2024-1

1. Let $f(x)$ be a continuously differentiable function on the interval $(0, \infty)$ such that $f(1) = 2$ and

$$\lim_{t \to x} \frac{t^{10} f(x) - x^{10} f(t)}{t^9 - x^9} = 1$$

for each $x > 0$. Then, for all $x > 0$, $f(x)$ is equal to

   (a) $\frac{31}{11x} - \frac{9}{11} x^{10}$

   (b) $\frac{9}{11x} + \frac{13}{11} x^{10}$

   (c) $\frac{-9}{11x} + \frac{31}{11} x^{10}$

   (d) $\frac{13}{11x} + \frac{9}{11} x^{10}$

2. A student appears for a quiz consisting of only true-false type questions and answers all the questions. The student knows the answers of some questions and guesses the answers for the remaining questions. Whenever the student knows the answer of a question, he gives the correct answer. Assume that the probability of the student giving the correct answer for a question, given that he has guessed it, is $\frac{1}{2}$. Also assume that the probability of the answer for a question being guessed, given that the student's answer is correct, is $\frac{1}{6}$. Then the probability that the student knows the answer of a randomly chosen question is

   (a) $\frac{1}{12}$

   (b) $\frac{1}{7}$

   (c) $\frac{5}{7}$

   (d) $\frac{5}{12}$

3. Let $\frac{\pi}{2} < x < \pi$ be such that $\cot x = \frac{-5}{\sqrt{11}}$. Then

$$\left(\sin \frac{11x}{2} \right) (\sin 6x - \cos 6x) + \left(\cos \frac{11x}{2} \right) (\sin 6x + \cos 6x)$$

is equal to

(a) $\frac{\sqrt{11}-1}{2\sqrt{3}}$

(b) $\frac{\sqrt{11}+1}{2\sqrt{3}}$

(c) $\frac{\sqrt{11}+1}{3\sqrt{2}}$

(d) $\frac{\sqrt{11}-1}{3\sqrt{2}}$

4. Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Let $S(p,q)$ be a point in the first quadrant such that

$$\frac{p^2}{9} + \frac{q^2}{4} > 1.$$

Two tangents are drawn from S to the ellipse, of which one meets the ellipse at one end point of the minor axis and the other meets the ellipse at a point T in the fourth quadrant. Let R be the vertex of the ellipse with positive x-coordinate and O be the center of the ellipse. If the area of the triangle $\triangle ORT$ is $\frac{3}{2}$, then which of the following options is correct?

(a) $q = 2, \quad p = 3\sqrt{3}$

(b) $q = 2, \quad p = 4\sqrt{3}$

(c) $q = 1, \quad p = 5\sqrt{3}$

(d) $q = 1, \quad p = 6\sqrt{3}$

5. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, $T_1 = \{(-1 + \sqrt{2})^n : n \in \mathbb{N}\}$, and $T_2 = \{(1 + \sqrt{2})^n : n \in \mathbb{N}\}$. Then which of the following statements is (are) TRUE?

(a) $\mathbb{Z} \cup T_1 \cup T_2 \subseteq S$

(b) $T_1 \cap \left(0, \frac{1}{2024}\right) = \phi$, where ϕ denotes the empty set.

(c) $T_2 \cap (2024, \infty) \neq \phi$

(d) For any given $a, b \in \mathbb{Z}$, $\cos\left(\pi(a + b\sqrt{2})\right) + i\sin\left(\pi(a + b\sqrt{2})\right) \in \mathbb{Z}$ if and only if $b = 0$, where $i = \sqrt{-1}$.

6. Let $\mathbb{R}^2$ denote the Euclidean space. Let $S = \{(a, b, c) : a, b, c \in \mathbb{R}, ax^2 + 2bxy + cy^2 > 0$ for all $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}\}$. Then which of the following statements is (are) TRUE?

(a) $\left(2, \frac{7}{2}, 6\right) \in S$

(b) If $\left(3, b, \frac{1}{12}\right) \in S$, then $|2b| < 1$.

(c) For any given $(a, b, c) \in S$, the system of linear equations

$$ax + by = 1 \quad bx + cy \qquad\qquad = -1$$

has a unique solution.

(d) For any given $(a, b, c) \in S$, the system of linear equations

$$(a + 1)x + by = 0$$
$$bx + (c + 1)y = 0$$

has a unique solution.

7. Let $\mathbb{R}^3$ denote the three-dimensional space. Take two points $P = (1, 2, 3)$ and $Q = (4, 2, 7)$. Let $dist(X, Y)$ denote the distance between two points X and Y in $\mathbb{R}^3$. Let

$$S = \left\{ X \in \mathbb{R}^3 \ : \ (dist(X, P))^2 - (dist(X, Q))^2 = 50 \right\}$$

$$T = \left\{ Y \in \mathbb{R}^3 \ : \ (dist(Y, Q))^2 - (dist(Y, P))^2 = 50 \right\}$$

Then which of the following statements is (are) TRUE?

(a) There is a triangle whose area is 1 and all of whose vertices are from S.

(b) There are two distinct points L and M in T such that each point on the line segment LM is also in T.

(c) There are infinitely many rectangles of perimeter 48, two of whose vertices are from S and the other two vertices are from T.

(d) There is a square of perimeter 48, two of whose vertices are from S and the other two vertices are from T.

8. Let $a = 3\sqrt{2}$ and $b = \frac{1}{5^{1/6}\sqrt{6}}$. If $x, y \in \mathbb{R}$ are such that

$$3x + 2y = \log_a \left(18^{5/4} \right)$$

$$2x - y = \log_b \left(\sqrt{1080} \right)$$

then $4x + 5y$ is equal to _____.

9. Let $f(x) = x^4 + ax^3 + bx^2 + c$ be a polynomial with real coefficients such that $f(1) = -9$. Suppose that $i\sqrt{3}$ is a root of the equation

$$4x^3 + 3ax^2 + 2bx = 0, \quad \text{where } i = \sqrt{-1}.$$

If $\alpha_1, \alpha_2, \alpha_3$, and α_4 are all the roots of the equation $f(x) = 0$, then

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2$$

is equal to ____.

10. Let $S = \left\{ A = \begin{pmatrix} 0 \end{pmatrix} 1c1ad1be \right.$

: a, b, c, d, e $\in \{0, 1\}$ and $|A| \in \{-1, 1\}$, where $|A|$ denotes the determinant of A. Then the number of elements in S is ____.

11. A group of 9 students, $s_1, s_2, \ldots, s_9$, is to be divided into three teams X, Y, Z of sizes 2, 3, and 4, respectively. Suppose that s_1 cannot be selected for the team X, and s_2 cannot be selected for the team Y. Then the number of ways to form such teams is ____.

12. Let

$$\overrightarrow{OP} = \frac{\alpha - 1}{\alpha}\hat{i} + \hat{j} + \hat{k}, \quad \overrightarrow{OQ} = \hat{i} + \frac{\beta - 1}{\beta}\hat{j} + \hat{k}, \quad \overrightarrow{OR} = \hat{i} + \hat{j} + \frac{1}{2}\hat{k}$$

be three vectors, where $\alpha, \beta \in \mathbb{R} - \{0\}$ and O denotes the origin.

If

$$\left(\overrightarrow{OP} \times \overrightarrow{OQ} \right) \cdot \overrightarrow{OR} = 0$$

and the point $(\alpha, \beta, 2)$ lies on the plane

$$3x + 3y - z + l = 0,$$

then the value of l is ____.

13. Let X be a random variable, and let $P(X = x)$ denote the probability that X takes the value x.

Suppose that the points $(x, P(X = x))$, where $x = 0, 1, 2, 3, 4$, lie on a fixed straight line in the xy-plane, and $P(X = x) \neq 0$ for all $x \in \mathbb{R} - \{0, 1, 2, 3, 4\}$.

If the mean of X is $\frac{5}{2}$, and the variance of X is α, then the value of 24α is

____.

14. Let α and β be the distinct roots of the equation $x^2 + x - 1 = 0$. Consider the set $T = \{1, \alpha, \beta\}$.

For a 3×3 matrix $M = (a_{ij})_{3\times3}$, define $R_i = a_{i1} + a_{i2} + a_{i3}$ and $C_j = a_{1j} + a_{2j} + a_{3j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Match each entry in List-I to the correct entry in List-II.

List-I   **List-II**

(P) The number of matrices $M = (a_{ij})_{3\times3}$ with all entries in T such that $R_i = C_j = 0$ for all i, j, is   [1] 1

(Q) The number of symmetric matrices $M = (a_{ij})_{3\times3}$ with all entries in T such that $C_j = 0$ for all j, is   [2] 12

(R) Let $M = (a_{ij})_{3\times3}$ be a skew-symmetric matrix such that $a_{ij} \in T$ for $i > j$. Then the number of elements in the set   [3] infinite

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R}, M = \begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & a_{23} \\ 0 & -a_{23} & 0 \end{pmatrix} \right\} \text{ is}$$

(S) Let $M = (a_{ij})_{3\times3}$ be a matrix with all entries in T such that $R_i = 0$ for all i. Then the absolute value of the determinant of M is   [4] 6

                                                                            [5] 0

The correct option is

    (a) $P \rightarrow 4$, $Q \rightarrow 2$, $R \rightarrow 5$, $S \rightarrow 1$

    (b) $P \rightarrow 2$, $Q \rightarrow 4$, $R \rightarrow 1$, $S \rightarrow 5$

    (c) $P \rightarrow 2$, $Q \rightarrow 4$, $R \rightarrow 3$, $S \rightarrow 5$

    (d) $P \rightarrow 1$, $Q \rightarrow 5$, $R \rightarrow 3$, $S \rightarrow 4$

15. Let the straight line $y = 2x$ touch a circle with center $(0, a)$, $a > 0$, and radius r at a point A_1. Let B_1 be the point on the circle such that the line segment $A_1 B_1$ is a diameter of the circle. Let $a + r = 5 + \sqrt{5}$.

Match each entry in List-I to the correct entry in List-II.

List-I   **List-II**

(P) a equals   (1) $(-2, 4)$

(Q) r equals   (2) $\sqrt{5}$

(R) A_1 equals   (3) $(-2, 6)$

(S) B_1 equals   (4) 5

(5) $(2, 4)$

The correct option is

(a) $(P) \to (4)$ $(Q) \to (2)$ $(R) \to (1)$ $(S) \to (3)$

(b) $(P) \to (2)$ $(Q) \to (4)$ $(R) \to (1)$ $(S) \to (3)$

(c) $(P) \to (4)$ $(Q) \to (2)$ $(R) \to (5)$ $(S) \to (3)$

(d) $(P) \to (2)$ $(Q) \to (4)$ $(R) \to (3)$ $(S) \to (5)$

16. Let $\gamma \in \mathbb{R}$ be such that the lines

$$L_1 : \frac{x + 11}{1} = \frac{y + 21}{2} = \frac{z + 29}{3}$$

and

$$L_2 : \frac{x + 16}{3} = \frac{y + 11}{2} = \frac{z + 4}{\gamma}$$

intersect. Let R_1 be the point of intersection of L_1 and L_2. Let $O = (0, 0, 0)$ and $\hat{n}$ denote a unit normal vector to the plane containing both the lines L_1 and L_2.

Match each entry in List-I to the correct entry in List-II.

List-I

(P) γ equals

(Q) A possible choice for $\hat{n}$ is

(R) $\mathbf{OR}_1$ equals

(S) A possible value of $\mathbf{OR}_1 \cdot \hat{n}$ is

List-II

(1) $-\hat{i} - \hat{j} + \hat{k}$

(2) $\sqrt{\frac{3}{2}}$

(3) 1

(4) $\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$

(5) $\sqrt{\frac{2}{3}}$

The correct option is

(a) $(P) \to (3)$ $(Q) \to (4)$ $(R) \to (1)$ $(S) \to (2)$

(b) $(P) \to (5)$ $(Q) \to (4)$ $(R) \to (1)$ $(S) \to (2)$

(c) $(P) \to (3)$ $(Q) \to (4)$ $(R) \to (1)$ $(S) \to (5)$

(d) $(P) \to (3)$ $(Q) \to (1)$ $(R) \to (4)$ $(S) \to (5)$

17. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$g(x) = \begin{cases} 1 - 2x, & 0 \le x \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a, b, c, d \in \mathbb{R}$. Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = af(x) + b(g(x) + g(1-x)) + c(x - g(x) + g(1-x)) + dg(x), \quad x \in \mathbb{R}.$$

Match each entry in List-I to the correct entry in List-II.

List-I	List-II
(P) If $a = 0, b = 1, c = 0$, and $d = 0$, then	(1) h is one-one.
(Q) If $a = 1, b = 0, c = 0$, and $d = 0$, then	(2) h is onto.
(R) If $a = 0, b = 0, c = 1$, and $d = 0$, then	(3) h is differentiable on $\mathbb{R}$.
(S) If $a = 0, b = 0, c = 0$, and $d = 1$, then	(4) The range of h is $[0, 1]$.
	(5) The range of h is $\{0, 1\}$.

The correct option is

(a) $(P) \to (4)$ $(Q) \to (3)$ $(R) \to (1)$ $(S) \to (2)$

(b) $(P) \to (5)$ $(Q) \to (2)$ $(R) \to (4)$ $(S) \to (3)$

(c) $(P) \to (5)$ $(Q) \to (3)$ $(R) \to (2)$ $(S) \to (4)$

(d) $(P) \to (4)$ $(Q) \to (2)$ $(R) \to (1)$ $(S) \to (3)$