A Risk-Sensitive Finite-Time Reachability Problem for Safety of Stochastic Dynamic Systems

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Abstract*— A classic reachability problem for safety of dynamic systems is to compute the set of initial states from which the state trajectory is guaranteed to stay inside a given constraint set over some time horizon. In this paper, we leverage existing theory of reachability analysis and risk measures to formulate a *risk-sensitive* reachability problem for safety of stochastic dynamic systems under non-adversarial disturbances over a finite time horizon. We provide two key contributions to the reachability literature. First, our formulation quantifies the distance between the boundary of the constraint set and the state trajectory for a stochastic dynamic system. In the literature, Hamilton-Jacobi (HJ) reachability methods quantify this distance for non-deterministic systems subject to adversarial disturbances, while stochastic reachability methods reduce the distance to a binary random variable in order to quantify the probability of safety. Second, our formulation accounts for rare high-consequence events by posing the optimal control problem in terms of a risk measure, called *Conditional Value-at-Risk* (CVaR). HJ reachability assumes that high-consequence events occur always, which may yield overly conservative solutions in practice, whereas stochastic reachability does not explicitly account for rare high-consequence events, since the optimal control problem is posed in terms of the expectation operator. We define a *risk-sensitive safe set* as the set of initial states from which the risk of extreme constraint violations can be made small via an appropriate control policy, where risk is quantified using CVaR. We show that certain risk-sensitive safe sets enjoy probabilistic safety guarantees. We provide a dynamic programming algorithm to compute under-approximations of risk-sensitive safe sets and prove the correctness of the algorithm for finite probability spaces. Our proof is a novel contribution, as it does not require the assumption of strong duality, which was required in a previous paper. Finally, we demonstrate the utility of risk-sensitive reachability analysis on a numerical example.

I. Introduction

Reachability analysis is a formal verification method based on optimal control theory that is used to prove safety or performance properties of dynamic systems [1]. A classic reachability problem for safety is to compute the set of initial states from which the state trajectory is guaranteed to stay inside a given constraint set over some time horizon. This problem was first considered for discrete-time dynamic systems by Bertsekas and Rhodes under the assumption that disturbances are uncertain but belong to known sets [2], [3], [4]. In this context, the problem is solved using a minimax formulation, in which disturbances behave adversarially and safety is described as a binary notion based on set membership [2], [3], [4].[1]

In practice, minimax formulations can yield overly conservative solutions, particularly because disturbances are not often adversarial. Most storms do not cause major floods, and most vehicles are not involved in pursuit-evader games. If there are enough observations of the system, one can estimate a probability distribution for the disturbance, and then assess safety properties of the system in a more realistic context.[2] For stochastic discrete-time dynamic systems, Abate et al. developed an algorithm that computes the set of initial states from which the probability of safety of the state trajectory can be made large by an appropriate control policy [6].[3] Summers and Lygeros extended the algorithm of Abate et al. to quantify the probability of safety and performance of the state trajectory, by specifying that the state trajectory should also reach a target set [7].

Both the stochastic reachability methods [6], [7] and the minimax reachability methods [2], [3], [4] for discrete-time dynamic systems describe safety as a binary notion based on set membership. In Abate et al., for example, the probability of safety to be optimized is the expectation of the product (or maximum) of indicator functions, where each indicator encodes the event that the state at a particular time point is inside a given set [6]. The stochastic reachability methods [6], [7] do not generalize to quantify the random distance between the state trajectory and the boundary of the constraint set, since they use indicator functions to convert probabilities to expectations to be optimized.

In contrast, Hamilton-Jacobi (HJ) reachability methods quantify the deterministic analogue of this distance for continuous-time systems subject to adversarial disturbances (e.g., see [1], [8], [9], [10]). Quantifying the distance between the state trajectory and the boundary of the constraint set in a non-binary fashion may be important in applications where the boundary is not known exactly, or where mild constraint violations are inevitable, but extreme constraint violations must be avoided.

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[1]in ref. [4], see Sec. 3.6.2, "Control within a Target Tube"

[2]Ref. [5] presents methods for estimating probability distributions.

[3]Safety of the state trajectory is the event that the state trajectory stays in the constraint set over a finite time horizon.

It is imperative that reachability methods for safety take into account the possibility that rare events can occur with potentially damaging consequences. Reachability methods that assume adversarial disturbances (e.g., [1], [3]) suppose that harmful events can always occur, which may yield solutions with limited practical utility, especially in applications with large uncertainty sets. Stochastic reachability methods [6], [7] do not explicitly account for rare high-consequence events because the optimal control problem is expressed as an expectation.

In contrast, we leverage existing results on *risk measures* to formulate an optimal control problem that explicitly encodes a realistic viewpoint on the possibility of rare high-consequence events: harmful events are likely to occur at some point, but they are unlikely to occur always. A *risk measure* is a function that maps a random variable, Z, representing loss into the real line, according to the risk associated with Z (see [11], Sec. 6.3; see [12], Sec. 2.2). Risk-sensitive optimization is being studied in applied mathematics [13], reinforcement learning [14], [15], [16], and optimal control [17].[4] Risk-sensitive formulations have the potential to inform practical decision-making that also protects against damaging outcomes [18], where the level of conservatism can be modified as needed.

We use a particular risk measure, called *Conditional Value-at-Risk* (CVaR), in this paper. If Z is a random cost with finite expectation, then the Conditional Value-of-Risk of Z at confidence level $\alpha \in (0,1)$ is,

$$\text{CVaR}_\alpha[Z] = \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha}\mathbb{E}\big[\max\{Z - t, 0\}\big] \right\}; \quad (1)$$

see [11], Equation 6.22.[5] Note that $\text{CVaR}_\alpha[Z]$ increases from $\mathbb{E}[Z]$ to $\sup Z$, as α decreases from 1 to 0.[6] Further, there is a well-established relationship between CVaR and chance constraints that we use to obtain probabilistic safety guarantees. Chow et al. provides tractable methods to compute the CVaR of a cumulative cost incurred by a Markov Decision Process [15] that we also leverage. CVaR has additional desirable properties that are of particular interest to researchers in financial risk management and are summarized in ref. [18].

The key contributions of this paper follow. We formulate a risk-sensitive reachability problem for safety of stochastic dynamic systems under non-adversarial disturbances over a finite time horizon. In particular, our formulation quantifies the non-binary distance between the boundary of the constraint set and the state trajectory for a stochastic dynamic system. This is an extension of stochastic reachability methods (e.g., [6], [7]), which reduce this distance to a binary random variable. Further, in contrast to stochastic reachability methods, our formulation explicitly accounts for rare high-consequence events by posing the optimal control problem in terms of Conditional Value-at-Risk instead of expectation. This is the first use of risk measures in the reachability literature to our knowledge. In Sec. II, we define the notion of a *risk-sensitive safe set* and formalize the problem statement. Sec. III summarizes properties of risk-sensitive safe sets, including their relation to probabilistic safety. Sec. IV provides a dynamic programming algorithm to compute under-approximations of risk-sensitive safe sets. In Sec. V, we provide a numerical example in the context of the design of stormwater infrastructure. Sec. VI provides steps for future work.

II. Problem Statement

We consider a stochastic discrete-time dynamic system over a finite time horizon,[7]

$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \dots, N-1, \quad (2)$$

such that $x_k \in \mathbb{R}^n$ is the state of the system at time k, $u_k \in U$ is the control at time k, and $w_k \in D$ is the random disturbance at time k. Both U and D are finite sets. $\mathbb{P}[w_k = d_j] = p_j$ is the probability that the disturbance has the value $d_j \in D$ at time k, where $0 \leq p_j \leq 1$ and $\sum_{j=1}^{W} p_j = 1$.[8] The control input is not random, but the state is random generally because it depends on random disturbances. The initial condition, x_0, is not random for simplicity.

III. Properties of Risk-Sensitive Safe Sets

IV. Computational Method

V. Numerical Example

VI. Conclusion

VII. System Model

The system model is a special case of the model given by [4] in Sec. 1.2.

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$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \dots, N-1, \quad (3)$$

such that $x_k \in \mathbb{R}^n$ is the state of the system at time k, u_k is the control input at time k, and $w_k \in D_k = \{d_1^k, \dots, d_N^k\}$ is the random disturbance input at time k defined over a finite probability space. The control input is not random, but the state generally is random because it depends on random disturbances. The initial condition, x_0, is not random for simplicity. The collection of admissible control policies is,

$$\Pi = \{(\mu_0, \mu_1, \dots, \mu_{T-1}), \text{ such that } \mu_k : S \to C\}. \quad (4)$$

The random disturbance at time k, w_k, is characterized by a time-dependent probability mass function that is independent

[4] In risk-sensitive optimization, the risk of a cost is minimized, where risk is quantified using a risk measure. Conversely, in stochastic optimization, we usually minimize the expected value of a cost.

[5] Conditional Value-at-Risk is also called *Average Value-at-Risk*, which is abbreviated as AV@R in [11].

[6] Technically, $\text{CVaR}_\alpha[Z] \to \text{ess}\sup Z$ as $\alpha \to 0$, where $\text{ess}\sup Z$ is the *essential supremum* of Z. Informally, essential supremum is a supremum for random variables.

[7] The system model is a special case of the model given by [4] in Sec. 1.2.

[8] We also assume that w_k is independent of x_k, u_k, and disturbances at any other times.

of any control policy, $\pi \in \Pi$, and other disturbances, $w_{\not k} = (w_0, \dots, w_{k-1}, w_{k+1}, \dots, w_{T-1})$.[9] Formally, we have

$$\begin{aligned} P_k(w_k = d_j^k | x_k) = p_j^k, \\ \textstyle\sum_{j=1}^{N} p_j^k = 1,\ p_j^k \geq 0, \\ P_k(w_k = d_j^k | x_k, \pi, w_{\not k}) = P_k(w_k = d_j^k | x_k), \end{aligned} \tag{5}$$

for each disturbance sample $j = 1, \dots, N$ and each time point $k = 0, 1, \dots, T-1$. We are given a (non-empty) constraint set, $\mathcal{K} \subset S$, and the safety criterion that the state of the system should stay inside $\mathcal{K}$ over time. For example, if our application is the flow of water through a network of ponds and streams, $\mathcal{K}$ may indicate that the water does not overflow the banks during a storm event.

VIII. Problem Statement

The goal of this paper is to design an algorithm that computes a *risk-sensitive safe set* for a system of the form specified in Sec. VII. A risk-sensitive safe set is, informally, the set of initial conditions of the system, from which there is small risk of large constraint violations over time.

We quantify risk using the well-established risk measure, *Conditional Value-at-Risk* (CVaR), which is equal to,

$$\text{CVaR}_\delta(Z) = \min_{c \in \mathbb{R}} \Big\{ c + \frac{1}{\delta} \mathbb{E}\big[\max\{Z - c, 0\} \big] \Big\}, \tag{6}$$

where $\delta \in (0,1)$, and Z is a random variable representing loss [19].[10]

If Z is a continuous random variable, then $\text{CVaR}_\delta(Z)$ is the expected value of Z over large realizations of Z, where the meaning of large is based on δ (Fig. 1).

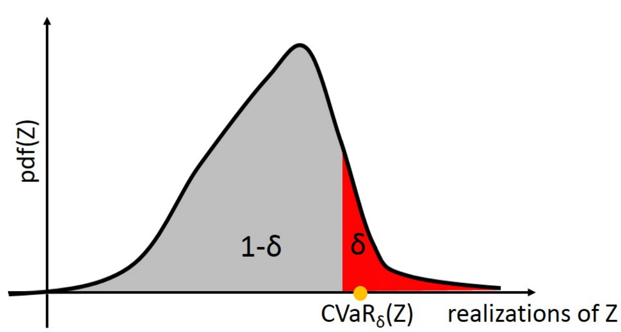

Fig. 1. An illustration of $\text{CVaR}_\delta(Z) \in \mathbb{R}$, if Z is a continuous random variable. The graph shows the probability density function of Z versus the realizations of Z. The area of the right portion under the curve, shown in red, is $\delta \in (0,1)$. The area of the left portion under the curve, shown in grey, is $1-\delta$. $\text{CVaR}_\delta(Z)$ is the expectation of the values along the right portion under the curve, indicated by a yellow circle.

We quantify the extent of constraint violation via a surface function that characterizes the constraint set, $\mathcal{K}$. Let $g : S \to \mathbb{R}$ satisfy,

$$x \in \mathcal{K} \iff g(x) < 0, \tag{7}$$

where we adopt the convention provided by [9] in Eq. (2.3). The particular form of g is chosen based on how safety of the system changes with distance to the boundary of $\mathcal{K}$ for the application at hand. For example, if the relationship between safety and distance to the boundary of $\mathcal{K}$ is linear, then the signed distance function for $\mathcal{K}$ is a suitable choice for g (Fig. 2, dotted). However, if the relationship between safety and distance to the boundary of $\mathcal{K}$ is non-linear, then a quadratic function may be more appropriate (Fig. 2, solid).

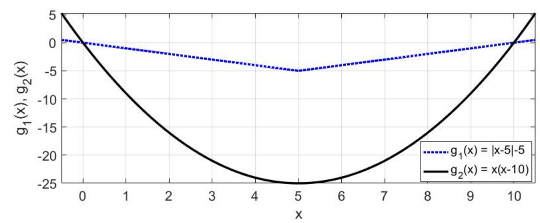

Fig. 2. Different choices for the particular form of g, see (7), for an example constraint set, $\mathcal{K} = (0, 10)$. The relationship between safety and distance to the boundary of $\mathcal{K}$ is linear for $g_1(x) = |x-5| - 5$ (dotted), and non-linear for $g_2(x) = x(x-10)$ (solid). For example, the degree of safety at $x = 6$ characterized by the linear relationship is, $g_1(6) = -4$, since the closest distance from $x = 6$ to the boundary of $\mathcal{K}$ is 4, and $x = 6$ is inside $\mathcal{K}$.

We are now ready to define the risk-sensitive safe set formally. Let $\xi_y^\pi(k) \in S$ be the random state of the system at time k that satisfies (3) under a given control policy $\pi \in \Pi$, starting from a given (non-random) state $y \in S$ at time 0. The maximum extent of constraint violation attained by the system under policy $\pi \in \Pi$, starting from initial condition $y \in S$, is given by the random variable,

$$X_y^\pi = \max_{k \in \{0,\dots,T\}} \Big\{ g\big(\xi_y^\pi(k)\big) \Big\}, \tag{8}$$

where g satisfies (7). For any $\delta \in (0,1)$, the risk-sensitive safe set is,

$$\begin{aligned} \mathcal{U}_\delta :=& \Big\{ y \in S \mid \exists \pi \in \Pi \text{ such that } \text{CVaR}_\delta\left(X_y^\pi\right) < 0 \Big\} \\ =& \Big\{ y \in S \mid \min_{\pi \in \Pi} \left\{ \text{CVaR}_\delta\left(X_y^\pi\right) \right\} < 0 \Big\}, \end{aligned} \tag{9}$$

where the random variable, X_y^π, is defined in (8), and the conditional value-at-risk is taken with respect to the probability distribution of $(w_0, \dots, w_{T-1})$. To summarize, the risk-sensitive safe set is the set of initial conditions from which the risk of large constraint violations can be made small by an appropriate control policy. The problem addressed in this paper is how to compute (9).

IX. Properties of $\mathcal{U}_\delta$

Here we present two key properties of the risk-sensitive safe set. The first property is that every state in $\mathcal{U}_\delta$ enjoys a probabilistic safety guarantee. To prove this property, we need the following result adopted from [11].

SJ: The simplified proof is using $\delta = 1 - \alpha$. I suggest we use this notation throughout the paper.

Lemma 1: Let $\delta \in (0,1)$, and Z be a random variable. If $\text{CVaR}_\delta(Z) < 0$, then $\mathbb{P}[Z \geq 0] < \delta$ (see [11], Sec. 6.2.4).[11]

[9]The probability mass function may be state-dependent as well.

Proof: SJ: simplifying the proof below.

$$
\begin{aligned}
&\mathrm{CVaR}_{1-\alpha}(Z) < 0 \\
\iff &\frac{1}{1-\alpha}\mathbb{E}\big[\max\{Z-c,0\}\big] < -c \\
\iff &\exists c \in \mathbb{R} \;\; c + \frac{1}{1-\alpha}\mathbb{E}\big[\max\{Z-c,0\}\big] < 0 \text{ [using (6)]} \\
\iff &\mathbb{E}\big[\max\{Z-c,0\}\big] < -c(1-\alpha) \qquad (10)
\end{aligned}
$$

Now, the LHS of the inequality, $\mathbb{E}\big[\max\{Z-c,0\}\big] \geq 0$ because the expectation of non-negative values cannot be negative. Consequently, the RHS of the inequality must also be non-negative, that is, $-c(1-\alpha) \geq 0$, that is, $c \leq 0$ since $1-\alpha \geq 0$. So, we can rewrite inequality (10) using $a = -c \geq 0$ as follows:

$$
\begin{aligned}
&\mathbb{E}\big[\max\{Z+a,0\}\big] < a(1-\alpha), \text{ where } a \geq 0 \\
\iff &\frac{1}{a}\mathbb{E}\big[\max\{Z+a,0\}\big] < 1-\alpha, \text{ where } a \geq 0 \qquad (11)
\end{aligned}
$$

Using Markov's Inequality, $\mathbb{P}\big[\max\{Z+a,0\} \geq a\big] \leq \frac{1}{a}\mathbb{E}\big[\max\{Z+a,0\}\big]$. Combining with inequality (11),

$$
\begin{aligned}
&\mathbb{P}\big[\max\{Z+a,0\} \geq a\big] \leq \frac{1}{a}\mathbb{E}\big[\max\{Z+a,0\}\big] < 1-\alpha \\
\Rightarrow \; &\mathbb{P}\big[\max\{Z+a,0\} \geq a\big] < 1-\alpha \qquad (12)
\end{aligned}
$$

Now, $Z \geq 0 \iff Z + a \geq a \iff \max\{Z+a,0\} \geq a$ since $a \geq 0$, and so,
$\mathbb{P}\big[Z \geq 0\big] = \mathbb{P}\big[\max\{Z+a,0\} \geq a\big]$. Combining with the inequality (12),

$$
\begin{aligned}
&\mathbb{P}\big[Z \geq 0\big] = \mathbb{P}\big[\max\{Z+a,0\} \geq a\big] < 1-\alpha \\
\iff &\mathbb{P}\big[Z \geq 0\big] < 1-\alpha
\end{aligned}
$$

■

The only one-sided implication is in the use of Markov's Inequality to get inequality (11), and this corresponds to the approximation gap in using $\mathrm{CVaR}_{1-\alpha}(Z) < 0$ to approximate $\mathbb{P}[Z \geq 0] < 1-\alpha$.

The next corollary indicates that every state in $\mathcal{U}_\delta$ enjoys a probabilistic safety guarantee.

Corollary 1: $\mathcal{U}_\delta$, as defined in (9), is a subset of $\mathcal{S}_\delta$,

$$
\mathcal{S}_\delta := \Big\{ y \in S \mid \exists \pi \in \Pi, \, \mathbb{P}\left[\forall k \in \mathbb{T}, \, \xi_y^\pi(k) \in \mathcal{K}\right] > 1-\delta \Big\}, \qquad (13)
$$

where $\mathbb{P}$ is the probability measure for the state trajectory, and $\mathbb{T} = \{0, 1, \dots, T\}$ is the time horizon.

Proof: may want to remove this proof b/c it's not very important? Take $y \in \mathcal{U}_\delta$. Then, there exists $\pi \in \Pi$ such that $\mathrm{CVaR}_\delta(X_y^\pi) < 0$, which implies $\mathbb{P}[X_y^\pi \geq 0] < \delta$ by Lemma 1. After some algrebra using (7) and (8),

$$
\mathbb{P}[X_y^\pi \geq 0] = 1 - \mathbb{P}\left[\forall k \in \mathbb{T}, \, \xi_y^\pi(k) \in \mathcal{K}\right]. \qquad (14)
$$

[11]Ref. [11] indicates that the constraint, $\mathrm{CVaR}_\delta(Z) \leq 0$, gives a conservative approximation of the chance constraint, $\mathbb{P}[Z > 0] \leq \delta$. $\mathrm{CVaR}_\delta(Z) \leq 0$ is written as "$\mathrm{AV@R}_\alpha(Z_x) \leq 0$" in [11], see (6.24). $\mathbb{P}[Z > 0] \leq \delta$ is equivalent to $\mathbb{P}[Z \leq 0] \geq 1-\delta$, which is written as "$\Pr(Z_x \leq 0) \geq 1-\alpha$" in [11], see text below (6.18).

TABLE I

Symbol	Definition	Expression (if ap
g	Surface function that characterizes the constraint set, $\mathcal{K}$	$x \in \mathcal{K} \iff g(x$
C	Set of possible values for the control input	
D_k	Sample space for the random disturbance input at time k	$D_k := \{d_1^k, d_2^k, .$
S	Set of (continuous) states	$S := \mathbb{R}^n$
$\mathcal{K}$	Constraint set	$\mathcal{K} \subset S$
Π	Set of admissible control policies	$\Pi := \{(\mu_0, \mu_1, .$
$\mathbb{P}$	The probability measure with respect to $(w_0, w_1, \dots, w_{T-1})$	
$\mathbb{T}$	Finite discrete time horizon	$\mathbb{T} := \{0, 1, \dots, $
$\xi_x^\pi(k)$	Random state at time k under (fixed) policy π, starting from (fixed) initial condition, $x \in S$, at time 0	

So, $\exists \pi \in \Pi$ such that $\mathbb{P}\left[\forall k \in \mathbb{T}, \, \xi_y^\pi(k) \in \mathcal{K}\right] > 1-\delta$, implying that $y \in \mathcal{S}_\delta$. ■

The second key property of the risk-sensitive safe set, $\mathcal{U}_\delta$, is, as δ decreases, the states of $\mathcal{U}_\delta$ enjoy a stronger probabilistic safety guarantee, and $\mathcal{U}_\delta$ becomes smaller.

Lemma 2: Let $1 > \delta_1 \geq \delta_2 > 0$. Then, $\mathcal{S}_{\delta_2} \supset \mathcal{S}_{\delta_1}$, and $\mathcal{U}_{\delta_2} \supset \mathcal{U}_{\delta_1}$, where $\mathcal{U}_\delta$ is given by (9) and $\mathcal{S}_\delta$ is given by (13).

Remark 1: Remark

Please see Table I for a summary of relevant notation.

Problem 1. An important problem is to compute the set of initial states for which there exists an admissible control policy that keeps the system inside the constraint set over time with sufficiently high probability. The *safe set* with confidence $1-\delta \in (0,1)$ is defined as,

$$
\mathcal{S}(\delta) := \{x \in S \mid \exists \pi \in \Pi \text{ such that } \mathbb{P}[\forall k \in \mathbb{T}, \, \xi_x^\pi(k) \in \mathcal{K}] > 1-\delta\}. \qquad (15)
$$

X. Relation between Probablistic Safety and CVaR safety

XI. Conclusion

-inform the cost-effective design of infrastructure that must withstand rare extreme storms, -possible other applications: to reduce overly conservative error bounds that arise in safe dynamic motion planning (e.g., [8]), and to increase the amount of time that an autonomous vehicle can operate safely while simultaneously optimizing for performance.

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