see Shapiro Sec. 6.3)

Let $Z \in \underbrace{\mathcal{L}_1(\Omega, \mathcal{F}, P)}_{\mathcal{Z}_1}$ be a random variable with finite expectation on the probability space, $(\Omega, \mathcal{F}, P)$,

set of possible outcomes — set of events "σ-algebra"

and denote $\mathcal{Z}_\infty := \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ the set of bounded random variables.

since $Z \in \mathcal{L}_1$, $CVaR_\alpha[Z] \in \mathbb{R}$, see Shapiro Sec. 6.3

$$CVaR_\alpha[Z] = \sup_{V \in A} \int_{\omega \in \Omega} V(\omega) Z(\omega)\, dP(\omega) =: \sup_{V \in A} \mathbb{E}[V \cdot Z] = \max_{V \in A} \mathbb{E}[V \cdot Z]$$

def of expectation w.r.t prob. measure P

$$A := \left\{ V \in \mathcal{Z}_\infty \;\middle|\; V(\omega) \in [0, \tfrac{1}{\alpha}] \text{ for almost every } \omega \in \Omega,\ \mathbb{E}[V] := \int_{\omega \in \Omega} V(\omega)\, dP(\omega) = 1 \right\}$$

Example: Do 2 coin flips

$\Omega := \{HH, HT, TT, TH\}$

E_1 = "at least 1 head" = $\{HH, HT, TH\}$

E_2 = "no more than 1 head" = $\{HT, TH, TT\}$

$E_1 \in \mathcal{F}$, $E_2 \in \mathcal{F}$, $E_3 \in \mathcal{F}$ example events

$E_3 = \{HH\}$.

Decomposition of CVaR

(Chow 2015; Pflug & Pichler 2015)

Let $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_s \subset \mathcal{F}$ be a sub-σ-algebra.

$CVaR_\alpha[Z] \in \mathbb{R}$, so sup is attained

restriction to a smaller collection of events

dependence of $\mathcal{F}$ is implicit

$$CVaR_\alpha[Z] = CVaR_\alpha[Z \mid \mathcal{F}] = \max_{V \in \mathcal{B}} \mathbb{E}\left[V \cdot CVaR_{\alpha V}[Z \mid \mathcal{F}_s] \;\middle|\; \mathcal{F} \right] = \max_{V \in \mathcal{B}} \mathbb{E}\left[V \cdot CVaR_{\alpha V}[Z \mid \mathcal{F}_s] \right].$$

Making dependence on $\mathcal{F}$ explicit

$$\mathcal{B} := \left\{ V \in \mathcal{L}_1(\Omega, \mathcal{F}_s, P) \;\middle|\; V(\omega) \in [0, \tfrac{1}{\alpha}] \text{ a.e. } \omega \in \Omega,\ \mathbb{E}[V] = 1 \right\}$$

random variable on the smaller collection of events

Please look closely here. Is this correct?

Example:

lives in some $\mathcal{L}_1$ space there's a $\mathcal{F}$ for it and a probability measure P, and a Ω

$$CVaR_\alpha\left[\sum_{i=k}^{N} c(x_i) \;\middle|\; x_k, \pi^k \right] = \max_{V \in \mathcal{B}} \mathbb{E}\left[V \cdot CVaR_{\alpha V}\left[\sum_{i=k}^{N} c(x_i) \;\middle|\; x_{k+1}, \pi^k \right] \;\middle|\; x_k, \pi^k \right]$$

associated with the outcomes of the system from time k to time N, "$\mathcal{F}$"

associated with the outcomes of the system from time k to time N, "$\mathcal{F}$"

associated with the outcomes of the system from time k+1 to time N, "$\mathcal{F}_s$"

<u>Remark</u> Pflug & Pichler 2015 defines the CVaR decomposition in terms of sigma-algebras.

Chow 2015 " " " the data process.

Shapiro (see just below eq. 6.269 in Sec. 6.7 Multistage Risk-Averse Optimization) explains how either sigma-algebras or the data process can be used to define conditional expectations and conditional risk-measures.

<u>DP Algorithm</u>

Define the functions, $J_{N-1}, \dots, J_0$, recursively as follows, $\forall\, x_k \in S,\ \forall\, y_k \in L$ (finite subset of $(0,1)$)

$$J_k(x_k, y_k) := \min_{u_k} \left\{ c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, u_k \right] \right\} \quad k = N-1, \dots, 0, \tag{3}$$

$R \cdot J_{k+1}(x_{k+1}, y_k R)$: a random variable on the event space associated with the outcomes of the system from time k to time N

$$\mathcal{B}(y_k, k+1) := \left\{ \begin{array}{l} \text{random variable assoc. on the event space of the} \\ \text{with the outcomes of the system from time } k+1 \text{ to time } N, \\ R \in [0, \tfrac{1}{y_k}],\ \mathbb{E}[R] = 1 \end{array} \right\}$$

with initialization $J_N(x, y) = c(x)\ \forall\, x \in S,\ \forall\, y \in L.$

Assume that the functions J_k $k = 0, \dots, N$ are well-defined and finite. (This is true in particular if each D_k is a finite set (see Bertsekas, Sec.1.5, just above eq. 1.16).)

Then, $$J^*(x_0, \alpha) \overset{(2)}{:=} \min_{\pi \in \Pi} \mathrm{CVaR}_\alpha\left[\sum_{k=0}^{N} c(x_k) \,\middle|\, x_0, \pi \right] = J_0(x_0, \alpha).$$

$J_0(x_0, \alpha)$: function at the last step of the recursion evaluated at state x_0 and level α.

Also, if $u_k^* = \mu_k^*(x_k, y_k)$ is a minimizer of (3) for each (x_k, y_k) and each $k = 0, \dots, N-1$,

then $\{\mu_0^*, \dots, \mu_{N-1}^*\}$ is an optimal policy.

Remark

The DP algorithm is an extension of the one provided by Bertsekas 2000 (see Proposition 1.3.1, in Sec. 1.3), and is the finite-time analog of the algorithm provided by Chow 2015.

Our proof is built on the one provided by Bertsekas 2000 (see Sec. 1.5).

Proof idea

Define a sub-optimal policy and a sub-optimal cost, and use these to show that J_k (output from DP) and J_k^* (see p. 1) are very close for each k by induction. Showing $J_k = J_k^*$ directly is hard because it is not obvious why we are allowed to exchange the order of the maximum and the minimum.

Proof details

Let $\varepsilon > 0$. For all $k = 0, \dots, N-1$ and $(x_k, y_k) \in S \times L$, let $\mu_k^\varepsilon : S \times L \to C$ be "ε-optimal", i.e.,

$$c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^\varepsilon(x_k)\right] \leq J_k(x_k, y_k) + \varepsilon . \quad (4)$$

$$\mathcal{B}(y_k, k+1) := \left\{ \begin{array}{l} \text{rand. var. on event space} \\ \text{assoc. w/ outcomes of system} \\ \text{from time } k+1 \text{ to } N \end{array}, \; R \in \left[0, \tfrac{1}{y_k}\right], \; \mathbb{E}[R] = 1 \right\}$$

Let $J_k^\varepsilon(x_k, y_k)$ be the "ε-optimal cost", i.e.,

$$J_k^\varepsilon(x_k, y_k) := \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k^\varepsilon \right], \quad \text{where } \pi_k^\varepsilon := (\mu_k^\varepsilon, \mu_{k+1}^\varepsilon, \ldots, \mu_{N-1}^\varepsilon).$$

Recall the definitions of J_k^* (see p. 1) and J_k (see p. 3).

We will show by induction for all $(x_k, y_k) \in S \times L$ and $k = N-1, \ldots, 0$, the following inequalities,

$$J_k(x_k, y_k) \le J_k^\varepsilon(x_k, y_k) \le J_k(x_k, y_k) + (N-k)\varepsilon$$

$$J_k^*(x_k, y_k) \le J_k^\varepsilon(x_k, y_k) \le J_k^*(x_k, y_k) + (N-k)\varepsilon$$

$$J_k(x_k, y_k) = J_k^*(x_k, y_k),$$

following the example of Bertsekas (Sec. 1.5).

<u>Base case</u> $k = N-1$

$$J_{N-1}(x_{N-1}, y) := \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot \overset{c(x_N)}{\overset{\|}{J_N(x_N, yR)}} \,\middle|\, x_{N-1}, u_{N-1} \right] \right\}$$

random variable on the event space assoc. w/ outcomes of system at time N, $R \in [0, \frac{1}{y}]$, $\mathbb{E}[R] = 1$

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot c(x_N) \,\middle|\, x_{N-1}, u_{N-1} \right] \right\}$$

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot \overset{= c(x_N)}{\mathrm{CVaR}_{yR}\left[c(x_N) \mid x_N, u_{N-1} \right]} \;\middle|\; x_{N-1}, u_{N-1} \right] \right\}$$

please verify

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \mathrm{CVaR}_{y}\left[c(x_N) \mid x_{N-1}, u_{N-1} \right] \right\}$$

by CVaR Decomposition Thm

$$= \min_{u_{N-1}} \left\{ \mathrm{CVaR}_{y}\left[c(x_{N-1}) + c(x_N) \mid x_{N-1}, u_{N-1} \right] \right\}$$

$a \in \mathbb{R}$, $a + \mathrm{CVaR}(Z) = \mathrm{CVaR}(a+Z)$

$$= \min_{\pi_{N-1} \in \Pi_{N-1}} \left\{ \mathrm{CVaR}_{y}\left[c(x_{N-1}) + c(x_N) \mid x_{N-1}, \pi_{N-1} \right] \right\}$$

$\hookrightarrow = \{ \mu_{N-1} : S \times L \to C \}$

$$= J^*_{N-1}(x_{N-1}, y)$$

<u>Next</u>: For completeness, show the inequalities for $k = N-1$ (or do we get these for free from $J^*_{N-1} = J_{N-1}$?