



## Stochastics and Statistics

## Time-inconsistent multistage stochastic programs: Martingale bounds ☆☆☆

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## ABSTRACT

Multistage stochastic programs show time-inconsistency in general, if the objective is neither the expectation nor the maximum functional.

This paper considers distortion risk measures (in particular the Average Value-at-Risk) at the final stage of a multistage stochastic program. Such problems are not time consistent. However, it is shown that by considering risk parameters at random level and by extending the state space appropriately, the value function corresponding to the optimal decisions evolves as a martingale and a dynamic programming principle is applicable. In this setup the risk profile has to be accepted to vary over time and to be adapted dynamically. Further, a verification theorem is provided, which characterizes optimal decisions by sub- and supermartingales. These enveloping martingales constitute a lower and an upper bound of the optimal value function.

The basis of the analysis is a new decomposition theorem for the Average Value-at-Risk, which is given in a time consistent formulation.

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## 1. Introduction

Coherent risk measures have been introduced by Artzner et al. in the pioneering papers (Artzner, Delbaen, & Heath, 1997) and (Artzner, Delbaen, Eber, & Heath, 1999). The set of axioms, which are proposed there, is widely accepted nowadays. Approximately 10 years later the same group of authors considered risk measures again in a multistage framework in Artzner, Delbaen, Eber, Heath, and Ku (2007). These authors notice that the Average Value-at-Risk, the most important risk measure, is not time consistent in the sense specified in their paper. Importantly, they relate time-consistency to the fundamental dynamic programming principle, known as Bellman's principle (cf. Fleming & Soner, 2006).

For giving a precise definition of time-consistency (time inconsistency, respectively) one has to distinguish between time-inconsistent risk measures and time-inconsistent decision problems.

Artzner et al. consider the following notion of time-inconsistency in Artzner et al. (2007): a risk measure  $\rho$  applied to a random vari-

able  $Y$  is said to be time-consistent, if knowing the value  $\rho(Y|\mathcal{F})$  for all conditional distributions for any conditioning  $\sigma$ -algebra  $\mathcal{F}$  is sufficient to calculate its unconditional value  $\rho(Y)$ . Their counterexample, showing that the Average Value-at-Risk is time-inconsistent in this sense, is reconsidered and resolved in this paper below (Fig. 2).

The notion of time-consistent decision problems is related, but slightly different: a multistage stochastic decision problem is time-consistent, if resolving the problem at later stages (i.e., after observing some random outcomes), the original solutions remain optimal for the later stages.

Shapiro (2009, p. 144), referring to a tree-structured problem, remarks that for time-consistency of a problem the solution at each stage is not allowed to depend on random parameters, which cannot follow this stage (i.e., in the language of trees, lie in other subtrees): “It is natural to consider the conceptual requirement that an optimal decision at state  $\xi_t$  should not depend on states which do not follow  $\xi_t$ , i.e., cannot happen in the future. That is, optimality of our decision at state  $\xi_t$  should only involve future children nodes of state  $\xi_t$ . We call this principle time consistency.”

Carpentier, Chancelier, Cohen, de Lara, and Girardeau (2012, p. 249) formulate the property as follows: “The sequence of optimization problems is said to be dynamically consistent if the optimal strategies obtained when solving the original problem at time  $t_0$  remain optimal for all subsequent problems. In other words, dynamic

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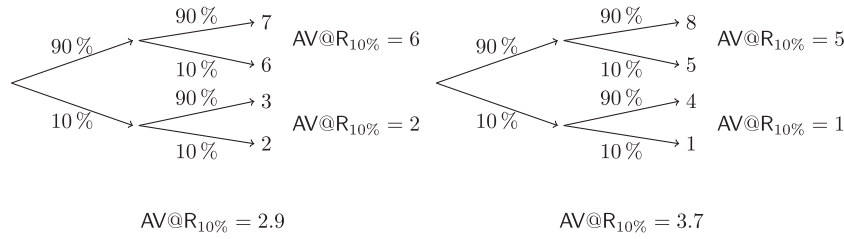


Fig. 1. Counterexample: the Average Value-at-Risk, conditioned on the subtrees, is higher on the left tree. But the Average Value-at-Risk at the initial stage is higher on the right tree.

consistency means that strategies obtained by solving the problem at the very first stage do not have to be questioned later on."

In order to enforce time-consistency for decision problems significant efforts and investigations have been initiated to identify classes of multiperiod risk measures which lead to time-consistency and allow stagewise decompositions. To this end, nested conditional risk measures have been proposed by Shapiro and Ruszczyński (Ruszczyński, 2010; Ruszczyński & Shapiro, 2006; Shapiro, 2009) (cf. as well Shapiro, 2012) and the recent paper by Carpentier et al. (2012). Here, we follow a different path.

First, we will maximize the acceptability rather than minimize risk. This is for ease of presentation of the results. Since acceptability measures can be seen as the negatives of risk measures this is no restriction. Secondly, we stick to the problem of maximizing the acceptability of the final outcome, where the acceptability is measured by a plain distortion measure, not by a nested one. We believe that nested risk or acceptability measures are difficult to interpret and are not what decision makers would understand under multistage risk or acceptability.

As a measure for acceptability we use initially the Average Value-at-Risk (AV@R) defined below in (1), and explain later how the results extend to general distortion measures. As illustration of time-inconsistency of decision problems involving AV@R consider the example displayed in Fig. 1. Suppose that the decision has to be made, which out of the two tree processes is to be selected for the criterion to maximize the final AV@R<sub>10 percent</sub>. It can be easily seen that looking at the conditional AV@R's at intermediate time 1, the left tree has to be preferred to the right one. Looking at the same criterion, but at time 0, the preference is opposite. Therefore, this introductory example confirms the fact that in general the maximization of the AV@R of the final outcome leads to time-inconsistency of decisions.

This paper is based on a new decomposition of the AV@R and related measures. The decomposition measures risk on conditional level only, and it recovers the initial risk measure by collecting the conditional risk measures via an expectation. In this setup the risk profile has to be adapted conditionally, such that the conditional risk profile is not static any longer.

Additional information changes the perception of risk. The adaptive choice of appropriate measures of risk complies with the course of action of a risk manager who adjusts the preferences whenever additional information is available. The decision maker is less reluctant, if an observation reveals that the future will be bright, but conversely she or he will be more strict if losses at the end become more likely. This gives rise to defining an extended notion of a conditional risk measure, which is not just the same risk measure applied to conditional distributions, but which may be a different functional for different conditional distributions, depending on its respective history.

By involving adapted conditional risk measures it is possible to recover dynamic programming principles for multistage stochastic programs. Moreover verification theorems, which are central in dynamic control, are established here for multistage stochastic programs. The dynamic programming equations presented are based on the dual representation of the risk measure, and different to those provided by Shapiro (2009). The presented approach allows a characterization

of optimal solutions of a multistage stochastic program in terms of enveloping sub- and supermartingales. It is shown that a solution of a multistage problem evolves as a martingale over time, where different risk measures are encountered at each stage.

Dynamic programming equations notably cannot remove the time inconsistency, which is inherent to these problems. But these equations come along with verification theorems, and it is their purpose to enable checking, if a given policy is optimal. By assessing the enveloping sub- and supermartingales it is moreover possible to provide upper and lower bounds, such that the quality of a given multistage policy can be assessed with these sub- and supermartingales as well.

### 1.1. Outline of the paper

Section 2 provides the setting for the Average Value-at-Risk, as this risk measure is basic for the presentation. Next, the conditional version is considered. The decomposition theorem, the central statement of this paper, is contained in Section 4 and Section 5 characterizes its properties. Section 6 introduces the multistage optimization problem. Section 7 exposes the dynamic programming formulation, while the subsequent Section 8 introduces the martingale representations, which are in line with dynamic programming.

## 2. Representations of the genuine risk measure

We reduce the conceptual description of the problem to the Average Value-at-Risk, AV@R. This reduction is justified, as more general coherent risk measures—distortion risk measures—are composed in a linear way of Average Value-at-Risks at different levels. Further, Kusuoka's theorem provides all version independent (also known as law invariant) risk measures via distortion risk measures (cf., for example, Pflug & Römisch, 2007), such that this reduction is without loss of generality.

The Average Value-at-Risk is considered in its concave variant involving the lower quantiles of the distribution function  $F_Y$  of the random variable  $Y$ ,

$$AV@R_\alpha(Y) := \frac{1}{\alpha} \int_0^\alpha F_Y^{-1}(u) du \quad (0 < \alpha \leq 1), \quad (1)$$

where  $\alpha$  is called *level*. In this setting AV@R accounts for profits, which are subject to maximization. Throughout this paper we shall assume that the profit variable  $Y$  is a  $\mathbb{R}$ -valued random variable defined on a general, filtered probability space  $(\Omega, (\mathcal{F}_t)_{t \in \{0,1,\dots,T\}}, P)$ . For convenience of presentation we assume that  $Y \in L^\infty(\mathcal{F}_T, P)$  ( $L^1(\mathcal{F}_T, P)$  could be chosen in many, but not in all situations).

The dual representation of the Average Value-at-Risk at level  $\alpha$  is  $AV@R_\alpha(Y) = \inf \{ \mathbb{E}(YZ) : 0 \leq Z, \alpha Z \leq 1 \text{ and } \mathbb{E}(Z) = 1 \}$ , (2)

where the expectation is with respect to the measure  $P$ , the infimum in (2) is among all positive random variables  $Z \geq 0$  with expectation  $\mathbb{E}(Z) = 1$  (i.e.,  $Z$  are densities), satisfying the additional truncation constraint  $\alpha Z \leq 1$ , as indicated. The infimum is attained if  $\alpha > 0$ , and in this case the optimal random variable  $Z$  in (2) is coupled in an anti-monotone way with  $Y$  (cf. Nelsen, 1998).

The equivalent relation<sup>1</sup>

$$\text{AV@R}_\alpha(Y) = \max_{q \in \mathbb{R}} q - \frac{1}{\alpha} \mathbb{E}(q - Y)_+ \quad (3)$$

was elaborated by Rockafellar and Uryasev (2002). This representation replaces the infimum in (2) by a maximum, which is very helpful in the context of profit maximization. The maximum in (3) is attained at some  $q^* \in \mathbb{R}$  satisfying the quantile condition  $P(Y < q^*) \leq \alpha \leq P(Y \leq q^*)$ . For the sake of completeness we mention that the Average Value-at-Risk at level  $\alpha = 0$  is defined as

$$\text{AV@R}_0(Y) = \lim_{\alpha \rightarrow 0} \text{AV@R}_\alpha(Y) = \text{ess inf}(Y).$$

### 2.1. Concave-convexity or the saddle point property

The mapping

$$Y \mapsto \text{AV@R}_\alpha(Y)$$

is concave in the present setting, while the Lorentz curve

$$\alpha \mapsto \alpha \cdot \text{AV@R}_\alpha(Y) = \int_0^\alpha F_Y^{-1}(u) du \quad (4)$$

is convex in its parameter  $\alpha$ . These properties will be exploited to characterize optimal solutions via saddle points.

### 3. The conditional Average Value-at-Risk at random level

The Average Value-at-Risk, as defined above, is a real-valued function on  $L^\infty(\mathcal{F}_T)$ , where  $\mathcal{F}_T$  is the sigma algebra measuring the information at the time horizon  $T$ . The Average Value-at-Risk quantifies the entire future risk associated with  $Y$  at stage 0 in the single value  $\text{AV@R}(Y)$ . Having multistage stochastic decision problems in mind it is desirable to have an idea of the accumulated risk at an intermediate stage  $t$  as well ( $0 < t < T$ ). To describe this evolution sigma algebras  $\mathcal{F}_t \subset \mathcal{F}_T$  are considered, which assemble the information up to time  $t$ .

#### 3.1. The conditional Average Value-at-Risk

Attempts to define a conditional Average Value-at-Risk for a smaller sigma algebra  $\mathcal{F}_t \subset \mathcal{F}_T$  are contained in Ruszczyński and Shapiro (2006) and in Pflug and Ruszczyński (2005). An extension to this initial definition considers a level parameter  $\alpha$ , which depends on the history (we are grateful to Werner Römisch for pointing out the references Cheridito and Kupper (2009, 2011, Section 2.3.1) and Detlefsen and Scandolo (2005)). This setting is broad enough to allow a decomposition.

Here the level  $\alpha$  is not considered fixed and constant any longer, but  $\mathcal{F}_t$ -measurable instead. We write  $\alpha_t \triangleleft \mathcal{F}_t$  to express that  $\alpha_t$  is measurable with respect to  $\mathcal{F}_t$ . For the trivial sigma algebra  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  the statement  $\alpha \triangleleft \mathcal{F}_0$  notably expresses that  $\alpha$  is deterministic, i.e., a constant.

**Definition 1** (Conditional Average Value-at-Risk at random level). The conditional Average Value-at-Risk at random level  $\alpha_t \triangleleft \mathcal{F}_t$  ( $0 \leq \alpha_t \leq 1$ ) of a random variable  $Y \in L^\infty(\mathcal{F}_T, P)$  is the  $\mathcal{F}_t$ -random variable

$$\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) := \text{ess inf} \left\{ \mathbb{E}(YZ|\mathcal{F}_t) \mid \begin{array}{l} Z \in L^\infty(\mathcal{F}_T), 0 \leq Z, \alpha_t Z \leq 1 \\ \mathbb{E}(Z|\mathcal{F}_t) = 1 \end{array} \right\}, \quad (5)$$

where  $1$  is the random variable being identically 1.<sup>3</sup>

**Remark 2.** It is enough to employ  $Z \in L^\infty$  in (5), as  $L^\infty$  is dense in the dual. For a complete treatment of the domain and the dual consider Pichler (2013).

Although the level  $\alpha_t$  is random it should be noted that it is measurable with respect to  $\mathcal{F}_t$ , but not with respect to the larger  $\mathcal{F}_T$ .  $\alpha_t \triangleleft \mathcal{F}_t$  depends on the history only, and this is an important limitation in comparison with  $Y \triangleleft \mathcal{F}_T$ .

**Remark 3.**  $\mathbb{E}(YZ|\mathcal{F}_t)$  is a random variable, the essential infimum in (5) thus is an infimum over a family of random variables, and the resulting  $\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t)$  is a  $\mathcal{F}_t$ -random variable itself.

#### 3.2. Characterization

The following characterization, used in Pflug and Römisch (2007) to define the conditional Average Value-at-Risk in a simpler context, extends to the situation  $\alpha_t \triangleleft \mathcal{F}_t$ , but replaces the essential infimum by a usual infimum.

**Proposition 4** (Characterization of the Average Value-at-Risk). Suppose that  $\alpha_t \triangleleft \mathcal{F}_t$ .

- (i) The conditional Average Value-at-Risk at random level  $\alpha_t$  is a  $\mathcal{F}_t$ -random variable satisfying

$$\begin{aligned} \mathbb{E}[\mathbb{1}_B \cdot \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t)] \\ = \inf\{\mathbb{E}(YZ) : 0 \leq Z, \alpha_t Z \leq \mathbb{1}_B, \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}_B\} \end{aligned}$$

for every set  $B \in \mathcal{F}_t$ .

- (ii) Moreover the conjugate duality relation

$$\begin{aligned} \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) \\ = \text{ess inf}_{Z \in L^\infty} \{\mathbb{E}(YZ|\mathcal{F}_t) - \text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t)\} \\ = \text{ess inf} \{\mathbb{E}(YZ|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z \text{ and } \alpha_t Z \leq 1\} \end{aligned}$$

holds, where the conjugate function is the indicator function

$$\text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) = \begin{cases} 0 & \text{if } \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z \text{ and } \alpha_t Z \leq 1, \\ -\infty & \text{else.} \end{cases} \quad (6)$$

**Remark 5.** Notably  $\alpha_t$ ,  $Z$  and  $\mathbb{E}(Z|\mathcal{F}_t)$  may have various versions. The defining equation (6) is understood to provide a version of  $\text{AV@R}_{\alpha_t}^*$  for any such version and  $\text{AV@R}_{\alpha_t}^*$  thus is well-defined.

**Proof.** The essential infimum  $\text{ess inf}$ , by the characterizing theorem (Appendix A in Karatzas and Shreve (1998) or Dunford and Schwartz (1957)), is a density provided by the Radon-Nikodým theorem satisfying

$$\begin{aligned} \int_B \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) dP \\ = \inf \left\{ \mathbb{E} \left[ \sum_{k=1}^K \mathbb{1}_{B_k} \mathbb{E}(YZ_k|\mathcal{F}_t) \right] : 0 \leq Z_k, \alpha_t Z_k \leq 1, \mathbb{E}(Z_k|\mathcal{F}_t) = 1 \right\}, \end{aligned}$$

where the infimum is among feasible  $Z_k$  and pairwise disjoint sets  $B_k \in \mathcal{F}_t$  with  $B = \bigcup_{k=1}^K B_k$ . The random variable  $Z := \sum_{k=1}^K \mathbb{1}_{B_k} Z_k$  satisfies  $Z = \mathbb{1}_B \cdot Z$ , and the equation in the latter display thus rewrites as

$$\begin{aligned} \mathbb{E}[\mathbb{1}_B \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t)] \\ = \inf\{\mathbb{E}YZ : 0 \leq Z, \alpha_t Z \leq \mathbb{1}_B, \mathbb{E}(Z|\mathcal{F}_t) = \mathbb{1}_B\}, \end{aligned}$$

which is the desired assertion.

The second assertion is the conditional equivalent to (2). For this recall the Fenchel-Moreau-Rockafellar duality theorem (cf. Ruszczyński and Shapiro (2006) or Pflug and Römisch (2007, Theorem 2.51) on conditional risk mappings), which states that

$$\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) = \text{ess inf}_{Z \in L^\infty} \{\mathbb{E}(YZ|\mathcal{F}_t) - \text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t)\},$$

<sup>1</sup>  $x_+$  is the positive part of  $x$ , that is  $x_+ = x$  if  $x \geq 0$ , and  $x_+ = 0$  else.

<sup>2</sup> The restriction to  $Y \in L^\infty(\mathcal{F}_T, P)$  is made for convenience of presentation. The conditional Average Value-at-Risk is well defined on  $L^1(\mathcal{F}_T, P)$  whenever  $\alpha_t \leq 1$ .

<sup>3</sup> All pointwise relations are required to hold almost surely.

where

$$\text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) = \text{ess inf}_{Y \in L^\infty} \mathbb{E}(YZ|\mathcal{F}_t) - \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t).$$

Thus

$$\begin{aligned} \text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) &\leq \text{ess inf}_{Y \in \mathbb{R}} \mathbb{E}[(Y-1)Z|\mathcal{F}_t] - \text{AV@R}_{\alpha_t}(Y-1|\mathcal{F}_t) \\ &= \text{ess inf}_{Y \in \mathbb{R}} Y \mathbb{E}[Z|\mathcal{F}_t] - 1 \end{aligned}$$

and hence  $\text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) = -\infty$  on the  $\mathcal{F}_t$ -set  $\{\mathbb{E}[Z|\mathcal{F}_t] \neq 1\}$ .

Next suppose that  $B := \{Z < 0\}$  has positive measure, then  $\mathbb{E}(Z \mathbb{1}_B|\mathcal{F}_t) < 0$  on  $B$ . Thus

$$\begin{aligned} \text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) &\leq \text{ess inf}_{Y > 0} \mathbb{E}(Y \mathbb{1}_B Z|\mathcal{F}_t) - \text{AV@R}_{\alpha_t}(Y \mathbb{1}_B|\mathcal{F}_t) \\ &\leq \text{ess inf}_{Y > 0} Y \mathbb{E}(Z \mathbb{1}_B|\mathcal{F}_t) = -\infty \end{aligned}$$

on  $B$ . Finally suppose that  $C := \{\alpha_t Z > 1\}$  has positive measure, so

$$\begin{aligned} \text{AV@R}_{\alpha_t}^*(Z|\mathcal{F}_t) &\leq \text{ess inf}_{Y > 0} \mathbb{E}(-Y \alpha_t \mathbb{1}_C Z|\mathcal{F}_t) - \text{AV@R}_{\alpha_t}(-Y \alpha_t \mathbb{1}_C|\mathcal{F}_t) \\ &\leq \text{ess inf}_{Y > 0} -Y \mathbb{E}(\alpha_t Z \mathbb{1}_C|\mathcal{F}_t) + Y \mathbb{E}(\mathbb{1}_C|\mathcal{F}_t) \\ &= \text{ess inf}_{Y > 0} -Y (\mathbb{E}[(\alpha_t Z - 1) \mathbb{1}_C|\mathcal{F}_t]) = -\infty \end{aligned} \quad (7)$$

on  $C$  by the same reasoning. Combining all three ingredients gives the statement, as they constitute all conditions for the Average Value-at-Risk in (5).

The first inequality in (7) is immediate by choosing the particular random variable  $Y = -Y \alpha_t \mathbb{1}_C$ . The second inequality follows from the monotonicity property, as is detailed further in Theorem 11 below.  $\square$

#### 4. The decomposition theorem

Given the Average Value-at-Risk conditionally on  $\mathcal{F}_t$ , how can one reassemble the Average Value-at-Risk at time 0? This is the content of the next theorem, which contains a central result on the Average Value-at-Risk in multistage situations. It is the basis for the martingale representation and the verification theorems in multistage stochastic optimization, which are presented in Section 8 then.

**Theorem 6** (Decomposition of the Average Value-at-Risk). *Let  $Y \in L^\infty(\mathcal{F}_T)$  and  $\mathcal{F}_t \subset \mathcal{F}_T$ .*

- (i) *For a (deterministic) constant  $\alpha \in [0, 1]$  the Average Value-at-Risk obeys the decomposition*

$$\text{AV@R}_\alpha(Y) = \inf \mathbb{E}[Z_t \cdot \text{AV@R}_{\alpha Z_t}(Y|\mathcal{F}_t)], \quad (8)$$

where the infimum is among all densities  $Z_t \triangleleft \mathcal{F}_t$  with  $0 \leq Z_t$ ,  $\alpha Z_t \leq 1$  and  $\mathbb{E}(Z_t) = 1$ . For  $\alpha > 0$  the infimum in (8) is attained.

- (ii) *Moreover, if  $Z$  is the optimal dual density for (2), then*

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t] \quad (9)$$

is the best choice in (8).

- (iii) *Let  $\mathcal{F}_t \subset \mathcal{F}_\tau \subset \mathcal{F}_T$  be nested sigma algebras. The conditional Average Value-at-Risk at random level  $\alpha_t \triangleleft \mathcal{F}_t$  ( $0 \leq \alpha_t \leq 1$ ) has the recursive (nested) representation*

$$\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) = \text{ess inf}_{Z_\tau} \mathbb{E}[Z_\tau \cdot \text{AV@R}_{\alpha_t Z_\tau}(Y|\mathcal{F}_\tau)|\mathcal{F}_t], \quad (10)$$

where the infimum is among all densities  $Z_\tau \triangleleft \mathcal{F}_\tau$  with  $0 \leq Z_\tau$ ,  $\alpha_t Z_\tau \leq 1$  and  $\mathbb{E}[Z_\tau|\mathcal{F}_t] = 1$ .

**Remark 7.** Note that  $\alpha \cdot Z_t$  in the index of the inner  $\text{AV@R}_{\alpha Z_t}$  is a  $\mathcal{F}_t$  random variable satisfying  $0 \leq \alpha \cdot Z_t \leq 1$ , which means that  $\text{AV@R}_{\alpha Z_t}(Y|\mathcal{F}_t)$  is indeed available and almost everywhere well-defined.

**Remark 8.** One might think that a nested decomposition of the  $\text{AV@R}$  might be a consequence of the fact that it can be written in terms of utility functions (having in mind that expected utility allows always a dynamic decomposition). By introducing the family of concave utility functions  $U_q^\alpha(y) := q - \frac{1}{\alpha}(q - y)_+$ , the  $\text{AV@R}_\alpha$  can be written, following (3), as

$$\text{AV@R}_\alpha(Y) = \max_{q \in \mathbb{R}} \mathbb{E} U_q^\alpha(Y).$$

Obviously, for the conditional distributions  $(Y|\mathcal{F}_t)$  the maximizing  $q$  depends on  $\mathcal{F}_t$ , but this is not the crucial point: in fact, as is the content of the decomposition theorem (Theorem 6), only the extension of the class  $(U_q^\alpha)_{q \in \mathbb{R}}$  to the much larger class  $(U_q^\alpha)_{q \in \mathbb{R}, \alpha \in [0, 1]}$  of utility functions with different  $q$ 's and  $\alpha$ 's on every atom of  $\mathcal{F}_t$  allows a decomposition.

**Corollary 9** (Bounds). *For any (deterministic) level  $0 \leq \alpha \leq 1$  it holds that*

$$\text{AV@R}_\alpha(Y) \leq \mathbb{E}[\text{AV@R}_\alpha(Y|\mathcal{F}_t)] \leq \mathbb{E}(Y);$$

for any random  $\alpha_t \triangleleft \mathcal{F}_t$  ( $0 \leq \alpha_t \leq 1$ ), moreover

$$\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) \leq \mathbb{E}[\text{AV@R}_{\alpha_t}(Y|\mathcal{F}_\tau)|\mathcal{F}_t] \leq \mathbb{E}(Y|\mathcal{F}_t).$$

**Proof.** The decomposition theorem, Theorem 6. We shall assume first that  $\alpha > 0$ .

Let  $Z \triangleleft \mathcal{F}_t$  be a simple function (i.e., a step function with finitely many outcomes) with  $Z \geq 0$  and  $\mathbb{E}Z = 1$ , i.e.,  $Z = \sum_i b_i \mathbb{1}_{B_i}$  where  $b_i \geq 0$ ,  $B_i \in \mathcal{F}_t$  and  $B_i \cap B_j = \emptyset$  whenever  $i \neq j$ . Then, by the characterization (Theorem 4),

$$\begin{aligned} \mathbb{E}[Z \text{AV@R}_{\alpha Z}(Y|\mathcal{F}_t)] &= \sum_i b_i \mathbb{E}[\mathbb{1}_{B_i} \text{AV@R}_{\alpha Z}(Y|\mathcal{F}_t)] \\ &= \sum_i b_i \inf\{\mathbb{E}[YX_i] : 0 \leq X_i, \alpha b_i \mathbb{1}_{B_i} X_i \leq \mathbb{1}_{B_i}, \mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i}\} \\ &= \inf \left\{ \sum_i b_i \mathbb{E}[YX_i] : 0 \leq X_i, \alpha b_i \mathbb{1}_{B_i} X_i \leq \mathbb{1}_{B_i}, \mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i} \right\}. \end{aligned}$$

As  $\mathbb{E}[X_i|\mathcal{F}_t] = \mathbb{1}_{B_i}$ , together with the additional constraint  $X_i \geq 0$ , one infers that  $X_i = 0$  on the complement of  $B_i$ , that is to say  $X_i \mathbb{1}_{B_i} = X_i$ .

Define  $X := \sum_i \mathbb{1}_{B_i} X_i$ , thus

$$ZX = \sum_{i,j} b_i \mathbb{1}_{B_i} \mathbb{1}_{B_j} X_j = \sum_i b_i \mathbb{1}_{B_i} X_i = \sum_i b_i X_i$$

and

$$\mathbb{E}[XYZ] = \sum_i b_i \mathbb{E}[YX_i],$$

such that we further obtain by assembling on the mutually disjoint sets  $B_i$

$$\begin{aligned} \mathbb{E}[Z \cdot \text{AV@R}_{\alpha Z}(Y|\mathcal{F}_t)] &= \inf\{\mathbb{E}[YZX] : 0 \leq X, \alpha ZX \leq \mathbb{1}, \mathbb{E}[X|\mathcal{F}_t] = \mathbb{1}\}. \end{aligned} \quad (11)$$

Note next that  $\mathbb{E}[XZ] = \mathbb{E}[Z \cdot \mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[Z \cdot \mathbb{1}] = 1$ , and hence (associate  $\tilde{Z}$  with  $XZ$ )

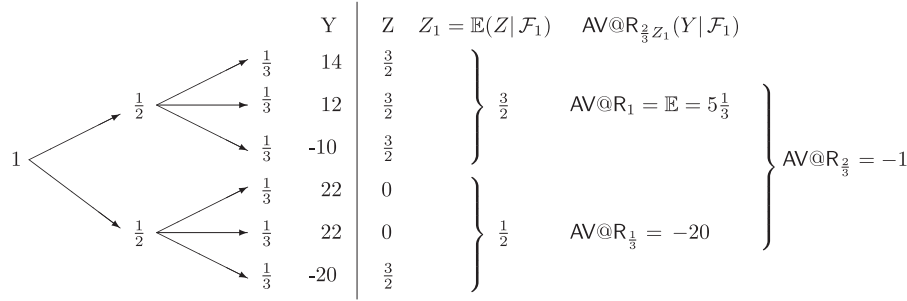
$$\begin{aligned} \mathbb{E}[Z \cdot \text{AV@R}_{\alpha Z}(Y|\mathcal{F}_t)] &\geq \inf\{\mathbb{E}[Y\tilde{Z}] : 0 \leq \tilde{Z}, \alpha \tilde{Z} \leq \mathbb{1}, \mathbb{E}[\tilde{Z}] = 1\} \\ &= \text{AV@R}_\alpha(Y). \end{aligned}$$

It follows by semi-continuity that  $\mathbb{E}[Z \cdot \text{AV@R}_{\alpha Z}(Y|\mathcal{F}_t)] \geq \text{AV@R}_\alpha(Y)$  for all  $Z \geq 0$  with  $\mathbb{E}Z = 1$  and  $\alpha Z \leq 1$ .

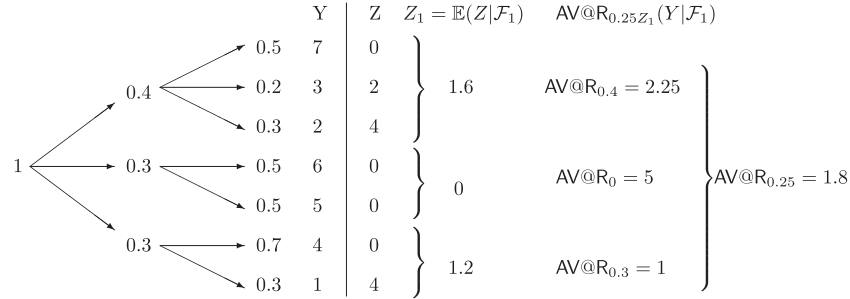
To obtain equality it remains to be shown that there is a  $Z_t \triangleleft \mathcal{F}_t$  such that  $\text{AV@R}_\alpha(Y) = \mathbb{E}[Z_t \text{AV@R}_{\alpha Z_t}(Y|\mathcal{F}_t)]$ . For this let  $Z$  be the optimal dual variable in Eq. (2), that is  $\text{AV@R}_\alpha(Y) = \mathbb{E}YZ$  with  $Z \geq 0$ ,  $\alpha Z \leq 1$  and  $\mathbb{E}Z = 1$ , and define

$$Z_t := \mathbb{E}[Z|\mathcal{F}_t].$$





**Fig. 2.** (cf. Artzner et al., 2007 and Example 10) Nested computation of  $AV@R_\alpha(Y)$  with  $\alpha = \frac{2}{3}$  and outcomes with equal probabilities. The intriguing and misleading fact is that the conditional Average Value-at-Risk, computed with the initial constant  $\alpha = \frac{2}{3}$ , is +1 at each in-between node ( $AV@R_{\frac{2}{3}}(Y|F_t) = +1$ ), but  $AV@R_{\frac{2}{3}}(Y) = -1$ .



**Fig. 3.** Nested computation of  $AV@R_\alpha(Y)$  with  $\alpha = 0.25$ ; in this tree-example with 11 nodes and 7 leaves the transitional probabilities are indicated. It holds true that  $AV@R_{0.25}(Y) = 1.8 = \mathbb{E}YZ = \mathbb{E}[Z_1 \cdot AV@R_{0.25}(Y|F_t)]$ .

$Z_t \triangleleft F_t$  is feasible, as  $0 \leq Z_t$ ,  $\alpha Z_t \leq 1$  and  $\mathbb{E}Z_t = 1$ . Define  $X := \begin{cases} \frac{Z}{Z_t} & \text{if } Z_t > 0 \\ 1 & \text{if } Z_t = 0 \end{cases}$  and observe that  $X$  is  $P$ -a.e. well-defined. Moreover  $0 \leq X$  (as  $0 \leq Z$ ),  $\alpha Z_t X = \alpha Z \leq 1$  and  $\mathbb{E}[X|F_t] = 1$ , such that  $X$  is feasible for (11). One deduces that

$$\begin{aligned} \mathbb{E}[Z_t \cdot AV@R_{\alpha Z_t}(Y|F_t)] \\ = \inf\{\mathbb{E}[YZ_t X] : 0 \leq X, \alpha Z_t X \leq 1, \mathbb{E}[X|F_t] = 1\} \\ \leq \mathbb{E}YZ_t \frac{Z}{Z_t} = \mathbb{E}YZ = AV@R_\alpha(Y). \end{aligned}$$

This is the converse inequality such that assertion (8) follows. The minimum is thus indeed attained for  $Z_t = \mathbb{E}[Z|F_t]$ , where  $Z$  is the optimal dual variable for the  $AV@R_\alpha$ , which exists for  $\alpha > 0$ .

As for  $\alpha = 0$  recall that  $AV@R_0(Y) = \text{ess inf } Y$  and  $AV@R_0(Y) \leq AV@R_0(Y|F_t)$ , and thus

$$\begin{aligned} AV@R_0(Y) &= \mathbb{E}Z_t AV@R_0(Y) \leq \mathbb{E}Z_t AV@R_0(Y|F_t) \\ &= \mathbb{E}Z_t AV@R_{0 Z_t}(Y|F_t). \end{aligned}$$

For the converse inequality choose  $Z^\varepsilon \geq 0$  with  $\mathbb{E}Z^\varepsilon Y \leq AV@R_0(Y) + \varepsilon$ . By the conditional  $L^1 - L^\infty$ -Hölder inequality it holds that

$$\begin{aligned} AV@R_0(Y) + \varepsilon &\geq \mathbb{E}Z^\varepsilon Y \geq \mathbb{E}(\mathbb{E}[Z^\varepsilon|F_t] AV@R_0(Y|F_t)) \\ &\geq \mathbb{E}(\mathbb{E}[Z^\varepsilon|F_t] AV@R_0(Y)) = AV@R_0(Y), \end{aligned}$$

hence

$$AV@R_0(Y) \geq \mathbb{E}Z_t^\varepsilon AV@R_{0 Z_t^\varepsilon}(Y|F_t) - \varepsilon$$

for  $Z_t^\varepsilon := \mathbb{E}[Z^\varepsilon|F_t]$ .

The proof for the remaining statement (iii) of the theorem reads along the same lines as above, but conditioned on  $F_t$ .  $\square$

**Example 10.** Both, Figs. 2 and 3, depict a typical, simple situation with two stages in time, the increasing sigma algebras are visualized via the tree structure.

The example in Fig. 2 is due to Artzner et al. (2007). The Average Value-at-Risk of the random variable  $Y$  in Fig. 2 is  $AV@R_{\frac{2}{3}}(Y) = -1$ . The intriguing fact here is that the conditional Average Value-at-Risk,

computed with the initial  $\alpha = \frac{2}{3}$ , is  $AV@R_{\frac{2}{3}}(Y|F_t) = +1$ , which is in conflicting contrast to  $AV@R_{\frac{2}{3}}(Y) = -1$ .

However, the decomposition Theorem 6 eliminates the discrepancy by involving the conditional Average Value-at-Risk at random level  $\alpha_t \triangleleft F_t$ .

Both figures display the optimal variables  $Z$  and  $Z_1$ :  $Z$  is the optimal dual for (2), and  $Z_1 = \mathbb{E}[Z|F_t]$  is the optimal dual for the decomposition at  $t = 1$  according (9).

## 5. Properties of the conditional Average Value-at-Risk

This section elaborates that the conditional Average Value-at-Risk at random level basically preserves all properties of the usual Average Value-at-Risk. The properties are essential in the following section for multistage stochastic optimization, where the problem is reconsidered for all sigma algebras in the filtration.

**Theorem 11.** For the conditional Average Value-at-Risk at random level  $\alpha_t \triangleleft F_t$  ( $0 \leq \alpha_t \leq 1$ ) the following hold true:

- PREDICTABILITY:**  $AV@R_{\alpha_t}(Y|F_t) = Y$  if  $Y \triangleleft F_t$ ;
- TRANSLATION EQUIVARIANCE:**<sup>4</sup>  $AV@R_{\alpha_t}(Y + c|F_t) = AV@R_{\alpha_t}(Y|F_t) + c$  if  $c \triangleleft F_t$ ;
- POSITIVE HOMOGENEITY:**  $AV@R_{\alpha_t}(\lambda Y|F_t) = \lambda AV@R_{\alpha_t}(Y|F_t)$  whenever  $\lambda$  is nonnegative, bounded and  $F_t$ -measurable,  $\lambda \geq 0$  and  $\lambda \triangleleft F_t$ ;
- MONOTONICITY:**  $AV@R_{\alpha_t}(Y_1|F_t) \leq AV@R_{\alpha_t}(Y_2|F_t)$  whenever  $Y_1 \leq Y_2$  and  $\alpha_t^1 \leq \alpha_t^2$  almost surely;
- CONCAVITY:**  $AV@R_{\alpha_t}((1 - \lambda)Y_0 + \lambda Y_1|F_t) \geq (1 - \lambda)AV@R_{\alpha_t}(Y_0|F_t) + \lambda AV@R_{\alpha_t}(Y_1|F_t)$  for  $\lambda \triangleleft F_t$  and  $0 \leq \lambda \leq 1$  almost surely;
- LOWER AND UPPER BOUNDS:**  $AV@R_0(Y) \leq AV@R_0(Y|F_t) \leq AV@R_{\alpha_t}(Y|F_t) \leq \mathbb{E}(Y|F_t)$ .

<sup>4</sup> In an economic or monetary environment this is often called CASH INVARIANCE instead.

**Proof.** As for the PREDICTABILITY just observe that

$$\begin{aligned} \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) &= \text{ess inf}\{Y \cdot \mathbb{E}(Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z, \alpha_t Z \leq 1\}, \\ &= \text{ess inf}\{Y \cdot 1 : \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z, \alpha_t Z \leq 1\} = Y \end{aligned}$$

whenever  $Y \triangleleft \mathcal{F}_t$ , and TRANSLATION EQUIVARIANCE follows from

$$\begin{aligned} \text{AV@R}_{\alpha_t}(Y + c|\mathcal{F}_t) &= \text{ess inf}\{\mathbb{E}(YZ|\mathcal{F}_t) + c \mathbb{E}(Z|\mathcal{F}_t) : \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z, \alpha_t Z \leq 1\} \\ &= \text{ess inf}\{\mathbb{E}(YZ|\mathcal{F}_t) + c : \mathbb{E}(Z|\mathcal{F}_t) = 1, 0 \leq Z, \alpha_t Z \leq 1\} \\ &= \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t) + c. \end{aligned}$$

To accept that the conditional Average Value-at-Risk is POSITIVELY HOMOGENEOUS observe that the assertion is correct for  $\lambda = 1_A$  ( $A \in \mathcal{F}_t$ ); by passing to the limit one gets the assertion for simple functions (step-functions) first, then for any nonnegative function  $\lambda \in L^\infty(\mathcal{F}_t)$ .

To prove CONCAVITY as stated observe that

$$((1 - \lambda)\mathbb{E}(Y_0 Z|\mathcal{F}_t) + \lambda \mathbb{E}(Y_1 Z|\mathcal{F}_t)) = \mathbb{E}(((1 - \lambda)Y_0 + \lambda Y_1)Z|\mathcal{F}_t)$$

by the measurability assumption  $\lambda \triangleleft \mathcal{F}_t$ , hence

$$\begin{aligned} \text{AV@R}_{\alpha_t}((1 - \lambda)Y_0 + \lambda Y_1|\mathcal{F}_t) &= \text{ess inf}_Z((1 - \lambda)\mathbb{E}(Y_0 Z|\mathcal{F}_t) + \lambda \mathbb{E}(Y_1 Z|\mathcal{F}_t)) \\ &\geq \text{ess inf}_{Z_0, Z_1}((1 - \lambda)\mathbb{E}(Y_0 Z_0|\mathcal{F}_t) + \lambda \mathbb{E}(Y_1 Z_1|\mathcal{F}_t)) \\ &\geq (1 - \lambda) \text{ess inf}_{Z_0} \mathbb{E}(Y_0 Z_0|\mathcal{F}_t) + \lambda \text{ess inf}_{Z_1} \mathbb{E}(Y_1 Z_1|\mathcal{F}_t) \\ &= (1 - \lambda) \text{AV@R}_{\alpha_t}(Y_0|\mathcal{F}_t) + \lambda \text{AV@R}_{\alpha_t}(Y_1|\mathcal{F}_t), \end{aligned}$$

where  $Z_0 \geq 0$  and  $Z_1 \geq 0$  are chosen to satisfy  $\mathbb{E}(Z_i|\mathcal{F}_t) = 1$  and  $\alpha_t Z_i \leq 1$  each.

To observe the MONOTONICITY property recall that  $\alpha_t^1 \leq \alpha_t^2$ , hence

$$\begin{aligned} \text{AV@R}_{\alpha_t^1}(Y_1|\mathcal{F}_t) &= \text{ess inf}_Z \left\{ \mathbb{E}(ZY_1|\mathcal{F}_t) : Z \geq 0, \alpha_t^1 Z \leq 1, \mathbb{E}(Z|\mathcal{F}_t) = 1 \right\} \\ &\leq \text{ess inf}_Z \left\{ \mathbb{E}(ZY_2|\mathcal{F}_t) : Z \geq 0, \alpha_t^1 Z \leq 1, \mathbb{E}(Z|\mathcal{F}_t) = 1 \right\} \\ &\leq \text{ess inf}_Z \left\{ \mathbb{E}(ZY_2|\mathcal{F}_t) : Z \geq 0, \alpha_t^2 Z \leq 1, \mathbb{E}(Z|\mathcal{F}_t) = 1 \right\} \\ &= \text{AV@R}_{\alpha_t^2}(Y_2|\mathcal{F}_t). \end{aligned}$$

The UPPER BOUND finally becomes evident because  $Z = 1$  is feasible for (5), the lower bounds already have been used in (7) (the characterization, Theorem 4) above.  $\square$

### 5.1. Concave-convexity or the saddle point property

The Average Value-at-Risk, in its respective variable, is convex and concave:

- (i) CONCAVITY of the Average Value-at-Risk

$$Y \mapsto \text{AV@R}_{\alpha_t}(Y|\mathcal{F}_t)$$

was elaborated in Theorem 11.

- (ii) CONVEXITY of the Average Value-at-Risk, for the deterministic level parameter  $\alpha$ , is mentioned in (4). The following Theorem 12 extends this observation for measurable level parameters.

**Theorem 12** (Convexity of the AV@R in its level parameter). *Let the random variables  $\alpha_t$ ,  $Z_0$ ,  $Z_1$  and  $\lambda$  be  $\mathcal{F}_t$  measurable with  $0 \leq \alpha_t \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 \leq \alpha_t Z_0 \leq 1$  and  $0 \leq \alpha_t Z_1 \leq 1$ , then*

$$\begin{aligned} Z_\lambda \cdot \text{AV@R}_{\alpha_t, Z_\lambda}(Y|\mathcal{F}_t) &\leq (1 - \lambda)Z_0 \cdot \text{AV@R}_{\alpha_t, Z_0}(Y|\mathcal{F}_t) \\ &\quad + \lambda Z_1 \cdot \text{AV@R}_{\alpha_t, Z_1}(Y|\mathcal{F}_t), \end{aligned}$$

where  $Z_\lambda$  is the convex combination  $Z_\lambda = (1 - \lambda)Z_0 + \lambda Z_1$ .

**Proof.** Recall that by the definition of the conditional Average Value-at-Risk we have that

$$\begin{aligned} (1 - \lambda)Z_0 \cdot \text{AV@R}_{\alpha_t, Z_0}(Y|\mathcal{F}_t) &+ \lambda Z_1 \cdot \text{AV@R}_{\alpha_t, Z_1}(Y|\mathcal{F}_t) \\ &= \text{ess inf} \mathbb{E}(Y(1 - \lambda)Z_0 f_0|\mathcal{F}_t) + \text{ess inf} \mathbb{E}(Y\lambda Z_1 f_1|\mathcal{F}_t) \\ &= \text{ess inf} \mathbb{E}(Y((1 - \lambda)Z_0 f_0 + \lambda Z_1 f_1)|\mathcal{F}_t), \end{aligned}$$

where  $f_0 \geq 0$ ,  $f_1 \geq 0$ ,  $\mathbb{E}(f_0|\mathcal{F}_t) = 1$ ,  $\mathbb{E}(f_1|\mathcal{F}_t) = 1$ , and moreover  $\alpha_t Z_0 f_0 \leq 1$  and  $\alpha_t Z_1 f_1 \leq 1$  in the latter lines of the last display. It follows that  $\alpha_t((1 - \lambda)Z_0 f_0 + \lambda Z_1 f_1) \leq 1$  and hence  $\alpha_t Z_\lambda f \leq 1$  for  $f := \frac{(1 - \lambda)Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}$ . Notice that  $f$  is nonnegative as well, and  $\mathbb{E}(f|\mathcal{F}_t) = \mathbb{E}(\frac{(1 - \lambda)Z_0 f_0 + \lambda Z_1 f_1}{Z_\lambda}|\mathcal{F}_t) = \frac{(1 - \lambda)Z_0 + \lambda Z_1}{Z_\lambda} = 1$ . The latter display continues thus as

$$\begin{aligned} (1 - \lambda)Z_0 \cdot \text{AV@R}_{\alpha_t, Z_0}(Y|\mathcal{F}_t) &+ \lambda Z_1 \cdot \text{AV@R}_{\alpha_t, Z_1}(Y|\mathcal{F}_t) \\ &\geq \text{ess inf} \mathbb{E}(YZ_\lambda f|\mathcal{F}_t), \end{aligned}$$

where the essential infimum is among all random variables  $f \geq 0$  with  $\mathbb{E}(f|\mathcal{F}_t) = 1$  and  $\alpha_t Z_\lambda f \leq 1$ . Hence

$$\begin{aligned} (1 - \lambda)Z_0 \cdot \text{AV@R}_{\alpha_t, Z_0}(Y|\mathcal{F}_t) &+ \lambda Z_1 \cdot \text{AV@R}_{\alpha_t, Z_1}(Y|\mathcal{F}_t) \\ &\geq \text{AV@R}_{\alpha_t, Z_\lambda}(YZ_\lambda|\mathcal{F}_t) \\ &= Z_\lambda \text{AV@R}_{\alpha_t, Z_\lambda}(Y|\mathcal{F}_t) \end{aligned}$$

by positive homogeneity, and this is the desired assertion.  $\square$

### 5.2. Representation as a maximum

In addition to the defining equation (5) for the Average Value-at-Risk at random level there is a further representation, which follows the classical equivalence of (2) and (3) (cf. Ruszczyński, 2010, p. 242). The proof is along the lines of the classical equivalence and rather technical.

**Theorem 13.** *The Average Value-at-Risk at strictly positive random level  $\alpha > 0$  has the additional representation*

$$\text{AV@R}_\alpha(Y|\mathcal{F}_t) = \text{ess sup} \left\{ Q - \frac{1}{\alpha} \mathbb{E}[(Q - Y)_+|\mathcal{F}_t] : Q \triangleleft \mathcal{F}_t \right\}$$

where the essential supremum is among all bounded random variables  $Q \in L^\infty(\mathcal{F}_t)$  ( $Q \triangleleft \mathcal{F}_t$ ).

## 6. Multistage optimization: problem formulation

It is the purpose of this and the following sections to make time inconsistent stochastic optimization problems, which involve the Average Value-at-Risk or an acceptability functional, available for dynamic programming. The problem we consider here incorporates—in the sense of integrated risk management—the acceptability functional in the objective such as

$$\begin{aligned} &\text{maximize} \quad \mathbb{E}Y + \gamma \cdot \text{AV@R}_\alpha(Y) \\ &\text{subject to} \quad Y \in \mathcal{Y}. \end{aligned} \quad (12)$$

$\alpha > 0$  and  $\gamma \geq 0$  are positive, deterministic parameters to account for the emphasis that should be given to risk:  $\gamma$  is the risk appetite, the degree of uncertainty the investor is willing to accept in respect of negative changes to its assets, which are described by the Average Value-at-Risk at the level  $\alpha$ .

The problem formulation (12) applies for optimal investment problems, it can be found in multistage decision models for electricity management as well. In applications (cf. Eichhorn, Heitsch, & Römis, 2010) a reformulation of (12) to account for the multistage situation is typically considered, which involves a stochastic process  $\xi_t$  ( $t \in \mathbf{T} := \{0, 1, \dots, T\}$ ) with values in  $\Xi_t$ .

The multistage reformulation is

$$\begin{aligned} & \text{maximize} && \mathbb{E} H(x, \xi) + \gamma \cdot \text{AV@R}_\alpha(H(x, \xi)) \\ & \text{subject to} && x \triangleleft \mathcal{F}, \\ & && x \in \mathcal{X}, \end{aligned} \quad (13)$$

where  $H: \mathcal{X}_0 \times \dots \times \mathcal{X}_T \times \Xi_0 \times \dots \times \Xi_T \rightarrow \mathbb{R}$  is a function defined on appropriate spaces  $\Xi_t$  and  $\mathcal{X}_t$ . We write  $x \in \mathcal{X}$  (or  $x_t \in \mathcal{X}_t$ ) to express that the component  $x_t: \Omega \rightarrow \mathcal{X}_t$  of the process  $x = (x_t)_{t \in T}$  has values in  $\mathcal{X}_t$ , which itself is a **convex subset of a vector space** ( $\mathcal{X}_t \subset \mathbb{R}^{d_t}$  for some dimension  $d_t$ , e.g. The sets  $\mathcal{X}_t$  themselves are deterministic). The constraint  $x \triangleleft \mathcal{F}$  is the *nonanticipativity* constraint, that is  $x_t \triangleleft \mathcal{F}_t$  for all  $t \in T$ .

Note that for the sigma algebras  $\mathcal{F}_t = \sigma(\xi_0, \xi_1, \dots, \xi_t)$  generated by the underlying process  $\xi$ , the nonanticipativity constraint  $x \triangleleft \mathcal{F}$  forces  $x_t$  to be a function of the process  $\xi_t$ ,  $x_t = x_t(\xi_t)$ , as follows from the Doob–Dynkin lemma (cf. Shiryayev, 1996, Theorem II.4.3). **This reflects the fact that the decisions  $x_t$  have to be fixed without knowledge of the future outcomes.**

We shall require the real-valued function  $H$  to be concave in  $x$  for  $x \in \mathcal{X}$ , such that

$$H((1-\lambda)x' + \lambda x'', \xi) \geq (1-\lambda)H(x', \xi) + \lambda H(x'', \xi)$$

for **any fixed state  $\xi$** .

By the monotonicity property and concavity of the acceptability functional it holds thus that

$$\begin{aligned} & \text{AV@R}_\alpha(H((1-\lambda)x' + \lambda x'', \xi)) \\ & \geq \text{AV@R}_\alpha((1-\lambda)H(x', \xi) + \lambda H(x'', \xi)) \\ & \geq (1-\lambda)\text{AV@R}_\alpha(H(x', \xi)) + \lambda \text{AV@R}_\alpha(H(x'', \xi)), \end{aligned} \quad (14)$$

which means that the mapping  $x \mapsto \text{AV@R}_\alpha(H(x, \xi))$  is concave as well. Concavity and (14) hold for distortion functionals and their conditional variants, **in particular for the Average Value-at-Risk** and the conditional Average Value-at-Risk at random level.

**Remark 14** (Notational convention). We shall write  $H(x)$  for the random variable  $H(x)$  given by  $H(x)(\cdot) := H(x, \cdot)$ . For notational convenience we shall use the straight forward abbreviation  $\xi_{i:j}$  for the sub-string  $\xi_{i:j} = (\xi_i, \xi_{i+1}, \dots, \xi_j)$ ; in particular  $\xi_{i:i} = (\xi_i)$ , and  $x_{i:i-1} = ()$ , the empty string.

**Remark 15** (Multi-period acceptability functionals). Some papers exclusively treat functions of the form  $H(x) = \sum_{t=0}^T H_t(x_{0:t})$  in the present setting. **This particular setting is just a special case and included in our general formulation and framework of problem (13).**

## 7. Dynamic programming formulation

The *dynamic programming principle* is the basis of the solution technique developed by Bellman (1957) in the 1950s for deterministic optimal control problems. They have been extended later to account for stochastic problems as well, where typically

- (i) the objective is an expectation and
- (ii) the transition does not depend on the history, but just on the current state of the system—that is to say for Markov chains.

The decomposition of the Average Value-at-Risk elaborated in Theorem 6 is the key which allows to define—in line with the classical dynamic programming principle—a value function with properties analogous to the classical theory. **The new value function overcomes the restrictions (i) and (ii), as we shall employ a risk measure in the objective, and the value function explicitly depends on the history.** The value function is then used in Section 8 to state the verification theorems.

The theory developed below applies to more general acceptability functionals (risk functions), other than the Average Value-at-Risk, it

includes in particular all approximations of law invariant acceptability functionals by Kusuoka's theorem of the form

$$\mathcal{A} = \sum_k \gamma_k \text{AV@R}_{\alpha_k},$$

and acceptability functionals of the type

$$\mathcal{A}(Y) = \sum_t \mathbb{E} \gamma_t \cdot \text{AV@R}_{\alpha_t}(Y | \mathcal{F}_t) \quad (15)$$

for some  $\mathcal{F}_t$ -measurable  $\alpha_t$  and  $\gamma_t$  ( $0 \leq \alpha_t, \gamma_t \triangleleft \mathcal{F}_t$ ). However, as these more general acceptability functionals are to be treated analogously we may continue with the simple Average Value-at-Risk in lieu of the more general setting (15).

**For Markov processes the value function is—in a natural way—a function of time and the current status of the system. In order to derive dynamic programming equations for the general multistage problem it is necessary to carry the entire history of earlier decisions.** This is respected by the following definition.

**Definition 16** (Value function). Let  $\alpha_t \triangleleft \mathcal{F}_t$  and  $\gamma_t \triangleleft \mathcal{F}_t$  be measurable with  $0 \leq \alpha_t \leq 1$  and  $0 \leq \gamma_t$ . The *value function* at stage  $t$  is the process

$$\begin{aligned} \mathcal{V}_t(x_{0:t-1}, \alpha_t, \gamma_t) &:= \text{ess sup}_{(x_{0:t-1}, \alpha_t, \gamma_t) \in \mathcal{X}} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_t] \\ &\quad + \gamma_t \cdot \text{AV@R}_{\alpha_t}(H(x_{0:T}) | \mathcal{F}_t), \end{aligned} \quad (16)$$

where  $x_{t:T}$  in (16) is chosen such that the concatenated string  $x_{0:T} = (x_{0:t-1}, x_{t:T})$  satisfies  $x_{0:T} \in \mathcal{X}$ .

The value function  $\mathcal{V}_t$  in (16) is measurable with respect to  $\mathcal{F}_t$  for every  $t \in T$ . It depends on

- **the decisions  $x_{0:t-1}$  up to time  $t-1$**  and
- the random model parameters  $\alpha_t \triangleleft \mathcal{F}_t$  and  $\gamma_t \triangleleft \mathcal{F}_t$ .

**Remark 17.** To simplify notation we shall write  $\sup_{x_{t:T}}$  instead of  $\sup_{x_{t:T} \in \mathcal{X}_{t:T}}$ , etc., in what follows.

**The initial stage  $t = 0$ .** The initial problem (13) can be expressed by the value function at initial time  $t = 0$  and assuming that the sigma-algebra  $\mathcal{F}_0$  is trivial. In this case  **$\alpha_0 = \alpha$  and  $\gamma_0 = \gamma$  are deterministic**, and

$$\begin{aligned} (13) &= \sup_{x_{0:T} \in \mathcal{X}} \mathbb{E} H(x_{0:T}) + \gamma \cdot \text{AV@R}_\alpha(H(x_{0:T})) \\ &= \text{ess sup}_{x_{0:T} \in \mathcal{X}} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_0] + \gamma \cdot \text{AV@R}_\alpha(H(x_{0:T}) | \mathcal{F}_0) \\ &= \mathcal{V}_0((), \alpha, \gamma), \end{aligned} \quad (17)$$

which expresses the initial problem (13) in terms of the value function.

**The intermediate stages  $t = 1 \dots T$ .** The decomposition theorem (Theorem 6) above allows to relate the value function at different stages. The recursion obtained can be considered as **generalized dynamic programming principle**.

**Theorem 18** (Dynamic programming principle). Assume that  $\alpha > 0$ , and  $H$  is random upper semi-continuous with respect to  $x$  and  $\xi$  evaluates in some convex, compact subset of  $\mathbb{R}^n$ . Then the following hold true.

- (i) At terminal time  $T$  the value function evaluates to

$$\mathcal{V}_T(x_{0:T-1}, \alpha_T, \gamma_T) = (1 + \gamma_T) \text{ess sup}_{x_T} H(x_{0:T}).$$

- (ii) **For any  $t < T$  ( $t, \tau \in T$ ) the recursive relation**

$$\begin{aligned} & \mathcal{V}_t(x_{0:t-1}, \alpha_t, \gamma_t) \\ &= \text{ess sup}_{x_{t:T-1}} \text{ess inf}_{Z_{t:\tau}} \mathbb{E}[\mathcal{V}_\tau(x_{0:\tau-1}, \alpha_\tau \cdot Z_{t:\tau}, \gamma_\tau \cdot Z_{t:\tau}) | \mathcal{F}_t], \end{aligned} \quad (18)$$

holds true, where  $Z_{t:\tau} \triangleleft \mathcal{F}_\tau$ ,  $0 \leq Z_{t:\tau}$ ,  $\alpha_t Z_{t:\tau} \leq 1$  and  $\mathbb{E}[Z_{t:\tau} | \mathcal{F}_t] = 1$ .

**Proof.** A direct evaluation at terminal time  $t = T$  gives

$$\begin{aligned} \mathcal{V}_T(x_{0:T-1}, \alpha_T, \gamma_T) &= \operatorname{ess\,sup}_{x_{T:T}} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_T] + \gamma_T \cdot \operatorname{AV@R}_{\alpha_T}(H(x_{0:T}) | \mathcal{F}_T) \\ &= \operatorname{ess\,sup}_{x_T} H(x_{0:T}) + \gamma_T \cdot H(x_{0:T}) \\ &= (1 + \gamma_T) \operatorname{ess\,sup}_{x_T} H(x_{0:T}), \end{aligned}$$

because the random variables, conditionally on the entire observations  $\xi_{0:T}$ , are constant. The final maximizations over  $x_{T:T} = x_T(\xi_{0:T})$  are deterministic, because all stochastic observations are available at this final stage.

As for the recursion at an intermediate time ( $t < T$ ) observe that

$$\begin{aligned} \mathcal{V}_t(x_{0:t-1}, \alpha_t, \gamma_t) &= \operatorname{ess\,sup}_{x_{t:T}} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_t] + \gamma_t \cdot \operatorname{AV@R}_{\alpha_t}(H(x_{0:T}) | \mathcal{F}_t) \\ &= \operatorname{ess\,sup}_{x_{t:T}} \operatorname{ess\,inf}_{Z_{t:t+1}} \mathbb{E} \left[ \begin{array}{c} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_{t+1}] \\ + \gamma_t \cdot Z_{t:t+1} \cdot \operatorname{AV@R}_{\alpha_t \cdot Z_{t:t+1}}(H(x_{0:T}) | \mathcal{F}_{t+1}) \\ \times (H(x_{0:T}) | \mathcal{F}_{t+1}) \end{array} \middle| \mathcal{F}_t \right] \end{aligned}$$

due to the nested decomposition (10) of the Average Value-at-Risk at random level, elaborated in Theorem 6. The  $\operatorname{ess\,inf}$  is among all random variables  $Z_{t:t+1} \triangleleft \mathcal{F}_{t+1}$  satisfying  $\alpha_t Z_{t:t+1} \leq 1$  and  $\mathbb{E}(Z_{t:t+1} | \mathcal{F}_t) = 1$ . By the discussions in the preceding sections the inner expression is concave in  $x_{0:T}$  and convex in  $Z_{t:t+1}$ .  $Z_{t:t+1}$  is moreover chosen from the  $\sigma(L^\infty, L^1)$  compact set  $Z_{t:t+1} \in \{Z \in L^\infty : 0 \leq Z \leq \frac{1}{\alpha}\}$ . By Sion's minimax theorem (cf. Sion, 1958 and Komiyama, 1988) one may thus interchange the min and max to obtain

$$\begin{aligned} \mathcal{V}_t(x_{0:t-1}, \alpha_t, \gamma_t) &= \operatorname{ess\,sup}_{x_t} \operatorname{ess\,inf}_{Z_{t:t+1}} \operatorname{ess\,sup}_{x_{t+1:T}} \mathbb{E} \left[ \begin{array}{c} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_{t+1}] \\ + \gamma_t \cdot Z_{t:t+1} \cdot \operatorname{AV@R}_{\alpha_t \cdot Z_{t:t+1}}(H(x_{0:T}) | \mathcal{F}_{t+1}) \\ \times (H(x_{0:T}) | \mathcal{F}_{t+1}) \end{array} \middle| \mathcal{F}_t \right]. \end{aligned}$$

As  $H$  is upper semi-continuous by assumption one may further apply the interchangeability principle (cf. Rockafellar & Wets, 1997, Theorem 14.60, or Shapiro, Dentcheva, & Ruszczyński, 2009, p. 405) such that

$$\begin{aligned} \mathcal{V}_t(x_{0:t-1}, \alpha_t, \gamma_t) &= \operatorname{ess\,sup}_{x_t} \operatorname{ess\,inf}_{Z_{t:t+1}} \mathbb{E} \left[ \begin{array}{c} \mathbb{E}[H(x_{0:T}) | \mathcal{F}_{t+1}] \\ \operatorname{ess\,sup}_{x_{t+1:T}} + \gamma_t \cdot Z_{t:t+1} \cdot \operatorname{AV@R}_{\alpha_t \cdot Z_{t:t+1}}(H(x_{0:T}) | \mathcal{F}_{t+1}) \\ \times (H(x_{0:T}) | \mathcal{F}_{t+1}) \end{array} \middle| \mathcal{F}_t \right] \\ &= \operatorname{ess\,sup}_{x_t} \operatorname{ess\,inf}_{Z_{t:t+1}} \mathbb{E}[\mathcal{V}_{t+1}(x_{0:t}, \alpha_t \cdot Z_{t:t+1}, \gamma_t \cdot Z_{t:t+1}) | \mathcal{F}_t], \end{aligned}$$

which is the desired relation for  $\tau = t + 1$ . Repeating the computation from above  $t - \tau$  times, or conditioning on  $\mathcal{F}_\tau$  instead of  $\mathcal{F}_{t+1}$  reveals the general result.  $\square$

## 8. Martingale representation and the verification theorems

The value function  $\mathcal{V}_t$  introduced in (16) is a function of some general  $\alpha_t \triangleleft \mathcal{F}_t$  and  $\gamma_t \triangleleft \mathcal{F}_t$ . To specify for the right and optimal parameters assume that the optimal policy  $\mathbf{x} = x_{0:T}$  of problem (13) exists.<sup>5</sup> Theorem 18 then gradually reveals the optimal dual variables  $\mathbf{Z}_T, \mathbf{Z}_{T-1}, \dots$  and finally  $\mathbf{Z}_0$  (assuming again that the respective argmins of the essential infimum  $\operatorname{ess\,inf}$  exist). The conditions  $\mathbb{E}(\mathbf{Z}_\tau | \mathcal{F}_t) = 1$  ( $\tau > t$ ) imposed on the dual variables suggest to compound the densities and to consider the densities  $\mathbf{Z}_{t:\tau} := \mathbf{Z}_t \cdot \mathbf{Z}_{t+1} \cdot \dots \cdot \mathbf{Z}_\tau$  such that  $\mathbb{E}(\mathbf{Z}_{t:\tau} | \mathcal{F}_t) = \mathbf{Z}_t$  and  $\mathbb{E}(\mathbf{Z}_{0:\tau} | \mathcal{F}_t) = \mathbf{Z}_{0:t}$ . With this setting the process  $\mathbf{Z} := (\mathbf{Z}_{0:t})_{t \in \mathbf{T}}$  is a martingale, satisfying moreover

$0 \leq \mathbf{Z}_t$  and  $\alpha \mathbf{Z}_t \leq 1$  during all times  $t \in \mathbf{T}$ . The optimal pair  $(\mathbf{x}, \mathbf{Z})$  is a saddle point for the Lagrangian corresponding to the initial problem (13).

This gives rise for the following definition.

**Definition 19.** Let  $\alpha \in [0, 1]$  be a fixed level.

- (i)  $\mathbf{Z} = (\mathbf{Z}_t)_{t \in \mathbf{T}}$  is a feasible (for the nonanticipativity constraints) process of densities if
  - (a)  $\mathbf{Z}_t$  is a martingale with respect to the filtration  $\mathcal{F}_t$  and
  - (b)  $0 \leq \mathbf{Z}_t, \alpha \mathbf{Z}_t \leq 1$  and  $\mathbb{E}(\mathbf{Z}_t) = 1$  for all  $t \in \mathbf{T}$ .
- (ii) The intermediate densities are

$$\mathbf{Z}_{t:\tau} := \begin{cases} \frac{\mathbf{Z}_\tau}{\mathbf{Z}_{t-1}} & \text{if } \mathbf{Z}_{t-1} > 0 \\ 0 & \text{else} \end{cases} \quad (0 < t < \tau),$$

$$\text{and } \mathbf{Z}_{0:\tau} := \mathbf{Z}_\tau.$$

For feasible  $\mathbf{x}$  and  $\mathbf{Z}$  we consider the stochastic process

$$M_t(\mathbf{x}, \mathbf{Z}) := \mathcal{V}_t(x_{0:t-1}, \alpha \mathbf{Z}_{0:t}, \gamma \mathbf{Z}_{0:t}) \quad (t \in \mathbf{T})$$

where  $\alpha$  and  $\gamma$  are simple real numbers.

Recall from (17) that  $M_0$  is a constant (as  $\mathcal{F}_0$  is trivial) solving the original problem (13) if  $(\mathbf{x}, \mathbf{Z})$  are optimal. Above that we shall prove in the next theorem that  $M_t(\mathbf{x}, \mathbf{Z})$  is a martingale in this case (we refer to the papers (Rockafellar & Wets, 1976, 1978) by Rockafellar and Wets for very early occurrences of martingales in a related context).

**Theorem 20** (Martingale property). *Given that  $\mathbf{x}$  and  $\mathbf{Z}$  are optimal, then the process  $M_t(\mathbf{x}, \mathbf{Z})$  is a martingale with respect to the filtration  $\mathcal{F}_t$ .*

*Conversely, if  $M_t(\mathbf{x}, \mathbf{Z})$  is a martingale and the argmax sets (for  $\mathbf{x}$ ) and argmin sets (for  $\mathbf{Z}$ ) in (18) are nonempty, then  $\mathbf{x}$  and  $\mathbf{Z}$  are optimal.*

**Proof.** By the dynamic programming equation (18) and the respective maximality of  $\mathbf{Z}_{t+1}$  and  $\mathbf{x}_{t+1}$  we have that

$$\begin{aligned} M_t(\mathbf{x}, \mathbf{Z}) &= \mathcal{V}_t(x_{0:t-1}, \alpha \mathbf{Z}_{0:t}, \gamma \mathbf{Z}_{0:t}) \\ &= \operatorname{ess\,sup}_{x_{t:t}} \operatorname{ess\,inf}_{Z_{t+1}} \mathbb{E}[\mathcal{V}_{t+1}(\mathbf{x}_{0:t+1}, x_t, \alpha \cdot \mathbf{Z}_{0:t+1}, \gamma \cdot \mathbf{Z}_{0:t+1}) | \mathcal{F}_t] \\ &= \operatorname{ess\,sup}_{x_{t:t}} \mathbb{E}[\mathcal{V}_{t+1}(\mathbf{x}_{0:t+1}, x_t, \alpha \cdot \mathbf{Z}_{0:t+1}, \gamma \cdot \mathbf{Z}_{0:t+1}) | \mathcal{F}_t] \\ &= \mathbb{E}[\mathcal{V}_{t+1}(\mathbf{x}_{0:t}, \alpha \cdot \mathbf{Z}_{0:t+1}, \gamma \cdot \mathbf{Z}_{0:t+1}) | \mathcal{F}_t] \\ &= \mathbb{E}[M_{t+1}(\mathbf{x}, \mathbf{Z}) | \mathcal{F}_t] \end{aligned}$$

again by the interchangeable principle.  $M_t$ , hence, is a martingale with respect to the filtration  $\mathcal{F}_t$ .

The converse follows from the following corollary.  $\square$

### 8.1. Verification theorems

Verification theorems characterize optimal decisions in Bellman's principle. For multistage stochastic optimization verification theorems are accessible as well, they are provided by the following corollary.

**Corollary 21** (Verification theorem). *Let  $\mathbf{x}$  be feasible for (13), and  $\mathbf{Z}$  be feasible according Definition 19.*

- (i) Suppose that  $\mathcal{W}$  satisfies

$$\begin{aligned} \mathcal{W}_t(x_{0:T-1}, \alpha \mathbf{Z}_{0:T}, \gamma \mathbf{Z}_{0:T}) &\geq (1 + \gamma \mathbf{Z}_{0:T}) H(x_{0:T}(\xi_{0:T})) \quad \text{and} \\ \mathcal{W}_t(x_{0:t-1}, \alpha \mathbf{Z}_{0:t}, \gamma \mathbf{Z}_{0:t}) &\geq \operatorname{ess\,sup}_{x_t} \mathbb{E}[\mathcal{W}_{t+1}(x_{0:t}, \alpha \cdot \mathbf{Z}_{0:t+1}, \gamma \cdot \mathbf{Z}_{0:t+1}) | \mathcal{F}_t], \end{aligned}$$

then the process  $\mathcal{W}_t(x_{0:t-1}, \alpha \mathbf{Z}_{0:t}, \gamma \mathbf{Z}_{0:t})$  ( $t \in \mathbf{T}$ ) is a supermartingale dominating  $\mathcal{V}(x_{0:t-1}, \alpha \mathbf{Z}_{0:t}, \gamma \mathbf{Z}_{0:t})$ ,  $\mathcal{V} \leq \mathcal{W}$ .

<sup>5</sup> Optimal decisions  $\mathbf{x}$ , and the corresponding optimal dual variables  $\mathbf{Z}$  are displayed in bold letters.



(ii) Let  $\mathcal{U}$  satisfy

$$\begin{aligned} \mathcal{U}_T(x_{0:T-1}, \alpha Z_{0:T}, \gamma Z_{0:T}) &\leq (1 + \gamma Z_{0:T})H(x_{0:T}(\xi_{0:T})) \quad \text{and} \\ \mathcal{U}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}) \\ &\leq \operatorname{ess\,inf}_{Z_{t+1}} \mathbb{E}[\mathcal{U}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t], \end{aligned}$$

then the process  $\mathcal{U}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t})$  is a submartingale dominated by  $\mathcal{V}(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t})$ ,  $\mathcal{U} \leq \mathcal{V}$ .

**Proof.** The proof is by induction on  $t$ , starting at the final stage  $T$ . Observe first that  $\mathcal{U}_T \leq \mathcal{V}_T \leq \mathcal{W}_T$  by assumption and (14). Then

$$\begin{aligned} \mathcal{U}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}) \\ &\leq \operatorname{ess\,inf}_{Z_{t+1}} \mathbb{E}[\mathcal{U}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &\leq \operatorname{ess\,sup}_{x_t} \operatorname{ess\,inf}_{Z_{t+1}} \mathbb{E}[\mathcal{V}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &= \mathcal{V}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}), \end{aligned}$$

and thus  $\mathcal{U} \leq \mathcal{V}$ .

As for  $\mathcal{W}_t$  observe that

$$\begin{aligned} \mathcal{W}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}) \\ &\geq \operatorname{ess\,sup}_{x_t} \mathbb{E}[\mathcal{W}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &\geq \operatorname{ess\,inf}_{Z_{t+1}} \operatorname{ess\,sup}_{x_t} \mathbb{E}[\mathcal{V}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &\geq \operatorname{ess\,sup}_{x_t} \operatorname{ess\,inf}_{Z_{t+1}} \mathbb{E}[\mathcal{V}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &= \mathcal{V}_t(x_{0:t-1}, \alpha \cdot Z_{0:t}, \gamma \cdot Z_{0:t}), \end{aligned}$$

because it always holds true that  $\inf_z \sup_x L(x, z) \geq \sup_x \inf_z L(x, z)$ .  $\mathcal{W}$  is a supermartingale, because

$$\begin{aligned} \mathcal{W}_t(x_{0:t-1}, \alpha Z_{0:t}, \gamma Z_{0:t}) \\ &\geq \operatorname{ess\,sup}_{x_t} \mathbb{E}[\mathcal{W}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t] \\ &\geq \mathbb{E}[\mathcal{W}_{t+1}(x_{0:t}, \alpha \cdot Z_{0:t+1}, \gamma \cdot Z_{0:t+1}) | \mathcal{F}_t], \end{aligned}$$

which is the characterizing property. The proof that  $\mathcal{U}$  is a submartingale is analogous.  $\square$

## 9. Concluding remarks and summary

Among influential papers and attempts to obtain dynamic programming equations for multistage programming are the papers by Shapiro (2009) and Römisch and Guigues (2012), which address the time consistency aspect. A focus on Bellman's principle is given in Artzner et al. (2007).

In this paper we demonstrate by use of an example that a naïve composition of risk measures is not time consistent. We introduce the conditional Average Value-at-Risk at random risk level. The central result is a decomposition, which allows to reassemble the Average Value-at-Risk given just the conditional risk observations. For this purpose it is necessary to give up the constant risk level and to accept a random risk level instead. The random risk level is adapted for each partial observation and reflects the fact that risk has to be quantified by adapted means, whenever information already is available.

The risk levels, which have to be applied at different levels, are not known a priori, they come along with the solution of the entire problem. This is of course in line with time inconsistency, which is intrinsic to these types of problems. However, dynamic programming principles can still be derived, they can be stated as verification theorems. Those verification theorems are formulated by employing enveloping super- and submartingales. They can be used to check, if

a given policy for a stochastic program is optimal or not. Further, the sub- and supermartingales provide useful lower and upper bounds for the objective of the stochastic program.

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