

# A Risk-Sensitive Finite-Time Reachability Approach for Safety of Stochastic Dynamic Systems

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**Abstract**—A classic reachability problem for safety of dynamic systems is to compute the set of initial states from which the state trajectory is guaranteed to stay inside a given constraint set over some time horizon. In this paper, we leverage existing theory of reachability analysis and risk measures to devise a *risk-sensitive* reachability approach for safety of stochastic dynamic systems under non-adversarial disturbances over a finite time horizon. Specifically, we first introduce the notion of a *risk-sensitive safe set* as the set of initial states from which the risk of extreme constraint violations can be made small via an appropriate control policy, where risk is quantified using the *Conditional Value-at-Risk* (CVaR) measure. Second, we show how the computation of a risk-sensitive safe set can be reduced to the solution to a Markov Decision Process (MDP), where cost is assessed according to CVaR. Third, leveraging this reduction, we devise a tractable algorithm to approximate a risk-sensitive safe set, and provide theoretical arguments about its correctness. Finally, we present numerical experiments that demonstrate the utility of risk-sensitive reachability analysis. In particular, our approach allows a practitioner to tune the level of risk sensitivity from worst-case (which is typical for Hamilton-Jacobi reachability analysis) to risk-neutral (which is the case for stochastic reachability analysis).

## I. INTRODUCTION

Reachability analysis is a formal verification method based on optimal control theory that is used to prove safety or performance properties of dynamic systems [1]. A classic reachability problem for safety is to compute the set of initial states from which the state trajectory is guaranteed to stay inside a given constraint set over some time horizon. This problem was first considered for discrete-time dynamic systems by Bertsekas and Rhodes under the assumption that disturbances are uncertain but belong to known sets [2], [3], [4]. In this context, the problem is solved using a minimax formulation, in which disturbances behave adversarially and safety is described as a binary notion based on set membership [2], [3], [4, Sec. 3.6.2].

In practice, minimax formulations can yield overly conservative solutions, particularly because disturbances are not often adversarial. Most storms do not cause major floods, and most vehicles are not involved in pursuit-evader games. If there are enough observations of the system, one can estimate a probability distribution for the disturbance (e.g., see [5]), and then assess safety properties of the system in a more realistic context. For stochastic discrete-time dynamic systems, Abate et al. [6] developed an algorithm that computes the set of initial states from which the probability of safety of the state trajectory can be made large by an appropriate control policy.<sup>1</sup> Summers and Lygeros [7] extended the algorithm of Abate et al. to quantify the probability of safety and performance of the state trajectory, by specifying that the state trajectory should also reach a target set.

Both the stochastic reachability methods [6], [7] and the minimax reachability methods [2], [3], [4] for discrete-time dynamic systems describe safety as a binary notion based on set membership. In Abate et al., for example, the probability of safety to be optimized is formulated as the expectation of the product (or maximum) of indicator functions, where each indicator encodes the event that the state at a particular time point is inside a given set [6]. Thus, the stochastic reachability methods [6], [7] do not generalize to quantify the random distance between the state trajectory and the boundary of the constraint set, since they use indicator functions to convert probabilities to expectations to be optimized.

In contrast, Hamilton-Jacobi (HJ) reachability methods quantify the deterministic analogue of this distance for continuous-time systems subject to adversarial disturbances (e.g., see [1], [8], [9], [10]). Quantifying the distance between the state trajectory and the boundary of the constraint set in a non-binary fashion may be important in applications where the boundary is not known exactly, or where mild constraint violations are inevitable, but extreme constraint violations must be avoided.

It is imperative that reachability methods for safety take into account the possibility that rare events can occur with potentially damaging consequences. Reachability methods that assume adversarial disturbances (e.g., [1], [3]) suppose that harmful events can always occur, which may yield solutions with limited practical utility, especially in applications with large uncertainty sets. Stochastic reachability methods [6], [7] do not explicitly account for rare high-

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<sup>1</sup>Safety of the state trajectory is the event that the state trajectory stays in the constraint set over a finite time horizon.

consequence events, as costs are evaluated according to a risk-neutral expectation.

In contrast, in this paper, we harness the tool of *risk measure theory* to formulate a reachability analysis approach that explicitly accounts for the possibility of rare events with negative consequences: harmful events are likely to occur at some point, but they are unlikely to occur always. Specifically, a *risk measure* is a function that maps a random variable,  $Z$ , representing a loss, or a cost, into the real line, according to the risk associated with  $Z$  [11, Sec. 6.3], [12, Sec. 2.2]. Risk-sensitive optimization has been studied in applied mathematics [13], reinforcement learning [14], [15], [16], and optimal control [17]. Risk-sensitive formulations have the potential to inform practical decision-making that also protects against damaging outcomes, where the level of conservatism can be modified as needed.

In this paper, we use a particular risk measure, called *Conditional Value-at-Risk* (CVaR). If  $Z$  is a random cost with finite expectation, then the Conditional Value-at-Risk of  $Z$  at the confidence level,  $\alpha \in (0, 1]$ , is defined as [11, Equation 6.22],<sup>2</sup>

$$\text{CVaR}_\alpha[Z] := \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha} \mathbb{E}[\max\{Z - t, 0\}] \right\}. \quad (1)$$

Note that  $\text{CVaR}_\alpha[Z]$  increases from  $\mathbb{E}[Z]$  to  $\text{ess sup } Z$ , as  $\alpha$  decreases from 1 to 0. Thus, CVaR captures a full spectrum of risk assessments, from risk-neutral to worst-case, according to the value of  $\alpha$ . Further, there is a well-established relationship between CVaR and chance constraints that we will use to obtain probabilistic safety guarantees.

*Statement of Contributions.* This paper introduces a *risk-sensitive reachability* approach for safety of stochastic dynamic systems under non-adversarial disturbances over a finite-time horizon. Specifically, the contributions are four-fold. First, we introduce the notion of a *risk-sensitive safe set* as the set of initial states from which the risk of extreme constraint violations can be made small via an appropriate control policy, where risk is quantified using the *Conditional Value-at-Risk* (CVaR) measure. Our formulation explicitly assesses the distance between the boundary of the constraint set and the state trajectory of a stochastic dynamic system. This is an extension of stochastic reachability methods (e.g., [6], [7]), which replace this distance with a binary random variable. Further, in contrast to stochastic reachability methods, our formulation explicitly accounts for rare high-consequence events, by posing the optimal control problem in terms of CVaR, instead of a risk-neutral expectation. To the best of our knowledge, this paper is the first to use the Conditional Value-at-Risk measure in the reachability literature. Second, we show how the computation of a risk-sensitive safe set can be reduced to the solution to a Markov Decision Process (MDP), where cost is assessed according to CVaR. Third, leveraging this reduction, we devise a tractable algorithm to approximate a risk-sensitive safe set, and provide theoretical arguments about its correctness. Finally, we

present numerical experiments that demonstrate the utility of risk-sensitive reachability analysis.

*Organization.* The rest of this paper is organized as follows. We present the problem formulation and define risk-sensitive safe sets in Sec. II. In Sec. III, we show how the computation of a risk-sensitive safe set can be reduced to the solution to a CVaR-MDP problem, i.e., an MDP where cost is assessed according to CVaR. In Sec. IV, we present a value-iteration algorithm to approximate risk-sensitive safe sets, along with theoretical arguments that support its correctness. In Sec. V, we provide a numerical example in the domain of stormwater infrastructure design. Finally, in Sec. VI, we draw conclusions and discuss directions for future work.

## II. PROBLEM FORMULATION

### A. System Model

We consider a fully observable stochastic discrete-time dynamic system over a finite-time horizon [4, Sec. 1.2],

$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \dots, N-1, \quad (2)$$

such that  $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$  is the state of the system at time  $k$ ,  $u_k \in U$  is the control at time  $k$ , and  $w_k \in D$  is the random disturbance at time  $k$ . The control space,  $U$ , and disturbance space,  $D$ , are finite sets of real-valued vectors. The function,  $f : \mathcal{X} \times U \times D \rightarrow \mathcal{X}$ , is bounded and Lipschitz continuous. The probability that the disturbance equals  $d_j \in D$  at time  $k$  is,  $\mathbb{P}[w_k = d_j] = p_j$ , where  $0 \leq p_j \leq 1$  and  $\sum_{j=1}^W p_j = 1$ . We assume that  $w_k$  is independent of  $x_k$ ,  $u_k$ , and disturbances at any other times. The only source of randomness in the system is the disturbance. In particular, the initial state,  $x_0$ , is not random. The set of *admissible, deterministic, history-dependent control policies* is,

$$\Pi := \{(\mu_0, \mu_1, \dots, \mu_{N-1}) \mid \mu_k : H_k \rightarrow U\}, \quad (3)$$

where  $H_k := \underbrace{\mathcal{X} \times \dots \times \mathcal{X}}_{(k+1) \text{ times}}$  is the set of state histories up to time  $k$ . We are given a constraint set,  $\mathcal{K} \subseteq \mathcal{X}$ , and the *safety criterion* that the state of the system should stay inside  $\mathcal{K}$  over time. For example, if the system is a pond, then  $x_k$  may be the water level of the pond at time  $k$ , and  $\mathcal{K} := [0, 5\text{ft})$  indicates that the pond overflows if the water level exceeds 5ft. We quantify the extent of constraint violation/satisfaction using a surface function that characterizes the constraint set. Specifically, similar to [9, Eq. 2.3], let  $g : \mathcal{X} \rightarrow \mathbb{R}$  satisfy,

$$x \in \mathcal{K} \iff g(x) < 0. \quad (4)$$

For example, we may choose  $g(x) := x - 5$  to characterize  $\mathcal{K} := [0, 5\text{ft})$  on the state space,  $\mathcal{X} = [0, \infty)$ .

### B. Risk-Sensitive Safe Sets

A *risk-sensitive safe set* is a set of initial states from which the risk of extreme constraint violations over time can be made small via an admissible control policy, where risk is quantified using the CVaR measure. We use the term, *risk level*, to mean the allowable level of risk of constraint violations. Formally, the risk-sensitive safe set at

<sup>2</sup>Conditional Value-at-Risk is also called *Average Value-at-Risk*, which is abbreviated as AV@R in [11].

the confidence level,  $\alpha \in (0, 1]$ , and the risk level,  $r \in \mathbb{R}$ , is defined as,

$$\mathcal{S}_\alpha^r := \{x \in \mathcal{X} \mid W_0^*(x, \alpha) \leq r\}, \quad (5a)$$

where

$$\begin{aligned} W_0^*(x, \alpha) &:= \min_{\pi \in \Pi} \text{CVaR}_\alpha[Z_x^\pi], \\ Z_x^\pi &:= \max \{g(x_k) \mid k = 0, \dots, N\}, \end{aligned} \quad (5b)$$

such that the state trajectory,  $(x_0, x_1, \dots, x_N)$ , evolves according to the dynamics model (2) with the initial state,  $x_0 := x$ , under the policy,  $\pi \in \Pi$ . The surface function,  $g$ , characterizes distance to the constraint set,  $\mathcal{K}$ , according to (4). Note that the minimum in the definition of  $W_0^*(x, \alpha)$  is attained, as the next lemma states.

*Lemma 1 (Existence of a minimizer):* For any initial state,  $x_0 \in \mathcal{X}$ , and any confidence level,  $\alpha \in (0, 1]$ , there exists a policy,  $\pi^* \in \Pi$ , such that

$$\text{CVaR}_\alpha[Z_x^{\pi^*}] = \inf_{\pi \in \Pi} \text{CVaR}_\alpha[Z_x^\pi] = \min_{\pi \in \Pi} \text{CVaR}_\alpha[Z_x^\pi].$$

*Proof:* Fix the initial state,  $x_0$ . Since the control and disturbance spaces are finite, the set of states that could be visited (starting from  $x_0$ ) is finite. Therefore, the corresponding set of policies,  $\Pi$ , is finite. Hence, the infimum must be attained by some policy,  $\pi^*$ . ■

In the next sections, we will present a tractable algorithm to approximately compute risk-sensitive safe sets at different levels of confidence and risk.

### C. Discussion

Computing risk-sensitive safe sets, as defined by (5), is well-motivated for several reasons. Our formulation incorporates different confidence levels and non-binary distance to the constraint set. In contrast, the stochastic reachability problem addressed by Abate et al. [6] uses a single confidence level and an indicator function to measure distance to the constraint set, in order to quantify the probability of constraint violation. Specifically, let  $\epsilon \in [0, 1]$  be the maximum tolerable probability of constraint violation (called *safety level* in [6]), and choose  $\alpha := 1$ ,  $r := \epsilon - \frac{1}{2}$ , and  $g(x) := \mathbf{1}_{\mathcal{K}}(x) - \frac{1}{2}$ , where

$$\mathbf{1}_{\mathcal{K}}(x) := \begin{cases} 1 & \text{if } x \notin \mathcal{K} \\ 0 & \text{if } x \in \mathcal{K} \end{cases}. \quad (6a)$$

Then, the risk-sensitive safe set (5) is equal to,

$$\left\{x \in \mathcal{X} \mid \min_{\pi \in \Pi} \mathbb{E} \left[ \max_{k=0, \dots, N} \mathbf{1}_{\mathcal{K}}(x_k) \right] \leq \epsilon \right\}, \quad (6b)$$

which is the *maximal probabilistic safe set* at the  $\epsilon$ -safety level [6, Eqs. 11 and 13], if we consider non-hybrid dynamic systems that evolve under history-dependent policies.<sup>3</sup>

Risk-sensitive safe sets have two desirable mathematical properties. The first property is that  $\mathcal{S}_\alpha^r$  shrinks as the risk level,  $r$ , or the confidence level,  $\alpha$ , decrease. Since  $\mathcal{S}_\alpha^r$  is an

$r$ -sublevel set, and  $\text{CVaR}_\alpha$  increases as  $\alpha$  decreases, one can show that

$$\begin{aligned} \mathcal{S}_{\alpha_2}^{r_2} &\subseteq \mathcal{S}_{\alpha_1}^{r_2} \subseteq \mathcal{S}_{\alpha_1}^{r_1}, \text{ and} \\ \mathcal{S}_{\alpha_2}^{r_2} &\subseteq \mathcal{S}_{\alpha_2}^{r_1} \subseteq \mathcal{S}_{\alpha_1}^{r_1} \end{aligned} \quad (7)$$

hold for any  $r_1 \geq r_2$  and  $1 \geq \alpha_1 \geq \alpha_2 > 0$ . In other words, as the allowable level of risk of constraint violation,  $r$ , decreases, or as the fraction of damaging outcomes that are not fully addressed,  $\alpha$ , decreases,  $\mathcal{S}_\alpha^r$  encodes a higher degree of safety.

The second property is that risk-sensitive safe sets at the risk level,  $r := 0$ , enjoy probabilistic safety guarantees.

*Lemma 2 (Link to a probabilistic safety guarantee):*

If  $x \in \mathcal{S}_\alpha^0$ , then the probability that the state trajectory initialized at  $x$  exits the constraint set can be made less than or equal to  $\alpha$ , by an admissible control policy.

*Proof:* The proof follows from the fact that  $\text{CVaR}_\alpha[Z_x^\pi] \leq 0 \implies \mathbb{P}[Z_x^\pi \geq 0] \leq \alpha$  [11, Sec. 6.2.4, pp. 257-258]. The event,  $Z_x^\pi \geq 0$ , is equivalent to the event that there is a state,  $x_k$ , of the associated trajectory that exits the constraint set, since  $g(x_k) \geq 0 \iff x_k \notin \mathcal{K}$ . ■

Lemma 2 indicates that  $\mathcal{S}_\alpha^0$  is a subset of the *maximal probabilistic safe set* at the safety level,  $\alpha \in (0, 1]$ , if we consider non-hybrid dynamic systems that evolve under history-dependent policies [6, Eqs. 9 and 11].

## III. REDUCTION OF RISK-SENSITIVE-SAFE-SET COMPUTATION TO CVAR-MDP

Computing risk-sensitive safe sets is challenging since the computation involves a maximum of costs (as opposed to a summation of costs) and the Conditional Value-at-Risk measure (as opposed to an expectation). In this section, we show how computing an under-approximation of a risk-sensitive safe set can be reduced to solving a CVaR-MDP, which has been studied, for example, by [15] and [18]. Such a reduction will be leveraged in Section IV to devise a value-iteration algorithm to compute tractable approximations of risk-sensitive safe sets.

### A. Preliminaries

The reduction procedure is inspired by Chow et al. [15]. Specifically, we consider an augmented state space,  $\mathcal{X} \times \mathcal{Y}$ , that consists of the original state space,  $\mathcal{X}$ , and the space of confidence levels,  $\mathcal{Y} := (0, 1]$ . The under-approximations of risk-sensitive safe sets will be defined in terms of the dynamics of the augmented state,  $(x, y) \in \mathcal{X} \times \mathcal{Y}$ , as explained next.

Let  $(x_0, y_0) := (x, \alpha)$  be a given initial condition. The augmented state at time  $k + 1$ ,  $(x_{k+1}, y_{k+1})$ , depends on the augmented state at time  $k$ ,  $(x_k, y_k)$ , as follows. Given a control,  $u_k \in U$ , and a sampled disturbance,  $w_k \in D$ , the next state,  $x_{k+1} \in \mathcal{X}$ , satisfies the dynamics model (2). The next confidence level,  $y_{k+1} \in \mathcal{Y}$ , is given by,

$$y_{k+1} = \bar{R}_{x_k, y_k}(w_k) \cdot y_k, \quad (8)$$

where  $\bar{R}_{x_k, y_k} : D \rightarrow (0, \frac{1}{y_k}]$  is a known deterministic function, which we will specify in Lemma 3. The augmented state space,  $\mathcal{X} \times \mathcal{Y}$ , is fully observable. Indeed, the history of

<sup>3</sup>Abate et al. [6] considers hybrid dynamic systems that evolve under Markov policies.

states and actions,  $(x_0, u_0, \dots, x_{k-1}, u_{k-1}, x_k)$ , is available at time  $k$  by (2). Also, the history of confidence levels,  $(y_0, \dots, y_k)$ , is available at time  $k$ , since the functions,  $\bar{R}_{x_k, y_k}$ , and the initial confidence level,  $y_0 = \alpha$ , are known.

We define the set of *deterministic, Markov* control policies in terms of the augmented state space as follows,

$$\bar{\Pi}_t := \{(\bar{\mu}_t, \bar{\mu}_{t+1}, \dots, \bar{\mu}_{N-1}) \mid \bar{\mu}_k : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{U}\}, \quad (9)$$

$$t = 0, \dots, N-1.$$

There is an important distinction between the set of policies,  $\bar{\Pi}_0$ , as defined above, and the set of policies,  $\Pi$ , as defined in (3). Given  $\bar{\pi}_0 \in \bar{\Pi}_0$ , the control law at time  $k$ ,  $\bar{\mu}_k \in \bar{\pi}_0$ , only depends on the current state,  $x_k \in \mathcal{X}$ , and the current confidence level,  $y_k \in \mathcal{Y}$ . However, given  $\pi \in \Pi$ , the control law at time  $k$ ,  $\mu_k \in \pi$ , depends on the state history up to time  $k$ ,  $(x_0, \dots, x_k) \in H_k$ . In particular, the set of policies,  $\bar{\Pi}_0$ , is included in the set of policies,  $\Pi$ . This is because the augmented state at time  $k$  is uniquely determined by the initial confidence level and the state history up to time  $k$ .<sup>4</sup>

The benefits of considering  $\bar{\Pi}_0$  instead of  $\Pi$  are two-fold. First, the computational requirements are reduced when the augmented state at time  $k$ ,  $(x_k, y_k)$ , is processed instead of the initial confidence level and the state history up to time  $k$ ,  $(y_0, x_0, x_1, \dots, x_k)$ . Second, we are able to define an under-approximation of the risk-sensitive safe set given by (5), using  $\bar{\Pi}_0$ , which we explain below.

#### B. Under-Approximation of Risk-Sensitive Safe Set

Define the set,  $\mathcal{U}_\alpha^r \subseteq \mathcal{X}$ , at the confidence level,  $\alpha \in (0, 1]$ , and the risk level,  $r \in \mathbb{R}$ ,

$$\mathcal{U}_\alpha^r := \{x \in \mathcal{X} \mid J_0^*(x, \alpha) \leq \beta e^{m \cdot r}\}, \quad (10)$$

where

$$J_0^*(x, \alpha) := \min_{\pi \in \bar{\Pi}_0} \text{CVaR}_\alpha[Y_x^\pi], \quad (11)$$

$$Y_x^\pi := \sum_{k=0}^N c(x_k),$$

such that  $c : \mathcal{X} \rightarrow \mathbb{R}$  is a stage cost, and the augmented state trajectory,  $(x_0, y_0, \dots, x_{N-1}, y_{N-1}, x_N)$ , satisfies (2) and (8) with the initial condition,  $(x_0, y_0) := (x, \alpha)$ , under the policy,  $\pi \in \bar{\Pi}_0$ . The next theorem, whose proof is provided in the Appendix, states that if the stage cost takes a particular form, then  $\mathcal{U}_\alpha^r$  is an under-approximation of the risk-sensitive safe set,  $\mathcal{S}_\alpha^r$ .

**Theorem 1 (Reduction to CVaR-MDP):** Choose the stage cost,  $c(x) := \beta e^{m \cdot g(x)}$ , where  $\beta > 0$  and  $m > 0$  are constants, and  $g$  satisfies (4). Then,  $\mathcal{U}_\alpha^r$ , as defined in (10), is a subset of  $\mathcal{S}_\alpha^r$ , as defined in (5). Further, the gap between  $\mathcal{U}_\alpha^r$  and  $\mathcal{S}_\alpha^r$  can be reduced by increasing  $m$ .

In the definition of the state costs, the parameter,  $\beta$ , is included to counter numerical issues that may arise, if  $m$  is set to a very large number.

<sup>4</sup>More formally, there exists an injective function,  $h_k : \mathcal{Y} \times H_k \rightarrow \mathcal{X} \times \mathcal{Y}$ , such that  $h_k(y_0, x_0, x_1, \dots, x_k) = (x_k, y_k)$ ; see (2) and (8). Given  $\bar{\pi}_0 \in \bar{\Pi}_0$ , the control at time  $k$  is  $\bar{\mu}_k(x_k, y_k)$ , which equals  $\bar{\mu}_k(h_k(y_0, x_0, x_1, \dots, x_k))$ . Define  $\mu_k(x_0, x_1, \dots, x_k) := \bar{\mu}_k(h_k(y_0, x_0, x_1, \dots, x_k))$  for all  $y_0 \in \mathcal{Y}$ . Note that  $\mu_k$  is the control law at time  $k$  for a particular  $\pi \in \Pi$ . Thus, there is an injective function that maps  $\bar{\Pi}_0$  to  $\Pi$ .

#### IV. A VALUE-ITERATION ALGORITHM TO APPROXIMATE RISK-SENSITIVE SAFE SETS

By leveraging Theorem 1, one can use existing CVaR-MDP algorithms to compute under-approximations of risk-sensitive safe sets. In this paper, we adapt a value-iteration algorithm from Chow et al. [15] to compute tractable approximations of the risk-sensitive-safe-set under-approximations,  $\{\mathcal{U}_\alpha^r\}$ . We start by stating an existing temporal decomposition result for CVaR that will be instrumental to devising the value-iteration algorithm.

##### A. Temporal Decomposition of Conditional Value-at-Risk

In this section, we present an existing result (namely, Lemma 22 in [19]) that specifies how the Conditional Value-at-Risk of a sum of costs can be partitioned over time, and how the confidence level evolves over time, which motivates the choice of the update function (8).

**Lemma 3 (Temporal decomposition of CVaR):** At time  $k$ , suppose that the system (2) is at the state,  $x_k \in \mathcal{X}$ , with the confidence level,  $y_k \in \mathcal{Y}$ , and is subject to a policy,  $\pi_k := (\mu_k, \pi_{k+1}) \in \bar{\Pi}_k$ . Then,

$$\text{CVaR}_{y_k}[Z|x_k, \pi_k] = \max_{R \in \mathcal{R}(y_k, \mathbb{P})} C(R, Z; x_k, y_k, \pi_k),$$

$$C(R, Z; x_k, y_k, \pi_k) :=$$

$$\mathbb{E}_{w_k \sim \mathbb{P}}[R(w_k) \cdot \text{CVaR}_{y_k R(w_k)}[Z|x_{k+1}, \pi_{k+1}|x_k, \mu_k]], \quad (12a)$$

where

$$\mathcal{R}(y_k, \mathbb{P}) := \{R : D \rightarrow (0, \frac{1}{y_k}] \mid \mathbb{E}_{w_k \sim \mathbb{P}}[R(w_k)] = 1\},$$

$$Z := \sum_{i=k+1}^N c(x_i), \quad (12b)$$

such that  $c : \mathcal{X} \rightarrow \mathbb{R}$  is a stage cost. Further, given the current state,  $(x_k, y_k)$ , the current control,  $u_k := \mu_k(x_k, y_k)$ , and the next state,  $x_{k+1}$ , the function that was introduced in (8),  $\bar{R}_{x_k, y_k} : D \rightarrow (0, \frac{1}{y_k}]$ , is defined as,

$$\bar{R}_{x_k, y_k}(w_k) = \arg \max_{R \in \mathcal{R}(y_k, \mathbb{P})} C(R, Z; x_k, y_k, \pi_k). \quad (13)$$

**Remark 1:** The proof of Lemma 3 is a consequence of Lemma 22 in [19], and its proof is omitted for brevity.

**Remark 2:** If we do not have access to  $w_k$ , but only to  $(x_k, y_k, u_k, x_{k+1})$ , then the next confidence level is defined as,  $y_{k+1} := \bar{R}_{x_k, y_k}(w)$ , where  $w \in D$  is a disturbance that satisfies  $x_{k+1} = f(x_k, u_k, w)$ .

**Remark 3:**  $\text{CVaR}_{y_k}[Z|x_k, \pi_k]$  is the risk of the cumulative cost of the trajectory,  $(x_{k+1}, \dots, x_N)$ , that is initialized at the state,  $x_k$ , with the confidence level,  $y_k$ , and is subject to the policy,  $\pi_k \in \bar{\Pi}_k$ .

##### B. Value-Iteration Algorithm

Using Lemma 3, we will devise a dynamic programming value-iteration algorithm to compute an approximation,  $J_0$ , of  $J_0^*$ , and thus, an approximation of  $\mathcal{U}_\alpha^r$  at different levels of confidence,  $\alpha$ , and risk,  $r$ .

Specifically, compute the functions,  $J_{N-1}, \dots, J_0$ , recursively as follows: for all  $z_k := (x_k, y_k) \in \mathcal{X} \times \mathcal{Y}$ ,

$$J_k(z_k) := \min_{u \in \mathcal{U}} \left\{ c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R J_{k+1}(x', y_k R) | z_k, u] \right\}, \quad (14)$$

for  $k = N - 1, \dots, 0$ , where  $c(x) := \beta e^{m \cdot g(x)}$ ,  $J_N(x_N, y_N) := c(x_N)$ ,  $x' := x_{k+1}$  satisfies (2), the expectation is taken with respect to  $w_k \sim \mathbb{P}$ , and  $\mathcal{R}(y_k, \mathbb{P})$  is defined in (12). We then approximate the set  $\mathcal{U}_\alpha^r$  as  $\hat{\mathcal{U}}_\alpha^r := \{x \in \mathcal{X} \mid J_0(x, \alpha) \leq \beta e^{m \cdot r}\}$ , where we have replaced  $J_0^*(x, \alpha)$  with  $J_0(x, \alpha)$ , as computed by solving the value iteration algorithm in (14). We present in the Appendix theoretical arguments (inspired by [15]) that justify such an approximation; in particular, we provide theoretical evidence for the following conjecture,

**Conjecture (C):** Assume that functions  $J_{N-1}, \dots, J_0$  are computed recursively as per equation (14). Then, for any  $(x, \alpha) \in \mathcal{X} \times \mathcal{Y}$ ,

$$J_0(x, \alpha) = J_0^*(x, \alpha). \quad (15)$$

This conjecture is further corroborated by the numerical experiments presented next.

## V. NUMERICAL EXPERIMENTS

We demonstrate the utility of computing approximate risk-sensitive safe sets on a practical example: to evaluate the design of a stormwater retention pond. Stormwater management facilities, such as retention ponds, are required to operate safely in the presence of precipitation uncertainty, but must be designed within the scope of public resources (e.g., money, land). Standard design practices assess how empty ponds respond to a given *design storm*, which is a synthetic storm based on historical rainfall. In our prior work, we proposed using reachability analysis to augment existing design practices, as it can assess system behavior from a larger number of initial conditions, but we assumed that the surface runoff due to the design storm was deterministic [20]. Here we consider the first pond from the example in our prior work [20] as a stochastic discrete-time dynamic system,

$$\begin{aligned} x_{k+1} &= x_k + \frac{\Delta t}{A} (w_k - q_p(x_k, u_k)), \quad k = 0, \dots, N-1, \\ q_p(x_k, u_k) &:= \begin{cases} C_d \pi r^2 u_k \sqrt{2\eta(x - E)} & \text{if } x_k \geq E \\ 0 & \text{if } x_k < E, \end{cases} \end{aligned} \quad (16)$$

where  $x_k \geq 0$  is the water level of the pond in feet at time  $k$ ,  $u_k \in U := \{0, 1\}$  is the valve setting at time  $k$ , and  $w_k \in D := \{d_1, \dots, d_{10}\}$  is the random surface runoff in feet-cubed-per-second at time  $k$ . In Eq. (16),  $\eta = 32.2 \frac{\text{ft}}{\text{s}^2}$  is the acceleration due to gravity,  $\pi \approx 3.14$ ,  $r = \frac{1}{3} \text{ft}$  is the outlet radius,  $A = 28,292 \text{ft}^2$  is the pond surface area,  $C_d = 0.61$  is the discharge coefficient, and  $E = 1 \text{ft}$  is the elevation of the outlet. We estimated a finite probability distribution for  $w_k$  using the surface runoff samples that we previously generated from a time-varying design storm [20]. We averaged each sample over time and solved for a distribution that satisfied the empirical statistics of the time-averaged samples (Table I). We set  $\Delta t := 300$  seconds, and  $N := 48$  to yield a 4-hour horizon. We set the constraint set,  $\mathcal{K} := [0, 5 \text{ft}]$ , and  $g(x) := x - 5$ . We used MATLAB R2016b (The MathWorks, Inc., Natick, MA) and MOSEK (Copenhagen, Denmark) with CVX [21]

TABLE I

Sample moment	Value
Mean	12.16 ft <sup>3</sup> /s
Variance	3.22 ft <sup>6</sup> /s <sup>2</sup>
Skewness	1.68 ft <sup>9</sup> /s <sup>3</sup>
Disturbance sample, $d_j$ ft <sup>3</sup> /s	Probability, $\mathbb{P}[w_k = d_j]$
8.57	0.0236
9.47	$10^{-4}$
10.37	$10^{-4}$
11.26	0.5249
12.16	0.3272
13.06	$10^{-4}$
13.95	$10^{-4}$
14.85	$10^{-4}$
15.75	$10^{-4}$
16.65	0.1237

on a standard laptop (64-bit OS, 16.0 GB RAM, Intel® Core™ i7-4700MQ CPU @ 2.40GHz). Our code is available at [https://github.com/chapmanmp/Risk\\_Sensitive\\_Reachability\\_Project/tree/stormwater\\_example/MATLAB\\_Code](https://github.com/chapmanmp/Risk_Sensitive_Reachability_Project/tree/stormwater_example/MATLAB_Code).

We computed approximations of sets  $\mathcal{U}_\alpha^r$  by using the value-iteration algorithm (14), and approximations of  $\{\mathcal{S}_y^r\}$  using a Monte Carlo procedure. Fig. 1 shows these approximations at different levels of confidence and risk. We computed over a finite grid of states and confidence levels,  $G := G_s \times G_c$ , where  $G_s := \{0, 0.1 \text{ft}, \dots, 6.4 \text{ft}, 6.5 \text{ft}\}$ , and  $G_c := \{0.999, 0.95, 0.80, \dots, 0.20, 0.05, 0.001\}$ . Since the initial state,  $x_0$ , is non-negative and the smallest realization of  $w_k$  is about  $8.5 \frac{\text{ft}^3}{\text{s}}$ ,  $x_{k+1} \geq x_k$  for all  $k$ . If  $x_{k+1} > 6.5 \text{ft}$ , we set  $x_{k+1} := 6.5 \text{ft}$  to stay within the grid.

*Value-iteration implementation.* To implement the value iteration algorithm in equation (14), we used the interpolation method over the confidence levels proposed by Chow et al. [15] to approximate the expectation in (14) as a piecewise linear concave function, which we maximized by solving a linear program (LP). Further, at each  $\alpha \in G_c$ , we used multi-linear interpolation to approximate the value of  $J_{k+1}(x_{k+1}, \alpha)$ . We set  $J_{k+1}(x_{k+1}, \alpha) := \frac{(x_{k+1} - x_i) \cdot J_{k+1}(x_{i+1}, \alpha) + (x_{i+1} - x_{k+1}) \cdot J_{k+1}(x^i, \alpha)}{x_{i+1} - x_i}$ , where  $x^i \in G_s$  and  $x_{i+1} \in G_s$  are the two nearest grid points to  $x_{k+1}$  that satisfy  $x^i \leq x_{k+1} \leq x_{i+1}$ . We chose the stage cost,  $c(x) := \beta e^{m \cdot g(x)}$ , such that  $\beta := 10^{-3}$  and  $m := 10$ . Fig. 2 shows the approximation of  $J_0$ , generated by the value-iteration algorithm (14), over the grid,  $G$ , using the interpolations just described. The computation time was roughly 3h 6min, which we deem acceptable as (i) such computations only need to be performed *offline*, (2) the problem entailed a moderately large state space, with a number of grid points  $|G_s| \cdot |G_c| = 594$ , and (3) our implementation is not yet optimized.

*Monte Carlo implementation.* An optimal control policy was known *a priori* for our one-pond system, which made a Monte Carlo implementation feasible. (If prior information on an optimal policy was not available, then  $|U|^N = 2^{48}$  possible control signals would need to be simulated.) Since

$x_{k+1} \geq x_k$  for all  $k$ , and the only way to exit the constraint set is if  $x_k \geq 5\text{ft}$ , an optimal policy is to keep the valve open, regardless of the state history up to the current time. For each  $(x, \alpha) \in G$ , we sampled 100,000 trajectories starting from  $x_0 := x$ , subject to keeping the valve open over time. For each trajectory sample  $i$ , we computed the cost sample,  $z_i := \max\{g(x_k^i)\}$ , and estimated the Conditional Value-at-Risk of the 100,000 cost samples at the confidence level,  $\alpha$ . We used the CVaR estimator,  $\widehat{\text{CVaR}}_\alpha[Z] := \frac{1}{\alpha M} \sum_{i=1}^M z_i \mathbf{1}_{\{z_i \geq \hat{Q}_\alpha\}}$ , where  $\hat{Q}_\alpha$  is the  $(1-\alpha)$ -quantile of the empirical distribution of the samples,  $\{z_i\}_{i=1}^M$ , and  $M := 100,000$  is the number of samples; see [11], Sec. 6.5.1. Since the estimator is designed for continuous distributions, we added zero-mean Gaussian noise with a small standard deviation,  $\sigma := 10^{-12}$ , to each cost sample prior to computing the CVaR. Fig. 3 provides a Monte Carlo estimate of  $W_0^*$ , as defined in (5).

100,000 samples per grid point appears to be sufficient. Also using this number of samples, we estimated  $\min_{\pi \in \Pi} \text{CVaR}_\alpha[Y_x^\pi]$  via an analogous Monte Carlo procedure as just described, where  $Y_x^\pi$  is the random sum of stage costs (Fig. 4).<sup>5</sup> See that Fig. 4 and Fig. 2 are comparable in most regions of the grid. However, our Monte Carlo estimate does not provide the higher costs at the smallest confidence level,  $\alpha = 0.001$ , that are provided by our value-iteration estimate.

## VI. CONCLUSION

In this paper, we propose the novel idea of a risk-sensitive safe set that encodes the safety of a stochastic dynamic system in terms of an allowable level of risk of constraint violation,  $r$ , in the  $\alpha$ -fraction of the most damaging outcomes. We reduce the computation of risk-sensitive safe sets to the problem of optimizing the Conditional Value-at-Risk of the maximum extent of constraint violation over the state trajectory. Further, we provide a dynamic programming algorithm with theoretical justification to compute tractable approximations of risk-sensitive safe sets.

Risk-sensitive safe sets have potential to inform the design of safety-critical infrastructure systems, by revealing trade-offs between the risk of damaging outcomes and design choices, at different levels of confidence. We illustrate our risk-sensitive reachability formulation on a stormwater pond that must be designed to operate safely in the presence of uncertain rainfall. Our results reveal that the current design of the pond is likely undersized: even if the pond starts empty, there is a risk of at least 0.25ft of overflow, at most levels of confidence, under the random surface runoff of the design storm (see Fig. 1,  $r = 0.25$  plot at  $x = 0$ ).

On the methodological side, future steps include: 1) formally proving the correctness of the value iteration algorithm, 2) devise approximate value iteration algorithms to improve scalability, and 3) consider a broader class of risk measures. On the applications side, future steps include: 1) adjust the parameters of the dynamics model (e.g., outlet

<sup>5</sup>We added zero-mean Gaussian noise with a small standard deviation,  $\sigma := 10^{-7}$ , to each sample of  $Y_x^\pi$  prior to computing the CVaR. See that the magnitude of  $J_0$  (Fig. 2) is about  $10^5$ -times greater than the magnitude of  $W_0$  (Fig. 3).

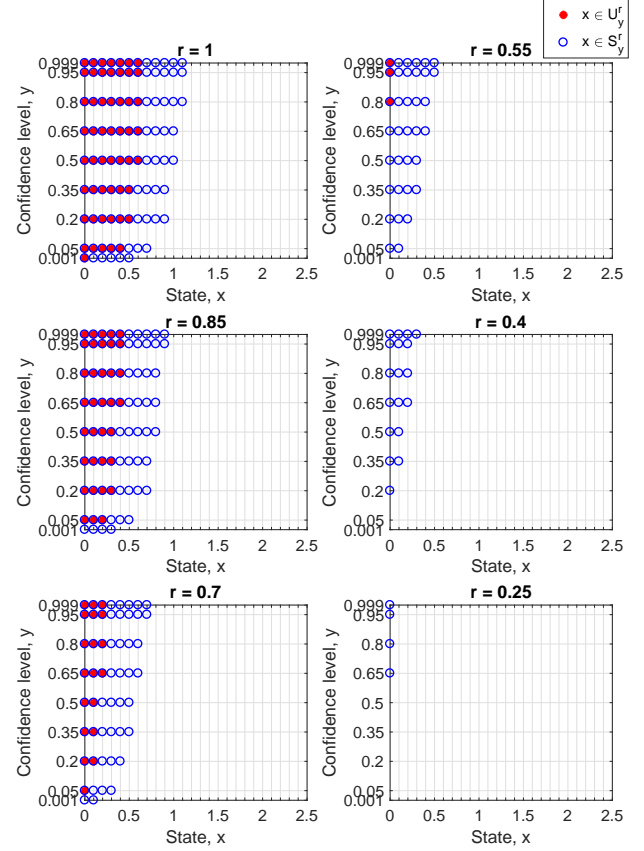


Fig. 1. Approximations of  $\{\hat{U}_y^r\}$  and  $\{S_y^r\}$  are shown for the pond system at various levels of confidence,  $y$ , and risk,  $r$ .  $\{\hat{U}_y^r\}$  were approximated using the value-iteration algorithm (14), and  $\{S_y^r\}$  were approximated via a Monte Carlo procedure.

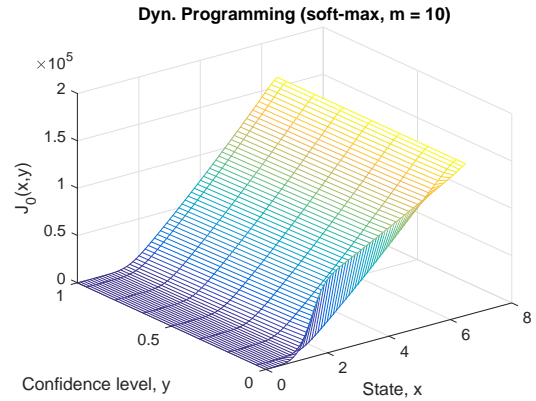


Fig. 2. Our value-iteration estimate of  $J_0(x, \alpha)$  versus  $(x, \alpha) \in G$  for the pond system, see (14).  $c(x) := \beta e^{m \cdot g(x)}$ ,  $\beta := 10^{-3}$ ,  $m := 10$ , and  $g(x) := x - 5$ .



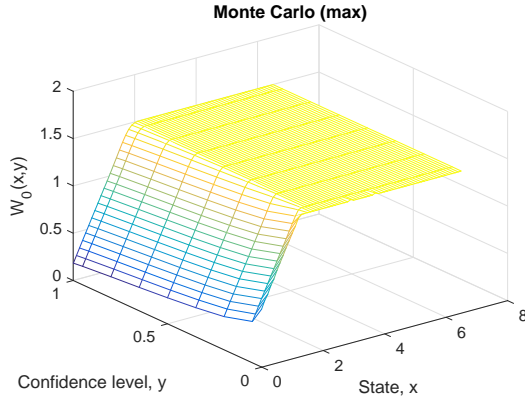


Fig. 3. A Monte Carlo estimate of  $W_0^*(x, \alpha)$ , as defined in (5), versus  $(x, \alpha) \in G$  for the pond system. 100,000 samples were generated per grid point, and  $g(x) := x - 5$ . The maximum is 1.5ft because the system state was prevented from exceeding 6.5ft.

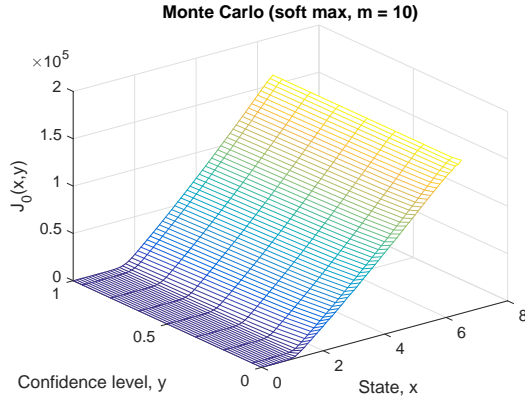


Fig. 4. A Monte Carlo estimate of  $\min_{\pi \in \Pi} \text{CVaR}_\alpha[Y_x^\pi]$  versus  $(x, \alpha) \in G$  for the pond system.  $Y_x^\pi := \sum_{k=0}^N \beta e^{m \cdot g(x_k)}$ ,  $x_0 := x$ ,  $\beta := 10^{-3}$ ,  $m := 10$ , and  $g(x) := x - 5$ . 100,000 samples were generated per grid point. See also Fig. 2.

radius) to reduce the risk of extreme overflows, 2) apply our method to a more realistic stormwater system that consists of two ponds in series on a larger grid, and 3) [develop an optimized toolbox for the computation of risk-sensitive sets](#). We are hopeful that with further development the concept of risk-sensitive safe sets will become a valuable tool for the design of safety-critical systems.

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## APPENDIX

*Proof:* [Proof of Theorem 1] The proof relies on two facts. The first fact is,

$$\begin{aligned} \max\{y_1, \dots, y_p\} &\leq \frac{1}{m} \log(e^{my_1} + \dots + e^{my_p}) \\ &\leq \max\{y_1, \dots, y_p\} + \frac{\log p}{m}, \end{aligned} \quad (17a)$$

for any  $y \in \mathbb{R}^p$ ,  $m > 0$ .<sup>6</sup> So, as  $m \rightarrow \infty$ ,

$$\frac{1}{m} \log(e^{my_1} + \dots + e^{my_p}) \rightarrow \max\{y_1, \dots, y_p\}. \quad (17b)$$

The second fact is that Conditional Value-at-Risk is a *coherent risk measure*, so it satisfies useful properties. In particular, CVaR is *positively homogeneous*,  $\text{CVaR}_\alpha[\lambda Z] = \lambda \text{CVaR}_\alpha[Z]$ , for any  $\lambda \geq 0$ , and *monotonic*,  $\text{CVaR}_\alpha[Y] \leq \text{CVaR}_\alpha[Z]$ , for any random variables,  $Y \leq Z$ .<sup>7</sup> Also, CVaR can be expressed as the supremum expectation over a particular set of probability density functions.<sup>8</sup> Using this property and the fact,  $\mathbb{E}[\log(Z)] \leq \log(\mathbb{E}[Z])$ , one can show,

$$\text{CVaR}_\alpha[\log(Z)] \leq \log(\text{CVaR}_\alpha[Z]), \quad (18)$$

for any random variable,  $Z$ , with finite expectation.

By monotonicity, positive homogeneity, (17), and (18),

$$\text{CVaR}_\alpha[Y_x^\pi] \leq \frac{1}{m} \text{CVaR}_\alpha[\log(\bar{Y}_x^\pi)] \leq \frac{1}{m} \log(\text{CVaR}_\alpha[\bar{Y}_x^\pi]), \quad (19)$$

where  $\bar{Y}_x^\pi := Y_x^\pi / \beta$ . Now, if  $x \in \mathcal{U}_\alpha^r$ , then

$$e^{m \cdot r} \geq \min_{\pi \in \Pi_0} \text{CVaR}_\alpha[Y_x^\pi / \beta] \geq \min_{\pi \in \Pi} \text{CVaR}_\alpha[Y_x^\pi / \beta],$$

since  $\bar{\Pi}_0$  is included in  $\Pi$ . By Lemma 1, there exists  $\pi \in \Pi$  such that,

$$r \geq \frac{1}{m} \log(\text{CVaR}_\alpha[Y_x^\pi / \beta]) \geq \text{CVaR}_\alpha[Z_x^\pi],$$

where the second inequality holds by (19). So,  $x \in \mathcal{S}_\alpha^r$ . ■

*Proof:* [Theoretical Justification of Conjecture (C)] Let  $\epsilon > 0$ . For all  $k = 0, \dots, N-1$  and  $z_k := (x_k, y_k) \in \mathcal{X} \times \mathcal{Y}$ , let  $\mu_k^\epsilon : \mathcal{X} \times \mathcal{Y} \rightarrow U$  satisfy,

$$c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R J_{k+1}(x_{k+1}, y_k R) | z_k, \mu_k^\epsilon] \leq J_k(z_k) + \epsilon. \quad (20)$$

Let  $J_k^\epsilon$  be a sub-optimal cost-to-go starting at time  $k$ ,

$$J_k^\epsilon(z_k) := \text{CVaR}_{y_k} \left[ \sum_{i=k}^N c(x_i) | z_k, \pi_k^\epsilon \right], \quad (21)$$

where  $\pi_k^\epsilon := (\mu_k^\epsilon, \dots, \mu_{N-1}^\epsilon) = (\mu_k^\epsilon, \pi_{k+1}^\epsilon)$ . Recall  $J_k$ , as defined in (14). Define  $J_k^*(z_k) := \min_{\pi \in \bar{\Pi}_k} \text{CVaR}_{y_k} \left[ \sum_{i=k}^N c(x_i) | z_k, \pi \right]$ . To prove the conjecture, we would like to show by induction that for all  $z_k := (x_k, y_k) \in \mathcal{X} \times \mathcal{Y}$  and  $k = N-1, \dots, 0$ ,

$$J_k(z_k) \leq J_k^\epsilon(z_k) \leq J_k(z_k) + (N-k)\epsilon, \quad (22a)$$

$$J_k^*(z_k) \leq J_k^\epsilon(z_k) \leq J_k^*(z_k) + (N-k)\epsilon, \quad (22b)$$

$$J_k(z_k) = J_k^*(z_k), \quad (22c)$$

<sup>6</sup>Use the log-sum-exp relation stated in [22], Sec. 3.1.5.

<sup>7</sup>See [12], Sec. 2.2

<sup>8</sup>See [11], Eqs. 6.40 and 6.70

which is the proof technique in [4], Sec. 1.5. One can show (22) for the base case,  $k := N - 1$ , since  $J_N$  is known. Assuming that (22) holds for index  $k + 1$  (induction hypothesis), we want to show that (22) holds for index  $k$  (induction step). The key idea is to use the following recursion,

$$J_k^\epsilon(z_k) = c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R \cdot J_{k+1}^\epsilon(x_{k+1}, y_k R) | z_k, \mu_k^\epsilon], \quad (23)$$

which we justify next. Let  $Z := \sum_{i=k+1}^N c(x_i)$ .

$$\begin{aligned} J_k^\epsilon(z_k) - c(x_k) &= \text{CVaR}_{y_k} [Z | z_k, \pi_k^\epsilon] \\ &= \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R \cdot \text{CVaR}_{y_k R} [Z | x_{k+1}, \pi_{k+1}^\epsilon] | z_k, \mu_k^\epsilon] \\ &= \max_{(a) R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R \cdot J_{k+1}^\epsilon(x_{k+1}, y_k R) | z_k, \mu_k^\epsilon], \end{aligned}$$

where we use (12), (21), and translation invariance<sup>9</sup>. The last equality (a) is the crux of our conjecture, as one needs to justify why the worst-case density  $R$  is equal to the a priori chosen density  $\bar{R}$  that defines the dynamics of the confidence level. Based on [15], we believe this equality to be correct, but we leave its formal proof for future research. Assuming the aforementioned equality is correct, then we show (22a) for index  $k$  using (23) and the induction hypothesis. Let  $\bar{\epsilon}_k := (N - k - 1)\epsilon$ , and  $x' := x_{k+1}$ .

$$\begin{aligned} J_k^\epsilon(z_k) &\leq c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R(J_{k+1}(x', y_k R) + \bar{\epsilon}_k) | z_k, \mu_k^\epsilon] \\ &= c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R J_{k+1}(x_{k+1}, y_k R) | z_k, \mu_k^\epsilon] + \bar{\epsilon}_k \\ &\leq J_k(z_k) + (N - k)\epsilon, \end{aligned}$$

since  $\mathbb{E}[R] = 1$ , and by (20). By (14), sub-optimality of  $\mu_k^\epsilon(z_k) \in U$ ,  $J_{k+1} \leq J_{k+1}^\epsilon$ , and (23),

$$\begin{aligned} J_k(z_k) &\leq c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R J_{k+1}^\epsilon(x_{k+1}, y_k R) | z_k, \mu_k^\epsilon] \\ &\leq J_k^\epsilon(z_k). \end{aligned}$$

The induction step for (22a) is complete. Next, we show (22b) for index  $k$ . By definition,  $J_k^*$  is the optimal risk-sensitive cost-to-go from stage  $k$ , thus,  $J_k^* \leq J_k^\epsilon$ . Let  $\hat{\epsilon}_k := (N - k)\epsilon$ ,  $x' := x_{k+1}$ ,  $y' := y_k R$ , and  $Z := \sum_{i=k+1}^N c(x_i)$ . For any  $\pi_k := (\mu_k, \pi') \in \bar{\Pi}_k$ ,

$$\begin{aligned} J_k^\epsilon(z_k) &\leq J_k(z_k) + \hat{\epsilon}_k \\ &\leq c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R J_{k+1}^*(x_{k+1}, y_k R) | z_k, \mu_k] + \hat{\epsilon}_k \\ &\leq c(x_k) + \max_{R \in \mathcal{R}(y_k, \mathbb{P})} \mathbb{E}[R \text{CVaR}_{y' R} [Z | x', \pi'] | z_k, \mu_k] + \hat{\epsilon}_k \\ &= c(x_k) + \text{CVaR}_{y_k} [Z | z_k, \pi_k] + \hat{\epsilon}_k \\ &= \text{CVaR}_{y_k} [\sum_{i=k}^N c(x_i) | z_k, \pi_k] + (N - k)\epsilon. \end{aligned}$$

The above statement implies

$$\begin{aligned} J_k^\epsilon(z_k) &\leq \min_{\pi \in \bar{\Pi}_k} \text{CVaR}_{y_k} [\sum_{i=k}^N c(x_i) | z_k, \pi_k] + (N - k)\epsilon \\ &= J_k^*(z_k) + (N - k)\epsilon, \end{aligned}$$

which completes the induction step for (22b).

We have shown that (22a) and (22b) hold for index  $k$ , for any  $\epsilon > 0$ . So, (22c) holds for index  $k$ . Assuming the conjectured equality is correct, this would complete the proof of the conjecture. ■

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<sup>9</sup>For  $a \in \mathbb{R}$ ,  $\text{CVaR}_\alpha[a + Z] = a + \text{CVaR}_\alpha[Z]$ ; see [12], Sec. 2.2.