Chapter 6

Momentum

6.1 Introduction

Momentum

The momentum of a mass m moving with velocity $\mathbf{v}$ is defined to be

$$\mathbf{p} = m\mathbf{v}. \tag{6.1}$$

Because velocity is a vector, momentum is also a vector. The momentum is what appears in Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. In the common case where m is constant, this law reduces to $\mathbf{F} = m\,d\mathbf{v}/dt \implies \mathbf{F} = m\mathbf{a}$.

Impulse

Consider the time integral of the force, which we shall define as the *impulse* $\mathbf{J}$:

$$\mathbf{J} \equiv \int \mathbf{F}\,dt \qquad \text{(impulse).} \tag{6.2}$$

This is just a definition, so there's no content here. But if we invoke Newton's second law, then we can produce some content. If we multiply both sides of $\mathbf{F} = d\mathbf{p}/dt$ by dt and then integrate, we obtain $\int \mathbf{F}\,dt = \Delta\mathbf{p}$. The left-hand side of this relation is just the impulse. We therefore see that the impulse $\mathbf{J}$ associated with a time interval Δt equals the total change in momentum $\Delta\mathbf{p}$ during that time:

$$\mathbf{J} = \Delta\mathbf{p}. \tag{6.3}$$

The impulse $\mathbf{J} \equiv \int \mathbf{F}\,dt$ is the area under the force vs. time curve (technically three different areas for the three different directions in 3-D). So an extended gradual force and a hard quick strike will impart the same momentum to an object if they have the same area under the force vs. time curve.

The above definition of impulse closely parallels the definition of work: $W = \int F\,dx$, or more generally $W = \int \mathbf{F}\cdot d\mathbf{x}$. The time integral of the force equals the change in momentum (by Newton's second law), while the space integral of the force equals the change in kinetic energy (by the work-energy theorem, which can be traced to Newton's second law).

Conservation of momentum

Consider an isolated system of two particles, labeled 1 and 2. If $\mathbf{F}_{ij}$ represents the force on particle i due to particle j, then Newton's third law tells us that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. But $\mathbf{F} = d\mathbf{p}/dt$, so we have

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt} \implies \frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = 0 \implies \mathbf{p}_1 + \mathbf{p}_2 = \text{Constant}. \tag{6.4}$$

In other words, the total momentum is conserved. This derivation wasn't much of a derivation, being only one line. We can therefore view Newton's third law as basically postulating conservation of momentum for two particles. If we have an isolated system of many particles, you can show that the forces cancel in pairs (by Newton's third law), so again the total momentum is conserved; see Problem 6.1.

If the system isn't isolated and there are external forces, then we can break up the force on the ith particle into external and internal forces: $d\mathbf{p}_i/dt = \mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$. Summing over all the particles gives

$$\frac{d(\sum \mathbf{p}_i)}{dt} = \sum \mathbf{F}_i^{\text{ext}} + \sum \mathbf{F}_i^{\text{int}} \implies \frac{d\mathbf{p}_{\text{total}}}{dt} = \mathbf{F}_{\text{total}}^{\text{ext}} + 0, \tag{6.5}$$

where we have used Newton's third law to say that the sum of the *internal* forces is zero since they cancel in pairs. We can therefore omit the "ext" superscript in this result, because the total force on a system of particles is the same as the total external force.

Collisions

Consider a collision between two particles. Assuming that there are no external forces, we know from Eq. (6.4) that the total momentum of the particles is conserved. Since momentum is a vector, this means that each component is conserved separately. So in N dimensions, we can write down N independent statements that must be true. Note that we don't have to worry about the messy specifics of what goes on during the collision. The momentum of an isolated system is conserved, period.

Is the total energy of the particles conserved? The answer to this question is technically yes, because energy is always conserved. However some of the energy may be "lost" to heat, in which case it doesn't show up in the form of $mv^2/2$ terms for the macroscopic particles in the system.[1] As mentioned in the discussion of heat on page 108, it is common in such a situation to say that energy isn't conserved, even though it is of course conserved when heat is taken into account. If no heat is created, then we call the collision *elastic*. If heat is created, so that energy "isn't" conserved, then we call the collision *inelastic*. In the first case, the sum of the $mv^2/2$ terms for the macroscopic particles in the system is the same before and after the collision. In the second case, it isn't. The degree to which it isn't depends on how inelastic the collision is. (The maximally inelastic case occurs when the particles stick together.) So in summary, in an isolated collision,

1. Momentum is always conserved.

2. Mechanical energy (by which we mean the $mv^2/2$ energies of the macroscopic particles involved; that is, excluding heat) is conserved only if the collision is elastic (by definition).

A helpful fact that is valid for 1-D elastic collisions is:

- In a 1-D elastic collision, the final relative velocity between the two particles equals the negative of the initial relative velocity.

That is, if one particle initially sees the other coming toward it with speed v, then after the collision, it sees the other moving away from it with the same speed v; see Problem 6.2 for a proof. This linear relation among the various velocities often simplifies calculations, because it can be used (in tandem with the linear conservation-of-p relation) as a substitute for the more complicated conservation-of-E relation, which is *quadratic* in the velocities.

[1] Imagine a ball of clay that is thrown at a wall and sticks to it. The "lost" energy shows up in $mv^2/2$ terms for the individual random motions of the microscopic molecules in the clay, not in the $mv^2/2$ term for the motion of the ball as a whole, because that v is now zero.

Center of mass

The location of the *center of mass*, or CM, of two objects lying along the x axis is defined to be

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{6.6}$$

This location has the property that the distances to the two masses are inversely proportional to the masses; see Problem 6.4. If one mass is ten times the other, then the CM is ten times closer to the larger mass. The analogous expressions hold for y_{CM} and z_{CM} in the general 3-D case. The definition generalizes to any number of particles; the vector location of the CM of many masses is

$$\mathbf{r}_{\text{CM}} = \frac{\sum m_i \mathbf{r}_i}{M}, \tag{6.7}$$

where $M \equiv \sum m_i$ is the total mass of the system. Basically, $\mathbf{r}_{\text{CM}}$ is the weighted average of the various positions, with each position being weighted by the associated mass. In the case of a continuous distribution of mass, we have

$$\mathbf{r}_{\text{CM}} = \frac{\int \mathbf{r}\, dm}{M}. \tag{6.8}$$

When calculating the location of the CM of an object, a useful fact that simplifies things is that a given subpart of the object can be replaced with a point mass (with the same mass as the subpart) located at the CM of the subpart. See Problem 6.5 for a proof.

Center of mass, v and a

By taking time derivatives of Eq. (6.7), the velocity and acceleration of the CM are

$$\mathbf{v}_{\text{CM}} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad \text{and} \qquad \mathbf{a}_{\text{CM}} = \frac{\sum m_i \mathbf{a}_i}{M}. \tag{6.9}$$

The first of these equations can be rewritten as $\sum m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}$, or equivalently,

$$\mathbf{p}_{\text{total}} = M\mathbf{v}_{\text{CM}}. \tag{6.10}$$

So the total momentum of a system is the same as if all the mass is lumped together and moves along with the velocity of the CM.

The second of the equations in Eq. (6.9) can be rewritten as $\sum m_i \mathbf{a}_i = M\mathbf{a}_{\text{CM}}$. But since $\mathbf{F} = m\mathbf{a}$, the left-hand side is just the sum of the forces on all of the masses. So we have

$$\mathbf{F}_{\text{total}} = M\mathbf{a}_{\text{CM}}. \tag{6.11}$$

In other words, the CM moves just as if all of the force were applied to a mass M located at the CM. As we mentioned above, all of the internal forces cancel in pairs, so we can write $\mathbf{F}_{\text{total}}$ alternatively as $\mathbf{F}_{\text{external}}$. Therefore, a corollary is that if there are no external forces, the CM moves with constant velocity, independent of how the various particles in the system may be moving with respect to each other, and independent of any complicated internal forces.

Collisions in the CM frame

When viewed in the CM frame (the frame that travels along with the CM), collisions are particularly simple. In a 1-D elastic collision, the masses head toward the (stationary) CM, and then after the collision they simply reverse direction and head out with the same speeds they originally had. This scenario satisfies conservation of momentum (the total momentum is zero both before and after the collision),[2] and it also satisfies conservation of energy (the speeds don't

[2]The total momentum is always zero in the CM frame, because we saw in Eq. (6.10) that $\mathbf{p}_{\text{total}} = M\mathbf{v}_{\text{CM}}$. And the velocity $\mathbf{v}_{\text{CM}}$ of the CM is zero in the CM frame, because the CM isn't moving in that frame, by definition.

change, so neither do the $mv^2/2$ kinetic energies). Since everything that needs to be conserved is in fact conserved, this scenario must be what happens.

In the more general case of a 2-D (or 3-D) elastic collision in the CM frame, the particles must still end up moving in opposite directions, with the same speeds they originally had, otherwise we wouldn't have conservation of both p and E; see Problem 6.6. But there is now freedom to choose the orientation of the line of the final velocities; see Fig. 6.1. This line can make any angle with respect to the original line, and both momentum and energy will still be conserved. To determine the direction (which can be specified by one angle in 2-D or two angles in 3-D), we need to be given more information about how exactly the particles collide.

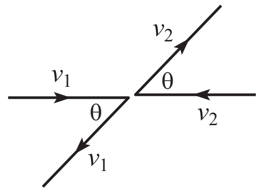

Figure 6.1

In the case of *inelastic* collisions in the CM frame, the only modification to the above results is that both speeds are scaled down by the *same* factor (this will keep the total momentum at zero). The size of this factor depends on how inelastic the collision is. If the collision is completely inelastic (so that the particles stick together), then the factor is zero.

If you want to solve a collision (let's assume it's elastic) by utilizing the CM frame, there are three steps to perform:

1. Assuming that the setup was given in the lab frame, you need to switch to the CM frame. This involves finding the velocity of the CM and then subtracting this velocity from the lab-frame velocities to obtain the CM-frame velocities.

2. Find the final velocities in the CM frame. In a 1-D elastic collision, this step is trivial; the velocities simply reverse. In a 2-D elastic collision, the final line containing the velocities may be different from the initial line. Additional information needs to be given in order to determine the direction of the line.

3. Assuming that the problem asks for the final velocities in the lab frame, you need switch back to the lab frame. This involves adding on the CM velocity to the velocities you found in the CM frame.

Variable mass

Since the momentum p equals mv (we'll work in just one dimension here), Newton's second law can be written as

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = ma + \frac{dm}{dt}v. \tag{6.12}$$

If the mass m of an object is constant, this reduces to $F = ma$. But if the mass changes, then we need to keep the $(dm/dt)v$ term. In the general case, the momentum can change because the velocity changes, or because the mass changes,[3] or both. For setups involving changing mass, it is important to label clearly the system that you are applying $F = dp/dt$ to, because the F and p here must apply to the *same* system.

6.2 Multiple-choice questions

6.1. A ping-pong ball and a bowling ball have the same momentum. Which one has the larger kinetic energy?

(a) the ping-pong ball

(b) the bowling ball

(c) They have the same kinetic energy.

[3]Imagine pushing a bucket at constant speed v, with someone dropping sand into it. You need to apply a force, and this force equals $(dm/dt)v$. This force doesn't speed up the sand that is already in the bucket, but rather it gives momentum to the new sand by suddenly bringing it up to speed v.

6.2. You are stuck in outer space and want to propel yourself as fast as possible in a certain direction by throwing an object in the opposite direction. You note that every object you've ever thrown in your life has had the *same kinetic energy*. Assuming that this trend continues, you should

(a) throw a small object with a large speed
(b) throw a large object with a small speed
(c) It doesn't matter.

6.3. How should you build a car in order to reduce the likelihood of injury in a head-on crash?

(a) Make the front bumper be rigid.
(b) Make the front of the car crumple when large forces are applied.
(c) Make the front of the car crumple easily when small forces are applied.
(d) Make the front of the car be rigid, so that it does not crumple at all.
(e) Install a set of springs in the front of the car, so that it bounces backward after the collision.

6.4. An apple falls from a tree. Which of the following does *not* explain why the apple speeds up as it falls?

(a) The momentum of the earth-apple system is conserved.
(b) There is a downward gravitational force acting on the apple.
(c) Gravity does positive work on the apple as it falls.
(d) The apple loses potential energy as it falls.

6.5. Two people stand on opposite ends of a long sled on frictionless ice. The sled is oriented in the east-west direction, and everything is initially at rest. The western person then throws a ball eastward toward the eastern person, who catches it. The sled

(a) moves eastward, and then ends up at rest
(b) moves eastward, and then ends up moving westward
(c) moves westward, and then ends up at rest
(d) moves westward, and then ends up moving eastward
(e) does not move at all

6.6. On a frictionless table, a mass m moving at speed v collides with another mass m initially at rest. The masses stick together. How much energy is converted to heat?

(a) 0 (b) $\frac{1}{4}mv^2$ (c) $\frac{1}{3}mv^2$ (d) $\frac{1}{2}mv^2$ (e) mv^2

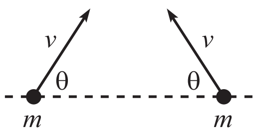

Figure 6.2

6.7. Two masses move toward each other as shown in Fig. 6.2. They collide and stick together. How much energy is converted to heat?

(a) mv^2
(b) $mv^2 \sin\theta$
(c) $mv^2 \sin^2\theta$
(d) $mv^2 \cos\theta$
(e) $mv^2 \cos^2\theta$

6.8. N balls with mass m lie at rest in a line on a frictionless table, with a small separation between adjacent balls. The first ball is given a kick and acquires a speed v. It collides and sticks to the second ball, and the resulting blob collides and sticks to the third ball, and so on. What is the final speed of the resulting blob of mass Nm?

(a) 0 (b) v/N (c) $v/\sqrt{N}$ (d) v (e) Nv