A Framework for Risk-Sensitive Reachability Analysis

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Abstract—

I. INTRODUCTION

Reachability analysis is a formal verification method based on optimal control theory that can be used to prove safety properties of dynamic systems [1].[1] A standard safety problem is to compute the set of initial conditions from which the state trajectory of the system is guaranteed to satisfy certain constraints over a pre-specified time horizon; this set is called the *safe set*. Typically, the dynamic system has a control signal to be designed to ensure constraint satisfaction and an uncertain disturbance signal that may try to prevent constraint satisfaction (i.e., is *adversarial*). The guarantees enjoyed by the states of the safe set are either deterministic or probabilistic in nature, depending on what we assume about the system dynamics.

For example, we may assume that the control and disturbance signals are bounded, and we may not specify their probability distributions; in this case, the dynamics are said to be *nondeterministic* [2]. If the dynamics are nondeterministic and the disturbance signal is adversarial, then the safe set may be defined as the set of states from which the system can start, such that for any disturbance signal, there exists a control signal that ensures constraint satisfaction (see [3], Sec. 2.2.1). The safety guarantee is deterministic in this case, which has been studied extensively and applied mainly to vehicles (see [1] and the references therein).

In contrast, we may assume that the state of the system is a random variable that evolves according to some probability distribution (e.g., see [4], Sec. 1.2); here the dynamics are said to be *stochastic*. If the dynamics are stochastic and the disturbance input is adversarial, then the safe set may be defined as the set of states from which the system can start, such that for any disturbance signal, there exists a control signal that ensures constraint satisfaction with sufficiently high probability (e.g., see [5]). If the dynamics are stochastic and the disturbance input is non-adversarial (i.e., behaves as random noise), then the safe set may be defined as the set of states from which the system can start, such that there exists a control signal that ensures constraint satisfaction with sufficiently high probability (e.g., see [6] and [7]). In these last two examples, the safety guarantees are probabilistic.

Whether the safety guarantee is deterministic or probabilistic in nature, its essential purpose is to inform decision-making in an uncertain world. The key distinction between deterministic and probabilistic safety guarantees is how the uncertainty is quantified, and whether we assume a pessimistic world, where the uncertainty is adversarial, or a realistic world, where the uncertainty behaves as random noise. In this paper, we develop a framework for reachability theory that lies on the spectrum between pessimism and realism, by leveraging the theory of risk from finance and mathematics.

Risk may be defined qualitatively as "danger, or the possibility of danger, defeat, or loss" [8]. To quantify risk, the mathematical concept of a *risk measure* has been developed. A risk measure is a function that maps a random variable, X, representing loss into the real line, according to the risk associated with X (see [9], Sec. 6.3; see [10], Sec. 2.2). Risk-sensitive optimization algorithms, which minimize the risk of predicted losses via a risk measure, have been receiving more attention in the communities of applied mathematics [11], reinforcement learning [12], [13], [14], and optimal control [15]. Optimization programs that appreciate risk are desirable due to the limitations of alternative methods. In particular, formulations that minimize worst-case losses under adversarial disturbances may produce conservative results with limited practical utility, and formulations that minimize expected losses under random disturbances do not account for low-probability extreme events [15], [16]. On the other hand, risk-sensitive formulations have the potential to generate decisions that can be used in practice and that also protect against particularly damaging outcomes [17].

In this paper, we leverage existing computational results for a particular risk measure, called *Conditional Value-at-Risk* (CVaR), to propose a framework for risk-sensitive reachability analysis. CVaR is a well-justified choice for several reasons. CVaR is a *coherent risk measure*, meaning that it satisfies several intuitive axioms, such as *subadditivity*, which can be interpreted as "diversification decreases risk" (see [10], Sec. 2.2). On finite probability spaces, coherent risk measures are expectations that have been maximized over a collection of perturbed probability distributions, or

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[1]Reachability analysis is also used to prove performance properties of dynamic systems. The focus of this paper is safety rather than performance for simplicity.

expectations that have been made more robust to large losses [15], [9], [13], [18]. Recent work [13] provides an algorithm to minimize the Conditional Value-at-Risk of total cost over time, which we leverage to compute risk-sensitive safe sets. Further, probabilistic safety guarantees and risk-sensitive safety guarantees are closely related, if the risk measure is CVaR, as we shall explain in Sec. IV.

We propose a formulation for risk-sensitive reachability with several desirable attributes. At a fixed confidence level for CVaR, our formulation partitions the state space into regions of varying degrees of safety quantified via the extent of constraint violation likely to be attained by the stochastic dynamic system. Quantification of varying degrees of safety is a feature of safety guarantees for non-deterministic systems (see [3], Eq. 2.3) but not for stochastic systems currently. Existing safety guarantees for stochastic systems are binary, meaning that they encode whether the system is likely to be inside or outside a given set (e.g., see [7], [6], and [5]). Our formulation, however, provides a non-binary quantification of safety for stochastic systems, which is particularly useful when constraint violation is not catastrophic (e.g., routine flooding of a pond after a large storm). Further, our formulation inherits the benefits of risk-sensitive optimization and the benefits of reachability theory. By using a risk measure, our formulation may protect against rare harmful outcomes, which are ignored by reachability formulations that provide safety guarantees in expectation (e.g., [7], [6], and [5]), and may also avoid unnecessary conservatism, which is a common limitation of deterministic safety guarantees (e.g., see [1]). Like existing reachability methods, our formulation provides a comprehensive characterization of the state space in terms of safety. This is not provided by recent work in risk-sensitive optimization, which computes optimal paths emanating from different initial conditions separately (e.g., see [15] and [13]). A comprehensive safety characterization of the state space may be used to inform the cost-effective design of infrastructure that must withstand rare extreme storms, to reduce overly conservative error bounds that arise in safe dynamic motion planning (e.g., [19]), and to increase the amount of time that an autonomous vehicle can operate safely while simultaneously optimizing for performance.

Our formulation also inherits the disadvantages of risk-sensitive optimization and reachability analysis. Since we evoke existing methods for risk-sensitive optimization, we are required to assume finite probability spaces. Since we are not yet learning probability mass functions on-line, we assume that estimates of these functions are available, which is the case for evaluating designs of stormwater infrastructure but not the case for real-time motion planning of a vehicle. Further, like existing methods for risk-sensitive optimization and reachability, our formulation generally requires a dynamic programming algorithm that is computationally expensive.

II. SYSTEM MODEL

The system model is a special case of the model given by [4] in Sec. 1.2. We consider a stochastic discrete-time dynamic system over a finite time horizon,

$$x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, \ldots, T-1, \quad (1)$$

such that $x_k \in S$ is the state of the system at time k, $u_k \in C$ is the control input at time k, and $w_k \in D_k = \{d_1^k, \ldots, d_N^k\}$ is the random disturbance input at time k defined over a finite probability space. The control input is not random, but the state generally is random because it depends on random disturbances. The initial condition, x_0, is not random for simplicity. The collection of admissible control policies is,

$$\Pi = \{(\mu_0, \mu_1, \ldots, \mu_{T-1}), \text{ such that } \mu_k : S \to C\}. \quad (2)$$

The random disturbance at time k, w_k, is characterized by a time-dependent probability mass function that is independent of any control policy, $\pi \in \Pi$, and other disturbances, $w_{\not k} = (w_0, \ldots, w_{k-1}, w_{k+1}, \ldots, w_{T-1})$.[2] Formally, we have

$$P_k(w_k = d_j^k | x_k) = p_j^k,$$
$$\sum_{j=1}^{N} p_j^k = 1, \ p_j^k \geq 0, \quad (3)$$
$$P_k(w_k = d_j^k | x_k, \pi, w_{\not k}) = P_k(w_k = d_j^k | x_k),$$

for each disturbance sample $j = 1, \ldots, N$ and each time point $k = 0, 1, \ldots, T-1$. We are given a (non-empty) constraint set, $\mathcal{K} \subset S$, and the safety criterion that the state of the system should stay inside $\mathcal{K}$ over time. For example, if our application is the flow of water through a network of ponds and streams, $\mathcal{K}$ may indicate that the water does not overflow the banks during a storm event.

III. PROBLEM STATEMENT

The goal of this paper is to design an algorithm that computes risk-sensitive sets for a system of the form specified in Sec. II. A *risk-sensitive set* is, informally, the set of initial conditions of the system, from which the risk of large constraint violations over time can be made small by an admissible control policy.

We quantify risk using the well-established risk measure, *Conditional Value-at-Risk* (CVaR), which is equal to,

$$\text{CVaR}_\delta(Z) = \min_{c \in \mathbb{R}} \left\{ c + \frac{1}{\delta} \mathbb{E}\big[\max\{Z - c, 0\}\big] \right\}, \quad (4)$$

where $\delta \in (0, 1)$, and Z is a random variable representing loss [20].[3] If Z is a continuous random variable, then $\text{CVaR}_\delta(Z)$ is the expected value of Z over large realizations of Z, where the meaning of large is determined by the value of δ (Fig. 1).

We quantify the extent of constraint violation via a surface function that characterizes the constraint set, $\mathcal{K}$. Let $g : S \to \mathbb{R}$ satisfy,

$$x \in \mathcal{K} \iff g(x) < 0, \quad (5)$$

where we adopt the convention provided by [3] in Eq. (2.3). The particular form of g is chosen based on how safety of the system changes with distance to the boundary of $\mathcal{K}$

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[2]The probability mass function may be state-dependent as well.
[3]Definitions of CVaR are presented in various forms. The original paper is [20]. Other references on CVaR include [17] (see Eq. (9)) and [10].
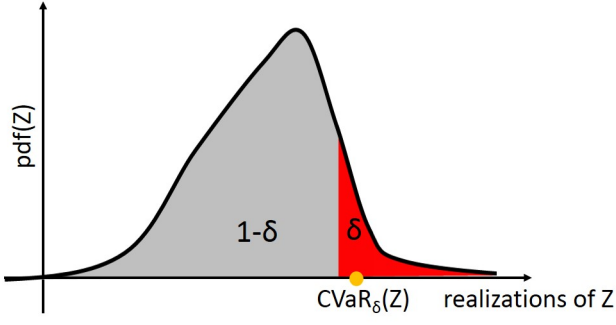
Fig. 1. An illustration of $\text{CVaR}_\delta(Z) \in \mathbb{R}$, if Z is a continuous random variable. The graph shows the probability density function of Z versus the realizations of Z. The area of the right portion under the curve, shown in red, is $\delta \in (0, 1)$. The area of the left portion under the curve, shown in grey, is $1 - \delta$. $\text{CVaR}_\delta(Z)$ is the expectation of the values along the right portion under the curve, indicated by a yellow circle.

for the application at hand. For example, if the relationship between safety and distance to the boundary of $\mathcal{K}$ is linear, then the signed distance function for $\mathcal{K}$ is a suitable choice for g (Fig. 2, dotted). However, if the relationship between safety and distance to the boundary of $\mathcal{K}$ is non-linear, then a quadratic function may be more appropriate (Fig. 2, solid).
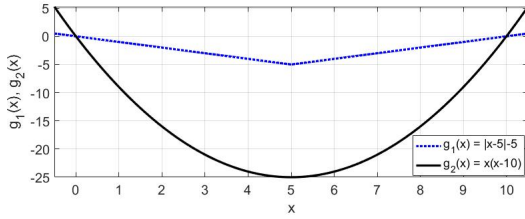

Fig. 2. Different choices for the particular form of g, see (5), for an example constraint set, $\mathcal{K} = (0, 10)$. The relationship between safety and distance to the boundary of $\mathcal{K}$ is linear for $g_1(x) = |x - 5| - 5$ (dotted), and non-linear for $g_2(x) = x(x - 10)$ (solid). For example, the degree of safety at $x = 6$ characterized by the linear relationship is, $g_1(6) = -4$, since the closest distance from $x = 6$ to the boundary of $\mathcal{K}$ is 4, and $x = 6$ is inside $\mathcal{K}$.

We are now ready to define a risk-sensitive set formally. Let $\xi_y^\pi(k) \in S$ be the random state of the system at time k that satisfies (1) under a given control policy $\pi \in \Pi$, starting from a given (non-random) state $y \in S$ at time zero. The maximum extent of constraint violation attained by the system under policy $\pi \in \Pi$, starting from initial condition $y \in S$, is given by the random variable,

$$X_y^\pi := \max_{k \in \{0, \dots, T\}} \left\{ g\big(\xi_y^\pi(k)\big) \right\}, \qquad (6)$$

where g satisfies (5), and the randomness in X_y^π comes from the disturbance sequence, $(w_0, w_1, \dots, w_{T-1})$. Given $\delta \in (0, 1)$, the *risk-sensitive r-sublevel set* is defined by,

$$\mathcal{U}_{\delta,r} := \left\{ y \in S \mid \exists \pi \in \Pi \text{ such that } \text{CVaR}_\delta\left(X_y^\pi\right) < r \right\}$$
$$= \left\{ y \in S \mid \min_{\pi \in \Pi} \left\{ \text{CVaR}_\delta\left(X_y^\pi\right) \right\} < r \right\}, \qquad (7)$$

where $r \in \mathbb{R}$, and the conditional value-at-risk is taken with respect to the probability distribution of $(w_0, w_1, \dots, w_{T-1})$. [Take out?: We give the risk-sensitive zero-sublevel set a particular name: the *risk-sensitive safe set*.] In this paper, we seek to compute $\mathcal{U}_{\delta,r}$ for a finite collection of $(\delta, r) \in (0, 1) \times \mathbb{R}$.

IV. PROPERTIES OF RISK-SENSITIVE SETS

Here we present key properties of risk-sensitive sets. The first property is that every state in the risk-sensitive zero-sublevel set, $\mathcal{U}_{\delta,0}$, enjoys a probabilistic safety guarantee. To prove this property, we need the following result adopted from [9].

Lemma 1: Let $\delta \in (0, 1)$, and Z be a random variable. If $\text{CVaR}_\delta(Z) < 0$, then $\mathbb{P}[Z \geq 0] < \delta$ (see [9], Sec. 6.2.4).[4]

Proof: Suppose $\text{CVaR}_\delta(Z) < 0$.
$\exists c \in \mathbb{R}$ such that $c + \frac{1}{\delta}\mathbb{E}\big[\max\{Z - c, 0\}\big] < 0$, by (1).
But, $\frac{1}{\delta}\mathbb{E}\big[\max\{Z - c, 0\}\big] \geq 0$, so $c < 0$.
Thus, $\exists c < 0$ such that $c + \frac{1}{\delta}\mathbb{E}\big[\max\{Z - c, 0\}\big] < 0$.
Set $\alpha := -c$.
Then, $\exists \alpha > 0$ such that

$$-\alpha + \frac{1}{\delta}\mathbb{E}\big[\max\{Z + \alpha, 0\}\big] < 0 \qquad \Longleftrightarrow$$
$$\mathbb{E}\big[\max\{Z + \alpha, 0\}\big] < \delta\alpha \qquad \Longleftrightarrow \qquad (8)$$
$$\frac{1}{\alpha}\mathbb{E}\big[\max\{Z + \alpha, 0\}\big] < \delta.$$

Since $\max\{Z + \alpha, 0\}$ is a non-negative random variable, we can use Markov's Inequality,

$$\mathbb{P}[\max\{Z + \alpha, 0\} \geq \alpha] \leq \frac{1}{\alpha}\mathbb{E}\big[\max\{Z + \alpha, 0\}\big]. \qquad (9)$$

Thus, $\exists \alpha > 0$ such that $\mathbb{P}[\max\{Z + \alpha, 0\} \geq \alpha] < \delta$.
If $\mathbb{P}[\max\{Z + \alpha, 0\} \geq \alpha] = \mathbb{P}[Z \geq 0]$, then $\mathbb{P}[Z \geq 0] < \delta$, which would complete the proof. We now prove this sufficient condition:

$\mathbb{P}[\max\{Z + \alpha, 0\} \geq \alpha]$
$= \mathbb{P}[\{Z + \alpha \geq \alpha \cap Z + \alpha \geq 0\} \cup \{0 \geq \alpha \cap Z + \alpha < 0\}]$
$= \mathbb{P}[\{Z + \alpha \geq \alpha\} \cap \{Z + \alpha \geq 0\}] + \mathbb{P}[\{0 \geq \alpha\} \cap \{Z + \alpha < 0\}]$
$= \mathbb{P}[\{Z + \alpha \geq \alpha\} \cap \{Z + \alpha \geq 0\}]$
$= \mathbb{P}[Z \geq 0].$
(10)

The first equality holds by the definition of $\max\{Z + \alpha, 0\}$. The second equality holds since the probability of a disjoint union of events is the sum of the probabilities of the individual events. The third and fourth equalities hold because $\alpha > 0$; $\{Z + \alpha \geq \alpha\} \cap \{Z + \alpha \geq 0\}$ is equivalent to $\{Z + \alpha \geq \alpha\}$, which is equivalent to $\{Z \geq 0\}$. ∎
The next result uses Lemma 1 to indicate that every state in $\mathcal{U}_{\delta,0}$ enjoys a probabilistic safety guarantee.

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[4]Ref. [9] indicates that the constraint, $\text{CVaR}_\delta(Z) \leq 0$, gives a conservative approximation of the chance constraint, $\mathbb{P}[Z > 0] \leq \delta$. $\text{CVaR}_\delta(Z) \leq 0$ is written as "AV@R$_\alpha(Z_x) \leq 0$" in [9], see (6.24). $\mathbb{P}[Z > 0] \leq \delta$ is equivalent to $\mathbb{P}[Z \leq 0] \geq 1 - \delta$, which is written as "Pr$(Z_x \leq 0) \geq 1 - \alpha$" in [9], see text below (6.18).

TABLE I

Symbol	Definition	Expression (if applicable)
g	Surface function that characterizes the constraint set, $\mathcal{K}$	$x \in \mathcal{K} \iff g(x) < 0$
C	Set of possible values for the control input	
D_k	Sample space for the random disturbance input at time k	$D_k := \{d_1^k, d_2^k, \ldots, d_N^k\}$
S	Set of (continuous) states	$S := \mathbb{R}^n$
$\mathcal{K}$	Constraint set	$\mathcal{K} \subset S$
Π	Set of admissible control policies	$\Pi := \{(u_0, u_1, \ldots, u_{T-1})\}$
$\mathbb{P}$	The probability measure with respect to $(w_0, w_1, \ldots, w_{T-1})$	
$\mathbb{T}$	Finite discrete time horizon	$\mathbb{T} := \{0, 1, \ldots, T\}$
$\xi_x^\pi(k)$	Random state at time k under (fixed) policy π, starting from (fixed) initial condition, $x \in S$, at time 0	

V. RELATION BETWEEN PROBABLISTIC SAFETY AND CVAR SAFETY

VI. CONCLUSION

Corollary 1: $\mathcal{U}_{\delta,0}$, as defined in (7) for $r = 0$, is a subset of $\mathcal{S}_\delta$,

$$\mathcal{S}_\delta := \left\{ y \in S \mid \exists \pi \in \Pi, \ \mathbb{P}\left[\forall k \in \mathbb{T}, \ \xi_y^\pi(k) \in \mathcal{K}\right] > 1 - \delta \right\}, \tag{11}$$

where $\mathbb{P}$ is the probability measure for the state trajectory, and $\mathbb{T} = \{0, 1, \ldots, T\}$ is the time horizon.

Proof: Take $y \in \mathcal{U}_\delta$. We need to show that $y \in \mathcal{S}_\delta$. Since $y \in \mathcal{U}_\delta$, there exists $\pi \in \Pi$ such that $\mathrm{CVaR}_\delta(X_y^\pi) < 0$. By Lemma 1, $\mathrm{CVaR}_\delta(X_y^\pi) < 0 \implies \mathbb{P}[X_y^\pi \geq 0] < \delta$. Thus, there exists $\pi \in \Pi$ such that

$$
\begin{aligned}
\mathbb{P}[X_y^\pi \geq 0] < \delta &\iff \\
\mathbb{P}\left[\max_{k \in \mathbb{T}}\left\{g\left(\xi_y^\pi(k)\right)\right\} \geq 0\right] < \delta &\iff \\
\mathbb{P}\left[\exists k \in \mathbb{T}, \ g\left(\xi_y^\pi(k)\right) \geq 0\right] < \delta &\iff \\
\mathbb{P}\left[\forall k \in \mathbb{T}, \ g\left(\xi_y^\pi(k)\right) < 0\right] > 1 - \delta &\iff \\
\mathbb{P}\left[\forall k \in \mathbb{T}, \ \xi_y^\pi(k) \in \mathcal{K}\right] > 1 - \delta.
\end{aligned}
\tag{12}
$$

The fourth line holds since for any event E with probability measure P, $P[E] > 1 - \delta \iff P[\neg E] < \delta$, where $\neg E$ is the logical negation of E. The last line holds because g satisfies (5). We have shown that there exists $\pi \in \Pi$ such that $\mathbb{P}\left[\forall k \in \mathbb{T}, \ \xi_y^\pi(k) \in \mathcal{K}\right] > 1 - \delta$, which means that $y \in \mathcal{S}_\delta$. ∎

The second key property of the risk-sensitive safe set, $\mathcal{U}_\delta$, is, as δ decreases, the states of $\mathcal{U}_\delta$ enjoy a stronger probabilistic safety guarantee, and $\mathcal{U}_\delta$ becomes smaller.

Lemma 2: Let $1 > \delta_1 \geq \delta_2 > 0$. Then, $\mathcal{S}_{\delta_2} \supset \mathcal{S}_{\delta_1}$, and $\mathcal{U}_{\delta_2} \supset \mathcal{U}_{\delta_1}$, where $\mathcal{U}_\delta$ is given by (**??**) and $\mathcal{S}_\delta$ is given by (11).

Remark 1: Remark

Please see Table I for a summary of relevant notation.

Problem 1. An important problem is to compute the set of initial states for which there exists an admissible control policy that keeps the system inside the constraint set over time with sufficiently high probability. The *safe set* with confidence $1 - \delta \in (0, 1)$ is defined as,

$$\mathcal{S}(\delta) := \{x \in S \mid \exists \pi \in \Pi \text{ such that } \mathbb{P}[\forall k \in \mathbb{T}, \ \xi_x^\pi(k) \in \mathcal{K}] > 1 - \delta\} \tag{13}$$

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