Chapter 3

Kinematics in 2-D (and 3-D)

3.1 Introduction

In this chapter, as in the previous chapter, we won't be concerned with the actual forces that cause an object to move the way it is moving. We will simply take the motion as given, and our goal will be to relate positions, velocities, and accelerations as functions of time. However, since we are now dealing with more general motion in two and three dimensions, we will give one brief mention of forces:

Motion in more than one dimension

Newton's second law (for objects with constant mass) is $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law (which is the topic of Chapter 4) is a *vector* equation. (See Appendix A in Section 13.1 for a review of vectors.) So it really stands for three different equations: $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$. In many cases, these three equations are "decoupled," that is, the x equation has nothing to do with what is going on in the y and z equations, etc. In such cases, we simply have three copies of 1-D motion (or two copies if we're dealing with only two dimensions). So we just need to solve for the three *independent* motions along the three coordinate axes.

Projectile motion

The classic example of independent motions along different axes is projectile motion. Projectile motion is the combination of two separate linear motions. The horizontal motion doesn't affect the vertical motion, and vice versa. Since there is no acceleration in the horizontal direction (ignoring air resistance), the projectile moves with constant velocity in the x direction. And since there is an acceleration of $-g$ in the vertical direction, we can simply copy the results from the previous chapter (in particular, Eq. (2.3) with $a_y = -g$) for the motion in the y direction. We therefore see that if the initial position is (X, Y) and the initial velocity is (V_x, V_y), then the acceleration components

$$a_x = 0 \qquad \text{and} \qquad a_y = -g \tag{3.1}$$

lead to velocity components

$$v_x(t) = V_x \qquad \text{and} \qquad v_y(t) = V_y - gt \tag{3.2}$$

and position components

$$x(t) = X + V_x t \qquad \text{and} \qquad y(t) = Y + V_y t - \frac{1}{2}gt^2. \tag{3.3}$$

Projectile motion is completely described by these equations for the velocity and position components.

Standard projectile results

The initial velocity **V** of a projectile is often described in terms of the initial speed v_0 (we'll use a lowercase v here, since it looks a little nicer) and the launch angle θ with respect to the horizontal. From Fig. 3.1, the initial velocity components are then $V_x = v_0 \cos\theta$ and $V_y = v_0 \sin\theta$, so the velocity components in Eq. (3.2) become

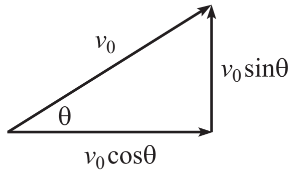


Figure 3.1

$$v_x(t) = v_0 \cos\theta \qquad \text{and} \qquad v_y(t) = v_0 \sin\theta - gt, \tag{3.4}$$

and the positions in Eq. (3.3) become (assuming that the projectile is fired from the origin, so that $(X,Y) = (0,0)$)

$$x(t) = (v_0 \cos\theta)t \qquad \text{and} \qquad y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \tag{3.5}$$

A few results that follow from these expressions are that the time to the maximum height, the maximum height attained, and the total horizontal distance traveled are given by (see Problem 3.1)

$$t_{\text{top}} = \frac{v_0 \sin\theta}{g}, \qquad y_{\max} = \frac{v_0^2 \sin^2\theta}{2g}, \qquad x_{\max} = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}. \tag{3.6}$$

The last of these results holds only if the ground is level (more precisely, if the projectile returns to the height from which it was fired). As usual, we are ignoring air resistance.

Motion along a plane

If an object slides down a frictionless plane inclined at angle θ, the acceleration down the plane is $g\sin\theta$, because the component of **g** (the downward acceleration due to gravity) that points along the plane is $g\sin\theta$; see Fig. 3.2. (There is no acceleration perpendicular to the plane because the normal force from the plane cancels the component of the gravitational force perpendicular to the plane. We'll discuss forces in Chapter 4.) Even though the motion appears to take place in 2-D, we really just have a (tilted) 1-D setup. We effectively have "freefall" motion along the tilted axis, with the acceleration due to gravity being $g\sin\theta$ instead of g. If $\theta = 0$, then the $g\sin\theta$ acceleration along the plane equals 0, and if $\theta = 90°$ it equals g (downward), as expected.

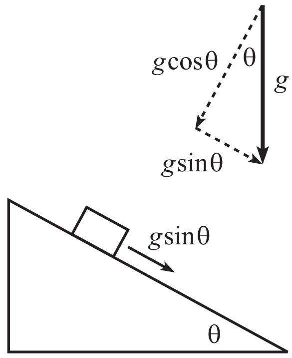


Figure 3.2

More generally, if a projectile flies through the air above an inclined plane, the object's acceleration (which is the downward-pointing vector **g**) can be viewed as the sum of its components along any choice of axes, in particular the $g\sin\theta$ acceleration along the plane and the $g\cos\theta$ acceleration perpendicular to the plane. This way of looking at the downward **g** vector can be very helpful when solving projectile problems involving inclined planes. See Section 13.1.5 in Appendix A for further discussion of vector components.

Circular motion

Another type of 2-D motion is circular motion. If an object is moving in a circle of radius r with speed v at a given instant, then the (inward) radial component of the acceleration vector **a** equals (see Problem 3.2(a))

$$a_{\text{r}} = \frac{v^2}{r}. \tag{3.7}$$

This radially inward acceleration is called the *centripetal* acceleration. If additionally the object is speeding up or slowing down as it moves around the circle, then there is also a tangential component of **a** given by (see Problem 3.2(b))

$$a_{\text{t}} = \frac{dv}{dt}. \tag{3.8}$$

This tangential component is the more intuitive of the two components of the acceleration; it comes from the change in the speed v, just as in the simple case of 1-D motion. The a_{r} component is the less intuitive one; it comes from the change in the *direction* of $\mathbf{v}$. Remember that the acceleration $\mathbf{a} \equiv d\mathbf{v}/dt$ involves the rate of change of the entire vector $\mathbf{v}$, not just the magnitude $v \equiv |\mathbf{v}|$. A vector can change because its magnitude changes or because its direction changes (or both). The former change is associated with a_t, while the latter is associated with a_r.

It is sometimes convenient to work with the *angular frequency* ω (also often called the *angular speed* or *angular velocity*), which is defined to be the rate at which the angle θ around the circle (measured in radians) is swept out. That is, $\omega \equiv d\theta/dt$. If we multiply both sides of this equation by the radius r, we obtain $r\omega = d(r\theta)/dt$. But $r\theta$ is simply the distance s traveled along the circle,[1] so the right-hand side of this equation is ds/dt, which is just the tangential speed v. Hence $r\omega = v \implies \omega = v/r$. In terms of ω, the radial acceleration can be written as

$$a_\mathrm{r} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2 r. \tag{3.9}$$

Similarly, we can define the *angular acceleration* as $\alpha \equiv d\omega/dt \equiv d^2\theta/dt^2$. If we multiply through by r, we obtain $r\alpha = d(r\omega)/dt$. But from the preceding paragraph, $r\omega$ is the tangential speed v. Therefore, $r\alpha = dv/dt$. And since the right-hand side of this equation is just the tangential acceleration, we have

$$a_\mathrm{t} = r\alpha. \tag{3.10}$$

We can summarize most of the results in the previous two paragraphs by saying that the "linear" quantities (distance s, speed v, tangential acceleration a_t) are related to the angular quantities (angle θ, angular speed ω, angular acceleration α) by a factor of r:

$$s = r\theta, \qquad v = r\omega, \qquad a_\mathrm{t} = r\alpha. \tag{3.11}$$

However, the radial acceleration a_r doesn't fit into this pattern.

3.2 Multiple-choice questions

3.1. A bullet is fired horizontally from a gun, and another bullet is simultaneously dropped from the same height. Which bullet hits the ground first? (Ignore air resistance, the curvature of the earth, etc.)

(a) the fired bullet
(b) the dropped bullet
(c) They hit the ground at the same time.

3.2. A projectile is fired at an angle θ with respect to level ground. Is there a point in the motion where the velocity is perpendicular to the acceleration?

Yes No

3.3. A projectile is fired at an angle θ with respect to level ground. Does there exist a θ such that the maximum height attained equals the total horizontal distance traveled?

Yes No

3.4. Is the following reasoning correct? If the launch angle θ of a projectile is increased (while keeping v_0 the same), then the initial v_y velocity component increases, so the time in the air increases, so the total horizontal distance traveled increases.

Yes No

[1] This is true by the definition of a radian. If you take a piece of string with a length of one radius and lay it out along the circumference of a circle, then it subtends an angle of one radian, by definition. So each radian of angle is worth one radius of distance. The total distance s along the circumference is therefore obtained by multiplying the number of radians (that is, the number of "radiuses") by the length of the radius.

3.5. A ball is thrown at an angle θ with speed v_0. A second ball is simultaneously thrown straight upward from the point on the ground directly below the top of the first ball's parabolic motion. How fast should this second ball be thrown if you want it to collide with the first ball?

(a) $v_0/2$ (b) $v_0/\sqrt{2}$ (c) v_0 (d) $v_0 \cos\theta$ (e) $v_0 \sin\theta$

3.6. A wall has height h and is a distance ℓ away. You wish to throw a ball over the wall with a trajectory such that the ball barely clears the wall at the top of its parabolic motion. What initial speed is required? (Don't solve this from scratch, just check special cases. See Problem 3.10 for a quantitative solution.)

(a) $\sqrt{2gh}$
(b) $\sqrt{4gh}$
(c) $\sqrt{g\ell^2/2h}$
(d) $\sqrt{2gh + g\ell^2/2h}$
(e) $\sqrt{4gh + g\ell^2/2h}$

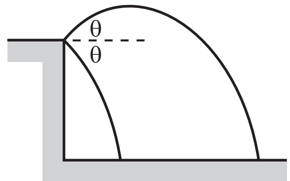


Figure 3.3

3.7. Two balls are thrown with the same speed v_0 from the top of a cliff. The angles of their initial velocities are θ above and below the horizontal, as shown in Fig. 3.3. How much farther along the ground does the top ball hit than the bottom ball? *Hint*: The two trajectories have a part in common. No calculations necessary!

(a) $2v_0^2/g$
(b) $2v_0^2 \sin\theta/g$
(c) $2v_0^2 \cos\theta/g$
(d) $2v_0^2 \sin\theta\cos\theta/g$
(e) $2v_0^2 \sin^2\theta\cos^2\theta/g$

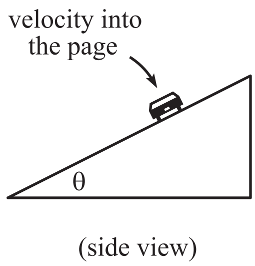


Figure 3.4

3.8. A racecar travels in a horizontal circle at constant speed around a circular banked track. A side view is shown in Fig. 3.4. (The triangle is a cross-sectional slice of the track; the car is heading into the page at the instant shown.) The direction of the racecar's acceleration is

(a) horizontal rightward
(b) horizontal leftward
(c) downward along the plane
(d) upward perpendicular to the plane
(e) The acceleration is zero.

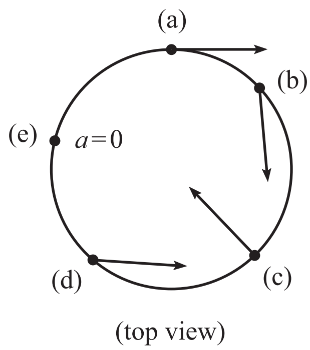


Figure 3.5

3.9. Which one of the following statements is *not* true for *uniform* (constant speed) circular motion?

(a) **v** is perpendicular to **r**.
(b) **v** is perpendicular to **a**.
(c) **v** has magnitude $R\omega$ and points in the **r** direction.
(d) **a** has magnitude v^2/R and points in the negative **r** direction.
(e) **a** has magnitude $\omega^2 R$ and points in the negative **r** direction.

3.10. A car travels around a horizontal circular track, *not* at constant speed. The acceleration vectors at five different points are shown in Fig. 3.5 (the four nonzero vectors have equal length). At which of these points is the car's speed the largest?

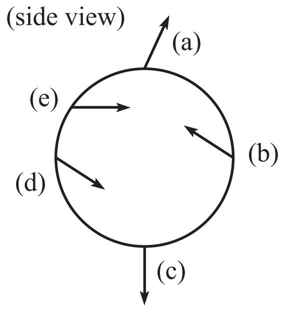


Figure 3.6

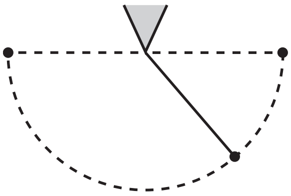
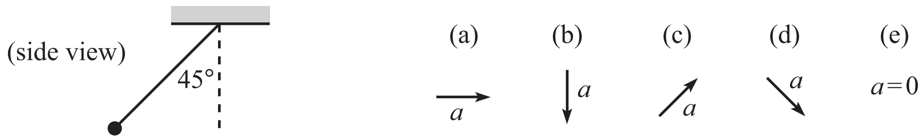


Figure 3.7

3.11. A bead is given an initial velocity and then circles indefinitely around a frictionless vertical hoop. Only one of the vectors in Fig. 3.6 is a possible acceleration vector at the given point. Which one?

3.12. A pendulum is released from rest at an angle of 45° with respect to the vertical, as shown below. Which vector shows the direction of the initial acceleration?



3.13. A pendulum swings back and forth between the two horizontal positions shown in Fig. 3.7. The acceleration is vertical (g downward) at the highest points, and is also vertical (upward) at the lowest point.

(a) There is at least one additional point where the acceleration is vertical.
(b) There is at least one point where the acceleration is horizontal.
(c) There is at least one point where the acceleration is zero.
(d) None of the above

3.3 Problems

The first three problems are foundational problems.

3.1. **A few projectile results**

On level ground, a projectile is fired at angle θ with speed v_0. Derive the expressions in Eq. (3.6). That is, find (a) the time to the maximum height, (b) the maximum height attained, and (c) the total horizontal distance traveled.

3.2. **Radial and tangential accelerations**

(a) If an object moves in a circle at constant speed v (uniform circular motion), show that the acceleration points radially inward with magnitude $a_{\rm r} = v^2/r$. Do this by drawing the position and velocity vectors at two nearby times and then making use of similar triangles.
(b) If the object speeds up or slows down as it moves around in the circle, then the acceleration also has a tangential component. Show that this component is given by $a_{\rm t} = dv/dt$.

3.3. **Radial and tangential accelerations, again**

A particle moves in a circle, not necessarily at constant speed. Its coordinates are given by $(x, y) = (R\cos\theta, R\sin\theta)$, where $\theta \equiv \theta(t)$ is an arbitrary function of t. Take two time derivatives of these coordinates to find the acceleration vector, and then explain why the result is consistent with the $a_{\rm r}$ and $a_{\rm t}$ magnitudes derived in Problem 3.2.

3.4. **Movie replica**

(a) A movie director wants to shoot a certain scene by building a detailed replica of the actual setup. The replica is 1/100 the size of the real thing. In the scene, a person jumps from rest from a tall building (into a net, so it has a happy ending). If the director films a tiny doll being dropped from the replica building, by what factor should the film be sped up or slowed down when played back, so that the falling person looks realistic to someone watching the movie? (Assume that the motion is essentially vertical.)

(b) The director now wants to have a little toy car zoom toward a cliff in the replica (with the same scale factor of 1/100) and then sail over the edge down to the ground below (don't worry, the story has the driver bail out in time). Assume that the goal is to have the movie viewer think that the car is traveling at 50 mph before it goes over the cliff. As in part (a), by what factor should the film be sped up or slowed down when played back? What should the speed of the toy car be as it approaches the cliff in the replica?

3.5. **Doubling gravity**

A ball is thrown with speed v at an angle θ with respect to the horizontal ground. At the highest point in the motion, the strength of gravity is somehow magically doubled. What is the total horizontal distance traveled by the ball?

3.6. **Ratio of heights**

From the standard $d = gt^2/2$ expression for freefall from rest, we see that if the falling time is doubled, the falling distance is quadrupled. Use this fact to find the ratio of the height of the top of projectile motion (point A in Fig. 3.8) to the height where the projectile would be if gravity were turned off (point B in the figure). Two suggestive distances are drawn.

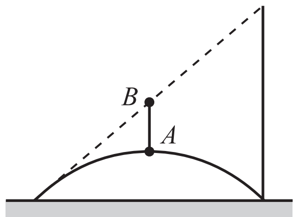


Figure 3.8

3.7. **Hitting horizontally**

A ball is thrown with speed v_0 at an angle θ with respect to the horizontal. It is thrown from a point that is a distance ℓ from the base of a cliff that has a height also equal to ℓ. What should θ and v_0 be so that the ball hits the corner of the cliff moving horizontally, as shown in Fig. 3.9?

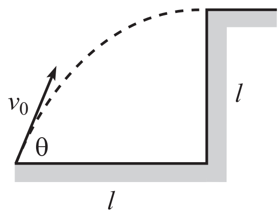


Figure 3.9

3.8. **Projectile and tube**

A projectile is fired horizontally with speed v_0 from the top of a cliff of height h. It immediately enters a fixed tube with length x, as shown in Fig. 3.10. There is friction between the projectile and the tube, the effect of which is to make the projectile decelerate with constant acceleration $-a$ (a is a positive quantity here). After the projectile leaves the tube, it undergoes normal projectile motion down to the ground.

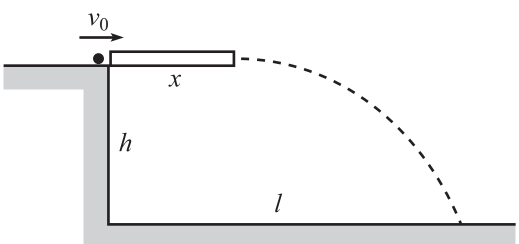


Figure 3.10

(a) What is the total horizontal distance (call it ℓ) that the projectile travels, measured from the base of the cliff? Give your answer in terms of x, h, v_0, g, and a.

(b) What value of x yields the maximum value of ℓ?

3.9. **Car in the mud**

A wheel is stuck in the mud, spinning in place. The radius is R, and the points on the rim are moving with speed v. Bits of the mud depart from the wheel at various random locations. In particular, some bits become unstuck from the rim in the upper left quadrant, as shown in Fig. 3.11. What should θ be so that the mud reaches the maximum possible height (above the ground) as it flies through the air? What is this maximum height? You may assume $v^2 > gR$.

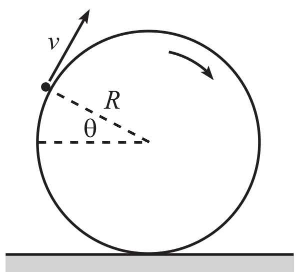


Figure 3.11

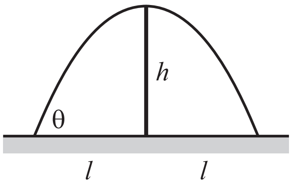


Figure 3.12

3.10. **Clearing a wall**

(a) You wish to throw a ball to a friend who is a distance 2ℓ away, and you want the ball to just barely clear a wall of height h that is located halfway to your friend, as shown in Fig. 3.12. At what angle θ should you throw the ball?

(b) What initial speed v_0 is required? What value of h (in terms of ℓ) yields the minimum v_0? What is the value of θ in this minimum case?

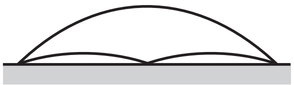


Figure 3.13

3.11. **Bounce throw**

A person throws a ball with speed v_0 at a 45° angle and hits a given target. How much quicker does the ball get to the target if the person instead throws the ball with the same speed v_0 but at the angle that makes the trajectory consist of two identical bumps, as shown in Fig. 3.13? (Assume unrealistically that there is no loss in speed at the bounce.)

3.12. **Maximum bounce**

A ball is dropped from rest at height h. At height y, it bounces elastically (that is, without losing any speed) off a board. The board is inclined at the angle (which happens to be 45°) that makes the ball bounce off horizontally. In terms of h, what should y be so that the ball hits the ground as far off to the side as possible? What is the horizontal distance in this optimal case?

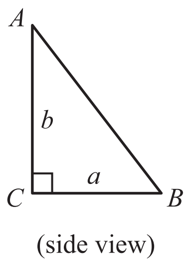


Figure 3.14

3.13. **Falling along a right triangle**

In the vertical right triangle shown in Fig. 3.14, a particle falls from A to B either along the hypotenuse, or along the two legs (lengths a and b) via point C. There is no friction anywhere.

(a) What is the time (call it t_H) if the particle travels along the hypotenuse?

(b) What is the time (call it t_L) if the particle travels along the legs? Assume that at point C there is an infinitesimal curved arc that allows the direction of the particle's motion to change from vertical to horizontal without any change in speed.

(c) Verify that $t_H = t_L$ when $a = 0$.

(d) How do t_H and t_L compare in the limit $b \ll a$?

(e) Excluding the $a = 0$ case, what triangle shape yields $t_H = t_L$?

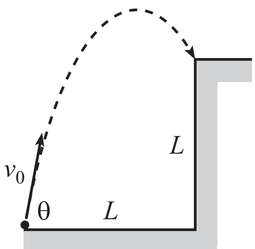


Figure 3.15

3.14. **Throwing to a cliff**

A ball is thrown at an angle θ up to the top of a cliff of height L, from a point a distance L from the base, as shown in Fig. 3.15.

(a) As a function of θ, what initial speed causes the ball to land right at the edge of the cliff?

(b) There are two special values of θ for which you can check your result. Check these.

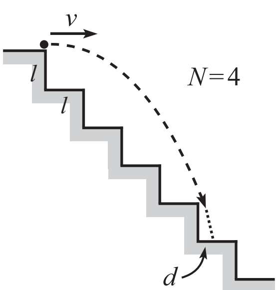


Figure 3.16

3.15. **Throwing from a cliff**

A ball is thrown with speed v at angle θ (with respect to horizontal) from the top of a cliff of height h. How far from the base of the cliff does the ball land? (The ground is horizontal below the cliff.)

3.16. **Throwing on stairs**

A ball is thrown horizontally with speed v from the floor at the top of some stairs. The width and height of each step are both equal to ℓ.

(a) What should v be so that the ball barely clears the corner of the step that is N steps down? Fig. 3.16 shows the case where $N = 4$.

(b) How far along the next step (the distance d in the figure) does the ball hit?

(c) What is d in the limit $N \to \infty$?

(d) Find the components of the ball's velocity when it grazes the corner, and then explain why their ratio is consistent with your answer to part (c).

3.17. **Bullet and sphere**

A bullet is fired horizontally with speed v_0 from the top of a fixed sphere with radius R, as shown in Fig. 3.17. What is the minimum value of v_0 for which the bullet doesn't touch the sphere after it is fired? (*Hint*: Find y as a function of x for the projectile motion, and also find y as a function of x for the sphere near the top where x is small; you'll need to make a Taylor-series approximation. Then compare your two results.) For the v_0 you just found, where does the bullet hit the ground?

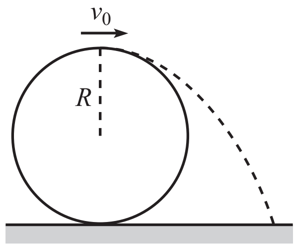


Figure 3.17

3.18. **Throwing on an inclined plane**

You throw a ball from a plane inclined at angle θ. The initial velocity is perpendicular to the plane, as shown in Fig. 3.18. Consider the point P on the trajectory that is farthest from the plane. For what angle θ does P have the same height as the starting point? (For the case shown in the figure, P is higher.) Answer this in two steps:

(a) Give a continuity argument that explains why such a θ should in fact exist.

(b) Find θ. In getting a handle on where (and when) P is, it is helpful to use a tilted coordinate system and to isolate what is happening in the direction perpendicular to the plane.

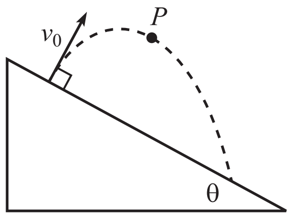


Figure 3.18

3.19. **Ball landing on a block**

A block is fired up along a frictionless plane inclined at angle β, and a ball is simultaneously thrown upward at angle θ (both β and θ are measured with respect to the horizontal). The objects start at the same location, as shown in Fig. 3.19. What should θ be in terms of β if you want the ball to land on the block at the instant the block reaches its maximum height on the plane? (An implicit equation is fine.) What is θ if β equals 45°? (You might think that we've forgotten to give you information about the initial speeds, but it turns out that you don't need these to solve the problem.)

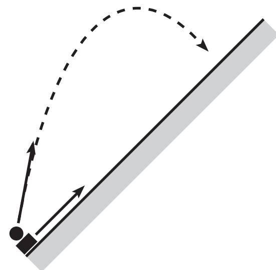


Figure 3.19

3.20. ***g*'s in a washer**

A typical front-loading washing machine might have a radius of 0.3 m and a spin cycle of 1000 revolutions per minute. What is the acceleration of a point on the surface of the drum at this spin rate? How many g's is this equivalent to?

3.21. **Acceleration after one revolution**

A car starts from rest on a circular track with radius R and then accelerates with constant tangential acceleration a_t. At the moment the car has completed one revolution, what angle does the total acceleration vector make with the radial direction? You should find that your answer doesn't depend on a_t or R. Explain why you don't have to actually solve the problem to know this.

3.22. **Equal acceleration components**

An object moves in a circular path of radius R. At $t = 0$, it has speed v_0. From this point on, the magnitudes of the radial and tangential accelerations are arranged to be equal at all times.

(a) As functions of time, find the speed and the distance traveled.

(b) If the tangential acceleration is positive (that is, if the object is speeding up), there is special value for t. What is it, and why is it special?