Chapter 10

Oscillations

10.1 Introduction

Hooke's law

Consider a force that depends on position according to

$$F(x) = -kx. \tag{10.1}$$

A force of this form (proportional to $-x$) is said to obey *Hooke's law*. The force is negative if x is positive, and positive if x is negative. So it is a restoring force; it is always directed back toward the equilibrium point (the origin). Since $F = -dU/dx$, the associated Hooke's-law potential energy is

$$U(x) = \frac{1}{2}kx^2. \tag{10.2}$$

Plots of $F(x)$ and $U(x)$ are shown in Fig. 10.1. Hooke's-law forces are extremely important because they are ubiquitous in nature, due to the fact that near an equilibrium point (which is where systems generally hang out), any potential-energy function looks essentially like a parabola; see Problem 10.1. So we can always approximate $U(x)$ as $kx^2/2$ for some value of k (although this approximation will, of course, break down for sufficiently large x). We'll often use a spring (see page 70) as an example of a Hooke's-law force, but there are countless other examples – pendulums, objects floating in water, electrical circuits, etc.

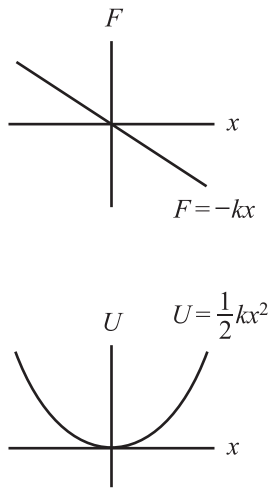

Figure 10.1

Simple harmonic motion

If we have a Hooke's-law force, $F = -kx$, then Newton's second law becomes

$$F = ma \implies -kx = ma \implies m\ddot{x} = -kx. \tag{10.3}$$

What is the solution, $x(t)$, to this equation? There are many ways to solve it, but the easiest way is to just note that we want to find a function whose second derivative is proportional to the negative of itself. And we know that sines and cosines have this property. So let's try a solution of the form,

$$x(t) = A\cos(\omega t + \phi). \tag{10.4}$$

Plugging this into $m\ddot{x} = -kx$ gives

$$m\big(-\omega^2 A\cos(\omega t + \phi)\big) = -kA\cos(\omega t + \phi) \implies \omega = \sqrt{\frac{k}{m}}, \tag{10.5}$$

where we have canceled the common factor of $A\cos(\omega t + \phi)$. We see that the expression for $x(t)$ in Eq. (10.4) is a solution to $F = ma$, provided that $\omega = \sqrt{k/m}$. A and ϕ can take on arbitrary values, and the solution is still valid. The sinusoidal motion in Eq. (10.4) is called *simple harmonic motion*. The quantity ω is a very important one:

- ω is the *angular frequency* of the oscillatory motion. The argument $\omega t + \phi$ of the cosine in Eq. (10.4) is an angle measured in radians (not degrees), and ω is the rate at which this angle increases. (So you could also call ω the "angular speed" or "angular velocity." But "angular frequency" is more common.)

You can quickly verify that $x(t + 2\pi/\omega) = x(t)$, which means that the motion repeats itself after every time interval of

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}\,. \tag{10.6}$$

The time T is the *period* of the oscillation. The *frequency* of the oscillation, in cycles per second (that is, in "hertz") is

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,. \tag{10.7}$$

We see that ω is larger than ν by a factor of 2π. This is due to the fact that there are 2π radians in each cycle. *Note*: The term "frequency," instead of the full "angular frequency," is often used when referring to ω. So the word "frequency" is somewhat ambiguous in practice. But as long as you write down the symbol ω or ν, it will be clear what you're referring to.

Initial conditions

Aside from the time t, there are three parameters in the expression for $x(t)$ in Eq. (10.4), namely A, ω, and ϕ. The angular frequency ω is determined by k and m via Eq. (10.5), and these are in turn determined by the setup; someone has to give you a particular spring and a particular mass. In contrast, the values of A and ϕ are *not* determined by k and m; that is, they are not determined by the setup. The $x(t)$ in Eq. (10.4) is a solution to Eq. (10.3) for *any* arbitrary values of A and ϕ. If we want to determine what the actual values of these two parameters are, we must specify two *initial conditions,* most commonly the initial position x_0 and the initial velocity v_0 at time $t = 0$. Differentiating Eq. (10.4), we see that the velocity is given by $v(t) = dx/dt = -\omega A\sin(\omega t + \phi)$. Letting $t = 0$ in this expression for $v(t)$ and also in the expression for $x(t)$ in Eq. (10.4), we find that the initial conditions at $t = 0$, namely $x_0 = x(0)$ and $v_0 = v(0)$, can be written as

$$x_0 = A\cos\phi \qquad \text{and} \qquad v_0 = -\omega A\sin\phi. \tag{10.8}$$

These two equations can be solved for A and ϕ (see Problem 10.2). These two parameters are:

- A is the *amplitude* of the motion. It is the largest value that $x(t)$ achieves; $x(t)$ bounces back and forth between A and $-A$.

- ϕ is the *phase* of the motion. It dictates where in the cycle of motion the mass is at $t = 0$. A larger value of ϕ means that the mass is further along in the cycle at $t = 0$. The phase ϕ depends on the choice of origin for time. If we pick a different instant when we set our clock equal to zero, we obtain a different value of ϕ for the same motion.

Other forms of *x(t)*

Equation (10.4) is one way to write the solution to Eq. (10.3). But there are others, for example,

$$x(t) = B\sin(\omega t + \psi) \qquad \text{or} \qquad x(t) = C\cos\omega t + D\sin\omega t. \tag{10.9}$$

These satisfy Eq. (10.3), provided that we again have $\omega = \sqrt{k/m}$. If $x(t)$ is written in the $C\cos\omega t + D\sin\omega t$ form, it isn't obvious what the amplitude of the motion is. But you can show in Problem 10.3 that the amplitude is $\sqrt{C^2 + D^2}$.

Note that no matter which form of $x(t)$ we use, there are always two free parameters (A and ϕ; or B and ψ; or C and D). These two parameters are determined by the two initial conditions. The two parameters in one expression are related to the two in another. For example, using the trig sum formula for sine, and comparing the two expressions in Eq. (10.9), we see that $C = B\sin\psi$ and $D = B\cos\psi$.

The $C\cos\omega t + D\sin\omega t$ form of $x(t)$ is the most amenable to the application of initial conditions. The $x(0) = x_0$ condition quickly gives $C = x_0$. And using $v(t) = dx/dt = -\omega C\sin\omega t + \omega D\cos\omega t$, the $v(0) = v_0$ condition quickly gives $\omega D = v_0 \implies D = v_0/\omega$. Depending on the setup and goal of a given problem, a particular form of $x(t)$ might make things easier than the other forms.

We know that sines and cosines are solutions to the $F = ma$ equation in Eq. (10.3). The fact that a sine plus a cosine (the second form given in Eq. (10.9)) is again a solution is due to the *linearity* of the $F = ma$ equation; x appears only to the first power (the number of derivatives doesn't matter). This linearity implies that any linear combination of two solutions is again a solution, as you can show in Problem 10.4.

Energy

Consider the standard system of a mass on a spring oscillating back and forth with amplitude A and angular frequency ω. The energy will slosh back and forth between being only potential (at the points of maximal stretch or compression, where the mass is instantaneously at rest) and being only kinetic (when the mass passes through the equilibrium point, where the spring is neither stretched nor compressed). Assuming that we have an ideal system with no damping forces, energy should be conserved. To determine what the constant value of the energy is, we can conveniently look at the $x = \pm A$ points, where the energy is all potential; the energy is therefore $kA^2/2$. And indeed, you can show in Problem 10.5 that the equation

$$\frac{kx^2}{2} + \frac{mv^2}{2} = \frac{kA^2}{2} \tag{10.10}$$

is satisfied for all values of t. That is, the sum of the potential and kinetic energies takes on the constant value of $kA^2/2$.

Generality

As mentioned above, Hooke's law is ubiquitous in nature because near an equilibrium point, any potential-energy function looks basically like a parabola. A spring is one example. Another common example is a pendulum. You can show in Problem 10.7 that the equation for the angle θ (measured relative to the vertical) of a point-mass pendulum with length ℓ is $\ddot{\theta} = -(g/\ell)\theta$, assuming that the amplitude of the oscillations is small. Having derived this equation, there is no need to waste any time solving it, because it takes exactly the same form as Eq. (10.3), if we rewrite that equation as $\ddot{x} = -(k/m)x$. The only difference is that k/m is replaced by g/ℓ (and x is replaced by θ). Therefore, since the $\ddot{x} = -(k/m)x$ equation has a solution of the form given in Eq. (10.4), with the condition that ω must be equal to $\sqrt{k/m}$, we conclude that the $\ddot{\theta} = -(g/\ell)\theta$ equation must also have a solution of the form in Eq. (10.4), but with ω now equal to $\sqrt{g/\ell}$.

In general, if you solve a problem that involves some variable z (by using $F = ma$, $\tau = I\alpha$, a conservation statement, or whatever), and if at the end of the day you arrive at an equation of the form,

$$\ddot{z} = -(\text{something})z, \tag{10.11}$$

then you can simply write down the answer with no further work: you know that $z(t)$ undergoes oscillatory motion given by Eq. (10.4), with an angular frequency ω equal to

$$\omega = \sqrt{\text{something}}. \tag{10.12}$$

10.2 Multiple-choice questions

10.1. The expression for the angle of an oscillating pendulum (with a small amplitude) is $\theta(t) = \theta_0 \cos(\omega t + \phi)$.

True or false: The angular velocity $\dot{\theta} \equiv d\theta/dt$ is equal to the angular velocity ω.

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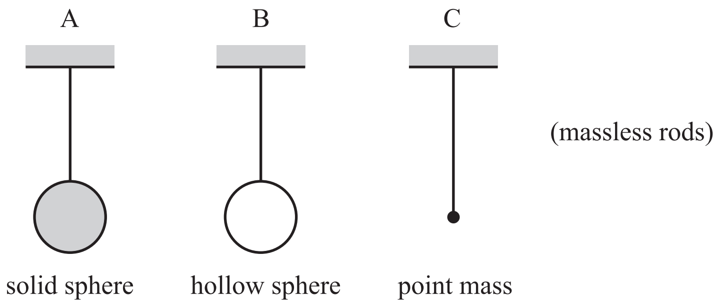
10.2. The position of a particle is given by $x(t) = x_0 \cos(\omega t + \pi/6)$, where $x_0 = 6\,\text{m}$ and $\omega = 2\ \text{s}^{-1}$. The maximum speed the particle achieves is

(a) 3 m/s (b) 6 m/s (c) 12 m/s (d) 24 m/s (e) 36 m/s

10.3. If you plot the functions $\cos\omega t$ and $\cos(\omega t + \pi/4)$, the $\cos(\omega t + \pi/4)$ curve will be shifted relative to the $\cos\omega t$ curve. Which way is it shifted?

(a) rightward (b) leftward

10.4. Which of the pendulums shown below has the largest frequency of small oscillations? The objects all have the same mass, and the CM's are all the same distance from the support. The (massless) rods are glued to each object to form rigid systems. The result from Problem 10.8 will come in handy (but don't look at the remark in the solution).



(a) A (b) B (c) C (d) They all have the same frequency.

10.3 Problems

The first eight problems are foundational problems.

10.1. **Quadratic potential near a minimum**

Use a Taylor series to show that near a local minimum, any (well-behaved) function, in particular the potential energy $U(x)$, looks essentially like a quadratic function (a parabola).

10.2. **Initial conditions**

Solve the equations in Eq. (10.8) for A and ϕ.

10.3. **Amplitude**

Show that the amplitude of the motion given by $x(t) = C\cos\omega t + D\sin\omega t$ equals $\sqrt{C^2 + D^2}$. You can do this by taking the derivative of $x(t)$ to find the maximum.

10.4. **Linearity**

Homogeneous linear differential equations have the property that the sum, or any linear combination, of two solutions is again a solution. ("Homogeneous" means that there is a zero on one side of the equation.) Consider, for example, the second-order linear differential equation (although the property holds for any order),

$$A\ddot{x} + B\dot{x} + Cx = 0. \tag{10.13}$$

Show that if $x_1(t)$ and $x_2(t)$ are solutions, then the sum $x_1(t) + x_2(t)$ is also a solution. Show that this property does *not* hold for the *nonlinear* differential equation, $A\ddot{x} + B\dot{x}^2 + Cx = 0$.

10.5. **Conservation of energy**

Using the form of $x(t)$ given in Eq. (10.4) (along with the derivative, $v(t) = -\omega A \sin(\omega t + \phi)$), show that the total energy (kinetic plus potential) of simple harmonic motion takes on the constant value of $kA^2/2$.

10.6. **Circular motion and simple harmonic motion**

(a) In the x-y plane, a particle moves in a circular path with radius A, centered at the origin, as shown in Fig. 10.2. The angular velocity ω is constant, so the angle with respect to the x axis is given by ωt (or technically $\omega t + \phi$, but the phase won't be important here). Using what you know about circular motion, determine the projections onto the x axis of the position **r**, velocity **v**, and acceleration **a** vectors.

(b) Show that the three projections you found are equal, respectively, to the $x(t)$, $v(t)$, and $a(t)$ functions for a particle undergoing simple harmonic motion, with position given by $x(t) = A\cos\omega t$. In other words, show that the projection of uniform circular motion onto a diameter is simple harmonic motion. (Imagine shining a light downward and looking at the shadow of the particle on the x axis.)

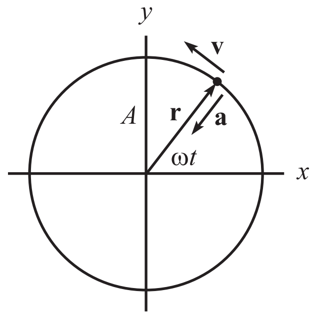


Figure 10.2

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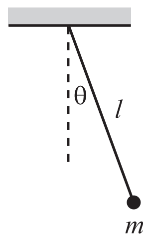

Figure 10.3

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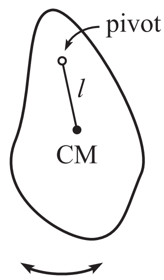

Figure 10.4

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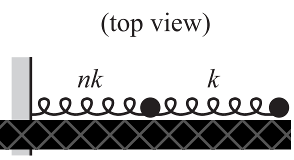

Figure 10.5

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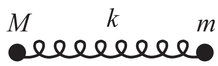

Figure 10.6