Chapter 2

Kinematics in 1-D

As mentioned in the preface, this book should not be thought of as a textbook. The introduction to each chapter is brief and is therefore no substitute for an actual textbook. You will most likely want to have a textbook on hand when reading the introductions.

2.1 Introduction

In this chapter and the next, we won't be concerned with the forces that cause an object to move in the particular way it is moving. We will simply take the motion as given, and our goal will be to relate positions, velocities, and accelerations as functions of time. Our objects can be treated like point particles; we will not be concerned with what they are actually made of. This is the study of *kinematics.* In Chapter 4 we will move on to *dynamics,* where we will deal with mass, force, energy, momentum, etc.

Velocity and acceleration

In one dimension, the *average* velocity and acceleration over a time interval Δt are given by

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad \text{and} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t}\,. \tag{2.1}$$

The *instantaneous* velocity and acceleration at a particular time t are obtained by letting the interval Δt become infinitesimally small. In this case we write the "Δ" as a "d," and the instantaneous v and a are given by

$$v = \frac{dx}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}\,. \tag{2.2}$$

In calculus terms, v is the derivative of x, and a is the derivative of v. Equivalently, v is the slope of the x vs. t curve, and a is the slope of the v vs. t curve. In the case of the velocity v, you can see how this slope arises by taking the limit of $v = \Delta x/\Delta t$, as Δt becomes very small; see Fig. 2.1. The smaller Δt is, the better the slope $\Delta x/\Delta t$ approximates the actual slope of the tangent line at the given point P.

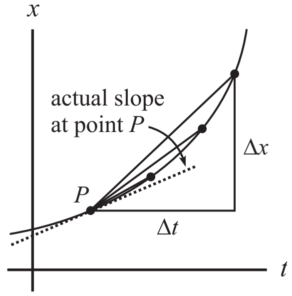


Figure 2.1

In 2-D and 3-D, the velocity and acceleration are vectors. That is, we have a separate pair of equations of the form in Eq. (2.2) for each dimension; the x components are given by $v_x = dx/dt$ and $a_x = dv_x/dt$, and likewise for the y and z components. The velocity and acceleration are also vectors in 1-D, although in 1-D a vector can be viewed simply as a number (which may be positive or negative). In any dimension, the *speed* is the magnitude of the velocity, which means the absolute value of v in 1-D and the length of the vector $\mathbf{v}$ in 2-D and 3-D. So the speed is a positive number by definition. The units of velocity and speed are m/s, and the units of acceleration are m/s^2.

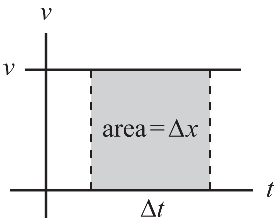


Figure 2.2

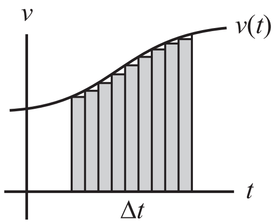


Figure 2.3

Displacement as an area

If an object moves with constant velocity v, then the displacement Δx during a time Δt is $\Delta x = v\Delta t$. In other words, the displacement is the area of the region (which is just a rectangle) under the v vs. t "curve" in Fig. 2.2. Note that the *displacement* (which is Δx by definition), can be positive or negative. The *distance* traveled, on the other hand, is defined to be a positive number. In the case where the displacement is negative, the v vs. t line in Fig. 2.2 lies below the t axis, so the (signed) area is negative.

If the velocity varies with time, as shown in Fig. 2.3, then we can divide time into a large number of short intervals, with the velocity being essentially constant over each interval. The displacement during each interval is essentially the area of each of the narrow rectangles shown. In the limit of a very large number of very short intervals, adding up the areas of all the thin rectangles gives exactly the total area under the curve; the areas of the tiny triangular regions at the tops of the rectangles become negligible in this limit. So the general result is:

- *The displacement (that is, the change in x) equals the area under the v vs. t curve.*

Said in a more mathematical way, the displacement equals the time integral of the velocity. This statement is equivalent (by the fundamental theorem of calculus) to the fact that v is the time derivative of x.

All of the relations that hold between x and v also hold between v and a. In particular, the change in v equals the area under the a vs. t curve. And conversely, a is the time derivative of v. This is summarized in the following diagram:

$$\boxed{x} \;\underset{\substack{\text{area}\\ \text{(integral)}}}{\overset{\substack{\text{slope}\\ \text{(derivative)}}}{\rightleftarrows}}\; \boxed{v} \;\underset{\substack{\text{area}\\ \text{(integral)}}}{\overset{\substack{\text{slope}\\ \text{(derivative)}}}{\rightleftarrows}}\; \boxed{a}$$

Motion with constant acceleration

For motion with constant acceleration a, we have

$$\begin{aligned} a(t) &= a, \\ v(t) &= v_0 + at, \\ x(t) &= x_0 + v_0 t + \frac{1}{2}at^2, \end{aligned} \tag{2.3}$$

where x_0 and v_0 are the initial position and velocity at $t = 0$. The above expressions for $v(t)$ and $x(t)$ are correct, because $v(t)$ is indeed the derivative of $x(t)$, and $a(t)$ is indeed the derivative of $v(t)$. If you want to derive the expression for $x(t)$ in a graphical manner, see Problem 2.1.

The above expressions are technically all you need for any setup involving constant acceleration, but one additional formula might make things easier now and then. If an object has a displacement d with constant acceleration a, then the initial and final velocities satisfy

$$v_\text{f}^2 - v_\text{i}^2 = 2ad. \tag{2.4}$$

See Problem 2.2 for a proof. If you know three out of the four quantities v_f, v_i, a, and d, then this formula quickly gives the fourth. In the special case where the object starts at rest (so $v_\text{i} = 0$), we have the simple result, $v_\text{f} = \sqrt{2ad}$.

Falling bodies

Perhaps the most common example of constant acceleration is an object falling under the influence of only gravity (that is, we'll ignore air resistance) near the surface of the earth. The constant nature of the gravitational acceleration was famously demonstrated by Galileo. (He mainly rolled balls down ramps instead of dropping them, but it's the same idea.) If we take the positive y axis to point upward, then the acceleration due to gravity is $-g$, where $g = 9.8\,\text{m/s}^2$. After every second, the velocity becomes more negative by 9.8 m/s; that is, the downward speed increases by 9.8 m/s. If we substitute $-g$ for a in Eq. (2.3) and replace x with y, the expressions become

$$\begin{aligned} a(t) &= -g, \\ v(t) &= v_0 - gt, \\ y(t) &= y_0 + v_0 t - \frac{1}{2}gt^2, \end{aligned} \tag{2.5}$$

For an object dropped from rest at a point we choose to label as $y = 0$, Eq. (2.5) gives $y(t) = -gt^2/2$.

In some cases it is advantageous to choose the positive y axis to point downward, in which case the acceleration due to gravity is g (with no minus sign). In any case, it is always a good idea to take g to be the *positive* quantity $9.8\,\text{m/s}^2$, and then throw in a minus sign by hand if needed, because working with quantities with minus signs embedded in them can lead to confusion.

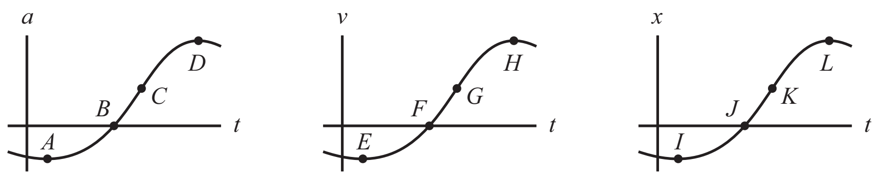
The expressions in Eq. (2.5) hold only in the approximation where we neglect air resistance. This is generally a good approximation, as long as the falling object isn't too light or moving too quickly. Throughout this book, we will ignore air resistance unless stated otherwise.

2.2 Multiple-choice questions

2.1. If an object has negative velocity and negative acceleration, is it slowing down or speeding up?

(a) slowing down
(b) speeding up

2.2. The first figure below shows the a vs. t plot for a certain setup. The second figure shows the v vs. t plot for a different setup. The third figure shows the x vs. t plot for a yet another setup. Which of the twelve labeled points correspond(s) to zero acceleration? Circle all that apply. (To repeat, the three setups have nothing to do with each other. That is, the v plot is *not* the velocity curve associated with the position in the x plot. etc.)



2.3. If the acceleration as a function of time is given by $a(t) = At$, and if $x = v = 0$ at $t = 0$, what is $x(t)$?

(a) $\dfrac{At^2}{2}$ (b) $\dfrac{At^2}{6}$ (c) At^3 (d) $\dfrac{At^3}{2}$ (e) $\dfrac{At^3}{6}$

2.4. Under what condition is the average velocity (which is defined to be the total displacement divided by the time) equal to the average of the initial and final velocities, $(v_\text{i} + v_\text{f})/2$?

(a) The acceleration must be constant.
(b) It is true for other motions besides constant acceleration, but not for all possible motions.
(c) It is true for all possible motions.

2.5. Two cars, with initial speeds of $2v$ and v, lock their brakes and skid to a stop. Assume that the deceleration while skidding is independent of the speed. The ratio of the distances traveled is

(a) 1 (b) 2 (c) 4 (d) 8 (e) 16

2.6. You start from rest and accelerate with a given constant acceleration for a given distance. If you repeat the process with twice the acceleration, then the time required to travel the same distance

(a) remains the same
(b) is doubled
(c) is halved
(d) increases by a factor of $\sqrt{2}$
(e) decreases by a factor of $\sqrt{2}$

2.7. A car travels with constant speed v_0 on a highway. At the instant it passes a stationary police motorcycle, the motorcycle accelerates with constant acceleration and gives chase. What is the speed of the motorcycle when it catches up to the car (in an adjacent lane on the highway)? *Hint*: Draw the v vs. t plots on top of each other.

(a) v_0 (b) $3v_0/2$ (c) $2v_0$ (d) $3v_0$ (e) $4v_0$

2.8. You start from rest and accelerate to a given final speed v_0 after a time T. Your acceleration need not be constant, but assume that it is always positive or zero. If d is the total distance you travel, then the range of possible d values is

(a) $d = v_0T/2$
(b) $0 < d < v_0T/2$
(c) $v_0T/2 < d < v_0T$
(d) $0 < d < v_0T$
(e) $0 < d < \infty$

2.9. You are driving a car that has a maximum acceleration of a. The magnitude of the maximum deceleration is also a. What is the maximum distance that you can travel in time T, assuming that you begin and end at rest?

(a) $2aT^2$ (b) aT^2 (c) $aT^2/2$ (d) $aT^2/4$ (e) $aT^2/8$

2.10. A golf club strikes a ball and sends it sailing through the air. Which of the following choices best describes the sizes of the position, speed, and acceleration of the ball at a moment in the middle of the strike? ("Medium" means a non-tiny and non-huge quantity, on an everyday scale.)

(a) x is tiny, v is medium, a is medium
(b) x is tiny, v is medium, a is huge
(c) x is tiny, v is huge, a is huge
(d) x is medium, v is medium, a is medium
(e) x is medium, v is medium, a is huge

2.11. Which of the following answers is the best estimate for the time it takes an object dropped from rest to fall a vertical mile (about 1600 m)? Ignore air resistance, as usual.

(a) 5 s (b) 10 s (c) 20 s (d) 1 min (e) 5 min

2.12. You throw a ball upward. After half of the time to the highest point, the ball has covered

(a) half the distance to the top
(b) more than half the distance
(c) less than half the distance
(d) It depends on how fast you throw the ball.

2.13. A ball is dropped, and then another ball is dropped from the same spot one second later. As time goes on while the balls are falling, the distance between them (ignoring air resistance, as usual)

(a) decreases
(b) remains the same
(c) increases and approaches a limiting value
(d) increases steadily

2.14. You throw a ball straight upward with initial speed v_0. How long does it take to return to your hand?

(a) $v_0^2/2g$ (b) v_0^2/g (c) $v_0/2g$ (d) v_0/g (e) $2v_0/g$

2.15. Ball 1 has mass m and is fired directly upward with speed v. Ball 2 has mass $2m$ and is fired directly upward with speed $2v$. The ratio of the maximum height of Ball 2 to the maximum height of Ball 1 is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4 (e) 8

2.3 Problems

The first three problems are foundational problems.

2.1. **Area under the curve**

At $t = 0$ an object starts with position x_0 and velocity v_0 and moves with constant acceleration a. Derive the $x(t) = x_0 + v_0 t + at^2/2$ result by finding the area under the v vs. t curve (without using calculus).

2.2. **A kinematic relation**

Use the relations in Eq. (2.3) to show that if an object moves through a displacement d with constant acceleration a, then the initial and final velocities satisfy $v_f^2 - v_i^2 = 2ad$.

2.3. **Maximum height**

If you throw a ball straight upward with initial speed v_0, it reaches a maximum height of $v_0^2/2g$. How many derivations of this result can you think of?

2.4. **Average speeds**

(a) If you ride a bike up a hill at 10 mph, and then down the hill at 20 mph, what is your average speed?
(b) If you go on a bike ride and ride for half the time at 10 mph, and half the time at 20 mph, what is your average speed?

2.5. **Colliding trains**

Two trains, A and B, travel in the same direction on the same set of tracks. A starts at rest at position d, and B starts with velocity v_0 at the origin. A accelerates with acceleration a, and B decelerates with acceleration $-a$. What is the maximum value of v_0 (in terms of d and a) for which the trains don't collide? Make a rough sketch of x vs. t for both trains in the case where they barely collide.