Chapter 11

Gravity

11.1 Introduction

Universal law of gravitation

Newton's *universal law of gravitation,* which is the basis for everything we will do in this chapter, gives the magnitude of the (attractive) gravitational force between two point masses m_1 and m_2, separated by a distance r, as

$$F = \frac{Gm_1 m_2}{r^2}\,. \tag{11.1}$$

Newton first wrote down this law in 1687. The important ingredients in the law are that the force is proportional to the product of the masses and inversely proportional to the square of the separation. The gravitational constant G depends on the system of units used. In our standard mks system, the value is

$$G = 6.674 \cdot 10^{-11}\,\frac{\text{m}^3}{\text{kg s}^2}\,. \tag{11.2}$$

In 1798, Henry Cavendish measured G to the impressive accuracy of one percent. His very delicate experiment involved determining the force between two everyday-sized objects with known masses. (The earth couldn't be used as one of the masses because its mass wasn't known until Cavendish did his experiment.) The smallness of G means that the force between two everyday-sized masses is barely noticeable, so Cavendish needed to take great pains to isolate the tiny force he was trying to measure. See Section 5.4.2 in Morin (2008) for a discussion of the experiment.

Relation between *G* and *g*

It turns out that Eq. (11.1) holds for spheres as well as point masses. That is, from the outside, a sphere can be treated like a point mass at the center, as far as the gravitational force is concerned (more on this below). So for a point mass m located near the surface of the earth, Eq. (11.1) gives the force between the mass and the earth as $F = GmM_\text{E}/R_\text{E}^2$. However, we already know what this force is; it is the standard $F = mg$ gravitational force, where $g = 9.8\,\text{m/s}^2$. Equating these two expressions for the force tells us that

$$g = \frac{GM_\text{E}}{R_\text{E}^2}\,. \tag{11.3}$$

And indeed, if we plug in the values of the various constants, we obtain

$$g = \frac{GM_\text{E}}{R_\text{E}^2} = \frac{(6.67 \cdot 10^{-11}\,\text{m}^3/\text{kg s}^2)(5.97 \cdot 10^{24}\,\text{kg})}{(6.37 \cdot 10^6\,\text{m})^2} = 9.81\ \text{m/s}^2, \tag{11.4}$$

as expected.

Equation (11.3) isn't exact, in that the actual value of g varies over the surface of the earth, from about 9.78 at the equator to about 9.83 at the poles. It depends in a fairly complicated way on the nonspherical shape of the earth, which bulges at the equator due to the rotation of the earth (which itself causes a slight centrifugal-force correction to the effective value of g for a dropped ball; see Multiple-Choice Question 12.4). Additionally, there are local variations due to variations in the mass density of the earth's crust. Measuring these small variations with a *gravimeter* (which has a remarkable sensitivity) can indicate what substances (oil, minerals, etc.) might be lurking deep in the ground.

Note that the relation in Eq. (11.3) by itself isn't enough to determine G. That is, Cavendish could not have determined the value of G by simply measuring the acceleration of a dropped ball. Equation (11.3) determines only the product GM_E. (More precisely, it determines GM_E/R_E^2, but the value of R_E was known reasonably accurately in Cavendish's time.) And for all we know, the value of G might be 10 times larger than what we thought, with M_E being 10 times smaller. Maybe the interior of the earth is made of styrofoam (not likely!). But once Cavendish determined G via an experiment with two *known* masses, he could then determine the value of M_E. This is why his experiment was known as "weighing the earth" (or rather, "massing the earth"). It is fascinating how the behavior of a few lead balls in a tabletop experiment can tell us what the mass of the earth is. The average density of the earth turns out to be about 5.5 times the density of water. So we conclude that the earth is certainly *not* made of styrofoam. Iron is a much better bet. Cavendish's tabletop experiment therefore tells us something about the earth's core, which we have no hope of investigating by direct access.

Potential energy

We know from Eqs. (5.4) and (5.7) that the potential energy associated with a force is $U(r) = -\int F\,dr$. (The radial direction is all that matters here.) Consider the potential energy U of a mass m located a distance r from a mass M. We can arbitrarily choose the reference point where we define U to be zero, but it is customary to take this reference point to be at infinity. So with $U = 0$ at $r = \infty$, the value of U at any other distance r from the given mass M is

$$U(r) = -\int_\infty^r F\,dr' = -\int_\infty^r \left(-\frac{GmM}{r'^2}\right) dr' = -\frac{GmM}{r'}\bigg|_\infty^r = -\frac{GmM}{r}. \tag{11.5}$$

We have used the fact that the gravitational force is attractive, which means that it points in the direction of decreasing r; hence the minus sign in F. We have put primes on the integration variable so that it isn't confused with the specific value of r that is the limit of integration. We see that the potential energy is negative. However, this negative nature isn't important; only differences in potential energy matter. Someone could add, say, 13 joules to the energy everywhere, and it would describe the same system. All that matters is that the potential energy decreases as we move in toward the mass M.

As mentioned in Chapter 5, a potential energy can be defined only if a force is conservative. The $F = -GmM/r^2$ force is indeed conservative, because when calculating the work done between two given points, only the starting and ending values of r matter; the path taken is irrelevant. (You can consider any path to consist of radial and tangential displacements, and no work is done along the tangential ones.) The particular $1/r^2$ nature of the (radial) force isn't important here. Any other function of r would still yield a conservative force.

Force from a sphere

The nice thing about the inverse-square law in Eq. (11.1) is that it leads to the following two facts:

- Outside a uniform hollow spherical shell with total mass m, the shell can be treated like a point mass m at the center, as far as the gravitational force is concerned.

- Inside a uniform hollow spherical shell with total mass m, the shell produces zero gravitational force; the shell effectively doesn't exist.

The first of these facts requires a bit of a calculation to demonstrate; see Problem 11.1. The second fact can be proved in a much simpler way; see Problem 11.2. Note that the shell needs to be spherical; the above facts don't apply to a cubical shell, for example.

If we have a solid sphere instead of a hollow shell, we can still make use of the above facts, because we can treat the solid sphere as the superposition of many thin hollow shells. We quickly conclude that if we are located at radius r inside a solid sphere of radius R, then the ball of mass inside radius r looks like a point mass at the center (with the same mass as the ball of radius r), while the mass between r and R effectively doesn't exist. This result holds even if the mass density isn't uniform throughout the sphere, provided that it depends only on radius (so that each thin hollow shell is uniform).

Kepler's laws

The motion of the planets around the sun is completely governed by the universal law of gravitation, Eq. (11.1) (assuming we neglect effects of general relativity). This law can be used to derive three other laws, knows as Kepler's laws:

1. The planets move in elliptical orbits, with the sun at one focus.
2. A planet sweeps out equal areas in equal times; see Fig. 11.1.
3. The square of the period of revolution, T, is proportional to the cube of the semimajor axis, a, of the elliptical orbit. More precisely:

$$T^2 = \frac{4\pi^2 a^3}{GM_{\text{sun}}}. \tag{11.6}$$

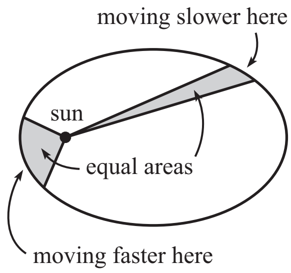

Figure 11.1

If we look at the historical order in which things developed, Kepler wrote down these laws in the early 1600's, about 75 years before Newton wrote down Eq. (11.1). Kepler arrived at his laws empirically (that is, by looking at observational data), which is quite a remarkable feat. But Newton showed that they are all consequences of a single universal law.

Kepler's second law can be derived quickly by showing that it is equivalent to the statement of conservation of angular momentum; see Problem 11.5. In contrast, the first and third laws require a lengthy calculation to derive (see Section 7.4 in Morin (2008)), although the third can be easily demonstrated in the special case of a circular orbit; see Problem 11.6.

Technically, Kepler's three laws hold only in the approximation where the sun is much more massive than the planets. Since this is a good approximation, we won't worry about the minor corrections to the laws.

11.2 Multiple-choice questions

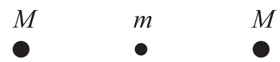

Figure 11.2

11.1. In the setup shown in Fig. 11.2, the two outside masses are glued in place. The center mass is initially located at the midway point. If it is displaced slightly to the right, the resulting net force on it is

(a) rightward (b) leftward (c) zero

11.2. Two planets have the same mass density, but one has twice the radius of the other. What is the ratio of the acceleration due to gravity on the larger planet to that on the smaller planet?

(a) 1/4 (b) 1/2 (c) 1 (d) 2 (e) 4

11.3. Two planets have the same mass density, but one has twice the radius of the other. What is the ratio of the potential energy (relative to infinity) of a mass m on the surface of the larger planet to that on the surface of the smaller planet?

(a) 1/4 (b) 1/2 (c) 1 (d) 2 (e) 4

11.4. Two identical planets have radius R and uniform mass density ρ. Their centers are a distance d apart. If all distances (both R and d) are scaled up by a factor of 2, while ρ is held constant, what is the ratio of the force between the planets in the new setup to the force in the original setup?

(a) 1/4 (b) 1 (c) 4 (d) 16 (e) 64

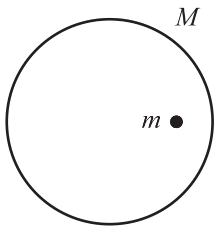

Figure 11.3

11.5. A mass m is located off-center in the interior of a uniform ring with mass M, as shown in Fig. 11.3. What is the direction of the gravitational force on m due to the ring? (You may want to look at the reasoning in Problem 11.2 first.)

(a) leftward
(b) rightward
(c) upward
(d) downward
(e) The force is zero inside the ring.

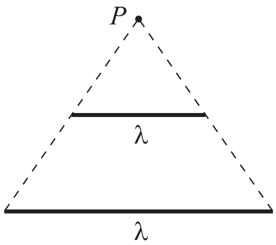

Figure 11.4

11.6. The two sticks shown in Fig. 11.4 have the same linear mass density λ, and they subtend the same angle from a given point P, as indicated. Their distances from P are ℓ and 2ℓ. Which stick creates a larger force on a mass at P? (This question is in the same spirit as the preceding one.)

(a) the top stick
(b) the bottom stick
(c) They produce equal forces at P.
(d) The relative size of the forces depends on ℓ.

11.7. A mass m is located very close to a very large (essentially infinite) flat sheet with surface mass density σ. Another mass m is located very close to a hollow spherical shell with radius R and the same surface mass density σ. What is the ratio of the gravitational force on the first m to the gravitational force on the second m? You can use the result from Problem 11.8 that the force from an infinite sheet is $F = 2\pi G\sigma m$.

(a) 1/4 (b) 1/2 (c) 1 (d) 2 (e) 4

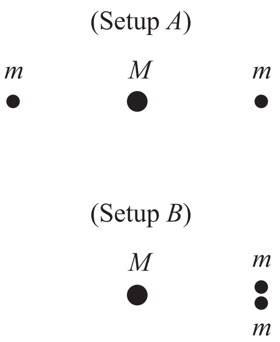

Figure 11.5

11.8. Let F_A and U_A be, respectively, the magnitudes of the gravitational force on, and the potential energy of, a mass M due to the two masses m shown in Setup A in Fig. 11.5. Likewise for F_B and U_B in Setup B. If all of the m's are the same distance from the M's, then

(a) $F_A = F_B$ and $U_A = U_B$
(b) $F_A = F_B$ and $U_A < U_B$
(c) $F_A < F_B$ and $U_A = U_B$
(d) $F_A < F_B$ and $U_A < U_B$
(e) $F_A > F_B$ and $U_A < U_B$

11.9. Relative to infinity, the gravitational potential energy of an object at the center of a hollow spherical shell is

(a) zero
(b) less than zero, but larger than the potential energy at the surface of the shell
(c) equal to the potential energy at the surface
(d) less than the potential energy at the surface, but greater than negative infinity
(e) negative infinity

11.3 Problems

The first nine problems are foundational problems.

11.1. **Force from a spherical shell**

A uniform hollow spherical shell has mass M and radius R. Find the force on a mass m at radius r due to the shell, both inside and outside. Do this by calculating the potential energy due to the shell, and then taking the derivative to obtain the force. *Hint*: Slice the shell into the rings shown in Fig. 11.6, and then integrate over the rings. You will need to use the law of cosines.

Note: The reason for finding the force via the potential energy is that the potential energy is a scalar quantity, whereas the force is a vector. If we tried to directly calculate the force, we would have to worry about forces pointing in all sorts of different directions. With the potential energy, we just need to add up some numbers.

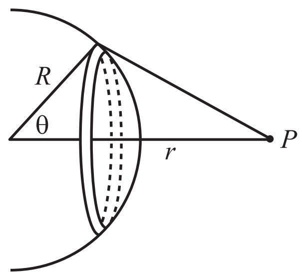

Figure 11.6

11.2. **Zero force inside a spherical shell**

Show that the gravitational force inside a uniform hollow spherical shell is zero by showing that the pieces of mass at the ends of the thin cones in Fig. 11.7 give canceling forces on a given mass at point P.

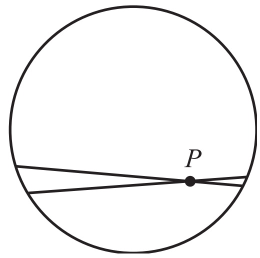

Figure 11.7

11.3. **Force inside a solid sphere**

What is the gravitational force on a mass m located at radius r inside a planet with radius R, mass M, and uniform mass density?

11.4. **Low-orbit and escape velocities**

(a) What is the speed of a low-orbit satellite circling around a planet with mass M and radius R, just above the surface?

(b) What is the escape velocity (or rather, escape speed) from the surface of a planet with mass M and radius R? That is, what minimum initial speed is required for a mass to end up infinitely far away from the planet (and thus refute the "What goes up must come down" claim)?

11.5. **Kepler's 2nd law**

Show that Kepler's second law (which says that a planet sweeps out equal areas in equal times) is equivalent to the statement of conservation of angular momentum.

11.6. **Kepler's 3rd law for circles**

For a circular orbit, use $F = ma$ to derive Kepler's third law, Eq. (11.6), from scratch.

11.7. **Force from a line**

A mass m is placed a distance ℓ away from an infinite straight line with mass density λ (kg/m). Show that the force on the mass is $F = 2G\lambda m/\ell$. *Hint*: If you use an angle as your variable, then you'll have to do some geometry, but the resulting integral should be easy. If you use the distance along the line as your variable, then a $\tan\theta$ trig substitution should simplify things.

11.8. **Force from a plane**

A mass m is placed a distance ℓ away from an infinite plane with mass density σ (kg/m^2). Show that the force on the mass is $F = 2\pi G\sigma m$. *Hint*: Imagine building up the plane from an infinite number of adjacent rods, and then integrate over these rods, using the result from Problem 11.7.