Chapter 6

Momentum

6.1 Introduction

Momentum

The momentum of a mass m moving with velocity $\mathbf{v}$ is defined to be

$$\mathbf{p} = m\mathbf{v}. \tag{6.1}$$

Because velocity is a vector, momentum is also a vector. The momentum is what appears in Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. In the common case where m is constant, this law reduces to $\mathbf{F} = m\,d\mathbf{v}/dt \implies \mathbf{F} = m\mathbf{a}$.

Impulse

Consider the time integral of the force, which we shall define as the *impulse* $\mathbf{J}$:

$$\mathbf{J} \equiv \int \mathbf{F}\,dt \qquad \text{(impulse)}. \tag{6.2}$$

This is just a definition, so there's no content here. But if we invoke Newton's second law, then we can produce some content. If we multiply both sides of $\mathbf{F} = d\mathbf{p}/dt$ by dt and then integrate, we obtain $\int \mathbf{F}\,dt = \Delta\mathbf{p}$. The left-hand side of this relation is just the impulse. We therefore see that the impulse $\mathbf{J}$ associated with a time interval Δt equals the total change in momentum $\Delta\mathbf{p}$ during that time:

$$\mathbf{J} = \Delta\mathbf{p}. \tag{6.3}$$

The impulse $\mathbf{J} \equiv \int \mathbf{F}\,dt$ is the area under the force vs. time curve (technically three different areas for the three different directions in 3-D). So an extended gradual force and a hard quick strike will impart the same momentum to an object if they have the same area under the force vs. time curve.

The above definition of impulse closely parallels the definition of work: $W = \int F\,dx$, or more generally $W = \int \mathbf{F}\cdot d\mathbf{x}$. The time integral of the force equals the change in momentum (by Newton's second law), while the space integral of the force equals the change in kinetic energy (by the work-energy theorem, which can be traced to Newton's second law).

Conservation of momentum

Consider an isolated system of two particles, labeled 1 and 2. If $\mathbf{F}_{ij}$ represents the force on particle i due to particle j, then Newton's third law tells us that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. But $\mathbf{F} = d\mathbf{p}/dt$, so we have

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt} \implies \frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = 0 \implies \mathbf{p}_1 + \mathbf{p}_2 = \text{Constant}. \tag{6.4}$$

In other words, the total momentum is conserved. This derivation wasn't much of a derivation, being only one line. We can therefore view Newton's third law as basically postulating conservation of momentum for two particles. If we have an isolated system of many particles, you can show that the forces cancel in pairs (by Newton's third law), so again the total momentum is conserved; see Problem 6.1.

If the system isn't isolated and there are external forces, then we can break up the force on the ith particle into external and internal forces: $d\mathbf{p}_i/dt = \mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$. Summing over all the particles gives

$$\frac{d(\sum \mathbf{p}_i)}{dt} = \sum \mathbf{F}_i^{\text{ext}} + \sum \mathbf{F}_i^{\text{int}} \quad \Longrightarrow \quad \frac{d\mathbf{p}_{\text{total}}}{dt} = \mathbf{F}_{\text{total}}^{\text{ext}} + 0, \tag{6.5}$$

where we have used Newton's third law to say that the sum of the *internal* forces is zero since they cancel in pairs. We can therefore omit the "ext" superscript in this result, because the total force on a system of particles is the same as the total external force.

Collisions

Consider a collision between two particles. Assuming that there are no external forces, we know from Eq. (6.4) that the total momentum of the particles is conserved. Since momentum is a vector, this means that each component is conserved separately. So in N dimensions, we can write down N independent statements that must be true. Note that we don't have to worry about the messy specifics of what goes on during the collision. The momentum of an isolated system is conserved, period.

Is the total energy of the particles conserved? The answer to this question is technically yes, because energy is always conserved. However some of the energy may be "lost" to heat, in which case it doesn't show up in the form of $mv^2/2$ terms for the macroscopic particles in the system.[1] As mentioned in the discussion of heat on page 108, it is common in such a situation to say that energy isn't conserved, even though it is of course conserved when heat is taken into account. If no heat is created, then we call the collision *elastic*. If heat is created, so that energy "isn't" conserved, then we call the collision *inelastic*. In the first case, the sum of the $mv^2/2$ terms for the macroscopic particles in the system is the same before and after the collision. In the second case, it isn't. The degree to which it isn't depends on how inelastic the collision is. (The maximally inelastic case occurs when the particles stick together.) So in summary, in an isolated collision,

1. Momentum is always conserved.

2. Mechanical energy (by which we mean the $mv^2/2$ energies of the macroscopic particles involved; that is, excluding heat) is conserved only if the collision is elastic (by definition).

A helpful fact that is valid for 1-D elastic collisions is:

- In a 1-D elastic collision, the final relative velocity between the two particles equals the negative of the initial relative velocity.

That is, if one particle initially sees the other coming toward it with speed v, then after the collision, it sees the other moving away from it with the same speed v; see Problem 6.2 for a proof. This linear relation among the various velocities often simplifies calculations, because it can be used (in tandem with the linear conservation-of-p relation) as a substitute for the more complicated conservation-of-E relation, which is *quadratic* in the velocities.

[1] Imagine a ball of clay that is thrown at a wall and sticks to it. The "lost" energy shows up in $mv^2/2$ terms for the individual random motions of the microscopic molecules in the clay, not in the $mv^2/2$ term for the motion of the ball as a whole, because that v is now zero.

Center of mass

The location of the *center of mass*, or CM, of two objects lying along the x axis is defined to be

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{6.6}$$

This location has the property that the distances to the two masses are inversely proportional to the masses; see Problem 6.4. If one mass is ten times the other, then the CM is ten times closer to the larger mass. The analogous expressions hold for y_{CM} and z_{CM} in the general 3-D case. The definition generalizes to any number of particles; the vector location of the CM of many masses is

$$\mathbf{r}_{\text{CM}} = \frac{\sum m_i \mathbf{r}_i}{M}, \tag{6.7}$$

where $M \equiv \sum m_i$ is the total mass of the system. Basically, $\mathbf{r}_{\text{CM}}$ is the weighted average of the various positions, with each position being weighted by the associated mass. In the case of a continuous distribution of mass, we have

$$\mathbf{r}_{\text{CM}} = \frac{\int \mathbf{r}\, dm}{M}. \tag{6.8}$$

When calculating the location of the CM of an object, a useful fact that simplifies things is that a given subpart of the object can be replaced with a point mass (with the same mass as the subpart) located at the CM of the subpart. See Problem 6.5 for a proof.

Center of mass, v and a

By taking time derivatives of Eq. (6.7), the velocity and acceleration of the CM are

$$\mathbf{v}_{\text{CM}} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad \text{and} \qquad \mathbf{a}_{\text{CM}} = \frac{\sum m_i \mathbf{a}_i}{M}. \tag{6.9}$$

The first of these equations can be rewritten as $\sum m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}$, or equivalently,

$$\mathbf{p}_{\text{total}} = M\mathbf{v}_{\text{CM}}. \tag{6.10}$$

So the total momentum of a system is the same as if all the mass is lumped together and moves along with the velocity of the CM.

The second of the equations in Eq. (6.9) can be rewritten as $\sum m_i \mathbf{a}_i = M\mathbf{a}_{\text{CM}}$. But since $\mathbf{F} = m\mathbf{a}$, the left-hand side is just the sum of the forces on all of the masses. So we have

$$\mathbf{F}_{\text{total}} = M\mathbf{a}_{\text{CM}}. \tag{6.11}$$

In other words, the CM moves just as if all of the force were applied to a mass M located at the CM. As we mentioned above, all of the internal forces cancel in pairs, so we can write $\mathbf{F}_{\text{total}}$ alternatively as $\mathbf{F}_{\text{external}}$. Therefore, a corollary is that if there are no external forces, the CM moves with constant velocity, independent of how the various particles in the system may be moving with respect to each other, and independent of any complicated internal forces.

Collisions in the CM frame

When viewed in the CM frame (the frame that travels along with the CM), collisions are particularly simple. In a 1-D elastic collision, the masses head toward the (stationary) CM, and then after the collision they simply reverse direction and head out with the same speeds they originally had. This scenario satisfies conservation of momentum (the total momentum is zero both before and after the collision),[2] and it also satisfies conservation of energy (the speeds don't

[2]The total momentum is always zero in the CM frame, because we saw in Eq. (6.10) that $\mathbf{p}_{\text{total}} = M\mathbf{v}_{\text{CM}}$. And the velocity $\mathbf{v}_{\text{CM}}$ of the CM is zero in the CM frame, because the CM isn't moving in that frame, by definition.

change, so neither do the $mv^2/2$ kinetic energies). Since everything that needs to be conserved is in fact conserved, this scenario must be what happens.

In the more general case of a 2-D (or 3-D) elastic collision in the CM frame, the particles must still end up moving in opposite directions, with the same speeds they originally had, otherwise we wouldn't have conservation of both p and E; see Problem 6.6. But there is now freedom to choose the orientation of the line of the final velocities; see Fig. 6.1. This line can make any angle with respect to the original line, and both momentum and energy will still be conserved. To determine the direction (which can be specified by one angle in 2-D or two angles in 3-D), we need to be given more information about how exactly the particles collide.

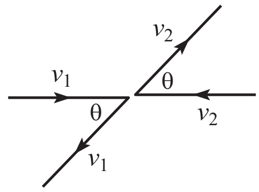


Figure 6.1

In the case of *inelastic* collisions in the CM frame, the only modification to the above results is that both speeds are scaled down by the *same* factor (this will keep the total momentum at zero). The size of this factor depends on how inelastic the collision is. If the collision is completely inelastic (so that the particles stick together), then the factor is zero.

If you want to solve a collision (let's assume it's elastic) by utilizing the CM frame, there are three steps to perform:

1. Assuming that the setup was given in the lab frame, you need to switch to the CM frame. This involves finding the velocity of the CM and then subtracting this velocity from the lab-frame velocities to obtain the CM-frame velocities.

2. Find the final velocities in the CM frame. In a 1-D elastic collision, this step is trivial; the velocities simply reverse. In a 2-D elastic collision, the final line containing the velocities may be different from the initial line. Additional information needs to be given in order to determine the direction of the line.

3. Assuming that the problem asks for the final velocities in the lab frame, you need switch back to the lab frame. This involves adding on the CM velocity to the velocities you found in the CM frame.

Variable mass

Since the momentum p equals mv (we'll work in just one dimension here), Newton's second law can be written as

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = ma + \frac{dm}{dt}v. \tag{6.12}$$

If the mass m of an object is constant, this reduces to $F = ma$. But if the mass changes, then we need to keep the $(dm/dt)v$ term. In the general case, the momentum can change because the velocity changes, or because the mass changes,[3] or both. For setups involving changing mass, it is important to label clearly the system that you are applying $F = dp/dt$ to, because the F and p here must apply to the *same* system.

6.2 Multiple-choice questions

6.1. A ping-pong ball and a bowling ball have the same momentum. Which one has the larger kinetic energy?

(a) the ping-pong ball

(b) the bowling ball

(c) They have the same kinetic energy.

[3]Imagine pushing a bucket at constant speed v, with someone dropping sand into it. You need to apply a force, and this force equals $(dm/dt)v$. This force doesn't speed up the sand that is already in the bucket, but rather it gives momentum to the new sand by suddenly bringing it up to speed v.

6.2. You are stuck in outer space and want to propel yourself as fast as possible in a certain direction by throwing an object in the opposite direction. You note that every object you've ever thrown in your life has had the *same kinetic energy*. Assuming that this trend continues, you should

(a) throw a small object with a large speed
(b) throw a large object with a small speed
(c) It doesn't matter.

6.3. How should you build a car in order to reduce the likelihood of injury in a head-on crash?

(a) Make the front bumper be rigid.
(b) Make the front of the car crumple when large forces are applied.
(c) Make the front of the car crumple easily when small forces are applied.
(d) Make the front of the car be rigid, so that it does not crumple at all.
(e) Install a set of springs in the front of the car, so that it bounces backward after the collision.

6.4. An apple falls from a tree. Which of the following does *not* explain why the apple speeds up as it falls?

(a) The momentum of the earth-apple system is conserved.
(b) There is a downward gravitational force acting on the apple.
(c) Gravity does positive work on the apple as it falls.
(d) The apple loses potential energy as it falls.

6.5. Two people stand on opposite ends of a long sled on frictionless ice. The sled is oriented in the east-west direction, and everything is initially at rest. The western person then throws a ball eastward toward the eastern person, who catches it. The sled

(a) moves eastward, and then ends up at rest
(b) moves eastward, and then ends up moving westward
(c) moves westward, and then ends up at rest
(d) moves westward, and then ends up moving eastward
(e) does not move at all

6.6. On a frictionless table, a mass m moving at speed v collides with another mass m initially at rest. The masses stick together. How much energy is converted to heat?

(a) 0 (b) $\frac{1}{4}mv^2$ (c) $\frac{1}{3}mv^2$ (d) $\frac{1}{2}mv^2$ (e) mv^2

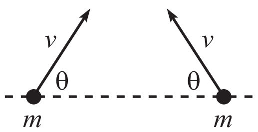


Figure 6.2

6.7. Two masses move toward each other as shown in Fig. 6.2. They collide and stick together. How much energy is converted to heat?

(a) mv^2
(b) $mv^2 \sin\theta$
(c) $mv^2 \sin^2\theta$
(d) $mv^2 \cos\theta$
(e) $mv^2 \cos^2\theta$

6.8. N balls with mass m lie at rest in a line on a frictionless table, with a small separation between adjacent balls. The first ball is given a kick and acquires a speed v. It collides and sticks to the second ball, and the resulting blob collides and sticks to the third ball, and so on. What is the final speed of the resulting blob of mass Nm?

(a) 0 (b) v/N (c) $v/\sqrt{N}$ (d) v (e) Nv

6.9. A very light ping-pong ball bounces elastically head-on off a very heavy bowling ball that is initially at rest. The fraction of the ping-pong ball's initial kinetic energy that is transferred to the bowling ball is approximately

(a) 0 (b) 1/4 (c) 1/2 (d) 3/4 (e) 1

6.10. A small marble with mass m moves with speed v toward a large block with mass M (assume $M \gg m$), which sits at rest on a frictionless table. In Scenario A the marble sticks to the block. In Scenario B the marble bounces elastically off the block and heads backward. The ratio of the block's resulting kinetic energy in Scenario B to that in Scenario A is approximately

(a) 1/4 (b) 1/2 (c) 1 (d) 2 (e) 4

6.11. A ball of clay is thrown at a wall and sticks to it. Virtually all of its momentum ends up in

(a) the ball
(b) the earth
(c) heat
(d) sound
(e) It doesn't end up anywhere, because momentum isn't conserved.

6.12. A ball of clay is thrown at a wall and sticks to it. Virtually all of its kinetic energy ends up in

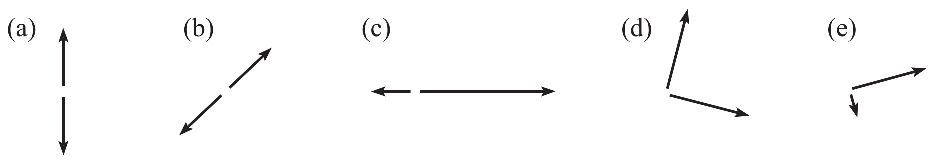
(a) kinetic energy of the ball
(b) kinetic energy of the earth
(c) heat
(d) sound
(e) It doesn't end up anywhere, because energy isn't conserved.

6.13. In a one-dimensional elastic collision, Ball 1 collides with Ball 2, which is initially at rest. Can the masses be chosen so that the final speed of Ball 2 is larger than the initial speed of Ball 1?

Yes No

6.14. A mass m moves with a given speed and collides (not necessarily head-on) elastically with another mass m that is initially at rest, as shown in Fig. 6.3. Which of the figures shows a possible outcome for the two velocities? (The velocities are drawn to scale.)

Figure 6.3



6.15. A paddle hits a ping-pong ball. The paddle moves with speed v to the right, and the ping-pong ball moves with speed $3v$ to the left. The collision is elastic. What is the resulting speed of the ping-pong ball? (Assume that the paddle is much more massive than the ping-pong ball.)

(a) $2v$ (b) $3v$ (c) $4v$ (d) $5v$ (e) $6v$

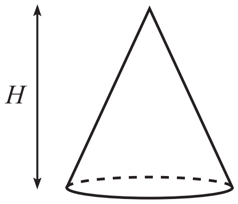


Figure 6.4

6.16. Let y be the height (above the base) of the center of mass of a hollow cone (like an ice cream cone) with total height H, as shown in Fig. 6.4. Then

(a) $y = H$
(b) $H/2 < y < H$
(c) $y = H/2$
(d) $0 < y < H/2$
(e) $y = 0$

6.17. A mass $2m$ moves rightward with speed v toward a mass m that is at rest. What is the speed of the mass $2m$ in the CM frame?

(a) 0 (b) $v/3$ (c) $v/2$ (d) $2v/3$ (e) v

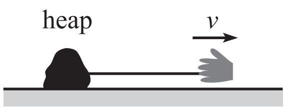


Figure 6.5

6.18. A heap of rope with mass density λ (per unit length) lies on a table. You grab one end and pull horizontally with constant speed v, as shown in Fig. 6.5. (Assume that the rope has no friction with itself in the heap.) The force that you must apply to maintain the constant speed v is

(a) 0
(b) λv
(c) λv^2
(d) $\lambda \ell g$, where ℓ is the length that you have pulled straight
(e) $\lambda v^4/g\ell$

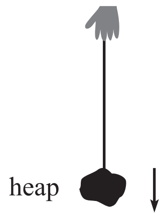


Figure 6.6

6.19. With your hand at a fixed position, you hold onto one end of a heap of rope with mass m and then let the heap fall, as shown in Fig. 6.6. The rope has no friction with itself in the heap. Eventually the heap runs out and you are left holding a vertical piece of rope with mass m. Is there any time during this process when the upward force you exert with your hand exceeds mg?

Yes No

6.20. A dustpan is accelerated with acceleration a across a frictionless floor, and it gathers up dust as it moves, as shown in Fig. 6.7. The mass of the dustpan itself is M, and the linear mass density of the dust on the floor is λ. If the dustpan starts empty at $x = 0$, then the force that must be applied to it later on when it is moving with speed v at position x is

(a) Ma
(b) $Ma + \lambda v^2$
(c) $(M + \lambda x)a$
(d) λv^2
(e) $(M + \lambda x)a + \lambda v^2$

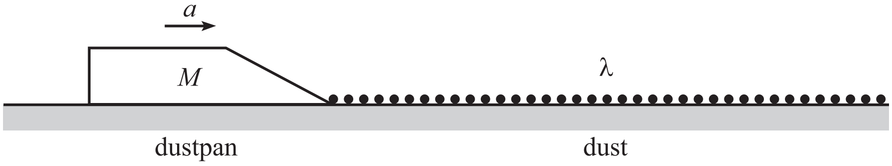


Figure 6.7

6.3 Problems

The first six problems are foundational problems.

6.1. **Conservation of momentum**

In an isolated system of many particles, show that the total momentum is conserved, by showing that the forces cancel in pairs (by Newton's third law).

6.2. **Same relative speed**

Show that in a 1-D elastic collision, the final relative velocity between the two masses equals the negative of the initial relative velocity.

Note: Although this can be shown very easily by working in the CM frame, demonstrate it here by working in the lab frame and using conservation of energy and momentum. One way to do this is to solve for the final two velocities in terms of the initial ones. But that gets very messy. A sneakier way is to put the terms associated with each particle on different sides of the conservation of E and p equations, and then divide the E equation by the p equation.

6.3. **1-D collision**

In a 1-D elastic collision, a mass M moving with velocity V collides with a mass m that is initially at rest. Show that the resulting velocities are

$$V_M = \frac{(M-m)V}{M+m} \qquad \text{and} \qquad v_m = \frac{2MV}{M+m}\,. \tag{6.13}$$

6.4. **Distances to the CM**

Show that the location of the center of mass, given in Eq. (6.6), has the property that the distances to the two masses are inversely proportional to the masses.

6.5. **Equivalent subparts**

Show that the location of the CM of the object in Fig. 6.8 can be found by replacing the two subparts S_1 and S_2 with point masses (with the same masses as the subparts) located at the CM's of the subparts.

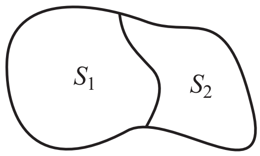


Figure 6.8

6.6. **Collision in the CM frame**

Consider an elastic collision (in any dimension), as viewed in the CM frame. Show that after the collision, the particles must move in opposite directions, with the same speeds they originally had.

6.7. **Hemispherical-shell CM**

Find the location of the CM of a hollow hemispherical shell with radius R and uniform surface mass distribution.

6.8. **Atwood's machine**

In Problem 4.5 we considered the Atwood's machine shown in Fig. 6.9. In the solution to that problem, we found that the accelerations of the masses and the tension in the upper rope are given by

$$a = g\frac{m_2 - m_1}{m_2 + m_1} \qquad \text{and} \qquad T = g\frac{4m_1m_2}{m_2+m_1}\,.$$

Let's assume $m_2 > m_1$ so that a is positive.

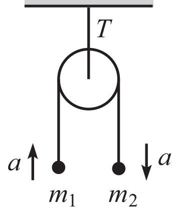


Figure 6.9

(a) After each mass has moved a distance d, find the potential and kinetic energies, and verify that energy is conserved. Assume that the masses start at rest.

(b) After a time t, verify that $P_{\text{total}} = F_{\text{total}}t$ (which is a consequence of integrating $F = dP/dt$ when F is constant). Be careful to include *all* the external forces acting on the system.

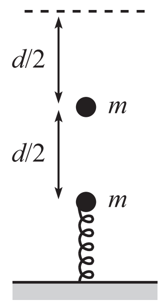


Figure 6.10

6.9. **Rising and colliding**

A vertical spring is compressed a distance d relative to its relaxed length. A mass m is attached to the top end and held at rest. Another mass m is suspended a distance $d/2$ above it, as shown in Fig. 6.10. The spring is released. The bottom mass rises up, smashes and sticks to the upper mass, and the resulting blob continues to rise up. What must the spring constant k be, so that the blob reaches its maximum height at the dotted line shown (where the spring is at its relaxed length)?

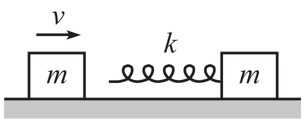


Figure 6.11

6.10. **Maximum compression**

A block with mass m slides with speed v along a frictionless table toward a stationary block that also has mass m. A massless spring with spring constant k is attached to the second block, as shown in Fig. 6.11. What is the maximum distance the spring gets compressed? *Hint*: How do the blocks' speeds compare at maximal compression?

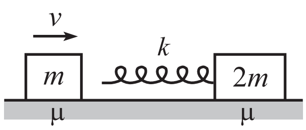


Figure 6.12

6.11. **Collision and a spring**

A spring with spring constant k is attached to a stationary mass $2m$, as shown in Fig. 6.12. A mass m travels to the right and sticks to the left end of the spring and compresses the spring. The coefficient of friction (both static and kinetic) between the masses and the ground is μ. Let v be the speed of m right before it hits the spring.

(a) What is the largest value of v for which the mass $2m$ never moves? (Don't forget that m feels the friction too.) Assume that the spring is sufficiently long so that m doesn't collide with $2m$.

(b) Assume that v takes on this largest value. The spring will compress and then expand back out and push m to the left. What will m's speed be by the time the spring expands back to its relaxed length? (Again, don't forget that m feels the friction.)

6.12. **Colliding balls**

A stream of N clay balls with mass m move with speed v in a line across a frictionless table. The spacing between them is ℓ. An additional ball with mass m sits at rest in front of them, as shown in Fig. 6.13. The front moving ball collides with the stationary ball and sticks to it and forms a blob of mass $2m$. Then the second ball collides with the blob and forms a blob of mass $3m$. And so on. How much time elapses between the instant shown below (when all the balls are separated by ℓ) and the last collision? Solve this by working in (a) the lab frame, and then (b) the frame in which the N balls are initially at rest (this is the quicker way). *Note*: As with any problem involving a general number N, it is often helpful to first work things out for small values of N.

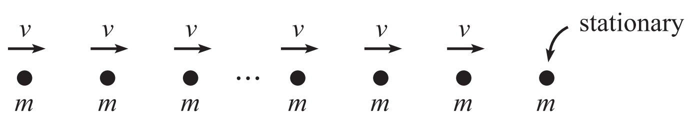


Figure 6.13

6.13. **Block and balls**

A block with large mass M moves with initial speed V_0 across frictionless ice. It encounters a long row of stationary balls, each with small mass m, and it collides elastically with each ball, one after the other; see Fig. 6.14.[4]

(a) If at a later time the block is moving with speed v right before it collides with a given ball, what is the block's speed right after the collision? Assume $m \ll M$, and give your answer to leading order in m. *Hint*: To save yourself some algebra, find the speed of the ball after the collision by working in the frame of the block.

(b) If the balls are essentially continuously distributed, with the row having mass density λ (kg/m), what is the decrease in the block's speed (in terms of v) during a small interval of time dt? Give your answer to leading order in dt.

(c) Find $v(t)$, assuming that $t = 0$ corresponds to the time right before the first collision, when the block has speed V_0.

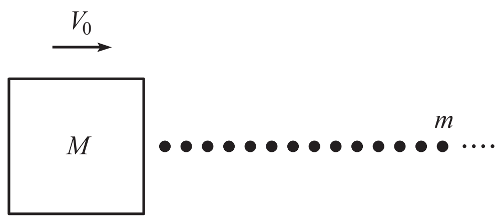


Figure 6.14

6.14. **Maximum final speed**

The result of Problem 6.3 was that in a 1-D elastic collision, if a mass M moving with velocity V collides with a stationary mass m, the resulting velocities are

$$V_M = \frac{(M-m)V}{M+m} \qquad \text{and} \qquad v_m = \frac{2MV}{M+m}. \tag{6.14}$$

(a) Given M and m, imagine putting a stationary mass (call it x) between M and m. If your goal is to have m pick up as large a speed as possible, what should x be in terms of M and m? Assume that both collisions are elastic.

(b) If you instead put two stationary masses, x and y, between M and m, what should they be so that m picks up as large a speed as possible? What about a general number of masses, say 10? In answering these questions, there is no need to do any calculations; your result from part (a) is all you need. A proof by contradiction is your best bet. (You are also encouraged to think about where the initial energy and momentum end up, in the $M \gg m$ limit with a very large number of optimally-chosen masses in between.)

6.15. **Throwing a block in pieces**

(a) You sit on a crate at rest on frictionless ice. The combined mass of you and the crate is m. You hold a block also of mass m and then throw it horizontally so that the relative speed between you and it is v_0. What is your resulting speed?

(b) Now assume that you cut the block in half and throw one half, and then wait a moment and throw the other half. What is your resulting speed? Assume that whenever you throw something, the relative speed between you and it is v_0.

(c) What is your resulting speed if you divide the block in thirds and throw the pieces successively? Or fourths? A pattern should emerge. In general, if you divide the block into n pieces, your resulting speed will take the form of $f(n)v_0$. What is the function $f(n)$?

[4]Assume that when each ball bounces off the block, it (the ball) magically passes through the other balls as it moves forward; so you don't have to worry about it bouncing back off the next ball and hitting the block. Equivalently, assume that the balls are displaced slightly from one another in the transverse direction.

(d) What is $f(100)$? If you have *Mathematica* or something similar handy, make a plot of $f(n)$ vs. n, for n from 1 to 100. (Take v_0 to be 1.) Make a guess for the limit of $f(n)$ as $n \to \infty$.

(e) The $n \to \infty$ limit is identical to rocket motion, because it involves a continuous ejection of mass. So we actually already know the final speed in this case, from Problem 6.22 below. What is the speed? How much does it differ from the speed in the $n = 100$ case?

6.16. **A collision in two frames**

Consider the following one-dimensional collision. A mass $4m$ moves to the right, and a mass m moves to the left, both with speed v. They collide elastically. Find their final lab-frame velocities. Solve this by working in (a) the lab frame, and (b) the CM frame.

6.17. **45-degree deflections**

A ball with mass m moves in the positive x direction with speed v, as shown in Fig. 6.15. It collides elastically with two other balls (also with mass m) which are situated so that they both move at equal speeds at equal angles of 45° with respect to the x axis after the collision. What are the final speeds of all three balls?

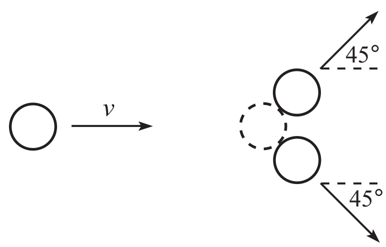


Figure 6.15

6.18. **Northward deflection 1**

A mass $2m$ moves to the east, and a mass m moves to the west, both with speed v_0. If they collide elastically, but not head-on, and if it is observed that the mass $2m$ ends up moving northward (that is, perpendicular to the original direction of motion), what is its speed?

6.19. **Northward deflection 2**

A mass m moving to the east with speed v_0 collides elastically, but not head-on, with a mass $2m$ at rest. If it is observed that the mass m ends up moving northward (that is, perpendicular to the original direction of motion), what angle does the resulting velocity of the mass $2m$ make with the east-west direction? Solve this by working in (a) the lab frame, and (b) the CM frame.

6.20. **Equal energies**

A mass m moving with speed v_0 collides elastically with a stationary mass $2m$, as shown in Fig. 6.16. Assuming that the final energies of the masses turn out to be equal, find the two final speeds, v_1 and v_2. Also, find the two angles of deflection, θ_1 and θ_2. *Hint*: The best way to solve for angles is usually to square equations (in appropriate form) and use the fact that $\sin^2\theta + \cos^2\theta = 1$.

6.21. **Equal speeds**

A mass m moving with speed v_0 collides elastically with a stationary mass $2m$, as shown in Fig. 6.17. Assuming that the final speeds of the masses turn out to be equal, find this speed, and also find the two angles of deflection, θ_1 and θ_2. (Same hint as in the preceding problem.)

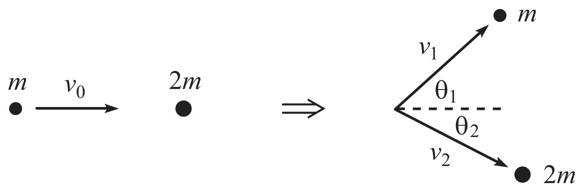


Figure 6.16

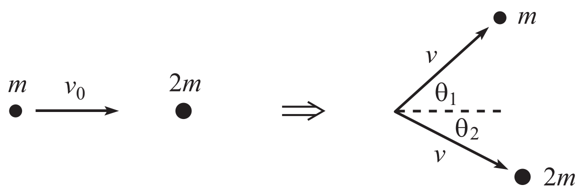


Figure 6.17

6.22. **Rocket motion**

A rocket ejects mass backward with a constant speed u relative to the rocket.[5] If the initial mass of the rocket is M, what is the rocket's speed at a later time when the mass is m? *Hint*: Consider a short interval of time, and determine the increase in speed after a small amount of mass has been ejected. This will yield a differential equation relating dv and dm, which you can then integrate.

6.23. **Hovering board**

(a) A hose shoots a stream of water vertically upward. The water leaves the hose at speed v_0 and at a mass rate R (kg/s). A horizontal board with mass m is placed a very small distance above the hose and then released. What should m be so that the board hovers at this height? Assume that when the water crashes into the board, it bounces off essentially sideways.

(b) If you break the board in half, so that its mass is now $m/2$, how high above the hose should it be located if you want it to hover in place?

(c) In part (a), what should m be if the stream of water is replaced by a stream of marbles that bounce off the board elastically (that is, they bounce off downward with the same speed v_0)?[6]

6.24. **Falling heap**

A rope with total length L and mass density λ (kg/m) is held in a heap, and you grab an end that protrudes a tiny bit out of the top. The heap is then released, and it falls downward. As a function of time, what is the force that your hand must apply to the top end of the rope, to keep it motionless? Assume that the rope has no friction with itself, so that the remaining part of the heap is always in freefall. The setup at a general later time is shown in Fig. 6.18.

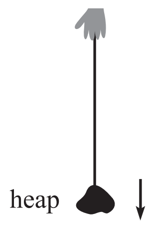


Figure 6.18

[5]To emphasize, u is the speed relative to the rocket. It wouldn't make sense to say "relative to the ground," because the rocket's engine shoots out the matter relative to itself, and the engine has no way of knowing how fast the rocket is moving with respect to the ground.

[6]Even though the marbles are discrete objects, assume that they have an essentially continuous mass rate equal to R. Also, assume that the downward-moving marbles that have bounced off the board somehow magically pass through the upward-moving ones without colliding.

6.25. **Bucket and chain**

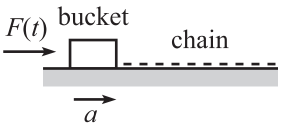


Figure 6.19

A massless bucket is initially at rest next to one end of a chain that lies in a straight line on the floor, as shown in Fig. 6.19. The chain has uniform mass density λ (kg/m). You push on the bucket (so that it gathers up the chain) with the force $F(t)$ that gives the bucket (plus whatever chain is inside) a constant acceleration a at all times. There is no friction between the bucket and the floor.

(a) What is $F(t)$?

(b) How much work do you do up to time t?

(c) How much energy is lost to heat up to time t? That is, by how much does the work you do exceed the kinetic energy of the system?

6.4 Multiple-choice answers

6.1. $\boxed{\text{a}}$ The kinetic energy can be written as $mv^2/2 = (mv)v/2 = pv/2$. Since the p's are equal, the object with the larger v has the larger energy. And equal p's imply that the ping-pong ball has the larger v because its mass is smaller. So in general we want to throw a small object with a large speed.

Alternatively, the kinetic energy can be written as $mv^2/2 = (mv)^2/2m = p^2/2m$. So if the p's are equal, the object with the smaller m (the ping-pong ball) has the larger energy.

Alternatively again, just pick some numbers: the combination of $m = 1$ and $v = 10$ yields the same momentum as the combination of $m = 10$ and $v = 1$, and the former has a larger energy $mv^2/2$.

Remark: Another way of seeing why the ping-pong ball has the larger energy is the following. Imagine both objects being decelerated with the same constant force F. Since $|\Delta p| = Ft$, the objects will take the same *time* to stop if they start with the same momentum. However, since the smaller object is moving faster at any given time, it travels for a larger *distance* x by the time it stops. But the work-energy theorem tells us that $|\Delta K| = Fx$. So the object with the larger x (which is the smaller object) must have started off with a larger kinetic energy. In short, the same impulse $\int F\,dt$ is imparted on both objects, but more work $\int F\,dx$ is done on the smaller object.

6.2. $\boxed{\text{b}}$ This question is basically the opposite of the previous one, where the momentum was assumed to be the same. In the present setup, conservation of momentum tells us that we want to give the object as much momentum as possible, so that you will have as much momentum as possible in the opposite direction. The kinetic energy $mv^2/2$ is assumed to be given. Since this energy can be written as $(mv)v/2 = pv/2$, a small v means a large p. And a small v in turn means a large m (since $mv^2/2$ is given). So we want to throw a large object with a small speed.

Alternatively, the energy can be written as $p^2/2m$. So if the energies are the same, the object with the larger m has the larger p.

Alternatively again, just pick some numbers: the combination of $m = 1$ and $v = 10$ yields the same energy as the combination of $m = 100$ and $v = 1$, and the latter has a larger momentum $p = mv$.

Remark: Is the assumption of every thrown object having the same kinetic energy a reasonable one? Certainly not in all cases. If a baseball pitcher throws a ball with 1/4 the mass of a baseball, he definitely isn't going to be able to throw it twice as fast. The limiting factor in this case is arm speed.

But what about large objects, for which a limiting arm speed isn't an issue? Ignoring myriad real-life complications, it's reasonable to make the approximation that you apply the same force on any large object, assuming that your hand is moving relatively slowly and that you are pushing as hard as you can. You apply this same force for the same distance, assuming that your throwing motion looks roughly the same for any large object. The work (force times distance) that you do is therefore the same for all objects, which means that the resulting kinetic energy is the same. So the assumption of equal kinetic energies stated in the question is reasonable in this case.

Note that this reasoning doesn't work with *momentum*, because you will accelerate a smaller (but still large) object more quickly, which means that you will spend less *time* applying a force to it, which in turn means that the *impulse* you apply will be smaller. Hence a smaller object will have less momentum than a larger object, even though their energies are the same.

6.3. $\boxed{\text{b}}$ The work done on you, which is $\int F\,dx$, equals the change in your kinetic energy. Since you end up at rest, this integral (which is the area under the F vs. x curve) takes on the given value $mv_0^2/2$ (in magnitude), independent of the specifics of the collision. To reduce injury, the goal is to have the force F be spread out evenly over the largest possible range of x, instead of being spiked at a small interval of x, because a large force will cause injury. Choice (b) correctly spreads out the force over the largest distance.

The other options are inferior because: Choice (a) involves only the front bumper (which is a small fraction of the total possible crumpling distance), so it is hardly different from a completely rigid car, (c) leads to a spike at the end of the collision, (d) leads to a spike at the beginning, and (e) actually causes the whole process to be repeated, because after you stop, the springs throw you backward. So you're even more likely to get injured. (If the spread-out force was on the threshold of seriously injuring you, making it last twice as long can't be good.) Additionally, the spring force isn't uniform, so it somewhat spikes halfway through the process.

Remark: You can also answer this question by working with momentum instead of energy. The impulse imparted on you, which is $\int F\,dt$, equals the change in your momentum, which is mv_0 (in magnitude). However, things are a little trickier now because while the collision has a specific maximum possible *distance*, it doesn't have a maximum possible *time*. The time could be long if you suddenly decelerate to a small speed and then continue to move forward for a while with this small speed. But this sudden deceleration will injure you. So the question reduces to: If you want the maximum value of $F(t)$ to be as small as possible, what should the function $F(t)$ look like if an object takes a given distance to go from a given speed v_0 to speed zero? The answer is that you want $F(t)$ to be a constant function. As an exercise, you can think about why.

6.4. $\boxed{\text{a}}$ Although momentum is indeed conserved, it doesn't explain why the apple speeds up. There are many different possible motions consistent with momentum conservation. The other three choices do in fact explain the acceleration. In choice (b), the force directly causes the acceleration. And in choices (c) and (d), the work and the loss in potential energy both imply an increase in kinetic energy, which means an increase in speed.

6.5. $\boxed{\text{c}}$ By conservation of momentum, the sled moves westward while the ball moves eastward in the air. Also by conservation of momentum, the whole system must be at rest at the end of the process, because nothing is moving with respect to anything else, and the total momentum must be zero, as it was at the start.

6.6. $\boxed{\text{b}}$ The initial kinetic energy is $mv^2/2$. Since the masses stick together to form a blob of mass $2m$, conservation of momentum gives the final speed as $mv = (2m)v_\text{f} \Longrightarrow v_\text{f} = v/2$. So the final kinetic energy is $(2m)(v/2)^2/2 = mv^2/4$. Therefore, $mv^2/2 - mv^2/4 = mv^2/4$ energy is converted to heat.

Alternatively, in the CM frame the masses head toward each other, each with speed $v/2$. The total kinetic energy of both masses in the CM frame is therefore $2 \cdot m(v/2)^2/2 = mv^2/4$. But all of this energy must end up as heat, because the resulting blob is at rest in the CM frame. And the heat is independent of the frame.

6.7. $\boxed{\text{e}}$ The kinetic energy can be written as $mv^2/2 = m(v_x^2 + v_y^2)/2$. In the given setup, the v_x components of the two velocities are brought to zero by the collision, while the v_y components are unchanged (by conservation of momentum). So the loss in kinetic energy is $2 \cdot mv_x^2/2 = 2m(v\cos\theta)^2/2 = mv^2\cos^2\theta$.

LIMITS: The $\theta \to 90°$ limit quickly eliminates choices (a), (b), and (c), because the energy loss is zero in this case, since the masses collide (or "touch" would be a better word) with zero relative velocity.

6.8. $\boxed{\text{b}}$ Momentum is conserved throughout the entire process. The initial momentum is mv, and the final momentum is $(Nm)v_\text{f}$. Setting these equal gives $v_\text{f} = v/N$.

REMARK: It is incorrect to use conservation of energy in this setup. That would give $mv^2/2 = (Nm)v_\text{f}^2/2 \Longrightarrow v_\text{f} = v/\sqrt{N}$, which is incorrect (it is too large) because energy is lost to heat in each of the (completely inelastic) collisions.

6.9. $\boxed{\text{a}}$ The ping-pong ball has nearly the same speed afterward as before, because the bowling ball is much more massive and essentially acts like a brick wall. So the ping-pong ball's kinetic energy remains essentially the same. Therefore, approximately none of the energy goes into the bowling ball.

REMARKS: If you want to solve this problem quantitatively, you can use the results from Problem 6.3. In the notation of that problem, the small ping-pong ball has mass M, and the large bowling ball has mass m. Using the final speed of m given in Eq. (6.13), the final kinetic energy of the bowling ball is $(1/2)m\big[2MV/(M+m)\big]^2$. This goes to zero in the $m \gg M$ limit, because there is an extra power of m in the denominator.

This zero-energy result is a general result that holds whenever a small object collides (elastically or inelastically) with a stationary large object. The large object picks up essentially zero energy, even though it picks up nonzero momentum. In the present case of an elastic collision, if the ping-pong ball's initial momentum is p, then its final momentum is essentially $-p$. The change is $-2p$, so the bowling ball's final momentum is $2p$. Consistent with this, its energy, which can be written as $(2p)^2/2m$, is essentially zero since m is very large. If the collision is instead completely inelastic and the ping-pong ball sticks to the bowling ball, then the bowling ball's final momentum is just p, and the same result of zero energy still holds.

The word "stationary" in the first sentence of the preceding paragraph is important. If a small object collides with a *moving* large object, some energy will be transferred between the two objects. This is true because (among other reasons) if the large object is moving, the speed of the small object will be different after the collision. This is most easily seen by utilizing the CM frame.

6.10. $\boxed{\text{e}}$ In the $M \gg m$ limit, the block's final momentum equals mv in scenario A, because the marble's momentum goes from mv to zero. But the block's final momentum equals $2mv$ in scenario B, because the marble's momentum goes from $+mv$ to $-mv$. The block therefore has twice the speed in B as in A. Since the kinetic energy $mv^2/2$ is proportional to v^2, the desired ratio is $2^2 = 4$. Note that the preceding multiple-choice question (where this question was basically answered in the remark given there) tells us that the two energies of the block will be small, whatever they are.

6.11. $\boxed{\text{b}}$ Momentum is conserved in the ball-earth system during the collision (or ball-wall-earth system, if you don't want to define the wall as part of the earth). The ball and the earth have the same final speed, so the earth has virtually all of the momentum because its mass is so much larger. Choices (c) and (d) can't be correct, because heat and sound don't have units of momentum.

6.12. $\boxed{\text{c}}$ As we saw in Multiple-Choice Question 6.9, the large object (the earth here) picks up essentially no energy. And the clay ball doesn't have any energy in the end, because it is at rest. But energy *is* always conserved, so it has to go somewhere. It goes into heat – the kinetic energy of the random motion of the molecules, mostly in the clay, but some in the wall. There is certainly also some energy in sound, since you will undoubtedly hear the splat on the wall. But this energy is much smaller than the heat energy.

6.13. $\boxed{\text{Yes}}$ For example, if $m_1 \gg m_2$ then the final speed of Ball 2 is twice the initial speed of Ball 1. This is true because the relative speed of the balls is unchanged by the elastic collision; see Problem 6.2. So if v is the initial rightward speed of Ball 1, then after the collision, Ball 2 must be moving rightward with speed v relative to Ball 1. But Ball 1 is still plowing forward with speed v relative to the ground, because it is essentially unaffected by the collision. So the speed of Ball 2 relative to the ground is $v + v = 2v$.

6.14. $\boxed{\text{e}}$ Choices (a) and (b) have zero total p_x, so they don't satisfy conservation of p_x. Choice (c) doesn't satisfy conservation of E, because the final energy is larger than the initial energy, due to the large velocity of the right mass. Choice (d) doesn't satisfy conservation of p_y (it has a nonzero final value); it also doesn't satisfy conservation of E (the final E is twice the initial). The correct answer must be (e), which does indeed satisfy conservation of p_x, p_y, and E (at least to the accuracy of how well the vectors are drawn).

Remark: The velocities in choice (e) appear to be orthogonal. And indeed, it can be shown that if the masses are equal and if one of them is initially at rest, then if the collision is elastic, the final velocities are always orthogonal, no matter how the masses glance off each other. See the example in Section 5.7.2 of Morin (2008) for a proof of this fact.

6.15. $\boxed{\text{d}}$ This problem is most easily solved by working in the reference frame of the paddle. In this frame the (heavy) paddle is at rest and the ball comes in with speed $v + 3v = 4v$ leftward. So the ball must go out with the same speed $4v$ (now rightward), because the paddle essentially doesn't move. This is the final speed of the ball with respect to the paddle. But in the frame of the ground, the paddle is still moving to the right with speed v. So the speed of the ball with respect to the ground is $v + 4v = 5v$.

6.16. $\boxed{\text{d}}$ If the cone is sliced into horizontal rings, the lower rings have larger radii and are therefore more massive. Each ring can effectively be replaced by a point mass at its center, so the cone is equivalent to a series of point masses (lying along the axis) that are larger the lower they are. The CM is therefore less than halfway to the top.

6.17. $\boxed{\text{b}}$ The speed of the CM is $((2m)v + 0)/3m = 2v/3$, so the speed of the mass $2m$ relative to the CM is $v - 2v/3 = v/3$.

Remark: More generally, if $2m$ is replaced by Nm, where N is a numerical factor, the answer becomes $v/(N + 1)$. This correctly goes to zero in the $N \to \infty$ limit (the CM coincides with the mass Nm as it moves along). And it correctly goes to v in the $N \to 0$ limit (the CM is located at the stationary mass m).

6.18. $\boxed{\text{c}}$ Your force is what adds momentum to the system, according to $\Delta p = F\,\Delta t$, or $F = dp/dt$ in derivative form. The new momentum shows up in the new mass that gets moving as it leaves the stationary heap and joins the moving straight part of the rope. If a new mass dm gets moving in a small time dt, the new momentum is $dp = (dm)v = (\lambda\, dx)v$. Therefore $F = dp/dt = \lambda v\, dx/dt = \lambda v^2$.

Equivalently, we can use Eq. (6.12). The ma term is zero because the rope isn't accelerating. So we have $F = (dm/dt)v = (\lambda\, dx/dt)v = \lambda v^2$.

Remarks: Note that you can rule out the other answers: (a) is incorrect because the force must be nonzero since the momentum of the rope is increasing. (b) has the wrong units. And (d) and (e) depend on g, but the problem doesn't involve vertical forces.

We should emphasize that the new momentum comes from the new mass that gets moving, and *not* from any increase in speed of the straight part. Imagine painting a dot on the straight part of the rope, and consider the part of the rope that lies between the dot and your hand. This part always moves with the same speed v, and its length (and hence mass) doesn't change, so its momentum is constant. The total force on this part must therefore be zero. And indeed, the rightward force from your hand is canceled by the leftward tension from the part of the rope just to the left of the dot. If we instead look at the rope to the *left* of the dot (which includes the heap plus a straight part), then the only horizontal force acting on it is the tension from the part of the rope just to the right of the dot. This force is what produces the new momentum of the rope, as sections from the stationary heap join the moving part.

6.19. $\boxed{\text{Yes}}$ In addition to balancing the weight of the straight part of the rope that hangs at rest, you must also provide the force needed to change the momentum of the pieces of rope as they make the transition from the moving heap to the stationary straight part. (From the reasoning in the preceding question, this force happens to be λv^2, where λ is the mass density. But the exact value isn't critical here. All we need to know is that the force isn't zero.) So at least right before the rope is straightened out (when the weight of the stationary hanging part is nearly mg), your force will need to exceed mg. See Problem 6.24 for a quantitative treatment.

6.20. $\boxed{\text{e}}$ The mass of the dustpan plus dust inside at a later time is $M + \lambda x$, so the momentum is $p = (M + \lambda x)\dot{x}$. The necessary force is therefore (using the product rule)

$$F = \frac{dp}{dt} = (M + \lambda x)\ddot{x} + \lambda \dot{x}^2 \equiv (M + \lambda x)a + \lambda v^2. \tag{6.15}$$

Physically, the first term in this result is the force needed to accelerate the dustpan plus the dust already inside it (this term depends on a, and not on v), while the second term is the force needed to get the new bits of dust moving (this term depends on v, and not on a). The applied force F makes the momentum of the system increase, and these are the two ways it increases.

6.5 Problem solutions

6.1. **Conservation of momentum**

The case of two particles was treated in Eq. (6.4). Consider now the case of three particles. If $\mathbf{F}_{ij}$ represents the force on particle i due to particle j, then the rate of change of the total momentum of the system equals

$$\begin{aligned}\frac{d(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)}{dt} &= \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_3}{dt} \\ &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= (\cancel{\mathbf{F}_{12}} + \cancel{\mathbf{F}_{13}}) + (\cancel{\mathbf{F}_{21}} + \cancel{\mathbf{F}_{23}}) + (\cancel{\mathbf{F}_{31}} + \cancel{\mathbf{F}_{32}}) \\ &= 0. \end{aligned} \tag{6.16}$$

We have used the fact that Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, tells us that the forces cancel in pairs, as indicated. This reasoning clearly extends to a general number N of particles. For every term $\mathbf{F}_{ij}$ in the generalization of the third line above, there will also be a term $\mathbf{F}_{ji}$. And the sum of these two terms is zero, by Newton's third law.

6.2. **Same relative speed**

Energy is conserved because we are assuming that the collision is elastic. With the notation shown in Fig. 6.20 (where some of the v's may be negative), the conservation of E and p equations are

$$\begin{aligned}\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 &= \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2, \\ m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2'. \end{aligned} \tag{6.17}$$

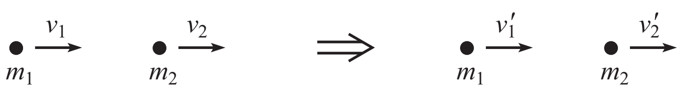


Figure 6.20

If we put all the "1" terms on the left and all the "2" terms on the right, these two equations become

$$\begin{aligned} m_1(v_1^2 - v_1'^2) &= m_2(v_2'^2 - v_2^2), \\ m_1(v_1 - v_1') &= m_2(v_2' - v_2). \end{aligned} \tag{6.18}$$

Dividing the first equation by the second gives

$$v_1 + v_1' = v_2' + v_2 \implies v_1' - v_2' = -(v_1 - v_2), \tag{6.19}$$

which is the desired result that the final relative velocity between the two masses equals the negative of the initial relative velocity.

REMARKS: In the CM frame, this result is clear. The masses simply head out with the same speeds they came in, because this satisfies conservation of both p (which is zero since we're dealing with the CM frame) and E. Therefore, because both velocities simply change sign, Eq. (6.19) is certainly true. And since this result holds in one frame, it holds in all others, because it involves only *differences* in velocities. Differences are frame independent, because both velocities shift by the same amount when going from one frame to another.

The linear equation in Eq. (6.19) can be combined with the linear conservation-of-p equation in Eq. (6.17) to solve for the final velocities, v_1' and v_2'. There will be only one solution to this system of linear equations. So you might wonder how we ended up with only one solution for v_1' and v_2', given that we started with a quadratic equation in Eq. (6.17). Quadratic equations should have two solutions. The explanation is that we lost the other solution when we divided the equations in Eq. (6.18). Another solution is clearly $v_1' = v_1$ and $v_2' = v_2$, because that makes all the terms in Eq. (6.18) be zero (in which case we divided by zero). This solution is simply the one involving the initial conditions, which are what we would end up with if the masses missed each other. The initial conditions certainly satisfy conservation of E and p with the initial conditions. A fine tautology, indeed.

6.3. **1-D collision**

FIRST SOLUTION: This solution involves some brute force; the other two solutions below will be shorter. Let's switch notation and have V' and v' be the final velocities. Since m is initially at rest, the conservation of E and p equations are

$$\frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \frac{1}{2}mv'^2 \qquad \text{and} \qquad MV = MV' + mv'. \tag{6.20}$$

Solving for v' in the p equation and substituting the result into the E equation (and then multiplying through by m/M) gives

$$\begin{aligned} &\frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \frac{1}{2}m\left(\frac{M(V - V')}{m}\right)^2 \\ &\implies 0 = (M + m)V'^2 - 2MVV' + (M - m)V^2. \end{aligned} \tag{6.21}$$

This quadratic equation for V' can be solved with the quadratic formula, or you can just note that it factors nicely:

$$0 = \Big((M + m)V' - (M - m)V\Big)(V' - V). \tag{6.22}$$

In retrospect, we know that in problems like this, $V' = V$ and $v' = v$ must always be a solution, because the initial conditions certainly satisfy conservation of E and p with the initial conditions. So that tells us that $(V - V')$ must be a factor of the above quadratic equation. However, we are concerned with the other (nontrivial) solution, $V' = (M - m)V/(M + m)$. Plugging this into the conservation-of-p equation then gives $v' = 2MV/(M + m)$, as desired.

LIMITS: If $M \ll m$, the final velocities of M and m are $-V$ and 0; M essentially bounces backward off a brick wall. If $M = m$, the final velocities are 0 and V; m picks up whatever velocity M had, and M ends up at rest (an outcome familiar to pool players). If $M \gg m$, the final velocities are V and $2V$; M plows forward with the same velocity, as expected, and m picks up twice this velocity (not obvious in the lab frame, but consistent with the relative-speed result from Problem 6.2). These limits were the subject of Problem 1.7.

Second solution: Instead of using both of the E and p conservation equations, we can use the p equation along with the relative-velocity relation from Problem 6.2, which says that the final relative velocity equals the negative of the initial relative velocity. That is, $v' - V' = -(0 - V)$. So $v' = V + V'$. Substituting this into the conservation-of-p equation in Eq. (6.20) gives

$$MV = MV' + m(V + V') \implies V' = \frac{(M-m)V}{(M+m)}. \tag{6.23}$$

We then have $v' = V + V' = 2MV/(M+m)$, as desired. This solution was quicker than the first one above, because it involved two linear equations instead of one linear and one quadratic.

Third solution: We can also solve this problem by working in the CM frame, following the strategy outlined on page 145.

1. Switch to the CM frame: The CM moves with velocity $MV/(M+m)$ with respect to the lab frame. So the velocities of M and m with respect to the CM are, respectively, $V - MV/(M+m) = mV/(M+m)$, and $0 - MV/(M+m)$.
2. Do the collision in the CM frame: In the CM frame, the velocities simply reverse themselves during the collision (see the discussion at the start of the "Collisions in the CM frame" section on page 144). So the final velocities of M and m in the CM frame are $-mV/(M+m)$ and $MV/(M+m)$.
3. Switch back to the lab frame: To get back to the original lab frame, we must add on the velocity of the CM, namely $MV/(M+m)$, to each of the velocities in the CM frame. The final velocities of M and m in the lab frame are therefore $V' = (M-m)V/(M+m)$ and $v' = 2MV/(M+m)$, as desired.

6.4. **Distances to the CM**

Assume without loss of generality that $x_2 > x_1$. Then the distance from the CM to x_1 is

$$d_1 = x_{\text{CM}} - x_1 = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} - x_1 = \frac{m_2(x_2 - x_1)}{m_1 + m_2}. \tag{6.24}$$

And the distance from the CM to x_2 is

$$d_2 = x_2 - x_{\text{CM}} = x_2 - \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1(x_2 - x_1)}{m_1 + m_2}. \tag{6.25}$$

Therefore, $d_1/d_2 = m_2/m_1$, as desired. If one mass is ten times the other, then the CM is ten times closer to the larger mass. That is, it is located $1/11$ of the way from the larger mass to the smaller one.

6.5. **Equivalent subparts**

The location of the CM of the entire object is given by Eq. (6.8), where the integral runs over the whole object. We can break the integral up into two integrals over the subparts S_1 and S_2. To demonstrate the desired result, we will judiciously multiply each integral by 1, in the form of a mass divided by itself. This gives

$$\begin{aligned}
\mathbf{r}_{\text{CM}} = \frac{1}{M}\int \mathbf{r}\,dm &= \frac{1}{M}\int_{S_1} \mathbf{r}\,dm + \frac{1}{M}\int_{S_2} \mathbf{r}\,dm \\
&= \frac{1}{M}\cdot m_1 \frac{\int_{S_1} \mathbf{r}\,dm}{m_1} + \frac{1}{M}\cdot m_2 \frac{\int_{S_2} \mathbf{r}\,dm}{m_2} \\
&= \frac{1}{M}(m_1\mathbf{r}_{\text{CM},1} + m_2\mathbf{r}_{\text{CM},2}),
\end{aligned} \tag{6.26}$$

where we have obtained the last line by using Eq. (6.8) for each subpart. By comparing this result with the CM for a discrete system (see Eq. (6.7)), we see that the given object can be treated like two point masses m_1 and m_2 located at the CM's of the two subparts. By repeated application of this result, we can subdivide the given object into an arbitrary number of subparts.

6.6. **Collision in the CM frame**

In the CM frame, the initial momentum (and hence the final momentum) is zero; see Footnote 2. The final momenta of the two particles must therefore be equal and opposite. Hence the particles must move in opposite directions.

The final speeds are the same as the initial speeds, for the following reason. Since the momenta are equal (and opposite), we know that the speeds are inversely proportional to the masses, as they are initially. So the final speeds must be in the same ratio as the initial speeds. This means that the only way they can possibly differ from the initial speeds is if they are both scaled up or scaled down by the same factor. But since kinetic energy is proportional to v^2, any such scaling would have the effect of scaling the total final kinetic energy up or down by the square of this factor, causing energy to not be conserved. The scaling (or lack thereof) factor must therefore simply be 1. That is, the final speeds must be the same as the initial speeds.

REMARKS: All of the above reasoning is valid in any dimension. But 2-D and 3-D, the orientation of the line containing the final velocities requires additional information about how exactly the particles collide.

Note that conservation of p and E are both required in the reasoning. Conservation of p alone would allow for a scaling of the speeds (which is what happens in an inelastic collision). And conservation of E alone would allow for increasing one speed and decreasing the other, in such a way that the total kinetic energy stays the same. Additionally, the particles could head off in arbitrary directions.

6.7. **Hemispherical-shell CM**

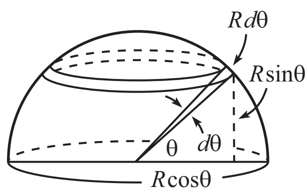


Figure 6.21

First note that the x coordinate of the CM in Fig. 6.21 is zero, by symmetry. Let the surface mass density be σ (kg/m^2). Consider a circular strip located at angle θ above the horizontal, subtending an angle $d\theta$, as shown. The radius of this circle is $R\cos\theta$ and the width of the strip is $R\,d\theta$, so the mass of the strip is

$$\begin{aligned} dm &= \sigma(\text{area}) = \sigma(\text{length})(\text{width}) \\ &= \sigma(2\pi R\cos\theta)(R\,d\theta) = 2\pi R^2\sigma\cos\theta\,d\theta. \end{aligned} \tag{6.27}$$

All points on the strip have a y value of $R\sin\theta$ (at least in the limit where $d\theta$ is infinitesimal), so the CM of the strip is located at the point $(0, R\sin\theta)$. From Problem 6.5, we can replace the circular strip with a point mass at $(0, R\sin\theta)$, with a mass given by the above dm. The hemisphere is therefore equivalent to a string of point masses located on the y axis. The total mass of the hemisphere is $\sigma(\text{area}) = \sigma(2\pi R^2)$, so Eq. (6.8) gives the height of the CM as

$$\begin{aligned} y_{\text{CM}} &= \frac{1}{M}\int y\,dm = \frac{1}{2\pi R^2\sigma}\int_0^{\pi/2}(R\sin\theta)(2\pi R^2\sigma\cos\theta\,d\theta) \\ &= R\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = R\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{R}{2}. \end{aligned} \tag{6.28}$$

REMARKS: The fact that the answer comes out to be so simple implies that there is probably a quicker way to figure it out. And indeed, consider slicing the hemisphere into circular strips with equal *heights dy* (instead of equal angular spans $d\theta$, although technically we never said anything about equal $d\theta$'s when doing the above integral). We claim that all of these strips have the *same mass*, which means that we effectively have equal masses evenly distributed on the y axis, which implies that the CM is located halfway up at $y = R/2$. The equality of the strips' masses follows from the facts that the radius of each strip is $R\cos\theta$, while the width (the slant height along the surface of the hemisphere) equals $dy/\cos\theta$, as you can verify. These factors of $\cos\theta$ cancel when calculating the area, so all strips spanning the same height dy have the same area, and hence the same mass.

As an exercise, you can find the height of the CM for a *solid* hemisphere, which involves slicing the hemisphere into disks instead of circular strips. By modifying the reasoning in the preceding paragraph, you should convince yourself that the height of the CM must be less than $R/2$. See the remark in the solution to Problem 7.12 for further discussion of the difference between 2-D and 3-D objects.

6.8. **Atwood's machine**

(a) The speed of each mass after it has moved a distance d is $v = \sqrt{2ad}$. (This kinematic result can be derived in many ways. For example, the time it takes an object with acceleration a to move a distance d is given by $d = at^2/2 \implies t = \sqrt{2d/a}$. The speed at this time is then $v = at = \sqrt{2ad}$.) The total kinetic energy of the masses is therefore

$$\begin{aligned} K &= \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 = \frac{1}{2}(m_1+m_2)(2ad) \\ &= \frac{1}{2}(m_1+m_2)\left(2\cdot g\frac{m_2-m_1}{m_2+m_1}\cdot d\right) = (m_2-m_1)gd. \end{aligned} \tag{6.29}$$

The potential energy of the masses (relative to their initial positions) is

$$U = m_1gy_1 + m_2gy_2 = m_1gd + m_2g(-d) = (m_1-m_2)gd. \tag{6.30}$$

Therefore, $K + U = 0$, so energy is indeed conserved.

(b) After a time t, the total momentum of the masses is (with upward taken to be positive)

$$\begin{aligned} P_{\text{total}} &= m_1v_1 + m_2v_2 = m_1at + m_2(-at) = (m_1-m_2)at \\ &= (m_1-m_2)\left(g\frac{m_2-m_1}{m_2+m_1}\right)t = -gt\frac{(m_2-m_1)^2}{m_2+m_1}. \end{aligned} \tag{6.31}$$

Let the system be defined to be the two masses and the pulley. The external forces acting on the system are the two weights *and* the tension T in the upper string. The tension in the string connecting the masses is an internal force, so we can ignore it. (If you instead define the system to be just the two masses, then the upward tensions $T/2$ acting on each mass are now relevant external forces, while the upper string is irrelevant. So the total force is the same.) The total external force is therefore

$$\begin{aligned} F_{\text{total}} &= -m_1g - m_2g + T = -(m_1+m_2)g + g\frac{4m_1m_2}{m_1+m_2} \\ &= g\frac{-(m_1+m_2)^2 + 4m_1m_2}{m_1+m_2} = -g\frac{(m_2-m_1)^2}{m_2+m_1}. \end{aligned} \tag{6.32}$$

Comparing this with Eq. (6.31), we see that P_{total} is indeed equal to $F_{\text{total}}t$, as desired.

6.9. **Rising and colliding**

Conservation of energy gives the kinetic energy of the bottom mass right before the collision as (taking zero height for the gravitational U to be at that start)

$$\begin{aligned} & K^{\text{i}} + U_{\text{s}}^{\text{i}} + U_{\text{g}}^{\text{i}} = K^{\text{f}} + U_{\text{s}}^{\text{f}} + U_{\text{g}}^{\text{f}} \\ \implies\ & 0 + \frac{1}{2}kd^2 + 0 = \frac{1}{2}mv^2 + \frac{1}{2}k\left(\frac{d}{2}\right)^2 + mg\frac{d}{2} \\ \implies\ & \frac{3}{8}kd^2 - mg\frac{d}{2} = \frac{1}{2}mv^2. \end{aligned} \tag{6.33}$$

During the collision, half of this kinetic energy is lost to heat, because conservation of momentum gives the speed of the resulting mass $2m$ as $v/2$, so the resulting kinetic energy is $(1/2)(2m)(v/2)^2 = mv^2/4$. Therefore, from Eq. (6.33) the kinetic energy of the mass $2m$ right after the collision is $3kd^2/16 - mgd/4$. Applying conservation of energy from this moment (which we will now take to be at zero height for the gravitational U) up to the moment when the mass $2m$ reaches its maximum height at the relaxed length of the spring, we obtain (using the fact that there is no kinetic energy or spring potential energy at the end)

$$\begin{aligned}
K^{\mathrm{i}} + U_{\mathrm{s}}^{\mathrm{i}} + U_{\mathrm{g}}^{\mathrm{i}} &= K^{\mathrm{f}} + U_{\mathrm{s}}^{\mathrm{f}} + U_{\mathrm{g}}^{\mathrm{f}} \\
\Longrightarrow \quad \left(\frac{3}{16}kd^2 - mg\frac{d}{4}\right) + \frac{1}{2}k\left(\frac{d}{2}\right)^2 + 0 &= 0 + 0 + (2m)g\frac{d}{2} \\
\Longrightarrow \quad \frac{5}{16}kd^2 &= \frac{5mgd}{4} \\
\Longrightarrow \quad k &= \frac{4mg}{d}\,. \qquad (6.34)
\end{aligned}$$

Remark: The following continuity argument shows that there must exist a value of k that makes the masses stop at the relaxed length of the spring. If k is small, then the masses won't make it back up to the dotted line. (If k is smaller than mg/d, then the bottom mass will actually fall when released.) And if k is very large, then we can essentially ignore gravity, so we can imagine that the setup lies on a horizontal table, in which case the mass $2m$ certainly travels beyond the relaxed position of the spring. The spring force must reverse direction if the mass is to slow down and stop.

6.10. **Maximum compression**

First solution: At maximum compression the speeds of the blocks are equal; that is, the relative speed is zero. This is true because if the relative speed weren't zero, then the distance between the blocks would either be decreasing to a smaller value, or increasing from a smaller value, contradicting the assumption of maximum compression. Let the common speed be u. Then conservation of momentum gives

$$mv = (2m)u \quad \Longrightarrow \quad u = \frac{v}{2}\,. \qquad (6.35)$$

And if the maximum compression is x, then conservation of energy gives

$$\frac{1}{2}mv^2 = \frac{1}{2}(2m)u^2 + \frac{1}{2}kx^2. \qquad (6.36)$$

Substituting $u = v/2$ into this equation gives the maximum compression as

$$\frac{1}{2}mv^2 = \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 + \frac{1}{2}kx^2 \quad \Longrightarrow \quad \frac{1}{4}mv^2 = \frac{1}{2}kx^2 \quad \Longrightarrow \quad x = v\sqrt{\frac{m}{2k}}\,. \qquad (6.37)$$

At maximum compression, we see that half of the initial kinetic energy $mv^2/2$ remains kinetic energy, and half ends up as potential energy of the spring.

Limits: x increases with m and decreases with k, as expected.

Remarks: After the spring stretches back to its relaxed length and loses contact with the left block, what do the motions of the blocks look like? You can show by writing down the initial/final conservation of E and p equations (in which the spring doesn't come into play, because it is uncompressed initially and finally) that the left block ends up at rest and the right block moves at speed v. So the velocities of the blocks are simply interchanged from their initial values. In retrospect, this must be what happens, because this scenario certainly satisfies conservation of E and p with the initial motion.

The above collision involving two blocks and a spring is a good model for a completely elastic collision between two balls. If a rubber ball collides with another one, they will both compress and then uncompress during the collision, just like the spring. The ball that was initially moving will now be at rest (in 1-D).

In a completely *inelastic* collision where the balls stick together, the energy that goes into the spring eventually ends up as heat (which is the internal kinetic energy of the random motion of the molecules in the balls). You can model the collision roughly as having the spring vibrate back and forth, with an amplitude that gradually decreases (eventually reaching zero, with the balls being at rest with respect to each other) due to some kind of friction force. This friction generates the heat.

SECOND SOLUTION: Consider the collision in the CM frame. In this frame the blocks move toward each other with equal speeds of $v/2$. At maximum compression the blocks are both instantaneously at rest, so all of the initial kinetic energy must end up as potential energy of the spring. That is,

$$2 \cdot \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{1}{2}kx^2 \implies x = v\sqrt{\frac{m}{2k}}, \tag{6.38}$$

in agreement with the first solution.

6.11. **Collision and a spring**

(a) Let x be the compression distance of the spring when m comes to rest. This is the maximum compression (assuming that $2m$ doesn't move), so this is the moment when the spring exerts the maximum force on the $2m$ block. If the $2m$ block is to remain at rest at all times, then we need kx to be less than or equal to the maximum friction force on $2m$, which is $\mu(2mg)$. So $kx \le 2\mu mg \implies x \le 2\mu mg/k$. This is the condition on x. We must now convert it to the desired condition on v.

We can relate x and v by conservation of energy. The m block ends up instantaneously at rest, so the initial kinetic energy $mv^2/2$ goes into the spring's potential energy $kx^2/2$ plus heat, which equals the magnitude of the work done by friction, $W_f = F_f x = (\mu mg)x$. So we have

$$\frac{mv^2}{2} = \frac{kx^2}{2} + (\mu mg)x. \tag{6.39}$$

Equivalently, if we multiply this equation through by -1, it is the statement that the change in kinetic energy of the m block (which is negative) equals the total work done by the spring and friction (both of which are negative).

There is no need to solve the quadratic equation in Eq. (6.39). Instead, we can simply take the $x \le 2\mu mg/k$ condition from above and plug it into Eq. (6.39). This gives the desired condition on v:

$$\begin{aligned} \frac{mv^2}{2} &\le \frac{k}{2}\left(\frac{2\mu mg}{k}\right)^2 + (\mu mg)\left(\frac{2\mu mg}{k}\right) \\ \implies \frac{mv^2}{2} &\le \frac{4\mu^2 m^2 g^2}{k} \implies v \le \sqrt{\frac{8\mu^2 mg^2}{k}}. \end{aligned} \tag{6.40}$$

You can check that this result has the correct units.

LIMITS: If $\mu = 0$ then the maximum v equals 0; that is, the $2m$ mass always slips, which is correct. Large μ, m, or g implies large v. These all make sense, although the m case requires a little thought because there are partially canceling effects; you can see the various effects of m in the first line of Eq. (6.40). And large k implies small v; the mass m stops very quickly, so the spring force would be very large if v weren't very small.

(b) If v takes on the value we just found, then the maximum compression is $x = 2\mu mg/k$. When m bounces back, the spring's potential energy goes into kinetic energy plus heat due to friction. So if u is the speed at the point when the spring reaches its relaxed length, we have

$$\begin{aligned} \frac{kx^2}{2} &= \frac{mu^2}{2} + (\mu mg)x \\ \Longrightarrow \quad \frac{k}{2}\left(\frac{2\mu mg}{k}\right)^2 &= \frac{mu^2}{2} + (\mu mg)\left(\frac{2\mu mg}{k}\right), \end{aligned} \tag{6.41}$$

which gives $u = 0$. So m barely makes it back to the position where it first came in contact with the spring.

Remark: If the mass of the right block is larger than $2m$, and if m comes in with the speed that yields the maximum compression without the right block moving, then as m bounces back, its speed as it passes through the relaxed length of the spring will be positive. Conversely, if the mass of the right block is smaller than $2m$, then in the analogous scenario, m won't reach the relaxed length of the spring as it bounces back. These results quickly follow by replacing the two $\mu(2m)g$ terms in Eq. (6.41) with the more general $\mu(nm)g$ term, where n is a numerical factor.

6.12. **Colliding balls**

(a) It takes the first moving ball a time of ℓ/v to reach the stationary ball and produce the blob of mass $2m$. After this collision, conservation of momentum gives the speed of the $2m$ blob as

$$mv + 0 = (2m)v_f \quad \Longrightarrow \quad v_f = v/2. \tag{6.42}$$

How long does it take the second moving ball (which is moving at speed v) to catch up with the $2m$ blob (which is moving at speed $v/2$)? It must close the gap of ℓ between them at a relative speed of $v - v/2 = v/2$, so the time is $\ell/(v/2) = 2\ell/v$. After it collides, conservation of momentum gives the speed of the resulting $3m$ blob as

$$mv + 2m(v/2) = (3m)v_f \quad \Longrightarrow \quad v_f = 2v/3. \tag{6.43}$$

How long does it take the third moving ball (which is moving at speed v) to catch up with the $3m$ blob (which is moving at speed $2v/3$)? It must close the gap of ℓ between them at a relative speed of $v - 2v/3 = v/3$, so the time is $\ell/(v/3) = 3\ell/v$. From conservation of momentum, you can show that the speed of the resulting $4m$ blob is $3v/4$.

Continuing in this manner, it takes the fourth moving ball a time of $4\ell/v$ to catch up with the blob in front of it, and the speed of the resulting $5m$ blob is $4v/5$.

In general, the blob of mass nm moves at speed $(n-1)v/n$. (In short, this is because it has mass nm and a total momentum of $(n-1)mv$ from the $n-1$ balls that were originally moving.) And it takes the nth moving ball a time of $n\ell/v$ to catch up with the nm blob, because the relative speed is $v - (n-1)v/n = v/n$. The total time for the entire process is therefore

$$\frac{\ell}{v}\Big(1 + 2 + 3 + \cdots + (N-1) + N\Big) = \frac{\ell}{v}\left(\frac{N(N+1)}{2}\right), \tag{6.44}$$

where we have used the formula for the sum of the first N integers.

Limits: If $N = 1$, the time equals ℓ/v, as expected. For large N, the time goes like N^2, which isn't intuitively obvious.

Remark: As an exercise, you can calculate the total distance the blob moves, between the first and last collisions. The quick way to do this is to subtract $N\ell$ from the total distance the leftmost ball moves (which is just v times the total time we found above). But you should verify that you obtain the same result by adding up the distances the blob moves between successive collisions.

(b) In the frame in which the N balls are initially at rest, one ball moves toward N stationary balls that lie in a line. From the instant indicated in Fig. 6.13, it takes a time of ℓ/v for the moving ball to reach the first stationary ball. After this collision, conservation of momentum gives the speed of the $2m$ blob as $v/2$. This blob therefore takes a time of $\ell/(v/2) = 2\ell/v$ to reach the second stationary ball.

After this collision, conservation of momentum gives the speed of the $3m$ blob as $v/3$. (This follows from $(2m)(v/2) + 0 = (3m)v_\text{f}$, or from simply noting that the total momentum is still just the mv from the initially moving ball, but the mass is now $3m$.) This blob therefore takes a time of $\ell/(v/3) = 3\ell/v$ to reach the third stationary ball.

Continuing in this manner, we see that after the collision with the $(n-1)$th stationary ball, the speed of the nm blob is v/n, and the time to reach the nth stationary ball is $n\ell/v$. The result holds for all n from 1 to N, so the total time is

$$\frac{\ell}{v}\Big(1 + 2 + 3 + \cdots + (N-1) + N\Big) = \frac{\ell}{v}\left(\frac{N(N+1)}{2}\right), \tag{6.45}$$

in agreement with the result in part (a).

6.13. **Block and balls**

(a) In the frame of the heavy block, the light ball comes in at speed v and bounces out at (essentially) speed v. The final relative speed is therefore v. But the relative speed doesn't depend on the frame, so it is also v in the original lab frame. And since the block keeps moving forward at (essentially) speed v, the final speed of the ball after the collision must be $2v$ in the lab frame. Conservation of momentum in the lab frame then gives the final speed v' of the block as

$$Mv + 0 = Mv' + m(2v) \implies v' = v - \frac{2mv}{M}. \tag{6.46}$$

The block's speed therefore decreases by $2mv/M$. You can also find v' by making a Taylor-series approximation to the V_M in Eq. (6.13).

(b) In a small time dt, the block effectively hits a "ball" (which is actually more like a little tube) with mass $dm = \lambda\, dx$, where $dx = v\, dt$ is the distance the block travels in time dt. So $dm = \lambda v\, dt$. Using this as the m in part (a), we see that the change in the block's speed is

$$dv = -\frac{2(dm)v}{M} = -\frac{2(\lambda v\, dt)v}{M} = -\frac{2\lambda v^2\, dt}{M}. \tag{6.47}$$

Remark: Note the v^2 dependence here. One power of v comes from the fact that if we make v larger, then a given ball bounces off with a larger speed, which means that it has a larger increase in momentum, which in turn means that the block has a larger decrease in momentum, and hence speed. The other power of v comes from the fact that the faster the block moves, the more balls it hits in a given time. This v^2 dependence is a general feature of drag forces where the actual motion of mass (shoving things out of the way) is the dominant effect. Even if the balls stuck to the block, we would still obtain the v^2 factor; we just wouldn't have the factor of 2 in Eq. (6.47).

(c) Equation (6.47) is a differential equation involving v and t. Separating variables and integrating yields (we won't bother putting primes on the integration variables)

$$\int_{V_0}^{v} \frac{dv}{v^2} = -\int_0^t \frac{2\lambda\, dt}{M} \implies -\frac{1}{v}\bigg|_{V_0}^{v} = -\frac{2\lambda t}{M}$$

$$\implies -\frac{1}{v} + \frac{1}{V_0} = -\frac{2\lambda t}{M} \implies v(t) = \frac{1}{\dfrac{1}{V_0} + \dfrac{2\lambda t}{M}}. \tag{6.48}$$

LIMITS: If $\lambda \to 0$ or $M \to \infty$ then $v(t) \to V_0$, as expected. If $t \to 0$ then $v \to V_0$, which is correct. And if $t \to \infty$ then $v \to M/2\lambda t$, which correctly goes to zero and which interestingly is independent of V_0. The total distance traveled by the block equals $\int v\,dt$, which diverges like $\ln(t)$ as $t \to \infty$.

6.14. **Maximum final speed**

(a) Using the second of the equations given in Eq. (6.14), the collision between M and x gives x a speed of $v_x = 2MV/(M + x)$. We then need to use this speed as the initial speed for the collision between x and m. This yields a speed of m equal to

$$v_m = \frac{2xv_x}{x+m} = \left(\frac{2x}{x+m}\right)\left(\frac{2MV}{M+x}\right). \tag{6.49}$$

Our goal is to maximize this, which means that we want to maximize the function $f(x) = x/(x + m)(x + M)$. Setting the derivative equal to zero (and ignoring the denominator of the result) gives

$$\begin{aligned} 0 &= (x+m)(x+M)(1) - x(2x+m+M) \\ \Longrightarrow \quad 0 &= mM - x^2 \quad \Longrightarrow \quad x = \sqrt{Mm}, \end{aligned} \tag{6.50}$$

which correctly has units of mass. We see that the optimal value of x is the geometric mean of the original masses, which is about as nice a result as we could hope for. The three masses therefore form a geometric progression. From Eq. (6.49) the maximum final speed of m equals $4MV/(\sqrt{m} + \sqrt{M})^2$.

LIMITS: If $m = M$ then $x = m = M$. This makes sense, because with three equal masses, M and x end up at rest. So m has all of the energy. And we can't do any better than that, by conservation of energy.

(b) With two masses inserted between M and m, we claim that the largest speed of m is obtained if the four masses form a geometric progression, that is, if the ratio of any two successive masses is the same. In other words, the masses should take the form of M, $M^{2/3}m^{1/3}$, $M^{1/3}m^{2/3}$, m, with the common ratio here being $(m/M)^{1/3}$. This claim can be proved by contradiction, as follows.

Assume that in the optimal case, the first three masses, M, x, and y, do *not* form a geometric progression. If this is the case, then from part (a), we can increase the speed of y (which would then increase the speed of m) by making x be the geometric mean of M and y. This contradicts our initial assumption of optimal-ness, so it must be the case that M, x, and y form a geometric progression. Likewise, x, y, and m must form a geometric progression, because otherwise we could increase the speed of m by making y be the geometric mean of x and m. Since these two progressions have the x/y ratio as overlap, all four masses must form a geometric progression, as we wanted to show. The values of x and y are therefore $M^{2/3}m^{1/3}$ and $M^{1/3}m^{2/3}$.

If we have a general number of masses, say 10, then they must all form a geometric progression. This follows from the fact that if this *weren't* the case, then we could take a group of three of the masses that aren't in geometric progression and increase the speed of the third mass by making the middle one be the geometric mean of the other two. This would then increase the speeds of all the masses to the right of this group, in particular the last one, m. This means that we didn't actually have the optimal scenario in the first place. In short, any sequence that isn't in geometric progression can be improved, so the only possibility for optimal-ness is the sequence where the masses are all in geometric progression.

REMARK: It turns out that if $M \gg m$, and if there is a very large number of masses between M and m (all in geometric progression), then essentially *all* of the *kinetic energy* ends up in m (so the other masses have essentially none), but essentially *none* of the *momentum* ends up in m (so the other masses have essentially all of it). In the more general case where M isn't much greater than m, the former of these statements is still true, but the latter isn't.

The basic reason for these divisions of the energy and momentum is that the final speeds of all the masses except the last one are small, but not *too* small. More precisely, on one hand the speeds are small enough so that when their *squares* are multiplied by the masses to obtain the kinetic energy, the smallness of the squares wins out over the fact that there is a large number of masses. So the energy ends up being essentially zero, which means that m must have essentially all of the energy. But on the other hand the speeds are large enough so that when their *first powers* are multiplied by the masses to obtain the momentum, the result isn't zero. And in fact it equals the initial momentum.

If you want to verify the above claims rigorously, the calculations get a bit messy. But the basic strategy is the following. Let the ratio of the masses be r, where r is very close to 1. Then up to a factor of M, the masses are 1, r, r^2, r^3, etc. You can use the expressions for the velocities given in Eq. (6.14) to show that the final speeds of all the masses except the last one are (up to a factor of V)

$$\frac{1-r}{1+r}, \quad \left(\frac{2}{1+r}\right)\frac{1-r}{1+r}, \quad \left(\frac{2}{1+r}\right)^2\frac{1-r}{1+r}, \quad \left(\frac{2}{1+r}\right)^3\frac{1-r}{1+r}, \quad \dots . \tag{6.51}$$

And the final speed of the last mass is $(2/(1+r))^{N-1}$ if there are N masses. You can then explicitly calculate all the energies and momenta and verify the above claims. In some of the calculations it will be advantageous to set $r \equiv 1-\epsilon$, where ϵ is very small. If you (quite reasonably) don't want to go through all the algebra, you are encouraged to at least check that things work out numerically. For example, if $m/M = 10^{-4}$ and $N = 10^4$ (which implies that $r = (10^{-4})^{1/9999} = 0.99908$), then you will find that the last mass has essentially all (99.8%, don't forget to square the v) of the kinetic energy and essentially none (1%) of the momentum.

One thing that is quick enough to do here is to verify that the above claims are at least consistent. That is, we can show that if $M \gg m$ and *if* all of the initial energy goes into m, *then* m has (essentially) no momentum. This can be done as follows. Let the ratio of the masses be $M/m = R \gg 1$. Then given the assumption of equal energies, the final speed of m must be $\sqrt{R}$ times the initial speed of M (because the kinetic energy is proportional to mv^2). The final momentum of m is then $mv = (M/R)(\sqrt{R}\,V) = MV/\sqrt{R}$, which goes to zero for large R.

6.15. **Throwing a block in pieces**

Figure 6.22

Let's first solve the general case where you throw a mass m_1, and where the mass of you plus the cart plus any other mass you are holding is m_2. Let the final speeds *with respect to the ground* be v_1 and v_2, as shown in Fig. 6.22.

The initial momentum is zero, so conservation of momentum gives $m_2v_2 - m_1v_1 = 0$. And we are told that the final relative speed is $v_1 + v_2 = v_0$. Solving this system of two equations and two unknowns (v_1 and v_2) quickly gives your final speed as $v_2 = m_1v_0/(m_1+m_2)$. Additionally, the speed of m_1 is $v_1 = m_2v_0/(m_1+m_2)$, although we won't need this. These two speeds correctly add up to v_0 and are inversely proportional to the masses.

In words: your final speed v_2 equals v_0 times the mass of the piece you threw, divided by the *total* mass of you plus what you threw. We can write this as

$$v_{\text{you}} \equiv v_2 = v_0\frac{m_{\text{throw}}}{M_{\text{total}}} . \tag{6.52}$$

This correctly equals zero when $m_{\text{throw}} = 0$ and correctly equals v_0 when $m_{\text{throw}} = M_{\text{total}}$; if an ant "throws" a rock with speed v_0, the ant is really just propelling itself backward with speed v_0. Eq. (6.52) allows us to quickly solve the various parts of this problem.

(a) In this case we have $m_{\text{throw}} = m$, and $M_{\text{total}} = 2m$, so your final speed is $v_0/2$.

(b) The first stage involves $m_{\text{throw}} = m/2$ and $M_{\text{total}} = 2m$. So you gain a speed of $v_0/4$. The second stage involves $m_{\text{throw}} = m/2$ and $M_{\text{total}} = 3m/2$. So you gain an additional speed of $v_0/3$, relative to how fast you were going after the first throw. (The speeds do indeed simply add, because we can apply conservation of momentum in a new inertial frame moving at your speed of $v_0/4$ after the first throw.) Your final speed is therefore $v_0(1/4 + 1/3) = 7v_0/12 \approx (0.583)v_0$.

(c) Let $r \equiv m_{\text{throw}}/M_{\text{total}}$. If the block is divided in thirds, the first stage has $r = 1/6$, the second has $r = 1/5$, and the third has $1/4$. So your final speed is $v_0(1/6 + 1/5 + 1/4) = 37v_0/60 \approx (0.617)v_0$. Similarly, for fourths we obtain a final speed of $v_0(1/8 + 1/7 + 1/6 + 1/5) \approx (0.635)v_0$. In general, for n pieces we end up with the sum from $1/2n$ to $1/(n+1)$, so

$$f(n) = \sum_{n+1}^{2n} \frac{1}{k}. \tag{6.53}$$

(d) The plot of $f(n)$ for n from 1 to 100 is shown in Fig. 6.23. $f(100)$ equals 0.69065, and $f(n)$ appears to be approaching a number just a hair above that.

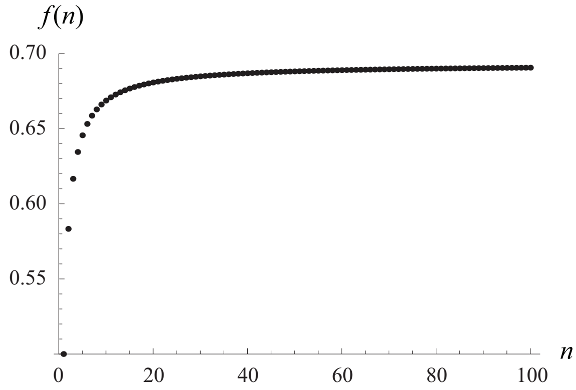


Figure 6.23

(e) For a continuous stream of particles, we have rocket motion. The initial mass is $2m$ and the final mass is m, so from Problem 6.22 the final speed is

$$v_{\text{final}} = v_0 \ln\left(\frac{m_{\text{initial}}}{m_{\text{final}}}\right) = (\ln 2)v_0 \approx (0.693)v_0. \tag{6.54}$$

This is very close to the $n = 100$ result. The difference is only $(0.69315)v_0 - (0.69065)v_0 = (0.0025)v_0$.

6.16. **A collision in two frames**

(a) First solution: Let the final velocities of $4m$ and m be v_4 and v_1, respectively, with rightward taken to be positive. Conservation of momentum gives

$$(4m)v + m(-v) = (4m)v_4 + mv_1 \implies 3v = 4v_4 + v_1. \tag{6.55}$$

And conservation of energy gives

$$\frac{1}{2}(4m)v^2 + \frac{1}{2}mv^2 = \frac{1}{2}(4m)v_4^2 + \frac{1}{2}mv_1^2 \implies 5v^2 = 4v_4^2 + v_1^2. \tag{6.56}$$

Solving for v_1 in Eq. (6.55) and plugging the result into Eq. (6.56) gives

$$\begin{aligned} 5v^2 = 4v_4^2 + (3v - 4v_4)^2 &\implies 0 = 20v_4^2 - 24vv_4 + 4v^2 \\ \implies 0 = 4(5v_4 - v)(v_4 - v) &\implies v_4 = \frac{v}{5}. \end{aligned} \tag{6.57}$$

(Another solution is $v_4 = v$, of course, because that is the initial velocity of $4m$.) Equation (6.55) then gives $v_1 = 3v - 4v_4 = 11v/5$. Both velocities are positive, so both masses move to the right.

SECOND SOLUTION: We can solve this problem by combining Eq. (6.55) with the relative-velocity statement from Problem 6.2, instead of with Eq. (6.56). This way, we won't need to deal with a quadratic equation. Eq. (6.19) gives

$$v_1 - v_4 = -((-v) - v) \implies v_1 - v_4 = 2v. \tag{6.58}$$

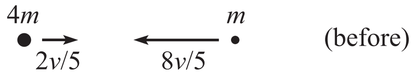
Plugging the v_1 from this equation into Eq. (6.55) gives

$$3v = 4v_4 + (2v + v_4) \implies v_4 = \frac{v}{5}. \tag{6.59}$$

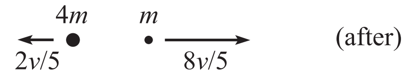
And then $v_1 = 2v + v_4 = 11v/5$.

(b) The CM frame's velocity with respect to the lab frame is given by $v_{\rm CM} = [(4m)v + m(-v)]/5m = 3v/5$. Our strategy will be to switch to the CM frame, then do the collision in the CM frame (which will be trivial), and then switch back to the lab frame.

- Switch to the CM frame: The initial velocities of the two masses in the CM frame are $v - v_{\rm CM} = 2v/5$ and $-v - v_{\rm CM} = -8v/5$, as shown:



- Do the collision in the CM frame: The masses simply reverse their velocities (because this satisfies conservation of p and E). So after the collision, the velocities are $-2v/5$ and $8v/5$, as shown:



- Switch back to the lab frame: To get back to the lab frame, we must add $v_{\rm CM}$ to the CM-frame velocities. So the final velocities in the lab frame are

$$\begin{aligned} v_4 &= -\frac{2v}{5} + v_{\rm CM} = -\frac{2v}{5} + \frac{3v}{5} = \frac{v}{5}, \\ v_1 &= \frac{8v}{5} + v_{\rm CM} = \frac{8v}{5} + \frac{3v}{5} = \frac{11v}{5}. \end{aligned} \tag{6.60}$$

6.17. **45-degree deflections**

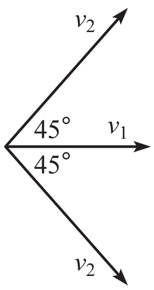


Figure 6.24

Let the final velocities be labeled as in Fig. 6.24. In drawing $\mathbf{v}_1$ horizontal, we have already used conservation of p_y (the initial p_y was zero). Conservation of p_x gives

$$mv = mv_1 + 2 \cdot mv_2 \cos 45^\circ \implies v = v_1 + \sqrt{2}v_2. \tag{6.61}$$

Conservation of E gives

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + 2 \cdot \frac{1}{2}mv_2^2 \implies v^2 = v_1^2 + 2v_2^2. \tag{6.62}$$

Plugging the v_1 from the p_x equation into the E equation gives

$$v^2 = (v - \sqrt{2}v_2)^2 + 2v_2^2 \implies 0 = -2\sqrt{2}vv_2 + 4v_2^2 \implies v_2 = \frac{v}{\sqrt{2}}. \tag{6.63}$$

(Technically, another solution is $v_2 = 0$, but this corresponds to the balls missing each other. This is simply the initial value of v_2, which of course is guaranteed to be a solution to the conservation equations.) We then find v_1 to be $v_1 = v - \sqrt{2}v_2 = 0$. So the ball that was initially moving ends up at rest.

REMARK: We've noted on various occasions that a ball will end up at rest if it collides elastically head-on with an identical ball; the second ball absorbs all of the energy and momentum of the first ball. The result of this problem therefore tells us that the right two balls look effectively like a single ball with mass m, as far as the left ball is concerned. That is, the right two balls absorb all of the energy and momentum of the left ball. As an exercise, you can show that if the right two balls scatter at equal angles of θ instead of 45°, and if they each have a mass of $m/(2\cos^2\theta)$, then the left ball will end up at rest. This mass correctly equals $m/2$ when $\theta = 0$, and ∞ when $\theta \to 90°$.

6.18. **Northward deflection 1**

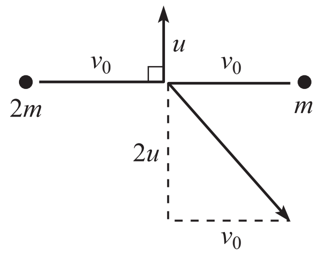


Figure 6.25

Let the northward direction point along the y axis, as shown in Fig. 6.25. Let u be the desired final velocity of the mass $2m$ in the y direction. Then conservation of p_y (which is initially zero) quickly gives the final y velocity of the mass m as $-2u$. Also, if v_x is the final x velocity of m, then conservation of p_x gives

$$2mv_0 + m(-v_0) = 2m(0) + mv_x \implies v_x = v_0, \tag{6.64}$$

as shown in the figure. We now have only one unknown, u, so we can use conservation of E to solve for u:

$$\begin{aligned}
&\frac{1}{2}(2m)v_0^2 + \frac{1}{2}mv_0^2 = \frac{1}{2}(2m)u^2 + \frac{1}{2}m\left[(2u)^2 + v_0^2\right] \\
&\implies 2v_0^2 + v_0^2 = 2u^2 + 4u^2 + v_0^2 \implies u = \frac{v_0}{\sqrt{3}}.
\end{aligned} \tag{6.65}$$

This is the final northward speed of the mass $2m$.

Remark: We can also solve this problem by working in the CM frame. We'll just sketch the solution here, by stating some facts you can justify. The velocity of the CM is $v_0/3$ eastward. So the velocities of the $2m$ and m masses in the CM frame are $2v_0/3$ eastward and $4v_0/3$ westward. The speeds are the same after the collision, although the directions change (but they are still antiparallel). The $2m$ mass's velocity in the CM frame must have a westward component of $v_0/3$, so that it has no east-west component when we transform back to the original frame by adding on the eastward $v_0/3$ velocity of the CM. So the velocity of $2m$ in the CM frame must look like the vector shown in Fig. 6.26, with magnitude $2v_0/3$ and westward component $v_0/3$. The vertical component, which is the northward velocity in both the CM frame and the original frame, is therefore $v_0/\sqrt{3}$. Additionally, since the final velocity of m in the CM frame is antiparallel to the vector in Fig. 6.26 and has twice the length, you can quickly show that the final eastward component of the velocity of m in the original frame equals v_0, consistent with Eq. (6.64).

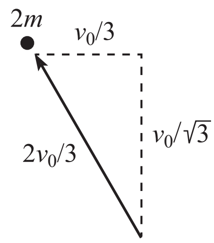


Figure 6.26

6.19. **Northward deflection 2**

(a) Lab frame: Let the final northward speed of m be u, as shown in Fig. 6.27. Conservation of p_y (which is initially zero) quickly tells us that the final southward speed of $2m$ is $u/2$. And conservation of p_x (which is initially mv_0) quickly tells us that the final eastward speed of $2m$ is $v_0/2$, as shown in the figure.

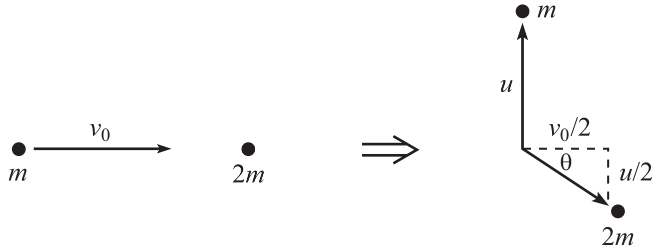


Figure 6.27

We now have only one unknown, u, so we can use conservation of E to solve for u:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mu^2 + \frac{1}{2}(2m)\left(\left(\frac{v_0}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right)$$
$$\Longrightarrow \quad v_0^2 = u^2 + \frac{v_0^2}{2} + \frac{u^2}{2} \quad \Longrightarrow \quad u = \frac{v_0}{\sqrt{3}}. \tag{6.66}$$

The desired angle is therefore

$$\tan\theta = \frac{u/2}{v_0/2} = \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad \theta = 30^\circ. \tag{6.67}$$

Additionally, the 30-60-90 triangle tells us that the final speed of $2m$ is $v_0/\sqrt{3}$. So the two masses end up with the same speeds.

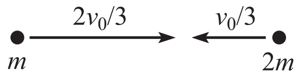


Figure 6.28

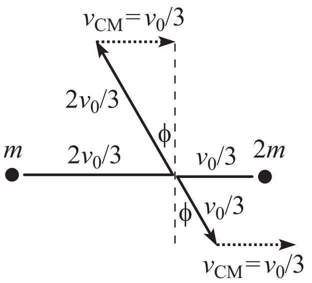


Figure 6.29

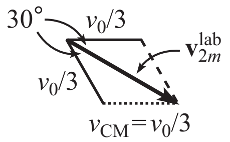


Figure 6.30

(b) CM FRAME: The CM moves with velocity $v_{CM} = mv_0/(m + 2m) = v_0/3$. So the velocities of m and $2m$ in the CM frame are, respectively, $v_0 - v_{CM} = 2v_0/3$ and $0 - v_{CM} = -v_0/3$. The situation is shown in Fig. 6.28.

In the CM frame, the speeds of the masses are unchanged by the elastic collision, and the velocities come off in antiparallel directions. The direction of m's final velocity must be such that when we add the vector $\mathbf{v}_{CM}$ back on, to get back to the lab frame, the result is a vertical (northward) velocity for m. So the final velocity of m in the CM frame must have a westward component of $v_0/3$, as shown in Fig. 6.29. From the right triangle shown, we see that the angle ϕ is given by $\sin\phi = 1/2 \Longrightarrow \phi = 30^\circ$.

The final velocity of $2m$ in the lab frame is obtained by adding $\mathbf{v}_{CM}$ to the diagonally downward vector with length $v_0/3$ in Fig. 6.29. The resulting lab-frame velocity of $2m$ is shown in Fig. 6.30. This shape is a rhombus, so the final lab-frame velocity of $2m$ bisects the 60° angle between the initial and final velocities of $2m$ in the CM frame. The desired angle with respect to the horizontal is therefore 30°.

Additionally, the 30° angles in the rhombus tell us that the length of the long diagonal is $\sqrt{3}$ times the length of a side. So the final speed of $2m$ in the lab frame is $v_0/\sqrt{3}$.

6.20. **Equal energies**

Since the final energies are equal, they must each be half of the initial energy. So we quickly obtain

$$\frac{1}{2}mv_1^2 = \frac{1}{2}\left(\frac{1}{2}mv_0^2\right) \quad \Longrightarrow \quad v_1 = \frac{v_0}{\sqrt{2}},$$
$$\frac{1}{2}(2m)v_2^2 = \frac{1}{2}\left(\frac{1}{2}mv_0^2\right) \quad \Longrightarrow \quad v_2 = \frac{v_0}{2}. \tag{6.68}$$

Conservation of p_x and p_y will allow us to solve for the angles θ_1 and θ_2. Conservation of p_x gives

$$mv_0 = m\cdot\frac{v_0}{\sqrt{2}}\cdot\cos\theta_1 + 2m\cdot\frac{v_0}{2}\cdot\cos\theta_2 \quad \Longrightarrow \quad 1 = \frac{1}{\sqrt{2}}\cos\theta_1 + \cos\theta_2. \tag{6.69}$$

And conservation of p_y gives

$$0 = m\cdot\frac{v_0}{\sqrt{2}}\cdot\sin\theta_1 - 2m\cdot\frac{v_0}{2}\cdot\sin\theta_2 \quad \Longrightarrow \quad 0 = \frac{1}{\sqrt{2}}\sin\theta_1 - \sin\theta_2. \tag{6.70}$$

Putting the θ_1 terms on the left-hand sides of the previous two equations, squaring and adding, and using $\sin^2\theta + \cos^2\theta = 1$, gives

$$1 - \sqrt{2}\cos\theta_1 + \frac{1}{2} = 1 \quad \Longrightarrow \quad \cos\theta_1 = \frac{1}{2\sqrt{2}} \quad \Longrightarrow \quad \theta_1 \approx 69.3^\circ. \tag{6.71}$$

Plugging $\cos\theta_1 = 1/(2\sqrt{2})$ into Eq. (6.69) then gives $\cos\theta_2 = 3/4 \Longrightarrow \theta_2 \approx 41.4°$. Note that the sum of the two angles isn't 90°, as it would be if the masses where equal.

Remark: If a marble collides with a stationary bowling ball, then no matter what the angle of deflection is, the marble will end up with essentially all of the energy, which means that the energies certainly can't be equal. Similarly, if a bowling ball collides with a stationary marble, then the bowling ball will end up with essentially all of the energy. So if we replace the $2m$ in this problem with a general mass Nm, then N must lie within a certain range in order for it to be possible for the two masses to end up with equal energies. As an exercise, you can show that this range is $3 - 2\sqrt{2} \le N \le 3 + 2\sqrt{2}$, that is, $0.17 \le N \le 5.8$. In the $N = 0.17$ case, the collision is head-on, and the mass m continues moving forward. In the $N = 5.8$ case, the collision is head-on, and the mass m bounces directly backward.

6.21. **Equal speeds**

Conservation of energy gives the common final speed as

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 \quad\Longrightarrow\quad v = \frac{v_0}{\sqrt{3}}. \tag{6.72}$$

Conservation of p_x and p_y will allow us to solve for the angles θ_1 and θ_2. Conservation of p_x gives

$$mv_0 = m\cdot\frac{v_0}{\sqrt{3}}\cdot\cos\theta_1 + 2m\cdot\frac{v_0}{\sqrt{3}}\cdot\cos\theta_2 \quad\Longrightarrow\quad \sqrt{3} = \cos\theta_1 + 2\cos\theta_2. \tag{6.73}$$

And conservation of p_y gives

$$0 = m\cdot\frac{v_0}{\sqrt{3}}\cdot\sin\theta_1 - 2m\cdot\frac{v_0}{\sqrt{3}}\cdot\sin\theta_2 \quad\Longrightarrow\quad 0 = \sin\theta_1 - 2\sin\theta_2. \tag{6.74}$$

Putting the θ_1 terms on the left-hand sides of the previous two equations, squaring and adding, and using $\sin^2\theta + \cos^2\theta = 1$, gives

$$3 - 2\sqrt{3}\cos\theta_1 + 1 = 4 \quad\Longrightarrow\quad \cos\theta_1 = 0 \quad\Longrightarrow\quad \theta_1 = 90°. \tag{6.75}$$

Plugging $\cos\theta_1 = 0$ into Eq. (6.73) then gives $\cos\theta_2 = \sqrt{3}/2 \Longrightarrow \theta_2 = 30°$. Note that the sum of the two angles isn't 90°, as it would be if the masses where equal.

Remark: As in Problem 6.20, if we replace the $2m$ in this problem with a general mass Nm, then N must lie within a certain range in order for it to be possible for the two masses to end up with equal speeds. As an exercise, you can show that this range is $0 < N \le 3$ (the 0 means that the stationary ball can be made arbitrarily small). In the $N = 3$ case, the collision is head-on, and the mass m bounces directly backward. The $N \approx 0$ case is a little tricker. A head-on collision will give the tiny stationary mass Nm a speed of $2v_0$ (as you can show). And a completely glancing collision, where m doesn't quite touch the tiny mass, will give it no speed at all. So by continuity there must be a particular glancing collision that causes the tiny mass Nm to come off at a specific intermediate angle with speed v_0 (which is the final speed of m, which just plows through with essentially the same speed). As an exercise, you can show that the $N = 0$ modification of the above solution leads to θ_2 equaling the nice angle of 60°. Alternatively, there is a quick way to derive this 60° result by working in the CM frame.

6.22. **Rocket motion**

Since the given u is a speed, it is defined to be positive. This means that the velocity of the particles ejected at a given instant is obtained by subtracting u from the velocity of the rocket at that instant. The rocket's initial mass is M, and m is the (decreasing) mass at a general later time. The rate of change of the rocket's mass is dm/dt, which is negative. So mass is ejected at a rate $|dm/dt| = -dm/dt$, which is positive. In other words, during a small time dt, a negative mass dm gets added to the rocket (so the rocket's mass decreases), and a positive mass $-dm$ gets shot out the back.[7]

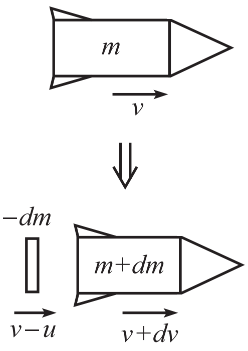


Figure 6.31

Consider a moment when the rocket has mass m and velocity v (with respect to the ground). Then at a time dt later (see Fig. 6.31), the rocket has mass $m + dm$ and velocity $v + dv$, while the exhaust has mass $-dm$ and velocity $v - u$ (which may be positive or negative, depending on the relative size of v and u).[8] There are no external forces, so the total momenta at these two times must be equal. Therefore,

$$mv = (m + dm)(v + dv) + (-dm)(v - u). \tag{6.76}$$

Ignoring the second-order term $dm\,dv$, this simplifies to $m\,dv = -u\,dm$.

REMARK: This relation is actually much easier to see in the inertial reference frame that coincides with the rocket's frame at the start of the dt time interval. In this frame the rocket is initially at rest, and then the ejected mass picks up a momentum of $u\,dm$ (which is negative) while the rocket picks up a momentum of $m\,dv$. (Technically these two quantities should be $(u - dv)\,dm$ and $(m + dm)\,dv$, but the corrections are of second order, and they cancel anyway). These momenta must be equal and opposite, hence $m\,dv = -u\,dm$.

Dividing the $m\,dv = -u\,dm$ equation by m and integrating from t_1 to t_2 gives

$$\int_{v_1}^{v_2} dv = -\int_{m_1}^{m_2} u\frac{dm}{m} \quad\Longrightarrow\quad v_2 - v_1 = u\ln\frac{m_1}{m_2}. \tag{6.77}$$

For the case where the initial mass m_1 is M and the initial speed v_1 is 0, we obtain

$$v = u\ln\left(\frac{M}{m}\right). \tag{6.78}$$

REMARKS: Note that we didn't assume anything about the ejection rate dm/dt in this derivation. There is no need for it to be constant; it can change in any way it wants. The only thing that matters (assuming that M and u are given) is the final mass m. In the special case where dm/dt is constant (call it $-\eta$, where η is positive), we have $m(t) = M - \eta t$, so $v(t) = u\ln[M/(M - \eta t)]$.

The log in the result in Eq. (6.78) is not very encouraging. If the mass of the metal in the rocket is m, and if the mass of the fuel is $9m$, then the final speed after all the fuel has been used up is only $u\ln 10 \approx (2.3)u$. If the mass of the fuel is increased by a factor of 11 up to $99m$ while holding the mass m of the metal constant (which is probably not even structurally possible),[9] then the final speed only doubles to $u\ln 100 = 2(u\ln 10) \approx (4.6)u$. So the factor of 11 in the fuel gives only a factor of 2 in the speed. How, then, do you make a rocket go significantly faster?

For concreteness, let's assume that it is impossible to build a structurally sound container that can hold fuel of more than, say, 19 times its mass. It would then seem like the limit for the speed of a rocket is $u\ln 20$. The strategy for beating this limit is to simply put a little rocket on top of another one. The final speed of the little rocket is the sum of the $u\ln 20$ limits for each rocket, which gives $2u\ln 20$. This is the same as having one rocket with a fuel-to-container mass ratio of $20^2 - 1 = 399$, which is huge. The point of using these "stages" is that if you jettison the container of the big rocket after its fuel is used up, then the little rocket doesn't have to keep accelerating it.

[7]If you wanted, you could define dm to be positive, and then *subtract* it from the rocket's mass, and have dm get shot out the back. However, you would then have to be careful to switch the order of the m_i limits of integration in Eq. (6.77) below.

[8]Technically the exhaust's velocity is $(v + dv) - u$, because we are told that the resulting speed relative to the rocket is u. But the distinction is irrelevant in the $dt \to 0$ limit, because the dv would lead to a second-order term $dm\,dv$ in Eq. (6.76) below. This term would be proportional to dt^2 and hence negligible in comparison with the first-order terms proportional to dt.

[9]The space shuttle's external fuel tank, just by itself, has a fuel-to-container mass ratio of only about 20.

6.23. **Hovering board**

(a) The magnitude of the upward force on the board due to the water equals (by Newton's third law) the magnitude of the downward force on the water due to the board. And Newton's second law, $F = dp/dt$, tells us that this force equals the rate of change of momentum of the water. The water is initially moving upward at speed v_0 and is then brought to rest (at least vertically). So the magnitude of the change in momentum of a little drop of water with mass dm is $|dp| = (dm)v_0$. The magnitude of the rate of change of momentum is therefore (with dt being the small time that it takes the small dm to crash into the board)

$$\frac{dp}{dt} = \frac{(dm)v_0}{dt} = \frac{dm}{dt}v_0 = Rv_0. \tag{6.79}$$

This is the downward force on the water, and hence also the upward force on the board. The board will hover in place if this force equals mg, so we want $m = Rv_0/g$. The units of R are kg/s, so the units of m are correct. And m correctly grows with R and v_0, and decreases with g.

(b) Since the mass (and hence weight) of the board is half of what it was in part (a), we want the force from the water to also be half. So we want the dp/dt of the water to be half of what it was. The water still hits the board at the same mass rate R (because if it didn't, mass would be piling up or magically appearing at some intermediate height). But it is going slower because it loses speed as it rises. (What happens is that although the water is moving slower higher up, its stream is wider, so the same amount of water crosses a given plane as at a lower height.) From the expression for dp/dt in Eq. (6.79), we want the speed to be $v_0/2$. At what height is the speed reduced to $v_0/2$? This can quickly be answered by using conservation of energy:

$$\frac{1}{2}mv_0^2 = mgh + \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 \implies h = \frac{3v_0^2}{8g}. \tag{6.80}$$

This height is 3/4 of the maximum height of $v_0^2/2g$ that the water would reach if the board weren't present. The height h correctly grows with v_0 and decreases with g.

Alternatively, you can also obtain h by using standard constant-acceleration kinematics. The time is given by $v_0 - gt = v_0/2 \implies t = v_0/2g$. Plugging this into $h = v_0 t - gt^2/2$ gives $h = 3v_0^2/8g$.

(c) The rate of change of momentum is now doubled, because each marble of mass dm goes from having momentum $(dm)v_0$ upward to $(dm)v_0$ downward. Therefore, the magnitude of dp is $2(dm)v_0$. The solution proceeds in exactly the same way as in part (a), except with an extra factor of 2, so we obtain a mass of $m = 2Rv_0/g$.

6.24. **Falling heap**

First solution: At time t, the distance the heap has fallen is $gt^2/2$, because we are assuming that it is always in freefall. Therefore, the length left in the heap is $L - gt^2/2$. The heap is moving with speed gt, so its momentum is $p = \lambda(L - gt^2/2)(-gt)$, with upward taken to be positive. This is the momentum of the entire rope, because only the heap is moving.

The net force on the entire rope is $F_{\text{hand}} - \lambda Lg$. (The entire weight is $(\lambda L)g$, and gravity doesn't care that part of the rope is moving and part of it isn't.) So $F = dp/dt$ gives

$$\begin{aligned} F_{\text{hand}} - \lambda Lg &= \frac{d}{dt}\left(-\lambda Lgt + \frac{1}{2}\lambda g^2 t^3\right) = -\lambda Lg + \frac{3}{2}\lambda g^2 t^2 \\ \implies F_{\text{hand}} &= \frac{3}{2}\lambda g^2 t^2. \end{aligned} \tag{6.81}$$

This result holds until the rope straightens out when $gt^2/2 = L \implies t = \sqrt{2L/g}$. Just before this time, Eq. (6.81) gives $F_{\text{hand}} = 3\lambda Lg$. And just after, F_{hand} simply equals the weight λLg of the stationary hanging rope. So F_{hand} drops abruptly from $3\lambda Lg$ to λLg.

SECOND SOLUTION: F_{hand} is responsible for holding up the straight part of the rope, which weighs $\lambda(gt^2/2)g$, and also for stopping the atoms that join the straight part. In a small time dt, a mass of $dm = \lambda\, dx = \lambda(v\, dt)$ joins the straight part. This mass initially has momentum with magnitude $(\lambda v\, dt)v$ downward, and then it comes to rest. So the change in momentum is $dp = +\lambda v^2\, dt$ (it increases from a negative quantity to zero). Hence, $dp/dt = \lambda v^2 = \lambda(gt)^2$. This much additional force must be supplied by your hand, so the total force you apply is

$$F_{\text{hand}} = \lambda\left(\frac{gt^2}{2}\right)g + \lambda(gt)^2 = \frac{3}{2}\lambda g^2 t^2. \tag{6.82}$$

6.25. **Bucket and chain**

(a) Since the acceleration a is constant, the position and speed of the bucket are $x = at^2/2$ and $v = at$. The mass of the chain that has been gathered up is $m = \lambda x$, so the momentum of the bucket (which is massless) plus whatever chain is inside is

$$p = mv = (\lambda x)v = \lambda\left(\frac{at^2}{2}\right)(at) = \frac{1}{2}\lambda a^2 t^3. \tag{6.83}$$

The force that you apply is what causes the change in this momentum, so

$$F = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{1}{2}\lambda a^2 t^3\right) = \frac{3}{2}\lambda a^2 t^2. \tag{6.84}$$

(You should verify that this answer is consistent with the answer to Multiple-Choice Question 6.20.) This force can also be written as $F = 3\lambda v^2/2$. Since λ has units of kg/m, F correctly has units of kg m/s^2.

REMARK: It would be incorrect to use $F = ma$ to say that $F = (\lambda \cdot at^2/2)a = \lambda a^2 t^2/2$, which equals $\lambda v^2/2$. This answer is too small by a factor of 3. The error is that your force is responsible for doing *two* things: It accelerates the mass that is already in the bucket (this yields ma), and it also gives momentum to the new bits of chain that are suddenly brought from speed 0 to speed v. Mathematically,

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v. \tag{6.85}$$

The first term here is simply ma, which we just showed equals $\lambda v^2/2$. The second term is $(d(\lambda x)/dt)v = \lambda(dx/dt)v = \lambda v^2$. The sum of these two terms gives the correct result of $3\lambda v^2/2$. Note that the second term is twice as large as the first term; the force needed to get the new bits of mass moving is twice as large as the force needed to accelerate the mass that is already in the bucket.

(b) The work that you do up to time t is (we won't bother putting a prime on the integration variable)

$$\begin{aligned} W = \int F\, dx = \int_0^t Fv\, dt &= \int_0^t \left(\frac{3}{2}\lambda a^2 t^2\right)(at)\, dt \\ &= \frac{3}{2}\lambda a^3 \int_0^t t^3\, dt = \frac{3}{8}\lambda a^3 t^4. \end{aligned} \tag{6.86}$$

Since λ has units of kg/m, you can quickly verify that F correctly has units of kg m^2/s^2.

(c) The kinetic energy of the chain inside the bucket at time t is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\left(\lambda\frac{at^2}{2}\right)(at)^2 = \frac{1}{4}\lambda a^3 t^4. \tag{6.87}$$

This is $\lambda a^3 t^4/8$ less than the $3\lambda a^3 t^4/8$ work you do (which is the energy that you put into the system). So this difference of $\lambda a^3 t^4/8$, which is 1/3 of the work you do, is what is lost to heat.