8/21/2018

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Proof of Dynamic Programming Algorithm for the CVaR of a sum over a finite-time horizon

- "Shapiro" refers to Lectures on Stochastic Programming Modeling and Theory
- "Chow 2015" refers to Risk-Sensitive and Robust Decision-Making: A CVaR Optimization Approach
- "Bertsekas 2000" refers to Dynamic Programming and Optimal Control, Vol I

Please see Sec. 1.5 "Some Mathematical Issues." Our proof will resemble the argument starting with equation (1.17) in this section.

System model

$$x_{k+1} = f_k(x_k, u_k, w_k) \qquad k = 0, 1, \dots, N-1 \qquad (1)$$

$x_k \in S$, $u_k \in C$, $w_k \in D_k$ countable

$P_k(w_k)$ is the probability that the random disturbance takes on the value w_k at time k

$w_0, \dots, w_{N-1}$ are independent but may not be identically distributed

$\Pi := \{(\mu_0, \dots, \mu_{N-1}),\ \mu_i : S \to C\}$ set of admissible control policies

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Goal: To compute $$J^*(x_0, \alpha) := \min_{\pi \in \Pi} \mathrm{CVaR}_\alpha\left[\sum_{k=0}^{N} c(x_k) \,\middle|\, x_0, \pi\right] \quad \forall x_0 \in S,\ \forall \alpha \in L \subset (0,1) \qquad (2)$$

$c : S \to \mathbb{R}$ bounded

random state at time k under policy π, subject to the dynamics (1), starting from non-random initial condition, $x_0 \in S$

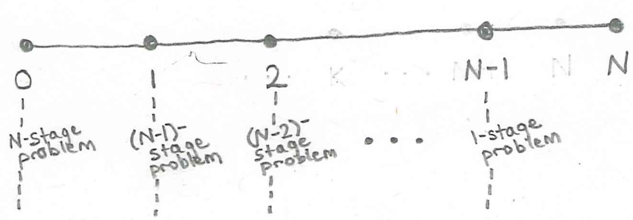
set of confidence levels

Next, we will develop machinery to compute (2).

Definition

$$J_k^*(x_k, y_k) := \min_{\pi_k \in \Pi_k} \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k\right]$$ is the optimal cost for the

$(N-k)$-stage problem that starts at state x_k, confidence level y_k, time k, and ends at time N.



$$\Pi_k := \{\overset{\pi_k}{(\mu_k, \dots, \mu_{N-1})},\ \mu_i : S \times L \to C\}$$

set of admissible control policies for the $(N-k)$-stage problem.

Decomposition of CVaR

Definition: $\mathrm{CVaR}_\alpha[Z] = \min_{c\in\mathbb{R}} \left\{ c + \frac{1}{\alpha}\mathbb{E}[\max(Z-c,0)] \right\}$, $\alpha \in (0,1)$, $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$

finite expectation

(in terms of σ-algebras; Pflug & Pichler 2015, Lemma 22)

Let $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_s$ be a sub-σ-algebra of $\mathcal{F}$.

sample space

σ-algebra (event space)

probability measure

Z has finite expectation so sup is attained

Making dependence on $\mathcal{F}$ implicit again

$$\mathrm{CVaR}_\alpha[Z] = \mathrm{CVaR}_\alpha[Z \mid \mathcal{F}] = \max_R \mathbb{E}\left[R \cdot \mathrm{CVaR}_{\alpha R}[Z \mid \mathcal{F}_s] \,\middle|\, \mathcal{F}\right] = \max_R \mathbb{E}\left[R \cdot \mathrm{CVaR}_{\alpha R}[Z \mid \mathcal{F}_s]\right].$$

Making dependence on $\mathcal{F}$ explicit

$R \in \mathcal{L}_1(\Omega, \mathcal{F}_s, P)$
$R(\omega) \in [0, \frac{1}{\alpha}]$ a.e. $\omega \in \Omega$
$\mathbb{E}[R] = 1$

(in terms of a stochastic process; Chow et al 2015, Theorem 2)

$$\mathrm{CVaR}_\alpha\left[\boxed{\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_k}, \pi_k\right] = \max_{R \in \mathcal{B}(\alpha,k+1)} \mathbb{E}\left[R \cdot \mathrm{CVaR}_{\alpha R}\left[\boxed{\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}}, \pi_k\right] \,\middle|\, x_k, \pi_k\right]$$

$\in \mathcal{L}_1(\Omega, \mathcal{F}_k, P)$
$\mathcal{F}_k$ is the σ-algebra assoc. with $x_k, x_{k+1}, \dots, x_N$

$\mathcal{F}_{k+1} \subset \mathcal{F}_k$

$R \in \mathcal{L}_1(\Omega, \mathcal{F}_{k+1}, P)$
$R(\omega) \in [0, \frac{1}{\alpha}]$ a.e. $\omega \in \Omega$
$\mathbb{E}[R] = 1$
the set of R satisfying the above is called $\mathcal{B}(\alpha, k+1)$

$\in \mathcal{L}_1(\Omega, \mathcal{F}_{k+1}, P)$
$\mathcal{F}_{k+1}$ is the σ-algebra associated with $x_{k+1}, x_{k+2}, \dots, x_N$

how do we specify these details accurately but simply?

$$= \max_R \sum_{x_{k+1}} R(x_{k+1}) \cdot \mathrm{CVaR}_{\alpha R(x_{k+1})}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, \overset{f_k(x_k,\mu_k(x_k),w_k)}{\overset{\parallel}{x_{k+1}}}, \pi_k\right] \cdot \overset{P_k(w_k)}{\overset{\parallel}{P_k(x_{k+1} \mid x_k, \pi_k)}}$$

discrete sample space

$$= \max_R \sum_{w_k \in D_k} R(w_k) \cdot \mathrm{CVaR}_{\alpha R(w_k)}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1} = f_k(x_k, \mu_k(x_k), w_k), \pi_k\right] \cdot P_k(w_k)$$

source of randomness is w_k in our model, see page 1

<u>Remark</u> Pflug & Pichler 2015 defines the CVaR decomposition in terms of sigma-algebras.

Chow 2015 " " " the data process.

Shapiro (see just below eq. 6.269 in Sec. 6.7 Multistage Risk-Averse Optimization) explains how either sigma-algebras or the data process can be used to define conditional expectations and conditional risk-measures.

<u>DP Algorithm</u> (Theorem)

Define the functions, $J_{N-1}, \ldots, J_0$, recursively as follows, $\forall\, x_k \in S$, $\forall\, y_k \in L$ (finite subset of $(0,1)$)

(3) $$J_k(x_k, y_k) := \min_{u_k} \left\{ c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, u_k \right] \right\} \qquad k = N-1, \ldots, 0,$$

$\mathcal{B}(y_k, k+1)$ is a set of random variables R that are measurable with respect to the sub-σ-algebra generated by the state at time $k+1$ (equivalently, by the disturbance at time k), where $R \in [0, \frac{1}{y_k}]$ and $\mathbb{E}[R] = 1$.

How do we say this better? More accurately without getting into measure theory?

with initialization $J_N(x, y) = c(x)$ $\forall\, x \in S$, $\forall\, y \in L$.

Assume that the functions J_k $k = 0, \ldots, N$ are well-defined and finite. (This is true in particular if each D_k is a finite set (see Bertsekas 2000, Sec. 1.5, just above eq. 1.16).)

Then, $$J^*(x_0, \alpha) \underset{(2)}{:=} \min_{\pi \in \Pi} \mathrm{CVaR}_\alpha \left[\sum_{k=0}^{N} c(x_k) \,\middle|\, x_0, \pi \right] = J_0(x_0, \alpha).$$

function at the last step of the recursion evaluated at state x_0 and level α.

Remark

Also, if $u_k^* = \mu_k^*(x_k, y_k)$ is a minimizer of (3) for each (x_k, y_k) and each $k = 0, \dots, N-1$,

then $\{\mu_0^*, \dots, \mu_{N-1}^*\}$ is an optimal policy.

Remark

The DP algorithm is an extension of the one provided by Bertsekas 2000 (see Proposition 1.3.1, in Sec. 1.3), and is the finite-time analog of the algorithm provided by Chow 2015.

Our proof is built on the one provided by Bertsekas 2000 (see Sec. 1.5).

Proof idea

Define a sub-optimal policy and a sub-optimal cost, and use these to show that J_k (output from DP) and J_k^* (see p. 1) are very close for each k by induction. Showing $J_k = J_k^*$ directly is hard because it is not obvious why we are allowed to exchange the order of the maximum and the minimum.

Proof details

Let $\varepsilon > 0$. For all $k = 0, \dots, N-1$ and $(x_k, y_k) \in S \times L$, let $\mu_k^\varepsilon : S \times L \to C$ be "ε-optimal", i.e.,

$$c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^\varepsilon \right] \leq J_k(x_k, y_k) + \varepsilon. \quad (4)$$

$\mathcal{B}(y_k, k+1)$ is defined on page 3.

Let $J_k^\varepsilon(x_k, y_k)$ be the "ε-optimal cost", i.e.,

$$J_k^\varepsilon(x_k, y_k) := \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k^\varepsilon\right], \quad \text{where } \pi_k^\varepsilon := (\mu_k^\varepsilon, \mu_{k+1}^\varepsilon, \ldots, \mu_{N-1}^\varepsilon).$$

Recall the definitions of J_k^* (see p. 1) and J_k (see p. 3).

We will show by induction for all $(x_k, y_k) \in S \times L$ and $k = N-1, \ldots, 0$, the following inequalities,

$$J_k(x_k, y_k) \le J_k^\varepsilon(x_k, y_k) \le J_k(x_k, y_k) + (N-k)\varepsilon \qquad \text{(a)}$$

$$J_k^*(x_k, y_k) \le J_k^\varepsilon(x_k, y_k) \le J_k^*(x_k, y_k) + (N-k)\varepsilon \qquad \text{(b)}$$

$$J_k(x_k, y_k) = J_k^*(x_k, y_k), \qquad \text{(c)}$$

following the example of Bertsekas (Sec. 1.5).

<u>Base case</u> $k = N-1$

initialization of the algorithm

$$J_{N-1}(x_{N-1}, y) := \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot \overset{c(x_N)}{\overset{\parallel}{J_N(x_N, yR)}} \,\middle|\, x_{N-1}, u_{N-1}\right] \right\}$$

$\mathcal{B}(y,N)$ is defined on page 3.

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot c(x_N) \,\middle|\, x_{N-1}, u_{N-1}\right] \right\}$$

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot \underbrace{\mathrm{CVaR}_{yR}\left[c(x_N) \middle| x_N, u_{N-1} \right]}_{=\, c(x_N)} \middle| x_{N-1}, u_{N-1} \right] \right\}$$

please verify
yes

$$= \min_{u_{N-1}} \left\{ c(x_{N-1}) + \mathrm{CVaR}_y\left[c(x_N) \middle| x_{N-1}, u_{N-1} \right] \right\} \quad \text{by CVaR Decomposition Thm,}$$

$$= \min_{u_{N-1}} \left\{ \mathrm{CVaR}_y\left[c(x_{N-1}) + c(x_N) \middle| x_{N-1}, u_{N-1} \right] \right\} \quad a \in \mathbb{R}, \ a + \mathrm{CVaR}(Z) = \mathrm{CVaR}(a+Z)$$

$$= \min_{\pi_{N-1} \in \Pi_{N-1}} \left\{ \mathrm{CVaR}_y\left[c(x_{N-1}) + c(x_N) \middle| x_{N-1}, \pi_{N-1} \right] \right\}$$

$\Pi_{N-1} = \{ \mu_{N-1} : S \times L \to C \}$

defined on page 1

$$= J^*_{N-1}(x_{N-1}, y) .$$

ⓒ for $k = N-1$

ⓐ for $k=N-1$

We have shown that $J_{N-1} = J^*_{N-1}$. Next, we will show $J_{N-1}(x_{N-1}, y) \le J^{\varepsilon}_{N-1}(x_{N-1}, y) \le J_{N-1}(x_{N-1}, y) + \varepsilon$.

$$J_{N-1}(x_{N-1}, y) = J^*_{N-1}(x_{N-1}, y) = \min_{\mu_{N-1}} \left\{ \mathrm{CVaR}_y\left[c(x_{N-1}) + c(x_N) \middle| x_{N-1}, \mu_{N-1} \right] \right\}$$

$\mu_{N-1} : S \times L \to C$

$$\le \mathrm{CVaR}_y\left[c(x_{N-1}) + c(x_N) \middle| x_{N-1}, \mu^{\varepsilon}_{N-1} \right]$$

$$= J^{\varepsilon}_{N-1}(x_{N-1}, y) .$$

defined on page 5

$$J_{N-1}^{\varepsilon}(x_{N-1}, y) = \mathrm{CVaR}_y\left[c(x_{N-1}) + c(x_N) \,\middle|\, x_{N-1}, \mu_{N-1}^{\varepsilon} \right]$$

$$= c(x_{N-1}) + \mathrm{CVaR}_y\left[c(x_N) \,\middle|\, x_{N-1}, \mu_{N-1}^{\varepsilon} \right]$$

$$= c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot \underbrace{\mathrm{CVaR}_{yR}\left[c(x_N) \,\middle|\, x_N, \mu_{N-1}^{\varepsilon} \right]}_{\substack{= c(x_N) \\ = J_N(x_N, yR)}} \,\middle|\, x_{N-1}, \mu_{N-1}^{\varepsilon} \right] \qquad \text{by CVaR Decomp Thm}$$

specified on page 3

$$= c(x_{N-1}) + \max_{R \in \mathcal{B}(y,N)} \mathbb{E}\left[R \cdot J_N(x_N, yR) \,\middle|\, x_{N-1}, \mu_{N-1}^{\varepsilon} \right]$$

$$\leq J_{N-1}(x_{N-1}, y) + \varepsilon . \qquad \text{by (4) for } k := N-1 \text{ (see page 4)}$$

We have shown $J_{N-1}(x_{N-1}, y) \leq J_{N-1}^{\varepsilon}(x_{N-1}, y) \leq J_{N-1}(x_{N-1}, y) + \varepsilon$. (a) for $k = N-1$

Since $J_{N-1} = J_{N-1}^{*}$, (c) for $k=N-1$, we also have $J_{N-1}^{*}(x_{N-1}, y) \leq J_{N-1}^{\varepsilon}(x_{N-1}, y) \leq J_{N-1}^{*}(x_{N-1}, y) + \varepsilon$. (b) for $k = N-1$

Thus, the $\underbrace{\text{inequalities (a)-(c)}}_{\text{defined on page 5}}$ have been proved for $k = N-1$.

Induction hypothesis Assume (a)-(c) hold for index $k+1$.

Induction step We need to show that (a)-(c) hold for index k.

The key idea is to use the following recursion for the ε-optimal cost,

$$J_k^{\varepsilon}(x_k, y_k) = c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}^{\varepsilon}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^{\varepsilon} \right] \tag{5}$$

$\mu_k^{\varepsilon} : S \times L \to C$ maps augmented state to control action

which requires proof. (The analagous expression in Bertsekas 2000, Sec. 1.5, is $J_k^{\varepsilon}(x_k) = \mathbb{E}_{w_k}\left\{ g_k(x_k) + J_{k+1}^{\varepsilon}(x_{k+1}) \right\}$.)

<u>Proof of (5)</u>

defined on page 5

$$J_k^{\varepsilon}(x_k, y_k) = \mathrm{CVaR}_{y_k}\left[c(x_k) + \sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_k, \pi_k^{\varepsilon} \right]$$

$$= c(x_k) + \mathrm{CVaR}_{y_k}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_k, \pi_k^{\varepsilon} \right]$$

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \mathrm{CVaR}_{y_k \cdot R}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}, \underbrace{\pi_k^{\varepsilon}}_{} \right] \,\middle|\, x_k, \pi_k^{\varepsilon} \right]$$

$(\mu_k^{\varepsilon}, \pi_{k+1}^{\varepsilon}) = \pi_k^{\varepsilon}$

by CVaR Decomp Thm

please verify

yes

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \underbrace{\mathrm{CVaR}_{y_k \cdot R}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}, \pi_{k+1}^{\varepsilon} \right]}_{= J_{k+1}^{\varepsilon}(x_{k+1}, y_k R) \ \leftarrow \text{ def on page 5}} \,\middle|\, x_k, \pi_k^{\varepsilon} \right]$$

because $\sum_{i=k+1}^{N} c(x_i)$ does not depend on μ_k^{ε},

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}^{\varepsilon}(x_{k+1}, y_k R) \,\middle|\, x_k, \pi_k^{\varepsilon} \right]$$

$\pi_k^{\varepsilon} = (\mu_k^{\varepsilon}, \mu_{k+1}^{\varepsilon}, \ldots, \mu_{N-1}^{\varepsilon})$

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}^{\varepsilon}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^{\varepsilon} \right]$$

The expectation is with respect to the state at time k+1, which depends on the action at time k and the state at time k but not on the actions at future times.

We have proved (5) and will use it in the induction step.

Now, we will show (a) for index k using (5) and the induction hypothesis. (a) is defined on page 5.

$$J_k^\varepsilon(x_k, y_k) = c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}^\varepsilon(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^\varepsilon\right]$$

by (5), shown on page 8

$$\leq c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \left(J_{k+1}(x_{k+1}, y_k R) + (N-k-1)\varepsilon\right) \,\middle|\, x_k, \mu_k^\varepsilon\right]$$

by induction hyp $J_{k+1}^\varepsilon \leq J_{k+1} + (N-k-1)\varepsilon$

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \left\{ \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^\varepsilon\right] + \mathbb{E}\left[R \cdot (N-k-1)\varepsilon \,\middle|\, x_k, \mu_k^\varepsilon\right] \right\}$$

by linearity of expectation

$$\mathbb{E}\left[R \cdot (N-k-1)\varepsilon \,\middle|\, x_k, \mu_k^\varepsilon\right] = \mathbb{E}\left[R \cdot (N-k-1)\varepsilon\right] = (N-k-1)\varepsilon \cdot \mathbb{E}[R] = (N-k-1)\varepsilon$$

Does not depend on R, so take it outside of max

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \left\{ \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^\varepsilon\right] \right\} + (N-k-1)\varepsilon$$

The expectation is taken with respect to the uncertainty of transitioning to the next state, starting from x_k at time k. The actions at the next time steps $(k+1, k+2, \dots)$ do not affect where the system goes at time k+1.

$$\leq J_k(x_k, y_k) + \varepsilon + (N-k-1)\varepsilon$$

by (4), see page 4

$$= J_k(x_k, y_k) + (N-k)\varepsilon.$$

To show the rest of (a) (which is defined on page 5),

$$J_k(x_k, y_k) = \min_{u_k} \left\{ c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, u_k \right] \right\}$$

control action

$$\leq c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^{\varepsilon} \right]$$

maps augmented state to control action

$$\leq c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}^{\varepsilon}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k^{\varepsilon} \right]$$

by induction hypothesis $J_{k+1} \leq J_{k+1}^{\varepsilon}$

$$= J_k^{\varepsilon}(x_k, y_k)$$

by (5), see page 8

All together, we have shown (a) for index k, $J_k(x_k, y_k) \leq J_k^{\varepsilon}(x_k, y_k) \leq J_k(x_k, y_k) + (N-k)\varepsilon$.

To show (b) for index k,

def on page 1   def on page 5

$$J_k^*(x_k, y_k) := \min_{\pi_k \in \Pi_k} \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right] \leq \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k^{\varepsilon} \right] =: J_k^{\varepsilon}(x_k, y_k).$$

So, $J_k^*(x_k, y_k) \leq J_k^{\varepsilon}(x_k, y_k)$.

Next, we will show $J_k^{\varepsilon}(x_k, y_k) \leq J_k^*(x_k, y_k) + (N-k)\varepsilon$, which is the rest of (b) for index k.

For all $\pi_k = (\mu_k, \dots, \mu_{N-1}) \in \Pi_k$,

$$J_k^{\varepsilon}(x_k, y_k) \le J_k(x_k, y_k) + (N-k)\varepsilon$$

by just proved, see page 10

$$= \min_{u_k} \left\{ c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, u_k \right] \right\} + (N-k)\varepsilon$$

by def of J_k, page 3

$$\le c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \mu_k \right] + (N-k)\varepsilon$$

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot J_{k+1}(x_{k+1}, y_k R) \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon$$

Conditioning on future actions does not affect the expectation.

$$J_{k+1}(x_{k+1}, y_k R) = J^*_{k+1}(x_{k+1}, y_k R)$$

see definition on page 1

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \min_{\pi_{k+1}} \mathrm{CVaR}_{y_k R}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}, \pi_{k+1} \right] \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon$$

by ind. hyp. $J^*_{k+1} = J_{k+1}$

$$\le c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \mathrm{CVaR}_{y_k R}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}, \pi_{k+1} \right] \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon$$

the policy π_{k+1} is arbitrary

b/c we're using inequalities, we can get rid of min, so we don't run into problematic min/max situation

$$= c(x_k) + \max_{R \in \mathcal{B}(y_k, k+1)} \mathbb{E}\left[R \cdot \mathrm{CVaR}_{y_k R}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_{k+1}, \pi_k \right] \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon$$

b/c x_{k+1} is given, conditioning on the action at time k does not change the CVaR

$$= c(x_k) + \mathrm{CVaR}_{y_k}\left[\sum_{i=k+1}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon$$

by CVaR Decomposition Thm, see page 2

$$= \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon \,.$$

$a + \mathrm{CVaR}[Z] = \mathrm{CVaR}[a + Z]$

We have shown that

$$J_k^{\varepsilon}(x_k, y_k) \leq \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right] + (N-k)\varepsilon \qquad \text{for all } \pi_k = (\mu_k, \ldots, \mu_{N-1}) \in \Pi_k$$

This implies that

$$J_k^{\varepsilon}(x_k, y_k) \leq \min_{\pi_k \in \Pi_k} \left\{ \mathrm{CVaR}_{y_k}\left[\sum_{i=k}^{N} c(x_i) \,\middle|\, x_k, \pi_k \right] \right\} + (N-k)\varepsilon$$

$$= \underbrace{J_k^{*}(x_k, y_k)}_{\text{defined on page 1}} + (N-k)\varepsilon.$$

So, we have shown (b) for index k, $J_k^{*}(x_k, y_k) \underset{\text{see page 10}}{\leq} J_k^{\varepsilon}(x_k, y_k) \underset{\text{see line above}}{\leq} J_k^{*}(x_k, y_k) + (N-k)\varepsilon.$

By the above analyses, we've shown that for any $\varepsilon > 0$,

$$J_k(x_k, y_k) \leq J_k^{\varepsilon}(x_k, y_k) \leq J_k(x_k, y_k) + (N-k)\varepsilon, \quad \text{((a) for index } k\text{), and}$$

$$J_k^{*}(x_k, y_k) \leq J_k^{\varepsilon}(x_k, y_k) \leq J_k^{*}(x_k, y_k) + (N-k)\varepsilon. \quad \text{((b) for index } k\text{).}$$

This implies that $J_k^{*}(x_k, y_k) = J_k(x_k, y_k)$. ((c) for index k).

Consequently, we have shown that (a)-(c) hold for index k, which completes the induction.

We have proved the theorem stated on page 3.