Chapter 5

Energy

5.1 Introduction

Work by a constant force

In one dimension, if a constant force F is directed parallel to the line of motion of an object, we define the *work* done by the force on the object to be the force times the displacement:

$$W = F\Delta x. \tag{5.1}$$

Work is a signed quantity, so if the force points opposite to the direction of motion (so that F and Δx have opposite signs), then W is negative. The units of work are

$$[W] = [F][\Delta x] = \frac{\text{kg m}}{\text{s}^2} \cdot \text{m} = \frac{\text{kg m}^2}{\text{s}^2}. \tag{5.2}$$

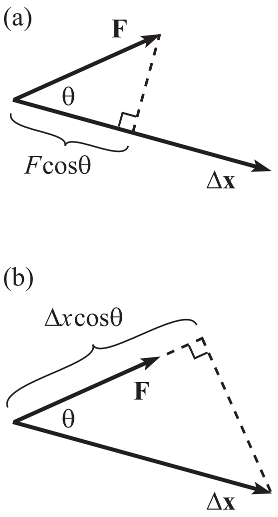

Figure 5.1

This combination of units is called a *joule* (J). Why do we define the work as $F\Delta x$ and not, say, $F^3(\Delta x)^2$ or something else? The reason is that the $F\Delta x$ combination of force and displacement happens to appear in a very useful relation, namely the "work-energy" theorem below.

In the more general case where the force (still assumed to be constant) does *not* point along the line of motion, as shown in Fig. 5.1, the work is defined to be

$$W = F\Delta x\, \cos\theta, \tag{5.3}$$

where θ is the angle between the force $\mathbf{F}$ and the displacement $\Delta\mathbf{x}$ (we'll use $\Delta\mathbf{x}$ to denote a general displacement in space, in any direction). You can write W as $(F\cos\theta)\,\Delta x$, which corresponds to the entire displacement times the component of the force along the displacement; see Fig. 5.1(a). Or you can write W as $F(\Delta x\, \cos\theta)$, which corresponds the entire force times the component of the displacement along the force; see Fig. 5.1(b). If you are familiar with the "dot product" (see Section 13.1.6 in Appendix A for the definition of the dot product), you can write W as $W = \mathbf{F}\cdot\Delta\mathbf{x}$ (which tells us that work is a scalar obtained from two vectors), but we won't need to use that form in this book.

Note that our physics definition of work isn't the same as the colloquial definition. If you hold up a heavy object and keep it at rest (or even move it horizontally at constant speed, so that your upward force is perpendicular to the horizontal displacement), then you are doing zero work by our physics definition, whereas you are certainly doing nonzero work in a colloquial sense.

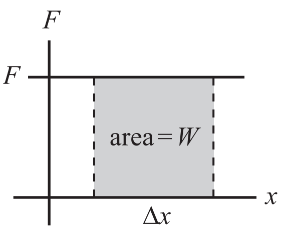

Figure 5.2

Work by a nonconstant force

What is the work done by a nonconstant force? (We'll work in just one dimension here.) In the simple case of a *constant* force, the work done is given in Eq. (5.1) as $F\Delta x$, which equals the area of the region (which is just a rectangle) under the F vs. x "curve" in Fig. 5.2. If the force is instead *nonconstant*, that is, if it varies with position as in Fig. 5.3, then we can divide the x axis into a large number of short intervals, with the force being essentially constant over each one. The work done in each interval is essentially the area of each of the narrow rectangles shown. In the limit of a very large number of very short intervals, adding up all the areas of all the thin rectangles gives exactly the total area under the curve. So the general result is: The work done is the area under the F vs. x curve. That is, the work is the integral of the force:

$$W = \int F\,dx. \tag{5.4}$$

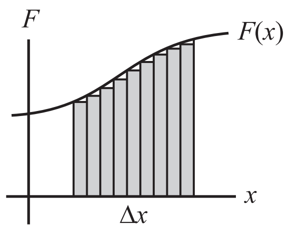

Figure 5.3

The task of Problem 5.1 is to calculate the work done between two given points, x_1 and x_2, by a Hooke's-law spring.

In higher dimensions, Eq. (5.4) becomes $W = \int F\Delta x\,\cos\theta$. If you want, you can write this in terms of the dot product as $W = \int \mathbf{F}\cdot d\mathbf{x}$. But however you want to write it, you're simply adding up the work done over a large number of tiny intervals.

Work-energy theorem

Define the kinetic energy of a particle as $K \equiv mv^2/2$. Then the *work-energy theorem* states:

- *Work-energy theorem:* The work done on a particle equals the change in kinetic energy of the particle. That is,
$$W = \Delta K. \tag{5.5}$$

See Problem 5.2 for a proof. From $F = ma$, we know that if the force points in the same direction as the velocity of the particle, then the speed increases, so the kinetic energy increases. That is, ΔK is positive. This is consistent with the fact that the work W is positive if the force points in the same direction as the velocity. Conversely, if the force points opposite to the velocity of the particle, then the speed decreases, so the kinetic energy decreases. This is consistent with the fact that the work is negative.

Work-energy theorem (general)

The above statement of the work-energy theorem actually is valid only in the special case where the object has no internal structure, that is, where it is a rigid featureless object. If the object *does* have internal structure (for example, subparts that can move or springs that can be compressed, etc.), then the general form of the work-energy theorem states that the work done equals the change in the *total* energy of the object. This energy comes in the form of not only the kinetic energy K of the object as a whole, but also the energy E_{int} of the internal constituents (this can include both kinetic and potential energies; see below for the definition of potential energy). So the general form of the work-energy theorem can be stated as

$$W = \Delta E_{\text{total}} = \Delta K + \Delta E_{\text{int}}. \tag{5.6}$$

ΔE_{int} includes heat, because heat is a measure of the kinetic energy of molecules vibrating on a microscopic scale.

If an object is deformable, we need to be careful about how we define work. The work is $F\Delta x$ (we'll deal with 1-D here, for simplicity), but if different parts of the object move different amounts, which displacement should we pick as Δx? The correct choice for Δx is the displacement of the point in the object where the force is applied. For contact forces (like pushing or pulling, as opposed to long-range forces like gravity) that don't involve any slipping, we can equivalently say that Δx is the displacement of the thing that is applying the force. If you walk up some stairs, then the stairs do no work on you, because they aren't moving.

When applying Eq. (5.6), the first thing you need to do is define what your *system* is, because your choice of system determines which forces are *external* (which in turn determines the work done) and which changes in energy are *internal*. Multiple-Choice Question 5.2 and Problem 5.5 discuss this issue.

Conservative forces

A *conservative* force can be defined in two equivalent ways:

1. A force is conservative if it does zero total work on an object during a round trip.
2. A force is conservative if the work done between two given points is independent of the path taken.

You can quickly verify that gravitational and spring forces are conservative, but kinetic friction is not. Kinetic friction always does negative work, because the force always points opposite to the velocity, which means that there can't exist the necessary cancelation of positive and negative contributions to make the total work be zero during a round trip.

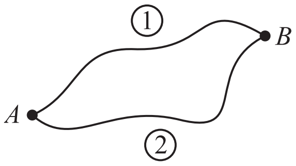

Figure 5.4

The above two definitions are equivalent for the following reason. Let's start with the second definition. From this definition, the work done by a conservative force in going from A to B in Fig. 5.4 is the same along the two paths. Label these works as W. Then the work done in going *from* point B *to* point A along path 2 is $-W$, because the $d\mathbf{x}$ in the $W = \int \mathbf{F} \cdot d\mathbf{x}$ integral changes sign. Therefore the total work done in the round trip from A to B along path 1, and then from B to A along path 2, equals $W + (-W) = 0$, which is consistent with the first definition.

Potential energy

We define the change in the *potential energy* (associated with a given force) of an object to be the negative of the work done by the force on the object:

$$\Delta U \equiv -W. \tag{5.7}$$

(Sometimes the letter V is used instead of U.) For a falling mass, gravity does *positive* work, so ΔU is *negative*, which makes intuitive sense; the object loses gravitational potential energy. Potential energy is defined only for conservative forces, because if a force isn't conservative, then the work is path dependent, so the ΔU between two points isn't well defined. The potential energies associated with two common conservative forces, gravitational and spring, are derived in Problem 5.3.

Note that the definition in Eq. (5.7) deals only with the *change* in U. It makes no sense to ask what U is for an object at a given point; it makes sense only to ask what U is for an object at a given point, *relative* to a given reference point. Only changes in U matter. If one person measures the gravitational U with respect to the floor, and another person measures it with respect to the ceiling, they will have different results for U at any given point. But they will always calculate the same difference between any two given points.

If we combine Eqs. (5.4) and (5.7), we see that ΔU is the negative integral of F. That is, $U(x) - U(x_0) = -\int_{x_0}^{x} F\,dx$. The fundamental theorem of calculus then tells us that F is the negative derivative of U:

$$F(x) = -\frac{dU}{dx}\,. \tag{5.8}$$

The general result in 3-D is that $\mathbf{F}$ is the negative gradient of U, that is, $\mathbf{F} = -\nabla U$. But we'll stick to one dimension here. Eq. (5.8) gives another explanation of why adding a constant to the potential energy doesn't change the system: Since the derivative of a constant is zero, an additive constant doesn't affect the force. So a particle will move in exactly the same way.

Conservation of energy

If we use $\Delta U \equiv -W$ (which is just a definition) to replace W with $-\Delta U$ in the work-energy theorem, $W = \Delta K$ (which is an actual theorem with content), we obtain

$$-\Delta U = \Delta K \implies \Delta(U + K) = 0. \tag{5.9}$$

This tells us that the quantity $U + K$ is constant (that is, *conserved*) throughout the motion. We call this quantity the total energy: $E \equiv U + K$. In general, U can come from various different forces – gravity, spring, electrical, and so on. Note that conservation of energy gives the velocity of an object (in 1-D) as

$$U + K = E \implies U(x) + \frac{mv^2}{2} = E \implies v(x) = \pm\sqrt{\frac{2(E - U(x))}{m}}. \tag{5.10}$$

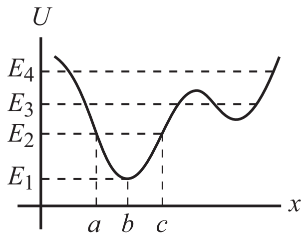

Figure 5.5

Conservation of energy lends itself to an easy graphical visualization. Fig. 5.5 shows an example of a potential-energy function, $U(x)$, relative to a chosen reference point. Various different choices of the total energy E are shown. (While $U(x)$ is determined by the given setup, the total energy E is determined by the initial speed and position of the object, which you are free to choose.) From Eq. (5.10), the object can't exist at values of x for which $U(x) < E$, because v would be imaginary. So, for example, if the energy in Fig. 5.5 is E_1, then the object will sit at rest at $x = b$. If the energy is E_2, then the object will oscillate back and forth between $x = a$ and $x = c$. If the energy is E_3, then the object will oscillate in *either* of two wells (whichever one you put it in at the start). And if the energy is E_4, then the object will oscillate (with a nontrivially varying speed) in one large well.

When using conservation of energy to solve a simple problem (say you want to find the speed of a ball dropped from rest), you can usually get by with saying something like, "the loss in potential energy shows up as kinetic energy," which will allow you to write down $mgh = mv^2/2$. But in more complicated problems, you are strongly advised to be systematic by writing down an $E_{\text{initial}} = E_{\text{final}}$ equation and then dealing with the various terms. For example, if a setup involves an object under the influence of gravity and a spring, then you should immediately write down:

$$K_{\text{i}} + U_{\text{i}}^{\text{grav}} + U_{\text{i}}^{\text{spring}} = K_{\text{f}} + U_{\text{f}}^{\text{grav}} + U_{\text{f}}^{\text{spring}}. \tag{5.11}$$

You can then gradually get a handle on what each term is and then solve for whatever unknown you're trying to solve for. See, for example, Problem 5.9.

Since conservation of energy follows from the work-energy theorem,[1] which in turn follows from $F = ma$ (see Problem 5.2), any problem that you can solve with conservation of energy you can also solve with $F = ma$. However, in many cases a conservation-of-energy approach makes for a much quicker solution, because the $F = ma$ solution would probably involve redoing the derivation of the work-energy theorem in Problem 5.2. This task has already been done once and for all if you use conservation of energy.

Along the same lines (and consistent with Footnote 1), if someone solves a problem by using conservation of energy, and another person uses the work-energy theorem, then they're essentially doing they same thing. A potential-energy term in the former solution will show up as a work term in the latter. See, for example, Problem 5.5.

Conservation of energy is one of the fundamental tools in physics. Two other conservation laws that permeate classical mechanics are conservation of momentum (discussed in Chapter 6) and conservation of angular momentum (discussed in Chapter 8).

Heat

The total energy of an isolated system is always conserved. In standard mechanics, the energy can take three basic forms: potential energy, kinetic energy on a macroscopic scale, and kinetic energy on a microscopic scale. The first two of these are commonly called "mechanical energy" (examples include the $kx^2/2$ potential energy of a spring and the $mv^2/2$ kinetic energy of an object), while the latter is called "thermal energy" or "heat."

Heat is just the sum of the $mv^2/2$ kinetic energies of the tiny molecules moving around inside an object. But the reason we split the total kinetic energy into a macroscopic piece and a microscopic piece is that it is generally easy to write down all the $mv^2/2$ terms for the former, but hopeless for the latter. However, it *is* often easy to get a handle on the *total* heat energy

[1] Instead of using the word "follows" here, it might be more accurate to say, "Since conservation of energy *is equivalent to* the work-energy theorem." The only thing separating these two results is the $\Delta U \equiv -W$ definition, which doesn't really have any content, being just a definition.

in a system. In problems in this book, the way that heat arises is from kinetic friction. More precisely, the heat generated equals the magnitude of the work done by kinetic friction.

(An aside, which isn't important for this book: The phrase "work done by kinetic friction" is riddled with subtleties; the exact nature of the friction force needs to be specified before we can determine how much work is done on any given object. See Problem 5.6 in Morin (2008) for a discussion of this. However, for the purpose of calculating the total heat generated by kinetic friction (without caring how it is divided between the objects), it is valid to say that the work equals the kinetic friction force times the distance that one object moves with respect to the other.)

When energy is "lost" to heat, people often say colloquially that energy isn't conserved. What they really mean is that the *mechanical* energy isn't conserved. The *total* energy, including heat, is always conserved. No energy is actually lost; it's just that some of it might change from a macroscopic form to a microscopic from and hence not show up in the overall motion of objects in the system.

Lest you think that there is no way that tiny molecules moving around inside an object can have a substantial amount of energy, remember that there are a lot of them, and that they can be moving very fast, even though you can't see the motion. It turns out that if you increase the temperature of a liter of water from just above freezing to just below boiling, then the energy you need to add is the same as the gravitational potential energy of a 1.5-ton car raised 30 meters, or equivalently the gravitational potential energy of the liter of water raised 40 kilometers, or equivalently the mechanical kinetic energy of the liter of water projected with a speed of 900 m/s! You can verify these claims by using the facts that the amount of energy required to raise one gram of water by one Celsius degree is one calorie, and that there are 4.2 joules in a calorie,

Power

Power is the rate at which work is done. If a force F in 1-D acts on an object for a small time dt, during which the object has a displacement dx, then the small amount of work done is $dW = F\,dx$. So the rate at which work is done (that is, the power) equals

$$P \equiv \frac{dW}{dt} = \frac{F\,dx}{dt} = F\frac{dx}{dt} = Fv. \tag{5.12}$$

In other words, power equals force times velocity. These are all signed quantities, so if the force and velocity have opposite signs, then the power is negative. More generally, in higher dimensions the $dW = \mathbf{F} \cdot d\mathbf{x}$ relation leads to $P = \mathbf{F} \cdot \mathbf{v}$, by the same reasoning. Since the unit of work is the joule, the units of power are joules/second, which are called a *watt* (W).

5.2 Multiple-choice questions

5.1. Which of the following forces can never, under *any* circumstances, do work? (Be careful!)

(a) gravity
(b) static friction
(c) kinetic friction
(d) tension
(e) normal force
(f) None of the above; they all can do work.

5.2. If you are driving down a road and you step on the gas, the car accelerates due to the static friction force between the ground and the tires. The car's speed increases, so its kinetic energy increases. Does the static friction force do any work?

Yes No

5.3. An escalator moves downward at constant speed. You walk up the escalator at this same speed, so that you remain at rest with respect to the earth. Are you doing any work?

Yes No

5.4. Fill in the blanks: If you walk up some stairs at constant speed, the net work done on your entire body (during some specific time interval) is _______ , and the net work done on just your head is _______ .

(a) negative, zero
(b) zero, zero
(c) zero, positive
(d) positive, zero
(e) positive, positive

5.5. A spring hangs from a ceiling. It is initially compressed by some distance. A mass is attached to the bottom end and then released from rest. Consider the lowest point in the mass's motion as it bounces up and down. At this point, the spring's potential energy is _______ its initial potential energy.

(a) equal to
(b) larger than
(c) smaller than
(d) The relative size cannot be determined from the given information.

5.6. A block with mass m initially has speed v_0 down a plane inclined at an angle θ. The block is attached to a spring with spring constant k, initially at its relaxed length; see Fig. 5.6. The coefficient of kinetic friction with the plane is μ.

True or false: A method for calculating the position where the block reaches its lowest point on the plane is to find the position where the net force on the block is zero.

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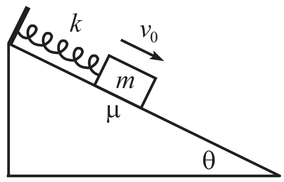

Figure 5.6

5.7. A block with mass m starts from rest and slides down a plane inclined at an angle θ. The coefficient of kinetic friction is μ. Which expression correctly yields the block's speed v after it has traveled a distance d down along the plane, assuming that it does indeed start sliding down? (d is a distance here, so it is a positive quantity.)

(a) $mgd\sin\theta + \mu mgd\cos\theta = \dfrac{mv^2}{2}$
(b) $mgd\sin\theta - \mu mgd\cos\theta = \dfrac{mv^2}{2}$
(c) $-mgd\sin\theta + \mu mgd\cos\theta = \dfrac{mv^2}{2}$
(d) $mgd\cos\theta + \mu mgd\sin\theta = \dfrac{mv^2}{2}$
(e) $mgd\cos\theta - \mu mgd\sin\theta = \dfrac{mv^2}{2}$

5.8. A cart with massless wheels contains sand. The cart starts at rest and then rolls (without any energy loss to friction) down into a valley and then up a hill on the other side. Let the initial height be h_1, and let the final height attained on the other side be h_2. If the cart leaks sand along the way, how does h_2 compare with h_1?

(a) $h_2 < h_1$ (b) $h_2 = h_1$ (c) $h_2 > h_1$

5.3 Problems

The first six problems are foundational problems.

5.1. **Work done by a spring**

If the force from a spring is given by Hooke's-law, $F(x) = -kx$, calculate the work done by the spring in going from x_1 to x_2.

5.2. **Work-energy theorem**

Show that the acceleration a of an object can be written as $a = v\,dv/dx$. Then combine this with $F = ma$ to prove the work-energy theorem: the work done on an object (with no internal structure) equals the change in its kinetic energy.

5.3. **Gravitational and spring U's**

(a) Find the gravitational potential energy of a mass m as a function of y (with upward taken to be positive), measured relative to a given point chosen as $y = 0$.

(b) Find the spring potential energy (for a spring with spring constant k) as a function of x, measured relative to the equilibrium point where $x = 0$.

5.4. **Work in different frames**

An object with mass m is initially at rest, and then a force is applied to it that causes it to undergo a constant rightward acceleration a for a time t.

(a) Calculate the work done on the object, and also the change in kinetic energy. Verify that $W = \Delta K$.

(b) Repeat the same tasks, but now by working in the reference frame moving to the left with constant speed v. (Equivalently, let the object start with speed v instead of starting at rest.)

5.5. **Raising a book**

Assume that you lift a book up a height h at constant speed (so there is no change in kinetic energy). The general work-energy theorem, Eq. (5.6), takes a different form depending on what you pick as your system, because your choice of system determines which forces are external (and thereby do work), and which changes in energy are internal. Write down Eq. (5.6) in the cases where your system is:

(a) the book

(b) the book plus the earth

(c) the book plus the earth plus you

Verify that the three different equations actually say the same thing.

5.6. **Hanging spring**

A massless spring with spring constant k hangs vertically from a ceiling, initially at its relaxed length. A mass m is then attached to the bottom and released.

(a) Calculate the total potential energy U (gravitational and spring) of the system, as a function of the height y (which is negative), relative to the initial position.

(b) Find y_0, the point at which the potential energy is minimum. Make a rough plot of $V(y)$.

(c) Rewrite the potential energy as a function of $z \equiv y - y_0$. Explain why your result shows that a hanging spring can be considered to be a spring in a world without gravity, provided that the new equilibrium point, y_0, is now called the "relaxed" length of the spring.

5.7. **A greener world?**

In an effort to make the world a greener place by reducing the burning of fossil fuels, consider the following alternative method of producing energy. This brilliant method applies the principle of hydroelectric power to normal solid matter. The plan is to blow up the sides of mountains and convert the loss in potential energy of the falling rock/dirt/etc. into electric power.

Your task is to find (very roughly) the volume of rock that needs to be converted daily to satisfy the entire world's need for power. You will need to look up the values of various things, such as the world's daily energy consumption. Assume that some ingenious method has been devised to capture all of the energy of the falling rock, and assume that on average the rock falls a height of one kilometer.

5.8. **Hoisting up**

A platform has a rope attached to it which extends vertically upward, over a pulley, and then back down. You stand on the platform. The combined mass of you and the platform is m.

(a) Some friends standing on the ground grab the other end of the rope and hoist you up a height h at constant speed. What is the tension in the rope? How much work do your friends do?

(b) Consider instead the scenario where you grab the other end of the rope and hoist yourself up a height h at constant speed. What is the tension in the rope? How much work do you do?

5.9. **Hanging block**

A block with mass m is attached to a ceiling by a spring with spring constant k and relaxed length ℓ. Initially, the spring is compressed to a length of $\ell/2$. If the block is released, at what distance below the ceiling will the block be brought to rest (instantaneously, at the lowest point) by the spring?

5.10. **Rising on a spring**

A spring with spring constant k and relaxed length ℓ stands vertically on the ground, with its bottom end attached to the ground. A mass m is attached to the top and then lowered down to the ground, so that the spring is compressed to zero length. The mass is then released from rest and accelerates vertically upward. What is the maximum height above the ground it reaches? You may assume that $k > mg/\ell$, so that the mass does indeed rise up off the ground. Note that at all times, the mass remains attached to the spring, and the spring remains attached to the ground.

5.11. **Jumping onto mattresses**

(a) You jump out of a window (with zero initial speed) and do a belly flop onto a mattress. Assume that the mattress can be treated like a spring with spring constant k. If you fall a distance h before hitting the mattress, what is the maximum compression distance of the mattress? What is the maximum force it applies to you? For simplicity, ignore the change in gravitational potential energy during the compression.

(b) Answer the same questions, but now with N identical mattresses stacked on top of each other. Assume that the height fallen before hitting the top mattress is still h. (You will need to use the generalization of the result from Problem 4.2.)

5.12. **Bungee jumping 1**

A bungee-jump cord has length ℓ and is initially folded back on itself, as shown in Fig. 5.7. The jumper has mass m. After she jumps (or rather, falls) off the platform, she is in freefall for a height ℓ. After that, the spring becomes stretched. Assume that it acts like an ideal

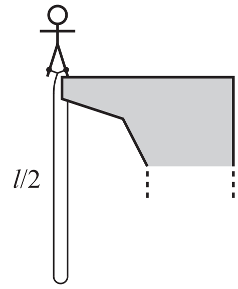

Figure 5.7

spring with a particular spring constant. (This applies only during the stretching motion, of course; the cord can't support compression like a normal spring.) It is observed that the lowest point the jumper reaches is a distance 2ℓ below the platform.

(a) What is the spring constant?

(b) What is the jumper's acceleration at the lowest point?

(c) At what position (specify the distance below the platform) is the jumper's speed maximum? What is this speed?

5.13. **Bungee jumping 2**

Consider again the setup in Fig. 5.7, with a bungee-jump cord of length ℓ. Assume that the spring constant takes on a particular value k.

(a) What is the lowest point the jumper achieves?

(b) What is the tension in the cord at the lowest point?

(c) If the jumper cuts the cord in half and uses one of the halves for the jump, what is the tension at the lowest point? (You will need to use the result from Problem 4.2.)

The following five problems have a common theme.

5.14. **Mass on a spring**

A block with mass m is located at position $x = 0$ on a horizontal table. A spring with spring constant k and relaxed length *zero* is connected to it and has its other end anchored at position $x = \ell$, as shown in Fig. 5.8. The coefficient of friction (both static and kinetic) between the block and the table depends on position according to $\mu = Ax$, where A is a constant. Assume that the block is small enough so that it touches the table at essentially only one value of x.

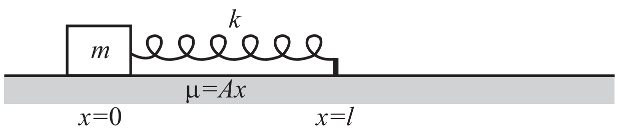

Figure 5.8

(a) The block is released from rest at $x = 0$. Where does it come to rest for the first time? What is the condition on A for which the stopping point is to the right of the $x = \ell$ anchor point? (Assume that the block can somehow pass through the anchor.)

(b) If the stopping point is to the right of the anchor, what is the condition on A for which the block starts moving leftward after it instantaneously comes to rest? In the cutoff case where it barely starts moving again, where is this (first) stopping point?

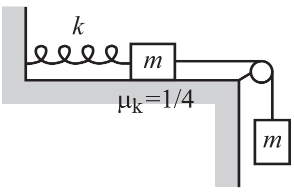

Figure 5.9

5.15. **Falling with a spring**

Consider the system shown in Fig. 5.9, with two equal masses m and a spring with spring constant k. The coefficient of kinetic friction between the left mass and the table is $\mu = 1/4$, and the pulley is frictionless. The system is held with the spring at its relaxed length and then released.

(a) How far does the spring stretch before the masses come to rest?

(b) What is the minimum value of the coefficient of *static* friction for which the system remains at rest once it has stopped?

(c) If the string is then cut, what is the maximal compression of the spring during the resulting motion?

Note: In parts (a) and (c), you can give the left mass a tiny kick to get it going if the coefficient of static friction happens to be large enough to make this necessary.

5.16. **Bead, spring, and rail**

Fig. 5.10 shows a setup involving a spring and two angled rails. Assume that the setup is located in deep space, so that you can ignore gravity in this problem. A bead with mass m is constrained to move along one of the rails; this rail has friction, and the coefficient of kinetic friction with the bead is μ. The bead is connected to a spring, the other end of which is constrained to move along a *frictionless* rail. The spring is always perpendicular to this rail (this is a consequence of the facts that the spring is massless and the rail is frictionless). The relaxed length of the spring is *zero*.

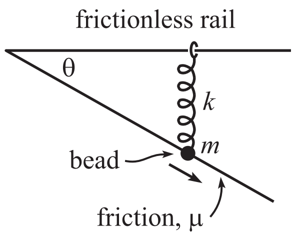

Figure 5.10

The bead starts out at the vertex of the rails (with the spring unstretched at its relaxed length of zero) and is given a kick so that its initial speed is v_0 (so the figure shows a general later time).

(a) Draw the free-body diagram for the bead at a general later time. If x is the distance the bead has traveled along the rail, what are all the forces in terms of x?

(b) How far does the bead travel along the rail before it comes to rest?

(c) Under what condition does the bead start moving again, back toward the vertex? (Assume that the coefficient of static friction is also μ.)

(d) Assuming that the bead does indeed start moving again, what is its speed when it arrives back at the vertex?

5.17. **Ring on a pole**

A spring with spring constant k and relaxed length *zero* has one end attached to a wall and the other end attached to a ring with mass m. The ring is pulled to the side and is slipped over a vertical pole that is fixed at a distance ℓ from the wall, as shown in Fig. 5.11. The coefficient of friction (both static and kinetic) between the ring and the pole is μ. The ring is held with the spring horizontal and is then released.

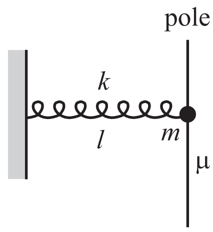

Figure 5.11

(a) Draw the free-body diagram for the ring at a general later time. What is the normal force between the ring and the pole?

(b) How far down the pole does the ring fall before bouncing back up?

(c) What is the cutoff value of μ, below which the ring does indeed:

   i. fall when it is released?

   ii. bounce back up at the bottom of its motion?

5.18. **Block on a plane**

A block with mass m lies on a plane inclined at angle θ. The coefficient of friction (both kinetic and static) between the block and the plane is $\mu = 1$. A massless spring with spring constant k is placed on the plane, with its lower end held fixed. The block is attached to the top end of the spring (see Fig. 5.12) and then moved down until the spring is compressed a distance ℓ relative to its relaxed length. The block is then released.

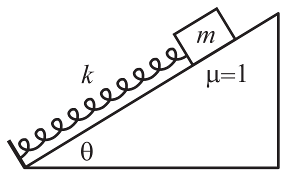

Figure 5.12

(a) For what value of ℓ does the block rise back up exactly to its original position (where the spring is uncompressed)?

(b) What is the condition on θ for which the block then starts to slide back down?

(c) Assuming that the block does indeed slide back down, how far down the plane does it go?

(d) What is the condition on θ for which the block then starts to move back up?

5.19. **Tangential acceleration**

A bead is initially at rest at the top of a fixed frictionless hoop with radius R that lies in a vertical plane. The bead is then given an infinitesimal push so that it slides down and around the hoop.

(a) What is the speed of the bead after it has fallen through an angle θ (measured relative to the vertical)?

(b) Take the time derivative of your result (don't forget to use the chain rule) to verify that the tangential acceleration dv/dt equals the tangential component of gravity, namely $g \sin\theta$.

5.20. **Comparing the tensions**

A pendulum with mass m and length ℓ swings back and forth between the two horizontal positions shown on the left in Fig. 5.13. Let the tension in the string as a function of θ be $T_1(\theta)$. The mass is then stopped at an angle θ and held in place with a horizontal rope, as shown on the right in Fig. 5.13. Let the tension in the string (the pendulum's string, not the rope) as a function of θ be $T_2(\theta)$.

(a) Find $T_1(\theta)$ and $T_2(\theta)$. For what θ is $T_1(\theta) = T_2(\theta)$?

(b) Explain with a continuity argument why you can say (without doing any calculations) that there must indeed exist an angle θ for which $T_1(\theta) = T_2(\theta)$.

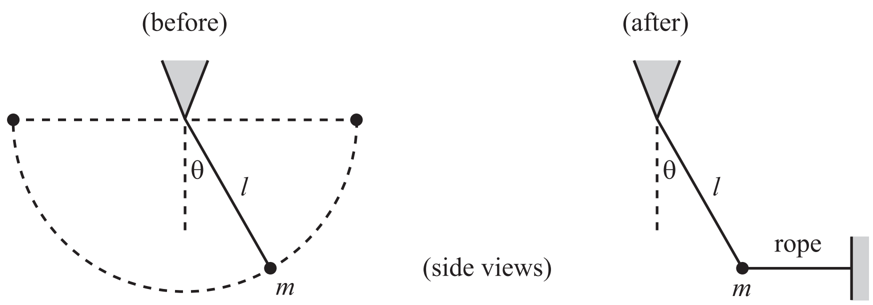

Figure 5.13

5.21. **Semicircular tube**

A frictionless tube is bent into the shape of a semicircle with radius R. The semicircle is tilted so that its diameter makes a fixed angle θ with respect to the vertical, as shown in Fig. 5.14. A small mass is released from rest at the top of the tube and slides down through it. When the mass leaves the tube, it undergoes projectile motion. Let d be the distance traveled in the projectile motion, up to the time when the mass returns to the height it had when it left the tube. What should θ be so that d is as large as possible?

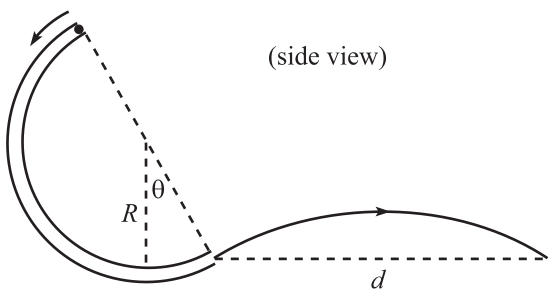

Figure 5.14

5.22. **Horizontal force**

A bead is constrained to move on a fixed frictionless vertical hoop with radius R. The bead starts at rest at the top and is given an infinitesimal kick. When the bead is at an angle θ below the horizontal, what is the normal force that the hoop applies to the bead? For what θ is the total force on the bead horizontal?

5.23. **Maximum vertical normal force**

A bead is constrained to move on a fixed frictionless vertical hoop with radius R. The bead is initially at rest at the top. It is then given a kick so that it suddenly acquires a speed v_0; see Fig. 5.15. It then travels around the hoop indefinitely. Let N_y be the vertical component of the bead's normal force on the hoop. (Note that we are talking about the force from the bead on the hoop, and not the hoop on the bead.) Let θ be the angle of the bead's position with respect to the vertical.

(a) What is N_y?

(b) For what θ does N_y achieve its maximum (upward) value? In answering this, you can assume that v_0 is relatively small. Relatively large v_0 is handled in part (c).

(c) There is a certain value of v_0 above which N_y achieves its maximum value at the top of the hoop. What is this value of v_0?

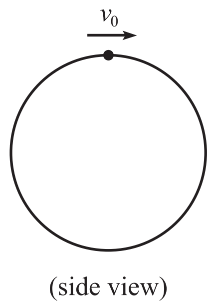

Figure 5.15

5.24. **Falling stick on a table**

A massless stick with length ℓ stands at rest vertically on a table, and a mass m is attached to its top end, as shown in Fig. 5.16. The coefficient of static friction between the stick and the table is μ. The mass is given an infinitesimal kick, and the stick-plus-mass system starts to fall over. At what angle (measured between the stick and the vertical) does the stick start to slip on the table? (Careful, the answer depends on whether μ is larger or smaller than a particular value.)

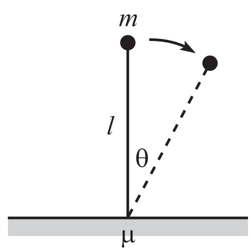

Figure 5.16

5.25. **Bead, spring, and hoop**

A massless spring with spring constant k and relaxed length *zero* has one end attached to a given point on fixed frictionless *horizontal* hoop of radius R, while the other end is attached to a bead with mass m that is constrained to lie on the hoop. The spring is initially stretched across a diameter, with the bead at rest. The bead is then given an infinitesimal kick, and it gets pulled around the hoop by the spring, as shown in Fig. 5.17.

(a) What is the normal force in the horizontal plane of the hoop (in other words, ignore gravity in this problem) that the hoop exerts on the bead at the moment the bead has gone a quarter of the way around the circle?

(b) Is there a point in the motion where the normal force (in the plane of the hoop) is zero? A simple yes or no, with proper reasoning, is sufficient; you don't have to solve for the normal force as a general function of the angle to answer this question.

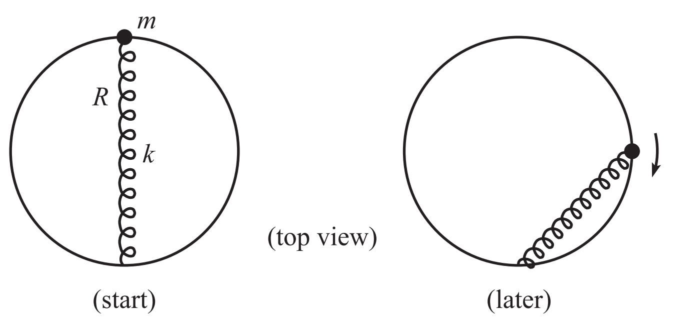

Figure 5.17